MATH 20510
Lecture 19

May 7, 2025

Theorem 0.1. *(Graded product rule) Let ω be a k-form and λ be an m-form, both of class C^1. Then*

$$d(\omega \wedge \lambda) = (d\omega) \wedge \lambda + (-1)^k \omega \wedge (d\lambda).$$

Proof. It suffices to show this on simple forms $\omega = f dx_I$, $\lambda = g dx_J$, $f, g \in C^1$. Then

$$\omega \wedge \lambda = fg dx_I \wedge dx_J.$$

If I and J have any common indices, then both sides equal 0 and the theorem holds. Now assume that I and J have no common terms. Then

$$\begin{aligned}
d(\omega \wedge \lambda) &= d(fg dx_I \wedge dx_J) \\
&= (-1)^\alpha d(fg dx_{[I,J]}) \\
&= (-1)^\alpha (f dg + g df) \wedge dx_{[I,J]} \\
&= (f dg + g df) \wedge dx_I \wedge dx_J \\
&= (-1)^k (f dx_I) \wedge dg \wedge dx_J + df \wedge dx_I \wedge (g dx_J) \\
&= (-1)^k \omega \wedge (d\lambda) + (d\omega) \wedge \lambda.
\end{aligned}$$

□

Theorem 0.2. *If ω is a k-form of class C^1, then*

$$d^2(\omega) = 0.$$

Proof.

$$d(dx_I) = 0 = d(1 dx_I) = d(1) \wedge dx_I.$$

Let $f \in C^2(E)$, $E \subseteq \mathbb{R}^n$. Then

$$\begin{aligned} d^2 f &= d\left(\sum_{i=1}^{n} (D_i f)(x) dx_i\right) \\ &= \sum_{i=1}^{n} d(D_i f) \wedge dx_i \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} D_{ij} f dx_j \wedge dx_i \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} -D_{ij} f dx_i \wedge dx_j. \end{aligned}$$

Thus $d^2 f = 0$. Then, for a k-form $\omega = f dx_I$,

$$d\omega = df \wedge dx_I$$

and

$$\begin{aligned} d^2\omega &= d^2 f \wedge dx_I + (-1) df \wedge d(dx_I) \\ &= 0 - 0 \\ &= 0. \end{aligned}$$

$\square$

Definition 0.3.

(i) A k-form ω is *closed* if $d\omega = 0$.
(ii) A k-form ω is *exact* if there exists a $(k-1)$-form α such that $d(\alpha) = \omega$.

Remark 0.4.

(i) If ω is a k-form that is of class C^2 then $d^2\omega = 0$ so $d\omega$ is exact.
(ii) Every exact form is closed ($d(d\omega) = 0$).

Question. Is the converse of (ii) true?
Answer. No.