MATH 20510. Analysis in $\mathbb{R}^n$ III (accelerated)

Based on lectures by Prof. Donald Stull

Notes taken by Andrew Hah

The University of Chicago

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1 Measure Theory

Definition 1.1. A family of sets $\mathcal{A}$ is called a *ring* if, for every $A, B \in \mathcal{A}$,

(i) $A \cup B \in \mathcal{A}$

(ii) $A \setminus B \in \mathcal{A}$

Definition 1.2. A ring $\mathcal{A}$ is called a *σ-ring* if for any $\{A_n\}_1^\infty \subseteq \mathcal{A}$,

$$\bigcup_1^\infty A_n \in \mathcal{A}.$$

Note. This implies that $\bigcap_1^\infty A_n \in \mathcal{A}$.

Definition 1.3. ϕ is a *set function* on a ring $\mathcal{A}$ if for every $A \in \mathcal{A}$,

$$\phi(A) \in [-\infty, \infty].$$

Definition 1.4. A set function ϕ is *additive* if for any $A, B \in \mathcal{A}$ such that $A \cap B = \emptyset$,

$$\phi(A \cup B) = \phi(A) + \phi(B).$$

Definition 1.5. A set function ϕ is *countably additive* if for any $\{A_n\} \subseteq \mathcal{A}$ such that $A_i \cap A_j = \emptyset$, $\forall i \neq j$,

$$\phi\left(\bigcup_1^n A_n\right) = \sum_1^n \phi(A_n).$$

In the last two we assume that there are no $A, B \in \mathcal{A}$ such that $\phi(A) = -\infty, \phi(B) = \infty$.

Remark 1.6. If ϕ is an additive set function,

(i) $\phi(\emptyset) = 0$.

(ii) If $A_1, \dots, A_n$ are pairwise disjoint then $\phi(\bigcup_1^n A_n) = \sum_1^n \phi(A_n)$.

(iii) $\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2)$.

(iv) If ϕ is nonnegative and $A_1 \subseteq A_2$ then $\phi(A_1) \leq \phi(A_2)$.

(v) If $B \subseteq A$ and $|\phi(B)| < \infty$ then $\phi(A \setminus B) = \phi(A) - \phi(B)$.

Theorem 1.7. *Let ϕ be a countably additive set function on a ring $\mathcal{A}$. Suppose $\{A_n\} \subseteq \mathcal{A}$ such that $A_1 \subseteq A_2 \subseteq \dots$ and $A = \bigcup_1^\infty A_n \in \mathcal{A}$. Then $\phi(A_n) \to \phi(A)$ as $n \to \infty$.*

Proof. Set $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$. Note

(i) $\{B_n\}$ is pairwise disjoint.

(ii) $A_n = B_1 \cup B_2 \cup \cdots \cup B_n$.

(iii) $A = \bigcup_1^\infty B_n$.

Hence $\phi(A_n) = \sum_1^\infty \phi(B_j)$, $\phi(A) = \sum_1^\infty \phi(B_j)$ and the conclusion follows. $\square$

Definition 1.8. An *interval* $I = \{(a_i, b_i)\}_1^n$ of $\mathbb{R}^n$ is the set of points $x = (x_1, \dots, x_n)$ such that $a_i \leq x_i \leq b_i$ or $a_i < x_i \leq b_i$, etc. where $a_i \leq b_i$.

Note. $\emptyset$ is an interval.

Definition 1.9. If A is the union of a finite number of intervals, we say A is *elementary*.

We denote the set of elementary sets by $\mathcal{E}$.

Definition 1.10. If I is an interval of $\mathbb{R}^n$, we define the volume of I by

$$\operatorname{vol}(I) = \prod_i^n (b_i - a_i).$$

If $A = I_1 \cup I_2 \cup \cdots \cup I_k$ is elementary, and the intervals are disjoint, then

$$\operatorname{vol}(A) = \sum_1^k \operatorname{vol}(I_j).$$

Remark 1.11. (i) $\mathcal{E}$ is a ring, but not a σ-ring.

(ii) If $A \in \mathcal{E}$, then A can be written as a finite union of disjoint intervals.

(iii) If $A \in \mathcal{E}$, then $\operatorname{vol}(A)$ is well-defined.

(iv) vol is an additive set function on $\mathcal{E}$, and $\operatorname{vol} \geq 0$.

Definition 1.12. A nonnegative set function ϕ on $\mathcal{E}$ is *regular* if $\forall A \in \mathcal{E}$, $\forall \varepsilon > 0$, $\exists$ open $G \in \mathcal{E}$, $G \supseteq A$ and closed $F \in \mathcal{E}$, $F \subseteq A$, such that

$$\phi(G) \leq \phi(A) + \varepsilon, \qquad \phi(A) \leq \phi(F) + \varepsilon.$$

Note. vol is regular.

Definition 1.13. A *countable open cover* of $E \subseteq \mathbb{R}^n$ is a collection of open elementary sets $\{A_n\}$ such that $E \subseteq \bigcup_1^\infty A_n$.

Definition 1.14. The *Lebesgue outer measure* of $E \subseteq \mathbb{R}^n$ is defined as

$$m^*(E) = \inf \sum_1^\infty \operatorname{vol}(A_n).$$

where inf is taken over all countable open covers of E.

Remark 1.15. (i) $\mathrm{m}^*(E)$ is well-defined.

(ii) $m^*(E) \geq 0$.

(iii) If $E_1 \subseteq E_2$ then $m^*(E_1) \leq m^*(E_2)$.

Theorem 1.16. *(i) If $A \in \mathcal{E}$, then $m^*(A) = \operatorname{vol}(A)$.*

(ii) If $E = \bigcup_1^\infty E_n$ then $m^*(E) \leq \sum_1^\infty m^*(E_n)$.

Proof. □

Definition 1.17. Let $A, B \subseteq \mathbb{R}^n$.

(i) $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

(ii) $d(A, B) = m^*(A \triangle B)$.

(iii) We say $A_n \to A$ if $\lim_{n\to\infty} d(A_n, A) = 0$.

Definition 1.18. If there is a sequence of elementary sets $\{A_n\}$ such that $A_n \to A$ then we say A is *finitely m-measurable* and we write $A \in \mathfrak{M}_F(m)$.

Definition 1.19. If A is the countable union of finitely m-measurable sets, we say that A is *m-measurable* (Lebesgue measurable) and we write $A \in \mathfrak{M}(m)$.

Theorem 1.20. *$\mathfrak{M}(m)$ is a σ-ring and m^* is countably additive on $\mathfrak{M}(m)$.*

Definition 1.21. The *Lebesgue measure* is the set function defined on $\mathfrak{M}(m)$ by

$$m(A) = m^*(A), \quad \forall A \in \mathfrak{M}(m).$$

To summarize,

set function	**domain**	**properties**
vol	$\mathcal{E}$	≥ 0, additive, $\mathcal{E}$-regular
m^*	$\subseteq \mathbb{R}^n$	≥ 0, $m^*(A) = \text{vol}(A)\ \forall A \in \mathcal{E}$, countably subadditive
m	$\mathcal{M}(m)$	≥ 0, $m(E) = m^*(E)\ \forall E \in \mathcal{M}(m)$, countable additivity(!)

Example 1.22. Fix $n \in \mathbb{N}$.

(i) If $A \in \mathcal{E}$ then $A \in \mathfrak{M}(m)$ since $m^*(A \triangle A) = m^*(\emptyset) = 0 \implies A \to A$.

(ii)