MATH 297. Proseminar in Mathematics

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Theorem 0.1. *(Multicolor triangle Ramsey theorem) For every positive integer r, there is some integer $N = N(r)$ such that if each edge of K_N is colored using on of r colors, then there is a monochromatic triangle.*

Proof. Zhao, GTAC, pg 3. □

Theorem 0.2. *(Graph Ramsey theorem) For every k and r there exists some $N = N(k, r)$ such that if each edge of K_N is colored using one of r colors, then there is a monochromatic K_k.*

Question. What is the maximum number of edges in a triangle-free n-vertex graph?

Definition 0.3. (Turán number) We write $\mathrm{ex}(n, H)$ for the maximum number of edges in an n-vertex H-free graph, where a graph is H-free if it does not contain H as a subgraph.

Theorem 0.4. *(Mantel's theorem) Every n-vertex triangle-free graph has at most $\lfloor n^2/4 \rfloor$, i.e., $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$.*

Proof. Proof ii, Zhao, GTAC, pg 13. □

Exercise L. et X and Y be independent and identically distributed random vectors in $\mathbb{R}^d$ according to some arbitrary probability distribution. Prove that

$$\mathbb{P}(|X + Y| \geq 1) \geq \frac{1}{2}\mathbb{P}(|X| \geq 1)^2.$$

Definition 0.5. The Turán graph $T_{n,r}$ is defined to be the complete n-vertex r-partite graph with part sizes differing by at most 1 (so each part has size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$.

Example 0.6. $T_{10,3} = K_{3,3,4}$.

Theorem 0.7. *(Turán's theorem) The Turán graph $T_{n,r}$ maximizes the number of edges among all n-vertex K_{r+1}-free graphs. It is also the unique maximizer.*

Corollary 0.8. $\mathrm{ex}(n, K_{r+1}) \leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2}$.

Definition 0.9. The edge density of a graph G is

$$\frac{e(G)}{\binom{v(G)}{2}}.$$

Proposition 0.10. *(Monotonicity of Turán numbers) For every graph H and positive integer n,*

$$\frac{\mathrm{ex}(n+1,H)}{n+12} \leq \frac{\mathrm{ex}(n,H)}{n2}.$$

For every fixed H, the sequence $\frac{\mathrm{ex}(n,H)}{n2}$ is nonincreasing and bounded between 0 and 1. It follows that it approaches a limit.

Definition 0.11. The *Turán density* of a graph H is defined to be

$$\pi(H) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{n2}.$$