MATH 20510
Lecture 14

April 25, 2025

Recall. $f : E \to \mathbb{R}$, $E \subseteq \mathbb{R}^n$ open, partials $D_1 f, \dots, D_n f$. If the partials are themselves differentiable then the second order derivatives of f are defined by

$$D_{ij} f = D_i D_j f, \quad (i, j = 1, \dots, n).$$

If these functions are continuous in E, we say f is C^2 in E.

Theorem 0.1. *If $f \in C^2$ in E then*

$$D_{ij} f = D_{ji} f, \quad \forall i, j.$$

Definition 0.2. If $f : E \to \mathbb{R}^n$, $E \subseteq \mathbb{R}^n$ open, f is differentiable at $x \in E$, the determinant of (the linear operator) $f'(x)$ is called the *Jacobian of f at x*

$$J_f(x) = \det f'(x)$$

Notation. We may also use $\frac{\partial(y_1, \dots, y_n}{\partial(x_1, \dots, x_n)}$; $f(x_1, \dots, x_n) = y_1, \dots, y_n$.

Definition 0.3. Let $k \in \mathbb{N}$. A *k-cell* in $\mathbb{R}^k$ is the set of points $I^k = \{x = (x_1, \dots, x_k)\}$ such that $a_i \leq x_i \leq b_i$, $\forall i = 1, \dots, k$.

Suppose I^k is a k-cell in $\mathbb{R}^k$ and $f : I^k \to \mathbb{R}$ is continuous. For every $j \leq k$, let I^j be the restriction of I^k to the first j components.

Define $g_k : I^k \to \mathbb{R}$ by $g_k = f$. Define $g_{k-1} : I^{k-1} \to \mathbb{R}$ by

$$g_{k-1}(x_1, \dots, x_{k-1}) = \int_{a_k}^{b_k} g_k(x_1, \dots, x_k) dx_k$$

Since g_k is uniformly continuous on I^k, g_{k-1} is (uniformly) continuous on I^{k-1}. Define $g_{k-2} : I^{k-2} \to \mathbb{R}$ by

$$g_{k-2}(x_1, \dots, x_{k-2}) = \int_{a_{k-1}}^{b_{k-1}} g_{k-1}(x_1, \dots, x_{k-1}) dx_{k-1}$$

We can repeat this process, ultimately arriving at a number

$$g_0 = \int_{a_1}^{b_1} g_1(x_1)dx_1$$

We say g_0 is the integral of f over I^k and we write

$$\int_{I^k} f(x)dx = g_0.$$

Example 0.4. Let $I^2 = [1,2] \times [0,1]$, $f(x_1, x_2) = 2x_1x_2^2$. What is $\int_{I^2} fdx$?

$$g_1(x_1) = \int_0^1 2x_1x_2^2 dx = \left[\frac{2}{3}x_1x_2^3\right]_0^1 = \frac{2}{3}x_1$$

$$\int_{I^2} fdx = g_0 = \int_1^2 g_1(x_1)dx_1 = \int_1^2 \frac{2}{3}x_1 dx_1 = \left[\frac{1}{3}x_1^2\right]_1^2 = 1$$

Question. Does this depend on the order of integration?

Answer. No (try the other direction in the example above).

Definition 0.5. If $f : \mathbb{R}^k \to \mathbb{R}$, the *support* of f is the closure of the set $\{x \in \mathbb{R}^k : f(x) \neq 0\}$.

If $f : \mathbb{R}^k \to \mathbb{R}$ is continuous with compact support, let I^k be any k-cell containing supp(f). We define

$$\int_{\mathbb{R}^k} fdx = \int_{I^k} fdx$$

Theorem 0.6. *(Change of variables) Let T be a 1-1, C^1 mapping of $E \subseteq \mathbb{R}^n$ open to $\mathbb{R}^n$. Also assume $J_T(x) \neq 0$ for all $x \in E$. If f is continuous on $\mathbb{R}^n$ with compact support that is contained in $T(E)$, then*

$$\int_{\mathbb{R}^n} f(y)dy = \int_{\mathbb{R}^n} f(T(x))|J_T(x)|dx.$$