MATH 20510. Analysis in $\mathbb{R}^n$ III (accelerated)

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Contents

1 Measure and Integration

Definition 1.1. A family of sets $\mathscr{A}$ is called a *ring* if, for every $A, B \in \mathscr{A}$,

(i) $A \cup B \in \mathscr{A}$
(ii) $A \setminus B \in \mathscr{A}$

Definition 1.2. A ring $\mathscr{A}$ is called a *σ-ring* if for any $\{A_n\}_1^\infty \subseteq \mathscr{A}$,

$$\bigcup_1^\infty A_n \in \mathscr{A}.$$

Definition 1.3. ϕ is a *set function* on a ring $\mathscr{A}$ if for every $A \in \mathscr{A}$,

$$\phi(A) \in [-\infty, \infty].$$

Definition 1.4. A set function ϕ is *additive* if for any $A, B \in \mathscr{A}$ such that $A \cap B = \emptyset$,

$$\phi(A \cup B) = \phi(A) + \phi(B).$$

Definition 1.5. A set function ϕ is *countably additive* if for any $\{A_n\} \subseteq \mathscr{A}$ such that $A_i \cap A_j = \emptyset$, $\forall i \neq j$,

$$\phi\left(\bigcup_1^n A_n\right) = \sum_1^n \phi(A_n).$$

In the last two we assume that there are no $A, B \in \mathscr{A}$ such that $\phi(A) = -\infty, \phi(B) = \infty$.

Remark 1.6. If ϕ is an additive set function,

(i) $\phi(\emptyset) = 0$.
(ii) If $A_1, \ldots, A_n$ are pairwise disjoint then $\phi(\bigcup_1^n A_n) = \sum_1^n \phi(A_n)$.

(iii) $\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2)$.
(iv) If ϕ is nonnegative and $A_1 \subseteq A_2$ then $\phi(A_1) \leq \phi(A_2)$.
(v) If $B \subseteq A$ and $|\phi(B)| < \infty$ then $\phi(A \setminus B) = \phi(A) - \phi(B)$.

Theorem 1.7. *Let ϕ be a countably additive set function on a ring $\mathscr{A}$. Suppose $\{A_n\} \subseteq \mathscr{A}$ such that $A_1 \subseteq A_2 \subseteq \dots$ and $A = \bigcup_1^\infty A_n \in \mathscr{A}$. Then $\phi(A_n) \to \phi(A)$ as $n \to \infty$.*

Proof. Set $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$. Note

(i) $\{B_n\}$ is pairwise disjoint.
(ii) $A_n = B_1 \cup B_2 \cup \cdots \cup B_n$.
(iii) $A = \bigcup_1^\infty B_n$.

Hence $\phi(A_n) = \sum_1^\infty \phi(B_j)$, $\phi(A) = \sum_1^\infty \phi(B_j)$ and the conclusion follows. □

Definition 1.8. An *interval* $I = \{(a_i, b_i)\}_1^n$ of $\mathbb{R}^n$ is the set of points $x = (x_1, \dots, x_n)$ such that $a_i \leq x_i \leq b_i$ or $a_i < x_i \leq b_i$, etc. where $a_i \leq b_i$.

Note. $\emptyset$ is an interval.

Definition 1.9. If A is the union of a finite number of intervals, we say A is *elementary.*

We denote the set of elementary sets by $\mathscr{E}$.

Definition 1.10. If I is an interval of $\mathbb{R}^n$, we define the volume of I by

$$\operatorname{vol}(I) = \prod_i^n (b_i - a_i).$$

If $A = I_1 \cup I_2 \cup \cdots \cup I_k$ is elementary, and the intervals are disjoint, then

$$\operatorname{vol}(A) = \sum_1^k \operatorname{vol}(I_j).$$

Remark 1.11.

(i) $\mathscr{E}$ is a ring, but not a σ-ring.
(ii) If $A \in \mathscr{E}$, then A can be written as a finite union of disjoint intervals.
(iii) If $A \in \mathscr{E}$, then $\operatorname{vol}(A)$ is well-defined.
(iv) vol is an additive set function on $\mathscr{E}$, and $\operatorname{vol} \geq 0$.

Definition 1.12. A nonnegative set function ϕ on $\mathscr{E}$ is *regular* if $\forall A \in \mathscr{E}$, $\forall \varepsilon > 0$, $\exists$ open $G \in \mathscr{E}$, $G \supseteq A$ and closed $F \in \mathscr{E}$, $F \subseteq A$, such that

$$\phi(G) \leq \phi(A) + \varepsilon, \qquad \phi(A) \leq \phi(F) + \varepsilon.$$

Note. vol is regular.

Definition 1.13. A *countable open cover* of $E \subseteq \mathbb{R}^n$ is a collection of open elementary sets $\{A_n\}$ such that $E \subseteq \bigcup_1^\infty A_n$.

Definition 1.14. The *Lebesgue outer measure* of $E \subseteq \mathbb{R}^n$ is defined as

$$m^*(E) = \inf \sum_1^\infty \mathrm{vol}(A_n).$$

where inf is taken over all countable open covers of E.

Remark 1.15.

(i) m$^*(E)$ is well-defined.
(ii) $m^*(E) \geq 0$.
(iii) If $E_1 \subseteq E_2$ then $m^*(E_1) \leq m^*(E_2)$.

Theorem 1.16.

(i) If $A \in \mathscr{E}$, then $m^*(A) = \mathrm{vol}(A)$.
(ii) If $E = \bigcup_1^\infty E_n$ then $m^*(E) \leq \sum_1^\infty m^*(E_n)$.

Proof. (i) Let $A \in \mathscr{E}$ and $\epsilon > 0$. Since vol is regular, $\exists$ open $G \in \mathscr{E}$ such that $A \subseteq G$ and $\mathrm{vol}(G) \leq \mathrm{vol}(A) + \epsilon$. Since $G \supseteq A$ and $G \in \mathscr{E}$ is open, $m^*(A) \leq \mathrm{vol}(G) \leq \mathrm{vol}(A) + \epsilon$. There also $\exists$ closed $F \in \mathscr{E}$ such that $F \subseteq A$ and $\mathrm{vol}(A) \leq \mathrm{vol}(F) + \epsilon$. By definition, $\exists$ collection $\{A_n\}$ of open elementary sets such that $A \subseteq \bigcup A_n$ and $\sum_1^\infty \mathrm{vol}(A_n) \leq m^*(A) + \epsilon$. Since $F \subseteq \bigcup A_n$ and F is compact, $F \subseteq A_1 \cup \cdots \cup A_N$ from some N.

$$\begin{aligned}
\mathrm{vol}(A) &\leq \mathrm{vol}(F) + \epsilon \\
&\leq \mathrm{vol}(A_1 \cup \cdots \cup A_N) + \epsilon \\
&\leq \sum_1^N \mathrm{vol}(A_n) + \epsilon \\
&\leq \sum_1^\infty \mathrm{vol}(A_n) + \epsilon \\
&\leq m^*(A) + \epsilon + \epsilon \\
&= m^*(A) + 2\epsilon
\end{aligned}$$

Since ϵ was arbitrary, $m^*(A) = \mathrm{vol}(A)$. □

Proof. (ii) If $m^*(E_n) = \infty$ for any $n \in \mathbb{N}$, then we are done. Assume not. Let $\epsilon > 0$. For every $n \in \mathbb{N}$, $\exists$ open cover of E_n, $\{A_{n,k}\}_{k=1}^\infty$ such that

$$\sum_{k=1}^\infty \mathrm{vol}(A_{n,k}) \leq m^*(E_n) + \epsilon/2^n$$

Then $E \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{n,k}$ and so

$$\begin{aligned} m^*(E) &\le \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \operatorname{vol}(A_{n,k}) \\ &\le \sum_{n=1}^{\infty} m^*(E_n) + \epsilon/2^n \\ &= \sum_{n=1}^{\infty} m^*(E_n) + \sum_{1}^{\infty} \epsilon/2^n \\ &= \sum_{1}^{\infty} m^*(E_n) + \epsilon \end{aligned}$$ □

Definition 1.17. Let $A, B \subseteq \mathbb{R}^n$.

(i) $A \triangle B = (A \setminus B) \cup (B \setminus A)$.
(ii) $d(A, B) = m^*(A \triangle B)$.
(iii) We say $A_n \to A$ if $\lim_{n\to\infty} d(A_n, A) = 0$.

Definition 1.18. If there is a sequence of elementary sets $\{A_n\}$ such that $A_n \to A$ then we say A is *finitely m-measurable* and we write $A \in \mathfrak{M}_F(m)$.

Definition 1.19. If A is the countable union of finitely m-measurable sets, we say that A is *m-measurable* (Lebesgue measurable) and we write $A \in \mathfrak{M}(m)$.

Theorem 1.20. *$\mathfrak{M}(m)$ is a σ-ring and m^* is countably additive on $\mathfrak{M}(m)$.*

Definition 1.21. The *Lebesgue measure* is the set function defined on $\mathfrak{M}(m)$ by

$$m(A) = m^*(A), \quad \forall A \in \mathfrak{M}(m).$$

To summarize,

set function	**domain**	**properties**
vol	$\mathcal{E}$	≥ 0, additive, $\mathcal{E}$-regular
m^*	$\subseteq \mathbb{R}^n$	≥ 0, $m^*(A) = \operatorname{vol}(A)\ \forall A \in \mathcal{E}$, countably subadditive
m	$\mathcal{M}(m)$	≥ 0, $m(E) = m^*(E)\ \forall E \in \mathcal{M}(m)$, countable additivity(!)

Example 1.22. Fix $n \in \mathbb{N}$.

(i) If $A \in \mathscr{E}$ then $A \in \mathfrak{M}(m)$ since $m^*(A \triangle A) = m^*(\emptyset) = 0 \implies A \to A$.
(ii) $\mathbb{R}^n \in \mathfrak{M}(m)$ since $\mathbb{R}^n = \bigcup_{N\in\mathbb{N}} [-N, N]^n \implies m(\mathbb{R}^n) = \infty$.
(iii) If $A \in \mathfrak{M}(m)$ then $A^c \in \mathfrak{M}(m)$.

(iv) $\forall x \in \mathbb{R}^n$, $\{x\} \in \mathfrak{M}(m)$ and $m(\{x\}) = 0$.

(v) $\forall x_1, \ldots, x_n \in \mathbb{R}^n$, $\{x_1, \ldots, x_n\} \in \mathfrak{M}(m)$ and $m(\{x_1, \ldots, x_n\}) = 0 \implies m(\mathbb{Q}^n) = 0$.

Definition 1.23. $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, \infty\}$ is *measurable* if $\{x \in \mathbb{R}^n : f(x) > a\} \in \mathfrak{M}$, $\forall a \in \mathbb{R}$, i.e. $f^{-1}((a, \infty]) \in \mathfrak{M}$, $\forall a \in \mathbb{R}$.

Example 1.24. f continuous $\implies$ f measurable.

Theorem 1.25. *The following are equivalent,*

(i) $\{x : f(x) > a\}$ *is measurable* $\forall a \in \mathbb{R}$.
(ii) $\{x : f(x) \geq a\}$ *is measurable* $\forall a \in \mathbb{R}$.
(iii) $\{x : f(x) < a\}$ *is measurable* $\forall a \in \mathbb{R}$.
(iv) $\{x : f(x) \leq a\}$ *is measurable* $\forall a \in \mathbb{R}$.

Proof. (i) $\implies$ (ii).

$$\{x : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \left\{x : f(x) > a - \frac{1}{n}\right\}$$

□

Theorem 1.26. *If f is measurable then $|f|$ is measurable.*

Proof. It suffices to show that $\{x : |f(x)| < a\} \in \mathfrak{M}$, $\forall a \in \mathbb{R}$.

$$\{x : |f(x)| < a\} = \{x : f(x) < a\} \cap \{x : f(x) > -a\}$$

□

Theorem 1.27. *Suppose $\{f_n\}$ is a sequence of measurable functions. Define*

$$g = \sup_n f_n \qquad \textit{and} \qquad h = \limsup_{n \to \infty} f_n$$

Then g, h are measurable.

Proof. $\{x : g(x) > a\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > a\}$ implies g is measurable. Similarly, $\inf_n f_n$ is measurable. Define $g_n = \sup_{n \geq m} f_n$ and note that g_n is measurable for all m. Since $h = \inf_m g_m$, h is measurable. □

Corollary 1.28. *If f, g are measurable then* $\max\{f, g\}$ *and* $\min\{f, g\}$ *are also measurable.*

Corollary 1.29. *Define* $f^+ = \max\{f, 0\}$, $f^- = -\min\{f, 0\}$. *Then if f is measurable, f^+, f^- are also measurable.*

Corollary 1.30. *If $\{f_n\}$ is a sequence of measurable functions such that f_n converges to f pointwise, then f is measurable.*

Theorem 1.31. *$f, g : \mathbb{R}^n \to \mathbb{R}$ measurable, $F : \mathbb{R}^2 \to \mathbb{R}$ continuous, and $h(x) = F(f(x), g(x))$. Then h is measurable. In particular, this tells us that $f + g$ and fg are measurable.*

Definition 1.32. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *simple* if range(f) is a finite set.

Example 1.33. Let $E \subseteq \mathbb{R}^n$. The characteristic function of E is

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

Suppose f is simple, so range$(f) = \{c_1, \ldots, c_m\}$. Let $E_i = \{x : f(x) = c_i\}$. Then

$$f = \sum_1^m \chi_{E_i} c_i$$

Theorem 1.34. *$f : \mathbb{R}^n \to \mathbb{R}$. There exists a sequence $\{f_n\}$ of simple functions such that $f_n \to f$ pointwise.*

(i) If f is measurable, $\{f_n\}$ can be chosen to be measurable.
(ii) If $f \geq 0$ then $\{f_n\}$ can be chosen to be monotonically increasing.

Proof. If $f \geq 0$, define the sets

$$E_{n,i} = \left\{ x : \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \right\}, \quad n \geq 1, i = 1, \ldots, n2^n$$

$$F_n = \{x \mid f(x) \geq n\}, \quad n \geq 1$$

Define

$$f_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}} + n\chi_{F_n}$$

We see that f_n is measurable. Fix $x \in \mathbb{R}^n$, let $\varepsilon > 0$, and let $N \in \mathbb{N}$ such that $N > f(x)$ and $2^{-N} < \varepsilon$. Let $n \geq N$. Note that $x \in E_{n,i}$ for some i. Since $f_n(x) = \frac{i-1}{2^n}$ and $f(x) \geq f_n(x)$, $f(x) - f_n(x) \leq \frac{1}{2^n} < \varepsilon$. Thus $f_n \to f$ pointwise. We now show $\{f_n\}$ is monotonically increasing.

(i) Case 1: $x \in F_n$. Then $f(x) \geq n$ and $f_n(x) = n$. If $x \in F_{n+1}$, then $f_{n+1}(x) = n + 1 > n = f_n(x)$. If $x \notin F_{n+1}$ then $x \in$ some $E_{n+1,i}$. Then $\frac{i-1}{2^{n+1}} \geq n \implies f_{n+1}(x) \geq n = f_n(x)$.

(ii) Case 2: $x \in E_{n,i}$ for some i. Then $f_n(x) = \frac{i-1}{2^n}$. Then there is some j such that $x \in E_{n+1,j} = \{x : \frac{j-1}{2^{n+1}} \leq f(x) \leq \frac{j}{2^{n+1}}\}$. Because $\frac{i-1}{2^n} \leq f(x)$, we have $\frac{j-1}{2^{n+1}} \geq \frac{i-1}{2^n}$ so $f_{n+1}(x) = \frac{j-1}{2^{n+1}} \geq \frac{i-1}{2^n} = f_n(x)$.

Thus in both cases, $\{f_n\}$ is monotonically increasing. We next consider the general case. Given f, we write $f^+(x) = \max\{f(x), 0\}$ and $f^- = -\min\{f(x), 0\}$ so that $f = f^+ - f^-$ and $f^+, f^- \geq 0$. By the previous part, there exist two sequences of nonnegative measurable simple functions $f_n^+ \to f^+$ and $f_n^- \to f^-$ each converging pointwise. Define $f_n(x) = f_n^+(x) - f_n^-(x)$. Then f_n is simple and measurable since it is the difference of two simple measurable functions, and converges pointwise. ∎

Definition 1.35. (Lebesgue Integration) Suppose $g = \sum_{i=1}^k c_i \chi_{E_i}$, $c_i > 0$ is measurable and $E \in \mathfrak{M}$. Define

$$I_E(g) = \sum_1^k c_i m(E_i \cap E)$$

Let f be a nonnegative measurable function, $E \in \mathfrak{M}$. Define

$$\int_E f dm = \sup I_E(g)$$

where sup is taken over all measurable simple functions g such that $0 \leq g \leq f$.

Remark 1.36.

(i) $\int_E f dm$ is the Lebesgue integral of f over E.
(ii) It can take value ∞.
(iii) If f is measurable, simple, and nonnegative, then

$$\int_E f dm = I_E(f)$$

Proof. of remark (iii). Suppose for the sake of contradiction that there exists g simple, nonnegative, and measurable such that $0 \leq g \leq f$ and $I_E(g) > I_E(f)$. Then

$$g = \sum_1^k c_i \chi_{E_i}, \quad f = \sum_1^k d_j \chi_{F_j}$$

and

$$I_E(g) = \sum_1^k c_i m(E_i \cap E) > I_E(f) = \sum_1^k d_j m(F_j \cap E)$$

Let $H_{i,j} = E_i \cap F_j$. Since $g \leq f$, $\forall i$, $E_i \subseteq \bigcup F_j$. Hence,

$$\begin{aligned} g &= \sum_{i=1}^k \sum_{j=1}^k c_i \chi_{E_i \cap F_j} \\ &= \sum_{n=1}^M c_n \chi_{H_n} \end{aligned}$$

Note that for every n, $\exists$ unique $F_j \supseteq H_n$. This implies $c_n \leq d_j$, contradiction. □

Definition 1.37. Let f be measurable, and consider $\int_E f^+ dm$ and $\int_E f^- dm$. If at least one is finite, define

$$\int_E f dm = \int_E f^+ dm - \int_E f^- dm$$

If both $\int_E f^+ dm$ and $\int_E f^- dm$ are finite, we say that f is *integrable* on E and write $f \in \mathscr{L}$ on E.

Remark 1.38.

(i) If $a \le f(x) \le b$ for all $x \in E \in \mathfrak{M}$ and $m(E) < \infty$, then $am(E) \le \int_E f dm \le bm(E)$.

(ii) If f is bounded on $E \in \mathfrak{M}$ and $m(E) < \infty$, then $f \in \mathscr{L}$ on E.

(iii) If $f, g \in \mathscr{L}$ on E and $f(x) \le g(x)$ for all $x \in E$, then $\int_E f dm \le \int_E g dm$.

(iv) If $f \in \mathscr{L}$ on $E \in \mathfrak{M}$ and $c \in \mathbb{R}$ then $cf \in \mathscr{L}$ on E and $\int_E cf dm = c \int_E f dm$.

(v) If $m(E) = 0$ then $\int_E f dm = 0$.

(vi) If $f \in \mathscr{L}$ on E, $A \in \mathfrak{M}$, $A \subseteq E$, then $f \in \mathscr{L}$ on A.

(vii) If f is Riemann integrable on $[a, b]$ then $f \in \mathscr{L}$ on $[a, b]$ and the values of the integrals agree.

Proof. of remark (i). Assume $a \ge 0$. $\int_E f dm = \sup \int_E g dm$ where sup is taken over all simple measurable g such that $0 \le g \le f$. Let $g = a$ on E. Then $\int_E f dm \ge \int_E g dm = am(E)$. Let g be a measurable simple function such that $0 \le g \le f$. Then $g = \sum_1^k c_i \chi_{E_i}$ for distinct c_i's and measurable E_i that are disjoint. Since $g \le f \le b$, $c_i \le b$ for all i. So

$$\begin{aligned} \int_E g dm &= \sum_1^k c_i m(E_i \cap E) \\ &\le b \sum_1^k m(E_i \cap E) \\ &\le bm(E) \end{aligned}$$

Hence, $\int_E f dm \le bm(E)$. □

Theorem 1.39.

(i) Suppose f is nonnegative and measurable. For $A \in \mathfrak{M}$ define

$$\phi(A) = \int_A f dm$$

Then ϕ is countably additive on $\mathfrak{M}$.

(ii) The same conclusion holds if $f \in \mathscr{L}$.

Proof. To prove (ii), it suffices to apply (i) to f^+ and f^-. Suppose $\{A_n\}$ is a sequence of measurable sets which are pairwise disjoint. Let $A = \bigcup A_n$.

Step 1 (Characteristic functions). Suppose $f = \chi_E$ for some $E \in \mathfrak{M}$. Then

$$\begin{aligned}
\phi(A) &= \int_A f dm \\
&= m(A \cap E) \\
&= m\left(\left(\bigcup_1^\infty A_n\right) \cap E\right) \\
&= m\left(\bigcup_1^\infty (A_n \cap E)\right) \\
&= \sum_1^\infty m(A_n \cap E) \\
&= \sum_1^\infty \int_{A_n} f dm \\
&= \sum_1^\infty \phi(A_n)
\end{aligned}$$

Step 2 (Simple functions). Suppose f is simple, measurable, and nonnegative, i.e., $f = \sum_1^k c_i \chi_{E_i}$ for disjoint E_i's in $\mathfrak{M}$. Then

$$\begin{aligned}
\phi(A) &= \int_A f dm \\
&= \sum_1^k c_i m(E_i \cap A) \\
&= \sum_1^k c_i \int_A \chi_{E_i} dm \\
&= \sum_1^k c_i \sum_1^\infty \int_{A_n} \chi_{E_i} dm \\
&= \sum_1^\infty \sum_1^k \int_{A_n} c_i \chi_{E_i} dm \\
&= \sum_1^\infty \int_{A_n} f dm
\end{aligned}$$

$$= \sum_1^\infty \phi(A_n)$$

Step 3. Let g be a measurable simple function such that $0 \le g \le f$. Then

$$\begin{aligned}\int_A g dm &= \sum_1^\infty \int_{A_n} g dm \\ &\le \sum_1^\infty \int_{A_n} f dm \\ &= \sum_1^\infty \phi(A_n)\end{aligned}$$

Hence $\phi(A) = \int_A f dm \le \sum_1^\infty \phi(A_n)$.
If $\phi(A_n) = \infty$ for any n, then we are done. Thus assume $\phi(A_n) < \infty$ for every n. Let $\epsilon > 0$, and choose measurable simple g such that $0 \le g \le f$ and $\int_{A_1} g dm \ge \int_{A_1} f dm - \epsilon, \dots, \int_{A_n} g dm \ge \int_{A_n} f dm - \epsilon$. Hence

$$\phi(A_1 \cup \cdots \cup A_n) \ge \phi(A_1) + \cdots + \phi(A_n) - n\epsilon$$

Since ϵ was arbitrary, $\forall n$, $\phi(A_1 \cup \cdots \cup A_n) \ge \phi(A_1) + \cdots + \phi(A_n)$. □

Corollary 1.40. *If $A, B \in \mathfrak{M}$, $m(A \setminus B) = 0$, and $B \subseteq A$, then*

$$\int_A f dm = \int_B f dm$$

for every $f \in \mathscr{L}$.

Theorem 1.41. *If $f \in \mathscr{L}$ on E, then $|f| \in \mathscr{L}$ on E and $|\int_E f dm| \le \int_E |f| dm$.*

Proof. Let $A = \{x \in E \mid f(x) \ge 0\}$ and $B = \{x \in E \mid f(x) < 0\}$. Note that $E = A \sqcup B$ and $A, B \in \mathfrak{M}$. Then

$$\int_E |f| dm = \int_A |f| dm + \int_B |f| dm = \int_E f^+ dm + \int_E f^- dm < \infty$$

Thus $|f| \in \mathscr{L}$. Since $f \le |f|$ and $-f \le |f|$, $\int_E f dm \le \int_E f dm \le \int_E |f| dm$, and $\int_E -f dm = -\int_E f dm \le \int_E |f| dm$ so

$$\left| \int_E f dm \right| \le \int_E |f| dm$$

□

Theorem 1.42. *(Lebesgue's monotone convergence theorem). Let $E \in \mathfrak{M}$ and $\{f_n\}$ a sequence of measurable functions such that*

$$0 \leq f_1(x) \leq f_2(x) \leq \dots \quad \forall (x \in E)$$

Define $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in E$. Then

$$\int_E f_n dm \to \int_E f dm \quad (n \to \infty)$$

Proof. Since $\{f_n\}$ is a monotone sequence of nonnegative measurable functions, $\{\int_E f_n dm\}$ is a monotone sequence of extended real numbers. Thus there must exist $\alpha \in \mathbb{R} \cup \{\pm\infty\}$ such that $\alpha = \lim_{n\to\infty} \int_E f_n dm$. Since $f_n \leq f$ for every n, $\alpha \leq \int_E f dm$. Let $0 < c < 1$ and g be a simple, measurable function such that $0 \leq g \leq f$. For every $n \geq 1$, define

$$E_n = \{x \in E \mid f_n(x) \geq cg(x)\}$$

Since $\{f_n\}$ is increasing, $E_1 \subseteq E_2 \subseteq \dots$. Since $f_n \to f$ pointwise, $E = \bigcup_1^\infty E_n$. For every n, $cg \leq f_n$ on E_n, so

$$c \int_{E_n} g dm = \int_{E_n} cg dm \leq \int_{E_n} f_n dm$$

As $n \to \infty$,

$$\int_{E_n} g dm \to \int_E g dm$$

Therefore, $\alpha \geq c \int_E g dm$. Since $c < 1$ was arbitrary, $\alpha \geq \int_E g dm$. By definition of integration, $\alpha \geq \int_E f dm$. □

Theorem 1.43. *Let $f = f_1 + f_2$, $f_1, f_2 \in \mathscr{L}$ on $E \in \mathfrak{M}$. Then $f \in \mathscr{L}$ on E and $\int_E f dm = \int_E f_1 dm + \int_E f_2 dm$.*

Proof. If f_1, f_2 are simple measurable functions, then the conclusion is immediate. Assume that $f_1, f_2 \geq 0$. Choose a monotonically increasing sequence of nonnegative measurable simple functions $\{g_n\}$ and $\{h_n\}$ converging to f_1 and f_2 respectively. Let $s_n = g_n + h_n$. Then $\forall n$,

$$\int_E s_n dm = \int_E g_n dm + \int_E h_n dm$$

Note. $\{s_n\}$ is a monotonically increasing sequence of simple nonnegative measurable functions converging to f. By the monotone convergence theorem,

$$\int_E f dm = \lim_{n\to\infty} \int_E s_n dm = \lim_{n\to\infty} \int_E g_n dm + \int_E h_n dm = \int_E f_1 dm + \int_E f_2 dm$$

Now assume $f_1 \geq 0, f_2 < 0$. Define

$$A = \{x \in E \mid f(x) \geq 0\} \quad \text{and} \quad B = \{x \in E \mid f(x) < 0\}$$

Note that both A and B are measurable. Since $f, f_1, -f_2 \geq 0$ on A and $f_1 = f + (-f_2)$,

$$\int_A f_1 dm = \int_A f dm + \int_A -f_2 dm = \int_A f dm - \int_A f_2 dm$$

i.e., $\int_A f dm = \int_A f_1 dm + \int_A f_2 dm$. Since $-f, f_1, -f_2 \geq 0$ on B,

$$\int_B -f_2 dm = \int_B -f dm + \int_B f_1 dm$$

i.e., $\int_B f dm = \int_B f_1 dm + \int_B f_2 dm$. Hence

$$\begin{aligned}
\int_{E = A \sqcup B} f dm &= \int_A f dm + \int_B f dm \\
&= \int_A f_1 dm + \int_A f_2 dm + \int_B f_1 dm + \int_B f_2 dm \\
&= \int_E f_1 dm + \int_E f_2 dm
\end{aligned}$$

Let

$$\begin{aligned}
E_1 &= \{x \in E \mid f_1(x) \geq 0, f_2(x) \geq 0\} \\
E_2 &= \{x \in E \mid f_1(x) \geq 0, f_2(x) < 0\} \\
E_3 &= \{x \in E \mid f_1(x) < 0, f_2(x) \geq 0\} \\
E_4 &= \{x \in E \mid f_1(x) < 0, f_2(x) < 0\}
\end{aligned}$$

Apply what we've proven to all four sets and then we get the generalized conclusion.

□

Lemma 1.44. *(Fatou's lemma) $E \in \mathfrak{M}$, $\{f_n\}$ nonnegative measurable functions. Let $f = \liminf_{n\to\infty} f_n$. Then*

$$\int_E f dm \leq \liminf_{n\to\infty} \int_E f_n dm$$

Proof. For every $n \geq 1$, define

$$g_n = \inf_{m \geq n} f_n$$

Note. the g_n's are measurable on E and

(i) $0 \leq g_1 \leq g_2 \leq \dots$.

(ii) $g_n \leq f_n$, $\forall n$.

(iii) $\lim_{n\to\infty} g_n(x) = f(x)$, $\forall x \in E$.

By the monotone convergence theorem,

$$\lim_{n\to\infty} \int_E g_n dm = \int_E f dm$$

By property (ii),

$$\int_E g_n dm \leq \int_E f_n dm \quad \forall n$$

Together, these two imply the conclusion. ☐

Theorem 1.45. *(Dominated convergence theorem) Suppose $E \in \mathfrak{M}$, $\{f_n\}$ measurable on E such that $f_n \to f$ pointwise on E. Suppose $\exists g \in \mathscr{L}$ on E such that $|f_n(x)| \leq g(x)$ for all $x \in E$. Then*

$$\int_E f dm = \lim_{n\to\infty} \int_E f_n dm$$

Proof. Note $f_n \in \mathscr{L}$ on E for all n and $f \in \mathscr{L}$ on E. Since $f_n + g \geq 0$ for all n, applying Fatou's Lemma gives

$$\int_E (f+g) dm \leq \liminf_{n\to\infty} \int_E (f_n + g) dm$$

Then

$$\begin{aligned} \int_E f dm + \int_E g dm &\leq \liminf_{n\to\infty} \left(\int_E f_n dm + \int_E g dm \right) \\ &= \left(\liminf_{n\to\infty} \int_E f_n dm \right) + \int_E g dm \end{aligned}$$

Thus

$$\int_E f dm \leq \liminf_{n\to\infty} \int_E f_n dm$$

Since $g - f_n \geq 0$, apply Fatou's Lemma to get

$$\int_E (g - f) dm \leq \liminf_{n\to\infty} \left(\int_E (g - f_n) dm \right)$$

By the same logic as above, we see that

$$-\int_E f dm \leq \liminf_{n\to\infty} -\int_E f_n dm$$

We conclude that

$$\int_E f \geq \limsup_{n\to\infty} \int_E f_n dm$$

Thus

$$\int_E f dm = \lim_{n\to\infty} \int_E f_n dm$$

☐

Lemma 1.46. *Nonmeasurable sets exist (assuming Axiom of Choice).*

Proof. For every $a \in [-1,1]$ define $\tilde{a} = \{c \in [-1,1] : a - c \in \mathbb{Q}\}$.

Claim 1. If $\tilde{a} \cap \tilde{b} \neq \emptyset$ then $\tilde{a} = \tilde{b}$.
Suppose $c \in \tilde{a} \cap \tilde{b}$. Then $a - c \in \mathbb{Q}$, $b - c \in \mathbb{Q}$, and therefore $a - b, b - a \in \mathbb{Q}$. Let $d \in \tilde{a}$, so $a - d \in \mathbb{Q}$. Then $a - d = (a - b) + (b - d)$ so $b - d \in \mathbb{Q}$, i.e., $d \in \tilde{b}$ and the claim follows.
Note. $[-1,1] = \bigcup_{a \in [-1,1]} \tilde{a}$. Let V be a set that contains exactly one element from every distinct $\tilde{a}$ (Axiom of Choice). Let $r_1, r_2, \ldots$ be an enumeration of $\mathbb{Q} \cap [-2,2]$.

Claim 2. $[-1,1] \subseteq \bigcup_{k=1}^{\infty} V + r_k$.
Let $d \in [-1,1]$, so $d \in \tilde{a}$ for some a. Let $c \in V$ s.t. $c \in \tilde{a}$. Then $c - d \in \mathbb{Q} \cap [-2,2]$ so $c - d = r_k$ for some k. Hence, $d \in V + r_k$.
By Claim 2,

$$2 = m^*([-1,1]) \leq m^*\left(\bigcup_1^{\infty} V + r_k\right) \leq \sum_1^{\infty} m^*(V + r_k) = \sum_1^{\infty} m^*(V)$$

Thus $m^*(V) > 0$.

Claim 3. $V + r_1, V + r_2, \ldots$ are disjoint.
Suppose ftsoc that $d \in (V + r_k) \cap (V + r_\ell)$. Then $d = v + r_k$, $v \in V$ and $d = v' + r_\ell$, $v' \in V$. In particular, $v - v' \in \mathbb{Q}$. By Claim 1, $v, v' \in \tilde{a}$. Contradiction.
For any $n \in \mathbb{N}$,

$$\bigcup_{k=1}^{n} V + r_k \subseteq [-3,3]$$

Hence,

$$m^*\left(\bigcup_1^{\infty} V + r_k\right) \leq 6$$

Let $n \in \mathbb{N}$ such that $nm^*(V) > 6$. Then

$$m^*\left(\bigcup_1^{n} V + r_k\right) < \sum_1^{n} m^*(V + r_k)$$

Which implies that $V + r_1, V + r_2, \ldots$ cannot all be measurable. Hence, V is not measurable. ☐

Definition 1.47. Let $E \in \mathfrak{M}$, f measurable. We write $f \in \mathscr{L}^2$ on E if

$$\int_E |f|^2 dm < \infty$$

Remark 1.48. $f \in \mathscr{L}$ on E ($\mathscr{L}^1$) if $\int_E |f| dm < \infty$.

Example 1.49.
(i) $E = (0, 1]$, $f(x) = x^{-1/2}$. $f \in \mathscr{L}^1, f \notin \mathscr{L}^2$.
(ii) $E = (1, \infty)$, $f(x) = \frac{1}{x}$. $f \notin \mathscr{L}^1, f \in \mathscr{L}^2$.

Theorem 1.50. *If $m(E) < \infty$, then $f \in \mathscr{L}^2 \implies f \in \mathscr{L}^1$.*

2 Fourier Analysis

Recall. Let $f : \mathbb{R} \to \mathbb{C}$. We can decompose f into its real and imaginary components,

$$f = f_{RE} + i f_{IM}$$

where $f_{RE}, f_{IM} : \mathbb{R} \to \mathbb{R}$.
We say $f \in \mathscr{R}$ (Riemann integrable) if $f_{RE}, f_{IM} \in \mathscr{R}$ and

$$\int_{-\infty}^{\infty} f dx = \int_{-\infty}^{\infty} f_{RE} dx + i \int_{-\infty}^{\infty} f_{IM} dx$$

Definition 2.1. A *trigonometric polynomial* is a function

$$f(x) = a_0 + \sum_{1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

where $a_0, \ldots, a_N, b_1, \ldots, b_N \in \mathbb{C}$.

Note. We can equivalently write this as

$$f(x) = \sum_{-N}^{N} c_n e^{inx}$$

where $c_{-N}, \ldots, c_N \in \mathbb{C}$.