MATH 20510. Accelerated Analysis III
Lecture 1

March 24, 2025

Definition 0.1. A family of sets A is called a *ring* if, for every $A, B \in A$,

(i) $A \cup B \in A$
(ii) $A \setminus B \in A$

Definition 0.2. A ring A is called a *σ-ring* if for any $\{A_n\}_1^\infty \subseteq A$,

$$\bigcup_1^\infty A_n \in A$$

Note. This implies that $\bigcap_1^\infty A_n \in A$.

Definition 0.3. ϕ is a *set function* on a ring A if for every $A \in A$,

$$\phi(A) \in [-\infty, \infty]$$

Definition 0.4. A set function ϕ is *additive* if for any $A, B \in A$ such that $A \cap B = \emptyset$,

$$\phi(A \cup B) = \phi(A) + \phi(B)$$

Definition 0.5. A set function ϕ is *countably additive* if for any $\{A_n\} \subseteq A$ such that $A_i \cap A_j = \emptyset$, $\forall i \neq j$,

$$\phi\left(\bigcup_1^n A_n\right) = \sum_1^n \phi(A_n)$$

In the last two we assume that there are no $A, B \in A$ such that $\phi(A) = -\infty, \phi(B) = \infty$.

Remark 0.6. If ϕ is an additive set function,

(i) $\phi(\emptyset) = 0$.
(ii) If $A_1, \ldots, A_n$ are pairwise disjoint then $\phi(\bigcup_1^n A_n) = \sum_1^n \phi(A_n)$.
(iii) $\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2)$.
(iv) If ϕ is nonnegative and $A_1 \subseteq A_2$ then $\phi(A_1) \leq \phi(A_2)$.
(v) If $B \subseteq A$ and $|\phi(B)| < \infty$ then $\phi(A \setminus B) = \phi(A) - \phi(B)$.

Theorem 0.7. *Let ϕ be a countably additive set function on a ring A. Suppose $\{A_n\} \subseteq A$ such that $A_1 \subseteq A_2 \subseteq \ldots$ and $A = \bigcup_1^\infty A_n \in A$. Then $\phi(A_n) \to \phi(A)$ as $n \to \infty$.*

Proof. Set $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$. Note

(i) $\{B_n\}$ is pairwise disjoint.

(ii) $A_n = B_1 \cup B_2 \cup \cdots \cup B_n$.

(iii) $A = \bigcup_1^\infty B_n$.

Hence $\phi(A_n) = \sum_1^\infty \phi(B_j)$, $\phi(A) = \sum_1^\infty \phi(B_j)$ and the conclusion follows. $\square$

Definition 0.8. An *interval* $I = \{(a_i, b_i)\}_1^n$ of $\mathbb{R}^n$ is the set of points $x = (x_1, \ldots, x_n)$ such that $a_i \leq x_i \leq b_i$ or $a_i < x_i \leq b_i$, etc. where $a_i \leq b_i$.

Note. $\emptyset$ is an interval.

Definition 0.9. If A is the union of a finite number of intervals, we say A is *elementary*.

We denote the set of elementary sets by E.

Definition 0.10. If I is an interval of $\mathbb{R}^n$, we define the volume of I by

$$\mathrm{vol}(I) = \prod_i^n (b_i - a_i)$$

If $A = I_1 \cup I_2 \cup \cdots \cup I_k$ is elementary, and the intervals are disjoint, then

$$\mathrm{vol}(A) = \sum_1^k \mathrm{vol}(I_j)$$