MATH 235. Markov Chains
Lecture 14

April 25, 2025

Recall. The t-time transition matrix P_t is the $N \times N$ matrix satisfying e^{tA} where A is the matrix with x, y entries: $\alpha(x, y)$, $x \neq y$; $-\alpha(x), x = y$.

Note 0.1. $\alpha(x) = \sum_{y \in S \setminus \{x\}} \alpha(x, y)$.

Note 0.2. $e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$. If A is diagonalizable, then we may write $A = QDQ^{-1}$, and so $e^{tA} = Qe^{tD}Q^{-1}$ because $(tQDQ^{-1})^n = t^n QD^n Q^{-1}$. If D is diagonal with entries $\lambda_1, \ldots, \lambda_n$ then $(tD)^n$ is diagonal with entries $(t\lambda_1)^n, \ldots, (t\lambda_n)^n$.

Example 0.3. Consider the continuous Markov chain with state space $S = \{0, 1\}$ and rates $\alpha(0, 1) = 2$, $\alpha(1, 0) = 3$. We have

$$A = \begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix}$$

We diagonalize A. The eigenvalues are $\lambda_1 = 0, \lambda_2 = -5$ with eigenvectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$. Thus

$$A = \begin{pmatrix} -2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -5 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 3 & 1 \end{pmatrix}^{-1}$$

Which gives

$$e^{tA} = \begin{pmatrix} -2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -e^{5t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 3 & 1 \end{pmatrix}^{-1} = \frac{1}{5} \begin{pmatrix} 3 + 2e^{-5t} & 2 - 2e^{-5t} \\ 3 - 3e^{-5t} & 2 + 3e^{-5t} \end{pmatrix}$$

In particular, $\mathbb{P}(X_t = 1 \mid X_0 = 0) = \frac{2 - 2e^{-5t}}{5}$.

Remark 0.4. The matrix A is called the *infinitesimal generator* for the Markov chain.

Definition 0.5. A continuous time Markov chain is *irreducible* if we can get from any state to any other state, i.e., $p_t(x, y) > 0$ for all x, y, for all $t > 0$.

Definition 0.6. A function $\pi : S \to [0,1]$ with $\sum_{x \in S} \pi_x = 1$ is a *stationary (invariant) distribution* for $\{X_t\}$ if the following is true: If the distribution of X_0 is π, then the distribution of X_t is π for all $t > 0$.

This means $\frac{d}{dt}\pi P_t = 0 \iff \frac{d}{dt}\pi e^{tA} = 0 \iff \pi A = 0$.

Proposition 0.7. *If $\{X_t\}$ is irreducible, then there exists a unique stationary distribution π for $\{X_t\}$. Moreover, $\lim_{t\to\infty} p_t(x,y) = \pi_y$ for all $x, y \in S$.*