MATH 20510
Lecture 16

April 30, 2025

Definition 0.1. A 2-*surface* is a C^1 map $\gamma : I^2 \to \mathbb{R}^n$.

Definition 0.2. (Informal) A 2-*form* on $\mathbb{R}^n$ is

(i) An object which can be integrated over any 2-surface.

(ii) A rule which assigns a real number to every oriented parallelogram in $\mathbb{R}^n$ in a "suitable" way.

Specify an oriented parallelogram in $\mathbb{R}^n$ based at $p \in \mathbb{R}^n$ by giving (v, w). We want every 2-form ω to satisfy the following for every $p \in \mathbb{R}^n$

(i) $\omega_p(tv_1, v_2) = \omega_p(v_1, tv_2) = t\omega_p(v_1, v_2)$.

(ii) $\omega_p(v_1, v_2 + v_3) = \omega_p(v_1, v_2) + \omega_p(v_1, v_3)$ and $\omega_p(v_1 + v_2, v_3) = \omega_p(v_1, v_3) + \omega_p(v_2, v_3)$.

(iii) $\omega_p(v_1, v_2) = -\omega_p(v_2, v_1)$.

Basic 2-forms on $\mathbb{R}^n$. $\forall v, w \in \mathbb{R}^n$,

(i) $(dx_1 \wedge dx_2)(v, w) = \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$.

(ii) $(dx_1 \wedge dx_3)(v, w) = \det \begin{pmatrix} v_1 & w_1 \\ v_3 & w_3 \end{pmatrix}$.

(iii) $(dx_i \wedge dx_j)(v, w) = \det \begin{pmatrix} v_i & w_i \\ v_j & w_j \end{pmatrix}$.

Remark 0.3. If ω_p satisfies $(i) - (iii)$ then ω_p can be expressed as

$$\omega_p = \sum_{i,j} A_{i,j}(p)(dx_i \wedge dx_j)$$

for constant $A_{i,j}$.

Definition 0.4. A 2-*form* in $\mathbb{R}^n$ is a rule assigning a real number to each oriented parallelogram in $\mathbb{R}^n$ that can be written as

$$\omega = \sum_{i,j} f_{i,j}(dx_i \wedge dx_j)$$

where $f_{i,j} : \mathbb{R}^n \to \mathbb{R}$ is C^2.

For any $p \in \mathbb{R}^n$, $v, w \in \mathbb{R}^n$,

$$\omega_p(v,w) = \sum_{i,j} f_{i,j}(p)(dx_i \wedge dx_j)(v,w).$$

Example 0.5. ω is a 2-form in $\mathbb{R}^2$,

$$\begin{aligned}\omega &= f_{1,1}\underbrace{(dx_1 \wedge dx_1)}_{=0} + f_{1,2}(dx_1 \wedge dx_2) + f_{2,1}\underbrace{(dx_2 \wedge dx_1)}_{=-(dx_1 \wedge dx_2)} + f_{2,2}\underbrace{(dx_2 \wedge dx_2)}_{=0} \\ &= (f_{1,2} - f_{2,1})(dx_1 \wedge dx_2)\end{aligned}$$

This implies that every 2-form in $\mathbb{R}^2$ can be written as $\omega = f(dx_1 \wedge dx_2)$ where f is C^2.

Example 0.6. ω is a 2-form in $\mathbb{R}^3$,

$$\omega = f_1(dx_1 \wedge dx_2) + f_2(dx_1 \wedge dx_3) + f_3(dx_2 \wedge dx_3).$$

Definition 0.7. Let $\gamma : I^2 \to \mathbb{R}^3$ be C^1, and $\omega = f_1(dx_1 \wedge dx_2) + f_2(dx_1 \wedge dx_3) + f_3(dx_2 \wedge dx_3)$ be a 2-form. Then

$$\begin{aligned}\int_\gamma \omega &= \int_{I^2} \omega_{\gamma(z)} \left(\frac{\partial \gamma}{\partial x_1}(z), \frac{\partial \gamma}{\partial x_2}(z) \right) dz \\ &= \int_{I^2} f_1(\gamma(z))(dx_1 \wedge dx_2) \left(\frac{\partial \gamma}{\partial x_1}(z), \frac{\partial \gamma}{\partial x_2}(z) \right) \\ &\qquad + f_2(\gamma(z))(dx_1 \wedge dx_3) \left(\frac{\partial \gamma}{\partial x_1}(z), \frac{\partial \gamma}{\partial x_2}(z) \right) + f_3(\gamma(z))(dx_2 \wedge dx_3) \left(\frac{\partial \gamma}{\partial x_1}(z), \frac{\partial \gamma}{\partial x_2}(z) \right) dz \\ &= \int_{I^2} f_1(\gamma(z)) \det \begin{pmatrix} D_1\gamma_1(z) & D_2\gamma_1(z) \\ D_1\gamma_2(z) & D_2\gamma_2(z) \end{pmatrix} \\ &\qquad + f_2(\gamma(z)) \det \begin{pmatrix} D_1\gamma_1(z) & D_2\gamma_1(z) \\ D_1\gamma_3(z) & D_2\gamma_3(z) \end{pmatrix} + f_3(\gamma(z)) \det \begin{pmatrix} D_1\gamma_2(z) & D_2\gamma_2(z) \\ D_1\gamma_3(z) & D_2\gamma_3(z) \end{pmatrix}\end{aligned}$$