# MATH 20510
# Lecture 15

April 28, 2025

**Definition 0.1.** (Informal) A *differential* 1-*form* on $\mathbb{R}^n$ is

(i) An object which can be integrated on any curve in $\mathbb{R}^n$.

(ii) A rule assigning a real number to every oriented line segment in $\mathbb{R}^n$ in a "suitable" way.

**Definition 0.2.** Let $p \in \mathbb{R}^n$. The *tangent space* to $\mathbb{R}^n$ at $p$ is $T_p\mathbb{R}^n = \{(p, v) : v \in \mathbb{R}^n\}$.

Notation. If $\alpha$ is a 1-form, $p \in \mathbb{R}^n$, write $\alpha_p$ to denote the restriction of $\alpha$ to $T_p\mathbb{R}^n$.- $\alpha_p(v)$ is the value $\alpha$ assigns to the (oriented) line segment from $p$ to $p + v$.

We require that $\alpha_p$ is a linear functional $\forall p \in \mathbb{R}^n$, that is

(i) $\alpha_p(tv) = t \cdot \alpha_p(v)$, $\forall t \in \mathbb{R}, \forall p, v \in \mathbb{R}^n$.

(ii) $\alpha_p(v + w) = \alpha_p(v) + \alpha_p(w)$, $\forall p, v, w \in \mathbb{R}^n$.

With differential forms, we denote the projection maps in $\mathbb{R}^n$ by $dx_1, \ldots, dx_n$, where

$$dx_i(v) = dx_i(v_1, \ldots, v_n) = v_i, \quad \forall i = 1, \ldots, n$$

These form a basis for the set of linear functionals. Therefore, for any 1-form $\alpha$, its restriction $\alpha_p$ can be written as

$$\begin{aligned} \alpha_p &= A_1 dx_1 + A_2 dx_2 + \cdots + A_n dx_n \\ &= A_1(p) dx_1 + \cdots + A_n(p) dx_n \end{aligned}$$

Last requirement: $A_i(p)$ must be sufficiently continuous with respect to $p$.

**Definition 0.3.** A *differential* 1-*form* $\alpha$ on $\mathbb{R}^n$ is a map from every tangent vector $(p, v)$ in $\mathbb{R}^n$ which can be expressed in the form

$$\alpha = f_1 dx_1 + \cdots + f_n dx_n$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ is $C^2$.

**Example 0.4.** $\alpha = ydx + dz$ on $\mathbb{R}^3$. Let $p = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $v = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$. Then

$$\alpha((p, v)) = \alpha_p(v)$$

$$= f_1(p)dx_1(v) + f_2(p)dx_2(v) + f_3(p)dx_3(v)$$
$$= 2 \cdot 4 + 0 + 1 \cdot 6$$
$$= 14$$

**Definition 0.5.** A *curve* (1-surface) in $\mathbb{R}^n$ is a $C^1$-mapping $\gamma : [a, b] \to \mathbb{R}^n$.

**Definition 0.6.** Let $\alpha = f_1 dx_1 + \cdots + f_n dx_n$ be a 1-form in $\mathbb{R}^n$ and let $\gamma : [a, b] \to \mathbb{R}^n$ be $C^1$.

$$\int_\gamma \alpha = \int_a^b (f_1(\gamma(t))\gamma_1'(t) + \cdots + f_n(\gamma(t))\gamma_n'(t))dt$$

**Example 0.7.** $\alpha = x^2 dx_1 + dx_2$ on $\mathbb{R}^2$. $\gamma(t) = (t, t^2)$, $t \in [0, 1]$. Then $\gamma_1'(t) = 1$, $\gamma_2'(t) = 2t$.

$$\begin{aligned}\int_\gamma \alpha &= \int_0^1 (f_1(\gamma(t))\gamma_1'(t) + f_2(\gamma(t))\gamma_2'(t)) \\ &= (t^2 \cdot 1 + \cdot 2t)dt \\ &= \frac{4}{3}\end{aligned}$$