MATH 23500
Lecture 22

May 14, 2025

Remark 0.1. Suppose $\{X_n\}_{n\in\mathbb{N}}$ is a discrete stochastic process, and $A \subseteq S$ has $\mathbb{P}(X_n \in A) = 0$ for all n. Then $\mathbb{P}(\exists n \text{ such that } X_n \in A) \leq \sum_{n=0}^{\infty} \mathbb{P}(X_n \in A) = 0$. This is not true for Brownian motion. $\mathbb{P}(B_t = 1) = 0$, while $\mathbb{P}(\exists t \text{ such that } B_t = 1) = 1$.

Example 0.2. Suppose B is a standard Brownian motion.

(i) Find $\mathbb{E}[B_4 \mid B_2 = 6]$. Note $B_4 = (B_4 - B_2) + B_2$. Thus we seek

$$\begin{aligned}
\mathbb{E}[(B_4 - B_2) + B_2 \mid B_2 = 6] &= \mathbb{E}[B_4 - B_2 \mid B_2 = 6] + \mathbb{E}[B_2 \mid B_2 = 6] \\
&= 0 + 6 \\
&= 6.
\end{aligned}$$

(ii) Find $\mathbb{E}[B_2^2 B_t^2]$ for $s \leq t$. Note

$$\begin{aligned}
B_t^2 &= ((B_t - B_s) + B_s)^2 \\
&= B_{t-s}^2 + 2B_s B_{t-s} + B_s^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}[B_s^2 B_t^2] &= \mathbb{E}[B_s^2 B_{t-s}^2] + \mathbb{E}[2B_s^3 B_{t-s}] + \mathbb{E}[B_s^4] \\
&= s(t-s) + 3s^2.
\end{aligned}$$

Assume $X \sim \mathcal{N}(0, \sigma^2)$. The moment generating function of X is given by

$$\begin{aligned}
M_X(t) &:= \mathbb{E}[e^{tX}] \\
&= e^{\frac{\sigma^2 t^2}{2}}.
\end{aligned}$$

The left hand side is

$$\begin{aligned}
\mathbb{E}[e^{tX}] &= \sum_{k=0}^{\infty} \frac{\mathbb{E}[(tx)^k]}{k!} \\
&= \sum_{k=0}^{\infty} \frac{\mathbb{E}[X^k]}{k!} t^k.
\end{aligned}$$

On the other hand, the right hand side is

$$
\begin{aligned}
e^{\frac{\sigma^2 t^2}{2}} &= \sum_{k=0}^{\infty} \frac{\left(\frac{\sigma^2 t^2}{2}\right)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{\sigma^{2k}}{2^k k!} t^{2k}.
\end{aligned}
$$

Comparing the coefficients of t^{2n}, we find that

$$
\frac{\mathbb{E}[X^{2n}]}{2n!} = \frac{\sigma^{2n}}{2^n n!} \implies \mathbb{E}[X^{2n}] = \frac{\sigma^{2n}(2n)!}{2^n n!} = (2n-1)!!\sigma^{2n}.
$$

Example 0.3. Suppose B is a standard Brownian motion. Find $\mathbb{P}(B_2 > B_1 > B_3)$. Let $X = B_1, Y = B_2 - B_1, Z = B_3 - B_2$. We seek

$$
\begin{aligned}
\mathbb{P}(X + Y > X > X + Y + Z) &= \mathbb{P}(Y > 0 > Y + Z) \\
&= \mathbb{P}(B_1 > 0, B_2 < 0) \\
&= \int_0^{\infty} \mathbb{P}(B_2 < 0 \mid B_1 = x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.
\end{aligned}
$$

Note

$$
\begin{aligned}
\mathbb{P}(B_2 < 0 \mid B_1 = x) &= \mathbb{P}(B_2 < -x \mid B_1 = 0) \\
&= \mathbb{P}(B_2 > x \mid B_1 = 0).
\end{aligned}
$$

Thus, the integral becomes

$$
\begin{aligned}
\int_0^{\infty} \left(\int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \right) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx &= \frac{1}{2\pi} \int_0^{\infty} \int_x^{\infty} e^{-\left(\frac{x^2+y^2}{2}\right)} dy dx \\
&= \int_0^{\infty} \int_{\pi/4}^{\pi/2} r e^{-r^2/2} d\theta dr \\
&= \frac{1}{8}.
\end{aligned}
$$