MATH 23500
Lecture 18

May 5, 2025

Example 0.1. Let $\{X_n\}$ be the simple random walk on $\mathbb{Z}$. Let $\tau = \min\{n : X_n = a \text{ or } X_n = -b\}$. We wish to compute $\mathbb{E}[\tau]$. Let $M_n = X_n^2 - n$. As shown before, M_n is a martingale with respect to $\mathcal{F}_n$.

Claim. $\mathbb{P}(\tau > n) \leq e^{-cn}$ for some $c > 0$. The random walk stopped when we hit a or $-b$ is a Markov chain on $\{-b, -b+1, \ldots, 0, \ldots, a-1, a\}$, and $\{-b+1, \ldots, a-1\}$ is a transient communication class. With this, we find

$$\mathbb{E}[\tau] = \sum_{n=0}^{\infty} \mathbb{P}(\tau > n) \leq \sum_{n=0}^{\infty} e^{-cn} < \infty.$$

Furthermore,

$$\mathbb{E}[|M_\tau|] \leq \mathbb{E}[X_\tau^2] + \mathbb{E}[\tau] \leq \max\{a^2, b^2\} + \mathbb{E}[\tau] < \infty.$$

Since

$$\mathbb{E}[|M_n| \mathbb{1}_{\{\tau > n\}}] \leq \left(\max\{a, b\}^2 + n\right) e^{-cn} \to 0.$$

we have

$$\begin{aligned}
\mathbb{E}[M_0] &= \mathbb{E}[M_\tau] \\
&= \mathbb{E}[X_\tau^2] - \mathbb{E}[\tau] \\
&= a^2 \mathbb{P}(X_\tau = a) + b^2 \mathbb{P}(X_\tau = -b) - \mathbb{E}[\tau] \\
&= \frac{a^2 b + b^2 a}{a + b} - \mathbb{E}[\tau] \\
&= 0.
\end{aligned}$$

Hence,

$$\mathbb{E}[\tau] = ab.$$

Example 0.2. At each toss of a coin, I bet \$1 on tails, and let X_n be my winning on the n-th toss. $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2$. Let T be the time at which the first tail occurs. Let $Y_n = X_1 + X_2 + \cdots + X_n$ be my total winnings after n tosses. Y is a martingale, and by OST,

$$0 = \mathbb{E}[Y_T] = \mathbb{E}\left[\sum_{n=1}^{T} X_n\right] = \mathbb{E}[(-1)(T-1) + 1].$$

Thus

$$\mathbb{E}[T] = 2.$$

Example 0.3. If the coin is unfair, play the same game, except you are compensated \$$1 - 2p$ for every \$ bet.

Example 0.4. At times $1, 2, \ldots$ a monkey types a capital letter uniformly at random. We wish to find the expected time until the monkey types ABRACADABRA.

Just before each time n, a new gambler arrives and bets \$1 that the n-th letter is A. If he loses, he leaves, and otherwise he receives \$26, all of which he bets on the $(n+1)$-th letter being B. Again, if he wins he bets his entire fortune of \$26 on the $(n+2)$-th letter being R, and so on. Let M_n^j be the winnings of the j-th letter at time n. Then

$$\mathbb{E}[M_n^j \mid \mathcal{F}_{n-1}] = 0 = M_{n-1}^j.$$

If he wins the first $n-1$ rounds, his fortune is 26^{n-1} and so

$$\mathbb{E}[M_n^j \mid \mathcal{F}_{n-1}] = 26^n \cdot \frac{1}{26} + \frac{25}{26} \cdot 0 = 26^{n-1} = M_{n-1}^j.$$

By linearity, $M_n = \sum_{j=1}^n M_n^j$ is a martingale with respect to $\mathcal{F}_n$.

Claim. OST applies to M_n (exercise).
Let T be the first time hte monkey types ABRACADABRA. Then,

$$\begin{aligned}
\mathbb{E}[M_0] &= \mathbb{E}[M_T] \\
&= \mathbb{E}\left[\sum_{j=1}^{T} M_n^j\right] \\
&= \mathbb{E}\left[(26^{11} - 1) + (26^4 - 1) + 25 + (-1)(T-3)\right].
\end{aligned}$$

Thus,

$$\mathbb{E}[T] = 26^{11} + 26^4 + 26.$$