MATH 23500
Lecture 21

May 12, 2025

Brownian motion. This is a continuous time, continuous space process $\{B_t\}_{t\geq 0}$. We should think of it as a random function $B : [0,\infty) \to \mathbb{R}$. We define B in terms of its properties. Assume $B_0 = 0$.

(i) Stationary increments. For every $t > s \geq 0$, $B_t - B_s$ has the same distribution as B_{t-s}.
(ii) Independent increments. For every $s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq s_k \leq t_k$, the increments $B_{t_j} - B_{s_j}$ for $j = 1, \ldots, k$ are independent.
(iii) Continuity. $t \mapsto B_t$ is a continuous function.

Theorem 0.1. *Let $B : [0,\infty) \to \mathbb{R}$ be a random continuous function with independent, stationary increments. Assume $B_0 = 0$. Then there exists $\mu \in \mathbb{R}$, $\sigma^2 > 0$ such that $B_t \sim \mathcal{N}(\mu t, \sigma^2 t)$. Furthermore, μ and σ^2 characterize B uniquely.*

Construction.

(i) Take a simple random walk $S_n = \sum_{k=1}^n X_i$. Then by the central limit theorem,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \leq x\right) \to \Phi(x),$$

i.e., the rescaled random walk converges to a $\mathcal{N}(0,1)$ random variable. Then

$$B_n^{(t)} := \frac{S_{\lfloor tn \rfloor}}{\sqrt{tn}} \to B_t,$$

i.e., these processes are $\mathcal{N}(0,t)$.

(ii) (Lévy construction) Consider a family of normal random variables indexed by a countable dense subset of $[0,1]$. Recursively define B_k using this set. Linearly interpolate and take limits.

Definition 0.2. Standard Brownian motion is the process $\{B_t\}_{t\geq 0}$ with $B_0 = 0$ satisfying the following

(i) B is continuous
(ii) For each $s < t$, $B_t - B_s \sim \mathcal{N}(0, t-s)$.
(iii) For each $s_1 \leq t_1 \leq \cdots \leq s_k \leq t_k$, the increments $B_{t_j} - B_{s_j}$ are independent.

Note. $X \sim \mathcal{N}(\mu, \sigma^2)$. The density of X is

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{(x-\mu)^2}{2\sigma^2}\right)}.$$

Example 0.3. Suppose B is a standard Brownian motion. Compute $\mathbb{P}(B_1 \geq 1, B_3 \geq B_1 + 1)$.

$$\mathbb{P}(B_1 \geq 1) = \frac{1}{\sqrt{2\pi}} \int_1^\infty e^{-\frac{x^2}{2}} dx$$

and

$$\mathbb{P}(B_3 \geq B_1 + 1) = \mathbb{P}(B_3 - B_1 \geq 1) = \frac{1}{\sqrt{4\pi}} \int_1^\infty e^{-\frac{x^4}{4}} dx.$$

By independence, we multiply to get the result.

Proposition 0.4. *Let B be a standard Brownian motion, and $c > 0$. Then $t \mapsto c^{-1/2} B_{ct}$ is a standard Brownian motion.*

Proof. Clearly the rescaled process is continuous and has independent increments. Note that for $s < t$, $B_{ct} - B_{cs} \sim \mathcal{N}(0, c(t-s))$. Thus,

$$c^{-1/2}(B_{ct} - B_{cs}) \sim \mathcal{N}(0, t-s).$$

□

Proposition 0.5. *For each fixed $t \geq 0$, it holds with probability 1 that B is not differentiable at t.*

Proof. For $\varepsilon > 0$, $\varepsilon^{-1/2}(B_{t+\varepsilon} - B_t) \sim \mathcal{N}(0,1)$. Thus, $\frac{B_{t+\varepsilon} - B_t}{\varepsilon}$ is a $\mathcal{N}(0,1)$ random variable Z, but rescaled by $1/\sqrt{\varepsilon}$. Hence,

$$\mathbb{P}\left(\frac{|B_{t+\varepsilon} - B_t|}{\varepsilon} \geq c\right) = \mathbb{P}\left(\frac{1}{\sqrt{\varepsilon}}|Z| \geq c\right) = \mathbb{P}(|Z| \geq \sqrt{\varepsilon} c) \to 1,$$

as $\varepsilon \to 0$. Thus, with probability 1, $\limsup_{\varepsilon \to 0} \frac{B_{t+\varepsilon} - B_t}{\varepsilon} = \infty$. □