MATH 20510
Lecture 17

May 2, 2025

Definition 0.1. The integral of a 2-form $\omega = \sum_{i,j} f_{i,j}(dx_i \wedge dx_j)$ over a 2-surface $\gamma : [a,b] \times [c,d] \to \mathbb{R}^n$ (which is C^1) is

$$\int_\gamma \omega = \int_a^b \left(\int_c^d \omega_{\gamma(t_1,t_2)} \left(\frac{\partial \gamma}{\partial t_1}, \frac{\partial \gamma}{\partial t_2} \right) dt_2 \right) dt_1$$

Definition 0.2. A *k-surface* in $\mathbb{R}^n$ is a C^1 map $\gamma : D \to \mathbb{R}^n$ where D is a k-cell.

Definition 0.3. (Informal) A k-form in $\mathbb{R}^n$, ω, is a rule that assigns a real number to every oriented k-dimensional parallelepiped in $\mathbb{R}^n$ in a "suitable" way.

Specify a k-dimensional oriented parallelepiped in $\mathbb{R}^n$ based at $p \in \mathbb{R}^n$ by giving an ordered list of vectors $v_1, \ldots, v_k \in T_p\mathbb{R}^n$. We require that for any $p \in \mathbb{R}^n$, a k-form ω satisfies

(i) $\omega_p(v_1, \ldots, tv_i, \ldots, v_k) = t\omega_p(v_1, \ldots, v_i, \ldots, v_k)$.
(ii) $\omega_p(v_1, \ldots, v_i + w_i, \ldots, v_k) = \omega(v_1, \ldots, v_i, \ldots, v_k) + \omega_p(v_1, \ldots, w_i, \ldots v_k)$.
(iii) $\omega_p(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\omega_p(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$.

Definition 0.4. A *multi-index* of length k in $\mathbb{R}^n$ is a list $I = (i_1, \ldots, i_k)$ of k integers between 1 and n.

Definition 0.5. Let $I = (i_1, \ldots, i_k)$ be a multi-index. Then $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is the k-form in $\mathbb{R}^n$ defined by

$$dx_I(v^1, \ldots, v^k) = \det \begin{pmatrix} v^1_{i_1} & v^2_{i_1} & \cdots & v^k_{i_1} \\ v^1_{i_2} & v^2_{i_2} & \cdots & v^k_{i_2} \\ \vdots & \vdots & \ddots & \vdots \\ v^1_{i_k} & v^2_{i_k} & \cdots & v^k_{i_k} \end{pmatrix}$$

Remark 0.6.

(i) If I contains a repeated index, then $dx_I(v^1, \ldots, v^k) = 0$.
(ii) For any I, if $v^1, \ldots, v^k$ contains a repeated vector, then $dx_I(v^1, \ldots, v^k) = 0$.
(iii) If J is obtained from I by swapping a single pair of indices, then $dx_I(v^1, \ldots, v^k) = -dx_J(v^1, \ldots, v^k)$.

Definition 0.7. A *differential k-form* in $\mathbb{R}^n$, ω, is a rule assigning a real number to each oriented parallelepiped of the form

$$\omega = \sum_I f_I dx_I$$

where the sum is taken over all multi-indices I of length k and $f_I : \mathbb{R}^n \to \mathbb{R}$ is C^2. If $p \in \mathbb{R}^n$, $v^1, \ldots, v^k \in \mathbb{R}^n$,

$$\omega_p(v^1, \ldots, v^k) = \sum_I f_I(p) dx_I(v^1, \ldots, v^k)$$

Definition 0.8. Let $\phi : D \to \mathbb{R}^n$ be a k-surface and $\omega = \sum_I f_I dx_I$ be a k-form.

$$\begin{aligned}
\int_\phi \omega &= \int_D \omega_{\phi(u)} \left(\frac{\partial \phi}{\partial u_1}, \ldots, \frac{\partial \phi}{\partial u_k} \right) du \\
&= \int_D \sum_I f_I(\phi(u)) dx_I \left(\frac{\partial \phi}{\partial u_1}, \ldots, \frac{\partial \phi}{\partial u_k} \right) du \\
&= \int_D \sum_I f_I(\phi(u)) \frac{\partial(x_{i_1}, \ldots, x_{i_k})}{\partial(u_1, \ldots, u_k)} du
\end{aligned}$$

where $\frac{\partial(x_{i_1}, \ldots, x_{i_k})}{\partial(u_1, \ldots, u_k)}$ is the Jacobian of the map $u_1, \ldots, u_k \mapsto \phi_{i_1}(u), \ldots, \phi_{i_k}(u)$.

Example 0.9. $\omega = x dy \wedge dz - y dx \wedge dz + z dx \wedge dy$ is a 2-form in $\mathbb{R}^3$. $\phi : [0,3] \times [0, 2\pi] \to \mathbb{R}^3$, $\phi(r, \theta) = (r \cos\theta, r \sin\theta, 5)$.