

MATH 297. Proseminar in Mathematics

Andrew Hah
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Theorem 0.1. (*Multicolor triangle Ramsey theorem*) For every positive integer r , there is some integer $N = N(r)$ such that if each edge of K_N is colored using one of r colors, then there is a monochromatic triangle.

Proof. Zhao, GTAC, pg 3. □

Theorem 0.2. (*Graph Ramsey theorem*) For every k and r there exists some $N = N(k, r)$ such that if each edge of K_N is colored using one of r colors, then there is a monochromatic K_k .

Question. What is the maximum number of edges in a triangle-free n -vertex graph?

Definition 0.3. (Turán number) We write $\text{ex}(n, H)$ for the maximum number of edges in an n -vertex H -free graph, where a graph is H -free if it does not contain H as a subgraph.

Theorem 0.4. (*Mantel's theorem*) Every n -vertex triangle-free graph has at most $\lfloor n^2/4 \rfloor$, i.e., $\text{ex}(n, K_3) = \lfloor n^2/4 \rfloor$.

Proof. Proof ii, Zhao, GTAC, pg 13. □

Exercise L. Let X and Y be independent and identically distributed random vectors in \mathbb{R}^d according to some arbitrary probability distribution. Prove that

$$\mathbb{P}(|X + Y| \geq 1) \geq \frac{1}{2} \mathbb{P}(|X| \geq 1)^2.$$

Definition 0.5. The Turán graph $T_{n,r}$ is defined to be the complete n -vertex r -partite graph with part sizes differing by at most 1 (so each part has size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$).

Example 0.6. $T_{10,3} = K_{3,3,4}$.

Theorem 0.7. (*Turán's theorem*) The Turán graph $T_{n,r}$ maximizes the number of edges among all n -vertex K_{r+1} -free graphs. It is also the unique maximizer.

Definition 0.8. The edge density of a graph G is

$$\frac{e(G)}{\binom{v(G)}{2}}.$$