MATH 20510
Lecture 20

May 9, 2025

Let $E \subseteq \mathbb{R}^n$ open, $V \subseteq \mathbb{R}^m$ open, and $T : E \to V$ be a C^1 function. Let ω be a k-form on V.

Notation. We say elements of E are $x \in E$ and elements of V are $y \in V$.

Then write $\omega = \sum_I b_I(y)dy$ in standard presentation.

$$T(x) = (t_1(x), \dots, t_m(x)) = (y_1, \dots, y_m) = y.$$

Then

$$dt_i = \sum_{j=1}^{n} (D_j t_i)(x) dx_j \qquad i \in [m].$$

Note. dt_i is a 1-form on E.

T will transform the k-form ω on E to a k-form ω_T on E. This is called the *pullback* form.

$$\omega_T(x) = \sum_I b_I(T(x)) dt_{i_1} \wedge dt_{i_2} \wedge \cdots \wedge dt_{i_k}.$$

Example 0.1. Let $\mathrm{id} = T : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x = y$, and $\omega(y) = \sum b_I(y) dy_I$. Then $t_i(x) = x_i$ so $dt_i = dx_i$. Thus

$$\omega_T(x) = \sum_I b_I(T(x)) dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Example 0.2. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$, $(x_1, x_2) \mapsto (x_2, x_1^2, x_1 + x_2)$, and $\omega(y_1, y_2, y_3) = y_1 dy_2 \wedge dy_3$ is a 2-form on $\mathbb{R}^3$. Then $dt_1 = dx_2$, $dt_2 = 2x_1 dx_1$, $dt_3 = dx_1 + dx_2$. Thus

$$\begin{aligned}
\omega_T(x_1, x_2) &= b_{\{2,3\}}(T(x_1, x_2)) dt_2 \wedge dt_3 \\
&= x_2 (2x_1 dx_1) \wedge (dx_1 + dx_2) \\
&= 2x_2 x_1 (dx_1 \wedge dx_1 + dx_1 \wedge dx_2) \\
&= 2x_2 x_1 dx_1 \wedge dx_2.
\end{aligned}$$

Lemma 0.3. *Let $f : V \to \mathbb{R}$ be a C^1 function and $f_T = f \circ T$. Then $d(f_T) = (df)_T$.*

Proof.

$$\begin{aligned}
d(f_T) &= \sum_{j=1}^{n} D_j f_T dx_j \\
&= \sum_{j=1}^{n} D_j (f \circ T) dx_j \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} (D_i f)(T) \cdot (D_j t_i) dx_j \\
&= \sum_{i=1}^{m} (D_i f)(T) dt_i \\
&= (df)_T.
\end{aligned}$$

□

Theorem 0.4. *Let ω be a k-form and λ be an l-form on V. Then*

(i) $(\omega + \lambda)_T = \omega_T + \lambda_T$ *if* $k = l$.
(ii) $(\omega \wedge \lambda)_T = \omega_T \wedge \lambda_T$.
(iii) $d(\omega_T) = (d\omega)_T$ *if* ω *is of class* C^1 *and* T *is of class* C^2.

(i) *Proof.*

$$\begin{aligned}
(\omega + \lambda)_T &= (\sum (b_I + c_I) dy_I)_T \\
&= \sum (b_I + c_I)(T) dt_I \\
&= \sum b_I(T) dt_I + \sum c_I(T) dt_I \\
&= \omega_T +_T .
\end{aligned}$$

□

(ii) HW

(iii) *Proof.*

$$\begin{aligned}
\omega &= dy_I \\
&= dy_{i_1} \wedge \cdots \wedge dy_{i_k},
\end{aligned}$$

and

$$\omega_T = dt_{i_1} \wedge \cdots \wedge dt_{i_k}.$$

We have

$$\begin{aligned}
d(\omega) &= \sum d(b_I) \wedge dy_I \\
&= 0.
\end{aligned}$$

Thus equivalently,

$$\begin{aligned} d(\omega)_T &= \left(\sum d(b_I) \wedge dy_I\right)_T \\ &= 0. \end{aligned}$$

We will finish proof next lecture. $\square$