Entropy and Graphons

Andrew Hah

Spring 2025

Abstract

This paper began with my curiosity about how ideas from information theory and analysis can be used to understand the behavior of large combinatorial structures. In particular, I explored two tools in depth: entropy, which captures uncertainty and structure, and graphons, which describe the limiting behavior of dense graphs. This paper surveys what I've learned so far. Beginning with entropy in finite combinatorics, I then introduce the theory of graph limits and graphons, and finally explore how entropy extends to this setting and informs recent results.

Contents

1 Extremal Graph Theory

1.1 Classical Results

A classical question in extremal graph theory is of the form, given a fixed forbidden subgraph H, what is the maximum number of edges a graph on n vertices can have without containing H as a (not necessarily induced) subgraph? This number is called the Turán number, denoted $\mathrm{ex}(n, H)$. What is often considered the first result in extremal graph theory is Mantel's theorem, which answers this question for the special case when $H = K_3$ is a triangle.

Theorem 1.1. *(Mantel's theorem) Every n-vertex triangle-free graph has at most $\lfloor n^2/4 \rfloor$ edges, i.e.,* $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$.

Proof. See [Zha23, Theorem 1.1.1] for two different proofs, or the original argument in [Man07]. □

We would like to generalize this theorem from triangles to arbitrary cliques. To do so, we first construct the Turán graph.

Definition 1.2. The *Turán graph* $T_{n,r}$ is defined to be the complete n-vertex r-partite graph with part sizes differing by at most 1 (so each part has size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$.

For example, $T_{3,1} = K_3$ and $T_{10,3} = K_{3,3,4}$.

Theorem 1.3. *(Turán's theorem) The Turán graph $T_{n,r}$ maximizes the number of edges among all n-vertex K_{r+1}-free graphs. It is also the unique maximizer.*

1.2 Homomorphism Densities

2 Entropy in Combinatorics

3 Motivating Graph Limits

4 Graphons

5 Entropy of Graphons

6 Flag Algebras

7 Open Questions

Acknowledgments

References

References

[Man07] W. Mantel. “Problem 28”. In: *Wiskundige Opgaven* 10 (1907), pp. 60–61.

[Zha23] Yufei Zhao. *Graph Theory and Additive Combinatorics: Exploring Structure and Randomness*. Cambridge University Press, 2023. ISBN: 9781009310956.