

# MATH 20510. Accelerated Analysis III

Based on lectures by Donald Stull

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Spring 2025

## 1 Measure Theory

**Definition 1.1.** A family of sets  $A$  is called a *ring* if, for every  $A, B \in A$ ,

- (i)  $A \cup B \in A$
- (ii)  $A \setminus B \in A$

**Definition 1.2.** A ring  $A$  is called a  $\sigma$ -*ring* if for any  $\{A_n\}_1^\infty \subseteq A$ ,

$$\bigcup_1^\infty A_n \in A$$

Note. This implies that  $\bigcap_1^\infty A_n \in A$ .

**Definition 1.3.**  $\phi$  is a *set function* on a ring  $A$  if for every  $A \in A$ ,

$$\phi(A) \in [-\infty, \infty]$$

**Definition 1.4.** A set function  $\phi$  is *additive* if for any  $A, B \in A$  such that  $A \cap B = \emptyset$ ,

$$\phi(A \cup B) = \phi(A) + \phi(B)$$

**Definition 1.5.** A set function  $\phi$  is *countably additive* if for any  $\{A_n\} \subseteq A$  such that  $A_i \cap A_j = \emptyset$ ,  $\forall i \neq j$ ,

$$\phi\left(\bigcup_1^n A_n\right) = \sum_1^n \phi(A_n)$$

In the last two we assume that there are no  $A, B \in A$  such that  $\phi(A) = -\infty, \phi(B) = \infty$ .

**Remark 1.6.** If  $\phi$  is an additive set function,

- (i)  $\phi(\emptyset) = 0$ .
- (ii) If  $A_1, \dots, A_n$  are pairwise disjoint then  $\phi(\bigcup_1^n A_n) = \sum_1^n \phi(A_n)$ .
- (iii)  $\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2)$ .
- (iv) If  $\phi$  is nonnegative and  $A_1 \subseteq A_2$  then  $\phi(A_1) \leq \phi(A_2)$ .

(v) If  $B \subseteq A$  and  $|\phi(B)| < \infty$  then  $\phi(A \setminus B) = \phi(A) - \phi(B)$ .

**Theorem 1.7.** Let  $\phi$  be a countably additive set function on a ring  $A$ . Suppose  $\{A_n\} \subseteq A$  such that  $A_1 \subseteq A_2 \subseteq \dots$  and  $A = \bigcup_1^\infty A_n \in A$ . Then  $\phi(A_n) \rightarrow \phi(A)$  as  $n \rightarrow \infty$ .

*Proof.* Set  $B_1 = A_1$  and  $B_n = A_n \setminus A_{n-1}$ . Note

- (i)  $\{B_n\}$  is pairwise disjoint.
- (ii)  $A_n = B_1 \cup B_2 \cup \dots \cup B_n$ .
- (iii)  $A = \bigcup_1^\infty B_n$ .

Hence  $\phi(A_n) = \sum_1^\infty \phi(B_j)$ ,  $\phi(A) = \sum_1^\infty \phi(B_j)$  and the conclusion follows.  $\square$

**Definition 1.8.** An *interval*  $I = \{(a_i, b_i)\}_1^n$  of  $\mathbb{R}^n$  is the set of points  $x = (x_1, \dots, x_n)$  such that  $a_i \leq x_i \leq b_i$  or  $a_i < x_i \leq b_i$ , etc. where  $a_i \leq b_i$ .

Note.  $\emptyset$  is an interval.

**Definition 1.9.** If  $A$  is the union of a finite number of intervals, we say  $A$  is *elementary*.

We denote the set of elementary sets by  $E$ .

**Definition 1.10.** If  $I$  is an interval of  $\mathbb{R}^n$ , we define the volume of  $I$  by

$$\text{vol}(I) = \prod_i^n (b_i - a_i)$$

If  $A = I_1 \cup I_2 \cup \dots \cup I_k$  is elementary, and the intervals are disjoint, then

$$\text{vol}(A) = \sum_1^k \text{vol}(I_j)$$

**Remark 1.11.** Test

**Remark 1.12.** This is technically test 3.

## 2 Fourier Analysis