MATH 20510
Lecture 18

May 5, 2025

Definition 0.1. If $I = (i_1, \dots, i_k)$ is a multi-index and $i_1 < \cdots < i_k$, we say I is an *increasing multi-index*. We say that dx_I is a basic k-form.

Remark 0.2. Every k-form can be represented in terms of basic k-forms.

Example 0.3. $dx_1 \wedge dx_5 \wedge dx_3 \wedge dx_2 = -dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5$.

Example 0.4. $dx_1 \wedge dx_3 \wedge dx_5 \wedge dx_2 = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5$.

Definition 0.5. If $\omega = \sum_I a_I dx_I$ is a k-form, we can convert each multi-index I into an increasing multi-index J, and we say that

$$\omega = \sum_J b_J dx_J$$

is in *standard presentation*.

Example 0.6.

$$\begin{aligned}
\omega &= x_1 dx_2 \wedge dx_1 - x_2 dx_3 \wedge dx_2 + x_3 dx_2 \wedge dx_3 + dx_1 \wedge dx_2 \\
&= -x_1 dx_1 \wedge dx_2 + x_2 dx_2 \wedge dx_3 + x_3 dx_2 \wedge dx_3 + dx_1 \wedge dx_2 \\
&= (1 - x_1) dx_1 \wedge dx_2 + (x_2 + x_3) dx_2 \wedge dx_3.
\end{aligned}$$

The last line is in standard presentation.

Definition 0.7. Suppose $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_q)$ are increasing multi-indices. The *product* of dx_I and dx_J is the $(p+q)$-form

$$dx_I \wedge dx_J = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}.$$

Note. If I and J have an element in common, $dx_I \wedge dx_J = 0$.

Notation. If I and J have no elements in common, we denote the increasing $(p+q)$ length multi-index obtained from rearranging the members of $I \cup J$ in increasing order by $[I, J]$.

$$dx_I \wedge dx_J = (-1)^\alpha dx_{[I,J]}$$

where α is the number of swaps needed to convert $I \cup J$ into an increasing multi-index.

Suppose ω, λ are p and q-forms respectively in $\mathbb{R}^n$ with standard representations

$$\omega = \sum_I b_I dx_I \quad \lambda = \sum_J c_J dx_J.$$

The product of ω and λ is the $(p+q)$-form

$$\omega \wedge \lambda = \sum_{I,J} b_I c_I (dx_I \wedge dx_J).$$

Remark 0.8.

(i) $(\omega_1 + \omega_2) \wedge \lambda = (\omega_1 \wedge \lambda) + (\omega_2 \wedge \lambda)$

(ii) $\omega \wedge (\lambda_1 + \lambda_2) = (\omega \wedge \lambda_1) + (\omega \wedge \lambda_2)$

(iii) $(\omega \wedge \lambda) \wedge \sigma = \omega \wedge (\lambda \wedge \sigma)$

Definition 0.9. A 0-form is a C^1 function.

Notation. The product of a 0-form f with a k-form $\omega = \sum_I b_I dx_I$ is

$$f\omega = \omega f = \sum_I (f b_I) dx_I.$$

Remark 0.10. $f(\omega \wedge \lambda) = f\omega \wedge \lambda = \omega \wedge f\lambda$.

Definition 0.11. (Differentiation of k-forms) Operator which associates a $(k+1)$-form, $d\omega$, to each k-form, ω.

(i) 0-forms in $\mathbb{R}^n$. $f : E \to \mathbb{R}$, $E \subseteq \mathbb{R}^n$.

$$\begin{aligned} df &= D_1 f dx_1 + \cdots + D_n f dx_n \\ &= \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n. \end{aligned}$$

(ii) k-forms in $\mathbb{R}^n$. Let $\omega = \sum_I b_I dx_I$ be given in standard presentation.

$$d\omega = \sum_I (db_I) \wedge dx_I.$$

Example 0.12. Let $\omega = \underbrace{x}_{f_1} dx + \underbrace{y^2}_{f_2} dz$ be a 1-form in $\mathbb{R}^3$.

$$\begin{aligned} d\omega &= (df_1) \wedge dx + (df_2) \wedge dz \\ &= (1dx + 0dy + 0dz) \wedge dx + (0dx + 2ydy + 0dz) \wedge dz \\ &= dx \wedge dx + 2ydy \wedge dz \\ &= 2ydy \wedge dz. \end{aligned}$$

Further,

$$\begin{aligned} d(d\omega) &= d(2ydy \wedge dz) \\ &= (df) \wedge (dy \wedge dz) \\ &= (2dy) \wedge (dy \wedge dz) \\ &= 0. \end{aligned}$$