MATH 20510
Lecture 22

May 14, 2025

Definition 0.1. Let $p_0, p_1, \dots, p_k \in \mathbb{R}^n$. The *oriented affine k-simplex* $\sigma = [p_0, p_1, \dots, p_k]$ is the k-surface in $\mathbb{R}^n$ with parameter domain Q^k given by the affine map

$$\sigma(\alpha_1 e_1 + \cdots + \alpha_k e_k) = p_0 + \sum_1^k \alpha_i (p_i - p_0).$$

Remark 0.2. $\sigma(u) = p_0 + Au$, $u \in Q^k$, where $A \in L(\mathbb{R}^k, \mathbb{R}^n)$. $Ae_i = p_i - p_0$, $\forall 1 \leq i \leq k$.

Remark 0.3. σ is called oriented to emphasize that the order of the points $p_0, p_1, \dots, p_k$ matters. If $\overline{\sigma} = [p_{i_0}, p_{i_1}, \dots, p_{i_k}]$ where $\{i_0, i_1, \dots, i_k\}$ is a permutation of $\{0, 1, \dots, k\}$, then

$$\overline{\sigma} = s(i_0, i_1, \dots, i_k)\sigma,$$

where $s(i_0, i_1, \dots, i_k) = (-1)^\alpha$, where α is the minimum number of swaps needed to transform $0, 1, \dots, k$ to $i_0, i_1, \dots, i_k$. If $s(i_0, i_1, \dots, i_k) = 1$, then we say σ and $\overline{\sigma}$ have the same orientation, and if $s(i_0, i_1, \dots, i_k) = -1$, we say σ and $\overline{\sigma}$ have opposite orientation.

Definition 0.4. An oriented 0-simplex is a point $p \in \mathbb{R}^n$ with a sign attached, and we write $\sigma = +p_0$ or $\sigma = -p_0$. If f is a 0-form, $\sigma = \varepsilon p_0, \varepsilon = \pm 1$,

$$\int_\sigma f = \varepsilon f(p_0).$$

Theorem 0.5. *If σ is an oriented k-simplex in an open set $E \subseteq \mathbb{R}^n$ and if $\overline{\sigma} = \varepsilon\sigma, \varepsilon = \pm 1$, then $\forall k$-forms ω on E,*

$$\int_\sigma \omega = \varepsilon \int_{\overline{\sigma}} \omega.$$

Definition 0.6. An *affine k-chain* Γ in an open set $E \subseteq \mathbb{R}^n$ is a collection of finitely many oriented affine k-simplexes $\sigma_1, \dots, \sigma_r$ in E.

Note. The simplexes need not be distinct.

Definition 0.7. If Γ is an an affine k-chain in an open set $E \subseteq \mathbb{R}^n$ and ω is a k-form on E,

$$\int_\Gamma \omega = \sum_1^r \int_{\sigma_i} \omega.$$

Notation. This suggests the following notation,

$$\Gamma = \sigma_1 + \cdots + \sigma_r = \sum_1^r \sigma_i.$$

Warning. This is *just* notation.

Example 0.8. $\sigma_1 = [p_0, p_1, p_2]$ and $\sigma_2 = [p_1, p_0, p_2]$, i.e., $\sigma_1 = -\sigma_2$. Then

$$\int_\Gamma \omega = \int_{\sigma_1} \omega + \int_{\sigma_2} \omega = \int_{\sigma_1} \omega - \int_{\sigma_1} \omega = 0.$$

Definition 0.9. For $k \geq 1$, the *boundary* of an oriented affine k-simplex $\sigma = [p_0, p_1, \ldots, p_k]$ is the affine $(k-1)$-chain

$$\partial\sigma = \sum_{j=0}^{k} (-1)^j [p_0, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k].$$

Example 0.10. $\sigma = [p_0, p_1, p_2]$.

$$\begin{aligned} \partial\sigma &= [p_1, p_2] - [p_0, p_2] + [p_0, p_1] \\ &= [p_1, p_2] + [p_2, p_0] + [p_0, p_1]. \end{aligned}$$

Example 0.11. $\sigma = [p_0, p_1, p_2, p_3]$.

$$\begin{aligned} \partial\sigma &= [p_1, p_2, p_3] - [p_0, p_2, p_3] + [p_0, p_1, p_3] - [p_0, p_1, p_2] \\ &= [p_1, p_2, p_3] + [p_2, p_0, p_3] + [p_0, p_1, p_3] + [p_1, p_0, p_2]. \end{aligned}$$