Entropy and Graphons

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Abstract

This paper began with my curiosity about how ideas from information theory and analysis can be used to understand the behavior of large combinatorial structures. In particular, I explored two tools in depth: entropy, which captures uncertainty and structure, and graphons, which describe the limiting behavior of dense graphs. This paper surveys what I've learned so far. Beginning with entropy in finite combinatorics, I then introduce the theory of graph limits and graphons, and finally explore how entropy extends to this setting and informs recent results.

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1 Extremal Graph Theory

1.1 Classical Results

1.2 Homomorphism Densities

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. An important question in many areas of graph theory is: how often does a fixed small graph H appear inside G?

To quantify this, we define the *homomorphism density* of a graph H in a graph G, denoted $t(H,G)$, as the probability that a uniformly random map $\phi : V(H) \to V(G)$ is a graph homomorphism; that is, for every edge $\{u,v\} \in E(H)$, the image $\{\phi(u), \phi(v)\} \in E(G)$. Formally,

$$t(H,G) := \frac{|\mathrm{Hom}(H,G)|}{|V(G)|^{|V(H)|}},$$

where $\mathrm{Hom}(H,G)$ is the set of all homomorphisms from H to G.

This quantity reflects the "density" of H in G. For example, if H is the triangle K_3, then $t(K_3, G)$ captures how likely it is for three randomly chosen vertices in G to form a triangle, when accounting for all mappings.

The homomorphism density is closely related to induced subgraph counts and plays a fundamental role in extremal graph theory. Many classical results — such as Turán's Theorem — can be rephrased in terms of maximizing or minimizing $t(H,G)$ under certain constraints.

In what follows, we will use homomorphism densities as a lens for understanding graph sequences, graph parameters, and the eventual need for limiting objects such as graphons.

2 Entropy in Combinatorics

3 Motivating Graph Limits

4 Graphons

5 Entropy of Graphons

6 Flag Algebras

7 Open Questions

Acknowledgments

References