MATH 20510
Lecture 21

May 12, 2025

Theorem 0.1. *T is a C^1 map of an open set $E \subseteq \mathbb{R}^n$ into an open set $V \subseteq \mathbb{R}^m$, S is a C^1 map of V into an open set $W \subseteq \mathbb{R}^\ell$, ω is a k-form on W (ω_S is a k-form on V, $(\omega_S)_T$ is a k-form on E, ω_{ST} is a k-form on E). Then*

$$(\omega_S)_T = \omega_{ST}.$$

Theorem 0.2. *Suppose ω is a k-form on an open set $E \subseteq \mathbb{R}^n$, ϕ is a k-surface in E with parameter domain $D \subseteq \mathbb{R}^k$ and Δ is the trivial k-surface, $\Delta : D \to \mathbb{R}^k$, $\Delta(u) = u$. Then*

$$\int_\phi \omega = \int_\Delta \omega_\phi.$$

Proof. It suffices to prove thsi in the case when

$$\omega = a dx_I = a dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Let $\phi_1, \ldots, \phi_n$ denote the components of ϕ. Then $\omega_\phi = a(\phi) d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k}$. It suffices to prove

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} = J(u) du_1 \wedge \cdots \wedge du_k \tag{1}$$

where $J(u) = \frac{\partial(x_{i_1}, \ldots, x_{i_k})}{\partial(u_1, \ldots, u_k)}$. Assuming (1),

$$\begin{aligned}
\int_\Delta \omega_\phi &= \int_\Delta a(\phi) d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} \\
&= \int_\Delta a(\phi) J(u) du_1 \wedge \cdots \wedge du_k \\
&= \int_D a(\phi(u)) J(u) du \\
&= \int_\phi \omega.
\end{aligned}$$

Let $[A]$ be the $k \times k$ matrix with entries

$$\alpha(p, q) = D_q \phi_{i_p}(u), \qquad p, q = 1, \ldots, k.$$

Note. $\det(A) = J(u)$.
Since $d\phi_{i_p} = \sum_q \alpha(p, q) du_q$, we have

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} = \sum \alpha(1, q_1) \ldots \alpha(k, q_k) du_{q_1} \wedge \cdots \wedge du_{q_k},$$

where the sum ranges over all $q_1, \ldots, q_k \in \{1, \ldots, k\}$. Rearranging each $duq_1 \wedge \cdots \wedge du_{q_k}$ we get

$$\begin{aligned} d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} &= \det(A) du_1 \wedge \cdots \wedge du_k \\ &= J(u) du_1 \wedge \cdots \wedge du_k. \end{aligned}$$

□

Theorem 0.3. *Suppose T is a C^1 map of an open set $E \subseteq \mathbb{R}^n$ into an open set $V \subseteq \mathbb{R}^n$, ϕ is a k-surface in E, ω is a k-form on V. Then*

$$\int_{T\phi} \omega = \int_{\phi} \omega_T.$$

Proof. Let D be the parameter domain of ϕ (and therefore of $T\phi$ as well). Let Δ be the trivial k-surface on D, i.e., $\Delta(u) = u$. Then

$$\int_{T\phi} \omega = \int_{\Delta} \omega_{T\phi} = \int_{\Delta} (\omega_T)_\phi = \int_{\phi} \omega_T.$$

□

Definition 0.4. A map f from a vector space X to a vector space Y is called *affine* if $f - f(0)$ is linear, i.e.,

$$f(x) = f(0) + Ax, \qquad A : X \to Y \text{ is linear.}$$

Remark 0.5. An affine map $f : \mathbb{R}^k \to \mathbb{R}^n$ is determined by $f(0)$ and $f(e_i)$ for $i = 1, \ldots, k$.

Definition 0.6. The k-simplex in $\mathbb{R}^k$ is $Q^k \subseteq \mathbb{R}^k$,

$$Q^k = \{x = (x_1, \ldots, x_k) : x_i \geq 0, x_1 + \cdots + x_k \leq 1\}.$$