MATH 20510. Analysis in $\mathbb{R}^n$ III (accelerated)

Based on lectures by Prof. Donald Stull

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1 Measure Theory

Definition 1.1. A family of sets $\mathcal{A}$ is called a *ring* if, for every $A, B \in \mathcal{A}$,

(i) $A \cup B \in \mathcal{A}$

(ii) $A \setminus B \in \mathcal{A}$

Definition 1.2. A ring $\mathcal{A}$ is called a *σ-ring* if for any $\{A_n\}_1^\infty \subseteq \mathcal{A}$,

$$\bigcup_1^\infty A_n \in \mathcal{A}.$$

Definition 1.3. ϕ is a *set function* on a ring $\mathcal{A}$ if for every $A \in \mathcal{A}$,

$$\phi(A) \in [-\infty, \infty].$$

Definition 1.4. A set function ϕ is *additive* if for any $A, B \in \mathcal{A}$ such that $A \cap B = \emptyset$,

$$\phi(A \cup B) = \phi(A) + \phi(B).$$

Definition 1.5. A set function ϕ is *countably additive* if for any $\{A_n\} \subseteq \mathcal{A}$ such that $A_i \cap A_j = \emptyset$, $\forall i \neq j$,

$$\phi\left(\bigcup_1^n A_n\right) = \sum_1^n \phi(A_n).$$

In the last two we assume that there are no $A, B \in \mathcal{A}$ such that $\phi(A) = -\infty, \phi(B) = \infty$.

Remark 1.6. If ϕ is an additive set function,

(i) $\phi(\emptyset) = 0$.

(ii) If $A_1, \ldots, A_n$ are pairwise disjoint then $\phi(\bigcup_1^n A_n) = \sum_1^n \phi(A_n)$.

(iii) $\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2)$.

(iv) If ϕ is nonnegative and $A_1 \subseteq A_2$ then $\phi(A_1) \leq \phi(A_2)$.

(v) If $B \subseteq A$ and $|\phi(B)| < \infty$ then $\phi(A \setminus B) = \phi(A) - \phi(B)$.

Theorem 1.7. *Let ϕ be a countably additive set function on a ring $\mathcal{A}$. Suppose $\{A_n\} \subseteq \mathcal{A}$ such that $A_1 \subseteq A_2 \subseteq \dots$ and $A = \bigcup_1^\infty A_n \in \mathcal{A}$. Then $\phi(A_n) \to \phi(A)$ as $n \to \infty$.*

Proof. Set $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$. Note

(i) $\{B_n\}$ is pairwise disjoint.

(ii) $A_n = B_1 \cup B_2 \cup \dots \cup B_n$.

(iii) $A = \bigcup_1^\infty B_n$.

Hence $\phi(A_n) = \sum_1^\infty \phi(B_j)$, $\phi(A) = \sum_1^\infty \phi(B_j)$ and the conclusion follows. □

Definition 1.8. An *interval* $I = \{(a_i, b_i)\}_1^n$ of $\mathbb{R}^n$ is the set of points $x = (x_1, \dots, x_n)$ such that $a_i \leq x_i \leq b_i$ or $a_i < x_i \leq b_i$, etc. where $a_i \leq b_i$.

Note. $\emptyset$ is an interval.

Definition 1.9. If A is the union of a finite number of intervals, we say A is *elementary.*

We denote the set of elementary sets by $\mathcal{E}$.

Definition 1.10. If I is an interval of $\mathbb{R}^n$, we define the volume of I by

$$\mathrm{vol}(I) = \prod_i^n (b_i - a_i).$$

If $A = I_1 \cup I_2 \cup \dots \cup I_k$ is elementary, and the intervals are disjoint, then

$$\mathrm{vol}(A) = \sum_1^k \mathrm{vol}(I_j).$$

Remark 1.11. (i) $\mathcal{E}$ is a ring, but not a σ-ring.

(ii) If $A \in \mathcal{E}$, then A can be written as a finite union of disjoint intervals.

(iii) If $A \in \mathcal{E}$, then $\mathrm{vol}(A)$ is well-defined.

(iv) vol is an additive set function on $\mathcal{E}$, and vol ≥ 0.

Definition 1.12. A nonnegative set function ϕ on $\mathcal{E}$ is *regular* if $\forall A \in \mathcal{E}$, $\forall \varepsilon > 0$, $\exists$ open $G \in \mathcal{E}$, $G \supseteq A$ and closed $F \in \mathcal{E}$, $F \subseteq A$, such that

$$\phi(G) \leq \phi(A) + \varepsilon, \qquad \phi(A) \leq \phi(F) + \varepsilon.$$

Note. vol is regular.

Definition 1.13. A *countable open cover* of $E \subseteq \mathbb{R}^n$ is a collection of open elementary sets $\{A_n\}$ such that $E \subseteq \bigcup_1^\infty A_n$.

Definition 1.14. The *Lebesgue outer measure* of $E \subseteq \mathbb{R}^n$ is defined as

$$m^*(E) = \inf \sum_1^\infty \mathrm{vol}(A_n).$$

where inf is taken over all countable open covers of E.

Remark 1.15. (i) $\mathrm{m}^*(E)$ is well-defined.
(ii) $m^*(E) \geq 0$.
(iii) If $E_1 \subseteq E_2$ then $m^*(E_1) \leq m^*(E_2)$.

Theorem 1.16. *(i) If $A \in \mathcal{E}$, then $m^*(A) = \mathrm{vol}(A)$.*
(ii) If $E = \bigcup_1^\infty E_n$ then $m^*(E) \leq \sum_1^\infty m^*(E_n)$.

Proof. (i) Let $A \in \mathcal{E}$ and $\epsilon > 0$. Since vol is regular, $\exists$ open $G \in \mathcal{E}$ such that $A \subseteq G$ and $\mathrm{vol}(G) \leq \mathrm{vol}(A) + \epsilon$. Since $G \supseteq A$ and $G \in \mathcal{E}$ is open, $m^*(A) \leq \mathrm{vol}(G) \leq \mathrm{vol}(A) + \epsilon$. There also $\exists$ closed $F \in \mathcal{E}$ such that $F \subseteq A$ and $\mathrm{vol}(A) \leq \mathrm{vol}(F) + \epsilon$. By definition, $\exists$ collection $\{A_n\}$ of open elementary sets such that $A \subseteq \bigcup A_n$ and $\sum_1^\infty \mathrm{vol}(A_n) \leq m^*(A) + \epsilon$. Since $F \subseteq \bigcup A_n$ and F is compact, $F \subseteq A_1 \cup \cdots \cup A_N$ from some N.

$$\begin{aligned}
\mathrm{vol}(A) &\leq \mathrm{vol}(F) + \epsilon \\
&\leq \mathrm{vol}(A_1 \cup \cdots \cup A_N) + \epsilon \\
&\leq \sum_1^N \mathrm{vol}(A_n) + \epsilon \\
&\leq \sum_1^\infty \mathrm{vol}(A_n) + \epsilon
\end{aligned}$$

$$\leq m^*(A) + \epsilon + \epsilon$$
$$= m^*(A) + 2\epsilon$$

Since ϵ was arbitrary, $m^*(A) = \text{vol}(A)$. $\square$

Proof. (ii) If $m^*(E_n) = \infty$ for any $n \in \mathbb{N}$, then we are done. Assume not. Let $\epsilon > 0$. For every $n \in \mathbb{N}$, $\exists$ open cover of E_n, $\{A_{n,k}\}_{k=1}^{\infty}$ such that

$$\sum_{k=1}^{\infty} \text{vol}(A_{n,k}) \leq m^*(E_n) + \epsilon/2^n$$

Then $E \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{n,k}$ and so

$$\begin{aligned} m^*(E) &\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \text{vol}(A_{n,k}) \\ &\leq \sum_{n=1}^{\infty} m^*(E_n) + \epsilon/2^n \\ &= \sum_{n=1}^{\infty} m^*(E_n) + \sum_{1}^{\infty} \epsilon/2^n \\ &= \sum_{1}^{\infty} m^*(E_n) + \epsilon \end{aligned}$$ $\square$

Definition 1.17. Let $A, B \subseteq \mathbb{R}^n$.

(i) $A \triangle B = (A \setminus B) \cup (B \setminus A)$.
(ii) $d(A, B) = m^*(A \triangle B)$.
(iii) We say $A_n \to A$ if $\lim_{n\to\infty} d(A_n, A) = 0$.

Definition 1.18. If there is a sequence of elementary sets $\{A_n\}$ such that $A_n \to A$ then we say A is *finitely m-measurable* and we write $A \in \mathfrak{M}_F(m)$.

Definition 1.19. If A is the countable union of finitely m-measurable sets, we say that A is *m-measurable* (Lebesgue measurable) and we write $A \in \mathfrak{M}(m)$.

Theorem 1.20. *$\mathfrak{M}(m)$ is a σ-ring and m^* is countably additive on $\mathfrak{M}(m)$.*

Definition 1.21. The *Lebesgue measure* is the set function defined on $\mathfrak{M}(m)$ by

$$m(A) = m^*(A), \quad \forall A \in \mathfrak{M}(m).$$

To summarize,

set function	**domain**	**properties**
vol	$\mathcal{E}$	≥ 0, additive, $\mathcal{E}$-regular
m^*	$\subseteq \mathbb{R}^n$	$\geq 0,\ m^*(A) = \mathrm{vol}(A)\ \forall A \in \mathcal{E}$, countably subadditive
m	$\mathcal{M}(m)$	$\geq 0,\ m(E) = m^*(E)\ \forall E \in \mathcal{M}(m)$, countable additivity(!)

Example 1.22. Fix $n \in \mathbb{N}$.

(i) If $A \in \mathcal{E}$ then $A \in \mathfrak{M}(m)$ since $m^*(A \triangle A) = m^*(\emptyset) = 0 \implies A \to A$.

(ii) $\mathbb{R}^n \in \mathfrak{M}(m)$ since $\mathbb{R}^n = \bigcup_{N \in \mathbb{N}} [-N, N]^n \implies m(\mathbb{R}^n) = \infty$.

(iii) If $A \in \mathfrak{M}(m)$ then $A^c \in \mathfrak{M}(m)$.

(iv) $\forall x \in \mathbb{R}^n$, $\{x\} \in \mathfrak{M}(m)$ and $m(\{x\}) = 0$.

(v) $\forall x_1, \ldots, x_n \in \mathbb{R}^n$, $\{x_1, \ldots, x_n\} \in \mathfrak{M}(m)$ and $m(\{x_1, \ldots, x_n\}) = 0 \implies m(\mathbb{Q}^n) = 0$.

Definition 1.23. $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, \infty\}$ is *measurable* if $\{x \in \mathbb{R}^n : f(x) > a\} \in \mathfrak{M}$, $\forall a \in \mathbb{R}$, i.e. $f^{-1}((a, \infty]) \in \mathfrak{M}$, $\forall a \in \mathbb{R}$.

Example 1.24. f continuous $\implies$ f measurable.

Theorem 1.25. *The following are equivalent,*

(i) $\{x : f(x) > a\}$ *is measurable* $\forall a \in \mathbb{R}$.

(ii) $\{x : f(x) \geq a\}$ *is measurable* $\forall a \in \mathbb{R}$.

(iii) $\{x : f(x) < a\}$ *is measurable* $\forall a \in \mathbb{R}$.

(iv) $\{x : f(x) \leq a\}$ *is measurable* $\forall a \in \mathbb{R}$.

Proof. (i) $\implies$ (ii).

$$\{x : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \left\{x : f(x) > a - \frac{1}{n}\right\}$$

□

Theorem 1.26. *If f is measurable then $|f|$ is measurable.*

Proof. It suffices to show that $\{x : |f(x)| < a\} \in \mathfrak{M}$, $\forall a \in \mathbb{R}$.

$$\{x : |f(x)| < a\} = \{x : f(x) < a\} \cap \{x : f(x) > -a\}$$

$\square$

Theorem 1.27. *Suppose $\{f_n\}$ is a sequence of measurable functions. Define*

$$g = \sup_n f_n \quad \text{and} \quad h = \limsup_{n\to\infty} f_n$$

Then g, h are measurable.

Proof. $\{x : g(x) > a\} = \bigcup_{n=1}^{\infty}\{x : f_n(x) > a\}$ implies g is measurable. Similarly, $\inf_n f_n$ is measurable. Define $g_n = \sup_{n\geq m} f_n$ and note that g_n is measurable for all m. Since $h = \inf_m g_m$, h is measurable. $\square$

Corollary 1.28. *If f, g are measurable then $\max\{f, g\}$ and $\min\{f, g\}$ are also measurable.*

Corollary 1.29. *Define $f^+ = \max\{f, 0\}$, $f^- = -\min\{f, 0\}$. Then if f is measurable, f^+, f^- are also measurable.*

Corollary 1.30. *If $\{f_n\}$ is a sequence of measurable functions such that f_n converges to f pointwise, then f is measurable.*

Theorem 1.31. *$f, g : \mathbb{R}^n \to \mathbb{R}$ measurable, $F : \mathbb{R}^2 \to \mathbb{R}$ continuous, and $h(x) = F(f(x), g(x))$. Then h is measurable. In particular, this tells us that $f + g$ and fg are measurable.*

Definition 1.32. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *simple* if $\text{range}(f)$ is a finite set.

Example 1.33. Let $E \subseteq \mathbb{R}^n$. The characteristic function of E is

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

Suppose f is simple, so $\text{range}(f) = \{c_1, \ldots, c_m\}$. Let $E_i = \{x : f(x) = c_i\}$. Then

$$f = \sum_1^m \chi_{E_i} c_i$$

Theorem 1.34. *$f : \mathbb{R}^n \to \mathbb{R}$. There exists a sequence $\{f_n\}$ of simple functions such that $f_n \to f$ pointwise.*

(i) If f is measurable, $\{f_n\}$ can be chosen to be measurable.

(ii) If $f \geq 0$ then $\{f_n\}$ can be chosen to be monotonically increasing.

Proof. If $f \geq 0$, define the sets

$$E_{n,i} = \left\{ x : \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \right\}, \quad n \geq 1, i = 1, \ldots, n2^n$$

$$F_n = \{x \mid f(x) \geq n\}, \quad n \geq 1$$

Define

$$f_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}} + n\chi_{F_n}$$

We see that f_n is measurable. Fix $x \in \mathbb{R}^n$, let $\varepsilon > 0$, and let $N \in \mathbb{N}$ such that $N > f(x)$ and $2^{-N} < \varepsilon$. Let $n \geq N$. Note that $x \in E_{n,i}$ for some i. Since $f_n(x) = \frac{i-1}{2^n}$ and $f(x) \geq f_n(x)$, $f(x) - f_n(x) \leq \frac{1}{2^n} < \varepsilon$. Thus $f_n \to f$ pointwise. We now show $\{f_n\}$ is monotonically increasing.

(i) Case 1: $x \in F_n$. Then $f(x) \geq n$ and $f_n(x) = n$. If $x \in F_{n+1}$, then $f_{n+1}(x) = n + 1 > n = f_n(x)$. If $x \notin F_{n+1}$ then $x \in$ some $E_{n+1,i}$. Then $\frac{i-1}{2^{n+1}} \geq n \implies f_{n+1}(x) \geq n = f_n(x)$.

(ii) Case 2: $x \in E_{n,i}$ for some i. Then $f_n(x) = \frac{i-1}{2^n}$. Then there is some j such that $x \in E_{n+1,j} = \{x : \frac{j-1}{2^{n+1}} \leq f(x) \leq \frac{j}{2^{n+1}}\}$. Because $\frac{i-1}{2^n} \leq f(x)$, we have $\frac{j-1}{2^{n+1}} \geq \frac{i-1}{2^n}$ so $f_{n+1}(x) = \frac{j-1}{2^{n+1}} \geq \frac{i-1}{2^n} = f_n(x)$.

Thus in both cases, $\{f_n\}$ is monotonically increasing. We next consider the general case. Given f, we write $f^+(x) = \max\{f(x), 0\}$ and $f^- = -\min\{f(x), 0\}$ so that $f = f^+ - f^-$ and $f^+, f^- \geq 0$. By the previous part, there exist two sequences of nonnegative measurable simple functions $f_n^+ \to f^+$ and $f_n^- \to f^-$ each converging pointwise. Define $f_n(x) = f_n^+(x) - f_n^-(x)$. Then f_n is simple and measurable since it is the difference of two simple measurable functions, and converges pointwise. $\square$