MATH 20510. Analysis in $\mathbb{R}^n$ III (accelerated)

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The University of Chicago – Spring 2025

Any proof or argument that has been filled in, expanded, or written out in detail by me is marked with a ■. All other material follows the lectures and any errors or omissions are entirely my own.

Contents

1 Measure and Integration

Definition 1.1. A family of sets $\mathscr{A}$ is called a *ring* if, for every $A, B \in \mathscr{A}$,

(i) $A \cup B \in \mathscr{A}$,
(ii) $A \setminus B \in \mathscr{A}$.

Definition 1.2. A ring $\mathscr{A}$ is called a *σ-ring* if for any $\{A_n\}_1^\infty \subseteq \mathscr{A}$,

$$\bigcup_1^\infty A_n \in \mathscr{A}.$$

Definition 1.3. ϕ is a *set function* on a ring $\mathscr{A}$ if for every $A \in \mathscr{A}$,

$$\phi(A) \in [-\infty, \infty].$$

Definition 1.4. A set function ϕ is *additive* if for any $A, B \in \mathscr{A}$ such that $A \cap B = \emptyset$,

$$\phi(A \cup B) = \phi(A) + \phi(B).$$

Definition 1.5. A set function ϕ is *countably additive* if for any $\{A_n\} \subseteq \mathscr{A}$ such that $A_i \cap A_j = \emptyset$, $\forall i \neq j$,

$$\phi\left(\bigcup_1^n A_n\right) = \sum_1^n \phi(A_n).$$

Note. In the last two definitions we assume that there are no $A, B \in \mathscr{A}$ such that $\phi(A) = -\infty, \phi(B) = \infty$.

Remark 1.6. If ϕ is an additive set function,

(i) $\phi(\emptyset) = 0$.
(ii) If $A_1, \ldots, A_n$ are pairwise disjoint then $\phi(\bigcup_1^n A_n) = \sum_1^n \phi(A_n)$.
(iii) $\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2)$.
(iv) If ϕ is nonnegative and $A_1 \subseteq A_2$ then $\phi(A_1) \leq \phi(A_2)$.
(v) If $B \subseteq A$ and $|\phi(B)| < \infty$ then $\phi(A \setminus B) = \phi(A) - \phi(B)$.

Theorem 1.7. *Let ϕ be a countably additive set function on a ring $\mathscr{A}$. Suppose $\{A_n\} \subseteq \mathscr{A}$ such that $A_1 \subseteq A_2 \subseteq \ldots$ and $A = \bigcup_1^\infty A_n \in \mathscr{A}$. Then $\phi(A_n) \to \phi(A)$ as $n \to \infty$.*

Proof. Set $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$. Note

(i) $\{B_n\}$ is pairwise disjoint.
(ii) $A_n = B_1 \cup B_2 \cup \cdots \cup B_n$.
(iii) $A = \bigcup_1^\infty B_n$.

Hence $\phi(A_n) = \sum_1^\infty \phi(B_j)$, $\phi(A) = \sum_1^\infty \phi(B_j)$ and the conclusion follows. □

Definition 1.8. An *interval* $I = \{(a_i, b_i)\}_1^n$ of $\mathbb{R}^n$ is the set of points $x = (x_1, \dots, x_n)$ such that $a_i \le x_i \le b_i$ or $a_i < x_i \le b_i$, etc. where $a_i \le b_i$.

Note. $\emptyset$ is an interval.

Definition 1.9. If A is the union of a finite number of intervals, we say A is *elementary.*

We denote the set of elementary sets by $\mathscr{E}$.

Definition 1.10. If I is an interval of $\mathbb{R}^n$, we define the volume of I by

$$\operatorname{vol}(I) = \prod_i^n (b_i - a_i).$$

If $A = I_1 \cup I_2 \cup \cdots \cup I_k$ is elementary, and the intervals are disjoint, then

$$\operatorname{vol}(A) = \sum_1^k \operatorname{vol}(I_j).$$

Remark 1.11.

(i) $\mathscr{E}$ is a ring, but not a σ-ring.
(ii) If $A \in \mathscr{E}$, then A can be written as a finite union of disjoint intervals.
(iii) If $A \in \mathscr{E}$, then $\operatorname{vol}(A)$ is well-defined.
(iv) vol is an additive set function on $\mathscr{E}$, and $\operatorname{vol} \ge 0$.

Definition 1.12. A nonnegative set function ϕ on $\mathscr{E}$ is *regular* if $\forall A \in \mathscr{E}$, $\forall \varepsilon > 0$, $\exists$ open $G \in \mathscr{E}$, $G \supseteq A$ and closed $F \in \mathscr{E}$, $F \subseteq A$, such that

$$\phi(G) \le \phi(A) + \varepsilon, \qquad \phi(A) \le \phi(F) + \varepsilon.$$

Note. vol is regular.

Definition 1.13. A *countable open cover* of $E \subseteq \mathbb{R}^n$ is a collection of open elementary sets $\{A_n\}$ such that $E \subseteq \bigcup_1^\infty A_n$.

Definition 1.14. The *Lebesgue outer measure* of $E \subseteq \mathbb{R}^n$ is defined as

$$m^*(E) = \inf \sum_1^\infty \operatorname{vol}(A_n).$$

where inf is taken over all countable open covers of E.

Remark 1.15.

(i) $\mathrm{m}^*(E)$ is well-defined.
(ii) $m^*(E) \ge 0$.

(iii) If $E_1 \subseteq E_2$ then $m^*(E_1) \leq m^*(E_2)$.

Theorem 1.16.

(i) If $A \in \mathscr{E}$, then $m^*(A) = \mathrm{vol}(A)$.

(ii) If $E = \bigcup_1^\infty E_n$ then $m^*(E) \leq \sum_1^\infty m^*(E_n)$.

Proof. (i) Let $A \in \mathscr{E}$ and $\varepsilon > 0$. Since vol is regular, $\exists$ open $G \in \mathscr{E}$ such that $A \subseteq G$ and $\mathrm{vol}(G) \leq \mathrm{vol}(A) + \varepsilon$. Since $G \supseteq A$ and $G \in \mathscr{E}$ is open, $m^*(A) \leq \mathrm{vol}(G) \leq \mathrm{vol}(A) + \varepsilon$. There also $\exists$ closed $F \in \mathscr{E}$ such that $F \subseteq A$ and $\mathrm{vol}(A) \leq \mathrm{vol}(F) + \varepsilon$. By definition, $\exists$ collection $\{A_n\}$ of open elementary sets such that $A \subseteq \bigcup A_n$ and $\sum_1^\infty \mathrm{vol}(A_n) \leq m^*(A) + \varepsilon$. Since $F \subseteq \bigcup A_n$ and F is compact, $F \subseteq A_1 \cup \cdots \cup A_N$ from some N.

$$\begin{aligned}
\mathrm{vol}(A) &\leq \mathrm{vol}(F) + \varepsilon \\
&\leq \mathrm{vol}(A_1 \cup \cdots \cup A_N) + \varepsilon \\
&\leq \sum_1^N \mathrm{vol}(A_n) + \varepsilon \\
&\leq \sum_1^\infty \mathrm{vol}(A_n) + \varepsilon \\
&\leq m^*(A) + \varepsilon + \varepsilon \\
&= m^*(A) + 2\varepsilon.
\end{aligned}$$

Since ε was arbitrary, $m^*(A) = \mathrm{vol}(A)$. □

Proof. (ii) If $m^*(E_n) = \infty$ for any $n \in \mathbb{N}$, then we are done. Assume not. Let $\varepsilon > 0$. For every $n \in \mathbb{N}$, $\exists$ open cover of E_n, $\{A_{n,k}\}_{k=1}^\infty$ such that

$$\sum_{k=1}^\infty \mathrm{vol}(A_{n,k}) \leq m^*(E_n) + \varepsilon/2^n.$$

Then $E \subseteq \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty A_{n,k}$, and so

$$\begin{aligned}
m^*(E) &\leq \sum_{n=1}^\infty \sum_{k=1}^\infty \mathrm{vol}(A_{n,k}) \\
&\leq \sum_{n=1}^\infty m^*(E_n) + \varepsilon/2^n \\
&= \sum_{n=1}^\infty m^*(E_n) + \sum_1^\infty \varepsilon/2^n \\
&= \sum_1^\infty m^*(E_n) + \varepsilon.
\end{aligned}$$

□

Definition 1.17. Let $A, B \subseteq \mathbb{R}^n$.

(i) $A \triangle B = (A \setminus B) \cup (B \setminus A)$.
(ii) $d(A, B) = m^*(A \triangle B)$.
(iii) We say $A_n \to A$ if $\lim_{n\to\infty} d(A_n, A) = 0$.

Definition 1.18. If there is a sequence of elementary sets $\{A_n\}$ such that $A_n \to A$ then we say A is *finitely* m*-measurable* and we write $A \in \mathfrak{M}_F(m)$.

Definition 1.19. If A is the countable union of finitely m-measurable sets, we say that A is m*-measurable* (Lebesgue measurable) and we write $A \in \mathfrak{M}(m)$.

Theorem 1.20. *$\mathfrak{M}(m)$ is a σ-ring and m^* is countably additive on $\mathfrak{M}(m)$.*

Definition 1.21. The *Lebesgue measure* is the set function defined on $\mathfrak{M}(m)$ by

$$m(A) = m^*(A), \quad \forall A \in \mathfrak{M}(m).$$

To summarize,

set function	**domain**	**properties**
vol	$\mathcal{E}$	≥ 0, additive, $\mathcal{E}$-regular.
m^*	$\subseteq \mathbb{R}^n$	≥ 0, $m^*(A) = \mathrm{vol}(A)\ \forall A \in \mathcal{E}$, countably subadditive.
m	$\mathcal{M}(m)$	≥ 0, $m(E) = m^*(E)\ \forall E \in \mathcal{M}(m)$, countably additive(!)

Example 1.22. Fix $n \in \mathbb{N}$.

(i) If $A \in \mathscr{E}$ then $A \in \mathfrak{M}(m)$ since $m^*(A \triangle A) = m^*(\emptyset) = 0 \implies A \to A$.
(ii) $\mathbb{R}^n \in \mathfrak{M}(m)$ since $\mathbb{R}^n = \bigcup_{N\in\mathbb{N}}[-N, N]^n \implies m(\mathbb{R}^n) = \infty$.
(iii) If $A \in \mathfrak{M}(m)$ then $A^c \in \mathfrak{M}(m)$.
(iv) $\forall x \in \mathbb{R}^n$, $\{x\} \in \mathfrak{M}(m)$ and $m(\{x\}) = 0$.
(v) $\forall x_1, \ldots, x_n \in \mathbb{R}^n$, $\{x_1, \ldots, x_n\} \in \mathfrak{M}(m)$ and $m(\{x_1, \ldots, x_n\}) = 0 \implies m(\mathbb{Q}^n) = 0$.

Definition 1.23. $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, \infty\}$ is *measurable* if $\{x \in \mathbb{R}^n : f(x) > a\} \in \mathfrak{M}$, $\forall a \in \mathbb{R}$, i.e. $f^{-1}((a, \infty]) \in \mathfrak{M}$, $\forall a \in \mathbb{R}$.

Example 1.24. f continuous $\implies f$ measurable.

Theorem 1.25. *The following are equivalent,*

(i) $\{x : f(x) > a\}$ *is measurable* $\forall a \in \mathbb{R}$.
(ii) $\{x : f(x) \geq a\}$ *is measurable* $\forall a \in \mathbb{R}$.
(iii) $\{x : f(x) < a\}$ *is measurable* $\forall a \in \mathbb{R}$.
(iv) $\{x : f(x) \leq a\}$ *is measurable* $\forall a \in \mathbb{R}$.

Proof. (i) $\implies$ (ii).

$$\{x : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \left\{x : f(x) > a - \frac{1}{n}\right\}. \qquad \square$$

Theorem 1.26. *If f is measurable then $|f|$ is measurable.*

Proof. It suffices to show that $\{x : |f(x)| < a\} \in \mathfrak{M}$, $\forall a \in \mathbb{R}$.

$$\{x : |f(x)| < a\} = \{x : f(x) < a\} \cap \{x : f(x) > -a\}. \qquad \square$$

Theorem 1.27. *Suppose $\{f_n\}$ is a sequence of measurable functions. Define*

$$g = \sup_n f_n \quad \text{and} \quad h = \limsup_{n\to\infty} f_n.$$

Then g, h are measurable.

Proof. $\{x : g(x) > a\} = \bigcup_{n=1}^{\infty}\{x : f_n(x) > a\}$ implies g is measurable. Similarly, $\inf_n f_n$ is measurable. Define $g_n = \sup_{n\geq m} f_n$ and note that g_n is measurable for all m. Since $h = \inf_m g_m$, h is measurable. $\square$

Corollary 1.28. *If f, g are measurable then* $\max\{f, g\}$ *and* $\min\{f, g\}$ *are also measurable.*

Corollary 1.29. *Define $f^+ = \max\{f, 0\}$, $f^- = -\min\{f, 0\}$. Then if f is measurable, f^+, f^- are also measurable.*

Corollary 1.30. *If $\{f_n\}$ is a sequence of measurable functions such that f_n converges to f pointwise, then f is measurable.*

Theorem 1.31. *$f, g : \mathbb{R}^n \to \mathbb{R}$ measurable, $F : \mathbb{R}^2 \to \mathbb{R}$ continuous, and $h(x) = F(f(x), g(x))$. Then h is measurable. In particular, this tells us that $f + g$ and fg are measurable.*

Definition 1.32. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *simple* if range(f) is a finite set.

Example 1.33. Let $E \subseteq \mathbb{R}^n$. The characteristic function of E is

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose f is simple, so range$(f) = \{c_1, \ldots, c_m\}$. Let $E_i = \{x : f(x) = c_i\}$. Then

$$f = \sum_{1}^{m} \chi_{E_i} c_i.$$

Theorem 1.34. $f : \mathbb{R}^n \to \mathbb{R}$. *There exists a sequence $\{f_n\}$ of simple functions such that $f_n \to f$ pointwise.*

(i) If f is measurable, $\{f_n\}$ can be chosen to be measurable.
(ii) If $f \geq 0$ then $\{f_n\}$ can be chosen to be monotonically increasing.

Proof. If $f \geq 0$, define the sets

$$E_{n,i} = \left\{x : \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}\right\}, \quad n \geq 1, i = 1, \ldots, n2^n,$$

and

$$F_n = \{x \mid f(x) \geq n\}, \quad n \geq 1.$$

Let

$$f_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}} + n\chi_{F_n}.$$

We see that f_n is measurable. Fix $x \in \mathbb{R}^n$, let $\varepsilon > 0$, and let $N \in \mathbb{N}$ such that $N > f(x)$ and $2^{-N} < \varepsilon$. Let $n \geq N$. Note that $x \in E_{n,i}$ for some i. Since $f_n(x) = \frac{i-1}{2^n}$ and $f(x) \geq f_n(x)$, $f(x) - f_n(x) \leq \frac{1}{2^n} < \varepsilon$. Thus $f_n \to f$ pointwise. We now show $\{f_n\}$ is monotonically increasing.

(i) Case 1: $x \in F_n$. Then $f(x) \geq n$ and $f_n(x) = n$. If $x \in F_{n+1}$, then $f_{n+1}(x) = n + 1 > n = f_n(x)$. If $x \notin F_{n+1}$ then $x \in$ some $E_{n+1,i}$. Then $\frac{i-1}{2^{n+1}} \geq n \implies f_{n+1}(x) \geq n = f_n(x)$.
(ii) Case 2: $x \in E_{n,i}$ for some i. Then $f_n(x) = \frac{i-1}{2^n}$. Then there is some j such that $x \in E_{n+1,j} = \{x : \frac{j-1}{2^{n+1}} \leq f(x) \leq \frac{j}{2^{n+1}}\}$. Because $\frac{i-1}{2^n} \leq f(x)$, we have $\frac{j-1}{2^{n+1}} \geq \frac{i-1}{2^n}$ so $f_{n+1}(x) = \frac{j-1}{2^{n+1}} \geq \frac{i-1}{2^n} = f_n(x)$.

Thus in both cases, $\{f_n\}$ is monotonically increasing. We next consider the general case. Given f, we write $f^+(x) = \max\{f(x), 0\}$ and $f^- = -\min\{f(x), 0\}$ so that $f = f^+ - f^-$ and $f^+, f^- \geq 0$. By the previous part, there exist two sequences of nonnegative measurable simple functions $f_n^+ \to f^+$ and $f_n^- \to f^-$ each converging pointwise. Define $f_n(x) = f_n^+(x) - f_n^-(x)$. Then f_n is simple and measurable since it is the difference of two simple measurable functions, and converges pointwise. ■

Definition 1.35. (Lebesgue Integration) Suppose $g = \sum_{i=1}^k c_i \chi_{E_i}$, $c_i > 0$ is measurable and $E \in \mathfrak{M}$. Define

$$I_E(g) = \sum_1^k c_i m(E_i \cap E).$$

Let f be a nonnegative measurable function, $E \in \mathfrak{M}$. Define

$$\int_E f dm = \sup I_E(g),$$

where sup is taken over all measurable simple functions g such that $0 \leq g \leq f$.

Remark 1.36.

(i) $\int_E f dm$ is the Lebesgue integral of f over E.
(ii) It can take value ∞.
(iii) If f is measurable, simple, and nonnegative, then

$$\int_E f dm = I_E(f).$$

Proof. of remark (iii). Suppose for the sake of contradiction that there exists g simple, nonnegative, and measurable such that $0 \le g \le f$ and $I_E(g) > I_E(f)$. Then

$$g = \sum_1^k c_i \chi_{E_i}, \quad f = \sum_1^k d_j \chi_{F_j},$$

and

$$I_E(g) = \sum_1^k c_i m(E_i \cap E) > I_E(f) = \sum_1^k d_j m(F_j \cap E).$$

Let $H_{i,j} = E_i \cap F_j$. Since $g \le f$, $\forall i$, $E_i \subseteq \bigcup F_j$. Hence,

$$\begin{aligned} g &= \sum_{i=1}^k \sum_{j=1}^k c_i \chi_{E_i \cap F_j} \\ &= \sum_{n=1}^M c_n \chi_{H_n}. \end{aligned}$$

Note that for every n, $\exists$ unique $F_j \supseteq H_n$. This implies $c_n \le d_j$, which is a contradiction. $\square$

Definition 1.37. Let f be measurable, and consider $\int_E f^+ dm$ and $\int_E f^- dm$. If at least one is finite, define

$$\int_E f dm = \int_E f^+ dm - \int_E f^- dm.$$

If both $\int_E f^+ dm$ and $\int_E f^- dm$ are finite, we say that f is *integrable* on E and write $f \in \mathscr{L}$ on E.

Remark 1.38.

(i) If $a \le f(x) \le b$ for all $x \in E \in \mathfrak{M}$ and $m(E) < \infty$, then $am(E) \le \int_E f dm \le bm(E)$.
(ii) If f is bounded on $E \in \mathfrak{M}$ and $m(E) < \infty$, then $f \in \mathscr{L}$ on E.
(iii) If $f, g \in \mathscr{L}$ on E and $f(x) \le g(x)$ for all $x \in E$, then $\int_E f dm \le \int_E g dm$.
(iv) If $f \in \mathscr{L}$ on $E \in \mathfrak{M}$ and $c \in \mathbb{R}$ then $cf \in \mathscr{L}$ on E and $\int_E cf dm = c \int_E f dm$.
(v) If $m(E) = 0$ then $\int_E f dm = 0$.

(vi) If $f \in \mathscr{L}$ on E, $A \in \mathfrak{M}$, $A \subseteq E$, then $f \in \mathscr{L}$ on A.
(vii) If f is Riemann integrable on $[a, b]$ then $f \in \mathscr{L}$ on $[a, b]$ and the values of the integrals agree.

Proof. of remark (i). Assume $a \geq 0$. $\int_E f dm = \sup \int_E g dm$ where sup is taken over all simple measurable g such that $0 \leq g \leq f$. Let $g = a$ on E. Then $\int_E f dm \geq \int_E g dm = am(E)$. Let g be a measurable simple function such that $0 \leq g \leq f$. Then $g = \sum_1^k c_i \chi_{E_i}$ for distinct c_i's and measurable E_i that are disjoint. Since $g \leq f \leq b$, $c_i \leq b$ for all i. So

$$\begin{aligned} \int_E g dm &= \sum_1^k c_i m(E_i \cap E) \\ &\leq b \sum_1^k m(E_i \cap E) \\ &\leq bm(E). \end{aligned}$$

Hence, $\int_E f dm \leq bm(E)$. □

Theorem 1.39.

(i) Suppose f is nonnegative and measurable. For $A \in \mathfrak{M}$ define

$$\phi(A) = \int_A f dm.$$

Then ϕ is countably additive on $\mathfrak{M}$.

(ii) The same conclusion holds if $f \in \mathscr{L}$.

Proof. To prove (ii), it suffices to apply (i) to f^+ and f^-. Suppose $\{A_n\}$ is a sequence of measurable sets which are pairwise disjoint. Let $A = \bigcup A_n$.

Step 1 (Characteristic functions). Suppose $f = \chi_E$ for some $E \in \mathfrak{M}$. Then

$$\begin{aligned} \phi(A) &= \int_A f dm \\ &= m(A \cap E) \\ &= m\left(\left(\bigcup_1^\infty A_n\right) \cap E\right) \\ &= m\left(\bigcup_1^\infty (A_n \cap E)\right) \\ &= \sum_1^\infty m(A_n \cap E) \end{aligned}$$

$$= \sum_1^\infty \int_{A_n} f dm$$

$$= \sum_1^\infty \phi(A_n).$$

Step 2 (Simple functions). Suppose f is simple, measurable, and nonnegative, i.e., $f = \sum_1^k c_i \chi_{E_i}$ for disjoint E_i's in $\mathfrak{M}$. Then

$$\begin{aligned}
\phi(A) &= \int_A f dm \\
&= \sum_1^k c_i m(E_i \cap A) \\
&= \sum_1^k c_i \int_A \chi_{E_i} dm \\
&= \sum_1^k c_i \sum_1^\infty \int_{A_n} \chi_{E_i} dm \\
&= \sum_1^\infty \sum_1^k \int_{A_n} c_i \chi_{E_i} dm \\
&= \sum_1^\infty \int_{A_n} f dm \\
&= \sum_1^\infty \phi(A_n).
\end{aligned}$$

Step 3. Let g be a measurable simple function such that $0 \le g \le f$. Then

$$\begin{aligned}
\int_A g dm &= \sum_1^\infty \int_{A_n} g dm \\
&\le \sum_1^\infty \int_{A_n} f dm \\
&= \sum_1^\infty \phi(A_n).
\end{aligned}$$

Hence $\phi(A) = \int_A f dm \le \sum_1^\infty \phi(A_n)$.

If $\phi(A_n) = \infty$ for any n, then we are done. Thus assume $\phi(A_n) < \infty$ for every n. Let $\varepsilon > 0$, and choose measurable simple g such that $0 \le g \le f$ and $\int_{A_1} g dm \ge \int_{A_1} f dm - \varepsilon, \ldots, \int_{A_n} g dm \ge \int_{A_n} f dm - \varepsilon$. Hence

$$\phi(A_1 \cup \cdots \cup A_n) \ge \phi(A_1) + \cdots + \phi(A_n) - n\varepsilon.$$

Since ε was arbitrary, $\forall n$, $\phi(A_1 \cup \dots \cup A_n) \geq \phi(A_1) + \dots + \phi(A_n)$. $\square$

Corollary 1.40. *If $A, B \in \mathfrak{M}$, $m(A \setminus B) = 0$, and $B \subseteq A$, then*

$$\int_A f dm = \int_B f dm$$

for every $f \in \mathscr{L}$.

Theorem 1.41. *If $f \in \mathscr{L}$ on E, then $|f| \in \mathscr{L}$ on E and $|\int_E f dm| \leq \int_E |f| dm$.*

Proof. Let $A = \{x \in E \mid f(x) \geq 0\}$ and $B = \{x \in E \mid f(x) < 0\}$. Note that $E = A \sqcup B$ and $A, B \in \mathfrak{M}$. Then

$$\int_E |f| dm = \int_A |f| dm + \int_B |f| dm = \int_E f^+ dm + \int_E f^- dm < \infty.$$

Thus $|f| \in \mathscr{L}$. Since $f \leq |f|$ and $-f \leq |f|$, $\int_E f dm \leq \int_E f dm \leq \int_E |f| dm$, and $\int_E -f dm = -\int_E f dm \leq \int_E |f| dm$, so

$$\left| \int_E f dm \right| \leq \int_E |f| dm. \quad \square$$

Theorem 1.42. *(Lebesgue's monotone convergence theorem). Let $E \in \mathfrak{M}$ and $\{f_n\}$ a sequence of measurable functions such that*

$$0 \leq f_1(x) \leq f_2(x) \leq \dots \quad \forall (x \in E).$$

Define $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in E$. Then

$$\int_E f_n dm \to \int_E f dm \quad \text{as } n \to \infty.$$

Proof. Since $\{f_n\}$ is a monotone sequence of nonnegative measurable functions, $\{\int_E f_n dm\}$ is a monotone sequence of extended real numbers. Thus there must exist $\alpha \in \mathbb{R} \cup \{\pm\infty\}$ such that $\alpha = \lim_{n\to\infty} \int_E f_n dm$. Since $f_n \leq f$ for every n, $\alpha \leq \int_E f dm$. Let $0 < c < 1$ and g be a simple, measurable function such that $0 \leq g \leq f$. For every $n \geq 1$, define

$$E_n = \{x \in E \mid f_n(x) \geq c g(x)\}.$$

Since $\{f_n\}$ is increasing, $E_1 \subseteq E_2 \subseteq \dots$. Since $f_n \to f$ pointwise, $E = \bigcup_1^\infty E_n$. For every n, $cg \leq f_n$ on E_n, so

$$c \int_{E_n} g dm = \int_{E_n} c g dm \leq \int_{E_n} f_n dm.$$

As $n \to \infty$,

$$\int_{E_n} g dm \to \int_E g dm.$$

Therefore, $\alpha \geq c \int_E g dm$. Since $c < 1$ was arbitrary, $\alpha \geq \int_E g dm$. By definition of integration, $\alpha \geq \int_E f dm$. $\square$

Theorem 1.43. *Let $f = f_1 + f_2$, $f_1, f_2 \in \mathscr{L}$ on $E \in \mathfrak{M}$. Then $f \in \mathscr{L}$ on E and $\int_E f dm = \int_E f_1 dm + \int_E f_2 dm$.*

Proof. If f_1, f_2 are simple measurable functions, then the conclusion is immediate. Assume that $f_1, f_2 \geq 0$. Choose a monotonically increasing sequence of nonnegative measurable simple functions $\{g_n\}$ and $\{h_n\}$ converging to f_1 and f_2 respectively. Let $s_n = g_n + h_n$. Then $\forall n$,

$$\int_E s_n dm = \int_E g_n dm + \int_E h_n dm.$$

Note. $\{s_n\}$ is a monotonically increasing sequence of simple nonnegative measurable functions converging to f. By the monotone convergence theorem,

$$\int_E f dm = \lim_{n\to\infty} \int_E s_n dm = \lim_{n\to\infty} \int_E g_n dm + \int_E h_n dm = \int_E f_1 dm + \int_E f_2 dm.$$

Now assume $f_1 \geq 0, f_2 < 0$. Define

$$A = \{x \in E \mid f(x) \geq 0\} \quad \text{and} \quad B = \{x \in E \mid f(x) < 0\}.$$

Note that both A and B are measurable. Since $f, f_1, -f_2 \geq 0$ on A and $f_1 = f + (-f_2)$,

$$\int_A f_1 dm = \int_A f dm + \int_A -f_2 dm = \int_A f dm - \int_A f_2 dm,$$

i.e., $\int_A f dm = \int_A f_1 dm + \int_A f_2 dm$. Since $-f, f_1, -f_2 \geq 0$ on B,

$$\int_B -f_2 dm = \int_B -f dm + \int_B f_1 dm,$$

i.e., $\int_B f dm = \int_B f_1 dm + \int_B f_2 dm$. Hence,

$$\begin{aligned}
\int_{E=A\sqcup B} f dm &= \int_A f dm + \int_B f dm \\
&= \int_A f_1 dm + \int_A f_2 dm + \int_B f_1 dm + \int_B f_2 dm \\
&= \int_E f_1 dm + \int_E f_2 dm.
\end{aligned}$$

Let

$$\begin{aligned}
E_1 &= \{x \in E \mid f_1(x) \geq 0, f_2(x) \geq 0\}, \\
E_2 &= \{x \in E \mid f_1(x) \geq 0, f_2(x) < 0\}, \\
E_3 &= \{x \in E \mid f_1(x) < 0, f_2(x) \geq 0\}, \\
E_4 &= \{x \in E \mid f_1(x) < 0, f_2(x) < 0\}.
\end{aligned}$$

Apply what we've proven to all four sets and we get the generalized conclusion. $\square$

Lemma 1.44. *(Fatou's lemma) $E \in \mathfrak{M}$, $\{f_n\}$ nonnegative measurable functions. Let $f = \liminf_{n\to\infty} f_n$. Then*

$$\int_E f dm \leq \liminf_{n\to\infty} \int_E f_n dm.$$

Proof. For every $n \geq 1$, define

$$g_n = \inf_{m\geq n} f_n.$$

Note. the g_n's are measurable on E, and

(i) $0 \leq g_1 \leq g_2 \leq \ldots$.
(ii) $g_n \leq f_n$, $\forall n$.
(iii) $\lim_{n\to\infty} g_n(x) = f(x)$, $\forall x \in E$.

By the monotone convergence theorem,

$$\lim_{n\to\infty} \int_E g_n dm = \int_E f dm.$$

By property (ii),

$$\int_E g_n dm \leq \int_E f_n dm \quad \forall n.$$

Together, these two imply the conclusion. □

Theorem 1.45. *(Dominated convergence theorem) Suppose $E \in \mathfrak{M}$, $\{f_n\}$ measurable on E such that $f_n \to f$ pointwise on E. Suppose $\exists g \in \mathscr{L}$ on E such that $|f_n(x)| \leq g(x)$ for all $x \in E$. Then*

$$\int_E f dm = \lim_{n\to\infty} \int_E f_n dm.$$

Proof. Note $f_n \in \mathscr{L}$ on E for all n and $f \in \mathscr{L}$ on E. Since $f_n + g \geq 0$ for all n, applying Fatou's Lemma gives

$$\int_E (f+g) dm \leq \liminf_{n\to\infty} \int_E (f_n + g) dm.$$

Then

$$\begin{aligned}\int_E f dm + \int_E g dm &\leq \liminf_{n\to\infty} \left(\int_E f_n dm + \int_E g dm\right) \\ &= \left(\liminf_{n\to\infty} \int_E f_n dm\right) + \int_E g dm.\end{aligned}$$

Thus

$$\int_E f dm \leq \liminf_{n\to\infty} \int_E f_n dm.$$

Since $g - f_n \geq 0$, we apply Fatou's Lemma to get

$$\int_E (g - f)dm \leq \liminf_{n\to\infty} \left(\int_E (g - f_n)dm \right).$$

By the same logic as above, we see that

$$-\int_E f dm \leq \liminf_{n\to\infty} -\int_E f_n dm.$$

We conclude that

$$\int_E f \geq \limsup_{n\to\infty} \int_E f_n dm.$$

Thus,

$$\int_E f dm = \lim_{n\to\infty} \int_E f_n dm. \qquad \square$$

Lemma 1.46. *Nonmeasurable sets exist (assuming Axiom of Choice).*

Proof. For every $a \in [-1, 1]$ define $\tilde{a} = \{c \in [-1, 1] : a - c \in \mathbb{Q}\}$.

Claim 1. If $\tilde{a} \cap \tilde{b} \neq \emptyset$ then $\tilde{a} = \tilde{b}$.
Suppose $c \in \tilde{a} \cap \tilde{b}$. Then $a - c \in \mathbb{Q}$, $b - c \in \mathbb{Q}$, and therefore $a - b, b - a \in \mathbb{Q}$. Let $d \in \tilde{a}$, so $a - d \in \mathbb{Q}$. Then $a - d = (a - b) + (b - d)$ so $b - d \in \mathbb{Q}$, i.e., $d \in \tilde{b}$ and the claim follows.
Note. $[-1, 1] = \bigcup_{a\in[-1,1]} \tilde{a}$. Let V be a set that contains exactly one element from every distinct $\tilde{a}$ (Axiom of Choice). Let $r_1, r_2, \ldots$ be an enumeration of $\mathbb{Q} \cap [-2, 2]$.

Claim 2. $[-1, 1] \subseteq \bigcup_{k=1}^{\infty} V + r_k$.
Let $d \in [-1, 1]$, so $d \in \tilde{a}$ for some a. Let $c \in V$ s.t. $c \in \tilde{a}$. Then $c - d \in \mathbb{Q} \cap [-2, 2]$ so $c - d = r_k$ for some k. Hence, $d \in V + r_k$.

By Claim 2,

$$2 = m^*([-1, 1]) \leq m^* \left(\bigcup_1^{\infty} V + r_k \right) \leq \sum_1^{\infty} m^*(V + r_k) = \sum_1^{\infty} m^*(V).$$

Thus $m^*(V) > 0$.

Claim 3. $V + r_1, V + r_2, \ldots$ are disjoint.
Suppose for the sake of contradiction that $d \in (V + r_k) \cap (V + r_\ell)$. Then $d = v + r_k$, $v \in V$ and $d = v' + r_\ell$, $v' \in V$. In particular, $v - v' \in \mathbb{Q}$. By Claim 1, $v, v' \in \tilde{a}$. This is a contradiction.

For any $n \in \mathbb{N}$,

$$\bigcup_{k=1}^{n} V + r_k \subseteq [-3, 3].$$

Hence,

$$m^* \left(\bigcup_{1}^{\infty} V + r_k \right) \leq 6.$$

Let $n \in \mathbb{N}$ such that $nm^*(V) > 6$. Then

$$m^* \left(\bigcup_{1}^{n} V + r_k \right) < \sum_{1}^{n} m^*(V + r_k),$$

which implies that $V + r_1, V + r_2, \ldots$ cannot all be measurable. Hence, V is not measurable. $\square$

Definition 1.47. Let $E \in \mathfrak{M}$, f measurable. We write $f \in \mathscr{L}^2$ on E if

$$\int_E |f|^2 dm < \infty.$$

Remark 1.48. $f \in \mathscr{L}$ on E ($\mathscr{L}^1$) if $\int_E |f| dm < \infty$.

Example 1.49.

(i) $E = (0, 1]$, $f(x) = x^{-1/2}$. Then $f \in \mathscr{L}^1, f \notin \mathscr{L}^2$.
(ii) $E = (1, \infty)$, $f(x) = \frac{1}{x}$. Then $f \notin \mathscr{L}^1, f \in \mathscr{L}^2$.

Theorem 1.50. *If $m(E) < \infty$, then $f \in \mathscr{L}^2 \implies f \in \mathscr{L}^1$.*

2 Fourier Analysis

Recall. Let $f : \mathbb{R} \to \mathbb{C}$. We can decompose f into its real and imaginary components,

$$f = f_{RE} + i f_{IM},$$

where $f_{RE}, f_{IM} : \mathbb{R} \to \mathbb{R}$.
We say $f \in \mathscr{R}$ (Riemann integrable) if $f_{RE}, f_{IM} \in \mathscr{R}$ and

$$\int_{-\infty}^{\infty} f dx = \int_{-\infty}^{\infty} f_{RE} dx + i \int_{-\infty}^{\infty} f_{IM} dx.$$

Definition 2.1. A *trigonometric polynomial* is a function

$$f(x) = a_0 + \sum_{1}^{N} a_n \cos(nx) + b_n \sin(nx),$$

where $a_0, \ldots, a_N, b_1, \ldots, b_N \in \mathbb{C}$.

Note. Using Euler's formula, we can equivalently write this as

$$f(x) = \sum_{-N}^{N} c_n e^{inx},$$

where $c_{-N}, \ldots, c_N \in \mathbb{C}$.
We discuss 2π-periodic functions defined on intervals $[a, b]$ of length 2π.

Definition 2.2. Let $f \in \mathscr{R}$ on $[a, a+2\pi]$, $n \in \mathbb{Z}$. The *n-th Fourier coefficient of f* is

$$\hat{f}(n) = \frac{1}{2\pi} \int_{a}^{a+2\pi} f(x) e^{-inx} dx.$$

Definition 2.3. The *Fourier series of f* is given (formally) by

$$f \sim \sum_{-\infty}^{\infty} \hat{f}(n) e^{inx}.$$

Definition 2.4. The *N-th partial sum of f* is

$$s_N(f) = \sum_{-N}^{N} \hat{f}(n) e^{inx}.$$

Note. If $n \in \mathbb{Z} - \{0\}$, e^{inx} is the derivative of $\frac{e^{inx}}{in}$ (which is 2π-periodic). Therefore,

$$\frac{1}{2\pi} \int_{a}^{a+2\pi} e^{inx} dx = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Example 2.5. Suppose $f(x) = \sum_{-N}^{N} c_n e^{inx}$. Let $|m| \leq N$. Then

$$\begin{aligned}
\hat{f}(m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{-N}^{N} c_n e^{inx}\right)e^{-imx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{-N}^{N} c_n e^{ix(n-m)}\right)dx \\
&= \frac{1}{2\pi}\sum_{-N}^{N} c_n \int_{-\pi}^{\pi} e^{ix(n-m)}dx \\
&= \frac{1}{2\pi}(c_m 2\pi) \\
&= c_m.
\end{aligned}$$

Note. If $|m| > N$ then $\hat{f}(m) = 0$.
Hence, $f(x) = \sum_{-\infty}^{\infty} \hat{f}(n)e^{inx} = \sum_{-N}^{N} \hat{f}(n)e^{inx} = s_N(f)$.

Question. In what sense does $s_N(f) \to f$ as $N \to \infty$?

Example 2.6. Let $f(x) = x$ on $[-\pi, \pi]$.

$$\hat{f}(0) = 0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x dx.$$

For $n \neq 0$,

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx} \\
&= \frac{1}{2\pi}\left[\frac{xe^{inx}}{in}\right]_{-\pi}^{\pi} + \frac{1}{2\pi in}\int_{-\pi}^{\pi} e^{-inx}dx \\
&= \frac{(-1)^{n+1}}{in}.
\end{aligned}$$

Fourier series of f is

$$\sum_{n\neq 0}\frac{(-1)^{n+1}}{in}e^{inx} = 2\sum_{1}^{\infty}(-1)^{n+1}\frac{\sin(nx)}{n}.$$

In this example, $s_N(f) \to f$ uniformly.

Overview.

(i) Let f be (Riemann) integrable in $[a, a+2\pi]$. Does $s_N(f) \to f$ pointwise? NO
(ii) What about if f is continuous (and periodic)? NO
(iii) What if $f \in C^1$ (and periodic)? YES

Motivating question. If f is 2π-periodic, when can we prove that $s_N(f) \to f$ pointwise (uniformly)?

Theorem 2.7. *Suppose $f \in \mathscr{R}$ on $[0, 2\pi]$, f is 2π-periodic, $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f(x) = 0$ $\forall x$ at which f is continuous.*

Corollary 2.8. *If f is continuous, 2π-periodic, and $\hat{f}(n) = 0$ $\forall n \in \mathbb{Z}$, then $f = 0$.*

Corollary 2.9. *If f, g are continuous, 2π-periodic, and $\hat{f}(n) = \hat{g}(n)$ $\forall n \in \mathbb{Z}$, then $f = g$.*

Corollary 2.10. *Suppose f is continuous, 2π-periodic, and the Fourier series of f converges absolutely, i.e.,*

$$\sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

Then $\lim_{N\to\infty} S_N(f)(x) = f(x)$ uniformly.

Proof. Since $\sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty$, the partial sums $S_N(f)$ converge uniformly. Define

$$g(x) = \sum_{-\infty}^{\infty} \hat{f}(n)e^{inx} = \lim_{N\to\infty} \sum_{-N}^{N} \hat{f}(n)e^{inx}.$$

Since g is the uniform limit of continuous functions, g is continuous. Moreover, $\forall n \in \mathbb{Z}$,

$$\begin{aligned}
\hat{g}(n) &= \frac{1}{2\pi}\int_0^{2\pi} g(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{-\infty}^{\infty} \hat{f}(m)e^{imx}\right) e^{-inx}dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{-\infty}^{\infty} \hat{f}(m)e^{ix(m-n)}\right) dx \\
&= \sum_{m=-\infty}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} \hat{f}(m)e^{ix(m-n)}dx \\
&= \sum_{m=-\infty}^{\infty} \frac{1}{2\pi}\hat{f}(m)\int_0^{2\pi} e^{ix(m-n)}dx \\
&= \hat{f}(n).
\end{aligned}$$

Hence $f = g$. $\square$

Lemma 2.11. *Suppose f is C^2 and 2π-periodic. Then $\exists c > 0$ such that for all sufficiently large $|n|$,*

$$|\hat{f}(n)| \leq \frac{c}{|n|^2},$$

i.e., $|\hat{f}(n)| = O\left(\frac{1}{n^2}\right)$.

Proof. By integration by parts (twice),

$$\begin{aligned}
2\pi\hat{f}(n) &= \int_0^{2\pi} f(x)e^{-inx}dx \\
&= f(x)\left[\frac{e^{inx}}{in}\right]_0^{2\pi} + \frac{1}{in}\int_0^{2\pi} f'(x)e^{-inx}dx \\
&= \frac{1}{in}\left[-f'(x)\frac{e^{-inx}}{in}\right]_0^{2\pi} + \frac{1}{(in)^2}\int_0^{2\pi} f''(x)e^{-inx}dx \\
&= -\frac{1}{n^2}\int_0^{2\pi} f''(x)e^{-inx}dx.
\end{aligned}$$

Hence,

$$2\pi\hat{f}(n) = \frac{1}{|n|^2}\left|\int_0^{2\pi} f''(x)e^{-inx}dx\right|. \tag{1}$$

Then

$$\begin{aligned}
\text{RHS of (1)} &\leq \frac{1}{|n|^2}\int_0^{2\pi} |f''(x)||e^{-inx}dx \\
&= \frac{1}{|n|^2}\int_0^{2\pi} |f''(x)|dx \\
&\leq \frac{1}{|n|^2}2\pi c,
\end{aligned}$$

where $c = \max_{x\in[0,2\pi]} |f''(x)|$. Therefore, $|\hat{f}(n)| \leq \frac{c}{|n|^2}$. $\square$

Note. We showed within the above proof that if f is C^1, $\hat{f'}(n) = in\hat{f}(n)$.
Next question. If f is 2π-periodic and $\int_0^{2\pi} |f|^2 dx$ exists, under what type of convergence does $s_N(f) \to f$?

Theorem 2.12. *Let f be a complex valued, 2π-periodic, (Riemann) integrable function. Then*

$$\lim_{N\to\infty}\int_0^{2\pi} |f(x) - s_N(f)(x)|^2 dx = 0.$$

Definition 2.13. A *vector space* over $\mathbb{C}$ is a set V of vectors, operations $\cdot, +$ such that $\forall x, y, z \in V$, $\forall \lambda_1, \lambda_2 \in \mathbb{C}$,

(i) $x + y \in V$.
(ii) $x + y = y + x$.
(iii) $x + (y + z) = (x + y) + z$.
(iv) $\lambda_1 x \in V$.
(v) $\lambda_1(x + y) = \lambda_1 x + \lambda_1 y$.
(vi) $(\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x$.
(vii) $\lambda_1(\lambda_2 x) = (\lambda_1 \lambda_2)x$.

In addition, $\exists 0 \in V$ such that $x + 0 = x$ $\forall x$. $\forall x \in V, \exists(-x) \in V$ such that $x + (-x) = 0$. $\exists 1 \in V$ such that $1 \cdot x = x$.

Definition 2.14. An *inner product* of a vector space V is a map $(\cdot, \cdot) : V \times V \to \mathbb{C}$ satisfying

(i) $(x, y) = \overline{(y, x)}$.
(ii) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$.
(iii) $(x, \alpha y + \beta z) = \overline{\alpha}(x, y) + \overline{\beta}(x, z)$.
(iv) $(x, x) \geq 0$.

Definition 2.15. Given an inner product $(\cdot, \cdot)$, we can define a *norm* on V,

$$\|x\| = (x, x)^{\frac{1}{2}}.$$

Definition 2.16. We say that x, y are *orthogonal* if $(x, y) = 0$, and we write $x \perp y$.

Example 2.17. $V = \mathbb{C}$, $(x, y) = x\overline{y}$.

Example 2.18. $V = \mathbb{R}^n$, $(x, y) = x \cdot y$.

Example 2.19. Let $\mathcal{R}$ be the set of complex-valued, 2π-periodic (Riemann) integrable functions. This is a vector space over $\mathbb{C}$.

(i) $(f + g)(x) = f(x) + g(x)$.
(ii) $(\lambda f)(x) = \lambda f(x)$.

Define the inner product

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)} dx,$$

so the norm is

$$\|f\| = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx \right)^{\frac{1}{2}}.$$

Three properties. Let V be an inner product space.

(i) Pythagorean Theorem. If $x \perp y$ then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

(ii) Cauchy-Schwarz. For any $x, y \in V$,

$$|(x,y)| \leq \|x\|\|y\|.$$

(iii) Triangle inequality. For any $x, y \in V$,

$$\|x+y\| \leq \|x\| + \|y\|.$$

Notation. In the rest of this section, we will write $e_n(x) = e^{inx}$.
Observation. The family $\{e_n\}_{n\in\mathbb{Z}}$ is *orthonormal*, i.e.,

$$(e_n, e_m) = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases}$$

So $e_n \perp e_m$ if $n \neq m$ and $\|e_n\| = 1$ $\forall n \in \mathbb{Z}$. Moreover, $\forall f \in \mathcal{R}$, $n \in \mathbb{Z}$,

$$\begin{aligned} (f, e_n) &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{e_n(x)}dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx \\ &= \hat{f}(n). \end{aligned}$$

Then

$$\begin{aligned} s_N(f) &= \sum_{-N}^{N} \hat{f}(n)e_n \\ &= \sum_{-N}^{N} (f, e_n)e_n. \end{aligned}$$

Note. $\forall |m| \leq N$,

$$(f - s_N(f)) \perp e_m.$$

We see this since

$$\begin{aligned} (f - s_N(f), e_m) &= (f, e_m) - (s_N(f), e_m) \\ &= (f, e_m) - \sum_{-N}^{N} ((f, e_n)e_n, e_m) \end{aligned}$$

$$
\begin{aligned}
&= (f, e_m) - \sum_{-N}^{N} (f, e_m)(e_n, e_m) \\
&= (f, e_m) - (f, e_m) \\
&= 0.
\end{aligned}
$$

Corollary 2.20. *For every* $\{e_n\}_{-N}^{N}$*,*

$$(f - s_N(f)) \perp \sum_{-N}^{N} c_n e_n.$$

Then, $f = f - s_N(f) + s_N(f)$, so by the Pythagorean theorem,

$$\|f\|^2 = \|f - s_N(f)\|^2 + \|s_N(f)\|^2.$$

Since $s_N(f) = \sum_{-N}^{N} \hat{f}(n) e_n$,

$$
\begin{aligned}
\|s_N(f)\|^2 &= \sum_{-N}^{N} \|\hat{f}(n) e_n\|^2 \\
&= \sum_{-N}^{N} \|\hat{f}(n)\|^2,
\end{aligned}
$$

and thus

$$\|f\|^2 = \|f - s_N(f)\|^2 + \sum_{-N}^{N} \|\hat{f}(n)\|^2. \tag{2}$$

Lemma 2.21. *(Best approximation)* $f \in \mathcal{R}$*. Then*

$$\|f - s_N(f)\| \leq \left\| f - \sum_{-N}^{N} c_n e_n \right\|$$

for any complex numbers $\{c_n\}_{-N}^{N}$*.*

Theorem 2.22. *If* $f \in \mathcal{R}$ *then*

$$\lim_{N \to \infty} \int_0^{2\pi} |f - s_N(f)|^2 dx = 0.$$

Proof. Let $f \in \mathcal{R}$ be continuous. By (a version of) the Stone-Weierstrass theorem, $\forall \varepsilon > 0$, $\exists$ trigonometric polynomial P such that $|f(x) - P(x)| < \varepsilon$, $\forall x \in [0, 2\pi]$.

$$\|f - P\| = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x) - P(x)|^2 dx \right)^{1/2}$$

$$
\begin{aligned}
&< \left(\frac{1}{2\pi} \int_0^{2\pi} \varepsilon^2 dx \right)^{1/2} \\
&= \varepsilon.
\end{aligned}
$$

Let M be the degree of P, i.e., $P = \sum_{-M}^{M} c_n e_n$. By the best approximation lemma, $\forall N \geq M$,

$$\|f - s_N(f)\| \leq \|f - P\| < \varepsilon.$$

Hence, $\forall \varepsilon > 0$, $\exists M$ such that $\forall N \geq M$, $\|f - s_N(f)\| < \varepsilon$. Now we drop the condition that f is continuous. For every $\varepsilon > 0$, $\exists$ continuous g such that

(i) $\sup_{x\in[0,2\pi]} |g(x)| \leq \sup_{x\in[0,2\pi]} |f(x)| = B$.
(ii) $\int_0^{2\pi} |f(x) - g(x)| dx < \varepsilon^2$.

Then

$$
\begin{aligned}
\|f - g\| &= \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x) - g(x)|^2 dx \right)^{1/2} \\
&= \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x) - g(x)||f(x) - g(x)| dx \right)^{1/2} \\
&\leq \left(\frac{B}{\pi} \int_0^{2\pi} |f(x) - g(x)| dx \right)^{1/2} \\
&< \left(\frac{B}{\pi} \varepsilon^2 \right)^{1/2} \\
&= \sqrt{\frac{B}{\pi}} \varepsilon.
\end{aligned}
$$

Since g is continuous, $\exists$ trigonometric polynomial P such that $\|g - P\| < \varepsilon$. Therefore,

$$
\begin{aligned}
\|f - P\| &\leq \|f - g\| + \|g - P\| \\
&< \varepsilon \sqrt{\frac{B}{\pi}} + \varepsilon \\
&= \varepsilon \left(1 + \sqrt{\frac{B}{\pi}} \right).
\end{aligned}
$$

By the best approximation lemma, $\forall N \geq \deg(P)$,

$$\|f - s_N(f)\| < \varepsilon \left(1 + \sqrt{\frac{B}{\pi}} \right). \qquad \square$$

Corollary 2.23. *(Parseval's Identity) $f \in \mathcal{R}$. Then*

$$\sum_{-\infty}^{\infty} |\hat{f}(n)|^2 = \|f\|^2.$$

Proof. For every n, $\|f\|^2 \geq \sum_{-N}^{N} |\hat{f}(n)|^2$ by (2). By the previous theorem, $\forall \varepsilon > 0$, $\exists M$ such that $\forall N \geq M$, $\|f - s_N(f)\| < \varepsilon$, so by (2) again,

$$\sum_{-N}^{N} |\hat{f}(n)|^2 \geq \|f\|^2 - \varepsilon. \qquad \square$$

Corollary 2.24. *(Riemann-Lebesgue) $f \in \mathcal{R}$. Then*

$$\lim_{|n| \to \infty} |\hat{f}(n)| = 0.$$

3 Differential Forms

Recall. $f : E \to \mathbb{R}$, $E \subseteq \mathbb{R}^n$ open, partials $D_1 f, \dots, D_n f$. If the partials are themselves differentiable then the second order derivatives of f are defined by

$$D_{ij} f = D_i D_j f, \quad (i, j = 1, \dots, n).$$

If these functions are continuous in E, we say f is C^2 in E.

Theorem 3.1. *If $f \in C^2$ in E then*

$$D_{ij} f = D_{ji} f, \quad \forall i, j.$$

Definition 3.2. If $f : E \to \mathbb{R}^n$, $E \subseteq \mathbb{R}^n$ open, f is differentiable at $x \in E$, the determinant of (the linear operator) $f'(x)$ is called the *Jacobian of f at x*

$$J_f(x) = \det f'(x).$$

Notation. We may also use $\frac{\partial(y_1,\dots,y_n)}{\partial(x_1,\dots,x_n)}$; $f(x_1, \dots, x_n) = y_1, \dots, y_n$.

Definition 3.3. Let $k \in \mathbb{N}$. A *k-cell* in $\mathbb{R}^k$ is the set of points $I^k = \{x = (x_1, \dots, x_k)\}$ such that $a_i \leq x_i \leq b_i$, $\forall i = 1, \dots, k$.

Suppose I^k is a k-cell in $\mathbb{R}^k$ and $f : I^k \to \mathbb{R}$ is continuous. For every $j \leq k$, let I^j be the restriction of I^k to the first j components.

Define $g_k : I^k \to \mathbb{R}$ by $g_k = f$. Define $g_{k-1} : I^{k-1} \to \mathbb{R}$ by

$$g_{k-1}(x_1, \dots, x_{k-1}) = \int_{a_k}^{b_k} g_k(x_1, \dots, x_k) dx_k.$$

Since g_k is uniformly continuous on I^k, g_{k-1} is (uniformly) continuous on I^{k-1}. Define $g_{k-2} : I^{k-2} \to \mathbb{R}$ by

$$g_{k-2}(x_1, \dots, x_{k-2}) = \int_{a_{k-1}}^{b_{k-1}} g_{k-1}(x_1, \dots, x_{k-1}) dx_{k-1}.$$

We can repeat this process, ultimately arriving at a number

$$g_0 = \int_{a_1}^{b_1} g_1(x_1) dx_1.$$

We say g_0 is the integral of f over I^k and we write

$$\int_{I^k} f(x) dx = g_0.$$

Example 3.4. Let $I^2 = [1,2] \times [0,1]$, $f(x_1, x_2) = 2x_1x_2^2$. What is $\int_{I^2} f dx$?

$$g_1(x_1) = \int_0^1 2x_1x_2^2 dx = \left[\frac{2}{3}x_1x_2^3\right]_0^1 = \frac{2}{3}x_1,$$

$$\int_{I^2} f dx = g_0 = \int_1^2 g_1(x_1)dx_1 = \int_1^2 \frac{2}{3}x_1 dx_1 = \left[\frac{1}{3}x_1^2\right]_1^2 = 1.$$

Question. Does this depend on the order of integration?

Answer. No (try the other direction in the example above).

Definition 3.5. If $f : \mathbb{R}^k \to \mathbb{R}$, the *support* of f is the closure of the set $\{x \in \mathbb{R}^k : f(x) \neq 0\}$.

If $f : \mathbb{R}^k \to \mathbb{R}$ is continuous with compact support, let I^k be any k-cell containing $\text{supp}(f)$. We define

$$\int_{\mathbb{R}^k} f dx = \int_{I^k} f dx.$$

Theorem 3.6. *(Change of variables) Let T be a 1-1, C^1 mapping of $E \subseteq \mathbb{R}^n$ open to $\mathbb{R}^n$. Also assume $J_T(x) \neq 0$ for all $x \in E$. If f is continuous on $\mathbb{R}^n$ with compact support that is contained in $T(E)$, then*

$$\int_{\mathbb{R}^n} f(y)dy = \int_{\mathbb{R}^n} f(T(x))|J_T(x)|dx.$$

Definition 3.7. (Informal) A *differential* 1*-form* on $\mathbb{R}^n$ is

(i) An object which can be integrated on any curve in $\mathbb{R}^n$.

(ii) A rule assigning a real number to every oriented line segment in $\mathbb{R}^n$ in a "suitable" way.

Definition 3.8. Let $p \in \mathbb{R}^n$. The *tangent space* to $\mathbb{R}^n$ at p is $T_p\mathbb{R}^n = \{(p, v) : v \in \mathbb{R}^n\}$.

Notation. If α is a 1-form, $p \in \mathbb{R}^n$, write α_p to denote the restriction of α to $T_p\mathbb{R}^n$. $\alpha_p(v)$ is the value α assigns to the (oriented) line segment from p to $p + v$.

We require that α_p is a linear functional $\forall p \in \mathbb{R}^n$, that is

(i) $\alpha_p(tv) = t \cdot \alpha_p(v)$, $\forall t \in \mathbb{R}, \forall p, v \in \mathbb{R}^n$.

(ii) $\alpha_p(v + w) = \alpha_p(v) + \alpha_p(w)$, $\forall p, v, w \in \mathbb{R}^n$.

We denote the projection maps in $\mathbb{R}^n$ by $dx_1, \ldots, dx_n$, where

$$dx_i(v) = dx_i(v_1, \ldots, v_n) = v_i, \quad i = 1, \ldots, n.$$

These form a basis for the set of linear functionals. Therefore, for any 1-form α, its restriction α_p can be written as

$$\alpha_p = A_1dx_1 + A_2dx_2 + \cdots + A_ndx_n$$

$$= A_1(p)dx_1 + \cdots + A_n(p)dx_n.$$

Last requirement: $A_i(p)$ must be sufficiently continuous with respect to p.

Definition 3.9. A *differential* 1-*form* α on $\mathbb{R}^n$ is a map from every tangent vector (p, v) in $\mathbb{R}^n$ which can be expressed in the form

$$\alpha = f_1 dx_1 + \cdots + f_n dx_n,$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ is C^2.

Example 3.10. $\alpha = ydx + dz$ on $\mathbb{R}^3$. Let $p = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $v = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$. Then

$$\begin{aligned} \alpha((p, v)) &= \alpha_p(v) \\ &= f_1(p)dx_1(v) + f_2(p)dx_2(v) + f_3(p)dx_3(v) \\ &= 2 \cdot 4 + 0 + 1 \cdot 6 \\ &= 14. \end{aligned}$$

Definition 3.11. A *curve* (1-surface) in $\mathbb{R}^n$ is a C^1-mapping $\gamma : [a, b] \to \mathbb{R}^n$.

Definition 3.12. Let $\alpha = f_1 dx_1 + \cdots + f_n dx_n$ be a 1-form in $\mathbb{R}^n$ and let $\gamma : [a, b] \to \mathbb{R}^n$ be C^1.

$$\int_\gamma \alpha = \int_a^b (f_1(\gamma(t))\gamma_1'(t) + \cdots + f_n(\gamma(t))\gamma_n'(t))dt.$$

Example 3.13. $\alpha = x^2 dx_1 + dx_2$ on $\mathbb{R}^2$. $\gamma(t) = (t, t^2)$, $t \in [0, 1]$. Then $\gamma_1'(t) = 1$, $\gamma_2'(t) = 2t$.

$$\begin{aligned} \int_\gamma \alpha &= \int_0^1 (f_1(\gamma(t))\gamma_1'(t) + f_2(\gamma(t))\gamma_2'(t)) \\ &= \int_a^b (t^2 \cdot 1 + 1 \cdot 2t)dt \\ &= \frac{4}{3}. \end{aligned}$$

Definition 3.14. A 2-*surface* is a C^1 map $\gamma : I^2 \to \mathbb{R}^n$.

Definition 3.15. (Informal) A 2-*form* on $\mathbb{R}^n$ is

(i) An object which can be integrated over any 2-surface.
(ii) A rule which assigns a real number to every oriented parallelogram in $\mathbb{R}^n$ in a "suitable" way.

Specify an oriented parallelogram in $\mathbb{R}^n$ based at $p \in \mathbb{R}^n$ by giving (v, w). We want every 2-form ω to satisfy the following for every $p \in \mathbb{R}^n$,

(i) $\omega_p(tv_1, v_2) = \omega_p(v_1, tv_2) = t\omega_p(v_1, v_2)$.

(ii) $\omega_p(v_1, v_2 + v_3) = \omega_p(v_1, v_2) + \omega_p(v_1, v_3)$ and $\omega_p(v_1 + v_2, v_3) = \omega_p(v_1, v_3) + \omega_p(v_2, v_3)$.

(iii) $\omega_p(v_1, v_2) = -\omega_p(v_2, v_1)$.

Basic 2-forms on $\mathbb{R}^n$. $\forall v, w \in \mathbb{R}^n$,

(i) $(dx_1 \wedge dx_2)(v, w) = \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$.

(ii) $(dx_1 \wedge dx_3)(v, w) = \det \begin{pmatrix} v_1 & w_1 \\ v_3 & w_3 \end{pmatrix}$.

(iii) $(dx_i \wedge dx_j)(v, w) = \det \begin{pmatrix} v_i & w_i \\ v_j & w_j \end{pmatrix}$.

Remark 3.16. If ω_p satisfies $(i) - (iii)$ then ω_p can be expressed as

$$\omega_p = \sum_{i,j} A_{i,j}(p)(dx_i \wedge dx_j),$$

for constant $A_{i,j}$.

Definition 3.17. A 2-*form* in $\mathbb{R}^n$ is a rule assigning a real number to each oriented parallelogram in $\mathbb{R}^n$ that can be written as

$$\omega = \sum_{i,j} f_{i,j}(dx_i \wedge dx_j),$$

where $f_{i,j} : \mathbb{R}^n \to \mathbb{R}$ is C^2.

For any $p \in \mathbb{R}^n$, $v, w \in \mathbb{R}^n$,

$$\omega_p(v, w) = \sum_{i,j} f_{i,j}(p)(dx_i \wedge dx_j)(v, w).$$

Example 3.18. ω is a 2-form in $\mathbb{R}^2$,

$$\begin{aligned} \omega &= f_{1,1} \underbrace{(dx_1 \wedge dx_1)}_{=0} + f_{1,2}(dx_1 \wedge dx_2) + f_{2,1} \underbrace{(dx_2 \wedge dx_1)}_{=-(dx_1 \wedge dx_2)} + f_{2,2} \underbrace{(dx_2 \wedge dx_2)}_{=0} \\ &= (f_{1,2} - f_{2,1})(dx_1 \wedge dx_2). \end{aligned}$$

This implies that every 2-form in $\mathbb{R}^2$ can be written as $\omega = f(dx_1 \wedge dx_2)$ where f is C^2.

Definition 3.19. Let $\gamma : I^2 \to \mathbb{R}^3$ be C^1, and $\omega = f_1(dx_1 \wedge dx_2) + f_2(dx_1 \wedge dx_3) + f_3(dx_2 \wedge dx_3)$ be a 2-form. Then

$$\int_\gamma \omega = \int_{I^2} \omega_{\gamma(z)} \left(\frac{\partial \gamma}{\partial x_1}(z), \frac{\partial \gamma}{\partial x_2}(z) \right) dz$$

$$
\begin{aligned}
&= \int_{I^2} f_1(\gamma(z))(dx_1 \wedge dx_2)\left(\frac{\partial\gamma}{\partial x_1}(z), \frac{\partial\gamma}{\partial x_2}(z)\right) \\
&\qquad + f_2(\gamma(z))(dx_1 \wedge dx_3)\left(\frac{\partial\gamma}{\partial x_1}(z), \frac{\partial\gamma}{\partial x_2}(z)\right) \\
&\qquad + f_3(\gamma(z))(dx_2 \wedge dx_3)\left(\frac{\partial\gamma}{\partial x_1}(z), \frac{\partial\gamma}{\partial x_2}(z)\right) dz \\
&= \int_{I^2} f_1(\gamma(z)) \det\begin{pmatrix} D_1\gamma_1(z) & D_2\gamma_1(z) \\ D_1\gamma_2(z) & D_2\gamma_2(z) \end{pmatrix} \\
&\qquad + f_2(\gamma(z)) \det\begin{pmatrix} D_1\gamma_1(z) & D_2\gamma_1(z) \\ D_1\gamma_3(z) & D_2\gamma_3(z) \end{pmatrix} + f_3(\gamma(z)) \det\begin{pmatrix} D_1\gamma_2(z) & D_2\gamma_2(z) \\ D_1\gamma_3(z) & D_2\gamma_3(z) \end{pmatrix}
\end{aligned}
$$

Definition 3.20. The integral of a 2-form $\omega = \sum_{i,j} f_{i,j}(dx_i \wedge dx_j)$ over a 2-surface $\gamma : [a,b] \times [c,d] \to \mathbb{R}^n$ (which is C^1) is

$$\int_\gamma \omega = \int_a^b \left(\int_c^d \omega_{\gamma(t_1,t_2)} \left(\frac{\partial\gamma}{\partial t_1}, \frac{\partial\gamma}{\partial t_2} \right) dt_2 \right) dt_1.$$

Definition 3.21. A *k-surface* in $\mathbb{R}^n$ is a C^1 map $\gamma : D \to \mathbb{R}^n$ where D is a k-cell.

Definition 3.22. (Informal) A k-form in $\mathbb{R}^n$, ω, is a rule that assigns a real number to every oriented k-dimensional parallelepiped in $\mathbb{R}^n$ in a "suitable" way.

Specify a k-dimensional oriented parallelepiped in $\mathbb{R}^n$ based at $p \in \mathbb{R}^n$ by giving an ordered list of vectors $v_1, \dots, v_k \in T_p\mathbb{R}^n$. We require that for any $p \in \mathbb{R}^n$, a k-form ω satisfies

(i) $\omega_p(v_1, \dots, tv_i, \dots, v_k) = t\omega_p(v_1, \dots, v_i, \dots, v_k)$.
(ii) $\omega_p(v_1, \dots, v_i + w_i, \dots, v_k) = \omega(v_1, \dots, v_i, \dots, v_k) + \omega_p(v_1, \dots, w_i, \dots v_k)$.
(iii) $\omega_p(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\omega_p(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$.

Definition 3.23. A *multi-index* of length k in $\mathbb{R}^n$ is a list $I = (i_1, \dots, i_k)$ of k integers between 1 and n.

Definition 3.24. Let $I = (i_1, \dots, i_k)$ be a multi-index. Then $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is the k-form in $\mathbb{R}^n$ defined by

$$dx_I(v^1, \dots, v^k) = \det \begin{pmatrix} v^1_{i_1} & v^2_{i_1} & \dots & v^k_{i_1} \\ v^1_{i_2} & v^2_{i_2} & \dots & v^k_{i_2} \\ \vdots & \vdots & \ddots & \vdots \\ v^1_{i_k} & v^2_{i_k} & \dots & v^k_{i_k} \end{pmatrix}.$$

Remark 3.25.

(i) If I contains a repeated index, then $dx_I(v^1, \dots, v^k) = 0$.

(ii) For any I, if $v^1, \dots, v^k$ contains a repeated vector, then $dx_I(v^1, \dots, v^k) = 0$.
(iii) If J is obtained from I by swapping a single pair of indices, then $dx_I(v^1, \dots, v^k) = -dx_J(v^1, \dots, v^k)$.

Definition 3.26. A *differential k-form* in $\mathbb{R}^n$, ω, is a rule assigning a real number to each oriented parallelepiped of the form

$$\omega = \sum_I f_I dx_I,$$

where the sum is taken over all multi-indices I of length k and $f_I : \mathbb{R}^n \to \mathbb{R}$ is C^2. If $p \in \mathbb{R}^n$, $v^1, \dots, v^k \in \mathbb{R}^n$,

$$\omega_p(v^1, \dots, v^k) = \sum_I f_I(p) dx_I(v^1, \dots, v^k).$$

Definition 3.27. Let $\phi : D \to \mathbb{R}^n$ be a k-surface and $\omega = \sum_I f_I dx_I$ be a k-form.

$$\begin{aligned}
\int_\phi \omega &= \int_D \omega_{\phi(u)} \left(\frac{\partial \phi}{\partial u_1}, \dots, \frac{\partial \phi}{\partial u_k} \right) du \\
&= \int_D \sum_I f_I(\phi(u)) dx_I \left(\frac{\partial \phi}{\partial u_1}, \dots, \frac{\partial \phi}{\partial u_k} \right) du \\
&= \int_D \sum_I f_I(\phi(u)) \frac{\partial(x_{i_1}, \dots, x_{i_k})}{\partial(u_1, \dots, u_k)} du,
\end{aligned}$$

where $\frac{\partial(x_{i_1}, \dots, x_{i_k})}{\partial(u_1, \dots, u_k)}$ is the Jacobian of the map $u_1, \dots, u_k \mapsto \phi_{i_1}(u), \dots, \phi_{i_k}(u)$.

Definition 3.28. If $I = (i_1, \dots, i_k)$ is a multi-index and $i_1 < \cdots < i_k$, we say I is an *increasing multi-index*, and we say that dx_I is a basic k-form.

Remark 3.29. Every k-form can be represented in terms of basic k-forms.

Example 3.30. $dx_1 \wedge dx_5 \wedge dx_3 \wedge dx_2 = -dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5$.

Example 3.31. $dx_1 \wedge dx_3 \wedge dx_5 \wedge dx_2 = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5$.

Definition 3.32. If $\omega = \sum_I a_I dx_I$ is a k-form, we can convert each multi-index I into an increasing multi-index J, and we say that

$$\omega = \sum_J b_J dx_J$$

is in *standard presentation*.

Example 3.33.

$$\begin{aligned}
\omega &= x_1 dx_2 \wedge dx_1 - x_2 dx_3 \wedge dx_2 + x_3 dx_2 \wedge dx_3 + dx_1 \wedge dx_2 \\
&= -x_1 dx_1 \wedge dx_2 + x_2 dx_2 \wedge dx_3 + x_3 dx_2 \wedge dx_3 + dx_1 \wedge dx_2 \\
&= (1 - x_1) dx_1 \wedge dx_2 + (x_2 + x_3) dx_2 \wedge dx_3.
\end{aligned}$$

The last line is in standard presentation.

Definition 3.34. Suppose $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ are increasing multi-indices. The *product* of dx_I and dx_J is the $(p+q)$-form

$$dx_I \wedge dx_J = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}.$$

Note. If I and J have an element in common, $dx_I \wedge dx_J = 0$.

Notation. If I and J have no elements in common, we denote the increasing $(p+q)$ length multi-index obtained from rearranging the members of $I \cup J$ in increasing order by $[I, J]$.

$$dx_I \wedge dx_J = (-1)^\alpha dx_{[I,J]},$$

where α is the number of swaps needed to convert $I \cup J$ into an increasing multi-index.

Suppose ω, λ are p and q-forms respectively in $\mathbb{R}^n$ with standard representations

$$\omega = \sum_I b_I dx_I \quad \lambda = \sum_J c_J dx_J.$$

The product of ω and λ is the $(p+q)$-form

$$\omega \wedge \lambda = \sum_{I,J} b_I c_I (dx_I \wedge dx_J).$$

Remark 3.35.

(i) $(\omega_1 + \omega_2) \wedge \lambda = (\omega_1 \wedge \lambda) + (\omega_2 \wedge \lambda)$.
(ii) $\omega \wedge (\lambda_1 + \lambda_2) = (\omega \wedge \lambda_1) + (\omega \wedge \lambda_2)$.
(iii) $(\omega \wedge \lambda) \wedge \sigma = \omega \wedge (\lambda \wedge \sigma)$.

Definition 3.36. A 0-form is a C^1 function.

Notation. The product of a 0-form f with a k-form $\omega = \sum_I b_I dx_I$ is

$$f\omega = \omega f = \sum_I (f b_I) dx_I.$$

Remark 3.37. $f(\omega \wedge \lambda) = f\omega \wedge \lambda = \omega \wedge f\lambda$.

Definition 3.38. (Differentiation of k-forms) Operator which associates a $(k+1)$-form, $d\omega$, to each k-form, ω.

(i) 0-forms in $\mathbb{R}^n$. $f : E \to \mathbb{R}$, $E \subseteq \mathbb{R}^n$.

$$\begin{aligned} df &= D_1 f dx_1 + \cdots + D_n f dx_n \\ &= \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n. \end{aligned}$$

(ii) k-forms in $\mathbb{R}^n$. Let $\omega = \sum_I b_I dx_I$ be given in standard presentation.

$$d\omega = \sum_I (db_I) \wedge dx_I.$$

Example 3.39. Let $\omega = \underbrace{x}_{f_1} dx + \underbrace{y^2}_{f_2} dz$ be a 1-form in $\mathbb{R}^3$.

$$\begin{aligned} d\omega &= (df_1) \wedge dx + (df_2) \wedge dz \\ &= (1dx + 0dy + 0dz) \wedge dx + (0dx + 2ydy + 0dz) \wedge dz \\ &= dx \wedge dx + 2ydy \wedge dz \\ &= 2ydy \wedge dz. \end{aligned}$$

Further,

$$\begin{aligned} d(d\omega) &= d(2ydy \wedge dz) \\ &= (df) \wedge (dy \wedge dz) \\ &= (2dy) \wedge (dy \wedge dz) \\ &= 0. \end{aligned}$$

Theorem 3.40. *(Graded product rule) Let ω be a k-form and λ be an m-form, both of class C^1. Then*

$$d(\omega \wedge \lambda) = (d\omega) \wedge \lambda + (-1)^k \omega \wedge (d\lambda).$$

Proof. It suffices to show this on simple forms $\omega = f dx_I$, $\lambda = g dx_J$, $f, g \in C^1$. Then

$$\omega \wedge \lambda = fg dx_I \wedge dx_J.$$

If I and J have any common indices, then both sides equal 0 and the theorem holds. Now assume that I and J have no common terms. Then

$$\begin{aligned} d(\omega \wedge \lambda) &= d(fg dx_I \wedge dx_J) \\ &= (-1)^\alpha d(fg dx_{[I,J]}) \\ &= (-1)^\alpha (fdg + gdf) \wedge dx_{[I,J]} \\ &= (fdg + gdf) \wedge dx_I \wedge dx_J \\ &= (-1)^k (f dx_I) \wedge dg \wedge dx_J + df \wedge dx_I \wedge (g dx_J) \\ &= (-1)^k \omega \wedge (d\lambda) + (d\omega) \wedge \lambda. \end{aligned}$$

□

Theorem 3.41. *If ω is a k-form of class C^1, then*

$$d^2(\omega) = 0.$$

Proof.

$$d(dx_I) = 0 = d(1dx_I) = d(1) \wedge dx_I.$$

Let $f \in C^2(E)$, $E \subseteq \mathbb{R}^n$. Then

$$\begin{aligned} d^2 f &= d\left(\sum_{i=1}^{n}(D_i f)(x) dx_i\right) \\ &= \sum_{i=1}^{n} d(D_i f) \wedge dx_i \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} D_{ij} f dx_j \wedge dx_i \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} -D_{ij} f dx_i \wedge dx_j. \end{aligned}$$

Thus $d^2 f = 0$. Then, for a k-form $\omega = f dx_I$,

$$d\omega = df \wedge dx_I,$$

and

$$\begin{aligned} d^2\omega &= d^2 f \wedge dx_I + (-1) df \wedge d(dx_I) \\ &= 0 - 0 \\ &= 0. \end{aligned}$$

□

Definition 3.42.

(i) A k-form ω is *closed* if $d\omega = 0$.
(ii) A k-form ω is *exact* if there exists a $(k-1)$-form α such that $d(\alpha) = \omega$.

Remark 3.43.

(i) If ω is a k-form that is of class C^2 then $d^2\omega = 0$ so $d\omega$ is exact.
(ii) Every exact form is closed ($d(d\omega) = 0$).

Question. Is the converse of (ii) true?

Answer. No.

Let $E \subseteq \mathbb{R}^n$ open, $V \subseteq \mathbb{R}^m$ open, and $T : E \to V$ be a C^1 function. Let ω be a k-form on V.

Notation. We will say that elements of E are $x \in E$ and elements of V are $y \in V$.

Then we write $\omega = \sum_I b_I(y)dy$ in standard presentation.

$$T(x) = (t_1(x), \dots, t_m(x)) = (y_1, \dots, y_m) = y.$$

Then

$$dt_i = \sum_{j=1}^{n} (D_j t_i)(x) dx_j \qquad i \in [m].$$

Note. dt_i is a 1-form on E.

T will transform the k-form ω on E to a k-form ω_T on E. This is called the *pullback* form.

$$\omega_T(x) = \sum_I b_I(T(x)) dt_{i_1} \wedge dt_{i_2} \wedge \dots \wedge dt_{i_k}.$$

Example 3.44. Let $\mathrm{id} = T : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x = y$, and $\omega(y) = \sum b_I(y)dy_I$. Then $t_i(x) = x_i$ so $dt_i = dx_i$. Thus

$$\omega_T(x) = \sum_I b_I(T(x)) dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

Example 3.45. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$, $(x_1, x_2) \mapsto (x_2, x_1^2, x_1 + x_2)$, and $\omega(y_1, y_2, y_3) = y_1 dy_2 \wedge dy_3$ is a 2-form on $\mathbb{R}^3$. Then $dt_1 = dx_2$, $dt_2 = 2x_1 dx_1$, $dt_3 = dx_1 + dx_2$. Thus

$$\begin{aligned} \omega_T(x_1, x_2) &= b_{\{2,3\}}(T(x_1, x_2)) dt_2 \wedge dt_3 \\ &= x_2(2x_1 dx_1) \wedge (dx_1 + dx_2) \\ &= 2x_2 x_1 (dx_1 \wedge dx_1 + dx_1 \wedge dx_2) \\ &= 2x_2 x_1 dx_1 \wedge dx_2. \end{aligned}$$

Lemma 3.46. *Let $f : V \to \mathbb{R}$ be a C^1 function and $f_T = f \circ T$. Then $d(f_T) = (df)_T$.*

Proof.

$$\begin{aligned} d(f_T) &= \sum_{j=1}^{n} D_j f_T dx_j \\ &= \sum_{j=1}^{n} D_j (f \circ T) dx_j \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} (D_i f)(T) \cdot (D_j t_i) dx_j \\ &= \sum_{i=1}^{m} (D_i f)(T) dt_i \\ &= (df)_T. \end{aligned}$$

□

Theorem 3.47. *Let ω be a k-form and λ be an l-form on V. Then*

(i) $(\omega+\lambda)_T = \omega_T + \lambda_T$ *if* $k = l$.
(ii) $(\omega \wedge \lambda)_T = \omega_T \wedge \lambda_T$.
(iii) $d(\omega_T) = (d\omega)_T$ *if ω is of class C^1 and T is of class C^2.*

(i) *Proof.*

$$\begin{aligned}(\omega+\lambda)_T &= \left(\sum (b_I + c_I) dy_I\right)_T \\ &= \sum (b_I + c_I)(T) dt_I \\ &= \sum b_I(T) dt_I + \sum c_I(T) dt_I \\ &= \omega_T + \lambda_T.\end{aligned}$$

□

(ii) *Proof.* Let $\omega = \sum_I b_I(y) dy_I$ and $\lambda = \sum_J c_J(y) dy_J$. Then

$$\begin{aligned}\omega \wedge \lambda &= \left(\sum_I b_I(y) dy_I\right) \wedge \left(\sum_J c_J(y) dy_J\right) \\ &= \sum_{I,J} b_I(y) c_J(y) (dy_I \wedge dy_J).\end{aligned}$$

The pullback is

$$(\omega \wedge \lambda)_T = \sum_{I,J} b_I(T(x)) c_J(T(x)) dt_{i_1} \wedge \cdots \wedge dt_{i_k} \wedge dt_{j_1} \wedge \cdots \wedge dt_{j_m}.$$

On the other hand,

$$\begin{aligned}\omega_T \wedge \lambda_T &= \left(\sum_I b_I(T(x)) dt_{i_1} \wedge \cdots \wedge dt_{i_k}\right) \wedge \left(\sum_J c_J(T(x)) dt_{j_1} \wedge \cdots \wedge dt_{j_m}\right) \\ &= \sum_{I,J} b_I(T(x)) c_J(T(x)) dt_{i_1} \wedge \cdots \wedge dt_{i_k} \wedge dt_{j_1} \wedge \cdots \wedge dt_{j_m},\end{aligned}$$

which shows the conclusion. □

(iii) *Proof.* Rudin, Theorem 10.22. □

Theorem 3.48. *T is a C^1 map of an open set $E \subseteq \mathbb{R}^n$ into an open set $V \subseteq \mathbb{R}^m$, S is a C^1 map of V into an open set $W \subseteq \mathbb{R}^\ell$, ω is a k-form on W (ω_S is a k-form on V, $(\omega_S)_T$ is a k-form on E, ω_{ST} is a k-form on E). Then*

$$(\omega_S)_T = \omega_{ST}.$$

Theorem 3.49. *Suppose ω is a k-form on an open set $E \subseteq \mathbb{R}^n$, ϕ is a k-surface in E with parameter domain $D \subseteq \mathbb{R}^k$ and Δ is the trivial k-surface, $\Delta : D \to \mathbb{R}^k$, $\Delta(u) = u$. Then*

$$\int_\phi \omega = \int_\Delta \omega_\phi.$$

Proof. It suffices to prove thsi in the case when

$$\omega = a dx_I = a dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Let $\phi_1, \ldots, \phi_n$ denote the components of ϕ. Then $\omega_\phi = a(\phi) d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k}$. It suffices to prove

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} = J(u) du_1 \wedge \cdots \wedge du_k, \tag{1}$$

where $J(u) = \frac{\partial(x_{i_1}, \ldots, x_{i_k})}{\partial(u_1, \ldots, u_k)}$. Assuming (1),

$$\begin{aligned} \int_\Delta \omega_\phi &= \int_\Delta a(\phi) d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} \\ &= \int_\Delta a(\phi) J(u) du_1 \wedge \cdots \wedge du_k \\ &= \int_D a(\phi(u)) J(u) du \\ &= \int_\phi \omega. \end{aligned}$$

Let $[A]$ be the $k \times k$ matrix with entries

$$\alpha(p, q) = D_q \phi_{i_p}(u), \qquad p, q = 1, \ldots, k.$$

Note. $\det(A) = J(u)$.
Since $d\phi_{i_p} = \sum_q \alpha(p, q) du_q$, we have

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} = \sum \alpha(1, q_1) \ldots \alpha(k, q_k) du_{q_1} \wedge \cdots \wedge du_{q_k},$$

where the sum ranges over all $q_1, \ldots, q_k \in \{1, \ldots, k\}$. Rearranging each $duq_1 \wedge \cdots \wedge du_{q_k}$ we get

$$\begin{aligned} d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} &= \det(A) du_1 \wedge \cdots \wedge du_k \\ &= J(u) du_1 \wedge \cdots \wedge du_k. \end{aligned}$$ □

Theorem 3.50. *Suppose T is a C^1 map of an open set $E \subseteq \mathbb{R}^n$ into an open set $V \subseteq \mathbb{R}^n$, ϕ is a k-surface in E, ω is a k-form on V. Then*

$$\int_{T\phi} \omega = \int_\phi \omega_T.$$

Proof. Let D be the parameter domain of ϕ (and therefore of $T\phi$ as well). Let Δ be the trivial k-surface on D, i.e., $\Delta(u) = u$. Then

$$\int_{T\phi} \omega = \int_\Delta \omega_{T\phi} = \int_\Delta (\omega_T)_\phi = \int_\phi \omega_T.$$ □

Definition 3.51. A map f from a vector space X to a vector space Y is called *affine* if $f - f(0)$ is linear, i.e.,

$$f(x) = f(0) + Ax, \qquad A : X \to Y \text{ is linear.}$$

Remark 3.52. An affine map $f : \mathbb{R}^k \to \mathbb{R}^n$ is determined by $f(0)$ and $f(e_i)$ for $i = 1, \ldots, k$.

Definition 3.53. The k-simplex in $\mathbb{R}^k$ is $Q^k \subseteq \mathbb{R}^k$,

$$Q^k = \{x = (x_1, \ldots, x_k) : x_i \geq 0, x_1 + \cdots + x_k \leq 1\}.$$

Definition 3.54. Let $p_0, p_1, \ldots, p_k \in \mathbb{R}^n$. The *oriented affine k-simplex* $\sigma = [p_0, p_1, \ldots, p_k]$ is the k-surface in $\mathbb{R}^n$ with parameter domain Q^k given by the affine map

$$\sigma(\alpha_1 e_1 + \cdots + \alpha_k e_k) = p_0 + \sum_1^k \alpha_i (p_i - p_0).$$

Remark 3.55. $\sigma(u) = p_0 + Au$, $u \in Q^k$, where $A \in L(\mathbb{R}^k, \mathbb{R}^n)$. $Ae_i = p_i - p_0$, $\forall 1 \leq i \leq k$.

Remark 3.56. σ is called oriented to emphasize that the order of the points $p_0, p_1, \ldots, p_k$ matters. If $\overline{\sigma} = [p_{i_0}, p_{i_1}, \ldots, p_{i_k}]$ where $\{i_0, i_1, \ldots, i_k\}$ is a permutation of $\{0, 1, \ldots, k\}$, then

$$\overline{\sigma} = s(i_0, i_1, \ldots, i_k)\sigma,$$

where $s(i_0, i_1, \ldots, i_k) = (-1)^\alpha$, where α is the minimum number of swaps needed to transform $0, 1, \ldots, k$ to $i_0, i_1, \ldots, i_k$. If $s(i_0, i_1, \ldots, i_k) = 1$, then we say σ and $\overline{\sigma}$ have the same orientation, and if $s(i_0, i_1, \ldots, i_k) = -1$, we say σ and $\overline{\sigma}$ have opposite orientation.

Definition 3.57. An oriented 0-simplex is a point $p \in \mathbb{R}^n$ with a sign attached, and we write $\sigma = +p_0$ or $\sigma = -p_0$. If f is a 0-form, $\sigma = \varepsilon p_0, \varepsilon = \pm 1$,

$$\int_\sigma f = \varepsilon f(p_0).$$

Theorem 3.58. *If σ is an oriented k-simplex in an open set $E \subseteq \mathbb{R}^n$ and if $\overline{\sigma} = \varepsilon\sigma, \varepsilon = \pm 1$, then $\forall k$-forms ω on E,*

$$\int_\sigma \omega = \varepsilon \int_{\overline{\sigma}} \omega.$$

Definition 3.59. An *affine k-chain* Γ in an open set $E \subseteq \mathbb{R}^n$ is a collection of finitely many oriented affine k-simplexes $\sigma_1, \ldots, \sigma_r$ in E.

Note. The simplexes need not be distinct.

Definition 3.60. If Γ is an an affine k-chain in an open set $E \subseteq \mathbb{R}^n$ and ω is a k-form on E,

$$\int_\Gamma \omega = \sum_1^r \int_{\sigma_i} \omega.$$

Notation. This suggests the following notation,

$$\Gamma = \sigma_1 + \cdots + \sigma_r = \sum_1^r \sigma_i.$$

Warning. This is *just* notation.

Example 3.61. $\sigma_1 = [p_0, p_1, p_2]$ and $\sigma_2 = [p_1, p_0, p_2]$, i.e., $\sigma_1 = -\sigma_2$. Then

$$\int_\Gamma \omega = \int_{\sigma_1} \omega + \int_{\sigma_2} \omega = \int_{\sigma_1} \omega - \int_{\sigma_1} \omega = 0.$$

Definition 3.62. For $k \geq 1$, the *boundary* of an oriented affine k-simplex $\sigma = [p_0, p_1, \ldots, p_k]$ is the affine $(k-1)$-chain

$$\partial\sigma = \sum_{j=0}^k (-1)^j [p_0, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k].$$

Example 3.63. $\sigma = [p_0, p_1, p_2]$.

$$\begin{aligned}\partial\sigma &= [p_1, p_2] - [p_0, p_2] + [p_0, p_1] \\ &= [p_1, p_2] + [p_2, p_0] + [p_0, p_1].\end{aligned}$$

Example 3.64. $\sigma = [p_0, p_1, p_2, p_3]$.

$$\begin{aligned}\partial\sigma &= [p_1, p_2, p_3] - [p_0, p_2, p_3] + [p_0, p_1, p_3] - [p_0, p_1, p_2] \\ &= [p_1, p_2, p_3] + [p_2, p_0, p_3] + [p_0, p_1, p_3] + [p_1, p_0, p_2].\end{aligned}$$

Definition 3.65. Let T be a C^2 map from an open set $E \subseteq \mathbb{R}^n$ into an open set $V \subseteq \mathbb{R}^m$. Let σ be an oriented affine k-simplex in E. The map $\phi = T \circ \sigma$ is a k-surface in V. We call ϕ an oriented k-simplex. A finite collection ψ of oriented k-simplexes $\phi_1, \ldots, \phi_r$ of clas C^2 in V is called a k-chain of class C^2 in V. If ω is a k-form on V define

$$\int_\psi \omega = \sum_1^r \int_{\phi_i} \omega.$$

Notation. $\psi = \sum_1^r \phi_i$.

Notation. If $\Gamma = \sum_1^r \sigma_i$ is an affine k-chain and $\phi_i = T \circ \sigma_i$, we write $\psi = T \circ \Gamma$.

Definition 3.66. The boundary $\partial\phi$ of an oriented k-simplex $\phi = T\sigma$ is defined to be the $(k-1)$-chain

$$\partial\phi = T \circ \partial\sigma.$$

Note. If ϕ is C^2, then so is $\partial\phi$.

Definition 3.67. The boundary of a k-chain $\psi = \sum \phi_i$ is the $(k-1)$-chain

$$\partial\psi = \sum_1^r \partial\phi_i.$$

Theorem 3.68. *(Stokes' Theorem) If ψ is a k-chain of class C^2 in an open set $V \subseteq \mathbb{R}^m$ and ω is a $(k-1)$-form of class C^1 on V, then*

$$\int_\psi d\omega = \int_{\partial\psi} \omega.$$

Remark 3.69.

(i) If $k = m = 1$, this is the fundamental theorem of calculus. $k = 1 \implies \omega = f \in C^1$ and $m = 1 \implies V \subseteq \mathbb{R}$. $\psi = \sigma = [a, b]$. Then

$$f(b) - f(a) = \int_{\partial\psi} \omega = \int_\psi d\omega = \int_a^b df(x)dx.$$

(ii) If $k = m = 2$, this is Green's theorem.

(iii) If $k = m = 3$, this is the divergence theorem.

(iv) If $k = 2, m = 3$, this is the original Stokes' theorem.

Theorem 3.70. *(Baby Stokes) Let $E \subseteq \mathbb{R}^k$ be an open set containing Q^k. Let $\sigma = [0, e_1, \ldots, e_k]$. Let λ be a $(k-1)$-form in E of class C^1. Then*

$$\int_\sigma d\lambda = \int_{\partial\sigma} \lambda.$$

Claim. Baby Stokes $\implies$ Stokes.

Proof. It suffices to prove Stokes' theorem when $\psi = \phi = \sigma$ where σ is an affine k-simplex. Suppose $\phi = T\sigma$.

$$\begin{aligned}\int_\psi d\omega &= \int_{T\sigma} d\omega \\ &= \int_\sigma (d\omega)_T \\ &= \int_{\partial\sigma} \omega_T\end{aligned}$$

$$
\begin{aligned}
&= \int_{T\partial\sigma} \omega \\
&= \int_{\partial\psi} \omega.
\end{aligned}
$$

It suffices to prove Stokes' theorem when $\sigma = [0, e_1, \dots, e_k]$. Let $\psi = T\sigma$, where T is affine. Then

$$
\begin{aligned}
\int_\psi d\omega &= \int_{T\sigma} d\omega \\
&= \int_\sigma (d\omega)_T \\
&= \int_{\partial\sigma} \omega_T \\
&= \int_{T\partial\sigma} \omega \\
&= \int_{\partial\psi} \omega.
\end{aligned}
$$

□

Proof of Baby Stokes. If $k = 1$, this follows by FTC. Let $k \geq 2$. Fix $r \in \mathbb{N}$, $1 \leq r \leq k$, $f \in C^1(E)$. It suffices to show that the conclusion holds when $\lambda = f(x)dx_1 \wedge \cdots \wedge dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_k$. By definition,

$$
\partial\sigma = [e_1, \dots, e_k] + \sum_{j=1}^{k} (-1)^j \tau_i,
$$

where $\tau_i = [0, e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_k]$. Let $\tau_0 = [e_r, e_1, \dots, e_{r-1}, e_{r+1}, \dots, e_k]$, so $[e_1, \dots, e_k] = (-1)^{r-1}\tau_0$. Note that the domain of $\tau_0, \tau_1, \dots, \tau_k$ is Q^{k-1}. Let $u \in Q^k$. Let $x = (x_1, \dots, x_k) = \tau_0(u)$.

$$
x_j = \begin{cases} u_j & \text{if } 1 \leq j < r, \\ 1 - (u_1 + \cdots + u_k) & \text{if } j = r, \\ u_{j-1} & \text{if } r < j \leq k. \end{cases}
$$

Let $x = \tau_i(u)$.

$$
x_j = \begin{cases} u_j & \text{if } 1 \leq j < i, \\ 0 & \text{if } j = i, \\ u_{j-1} & \text{if } i < j \leq k. \end{cases}
$$

Let J_i be the Jacobian of the map $(u_1, \dots, u_{k-1}) \mapsto (x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_k)$ induced by τ_i. Then,

(i) $i = 0, J_0 = 1$.

(ii) $i = r, J_r = 1$.

(iii) For all other i, $J_i = 0$.

Hence,

$$\begin{aligned}
\int_{\partial\sigma} \lambda &= \sum \int \lambda \\
&= (-1)^{r-1} \int_{\tau_0} \lambda + (-1)^r \int_{\tau_r} \lambda \\
&= (-1)^{r-1} \left(\int_{\tau_0} \lambda - \int_{\tau_r} \lambda \right) \\
&= (-1)^{r-1} \left[\int_{Q^{k-1}} f(\tau_0(u)) du - \int_{Q^{k-1}} f(\tau_r(u)) du \right] \\
&= (-1)^{r-1} \int_{Q^{k-1}} \left[f(\tau_0(u)) - f(\tau_r(u)) \right] du.
\end{aligned}$$

Since

$$\begin{aligned}
d\lambda &= (D_1 f dx_1 \wedge \cdots \wedge D_k f dx_k) \wedge dx_1 \wedge \cdots \wedge dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_k \\
&= D_r f dx_r \wedge dx_1 \wedge dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_k \\
&= (-1)^{r-1} D_r f dx_1 \wedge \cdots \wedge dx_k,
\end{aligned}$$

we have

$$\begin{aligned}
\int_\sigma d\lambda &= (-1)^{r-1} \int_\sigma D_r f dx_1 \wedge \cdots \wedge dx_k \\
&= (-1)^{r-1} \int_{Q^k} D_r f(x) dx \\
&= (-1)^{r-1} \int_{Q^{k-1}} \left(\int_0^{1-(x_1+\ldots x_{r-1}+x_{r+1}+\cdots+x_k)} D_r f(x) dx_r \right) dx_1, \ldots, dx_{r-1}, dx_{r+1}, \ldots, dx_k \\
&= (-1)^{r-1} \int_{Q^{k-1}} \left[f(\tau_0(x)) - f(\tau_r(x)) \right] dx_1, \ldots, dx_{r-1}, dx_{r+1}, \ldots, dx_k \\
&= \int_{\partial\sigma} \lambda. \qquad \square
\end{aligned}$$

4 Baire Category

Let X be a metric space with metric d. Let $E \subseteq X$.

Definition 4.1.

(i) The *interior* of E, E°, is the union of all open subsets of E.
(ii) The *closure* of E, $\overline{E}$, is the intersection of all closed sets containing E.
(iii) E is *dense* in X if $\overline{E} = X$.
(iv) E is *nowhere dense* if $(\overline{E})^\circ = \emptyset$.

Definition 4.2.

(i) E is *of first category (meager)* if E is the countable union of nowhere dense sets.
(ii) If E is not of first category, we say E is *of second category.*
(iii) If E^{c} is of first category, we say E is *generic (comeager).*

Theorem 4.3. *(Baire category theorem) Every complete metric space is of second category itself, i.e., cannot be written as a countable union of nowhere dense sets.*

Proof. Let X be a complete metric space. Assume for the sake of contradiction that $X = \bigcup_1^\infty F_n$ where each F_n is nowhere dense. Without loss of generality, we can assume that each F_n is closed by replacing with closures. We will prove that $\exists x \in X$ such that $x \notin \bigcup F_n$. Since F_1 is closed and $F_1 \neq X$, there is an open ball B_1 of radius $r_1 > 0$ such that $\overline{B_1} \subseteq F_1^{\mathsf{c}}$. Since F_2 is closed and nowhere dense, B_1 cannot be entirely contained in F_2. Since F_2 is closed, there is a ball B_2 of radius $r_2 > 0$ such that $\overline{B_2} \subseteq B_1$ and $\overline{B_2} \subseteq F_2^{\mathsf{c}}$. We can assume $r_2 < r_1/2$. Continuing like this, we obtain a sequence of balls $\{B_n\}$ such that the following conditions hold.

(i) The radius r_n of B_n goes to 0 as $n \to \infty$.
(ii) $B_{n+1} \subseteq B_n$.
(iii) $F_n \cap \overline{B_n} = \emptyset$.

For each $n \in \mathbb{N}$, choose $x_n \in B_n$. Then $\{x_n\}$ is a Cauchy sequence. Since X is complete, $x_n \to x \in X$. By (iii), $x \notin F_n$ for every n, i.e., $x \notin \bigcup F_n$. This is a contradiction. $\square$

Corollary 4.4. *If X is complete, any generic set of X is dense.*

Proof. Assume for the sake of contradiction that $E \subseteq X$ is generic but not dense. Then there is a closed ball $\overline{B}$ such that $\overline{B} \subseteq E^{\mathsf{c}}$. Since E is generic, i.e., E^{c} is of first category, $E^{\mathsf{c}} = \bigcup_1^\infty F_n$ where the F_n are nowhere dense. Hence,

$$\overline{B} = \bigcup_1^\infty (F_n \cap \overline{B}).$$

But $\overline{B}$ is a complete metric space, contradicting the Baire category theorem. $\square$

Let $X = C([0,1])$ be the set of continuous, real-valued functions on $[0,1]$. Define a metric d on X by

$$d(f,g) = \|f - g\| = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Then X is a complete metric space with d.

Theorem 4.5. *The set of $f \in X = C([0,1])$ that are nowhere differentiable is generic.*

Proof. Let $\mathcal{D} = \{f \in X : \exists x \in [0,1] \text{ at which } f \text{ is differentiable}\}$. It suffices to show that $\mathcal{D}$ is of first category. For every $N \in \mathbb{N}$, let E_N be the set of functions $f \in X$ such that $\exists x^* \in [0,1]$ with

$$|f(x) - f(x^*)| \leq N|x - x^*| \qquad \forall x \in [0,1]. \tag{1}$$

Remark. $\mathcal{D} \subseteq \bigcup_1^\infty E_N$.

Lemma 4.6. *E_N is closed.*

Proof. Let $\{f_n\}$ be a sequence of functions in E_N such that $\|f_n - f\| \to 0$. It suffices to show that $f \in E_N$. For every n, let x_n^* be a point such that (1) holds with respect to f_n. Choose a subsequence $\{x_{n_k}^*\}$ such that $x_{n_k}^* \to x^* \in [0,1]$. Then

$$|f(x) - f(x^*)| = |f(x) - f_{n_k}(x)| + |f_{n_k}(x) - f_{n_k}(x^*)| + |f_{n_k}(x^*) - f(x^*)|.$$

Since $\|f_n - f\| \to 0$, $\exists K$ such that $\forall k \geq K$, $\forall x \in [0,1]$, $|f(x) - f_{n_k}(x)| < \varepsilon/2$. Then $\forall \varepsilon > 0$ for all sufficiently large k,

$$|f(x) - f(x^*)| < \varepsilon + |f_{n_k}(x) - f_{n_k}(x^*)|.$$

Since

$$\begin{aligned} |f_{n_k}(x) - f_{n_k}(x^*)| &= |f_{n_k}(x) - f_{n_k}(x_{n_k}^*)| + |f_{n_k}(x_{n_k}^*) - f_{n_k}(x^*)| \\ &\leq N(|x - x_{n_k}^*| + |x_{n_k}^* - x^*|). \end{aligned}$$

For all sufficiently large k,

$$|f_{n_k}(x) - f_{n_k}(x^*)| \leq N|x - x^*| + \varepsilon,$$

i.e.,

$$|f(x) - f(x^*)| < 2\varepsilon + N|x - x^*|.$$

Hence,

$$|f(x) - f(x^*)| \leq N|x - x^*|,$$

and $f \in E_N$. $\Diamond$

Let $\mathcal{P} \subseteq X$ be the set of piecewise linear functions in $C([0,1])$. For every M, let $\mathcal{P}_M \subseteq \mathcal{P}$ be the set of piecewise linear functions in $\mathcal{P}$ such that the slope of every line segment in the graph of f is at least M or at most $-M$.

Note. If $M > N$, then $\mathcal{P}_M \cap E_N = \emptyset$.

Lemma 4.7. *For all $M > 0$, $\mathcal{P}_M$ is dense in X.*

Proof. We first show that $\forall \varepsilon > 0, \forall f \in X$, $\exists g \in \mathcal{P}$ such that $d(f,g) < \varepsilon$. Since f is continuous on $[0,1]$, it is uniformly continuous, so $\exists \delta > 0$ such that $\forall |x-y| < \delta$, $|f(x) - f(y)| < \varepsilon$. Let $n > 1/\delta$, define as a piecewise linear function such that $\forall k = 0, \ldots, n-1$,

$$g\left(\frac{k}{n}\right) = f\left(\frac{k}{n}\right), g\left(\frac{k+1}{n}\right) = f\left(\frac{k+1}{n}\right).$$

So $g \in \mathcal{P}$ and $d(f,g) < \varepsilon$. We now show that $\exists h \in \mathcal{P}_M$ such that $d(g,h) < \varepsilon$. For $k = 0$, let $\phi_\varepsilon(x) = g(x) + \varepsilon$, $\psi_\varepsilon(x) = g(x) - \varepsilon$. Beginning at $g(0)$, travel the line segment of slope M until we intersect ϕ_ε. We then travel the line segment of slope $-M$ until we intersect ψ_ε. We obtain $h \in \mathcal{P}_M$ such that $\phi_\varepsilon \leq h \leq \psi_\varepsilon$ on $[0, 1/n]$. Repeat this process $[1/n, 2/n]$, starting from $h(1/n)$. ◊

By Lemma 4.7, $E_N^\circ = \emptyset$ for all N. To see this, let $f \in E_N$, $\varepsilon > 0$, and $M > N$. Then $\exists h \in \mathcal{P}_M$ such that $d(f,h) < \varepsilon$. But $E_N \cap \mathcal{P}_M = \emptyset$. Hence, there is no open ball containing f which is entirely contained in E_N. By Lemma 4.6, E_N is closed, so E_N is nowhere dense for all N, which implies $\mathcal{D} = \bigcup_1^\infty (E_N \cap \mathcal{D})$. □

Theorem 4.8. *Let $\{f_n\}$ be a sequence of continuous real-valued functions on $\mathbb{R}$, and suppose that $f_n \to f$ pointwise. The set of points at which f is continuous is generic.*

Let $\mathcal{B}$ be the set of continuous complex-valued functions on $[-\pi, \pi]$.

Theorem 4.9. *The set of $f \in \mathcal{B}$ whose Fourier series diverge on a generic set in $[-\pi, \pi]$ is generic in $\mathcal{B}$.*