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THE EUCLIDIAN DISTANCE MATRIX COMPLETION PROBLEM*

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Abstract. Motivated by the molecular conformation problem, completions of partial Euclidian distance matrices are studied. It is proved that any partial distance matrix with a chordal graph can be completed to a distance matrix. Given any nonchordal graph G, it is shown that there is a partial distance matrix A with graph G such that A does not admit any distance matrix completions. Finally, the connection between distance matrix completions and positive semidefinite completions is outlined.

Key words. Euclidian distance matrix, partial matrix, completion, positive (semi)definite matrix, circumhypersphere

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1. Introduction. Let $|| \quad ||$ denote Euclidian length on $\mathbf{R}^k$. For two points A and B, we use $d(A,B)$ for $||A-B||$. The matrix $D=(d_{ij})_{i,j=1}^n$ is a (*Euclidian*) *distance matrix* if there exist $P_1,\ldots,P_n \in \mathbf{R}^k$ such that $d_{ij}=d(P_i,P_j)^2$. A great deal is known about distance matrices (e.g., [2], [7], [9]). For example in [9], a symmetric matrix $D=(d_{ij})_{i,j=1}^n$, with $d_{ii}=0$, $i=1,\ldots,n$, is a distance matrix if and only if D is negative semidefinite on the orthogonal complement of the vector $e=(1,1,\ldots,1)^T$. This is equivalent to the statement that the bordered matrix

$$\begin{pmatrix} 0 & e^T \\ e & D \end{pmatrix} \tag{1}$$

has only one positive eigenvalue or to the fact that the Schur complement of the upper left $2-by-2$ principal submatrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in (1) is negative semidefinite. Furthermore, the rank of this Schur complement is the minimum dimension k in which the points $P_1,\ldots,P_n$ may lie. In this case we say that D is a *distance matrix* in $\mathbf{R}^k$. In §1, we use these characterizations to recover a result concerning circumhyperspheres [7].

We call an $n-by-n$ array $A=(a_{ij})_{i,j=1}^n$ a *partial distance matrix* in $\mathbf{R}^k$ if

(i) every entry a_{ij} of A is either "specified" or "unspecified" (free to be chosen);

(ii) a_{ii} is specified as 0, $i=1,\ldots,n$, and a_{ji} is specified (and equal to a_{ij}) if and only a_{ij} is specified; and

(iii) every fully specified principal submatrix of A is itself a distance matrix in $\mathbf{R}^k$.

A *completion* of a partial distance matrix is a choice of values for each of the unspecified entries, resulting in a conventional matrix. The *distance matrix completion*

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problem then asks which partial distance matrices have distance matrix completions. It is clear that assumption (iii) above is necessary for this. This "inheritance property" is important in other previously studied completion problems, such as those involving positive definiteness, inertia, rank, and contractions.

Our principal result here is that if the undirected graph of the specified entries of a partial distance matrix is chordal, then it necessarily has a distance matrix completion. This is based upon an analysis of the case of partial distance matrices with one pair of symmetrically placed unspecified entries, the "one variable problem," together with one-step-at-a-time technology for chordal graphs developed in [8]. For any nonchordal graph there exists partial distance matrices without distance matrix completions.

Though it should also be of interest in the classical subject of distance geometry, we were motivated by the molecular mapping, or "conformation," problem. This is the problem of deducing the possible shapes of a molecule from partial (or inaccurate) information about interatomic distances. For many compounds, not all of the interatomic distances may be measured accurately, but the shape (essentially determined by the distance matrix) is crucial to understanding how the molecule functions.

For terminology and results concerning graph theory we essentially follow [6]. An *undirected graph* is a pair $G = (V, E)$ in which V, the *vertex set*, is a finite set (usually $V = \{1, \dots, n\}$), and the *edge set* E is a symmetric binary relation on V. The *adjacency set* of a vertex v is denoted by $\mathrm{Adj}(v)$, i.e., $w \in \mathrm{Adj}(v)$ if $\{v, w\} \in E$. Given a subset $S \subseteq V$, define the *subgraph induced* by S by $G_S = (S, E_S)$, in which $E_S = \{\{x, y\} \in E | x \in S \text{ and } y \in S\}$. A *complete graph* is one with the property that every pair of distinct vertices is adjacent. A subset $K \subseteq V$ is a *clique* if the induced graph on K is complete. The complement $G = (V, \bar{E})$ of a graph $G = (V, E)$ is defined by $\bar{E} = \{\{i, j\} | i \neq j, \textit{ and } \{i, j\} \notin E\}$.

A *path* $[v_1, \dots, v_k]$ is a sequence of vertices such that $\{v_j, v_{j+1}\} \in E$ for $j = 1, \dots, k-1$. A *cycle* of length $k > 2$ is a path $[v_1, \dots, v_k, v_1]$ in which $v_1, \dots, v_k$ are distinct. A graph G is called *chordal* if every cycle of length greater than three possesses a chord, i.e., an edge joining two nonconsecutive vertices of the cycle. A subset $S \subset V$ is called a $u - v$ *vertex separator* for the nonadjacent vertices u and v if the removal of S from the graph separates u and v into distinct connected components. If no proper subset of S contains a $u - v$ separator, then S is a *minimal* $u - v$ *separator*. It is known ([6, Thm. 4.1]) that an undirected graph is chordal if and only if every minimal vertex separator is a clique.

In §4 we are concerned with the connections between positive semidefinite completions and distance matrix completions.

A partial matrix $A = (a_{ij})_{i,j}^n$ is called (*combinatorially*) *symmetric* if

(i) a_{ii} is specified, $i = 1, \dots, n$, and

(ii) a_{ij} is specified if and only if a_{ji} is also specified.

All partial distance matrices are symmetric. With a symmetric partial matrix $A = (a_{ij})_{i,j=1}^n$ we associate the undirected graph $G = (V, E)$ with $V = \{1, 2, \dots, n\}$ and $E = \{\{i, j\} | a_{ij} \textit{ is specified}\}$.

A symmetric partial matrix A is called *partial positive semidefinite* if all fully specified principal submatrices of A are positive semidefinite. In [8] it has been proved that any partial positive semidefinite matrix, the graph of whose specified entries is chordal, can be completed to a positive semidefinite matrix. We translate this result into a distance problem among points on a hypersphere. We also treat by this approach the problem of the existence of a positive semidefinite completion of a partial positive semidefinite matrix having a nonchordal graph.

Throughout the paper, for a matrix $A = (a_{ij})_{i,j=1}^n$ and an index set $\alpha \subset \{1,\ldots,n\}$, $A(\alpha)$ denotes the principal submatrix of A whose rows and columns correspond to the index set α. The notation A^- represents the (unique Moore–Penrose) generalized inverse of the Hermitian matrix A.

2. Circumhyperspheres. Using the approach presented in the introduction, one easily recovers the results of [7] concerning the existence of a circumhypersphere for a set of points.

THEOREM 2.1. *Let D be a distance matrix corresponding to the points $P_1,\ldots,P_n$ in $\mathbf{R}^n$. Then, there is a circumhypersphere for $P_1,\ldots,P_n$ if and only if $e^TD^-e \neq 0$. This has radius given by $r^2 = (2e^TD^-e)^{-1}$.*

Proof. Using results on generalized Schur complements (see e.g. [3]), one obtains that the number $i_+(M)$ of positive eigenvalues of a partitioned matrix $M = \left(\begin{smallmatrix} A & B \\ B^* & C\end{smallmatrix}\right)$ is given by

$$i_+(M) = i_+(C) + i_+(A - BC^-B^*) + \operatorname{rank}(B|\ker C), \tag{2}$$

in which $B|\ker C$ means the restriction of B to the null space of C. Let D be a distance matrix corresponding to the points $P_1,\ldots,P_n$ in $\mathbf{R}^n$. By the observation in the introduction, the existence of $O \in \mathbf{R}^n$ such that $d(O,P_i) = r$ for $i = 1,\ldots,n$ is equivalent to the condition that the bordered matrix

$$\hat{D} = \begin{pmatrix} 0 & 1 & e^T \\ 1 & 0 & r^2e^T \\ e & r^2e & D \end{pmatrix}$$

has exactly one positive eigenvalue. Let $a = -e^TD^-e$, and then, by (2) we have that

$$i_+(\hat{D}) = i_+(D) + i_+\left(\begin{pmatrix} -a & 1-r^2a \\ 1-r^2a & -r^4a \end{pmatrix}\right) + \operatorname{rank}\left(\begin{pmatrix} e^T \\ r^2e^T \end{pmatrix} |\ker D\right).$$

Since

$$\begin{pmatrix} 0 & e^T \\ e & D \end{pmatrix}$$

has exactly one positive eigenvalue, we have that $i_+(D) = 1$ and $e^Th = 0$ for any $h \in \ker D$. Thus

$$\operatorname{rank}\left(\begin{pmatrix} e^T \\ r^2e^T \end{pmatrix} |\ker D\right) = 0,$$

and $i_+(\hat{D}) = 1$ if and only if

$$\begin{pmatrix} -a & 1-r^2a \\ 1-r^2a & -r^4a \end{pmatrix} \tag{3}$$

is negative semidefinite. This can be realized if and only if $a > 0$. We always have $a \geq 0$ since

$$i_+\left(\begin{pmatrix} 0 & e^T \\ e & D \end{pmatrix}\right) = 1.$$

The smallest r that makes (3) negative semidefinite is given by $r = 1/\sqrt{2a}$, and this completes the proof. $\square$

3. The main results. The following result is a consequence of Corollary 3.2 in [5]. For the sake of completeness we present a proof.

LEMMA 3.1. *Let*

$$R = \begin{pmatrix} a & B & x \\ B^T & C & D \\ x & D^T & f \end{pmatrix}$$

be a real partial positive semidefinite matrix, with x an unknown scalar and

$$\operatorname{rank}\begin{pmatrix} a & B \\ B^T & C \end{pmatrix} = p$$

and

$$\operatorname{rank}\begin{pmatrix} C & D \\ D^T & f \end{pmatrix} = q,$$

with (necessarily) $|p-q| \leq 1$. Then there is real positive semidefinite completion F of R such that $\operatorname{rank}F = \max\{p,q\}$. *Moreover, this completion is unique if and only if* $\operatorname{rank}C = p$ *or* $\operatorname{rank}C = q$.

Proof. Let U be an orthogonal matrix that diagonalizes C, namely, $U^TCU = Y$, in which Y is a positive semidefinite diagonal matrix. Let $\hat{U} = 1 \oplus U \oplus 1$ and

$$\hat{R} = \hat{U}^T R \hat{U} = \begin{pmatrix} a & \hat{B} & x \\ \hat{B}^T & Y & \hat{D} \\ x & \hat{D}^T & f \end{pmatrix}$$

in which $\hat{B} = BU$ and $\hat{D} = U^TD$. Since $\hat{U}$ is orthogonal, the set of numbers that make R positive semidefinite coincides with that making $\hat{R}$ positive semidefinite and $\operatorname{rank}R = \operatorname{rank}\hat{R}$. Since

$$\begin{pmatrix} a & \hat{B} \\ \hat{B}^T & Y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Y & \hat{D} \\ \hat{D}^T & f \end{pmatrix}$$

are positive semidefinite, all entries (in the rows and columns corresponding to diagonal entries of Y that are 0) equal zero. We may eliminate those rows and columns and then assume that Y is invertible. Then, $\hat{R}$ is positive semidefinite if and only if the Schur complement S of Y in $\hat{R}$ is positive semidefinite. But,

$$S = \begin{pmatrix} a - \hat{B}Y^{-1}\hat{B}^T & x - \hat{B}Y^{-1}\hat{D} \\ x - \hat{D}^TY^{-1}\hat{B}^T & f - \hat{D}^TY^{-1}\hat{D} \end{pmatrix}$$

and $\operatorname{rank}\hat{R} = \operatorname{rank}C + \operatorname{rank}S$. If $\operatorname{rank}C = p$ or $\operatorname{rank}C = q$ then $a - \hat{B}Y^{-1}\hat{B}^T = 0$, respectively $f - \hat{D}^TY^{-1}\hat{D}^T = 0$, and, thus, we must choose $x = \hat{B}Y^{-1}\hat{D}$. If $\operatorname{rank}C < p = q$, the problem has two solutions given by

$$|x - \hat{B}Y^{-1}\hat{D}|^2 = (a - \hat{B}Y^{-1}\hat{B}^T)(f - \hat{D}^TY^{-1}\hat{D})$$

that realize the completion of S to a rank-1 negative semidefinite matrix. □

LEMMA 3.2. *The partial distance matrix*

$$R = \begin{pmatrix} 0 & D_{12} & x \\ D_{12}^T & D_{22} & D_{23} \\ x & D_{23}^T & 0 \end{pmatrix}$$

admits at least one completion to a distance matrix F. Moreover, if

$$\begin{pmatrix} 0 & D_{12} \\ D_{12}^T & D_{22} \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} D_{22} & D_{23} \\ D_{23}^T & 0 \end{pmatrix}$$

are distance matrices in $\mathbf{R}^p$, respectively, $\mathbf{R}^q$, then x can be chosen so that F is a distance matrix in $\mathbf{R}^s$, $s = \max\{p, q\}$.

Proof. Without loss of generality, we may assume that R is at least $3 - by - 3$, since, otherwise, we may complete with any positive number. Thus, R has at least one fully specified row and column. Interchange the first two rows and the first two columns of R, and then we must complete the partial distance matrix

$$\tilde{R} = \begin{pmatrix} 0 & d_{12} & \tilde{D}_{13} & d_{14} \\ d_{12} & 0 & \tilde{D}_{23} & x \\ \tilde{D}_{13}^T & \tilde{D}_{23}^T & \tilde{D}_{33} & \tilde{D}_{34} \\ d_{14} & x & \tilde{D}_{34}^T & 0 \end{pmatrix}$$

to a distance matrix in $\mathbf{R}^s$. By the remark in the introduction, this latter problem is equivalent to finding completions of the partial matrix

$$\tilde{D} = \begin{pmatrix} 0 & 1 & 1 & e^T & 1 \\ 1 & 0 & d_{12} & \tilde{D}_{13} & d_{14} \\ 1 & d_{12} & 0 & \tilde{D}_{23} & x \\ e & \tilde{D}_{13}^T & \tilde{D}_{23}^T & \tilde{D}_{33} & \tilde{D}_{34} \\ 1 & d_{14} & x & \tilde{D}_{34}^T & 0 \end{pmatrix}$$

to a matrix in which the Schur complement of the upper left $2 - by - 2$ principal submatrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is negative semidefinite and has rank s. This latter Schur complement is of the form

$$S = \begin{pmatrix} a & B & x - d_{12} - d_{14} \\ B^T & C & D \\ x - d_{12} - d_{14} & D^T & f \end{pmatrix},$$

in which

$$\begin{pmatrix} a & B \\ B^T & C \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C & D \\ D^T & f \end{pmatrix}$$

are negative semidefinite and have ranks less than or equal to s. Then, any negative semidefinite completion of S of rank s given by Lemma 3.1 provides a solution to our distance completion problem. $\square$

Remark 1. From the proof of Lemma 3.2 and the uniqueness part of Lemma 3.1, we obtain that the partial distance matrix in $\mathbf{R}^k$,

$$R = \begin{pmatrix} 0 & D_{12} & x \\ D_{12}^T & D_{22} & D_{23} \\ x & D_{23}^T & 0 \end{pmatrix}$$

admits a unique completion to a distance matrix in $\mathbf{R}^k$ if and only if

$$\operatorname{rank}\left(\begin{pmatrix} 0 & e^T \\ e & D_{22} \end{pmatrix}\right) = k + 2.$$

Our main result is the following theorem.

THEOREM 3.3. *Every partial distance matrix in $\mathbf{R}^k$, the graph of whose specified entries is chordal, admits a completion to a distance matrix in $\mathbf{R}^k$.*

Proof. Let R be an $n-by-n$ partial distance matrix in $\mathbf{R}^k$ and assume that the graph $G=(V,E)$ of R is chordal. Then from [8], there exists a sequence of chordal graphs $G=G_0, G_1, \ldots, G_t = K_n$ (the complete graph on n vertices), such that each G_j is obtained by adding exactly one new edge $\{u_j, v_j\}$ to G_{j-1}. Moreover, each G_j, $j=1,\ldots,t$, has only one maximal clique V_j that is not a clique in G_{j-1}.

Consider first the partial submatrix $R(V_1)$, with one pair of unknowns, symmetrically placed on the (u_1, v_1) and (v_1, u_1) positions. Then, by Lemma 3.2, we can specify these entries and obtain a partial distance matrix in $\mathbf{R}^k$ having G_1 as the graph of its specified entries. Then we complete the partial submatrix corresponding to the index set V_2. We continue this one-entry-at-a-time completion procedure until we complete R to a distance matrix in $\mathbf{R}^k$. □

Example. Given any nonchordal graph $G=(V,E)$, $V=\{1,\ldots,n\}$, we show that there exists a partial distance matrix $R=(r_{ij})_{i,j=1}^n$ such that R has no completion to a distance matrix. Assume that the vertices $1,2,\ldots,k \geq 4$ form a chordless cycle in G. Define the partial distance matrix R by

$$r_{ij} = \begin{cases} 0 & \text{if } \{i,j\} \in E \quad \text{and} \quad k+1 \leq i,j \leq n, \\ 0 & \text{if } |i-j| = 1,\ 1 \leq i \leq j \leq k, \\ 1 & \text{for any other } \{i,j\} \in E. \end{cases}$$

Then any fully specified principal submatrix of R is either

$$0, \begin{pmatrix} 0 & e^T \\ e & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & e^T \\ 0 & 0 & e^T \\ e & e & 0 \end{pmatrix}, \text{ or } \begin{pmatrix} 0 & 1 & e^T \\ 1 & 0 & e^T \\ e & e & 0 \end{pmatrix},$$

each of them being a distance matrix. Thus R is a partial distance matrix, but R does not admit a completion to a distance matrix. Indeed, the upper left $k-by-k$ principal submatrix cannot be completed to a distance matrix since otherwise there exist points $P_1,\ldots,P_k$ such that $d(P_i,P_{i+1})=0$ for $i=1,\ldots,k-1$ and $d(P_1,P_k)=1$, a contradiction.

THEOREM 3.4. *Let R be a partial distance matrix in $\mathbf{R}^k$, the graph $G=(V,E)$ of whose specified entries is chordal, and let $\mathcal{S}$ be the collection of all minimal vertex separators of G. Then R admits a unique completion to a distance matrix in $\mathbf{R}^k$ if and only if*

$$\begin{pmatrix} 0 & e^T \\ e & R(S) \end{pmatrix} \text{ has rank } k+2 \text{ for any } S \in \mathcal{S}. \tag{4}$$

Proof. We prove the result by induction on the cardinality m of the complement of the edge set of G. If $m=1$, the result follows from Remark 1. Assume the result is true for any chordal graph whose complement has cardinality less than m. Let $G=(V,E)$ be a chordal graph such that $|\bar{E}|=m$ and let R be a partial distance matrix satisfying (4). Let S be an arbitrary minimal vertex separator of G. By Ex. 12, Chap. IV in [6], there exist vertices u and v belonging to different connected components of G_{V-S} with the property that $S \subset \text{Adj}(u)$ and $S \subset \text{Adj}(v)$.

We first prove that the graph $G'=(V, E \cup \{\{u,v\}\})$ is also chordal. Assume the contrary, which means that there exists a chordless cycle $[u, x_1, \ldots, x_k, v]$, $k \geq 2$, in

G'. By the definition of a minimal vertex separator, at least one $x_l \in S$, $1 \leq l \leq k$. This implies that $\{x_l, u\}, \{x_l, v\} \in E$, a contradiction, showing that G' is chordal.

Then $S \cup \{u, v\}$ is the unique maximal clique in G' that is not a clique in G. As in the proof of Theorem 3.3, consider the principal submatrix $R(S \cup \{u, v\})$ having only one pair of symmetrically placed unspecified entries. Complete $R(S \cup \{u, v\})$ to a distance matrix in $\mathbf{R}^k$ to obtain a partial distance matrix $\tilde{R}$ having G' as the graph of its specified entries.

If

$$\operatorname{rank}\left(\begin{pmatrix} 0 & e^T \\ e & R(S) \end{pmatrix}\right) < k + 2,$$

by Remark 1 $R(S \cup \{u, v\})$ has more than one completion to a distance matrix in $\mathbf{R}^k$ and so R admits more than one completion.

If R satisfies (4), then $\tilde{R}$ constructed above is uniquely determined. Since any minimal vertex separator of G' contains a minimal vertex separator of G, $\tilde{R}$ also satisfies condition (4). By the assumption made for $m - 1$, $\tilde{R}$ admits a unique completion to a distance matrix in $\mathbf{R}^k$. This implies that R also admits a unique completion to a distance matrix in $\mathbf{R}^k$. □

Let $0 < m < n$ be given integers. Since the graph $G = (V, E)$ with $E = \{\{i, j\} | 0 < |i - j| \leq m\}$ is chordal, Theorem 3.3 and Remark 1 have the following consequence in the "band" case.

COROLLARY 3.5. *Any partial distance matrix* $R = (r_{ij})_{i,j=1}^n$ *in* $\mathbf{R}^k$, *with* r_{ij} *specified if and only if* $|i - j| \leq m$, *admits a completion to a distance matrix in* $\mathbf{R}^k$. *Moreover, the completion is unique if and only if all the matrices*

$$\begin{pmatrix} 0 & e^T \\ e & R(l, \ldots, l + m - 1) \end{pmatrix}$$

have rank $k + 2$ *for any* $l = 1, \ldots, n - m + 1$.

4. Connections with positive semidefinite completions.

LEMMA 4.1. *Let* $A = (a_{ij})_{i,j=1}^n$ *be a symmetric matrix such that* $a_{ii} = 1$ *for* $i = 1, \ldots, n$. *Then* A *is positive semidefinite if and only if there are* n *points* $P_1, \ldots, P_n$ *on a hypersphere of radius* $\sqrt{2}/2$ *in* $\mathbf{R}^k$, $k = \operatorname{rank} A$, *such that* $d(P_i, P_j) = \sqrt{1 - a_{ij}}$ *for any* $i, j = 1, \ldots, n$.

Proof. As remarked in the introduction, the existence of the points $P_1, \ldots, P_n$ and O in $\mathbf{R}^k$ such that $d(P_i, P_j) = \sqrt{1 - a_{ij}}$ and $d(O, P_i) = \sqrt{2}/2$ is equivalent to the condition that the Schur complement of the upper left $2 - by - 2$ principal submatrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & \ldots & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 & 1 - a_{12} & \ldots & 1 - a_{1n} \\ 1 & \frac{1}{2} & 1 - a_{12} & 0 & \ldots & 1 - a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{1}{2} & 1 - a_{1n} & 1 - a_{2n} & \ldots & 0 \end{pmatrix}$$

is negative semidefinite and has rank k. A straightforward computation shows that this latter Schur complement equals $-A$. This completes the proof. □

By Lemma 4.1, the result in [8] on positive definite completion of partial positive definite matrices having a chordal graph can be translated into the following.

THEOREM 4.2. *Let $G = (V, E)$ be a chordal graph and $\{K_j\}_{j=1}^m$ the maximal cliques of G. Consider points $P_1, \dots, P_n$ in $\mathbf{R}^n$ satisfying the following conditions:*

(i) *The distances $d(P_i, P_j)$ are specified if and only if $\{i, j\} \in E$.*

(ii) *Each of the subsets $\{P_l\}_{l \in K_j}$, $j = 1, \dots, m$, lie on a hypersphere of radius R.*

Then the points $P_1, \dots, P_n$ can be chosen to lie on a hypersphere of radius R.

Remark 2. The conclusion of Theorem 4.2 is not valid when the graph G is not chordal. Consider G, for example, to be the simple cycle of length 4 and points A, B, C, D such that $d(A, B) = d(B, C) = d(C, D) = 2$ and $d(A, D) = 0$. Then each of the pairs $\{A, B\}$, $\{B, C\}$, $\{C, D\}$, and $\{A, D\}$ lie on a sphere of radius 1, but the smallest radius of a sphere on which $A = D$, B, and C may lie is $2\sqrt{3}/3 > 1$.

As a particular case of Theorem 4.2 we obtain the following corollary, analogous to the result on positive semidefinite completion of banded partial matrices in [4].

THEOREM 4.3. *Let $0 < m < n$ be given integers, and consider points $P_1, \dots, P_n$ in $\mathbf{R}^n$ satisfying the following conditions:*

(i) *The distances $d(P_i, P_j)$ are specified if and only if $|i - j| \leq m$.*

(ii) *Each of the subsets $\{P_k, \dots, P_{k+m-1}\}$, $k = 1, \dots, n - m + 1$, lie on a hypersphere of radius R.*

Then the points $P_1, \dots, P_n$ can be chosen to lie on a hypersphere of radius R.

We note that an elementary, purely geometric proof of Theorem 4.3 can be provided.

Let us also mention the following positive semidefinite completion problem, which is still unsolved.

(P_1) Given is a partial positive semidefinite matrix $A = (a_{ij})_{i,j=1}^n$ such that the graph of the specified entries of A is not chordal. Determine necessary and sufficient conditions on A in order that A admits at least one positive semidefinite completion.

Without loss of generality we may assume that A has a unit diagonal, since otherwise we may apply a diagonal congruence.

Consider now the following distance problem.

(P_2) Let $G = (V, E)$ be a nonchordal graph and let $\{K_1, \dots, K_m\}$ be the maximal cliques of G. Consider the points $P_1, \dots, P_n$ in $\mathbf{R}^n$ satisfying the following conditions.

(C_1) The distances $d(P_i, P_j) = d_{ij}$ are specified if and only if $\{i, j\} \in E$.

(C_2) Each of the subsets $\{P_l\}_{l \in K_j}$, $j = 1, \dots, m$, lie on a hypersphere of radius R.

Determine the hypersphere of minimum radius (if any) on which the points $P_1, \dots, P_n$ satisfying (C_1) and (C_2) may lie.

The problems (P_1) and (P_2) are equivalent. Indeed, without loss of generality, we may assume that $R = \sqrt{2}/2$. Consider the partial positive semidefinite matrix A satisfying the conditions of Problem (P_1) and then consider the points $P_1, \dots, P_n$ in R^n such that $d(P_i, P_j) = \sqrt{1 - a_{ij}}$ for any $(i, j) \in E$. Then, by Lemma 4.1, A admits a positive semidefinite completion if and only if the points $P_1, \dots, P_n$ may be chosen to lie on a hypersphere of radius $\sqrt{2}/2$.

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