MATH 20410. Analysis in $\mathbb{R}^n$ II (accelerated)

Based on lectures by Prof. Donald Stull
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Any proof or argument that has been filled in, expanded, or written out in detail by me is marked with a ■. All other material follows the lectures and any errors or omissions are entirely my own.

Contents

1 Differentiation

Definition 1.1. Let $f : [a,b] \to \mathbb{R}$ and $x \in [a,b]$. We say that f is *differentiable* at x if the limit

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

exists. If the limit exists, we say it is the *derivative* of f at x, denoted by $f'(x)$.

Extensions.

(i) $f : [a,b] \to \mathbb{C}$. $f = f_{\mathrm{RE}} + i f_{\mathrm{IM}}$ where $f_{\mathrm{RE}}, f_{\mathrm{IM}} : [a,b] \to \mathbb{R}$. Then f is differentiable at $x \in [a,b] \iff f_{\mathrm{RE}}, f_{\mathrm{IM}}$ are differentiable at x. If f is differentiable at x then $f'(x) = f'_{\mathrm{RE}}(x) + i f'_{\mathrm{IM}}(x)$

(ii) $f : [a,b] \to \mathbb{R}^n$. $f = (f_1, \ldots, f_n)$ where $f_1, \ldots, f_n : [a,b] \to \mathbb{R}$. Define the derivative of f at x by $f'(x) = \lim_{t \to x} \frac{f(t)-f(x)}{t-x}$. The limit is the vector limit. By Theorem 4.10 of Rudin, this limit exists $\iff$ the limit of each component exists, i.e. $f'(x) = (f'_1(x), \ldots, f'_n(x))$.

Theorem 1.2. *If $f : [a,b] \to \mathbb{R}$, $x \in [a,b]$, and $f'(x)$ exists, then f is continuous at x.*

Proof. Let $t \in [a,b]$, $t \neq x$. Then $f(t) - f(x) = \frac{f(t)-f(x)}{t-x} \cdot (t - x)$. As $t \to x$, the right hand side goes to $f'(x) \cdot 0 = 0$, so f is continuous at x. □

Differentiation rules.

(i) Let $f, g : [a,b] \to \mathbb{R}$, both differentiable at $x \in [a,b]$. Then $f + g, f \cdot g, \frac{f}{g} (g(x) \neq 0)$ are all differentiable at x. Moreover, $(f + g)'(x) = f'(x) + g'(x)$, $(fg)' = f'(x)g(x) + f(x)g'(x)$, and $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x)-g'(x)f(x)}{g^2(x)}$.

(ii) $f : [a,b] \to \mathbb{R}$, $g : [c,d] \to \mathbb{R}$, $f([a,b]) \subseteq [c,d]$. Let $x \in [a,b]$ s.t. $f'(x)$ exists and $g'(f(x))$ exists. Then $g \circ f$ is differentiable at x and $(g \circ f)'(x) = g'(f(x))f'(x)$.

Definition 1.3. Let (X, d) be a metric space and $f : X \to \mathbb{R}$. We say that f has a *local maximum* at $x \in X$ if there is an open ball $U \ni x$ such that $\forall y \in U$, $f(y) \leq f(x)$, and a *local minimum* if $\forall y \in U, f(y) \geq f(x)$.

Theorem 1.4. *Let $f : [a,b] \to \mathbb{R}$ have a local maximum or local minimum at $x \in [a,b]$. If $f'(x)$ exists, then $f'(x) = 0$.*

Proof. Suppose x is a local maximum and $f'(x)$ exists. Then if $t < x$, $\frac{f(t)-f(x)}{t-x} \geq 0$, and if $t > x$, $\frac{f(t)-f(x)}{t-x} \leq 0$. Thus $f'(x) = 0$. □

2 Differentiation in $\mathbb{R}^n$

3 Riemann-Stieltjes Integration