MATH 23500
Lecture 15

April 28, 2025

Suppose X and Y take values in countable sets $S, T \subset \mathbb{R}$. For $x \in S, y \in T$, let

$$f(x,y) = \mathbb{P}\{X = x, Y = y\}$$

Then

$$\mathbb{E}[X \mid Y = y] = \sum_{x \in S} x\mathbb{P}\{X = x \mid Y = y\} = \frac{\sum_{x \in S} x f(x,y)}{\sum_{x \in S} f(x,y)}$$

We define the random variable

$$\mathbb{E}[X \mid Y] = \frac{\sum_{x \in S} x f(x,Y)}{\sum_{x \in S} f(x,Y)}$$

Definition 0.1. Let X, Y be random variables with X taking values in $\mathbb{R}$, and $\mathbb{E}[|X|] < \infty$. The *conditional expectation* $\mathbb{E}[X \mid Y]$ is the unique random variable satisfying the following

(i) $\mathbb{E}[X \mid Y]$ is a function of Y.

(ii) If $F(Y)$ is a function of Y with $\mathbb{E}[|F(Y)|] < \infty$, then

$$\mathbb{E}[XF(Y)] = \mathbb{E}[\mathbb{E}[X \mid Y]F(Y)]$$

We typically consider a sequence $\{Y_n\}_{n \geq 0}$ of random variables and condition on $Y_0, Y_1, \dots, Y_n$ for some n.

Definition 0.2. We write $\mathcal{F}_n$ for the information contained in $Y_0, \dots, Y_n$, i.e., $\mathcal{F}_n$ is the smallest σ-algebra generated by $Y_0, \dots, Y_n$. Thus,

$$\mathbb{E}[X \mid \mathcal{F}_n] = \mathbb{E}[X \mid Y_0, \dots, Y_n], \quad \forall \text{r.v.'s } X$$

We say that X is $\mathcal{F}_n$-measurable if it is a function of $Y_0, \dots, Y_n$.

Proposition 0.3. $\mathbb{E}[X \mid \mathcal{F}_n]$ *is the unique random variable satisfying the following*

(i) $\mathbb{E}[X \mid \mathcal{F}_n]$ *is* $\mathcal{F}_n$*-measurable.*

(ii) If Z *is* $\mathcal{F}_n$*-measurable, then* $\mathbb{E}[XZ] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{F}_n]Z]$.

Proposition 0.4. *If X_1, X_2 are real-valued random variables and $a, b \in \mathbb{R}$ are non-random, then*

$$\mathbb{E}[aX_1 + bX_2 \mid \mathcal{F}_n] = a\mathbb{E}[X_1 \mid \mathcal{F}_n] + b\mathbb{E}[X_2 \mid \mathcal{F}_n]$$

Proof. Suppose Z is $\mathcal{F}_n$-measurable.

$$\mathbb{E}[\mathbb{E}[aX_1 + bX_2 \mid \mathcal{F}_n]Z] = \mathbb{E}[(aX_1 + bX_2)Z]$$

Also,

$$\begin{aligned}
\mathbb{E}[(\mathbb{E}[X_1 \mid \mathcal{F}_n] + b\mathbb{E}[X_2 \mid \mathcal{F}_n])Z] &= a\mathbb{E}[\mathbb{E}[X_1 \mid \mathcal{F}_n]Z] + b\mathbb{E}[\mathbb{E}[X_2 \mid \mathcal{F}_n]Z] \\
&= a\mathbb{E}[X_1 Z] + b\mathbb{E}[X_2 Z] \\
&= \mathbb{E}[(aX_1 + bX_2)Z]
\end{aligned}$$

□

Proposition 0.5. *Suppose X is $\mathcal{F}_n$-measurable. Then*

$$\mathbb{E}[X \mid \mathcal{F}_n] = X$$

Proposition 0.6. *Suppose X is independent of $\mathcal{F}_n$. Then*

$$\mathbb{E}[X \mid \mathcal{F}_n] = \mathbb{E}[X]$$

Example 0.7. Suppose $X_1, X_2, \ldots$ are i.i.d. with mean μ. Let $\mathcal{F}_n$ be the information contained in $X_1, \ldots, X_n$. Find

$$\mathbb{E}[X_1 + \cdots + X_n \mid \mathcal{F}_m]$$

If $m \geq n$,

$$\mathbb{E}[X_1 + \cdots + X_n \mid \mathcal{F}_m] = X_1 + \cdots + X_n$$

If $m < n$,

$$\begin{aligned}
\mathbb{E}[X_1 + \cdots + X_n \mid \mathcal{F}_m] &= \mathbb{E}[X_1 + \cdots + X_m \mid \mathcal{F}_m] + \mathbb{E}[X_{m+1} + \cdots + X_n \mid \mathcal{F}_m] \\
&= X_1 + \cdots + X_m + (n-m)\mu
\end{aligned}$$

Example 0.8. Suppose $X_1, X_2, \ldots$ are i.i.d. with mean 0, variance σ^2. Let $\mathcal{F}_n$ be the information contained in $X_1, \ldots, X_n$. Find

$$\mathbb{E}[(X_1 + \cdots + X_n)^2 \mid \mathcal{F}_m]$$

Let $S_n = X_1 + \ldots X_n$. Then

$$S_n^2 = (S_m + (S_n - S_m))^2 = S_m^2 + (S_n - S_m)^2 + 2S_m(S_n - S_m)$$

Thus,

$$\begin{aligned}
\mathbb{E}[S_n^2 \mid \mathcal{F}_m] &= \mathbb{E}[S_m^2 \mid \mathcal{F}_m] + \mathbb{E}[(S_n - S_m)^2 \mid \mathcal{F}_m] + \mathbb{E}[2S_m(S_n - S_m) \mid \mathcal{F}_m] \\
&= S_m^2 + (n-m)\sigma^2
\end{aligned}$$