MATH 23500
Lecture 17

May 2, 2025

Theorem 0.1. *(Optional stopping theorem) Let $\{M_n\}$ be a martingale with respect to $\{\mathcal{F}_n\}$, and let τ be a stopping time. Assume that τ is bounded, i.e., $\exists k$ such that $\mathbb{P}(\tau \leq k) = 1$. Then*

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$$

Proof. $M_\tau = \sum_{n=0}^{k} M_n \mathbb{1}_{\{\tau=n\}}$. Conditioning on $\mathcal{F}_{k-1}$, we obtain

$$\begin{aligned}
\mathbb{E}[M_\tau \mid \mathcal{F}_{k-1}] &= \sum_0^k \mathbb{E}[M_n \mathbb{1}_{\{\tau=n\}} \mid \mathcal{F}_{k-1}] \\
&= \sum_0^{k-1} \mathbb{E}[M_n \mathbb{1}_{\{\tau=n\}} \mid \mathcal{F}_{k-1}] + \mathbb{E}[M_k \mathbb{1}_{\{\tau=k\}} \mid \mathcal{F}_{k-1}] \\
&= M_{k-1} \mathbb{1}_{\{\tau>k-1\}} + \sum_0^{k-1} M_n \mathbb{1}_{\{\tau=n\}} \\
&= M_{k-1} \mathbb{1}_{\{\tau>k-2\}} + \sum_0^{k-2} M_n \mathbb{1}_{\{\tau=n\}}
\end{aligned}$$

Similarly, $\mathbb{E}[M_\tau \mid \mathcal{F}_{k-2}] = M_{k-2} \mathbb{1}_{\{\tau>k-3\}} + \sum_0^{k-3} M_n \mathbb{1}_{\{\tau=n\}}$. We repeat this k times to obtain

$$\mathbb{E}[M_\tau \mid \mathcal{F}_0] = M_0 \mathbb{1}_{\{\tau \geq 0\}} = M_0$$

Thus

$$\mathbb{E}[M_\tau] = \mathbb{E}[\mathbb{E}[M_\tau \mid \mathcal{F}_0]] = \mathbb{E}[M_0]$$

□

Example 0.2. Let $\{X_n\}$ be the random walk on $\mathbb{Z}$ starting at 0. Then $\{X_n\}$ is a martingale. Let $\tau = \min\{n \geq 0 : X_n = 1\}$. Then $\mathbb{E}[X_\tau] = 1 \neq 0 = \mathbb{E}[X_0]$.

What if τ is not bounded? We write $\tau \wedge n = \min\{\tau, n\}$. Then $\tau \wedge n$ is a bounded stopping time. Then

$$\begin{aligned}\mathbb{E}[M_0] &= \mathbb{E}[M_{\tau \wedge n}] \\ &= \mathbb{E}[M_\tau \mathbb{1}_{\{\tau \leq n\}}] + \mathbb{E}[M_\tau \mathbb{1}_{\{\tau > n\}}]\end{aligned}$$

Theorem 0.3. *(Dominated convergence theorem) Let $\{X_n\}$ be a sequence of real-valued random variables such that $X_n \to X$ with probability 1. Assume there exists a random variable Y such that $\mathbb{E}[|Y|] < \infty$ and $|X_n| \leq |Y|$ for all n. Then*

$$\mathbb{E}[X_n] \to \mathbb{E}[X]$$

Example 0.4. If $\mathbb{E}[|M_\tau|] < \infty$ and $\mathbb{P}(\tau < \infty) = 1$, we can apply the DCT with $X_n = M_n \mathbb{1}_{\{\tau \leq n\}}$ and $Y = M_\tau$. We know that as $n \to \infty$, $\mathbb{1}_{\{\tau \leq n\}} \to 1$ so

$$\mathbb{E}[M_n \mathbb{1}_{\{\tau \leq n\}}] \to \mathbb{E}[M_\tau]$$

Thus, $\mathbb{E}[M_0] = \mathbb{E}[M_\tau]$ provided that $\lim_{n\to\infty} \mathbb{E}[M_n \mathbb{1}_{\{\tau > n\}}] = 0$.

Theorem 0.5. *(Optional stopping theorem II) Let τ be a stopping time, and assume that*

(i) $\mathbb{P}(\tau < \infty) = 1$
(ii) $\mathbb{E}[|M_\tau|] < \infty$
(iii) $\lim_{n\to\infty} \mathbb{E}[M_n \mathbb{1}_{\{\tau > n\}}] = 0$

Then

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$$

Remark 0.6. If M_n is bounded, i.e., $\exists c$ such that $|M_n| \leq c$ for all n, then $\mathbb{E}[M_n \mathbb{1}_{\{\tau > n\}}] \leq c\mathbb{P}(\tau > n) \to 0$.

Example 0.7. Let $\{X_n\}$ be the simple random walk on $\mathbb{Z}$, started at 0. Let $\tau = \min\{n \geq 0 : X_n = a \text{ or } X_n = -b\}$. Then $\mathbb{P}(X_\tau < \infty) = 1$, $\mathbb{E}[|X_\tau|] < \infty$, and

$$\mathbb{E}[|X_n| \mathbb{1}_{\{\tau > n\}}] \leq \max\{a, b\}\mathbb{P}(\tau > n) \to 0$$

Hence, by OST II, $0 = \mathbb{E}[X_\tau] = a\mathbb{P}(X_\tau = a) + (-b)\mathbb{P}(X_\tau = -b)$, which gives

$$\mathbb{P}(X_\tau = a) = \frac{b}{a+b}$$