MATH 20510
Lecture 15

April 28, 2025

Definition 0.1. (Informal) A *differential* 1-*form* on $\mathbb{R}^n$ is

(i) An object which can be integrated on any curve in $\mathbb{R}^n$.

(ii) A rule assigning a real number to every oriented line segment in $\mathbb{R}^n$ in a "suitable" way.

Definition 0.2. Let $p \in \mathbb{R}^n$. The *tangent space* to $\mathbb{R}^n$ at p is $T_p\mathbb{R}^n = \{(p, v) : v \in \mathbb{R}^n\}$.

Notation. If α is a 1-form, $p \in \mathbb{R}^n$, write α_p to denote the restriction of α to $T_p\mathbb{R}^n$.- $\alpha_p(v)$ is the value α assigns to the (oriented) line segment from p to $p + v$.

We require that α_p is a linear functional $\forall p \in \mathbb{R}^n$, that is

(i) $\alpha_p(tv) = t \cdot \alpha_p(v)$, $\forall t \in \mathbb{R}, \forall p, v \in \mathbb{R}^n$.

(ii) $\alpha_p(v + w) = \alpha_p(v) + \alpha_p(w)$, $\forall p, v, w \in \mathbb{R}^n$.

We denote the projection maps in $\mathbb{R}^n$ by $dx_1, \dots, dx_n$, where

$$dx_i(v) = dx_i(v_1, \dots, v_n) = v_i, \quad \forall i = 1, \dots, n$$

These form a basis for the set of linear functionals. Therefore, for any 1-form α, its restriction α_p can be written as

$$\begin{aligned}\alpha_p &= A_1dx_1 + A_2dx_2 + \cdots + A_ndx_n \\ &= A_1(p)dx_1 + \cdots + A_n(p)dx_n\end{aligned}$$

Last requirement: $A_i(p)$ must be sufficiently continuous with respect to p.

Definition 0.3. A *differential* 1-*form* α on $\mathbb{R}^n$ is a map from every tangent vector (p, v) in $\mathbb{R}^n$ which can be expressed in the form

$$\alpha = f_1dx_1 + \cdots + f_ndx_n$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ is C^2.

Example 0.4. $\alpha = ydx + dz$ on $\mathbb{R}^3$. Let $p = \begin{pmatrix}1\\2\\3\end{pmatrix}$ and $v = \begin{pmatrix}4\\5\\6\end{pmatrix}$. Then

$$\alpha((p, v)) = \alpha_p(v)$$

$$= f_1(p)dx_1(v) + f_2(p)dx_2(v) + f_3(p)dx_3(v)$$

$$= 2 \cdot 4 + 0 + 1 \cdot 6$$

$$= 14$$

Definition 0.5. A *curve* (1-surface) in $\mathbb{R}^n$ is a C^1-mapping $\gamma : [a,b] \to \mathbb{R}^n$.

Definition 0.6. Let $\alpha = f_1 dx_1 + \cdots + f_n dx_n$ be a 1-form in $\mathbb{R}^n$ and let $\gamma : [a,b] \to \mathbb{R}^n$ be C^1.

$$\int_\gamma \alpha = \int_a^b (f_1(\gamma(t))\gamma_1'(t) + \cdots + f_n(\gamma(t))\gamma_n'(t))dt$$

Example 0.7. $\alpha = x^2 dx_1 + dx_2$ on $\mathbb{R}^2$. $\gamma(t) = (t, t^2)$, $t \in [0,1]$. Then $\gamma_1'(t) = 1$, $\gamma_2'(t) = 2t$.

$$\begin{aligned}\int_\gamma \alpha &= \int_0^1 (f_1(\gamma(t))\gamma_1'(t) + f_2(\gamma(t))\gamma_2'(t)) \\ &= \int_a^b (t^2 \cdot 1 + 1 \cdot 2t)dt \\ &= \frac{4}{3}\end{aligned}$$