PHY350

Electromagnetism

Chapters	New	Review
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Method of Eval

5 hw assignments 50%

Final exam 50%

Lateness policy : 2 days late 75%

Lecture 2:

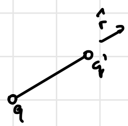
② No corrections to EM due to relativity. As it serves a fundational role. why? EM waves propagate without any medium.

③ No correction due to QM, except at short distance scales

④ Vital for quantum field theory and particle physics.

Review and construction

Coulombs law $\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qq'}{r^2}\hat{r}$



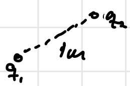
In the mks unit, the unit of charge is the amount of charge potential in one second of one ampere.

$[\epsilon_0] = \frac{[\text{coul}]^2}{\text{Newt-m}}$ → perm of free space.

$\frac{1}{4\pi\epsilon_0} = 9\times10^9$ by experiment.

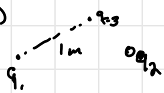
Superposition principle (Shown by experiment rather than deduced)

(a)



• q_3 very far away

(b)



• q_2 very far away

(c)

sum of forces a and b.

Force Lorentz law.

$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$ mag field

electric field; velocity charge q

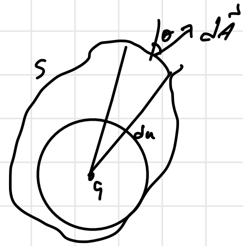
M.K.S units $[B]$ $\frac{\text{Newtons}}{\text{Amp-m}}$ = Tesla

C.G.S units $\vec{F} = q(\vec{E} + \frac{\vec{v}}{c}\times\vec{B})$ $[B]$ has units of gauss.

1 tesla = $1\cdot 10^4$ gauss

Some geo interpretations of experimental observations.

obs: coulombs laws



Geo perspective: equivalent to gauss's law $\oiint \vec{E}\cdot d\vec{A} = \frac{q}{\varepsilon_0}$

Flux through outer patch: $E_R\, dA \cos\theta$

Flux through inner patch: $E_r\, da$.

$$E_R\, dA\cos\theta = E\left(\frac{r}{R}\right)^2\left[da\left(\frac{R}{r}\right)^2\frac{1}{\cos\theta}\right]\cos\theta = E_r\, da$$

Flux through smaller sphere $\frac{q}{4\pi\varepsilon_0 r^2}(4\pi r^2) = q/\varepsilon_0$

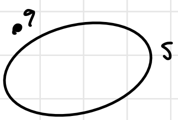
Apply superposition principle for general charge distribution.

$$d\Omega = \frac{dA\cdot\hat{r}}{r^2} \quad \text{by def}$$

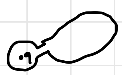
$$E\cdot d\vec{A} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}\cdot d\vec{A} = \frac{q}{4\pi\varepsilon_0}\,d\Omega$$

$$\therefore \oiint E\cdot d\vec{A} = \frac{q}{4\pi\varepsilon_0}\int d\Omega = q/\varepsilon_0$$

How do you know $\oiint_S \vec{E}\cdot d\vec{A} = 0$ if charge q is outside of surface S?



Try new surface

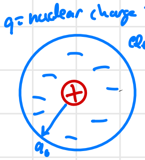


In many cases gauss's law and symmetry considerations offers a powerful tool.

Example 1: Semi-classical model of atom

q = nuclear charge = $|e|$

electron charge



Apply new ext field

Let ρ = charge density of cloud.

By symmetry ① E-field of e^- cloud is purely radial and depends only on r.

② Also charge between r and a_0 does not contribute to any spherical region of radius $\leq r$

③ By gauss's law $\vec{F} = -\frac{|e|\, q_{eff}}{x^2}\hat{x}$ where $q_{eff} = \left(\frac{x}{a_0}\right)^3 |e|$.

$\Rightarrow \vec{F} = -k_{eff}\,\hat{x}$ where $k_{eff} = \frac{e^2}{a_0^3}$ effective spring constant.

Force balance $|e|\vec{E}_{ext} - k_{eff}\vec{x} = 0$ at equi

"Induced dipole moment" $\vec{p} = |e|\vec{x} = \frac{|e|^2 \vec{E}_{ext}}{k_{eff}} = \alpha \vec{E}$

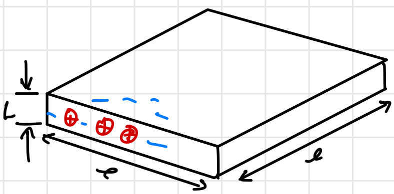
"Atomic Polarizability" $\alpha = a_0^3 \rightarrow$ M.K.S $\alpha = 4\pi\varepsilon_0 a_0^3$

Atom responds as simple harmonic oscillator with natural frequency $\omega = \sqrt{\frac{k_{eff}}{m}}$ $\approx e^-$ mass

for external fields of low frequency and amplitude.

[Reduced Mass: $\frac{1}{m} = \frac{1}{m_e} + \underbrace{\frac{1}{M_N}}_{\text{Nucleus.}} \approx \frac{1}{m_e}$]

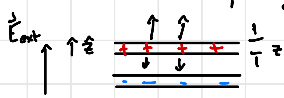
Example 2: Electrons in a metal



$N = \underbrace{n}_{\text{avg number density.}} L l^2$ total number of electrons and nuclei

"Jellium model:" Consider uniform positive and negative densities. (no E field).

Now consider displacing + and − components rigidly by an amount "z".



Two planar sheets of charge

By sym E-field is only in z-direction.

Surround top layer by gaussian box.

$$\oiint \vec{E} \cdot d\vec{A} = 2El^2 = \frac{1}{\varepsilon_0}(z l^2)|e| n.$$

(Turn off E_{ext}) $\Rightarrow E = \frac{\sigma}{2\varepsilon_0}$ where $\sigma = |e| n z$ is the "surface charge density"

Total electric field inside $= \frac{\sigma}{\varepsilon_0}$ (from both layers)

Restoring force on a single charge $|e|$ on the top layer

$$\vec{F} = -\hat{z}\frac{\sigma |e|}{\varepsilon_0} = -\hat{z}\, z \underbrace{\left(\frac{n e^2}{\varepsilon_0}\right)}_{\text{effective spring constant.}}$$

positive nuc very heavy and don't move much.

For electron mass "m" this leads to harmonic osc at frequency $\omega_p = \sqrt{\frac{k_{eff}}{m}}$

$\boxed{\omega_p^2 = \frac{n e^2}{\varepsilon_0 m}}$ known as plasma frequency.

Overview:

1. Magnetic field and its sources
   Biot - Savart law $\longleftrightarrow$ Ampère's law
2. Faraday's law of induction.
3. Conservation of charge
4. Maxwell's equations

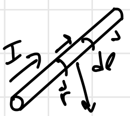
Recall previous observations:

Coulomb's law $\longleftrightarrow$ Gauss' law.
Superposition Principle
Lorentz force law.

<u>Experimental</u> <u>Facts/Observations</u> <u>continued</u>

<u>Biot Savart law (Sources of static Magnetic field)</u>

Magnetic analogue of coulombs law.



$\vec{B}$-field of steady current at point $d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \hat{r}}{r^2}$ (M.K.S)

$\mu \equiv$ permeability of free space $= 4\pi \times 10^{-7}\ N/A^2$

units are such that $[B]$ measured in $\frac{N}{A \cdot m} \equiv$ Tesla

In C.G.S units

$$d\vec{B} = \frac{I}{c} \frac{d\vec{l} \times \hat{r}}{r^2} \; ; \; \vec{B} = \frac{I}{c} \oint \frac{d\vec{l} \times \hat{r}}{r^2} \quad \text{(measured in Gauss)}$$

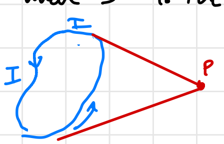
Valid only if: $\begin{cases} 1. & I \text{ is independent of time} \\ 2. & \text{Integrate over whole loop.} \end{cases}$

For steady stream of moving charges $d\vec{B} = \frac{dq}{c} \vec{v} \times \frac{\hat{r}}{r^2}$

<u>Geometrical perspective</u> (just for fun).

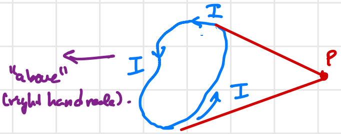
Theorem: $\vec{B} = \frac{I}{c} \oint \frac{d\vec{l} \times \hat{r}}{r^2}$ is equivalent to statement that $\vec{B} = -\frac{I}{c} \vec{\nabla} \Omega$

where Ω is the "solid angle" subtended by circuit at point P (in question).



Geometrical perspective (just for fun).

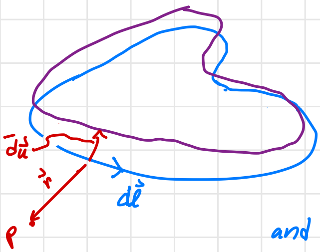
Theorem: $\vec{B} = \frac{I}{c} \oint \frac{d\vec{\ell} \times \hat{r}}{r^2}$ is equivalent to statement that $\vec{B} = -\frac{I}{c} \vec{\nabla} \Omega$ when Ω is the "solid angle" subtended by circuit at point P (in question).



Definition: $\Omega > 0$ if P is "above"
$\Omega < 0$ if P is "below"

Proof: Displace P by small amount $d\vec{u}$, then Ω changes by $d\Omega = d\vec{u} \cdot \vec{\nabla} \Omega$ ①

Now keep P fixed and displace current loop by $-d\vec{u}$.
This must give same $d\Omega$:



$d\Omega$ is the angle subtend by the area of the ribbon between the two loop positions.

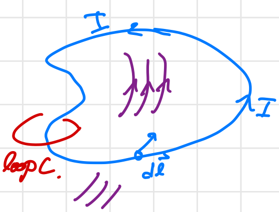
Differential element of area $-d\vec{u} \times d\vec{\ell}$ has magnitude of area and direction normal to the surface.

$$d\Omega = \oint \frac{-d\vec{u} \times d\vec{\ell} \cdot \hat{r}}{r^2} = -d\vec{u} \cdot \oint \frac{d\vec{\ell} \times \hat{r}}{r^2} \quad ②$$

(using vector identity $\vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C}$

① and ② $\Rightarrow$ $\boxed{\vec{\nabla} \Omega = -\oint \frac{d\vec{\ell} \times \hat{r}}{r^2}}$ Q.E.D.

Biot-Savart law is equivalent to Ampère's law.



$\oint_c \vec{B} \cdot d\vec{\ell} = \mu_0 I$ (M.K.S)

$\oint_c \vec{B} \cdot d\vec{\ell} = \frac{4\pi}{c} I$ (C.G.S)

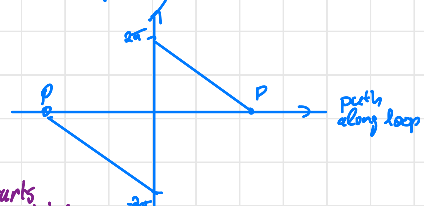
Proof of equivalence

Consider the whole current loop together with hypothetical closed loop C. that pierces it.

$$\int_c \vec{B} \cdot d\vec{\ell} = -\frac{I}{c} \oint \vec{\nabla}\Omega \cdot d\vec{\ell} = \frac{I}{c} \oint d\Omega = \begin{cases} 0 \\ \frac{4\pi I}{c} \end{cases} \text{(see below).}$$

If loop C contains the current loop then as $d\vec{\ell}$ passes through the current loop, the subtended solid angle changes continuously from -2π to 2π.

Then integrating $d\Omega$ around loop gives -4π



Starts at point where current loop subtends to zero.

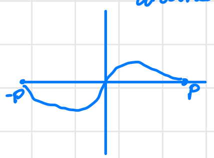
Away from discontinuity

$\vec\nabla\Omega\cdot d\vec l<0$ everywhere

$$\oint_c \vec\nabla\Omega\cdot d\vec l = -4\pi$$

If C does not pierce through current loop then Ω is cts around C and $\oint d\Omega = 0$

e.g.



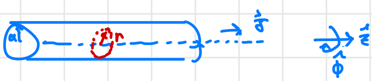
for a path that cuts plane of current loop twice at exterior of loop.

Above discussion is valid only for steady state (not for time varying fields).

How do we use Biot-Savart's & Ampere's law?

Example:

Magnetic field in a cylindrical wire with uniform charge density $\vec J$



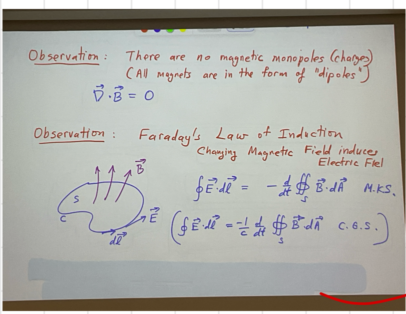
$\vec B$-field is purely tangential $= B\hat\phi$

Use Ampère's law for $r>a$

$$2\pi r B = \mu_0 J \pi r^2$$

$\therefore$ For $r<a$ $\quad \vec B = \frac{\mu_0}{2} J r\hat\phi$

For $r>a$, $2\pi r B = \mu_0 J\pi a^2 \Rightarrow \vec B = \frac{\mu_0}{2}\frac{Ja^2}{r}\hat\phi$



Now combine/summarize all previous observations with theorems of Stokes and Gauss

For general vector field $\vec A$ $\quad \oiint (\vec\nabla\times\vec A)\cdot\hat n\,dS = \oint \vec A\cdot d\vec l$

$\hat n$ normal to surface $\quad \iiint_V (\vec\nabla\cdot\vec A)\,d^3\vec r = \oint_S \vec A\cdot\hat n\,dS$

(Divergence thm).

to obtain four differential relations

① $\vec\nabla\cdot\vec E = \frac{1}{\varepsilon_0}\rho$ $\quad$ ③ $\vec\nabla\cdot\vec B = 0$ $\quad$ ρ = charge density

② $\vec\nabla\times\vec E = -\frac{\partial\vec B}{\partial t}$ $\quad$ ④ $\vec\nabla\times\vec B = \mu_0\vec J$ $\quad$ $\vec J$ = current density.

But Maxwell noticed an inconsistency.

Overview:

1. Maxwell's Hypothesis: A vital addition to previous emperical evidence.
2. Useful mathematical tools for Vector calc
3. Maxwell's prediction for wave prop in vacuum.
4. Vector and scalar potentials.
5. Decoupling of maxwell's equation for static charge and current densities.

Summary of Experimental observations in differential form.

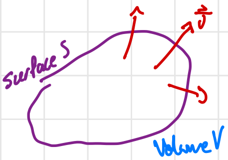
① $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ ③ $\vec{\nabla}\cdot\vec{B} = 0$ ρ = charge density

② $\vec{\nabla}\times\vec{E} = \frac{-\partial\vec{B}}{\partial t}$ ④ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ $\vec{J}$ = current density

Maxwell noticed an inconsistency:
Apply divergence to ④.

$$0 = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0\vec{\nabla}\cdot\vec{J}$$

But this means charge can never flow in or out of a volume of space.



$$0 = \int_V \vec{\nabla}\cdot\vec{J}\, d^3\vec{r} = \int_S \vec{J}\cdot d\vec{A}$$

But physically we know that charge must be able to enter and leave volume V.

For non-steady state situations, Maxwell hypothesized:

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$ modified from #4.

Why? let take divergence of left and right hand side.

$$\mu_0\vec{\nabla}\cdot\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E} = 0$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{J} = \frac{\partial\rho}{\partial t} = 0.}$$ Continuity equation for conserved charge.

This small modification leads to the propagation of E&M waves in a vacuum.

But first some mathematical tools:

1. Kronecker Delta Function $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$i = 1, 2, 3$

2. Levi-Civita Tensor (3rd rank)

Symbol: ϵ_{ijk} where $i, k, l = 1, 2, 3$

Definition: (i) $\epsilon_{123} \equiv 1$

(ii) Completely Antisymmetric $\epsilon_{ijk} = -\epsilon_{ikj}$

interchanging any pair of indices changes its sign.

e.g. $\epsilon_{ijk} = \epsilon_{jik}$ etc ....

Verify consistency $\epsilon_{ijk} = -\epsilon_{ikj} = +\epsilon_{kij} = -\epsilon_{kji}$

(iii) Corollary: if any two indices are the same, this tensor vanishes $\epsilon_{iik} = -\epsilon_{iik} = 0$

interchange by first two sides

Einstein summation convention: (a notation simplification and notation device).

Consider two vectors $\vec{A} = (A_1, A_2, A_3)$ and $\vec{B} = (B_1, B_2, B_3)$

$$\vec{A} \cdot \vec{B} = \sum_i^3 A_i B_i \xrightarrow{\text{rewrite}} A_i B_i$$ (summation $\sum_{i=1}^{3}$ is implied by repeated index).

<u>Cross product</u>:

$$(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k$$

check $(\vec{A} \times \vec{B})_1 = \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2 = A_2 B_3 - A_3 B_2$ etc ....

A very important Theorem:

$$\epsilon_{ikj} \epsilon_{ilm} = \delta_{kl} \delta_{jm} - \delta_{km} \delta_{jl}$$

You can verify case by case:

$$\epsilon_{i12} \epsilon_{i23} = \epsilon_{112} \epsilon_{123} + \epsilon_{212} \epsilon_{223} + \epsilon_{312} \epsilon_{323} = 0$$

$$\delta_{12} \delta_{23} - \delta_{13} \delta_{22} = 0$$

$$\epsilon_{i12} \epsilon_{i12} = \epsilon_{112} \epsilon_{112} + \epsilon_{212} \epsilon_{212} + \epsilon_{312} \epsilon_{312} = 1.$$

$$\delta_{11} \delta_{22} - \delta_{12} \delta_{21} = 1.$$ etc.

This enables easy derivation of Calculus formula.

Propagation of EM waves:

(2) $\Rightarrow \frac{\partial}{\partial t}\vec{\nabla}\times\vec{E} = \frac{-\partial^2 \vec{B}}{\partial t^2}$

But in free space, $\vec{J}=0$ and $\rho=0 \Rightarrow \vec{\nabla}\times\vec{B} = \underbrace{\mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}}_{\text{Due to maxwell modification}}$

$\therefore (\vec{\nabla}\times\vec{\nabla}\times\vec{B})_i = \epsilon_{ijk}\partial_j \epsilon_{klm}\partial_l B_m$

$= \epsilon_{kij}\epsilon_{klm}\partial_j\partial_l B_m$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l B_m = \partial_i\partial_j B_j - \partial_j\partial_j B_i$

$= \partial_i(\underbrace{\vec{\nabla}\cdot\vec{B}}_{0}) - \nabla^2 B_i$

$\therefore \vec{\nabla}\times\vec{\nabla}\times\vec{B} = \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B} \overset{\text{maxwell}}{=} \mu_0\epsilon_0 \frac{\partial^2 B}{\partial t^2}$

$$\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} = 0$$ free space wave equation where $\frac{1}{c^2}\equiv\mu_0\epsilon_0$

Since $\vec{\nabla}\cdot\vec{B}=0$, we can always write $\vec{B}=\vec{\nabla}\times\vec{A}$ for some "vector potential" $\vec{A}(\vec{r},t)$.
Recall electrostatics, we wrote $\vec{E}=-\vec{\nabla}\varphi$ for some "scalar potential" $\varphi(\vec{r},t)$.

But now we are allowing time variations
e.g. charge can enter or leave a region.

Make new guess:

Let $\vec{E} = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t}$ (Ansatz).

Plug into maxwell's equations.

(2) $\vec{\nabla}\times\vec{E} = \frac{-\partial}{\partial t}\vec{\nabla}\times\vec{A} = \frac{-\partial\vec{B}}{\partial t}$ ✓

(3) $\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$

So (2) and (3) are automatically satisfied.

(1) $\vec{\nabla}\cdot\vec{E}=\rho/\epsilon_0 \Rightarrow -\nabla^2\varphi - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \rho/\epsilon_0$

But there is some freedom of choice of A while keeping condition $\vec{B}=\vec{\nabla}\times\vec{A}$.

e.g. $\vec{A}' = \vec{A} + \vec{\nabla}\chi$ would still give the same $\vec{B}$, $\vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{\nabla}\chi$

We can use this dof to choose $\vec{\nabla}\cdot\vec{A}=0$

Then $-\nabla^2\varphi=\rho/\epsilon_0$ and we need to solve for φ as we did in electrostatics.

What about equation #4?

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \underbrace{\vec{\nabla}\times\vec{\nabla}\times\vec{A}}_{\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}^2\vec{A}} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\left[-\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t}\right]$$

This is still a complicated equation to solve, but $\underbrace{\text{at least its one equation, not four!}}_{\text{contains alot of physics. (mostly covered in PHY450)}}$

The simplest case is free space:

No currents: $\vec{J} = 0$

Also move all charges very far away (to ∞) : $\vec{\nabla}\varphi = 0$

Then $\boxed{\nabla^2\vec{A} = \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}}$ Wave equation.

Recall: Coulomb Gauge $\vec{\nabla}\cdot\vec{A} = 0$

In free space maxwell's equations $\Rightarrow \nabla^2\vec{A} = \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}$

Assume $\underset{\text{everywhere}}{\phi = 0}$ (all sources moved to ∞)

$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = 0$, has plane wave solutions.

Try $\vec{A}(\vec{r},t) = A_0\vec{\epsilon}_{\vec{k}}e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\nabla^2\vec{A} = -k^2\vec{A}$ where $k \equiv |\vec{k}|$

$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = \frac{\omega^2}{c^2}\vec{A}$

This is a solution iff $-k^2 + \frac{\omega^2}{c^2} = 0$, $k = \frac{\omega}{c}$

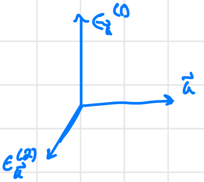
ω = frequency (rad/sec).

$\nu = \frac{\omega}{2\pi}$ = # of cycles/sec

$k = \frac{2\pi}{\lambda}$, λ = wavelength.

ϵ_k is a polarization vector.

Since $\vec{\nabla}\cdot\vec{A} = 0$, $\vec{k}\cdot\epsilon_{\vec{k}} = 0$ (transverse waves).



Electric field: $\vec{E} = -\frac{\partial\vec{A}}{\partial t} = iA_0\omega\vec{\epsilon}_{\vec{k}}e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

Magnetic field: $\vec{B} = \vec{\nabla}\times\vec{A}$

$\vec{B} = ikA_0\hat{k}\times\vec{\epsilon}_{\vec{k}}e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

where $k = \frac{\omega}{c}$

ith component $\epsilon_{ijk}\partial_j[(\vec{\epsilon}_{\vec{k}})_k e^{ik_n r_n}] = \epsilon_{ijk}(ik_j)(\vec{\epsilon}_{\vec{k}})_k e^{ik_n r_n} = i(\vec{k}\times\vec{\epsilon}_{\vec{k}})_i e^{ik_n r_n}$

continue next page.

$\vec{B} = \frac{1}{c}\hat{k}\times\vec{E}$ $\qquad |\vec{B}| = \frac{|\vec{E}|}{c}$ (in M.K.S units)

$\vec{E}\perp\vec{B}$ and wave propagates in direction of $\vec{E}\times\vec{B}$

Another special case: Re-introduce and charge current sources, but no time variation.

Maxwell's equations decouple:

$$\underbrace{① \ \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \qquad ② \ \vec{\nabla}\times\vec{E} = -\cancel{\frac{\partial\vec{B}}{\partial t}}\,0}_{\text{e statics}}$$

$$\underbrace{③ \ \vec{\nabla}\cdot\vec{B} = 0 \qquad ④ \ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\cancel{\frac{\partial\vec{E}}{\partial t}}\,0}_{\text{mag statics}}$$

But we know a solution of ① even before writing down ①:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^2}\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}\right)$$ } unit vector pointing from $\vec{r}'$ to r.

But $\vec{E} = \nabla\varphi$ potential

$\Rightarrow \nabla^2\varphi = -\rho/\varepsilon_0$ poissons eq.

Consider region with no known charge den

$\nabla^2\varphi = 0$ Laplace equation.

In cartesian coords: $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$, solutions are called harmonic functions.

An important thm:

If $\varphi(x,y,z)$ satisfies Laplaces equation, then the average value of φ over the surface of any sphere (not ness small) is equal to value of φ at center of sphere.

Intuitive proof:

(i) Consider a sphere S in the field of an external point charge q.

(ii) imagine a test charge q' spread uniformly over the sphere S.

(iii) Work required to bring q' from ∞ and distribute uniformly is $W = q'\varphi_{avg}$

where φ_{avg} is the average value of φ over sphere S, due to q.

(iv) But this must be the same work that would be needed to bring q from infinity if q' were already present.

(V) But (iv) must be same work as if the charge q' were concentrated at center of S

Intuitive proof:

(i) Consider a sphere S in the field of an external point charge q.
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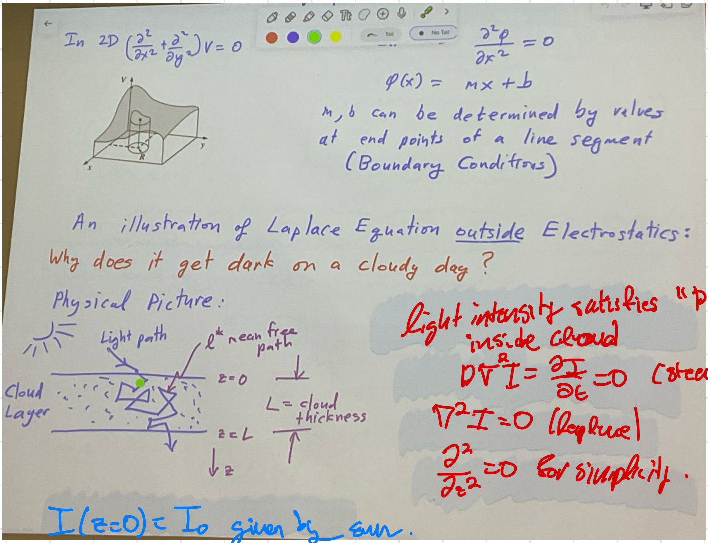
(iv) But this must be the same work that would be needed to bring q from infinity if q' were already present.

(v) But (iv) must be same work as if the charge q' were concentrated at center of S

(vi) This is same work needed to bring point charge q' from ∞ to center of S in presence of q.

(vii) Superposition principle: Since potentials of many sources are additive the above must be true for any system of sources lying outside of S.

Corollary: φ cannot have any local maxima or minima. The extreme values of φ must occur at boundaries.



light intensity satisfies "Diffusion equation" inside cloud

$D\nabla^2 I = \frac{\partial I}{\partial t} = 0$ (steady state).

$\nabla^2 I = 0$ (Laplace)

$\frac{\partial^2}{\partial z^2} = 0$ for simplicity.

$I(z=0) = I_0$ given by sun.

$\therefore I(z) = I_0 + az$ for some "a".

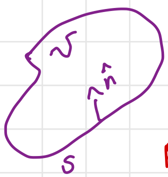
First uniqueness thm:

The solution of Laplace's equation $\nabla^2 \varphi = 0$ in some volume uniquely determined if φ is specified on the boundary surface.

Proof: Suppose there are two solutions φ_1 and φ_2 satisfying same boundary conditions.
Then $\nabla^2 \varphi_1 = 0$ and $\nabla^2 \varphi_2 = 0 \Rightarrow \nabla^2 \varphi = 0$, where $\varphi = \varphi_1 - \varphi_2$
But $\varphi = 0$ over the entire boundary surface.
Since φ cannot have any max or min inside this surface, $\varphi = 0$ everywhere inside.

General Uniqueness theorem:

If $\varphi(\vec{r})$ satisfies $\nabla^2 \varphi = -\rho/\varepsilon_0$ in a volume V with given boundary conditions, then it is the only solution.



By boundary condition, we mean either φ or $\hat{n}\cdot\vec{\nabla}\varphi$ is specified on surface

Proof: Suppose $\nabla^2 \varphi_1 = -\rho/\varepsilon_0$ and $\nabla^2 \varphi_2 = -\rho/\varepsilon_0$.
Then $\nabla^2 \varphi = 0$ where $\varphi = \varphi_1 - \varphi_2$
and on the surface S, either $\varphi = 0$ or $\hat{n}\cdot\vec{\nabla}\varphi = 0$.
Consider the vector function: $\vec{v} = \varphi \vec{\nabla} \varphi$

$$\vec{\nabla}\cdot\vec{v} = \varphi \underbrace{\nabla^2 \varphi}_{0} + (\vec{\nabla}\varphi)^2$$

$$\int d^3\vec{r}\,(\vec{\nabla}\cdot\vec{v}) = \underbrace{\int dS\, \hat{n}\cdot\vec{v}}_{\text{by divergence thm}} = \int dS\, \varphi(\hat{n}\cdot\vec{\nabla}\varphi) = 0$$

by our assumed boundary condition.

$$\parallel$$

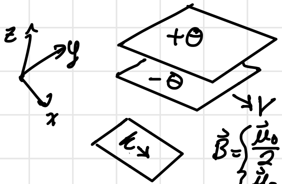
$\int d^3\vec{r}\,(\vec{\nabla}\varphi)^2 \leftarrow$ is positive definite $\Rightarrow \vec{\nabla}\varphi = 0$ throughout V.

$\therefore \varphi =$ constant throughout V.

For "Dirichlet B.C." $\varphi = 0 \Rightarrow \varphi = 0$ throughout V.
For "Neumann B.C." $\hat{n}\cdot\vec{\nabla}\varphi = 0$, $\varphi =$ const and so $\vec{\nabla}\varphi = 0$ throughout V.
In either case, the E field $\vec{E} = -\vec{\nabla}\varphi$ is unique.

Next week we'll look at ways of guessing the solution.

Q.5.18.



$\vec{B} = \begin{cases} \frac{\mu_0}{2} K \hat{y} & \text{above} \\ \frac{\mu_0}{2} K \hat{y} & \text{below} \end{cases}$

we can use superposition to add the two plates together.

$K_T = \sigma v \quad , \quad K_B = -\sigma v$

$\vec{B} = \begin{cases} 0 & \text{above} \\ K \mu_0 \hat{y} & \text{between.} \\ 0 & \text{below.} \end{cases}$ and $K = \sigma v$.

$\vec{F} = \int \vec{K} \times \vec{B}\, da \rightarrow f_m = \vec{K} \times \vec{B} = (\sigma v \hat{x}) \times (\frac{\mu_0}{2} \sigma v \hat{y})$

$\vec{E} = \frac{-\sigma}{2\epsilon_0} \Rightarrow f_e = \frac{-\sigma^2}{2\epsilon_0} \hat{z}$

$f_m = -f_e \Rightarrow v^2 \frac{\sigma^2}{2} \mu_0 \hat{z} = \frac{\sigma^2}{2\epsilon_0} \hat{z} \Rightarrow v^2 = \frac{1}{\mu_0 \epsilon_0}$

$\Rightarrow v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c.$

Q7.39:

$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$

$\nabla \cdot \vec{B} = 0$

$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \partial_t \vec{E}$

$\vec{E}(r,t) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \Theta(vt - r) \hat{r}, \ \vec{B}(r,t) = 0.$

$\nabla \cdot \vec{B} = 0$

$-\partial_t \vec{B} = 0 = \nabla \times \vec{B}.$

$\int_{-\infty}^{x} \delta(x')\, dx' = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} = \Theta(x) \qquad \partial_x \Theta(x) = \delta(x).$

$\rho = \epsilon_0 \nabla \cdot \vec{E} = \frac{q}{4\pi} \Theta(vt - r) \nabla \cdot (\frac{\hat{r}}{r^2}) + \frac{q}{4\pi r^2} \hat{r} \cdot \nabla \Theta(vt - r) = \frac{q}{4\pi} \delta^3(\vec{r}) \Theta(vt - r) + \frac{q}{4\pi r^2} \delta(vt - r)$

$= \frac{q}{4\pi} \delta^3(\vec{r}) \Theta(-vt - r) - \frac{q}{4\pi r^2} \delta(vt - r).$

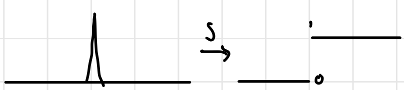
$\int_{B_R} \nabla \cdot (\frac{\hat{r}}{r^2})\, dV = \int_{S_R} \frac{\hat{r}}{r^2} \cdot \hat{r}\, da = \frac{1}{R^2} \int_{S_R} da = \frac{4\pi R^2}{R^2} = 4\pi.$

$\nabla \cdot (\frac{\hat{r}}{r^2}) = \delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$

$\vec{J} = -\epsilon_0 \partial_t \vec{E} = \frac{-q}{4\pi r^2} \delta(vt - r) \hat{r} v = \frac{-q}{4\pi r^2} v \delta(vt - r) \hat{r}$

$\frac{d}{dx} \Theta(x) = \delta(x)$, if $\int_{-\infty}^{x} f(x')\, dx' = F(x) \quad \frac{d}{dx} F(x) = f(x)$

$\Theta(x) = \int_{-\infty}^{x} \delta(x')\, dx' = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$



$\int_D \delta(x) f(x)\, dx = \begin{cases} f(0) & 0 \in D \\ 0 & 0 \notin D \end{cases}$

$\int_{\mathbb{R}} \Theta(x) f(x)\, dx = \int_{\mathbb{R}>0} f(x)\, dx$

$\rho(r) = kr$ (where k is constant).

$$W = \frac{1}{2}\int \rho V d\tau = \frac{k}{2}\int rV d\tau$$

note that $V = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(r')}{\eta} d\tau = \frac{1}{4\pi\varepsilon_0}\iiint \frac{kr'}{}$

Let $d\vec{a} = r^2 \sin\theta d\theta d\phi$

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow \oint E\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

$$\Rightarrow \oint E\cdot d\vec{a} = \int_0^{2\pi}\int_0^{\pi} Er^2 \sin\theta d\theta d\phi = Er^2\int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\phi$$

$$= Er^2$$

$$\int_0^{\pi}\sin\theta d\theta = \left(-\cos\theta\Big|_0^{\pi}\right) = (-\cos(\pi) + \cos(0)) = (1+1) = 2.$$

$$2Er^2\int_0^{2\pi} d\phi = 2(2\pi)Er^2 = 4\pi Er^2$$

We have $4\pi Er^2 = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow E = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}$. Now $Q_{enc} = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho(r) r^2 \sin\theta dr d\theta d\phi$.

$$= 4\pi\int_0^{r}\rho(r')r'^2 dr'$$

$$= 4\pi\int_0^{r} kr'^3 dr' = 4\pi k\int_0^{r} r'^3 dr' = \frac{4\pi k}{4}\left(r'^4\Big|_0^{r}\right)$$

$$= \pi k r^4$$

Therefore: $E_{in} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} = \frac{\pi k r^4}{4\pi\varepsilon_0 r^2} = \frac{kr^2}{4\varepsilon_0}$ for $(r_{in} < r_{out})$

and $E_{out} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} = \frac{\pi k r_{out}^4}{\pi k r^2}$ for $(r_{in} > r_{out})$.

$$V = -\int_{\infty}^{r} E\cdot dl = -\int_{\infty}^{R}\left(\frac{kR^4}{4\varepsilon_0 r^2}\right)dr - \int_R^{r}\left(\frac{kr^2}{4\varepsilon_0}\right)dr = \frac{-k}{4\varepsilon_0}\left(R^4\left(\frac{r^{-2+1}}{-2+1}\right)\Big|_{\infty}^{R} + \frac{r^3}{3}\Big|_R^{r}\right)$$

$$= \frac{-k}{4\varepsilon_0}\left(R^4\left(\frac{-1}{r}\right)\Big|_{\infty}^{R} + \frac{r^3}{3}\Big|_R^{r}\right)$$

$$= \frac{-k}{4\varepsilon_0}\left(-R^3 + \frac{r^3}{3} - \frac{R^3}{3}\right) = \frac{2k}{3k_0}\left(R^3 - \frac{r^3}{4}\right)$$

$$W = \frac{1}{2}\int \rho V d\tau = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} kr\left(\frac{2k}{3\varepsilon_0}\left(R^3 - \frac{r^3}{4}\right)\right)r^2\sin\theta dr d\theta d\phi = \frac{2k^2}{3\varepsilon_0}(4\pi)\left[\int_0^{R} r^3R^3 dr - \frac{1}{4}\int_0^{R} r^6 dr\right]$$

$$= \frac{8\pi k^2}{3\varepsilon_0}\left(\frac{R^4}{4}R^3 - \frac{1}{4}\frac{R^7}{7}\right) = \frac{8\pi k^2}{3\varepsilon_0}\left(\frac{R^7}{4} - \frac{1}{4}\frac{R^7}{7}\right)$$

$$= \frac{8\pi k^2}{3\varepsilon_0}\left(\frac{R^7}{4} - \frac{R^7}{28}\right) = \frac{7R^7 - R^7}{28}\left(\frac{8\pi k^2}{3\varepsilon_0}\right) = \frac{\pi k^2 R^7}{7\varepsilon_0}$$

We can also write: $W = \frac{\varepsilon_0}{2}\int E^2 d\tau = \frac{\varepsilon_0}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} E^2 r^2 \sin\theta \, dr \, d\theta \, d\phi$

$$\frac{\varepsilon_0}{2}\left(\frac{k^2}{16\varepsilon_0^2}\right)(4\pi) = \frac{\pi\varepsilon_0}{2}\left(\frac{k^2}{16\varepsilon_0^2}\right) = \frac{\pi\varepsilon_0}{2}\left(\frac{k^2}{4\varepsilon_0^2}\right) = \frac{k^2\pi}{8\varepsilon_0}$$

$$= \frac{\varepsilon_0}{2}\left(\int_0^R \left(\frac{kr^2}{4\varepsilon_0}\right)^2 4\pi r^2 dr + \int_R^\infty \left(\frac{kR^4}{4\varepsilon_0 r^2}\right)^2 4\pi r^2 dr\right).$$

$$= \frac{k^2\pi}{8\varepsilon_0}\left(\int_0^R r^6 dr + \int_R^\infty \frac{R^8}{r^2} dr\right)$$

$$= \frac{k^2\pi}{8\varepsilon_0}\left(\frac{r^7}{7}\Big|_0^R - \frac{R^8}{r}\Big|_R^\infty\right) = \frac{k^2\pi}{8\varepsilon_0}\left(\frac{R^7}{7} - R^7\right) = \frac{\pi k^2 R^7}{7\varepsilon_0}$$

We can see that the answers match! :).

Recall uniqueness theorem: If $\nabla^2\phi(\vec{r}) = -\rho(\vec{r})/\varepsilon_0$ in a specified volume with given boundary conditions on surface, the solution, $\phi(\vec{r})$, is unique (up to an overall constant).

Important implication: If we can guess a solution of $\nabla^2\phi = -\rho/\varepsilon_0$ satisfying BC's then it is the only sol.

Another interesting fact Earnshaw thm.

Another interesting fact (Earnshaw's Theorem):
$\nabla^2\phi = 0 \implies$ No charge can be held in stable equilibrium by the presence of other charges (by purely electrostatic forces)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = 0$$

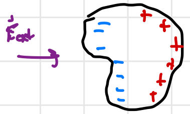
For stable equilibrium, ϕ must have a local minimum in x, y, and z-directions

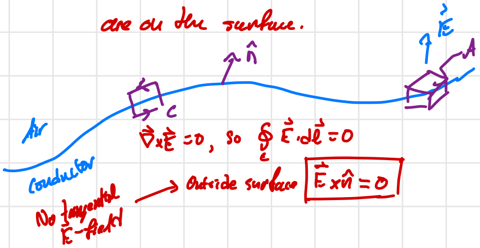
$$\frac{\partial^2\phi}{\partial x^2} > 0, \quad \frac{\partial^2\phi}{\partial y^2} > 0 \quad \text{and} \quad \frac{\partial^2\phi}{\partial z^2} > 0 \quad \text{(contradiction)}$$

Boundary conditions: "Conductor" provide a special class of boundary value problems. In conductor, e^- is free to move so that re-arrange themselves in response to ext $\vec{E}$ field. This arrangement is such that there is no internal $\vec{E}$ field.

i.e. $\phi(\vec{r})$ is const throughout conductor

$\vec{E} = 0 \Rightarrow \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 = 0$ and so charges that screen out external field are on the surface.





normal to surface
$EA = Q/\varepsilon_0 \Rightarrow E = \sigma/\varepsilon_0$
where $\sigma = Q/A$ surface charge density.
$\vec{E}\cdot\hat{n} = \sigma/\varepsilon_0$
So Neumann Boundary condition $-\hat{n}\cdot\vec{\nabla}\phi = \sigma/\varepsilon_0$

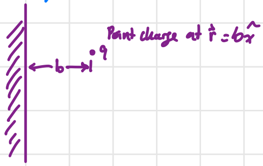
Surface charges are in stable equilibrium: Is Earnshaw's theorem violated?

No surface charges are prevented from leaving surface by non-electrostatic forces!

Guessing the solution for E static potential.

① Method of charges.

Example (1).



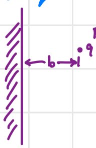
What happens physically inside conductor?

No surface charges are prevented from leaving surface by non-electrostatic forces!

Guessing the solution for the electrostatic potential.

① Method of charges.

Example (i).

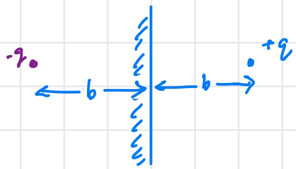


Point charge at $\vec r = b\hat x$

What happens physically inside conductor?

We need to make the plane $x=0$, an equipotential surface.

Try placing 'fictitious image charge' at $\vec r = -b\hat x$.



For $x>0$: $4\pi\epsilon_0 \varphi(\vec r) = \frac{q}{|\vec r - b\hat x|} - \frac{q}{|\vec r + b\hat x|}$

at $x=0$: $|\vec r \pm b\hat x|^2 = b^2+y^2+z^2$

$\therefore \varphi = 0$ along the plane $x=0$

Also $\varphi=0$ for $x>0$.

Surface charge induced: $\sigma = -\epsilon_0 \hat x \cdot \vec\nabla\varphi = -\epsilon_0 \frac{\partial \varphi}{\partial x}$

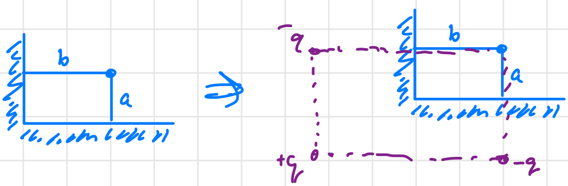
$$\frac{-\partial}{\partial x}\frac{1}{|\vec r - b\hat x|} = \frac{-\partial}{\partial x}\frac{1}{\sqrt{(x-b)^2+y^2+z^2}} = \left(\frac{1}{2}\right)\frac{2(x-b)}{[(x-b)^2+y^2+z^2]^{3/2}}$$

$$\therefore \sigma = \frac{q}{4\pi}\frac{(x-b)-(x+b)}{(b^2+y^2+z^2)^{3/2}} = \frac{-qb}{2\pi(b^2+y^2+z^2)^{3/2}}$$

Method of images: "A sol in search of problem" limited applicability.

Other special cases:

Example (ii): Conductor with a 90° bend



(3 fictitious image charges).

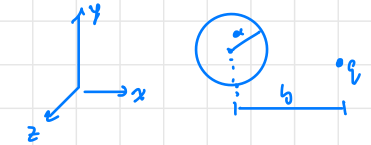
Examples (iii): Point charge between two parallel planes.

$a+b=L$ separation between conductors.

etc . . .

Need infinite number of fictitious charges.

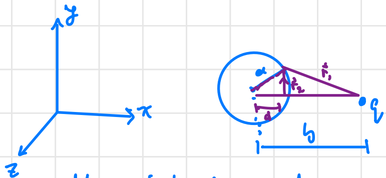
Example (iv): spherical conductor



Real charge q at $\vec{r} = b\hat{x}$. Neutral free standing-conductor.

Any suitable image charges?

Try fictious image charge q' at $\vec{r} = d\hat{x}$ $(x<a)$:



$r_2 = |-d\hat{x} + a\hat{r}|$

$r_1 = |-b\hat{x} + a\hat{r}|$

We want to choose d and q' such that $q/r_1 + q'/r_2 = 0$

If $q>0$, then clearly $q'<0$ $\quad Q \equiv -q'/q$

Then we require that $r_2^2 = d^2 + a^2 - 2da\hat{x}\cdot\hat{r} = Q^2 r_1^2 = Q^2(b^2 + a^2 - 2ba\hat{x}\cdot\hat{r})$. for any choice of $\hat{r}$.

$\therefore da = Q^2 da \Rightarrow Q^2 = d/b$

Also $d^2 + a^2 = Q^2(b^2 + a^2) \Rightarrow d^2 + a^2 = \frac{d}{b}(b^2 + a^2)$

Quadratic equation for the unknown "d."

Two solutions $d = b$ ⟷ this means trivially place image charge on top of real charge.

or $d = a^2/b$

$\Rightarrow q' = -Qq = -\sqrt{\frac{d}{b}}\, q = \sqrt{\frac{a^2}{b^2}}\, q.$

$q' = \frac{-aq}{b}$ Image charge.

(there is a problem here, if I were to put some fictitious charge in sphere then I have some charge residing on sphere.

How can we make conductor charge-neutral as required?

Answer unwanted charge can be cancelled by placing a second fictitious image charge @ center of sphere.

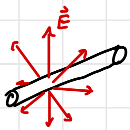
Image Charge methods only work for special situations where equipotential surfaces of a collection of fictitious and real charges conform to the shape of specified Boundary Conditions

2.53. Two inf long wires with uniform charge density $+\lambda$ and $-\lambda$.

a) Find the potential at any point (x,y,z), using your origin as your reference.

Recall that $V = -\int_{r_0}^{r} E(r')\,dr'$, so we can start by using Gauss's law to solve for the $\vec{E}$-field of these wires.

Gauss's law tells us that $\oint E \cdot da = \frac{Q_{enc}}{\varepsilon_0}$



In cylindrical coordinates: $d\vec{A} = s\,d\phi\,dz$

So we have $\Phi_{flux} = \iint E_s (s\,d\phi\,dz) = E_s \int_0^L \int_0^{2\pi} s\,d\phi\,dz = 2\pi s E \hat{s} \int_0^L dz = 2\pi s E L \hat{s}$

Since this is line charge with charge density $\pm\lambda$, $Q_{enc} = \rho_0 \int^L dz = \pm\lambda L$

Putting these together we are left with $\oint E \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow 2\pi s L E \hat{s} = \frac{\pm\lambda L}{\varepsilon_0} \Rightarrow \vec{E} = \frac{\pm\lambda L}{2\pi s L \varepsilon_0}\hat{s} = \frac{\pm\lambda}{2\pi s \varepsilon_0}\hat{s}$

$$V = -\int_{s_0}^{s} \vec{E}\,ds' = -\int_{s_0}^{s} \frac{\pm\lambda}{2\pi s \varepsilon_0}\hat{s}\,ds = \frac{\pm\lambda}{2\pi\varepsilon_0}\int_{s_0}^{s}\left(\frac{1}{s'}\right)\hat{s}\,ds = \frac{\pm\lambda}{2\pi\varepsilon_0}\left(\ln(s) - \ln(s_0)\right) = \frac{\pm\lambda}{2\pi\varepsilon_0}\ln\left(\frac{s}{s_0}\right)$$

For two wires shown in the figure we have the potential for $+\lambda : V_+ = \frac{-\lambda}{2\pi\varepsilon_0}\ln\left(\frac{s_+}{s_0}\right)$

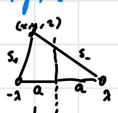
$-\lambda : V_- = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{s}{s_0}\right)$

So we may express potential as $V = V_+ + V_- = \frac{-\lambda}{2\pi\varepsilon_0}\ln\left(\frac{s_+}{s_0}\right) + \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{s_-}{s_0}\right)$

$$= \frac{-\lambda}{2\pi\varepsilon_0}\left(\ln(s_+) - \ln(s_0)\right) + \frac{\lambda}{2\pi\varepsilon_0}\left(\ln(s_-) - \ln(s_0)\right)$$

Now we need to express $s_\pm$ in terms of (x,y,z)

$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\ln(s_-) - \ln(s_+)\right) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{s_-}{s_+}\right)$$



note that I accidentally labelled $a = s_0$

$\Rightarrow s_\pm = \sqrt{(s_0 \pm y)^2 + z^2}$

$$\Rightarrow \boxed{V(x,y,z) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\sqrt{(a+y)^2+z^2}}{\sqrt{(a-y)^2+z^2}}\right) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(a+y)^2+z^2}{(a-y)^2+z^2}\right]}$$

2.53. b) Show that the equipotential surfaces are circular cylinders, and locate the axis and radius of the cylinder corresponding to a given potential V_0.

Recall from part a)

$$V(x,y,z) = \frac{\lambda}{4\pi\varepsilon_0} \ln\left[\frac{(a+y)^2+z^2}{(a-y)^2+z^2}\right]$$

And that the equation of a circle may be expressed as $(x-h)^2+(y-k)^2=r^2$ where the center is (h,k) and radius is r.

Assuming that that we have equipotential, we can write:

$$V_0 = \frac{\lambda}{4\pi\varepsilon_0} \ln\left[\frac{(a+y)^2+z^2}{(a-y)^2+z^2}\right]$$

And we observe that $\frac{4\pi\varepsilon_0 V_0}{\lambda} = \ln\left[\frac{(a+y)^2+z^2}{(a-y)^2+z^2}\right] \Rightarrow e^{\left(\frac{4\pi\varepsilon_0 V_0}{\lambda}\right)} = \frac{(a+y)^2+z^2}{(a-y)^2+z^2}$

$(a-y)(a-y) = a^2 - 2ay + y^2$

Setting this equal to some constant c, we can write:

$$c = \frac{(a+y)^2+z^2}{(a-y)^2+z^2} \Rightarrow c\left((a-y)^2+z^2\right) = (a+y)^2+z^2$$

$$\Rightarrow c(a^2-2ay+y^2+z^2) = a^2+2ay+y^2+z^2$$

$$\Rightarrow ca^2-a^2-2cay-2ay+cy^2-y^2+cz^2-z^2 = 0$$

$$\Rightarrow a^2(c-1)-2ay(c+1)+y^2(c-1)+z^2(c-1)=0$$

$$\Rightarrow a^2-2ay\left(\frac{c+1}{c-1}\right)+y^2+z^2=0$$

which can be written in the form of a circle.

$$(x-h)^2+(y-k)^2=r^2$$

$$\Rightarrow x^2+(y-h)^2-r^2=0$$

$$\Rightarrow x^2+y^2-2hy+h^2-r^2=0$$

combining the two

If we let $h = a\left(\frac{c+1}{c-1}\right) \Rightarrow a^2 = h^2-r^2 \Rightarrow a^2 = a^2\left(\frac{c+1}{c-1}\right)^2 - r^2 \Rightarrow r^2 = a^2\left(\frac{c+1}{c-1}\right)^2 - a^2$

$$r^2 = \frac{a^2(c+1)^2-a^2(c-1)^2}{(c-1)^2} = \frac{\cancel{a^2c^2}+2ac+\cancel{a^2}-\cancel{a^2c^2}+2ac-\cancel{a^2}}{c^2-2c-1} = \frac{4a^2}{(c-1)^2} \Rightarrow r^2 = \frac{4a^2}{(c-1)^2} \Rightarrow r = \frac{2a\sqrt{c}}{\sqrt{(c-1)^2}}$$

This implies that we have cylinder of radius $r = \frac{2a\sqrt{c}}{\sqrt{(c-1)^2}}$ at point $(h,0)$ where $h = a\left(\frac{C+1}{C-1}\right)$.

Problem 2.55. Imagine that new and extraordinarily precise measurements have revealed an error in Coulomb's law. The *actual* force of interaction between two point charges is found to be

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{\mathscr{r}^2}\left(1+\frac{\mathscr{r}}{\lambda}\right)e^{-(\mathscr{r}/\lambda)}\hat{\mathscr{r}},$$

where λ is a new constant of nature (it has dimensions of length, obviously, and is a huge number – say half the radius of the known Universe – so the correction is small, which is why no one ever noticed the discrepancy before). You are charged with the task of reformulating electrostatics to accommodate the new discovery. Assume the principle of superposition still holds.

(a) What is the $\vec{E}$-field charge distribution ρ (replacing eq 2.8)

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'.}$$

From the same logic of the jump from the force to descrete from (2.3 → 2.4) we can write

$$E(r) = \frac{F}{Q} \Rightarrow E(r) = \frac{F}{q_2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}^2}\left(1+\frac{\mathscr{r}}{\lambda}\right)e^{-(\mathscr{r}/\lambda)}\hat{\mathscr{r}}.$$

Using the logic the textbook used to go from 2.4 → 2.5 we have: $\boxed{E(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho}{\mathscr{r}^2}\left(1+\frac{\mathscr{r}}{\lambda}\right)e^{-(\mathscr{r}/\lambda)}\hat{\mathscr{r}}\,d\tau}$

(b) $\nabla\times E = 0$, so this $\vec{E}$-field should admit scalar potential.

c) Find scalar potential of point charge q.

$$V = -\int_{\mathcal{O}}^{r}\vec{E}\cdot d\vec{l} = \frac{-q}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{1}{\mathscr{r}^2}\left(1+\frac{\mathscr{r}}{\lambda}\right)e^{(-\mathscr{r}/\lambda)}\,d\mathscr{r} = \frac{q}{4\pi\varepsilon_0}\left(\int_r^{\infty}\frac{1}{\mathscr{r}}e^{-\mathscr{r}/\lambda}\,d\mathscr{r} + \frac{1}{\lambda}\int_r^{\infty}\frac{1}{\mathscr{r}^2}e^{-\mathscr{r}/\lambda}\,d\mathscr{r}\right)$$

int by parts $= \frac{q}{4\pi\varepsilon_0}\left(\left.\frac{e^{-\mathscr{r}/\lambda}}{\mathscr{r}}\right|_r^{\infty}\right) = \frac{q}{4\pi\varepsilon_0}\frac{e^{-r/\lambda}}{r}$

$$\boxed{V = \frac{q}{4\pi\varepsilon_0}\frac{e^{-r/\lambda}}{r}}$$

(d) For a point charge q at the origin, show that

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} + \frac{1}{\lambda^2} \int_{\mathcal{V}} V \, d\tau = \frac{1}{\epsilon_0} q,$$

where $\mathcal{S}$ is the surface and $\mathcal{V}$ the volume of any sphere centered at q.

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{q}{4\pi\varepsilon_0} \frac{1}{u^2} \left(1 + \frac{u}{\lambda}\right) e^{(-u/\lambda)} = \int_0^{2\pi} \int_0^{u} r \, dr \, d\theta = \frac{q}{\varepsilon_0} \left(1 + \frac{u}{\lambda}\right) e^{(-u/\lambda)}$$

and

$$\frac{1}{\lambda^2} \int_V V d\tau = \int_0^{2\pi} \int_0^{u} \cdots\cdots = \int_0^{u} \frac{rq}{\varepsilon_0} e^{-r/\lambda} dr = \frac{q}{\varepsilon_0} \int_0^{u} r e^{-r/\lambda} dr \overset{\text{sympy}}{=} \frac{q}{\varepsilon_0} \left(\frac{e^{-r/\lambda}}{(1/\lambda^2)} \left(-1 - \frac{r}{\lambda}\right) \right)\Bigg|_0^{u}$$

$$\frac{q}{\varepsilon_0} \left(\frac{e^{-u/\lambda}}{(1/\lambda)^2} \left(-1 - \frac{u}{\lambda}\right) - \frac{1}{(1/\lambda^2)} \left(-1 - \frac{0}{\lambda}\right) \right) = \frac{q\lambda^2}{\varepsilon_0} \left(-e^{-u} \left(1 + \frac{u}{\lambda}\right) + 1 \right)$$

So now, we can observe that:

$$\oint_S \vec{E} \cdot d\vec{a} + \frac{1}{\lambda^2} \int_V V d\tau = \frac{q}{\varepsilon_0} \left(1 + \frac{u}{\lambda}\right) e^{(-u/\lambda)} + \frac{1}{\lambda^2} \left(\frac{q\lambda^2}{\varepsilon_0} \left(-e^{-u} \left(1 + \frac{u}{\lambda}\right) + 1 \right) \right)$$

$$= \frac{q}{\varepsilon_0} \cancel{\left(1 + \frac{u}{\lambda}\right) e^{(-u/\lambda)}} + \frac{q}{\varepsilon_0} \left(\cancel{-e^{-u/\lambda} \left(1 + \frac{u}{\lambda}\right)} + 1 \right)$$

$$= \frac{q}{\varepsilon_0}$$ which is what we wanted to show.

(e) Show that this result generalizes:

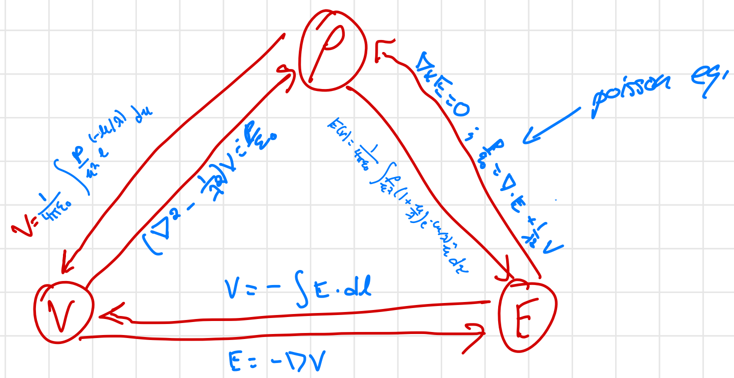
$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} + \frac{1}{\lambda^2} \int_{\mathcal{V}} V \, d\tau = \frac{1}{\epsilon_0} Q_{\text{enc}},$$

for *any* charge distribution. (This is the next best thing to Gauss's law, in the new "electrostatics.")

Now we can say that we have many point charges q_i, and the consitutent charges comprise the $\vec{E}$-field. Invoking the superposition principle, we can write the following.

$$\oint_S \vec{E} \cdot d\vec{a} + \frac{1}{\lambda^2} \int_V V d\tau = \frac{q}{\varepsilon_0} \Rightarrow \oint_S \vec{E}_{TOT} \cdot d\vec{a} + \frac{1}{\lambda^2} \int_V V_{TOT} d\tau = \sum_i^n \frac{q_i}{\varepsilon_0} = \frac{Q_{enc}}{\varepsilon_0}.$$

(f) Draw the triangle diagram (like Fig. 2.35) for this world, putting in all the appropriate formulas. (Think of Poisson's equation as the formula for ρ in terms of V, and Gauss's law (differential form) as an equation for ρ in terms of $\mathbf{E}$.)

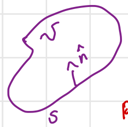


(g) Show that *some* of the charge on a conductor distributes itself (uniformly!) over the volume, with the remainder on the surface. [*Hint:* $\mathbf{E}$ is still zero, inside a conductor.]

This follows directly from General Uniqueness thm, which from lecture was:

General Uniqueness theorem:

If $\varphi(\vec{r})$ satisfies $\nabla^2\varphi = -\rho/\varepsilon_0$ in a volume $\mathcal{V}$ with given boundary conditions, then it is the only solution.



By boundary condition, we mean either φ or $\hat{n}\cdot\vec{\nabla}\varphi$ is specified on surface

Proof: Suppose $\nabla^2\varphi_1 = -\rho/\varepsilon_0$ and $\nabla^2\varphi_2 = -\rho/\varepsilon_0$.
Then $\nabla^2\varphi = 0$ where $\varphi = \varphi_1 - \varphi_2$
and on the surface S, either $\varphi = 0$ or $\hat{n}\cdot\vec{\nabla}\varphi = 0$.
Consider the vector function: $\vec{v} = \varphi\vec{\nabla}\varphi$
$\vec{\nabla}\cdot\vec{v} = \underbrace{\varphi\nabla^2\varphi}_{0} + (\vec{\nabla}\varphi)^2$

$$\int d^3\vec{r}\,(\vec{\nabla}\cdot\vec{v}) = \underbrace{\int dS\,\hat{n}\cdot\vec{v}}_{\text{by divergence thm}} = \int dS\,\varphi(\hat{n}\cdot\vec{\nabla}\varphi) = 0$$

by our assumed boundary condition.

$$\parallel$$

$\int d^3\vec{r}\,(\vec{\nabla}\varphi)^2$ ← is positive definite $\Rightarrow \vec{\nabla}\varphi = 0$ throughout V.

$\therefore \varphi =$ constant throughout $\mathcal{V}$.

For "Dirichlet B.C." $\varphi = 0 \Rightarrow \varphi = 0$ throughout $\mathcal{V}$.
For "Neumann B.C." $\hat{n}\cdot\vec{\nabla}\varphi = 0$, $\varphi =$ const and so $\vec{\nabla}\varphi = 0$ throughout $\mathcal{V}$.
In either case, the E field $\vec{E} = -\vec{\nabla}\varphi$ is unique.

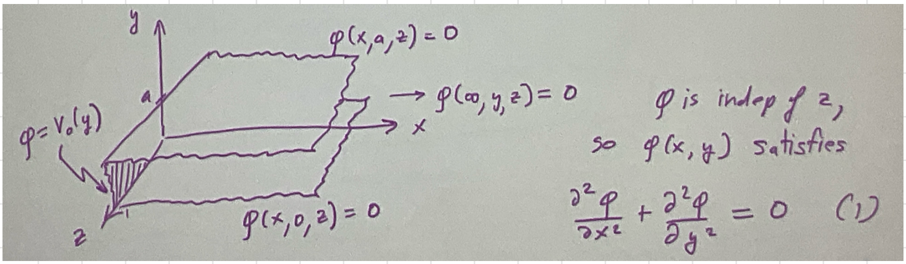
Solution of Electrostatic Problems by Method of seperation of vars.

- applicable given boundary conditions to conform to specific coordinate systems.

1. Cartesian coordinates
2. Cylindrical or Polar coordinates.
3. Spherical coordinates.

Similar to solution of schrodingers equation using orthogonal eigenfunctions.

Example 1: $\nabla^2 \varphi = 0$ with boundary conditions $\varphi = 0$ when $y=0$ and $y=a$. $\varphi = V_0(y)$ when $x=0$ and $\varphi = 0$ as $x \to \infty$



look for particular solutions of #1 and construct superposition to fullfill boundary conditions.

look for solutions of form: $\varphi(x,y) = X(x)Y(y) \Rightarrow X''(x)Y(y) + X(x)Y''(y) = 0$

Divide by XY: $\underbrace{\frac{X''(x)}{X}}_{\substack{f(x)\\ \text{indep of } y}} + \underbrace{\frac{Y''(y)}{Y}}_{\substack{g(y)\\ \text{indep of } x}} = 0$ for all x and y.

Suppose $f(x_1) \neq f(x_2)$, then $f(x_1) + g(y) \neq f(x_2) + g(y)$

$\therefore f(x) = C_1$, a constant

Similarly $g(y) = C_2$, another constant $\quad C_1 + C_2 = 0$

Let $C_1 = k^2$ and $C_2 = -k^2$, $X'' = k^2 X$ and $Y'' = -k^2 Y$

In general $X(x) = e^{-kx}$ to satisfy boundary conditions as $x \to \infty$, and $Y(y) = A\sin(ky+\delta)$

$\varphi(x,y) = Ae^{-kx}\sin(ky+\delta)$. Boundary conditions $\varphi(x,0) = \varphi(x,a) = 0$, require that $\delta = 0$ and $ka = n\pi$, $n = 1,2,3$

(like particle in box in QM).

General solution: $\varphi(x,y) = \sum_n A_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right)$

$\downarrow$

Orthogonality. $\underbrace{\int_b^a dy\, \Psi_n(y)\Psi_m(y)}_{\text{scalar product } (\Psi_n, \Psi_m)} = \frac{a}{2}\delta_{n,m}$ Kronecker Delta function.

Boundary condition $\varphi(0,y) = V_0(y)$ determines $\{A_n\}$

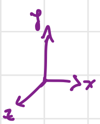
$$\varphi(0,y) = V_0(y) = \sum_n A_n \Psi_n(y)$$

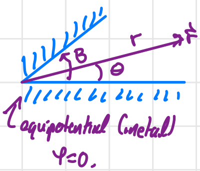
$$(\Psi_m, V_0) = \sum_n A_n(\Psi_m, \Psi_n) = \frac{a}{2}\sum_n A_n \delta_{nm} = \frac{a}{2}A_m$$

$$\therefore \boxed{A_m = \frac{2}{a}\int_0^a dy\, \sin\frac{m\pi y}{a}\, V_0(y)}$$

Since this satisfies $\nabla^2\varphi = 0$ and all boundary conditions, it is the unique solution.

Example 2: Conducting 2-D corner edge





Laplacian in cylindrical coordinates.

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = 0$$

Look for solutions of the form $\varphi(r,\theta) = R(r)\Psi(\theta)$, multiply by $\frac{r^2}{R\Psi}$:

$$\underbrace{\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)}_{\text{indep of } \theta} = \underbrace{-\frac{1}{\Psi}\frac{d^2\Psi}{d\theta^2}}_{\text{indep of } r} = \nu^2 \text{ (some constant)}$$

$\therefore$ independent of r and θ.

Try solutions of the form $R(r) = r^\nu$

$$\Psi(\theta) = A\sin(\nu\theta + \alpha)$$

Apply B.C. $\varphi(r,0) = 0 \Rightarrow \alpha = 0$

$\varphi(r,\beta) = 0 \Rightarrow \nu\beta = n\pi$ $\quad \nu = n\pi/\beta$

$n = 1, 2, 3 \ldots$

General form that satisfies BC's is

$$\varphi(r,\theta) = \sum_{n=1}^{\infty} a_n r^{n\pi/\beta}\sin(n\pi\theta/\beta)$$

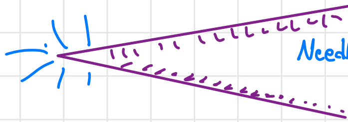
final choice $\{a_n\}$ is dictated by B.C's for large r $(0 < \theta < \beta)$

Consider behavior near tip $(r \to 0)$

$\varphi \sim a_1 r^{\pi/\beta}\sin(\pi\theta/\beta)$ (first term is dominant)

Electric field $\vec{E} = -\vec{\nabla}\varphi$ where $\vec{\nabla} \equiv \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta}$

$$= -a_1 r^{(\pi/\beta - 1)}\left[\hat{r}\,\frac{\pi}{\beta}\sin\left(\frac{\pi\theta}{\beta}\right) + \frac{\pi}{\beta}\,\hat{\theta}\cos(\pi\theta/\beta)\right]$$



Needle-like tip.

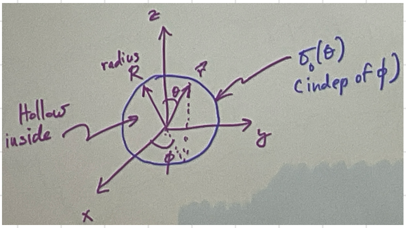
Note* that if $\beta > \pi$, then $\vec{E}$ becomes very large near tip!

Physical interpretation:

Charges on the surface of conductor try to spread out as much as possible to reduce $\vec{E}$ static energy. Tip is far away from most of the remaining surface even a small amount of charge on the tip gives a large amount of density.

$\Rightarrow$ large $\vec{E}$ $\rightarrow$ ionization of air surrounding tip.

Example 3: Sphere with fixed charge density $\sigma(\theta) = a\cos\theta$ "glued" on surface.



What is the potential φ everywhere?

$\nabla^2\varphi = 0$ away from surface.

In spherical coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\phi^2}$$

B.C. indep of ϕ ∴ we can choose φ indep of ϕ

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) = 0 \qquad (*)$$

Try Separable Solutions of the form $\varphi(r,\theta) = R(r)\psi(\theta)$

Divide $(*)$ by $\varphi(r,\theta)$:

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\text{indep of }\Theta} = \underbrace{\frac{-1}{\psi\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi}{d\theta}\right)}_{\text{indep of } r} = \text{const} \rightarrow l(l+1)$$

Radial equation: $\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R$

Try $R(r) = r^v$: $v\frac{d}{dr}\left(r^{v+1}\right) = v(v+1)r^v = l(l+1)r^v$

Quadratic equation: $v^2 + v - l(l+1) = 0$

$(v-l)(v+l+1) = 0$

$\therefore R(r) = Ar^l + \frac{B}{r^{l+1}}$ for arbitrary consts A, B.

Angular equation: $\frac{d}{d\theta}\left(\sin\theta \frac{d\varphi}{d\theta}\right) = -\ell(\ell+1)\sin\theta\,\varphi$

let $x \equiv \cos\theta$ $\qquad \frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx}$

$\therefore \; -\sin\theta \frac{d}{dx}\left(\sin\theta(-\sin\theta)\frac{d\varphi}{dx}\right) = -\ell(\ell+1)\sin\theta\,\varphi$

$\frac{d}{dx}\left[(1-x^2)\frac{d\varphi}{dx}\right] = -\ell(\ell+1)\varphi$ Legendre diff eq.

Non-singular solutions $|x| \le 1$ possible for $\ell = 0, 1, 2, \ldots$ are known as Legendre polynomials.

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = (3x^2-1)/2$

$P_3(x) = (5x^3-3x)/2$ etc....

ℓ^{th} order polynomials in x normalized such that $P_\ell(1) = 1$

These are orthogonal functions on the interval $-1 \le x \le 1$

$$\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = \frac{2\delta_{\ell,\ell'}}{2\ell+1}$$

Later will generalize

$\nabla^2\varphi = 0$

For problems where $\varphi(r,\theta\,\phi)$ is independent of ϕ, general solution of Laplace eq'n:

$$\varphi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

Apply boundary conditions:

① $\varphi(R_+,\theta) = \varphi(R_-,\theta)$ where $R_{+,-} \equiv R \pm \varepsilon$
(otherwise $\vec{E} = -\vec{\nabla}\varphi$ will diverge).

② Gauss's law $\left.\frac{\partial\varphi}{\partial r}\right|_{r=R_+} - \left.\frac{\partial\varphi}{\partial r}\right|_{R_-} = \frac{-1}{\varepsilon_0}\sigma_0(\theta)$

For $r<R$ $\quad \varphi(r,\theta) = \sum_\ell A_\ell r^\ell P_\ell \cos\theta$

For $r>R$ $\quad \varphi(r,\theta) = \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell \cos\theta$

① $\Rightarrow$ $A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}}$ since P_ℓ are orthogonal functions

② $\Rightarrow$ $\sum_\ell\left[\underbrace{\frac{-(\ell+1)B_\ell}{R^{\ell+2}}}_{-(\ell+1)A_\ell R^{\ell-1}} - \ell A_\ell R^{\ell-1}\right]P_\ell(\cos\theta) = -\frac{1}{\epsilon_0}\sigma_0(\theta)$
$= -\frac{k}{\epsilon_0}\cos\theta$

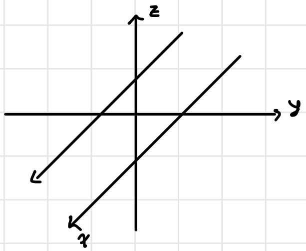
But $P_1(\cos\theta) = \cos\theta$

$\therefore \; A_\ell = 0$ for $\ell \ne 1$

and $3A_1 = k/\epsilon_0$

Solution: $\varphi(r,\theta) = \begin{cases} \frac{kr}{3\epsilon_0}\cos\theta & \text{for } r \le R \\ \frac{kR^3}{3\epsilon_0 r^2}\cos\theta & \text{for } r \ge R \end{cases}$

3.14: Refering to question 2.53. (Which we did in HW).



Gauss's law: $\oint E \cdot da = \frac{q_{enc}}{\varepsilon_0}$

$E\, 2\pi r L = \frac{\lambda L}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r}$

$V(B) - \cancel{V(A)}^{0} = -\int_A^B E \cdot dl$

$\Rightarrow V(B) = -\int_a^B \frac{\lambda}{2\pi\varepsilon_0 r} dl \Rightarrow V_+(s^+) = -\int_a^{s+} \frac{\lambda}{2\pi\varepsilon_0 r} dl = \frac{-\lambda}{2\pi\varepsilon_0} \ln\left(\frac{s^+}{a}\right)$

$V_-(s^-) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{s^-}{a}\right)$

$V_+(s^+) + V_-(s^-) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{s_-}{s_+}\right) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\sqrt{\frac{(y+a)^2+z^2}{(y-a)^2+z^2}}\right)$

b) Equipotential line V_0

$V_0 = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right) \Rightarrow e^{\frac{4\pi\varepsilon_0 V_0}{\lambda}} = \frac{(y+a)^2+z^2}{(y-a)^2+z^2}$

$\Rightarrow 0 = y^2 + a^2 + z^2 - \frac{k+1}{k-1} 2ay$

$(y-y_0)^2 + z^2 = R^2$

$y^2 - 2yy_0 + y_0^2 + z^2 - R^2 = 0$

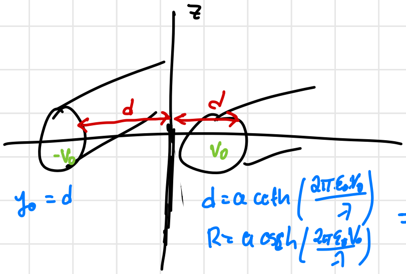
$a^2 = y_0^2 - R^2$

$2yy_0 = \frac{k+1}{k-1} 2ay \Rightarrow y_0 = \frac{k+1}{k-1} a$

$\Rightarrow R = \frac{2a\sqrt{k}}{|k-1|}$

So $k = e^{4\pi\varepsilon_0 V_0/\lambda} \Rightarrow \frac{e^{4\pi\varepsilon_0 V_0/\lambda}+1}{e^{4\pi\varepsilon_0 V_0/\lambda}-1} a = a\coth(2\pi\varepsilon_0 V_0/\lambda) \Rightarrow R = 2a \frac{e^{2\pi\varepsilon_0 V_0/\lambda}}{e^{4\pi\varepsilon_0 V_0/\lambda}-1} = \frac{a}{\sinh(2\pi\varepsilon_0 V_0/\lambda)} = a\,\mathrm{csch}\left(\frac{2\pi\varepsilon_0 V_0}{\lambda}\right).$

In problem 3.14. Instead of two line charges, you have two conducting cylinders of same potential $-V_0$ and V_0. So we may use the equations from problem 2.53.



$y_0 = d$

$d = a\coth\left(\frac{2\pi\varepsilon_0 V_0}{\lambda}\right)$

$R = a\,\mathrm{csch}\left(\frac{2\pi\varepsilon_0 V_0}{\lambda}\right)$

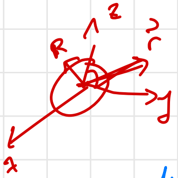
$\Rightarrow \frac{d}{R} = \cosh\left(\frac{2\pi\varepsilon_0 V_0}{\lambda}\right) \Rightarrow \lambda = \frac{1}{2\pi\varepsilon_0 V_0} a\cosh\left(\frac{d}{R}\right)$

$V = \frac{1}{4\pi\varepsilon_0 V} a\cosh\left(\frac{d}{R}\right) \ln\left(\frac{\sqrt{(y-a)^2+z^2}}{\sqrt{(y+a)^2+z^2}}\right).$

Dipole moments and multipole expansion.

Useful for describing electrostatic potential and electric fields far away from sources.

Recall:

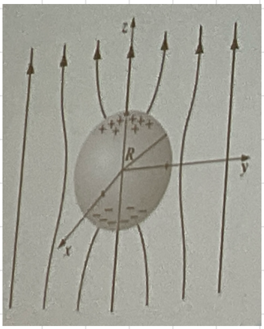


Surface charge density $\sigma_0(R,\theta) = a\cos\theta$ (glued to surface).

Resulting potential

$$\varphi(r,\theta) = \begin{cases} \frac{a}{3\epsilon_0} r\cos\theta & (r \le R) \\ \frac{aR^3}{3\epsilon_0}\frac{\cos\theta}{r^2} & (r > R) \end{cases}$$

Now consider an uncharged metal sphere of radius R in uniform external electric field $\vec{E}_{ext} = E_0\hat{z} \Rightarrow \varphi_{ext} = -E_0 z = -E_0 r\cos\theta$.



$$\varphi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Applying new boundary conditions.

① Sphere is equipotential; chose $\varphi(R,\theta) = 0$

② $\varphi \to -E_0 r\cos\theta$ for $r \gg R$.

① $\Rightarrow A_l R^l + \frac{B_l}{R^{l+1}} = 0$. $B_l = -A_l R^{2l+1}$

$\therefore \varphi(r,\theta) = \sum_{l=0}^{\infty} A_l\left(r^l - \frac{R^{2l+1}}{r^{l+1}}\right)P_l(\cos\theta)$ (for $r \ge R$)

② $\Rightarrow A_l = 0$ for $l \neq 1$.

For $r \gg R$ $\varphi \sim A_1 r\cos\theta \Rightarrow A_1 = -E_0$

$$\therefore \varphi(r,\theta) = \begin{cases} 0 & \text{for } r \le R \\ -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta & \text{for } r > R \end{cases}$$

due to e static potential

due to normal surface charges.

Induced surface charge.

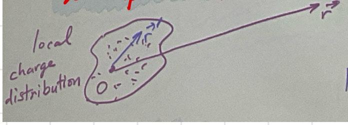
$$\frac{1}{\epsilon_0}\sigma(R,\theta) = -\left.\frac{\partial\varphi}{\partial r}\right|_{r=R_+} \quad \text{(Gauss)}$$

$$= E_0\left(1 + \frac{2R^3}{R^3}\right)\cos\theta = 3E_0\cos\theta$$

Suppose we only want to know the distribution of charges far away from sources (known charge distribution).

$\rightarrow$ A fancy word for Taylor exp.

Multipole expansion:



$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{d^3r' \rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$|\vec{r}| >> |\vec{r}'|$ with boundary condition $\varphi(\infty) = 0$

Expand $\frac{1}{|\vec{r}-\vec{r}'|}$ in a Taylor series $\vec{r}' = (x_1', x_2', x_3')$

Einstein convention (sum over repeated indices.)

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} - (\vec{r}'\cdot\nabla)\frac{1}{r} + \frac{1}{2!}(\vec{r}'\cdot\nabla)^2\frac{1}{r} + \ldots$$

$$= \frac{1}{r} - x_u' \frac{\partial}{\partial x_u}\frac{1}{r} + \frac{1}{2}x_j' x_u' \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_u}\frac{1}{r} + \ldots$$

$$\frac{\partial}{\partial x_u}\frac{1}{r} = \frac{\partial}{\partial x_u}\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}} = -\frac{1}{2}(x_1^2+x_2^2+x_3^2)^{-3/2}(2x_u) = \frac{-x_u}{r}\frac{1}{r^2}$$

$$\frac{\partial}{\partial x_j}\frac{1}{r^3} = \frac{-3}{2}(x_1^2+x_2^2+x_3^2)^{-5/2}(2x_j) = \frac{-3x_j}{r^5}$$

$$\therefore \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_u}\frac{1}{r} = \frac{-\delta_{ji}}{r^3} + \frac{3x_u x_j}{r^5} = \frac{1}{r^3}\left(\frac{3x_j x_u}{r^2} - \delta_{ju}\right)$$

$$\sum_{ij} x_i' x_j' \left(\frac{3x_i x_j}{r^2} - \delta_{ij}\right) = \frac{3x_i' x_j' x_i x_j}{r^2} - \underbrace{x_i' x_i'}_{(r')^2}$$

$$= \left(\frac{r'}{r}\right)^2\left[\frac{3x_i' x_j' x_i x_j}{(r')^2} - r^2\right] = \sum_{ij}\frac{x_i x_j}{r^2}\left[3x_i' x_j' - \delta_{ij}(r')^2\right]$$

So it takes on:

$$\boxed{4\pi\epsilon_0 \varphi(\vec{r}) = \frac{q}{r} + \frac{\hat{r}\cdot\vec{p}}{r^2} + \frac{1}{r^3}\sum_{ij}\frac{x_i x_j}{r^2}Q_{ij} + \ldots}$$

Where $q = \int d^3\vec{r}' \rho(\vec{r}')$ Total charge

$\vec{p} = \int d^3\vec{r}' \rho(\vec{r}')\vec{r}'$ Dipole moment

$Q_{ij} = \int d^3\vec{r}' \rho(\vec{r}')\frac{1}{2}(3x_i' x_j' - \delta_{ij}(r')^2)$ Quadrupole moment

In previous examples we found potentials corresponding to Dipole moments

Recall surface charge $\sigma_0(R,\theta) = k\cos\theta$, we found

$$\varphi(r,\theta) = \frac{kR^3}{3\epsilon_0}\frac{\cos\theta}{r^2} = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2} \quad \text{where } \vec{p} = \left(\frac{4}{3}\pi R^3\right)k\hat{z} \quad (r>R)$$

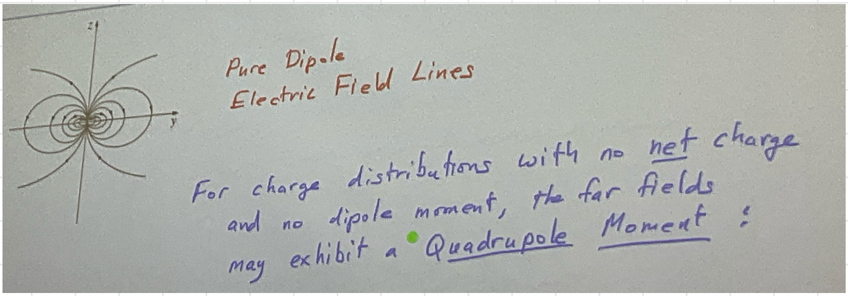
External E field app to uncharged conducting sphere. we found "induced potential":

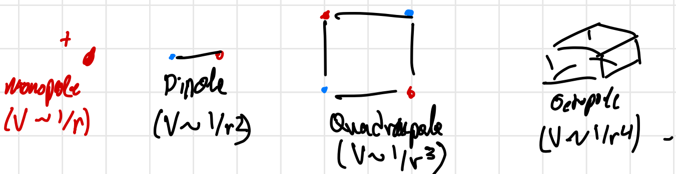
$$\varphi(r,\theta) = \frac{E_0 R^3}{r^2}\cos\theta = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^2} \quad \text{where } \vec{p} = (4\pi R^3) E_0 \varepsilon_0 \hat{z}.$$

Consider $\vec{E}$ field of a pure dipole: $\vec{E} = -\vec{\nabla}\varphi$

Consider
$$\frac{\partial}{\partial x_i}\left(\frac{\vec{r}\cdot\vec{p}}{r^3}\right) = \frac{\partial}{\partial x_i}\left(\frac{x_j p_j}{r^3}\right) = \frac{\partial x_j}{\partial x_i}\frac{p_j}{r^3} + x_j p_j \frac{\partial}{\partial x_i}\left(\frac{1}{r^3}\right)$$
$$= \frac{\delta_{ij} p_j}{r^3} - \frac{3}{2} x_j p_j \frac{2x_i}{r^5} = \frac{\vec{p}}{r^3} - \frac{3(\vec{p}\cdot\vec{r})\vec{r}}{r^5}$$
$$= \frac{\vec{p}}{r^3} - \frac{3(\vec{p}\cdot\hat{r})\hat{r}}{r^3}$$

$$\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0 r^3}\left(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right)$$





Quadrupole moment

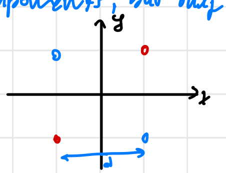
is a "2nd rank tensor" $Q_{ij} = \int d^3\vec{r}'\rho(\vec{r}')\frac{1}{2}\left(3x_i'x_j' - \delta_{ij}(r')^2\right)$

Properties of $Q_{ij} = Q_{ji}$ 1. symmetric.

2. $\mathrm{Tr}(Q) = \sum_{i=1}^{3} Q_{ii} = \frac{1}{2}\int d^3\vec{r}'\rho(\vec{r}')\left[3(\vec{r}')^2 - 3(r')^2\right] = 0.$

There are 9 components, but only 5 are independent. $Q_{11}, Q_{22}, Q_{12}, Q_{13}, Q_{23}$

Example 4 point charges



Charge locations $|x| = |y| = d/2$

$|z| = 0$.

$Q_{13} = Q_{23} = 0$ since $x_3' \equiv z = 0$ for all charges.

$Q_{33} = -\frac{1}{2}\int d^3\vec{r}'(r')^2\rho(\vec{r}') = ?$

Since $\rho(r')$ alternates in sign but $(r')^2$ is same for all charges.

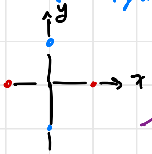
Similarly, $Q_{11} = Q_{22} = 0$ since $(x')^2$, $(y')^2$, and $(r')^2$ are the same for all charges

Only nonzero components are $Q_{12} = Q_{21}$

$$Q_{12} = \frac{3}{2}\int d^3r' \underbrace{x'y'}_{d^2/4}\rho(\vec{r}') = \frac{3}{2}\frac{d^2}{4}q\left[1 - 1(-1) + (-1)^2 - (-1)(1)\right] = \frac{3}{2}d^2 q$$

$$\therefore\ Q_{ij} = \frac{3}{2}qd^2\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Consider same configuration in rotated coordinated system:



Now for all charges $|z|=0$ and either $|x|$ or $|y|=0$

$\Rightarrow$ in $3x_i'x_j'$ terms all off-diagonal terms give zero.

Also $(r')^2 = \frac{d^2}{2}$ for all charges, so $\delta_{ij}(r')^2$ gives zero due to charge alternation

$$Q_{11} = \frac{3}{2}\int d^3\vec{r}'(x')^2\rho(\vec{r}') = \frac{3}{2}q\frac{d^2}{2}\left((1)^2 + (-1)^2\right) = 3q\,d^2/2$$

For Q_{22}, $q \to -q$ $\qquad Q_{ij} = \frac{3qd^2}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Relation to multipole expansion to Legendre's Polynomials

Rewrite: $\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r|\hat{r} - \frac{r'}{r}\hat{r}'|}$ $\qquad$ let $t \equiv r'/r$

$x \equiv \hat{r}\cdot\hat{r}' = \cos\theta$.

$= \frac{1}{r\sqrt{1 - 2xt + t^2}}$ Taylor expansion for $t<1$. $\qquad |x| \le 1$ and consider $r > r'$ $(t<1)$.

let $a \equiv t(t - 2x)$

$$(1 - 2xt + t^2)^{-1/2} = 1 + \left(-\tfrac{1}{2}\right)a + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}a^2 + \dots$$

$$= 1 + xt - \frac{1}{2}t^2 + \frac{3}{2}x^2t^2 + O(t^3) = \sum_{\ell=0}^{\infty} t^\ell P_\ell(x)$$

where $P_0(x) = 1$

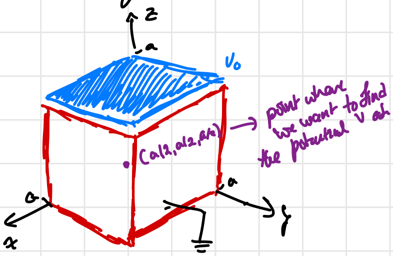
$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$ etc.

$\frac{1}{\sqrt{1 - 2xt + t^2}}$ is known as a generator for the Legendre polynomial.

Problem 3.18

A cubical box (side length a) consists of five metal plates, which are welded together and grounded. The top is made of a seperate sheet of metal, insulated from the others, and held at a constant potential V_0. Find the potential inside the box. [What should the potential at the center $(a/2, a/2, a/2)$ be? Check that your formula is consistent with this value.



Let us start by defining the boundary conditions.

$V=0$, when at the following postions:

On x-axis $x=0$ and $x=a$.

y-axis $y=0$ and $y=a$

z-axis $z=0$.

$V=V_0$ when at the following positions:

z-axis : $z=a$.

So our problem here is to solve laplace's equation such that $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$. subject to said boundary conditions we wrote above (in red).

Therefore, we'll solve for $V(x,y,z) = X(x)Y(y)Z(z)$

We can observe that

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = \frac{\partial^2(XYZ)}{\partial x^2}+\frac{\partial^2(XYZ)}{\partial y^2}+\frac{\partial^2(XYZ)}{\partial z^2}$$

$$= YZ\frac{\partial^2 X}{\partial x^2}+XZ\frac{\partial^2 Y}{\partial y^2}+XY\frac{\partial^2 Z}{\partial z^2}$$

$$= \frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}$$

Now we can take $C_1=\frac{1}{X}\frac{\partial^2 X}{\partial x^2}$, $C_2=\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$, $C_3=\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}$, leading to $C_1+C_2+C_3=0$

By the examples 3.3 and 3.5 indicates that we should set the region V_0 positive, with remaining regions negative. So we may set C_1 & C_2 negative with $C_1=-k^2$ and $C_2=-l^2$, which leaves us with $C_3=k^2+l^2$

This leaves us with the following set of differential equations:

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1 \Rightarrow \frac{d^2X}{dx^2} = C_1 X = -k^2 X, \quad \frac{d^2Y}{dy^2} = C_2 Y = -l^2 Y, \quad \frac{d^2Z}{dz^2} = C_3 Z = (k^2+l^2) Z$$

So the solutions are:

$$X(x) = A\sin kx + B\cos kx$$
$$Y(y) = C\sin ly + D\cos ly$$
$$Z(z) = Ee^{\sqrt{k^2+l^2}z} + Fe^{z\sqrt{k^2+l^2}}$$

The boundary conditions on x-axis implies that $k = \frac{n\pi}{a}$

But this would imply that $B=0$, otherwise boundary conds not met.

The bc on y-axis implies that similar to x axis $l = \frac{m\pi}{a} \Rightarrow D=0$

The bc on z-axis (where $z=0$ for $V=0$) we have $E+F=0 \Rightarrow E=-F$.

So it follows that

$$Z(z) = E\left(e^{z\sqrt{k^2+l^2}} - e^{-z\sqrt{k^2+l^2}}\right) = E\left(e^{z\sqrt{\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{a^2}}} - e^{-z\sqrt{\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{a^2}}}\right) = Ee^{-\pi z\sqrt{\frac{n^2\pi}{a^2}+\frac{m^2\pi}{a^2}}}$$

Now recall that $V(x,y,z)$ was reexpressed as $V(x,y,z) = X(x)Y(y)Z(z)$

$$\Rightarrow = ACE\sin\left(\frac{n\pi}{a}\right)\sin\left(\frac{m\pi}{a}\right)e^{-\pi z\sqrt{\frac{n^2\pi}{a^2}+\frac{m^2\pi}{a^2}}}$$

We are then left with

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\sin\left(\frac{n\pi}{a}\right)\sin\left(\frac{m\pi}{a}\right)e^{-\frac{\pi z}{a}\sqrt{n^2+m^2}}$$

And we want boundary condition:

$$V(x,y,a) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\sin\left(\frac{n\pi}{a}\right)\sin\left(\frac{m\pi}{a}\right)e^{-\frac{\pi z}{a}\sqrt{n^2+m^2}} = V_0$$

with the appropriate choice of constants $C_{n,m}$. Let us multiply by some $\sin\left(\frac{n'\pi}{a}\right)\sin\left(\frac{m'\pi}{a}\right)$, where n' and m' are arbitrary integers.

So we have: the following:

$$C_{n,m}e^{\frac{\pi z}{a}\sqrt{n^2+m^2}}\int_0^a \sin\left(\frac{n\pi}{a}\right)\sin\left(\frac{n'\pi}{a}\right)dx\int_0^a \sin\left(\frac{m\pi}{a}\right)\sin\left(\frac{m'\pi}{a}\right)dy = V_0\int_0^a\int_0^a \sin\left(\frac{n'\pi}{a}\right)\sin\left(\frac{m'\pi}{a}\right)dx\,dy$$

$$\Rightarrow C_{n,m}e^{-2\pi\sqrt{n^2+m^2}}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) = V_0\int_0^a\int_0^a \sin\left(\frac{n'\pi}{a}\right)\sin\left(\frac{m'\pi}{a}\right)dx\,dy.$$

$$\rightarrow C_{n,m}e^{-2\pi\sqrt{n^2+m^2}} = V_0\frac{4}{a^2}\int_0^a\int_0^a \sin\left(\frac{n'\pi}{a}\right)\sin\left(\frac{m'\pi}{a}\right)dx\,dy.$$

by example 3.5 from textbook, we'd have: $C_{n,m}e^{-2\pi\sqrt{n^2+m^2}} = \begin{cases} 0 & \text{if } n \text{ or } m \text{ is even} \\ \frac{16V_0}{\pi^2 nm} & \text{if both } n \text{ and } m \text{ are odd.} \end{cases}$

We know that if n and m must be odd for the potential $\neq 0$.
Therefore let us construct a set of natural numbers $\mathbb{N}^{odd}$, such that $(2n'+1) \in \mathbb{N}^{odd}$ for all $n' \in \mathbb{N}$. So we are finally left with:

$$e^{-\frac{z\pi}{a}\sqrt{n^2+m^2}} \quad e^{\pi\sqrt{n^2+m^2}}$$

$$= e^{\left(-\frac{z\pi}{a}+z\pi\right)\sqrt{n^2+m^2}}$$

$$V(x,y,z) = \frac{16V_0}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{nm} \sin\left(\frac{n\pi}{a}\right) \sin\left(\frac{m\pi}{a}\right) e^{\left(-\frac{z\pi}{a}+\pi\right)\sqrt{n^2+m^2}}$$

Now the question asks us to check for the potential inside the box at point $(a/2, a/2, a/2)$.

$$V(a/2, a/2, a/2) = \frac{16V_0}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{nm} \sin\left(\frac{n\pi}{a}\right) \sin\left(\frac{m\pi}{a}\right) e^{\left(\pi/2\right)\sqrt{n^2+m^2}}$$

Considering that only one side of the cube has potential V_0, by a geometric argument we'd expect that at point $(a/2, a/2, a/2)$, we'd have potential $V_0/6$.

Understand how method of images work for sphere's and planes.
(final exam PHY350).

Problem 3.21.

The potential at the surface of a sphere (radius R) is given by

$$V_0 = k\cos 3\theta$$

where k is constant. Find the potential inside and outside the sphere, as well as the surface charge density $\sigma(\theta)$ on the sphere. (Assume there is no charge outside the sphere.)

Using the hint provided, we may write:

Note that $P_1(x) = x$, $P_3(x) = (5x^3 - 3x)/2$

$$V_0(\theta) = k\cos\theta = k[4\cos^3\theta - 3\cos\theta]$$

$$\Rightarrow V_0(\theta) = k[4\cos^3\theta - 3\cos\theta] = k[C_3P_3(\cos\theta) + C_1P_1(\cos\theta)]$$

Solving for ten coefficients c_1 and c_3 we can observe that

$$c_3\left(\frac{5\cos^3\theta - 3\cos\theta}{2}\right) + c_1\cos\theta = \underbrace{\frac{5c_3}{2}\cos^3\theta} + \underbrace{\left(c_1 - \frac{3}{2}c_3\right)}\cos\theta$$

$$4 = \frac{5c_3}{2} \qquad c_1 - \frac{3}{2}\left(\frac{8}{5}\right) = -3$$

$$\Rightarrow \frac{8}{5} = c_3 \qquad \Rightarrow c_1 = \frac{3}{2}\left(\frac{8}{5}\right) - 3 = \frac{24}{10} - 3 = \frac{24-30}{10} = \frac{-6}{10} = \frac{-3}{5}$$

Substitute values c_3 and c_1 we can observe that:

$$V_0(\theta) = k\left[\frac{8}{5}P_3(\cos\theta) - \frac{3}{5}P_1(\cos\theta)\right]$$

From equation 3.65, we can express a general solution to laplaces equations for our legendre polynomials:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l\cos\theta \xrightarrow{\text{(for our bc's).}} V(r,\theta) = \begin{cases} \sum_{l=0}^{\infty}(A_l r^l)P_l(\cos\theta), & \text{for } r \le R \text{ (in)} \\ \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta), & \text{for } r > R \text{ (out)} \end{cases}$$

With these expressions, we can deduce the following:

$$V_{in} = A_1 r P_1(\cos\theta) + A_3 r^3 P_3(\cos\theta)$$

$$V_{out} = \frac{B_1}{r^2}P_1(\cos\theta) + \frac{B_3}{r^4}P_3(\cos\theta)$$

From example 3.6, they performed the following to find their respective A_l's:

$$A_l = \frac{2l+1}{2R^l}\int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta\, d\theta.$$

this looks like a funny k-delta function. Sum. :)

for $A_l = \frac{(2l+1)}{2R^l}\left[\frac{5}{k}\left(8\int_0^{\pi}P_3(\cos\theta)P_l(\cos\theta)\sin\theta\, d\theta - 3\int_0^{\pi}P_1(\cos\theta)P_l(\cos\theta)\sin\theta\, d\theta\right)\right]$

$$= \frac{(2l+1)}{2R^l}\left[\frac{k}{5}\left(8\frac{2}{2l+1}\delta_{l3} - 3\frac{2}{(2l+1)}\delta_{l1}\right)\right] = \frac{k}{5R^l}\left(8\delta_{l3} - 3\delta_{l1}\right)$$

$\Rightarrow$ when $l=1$ we have $A_1 = \frac{-3k}{5R}$ and when $l=3$ $A_3 = \frac{8k}{5R^3}$.

$V_{in} = A_1 r P_1(\cos\theta) + A_3 r^3 P_3(\cos\theta)$

$A_1 = \frac{-3k}{5R}$ $\qquad A_3 = \frac{8k}{5R^3}$

$$\Rightarrow \boxed{V_{in}(r,\theta) = \frac{-3}{5R} r P_1(\cos\theta) + \frac{8k}{5R^3} r^3 P_3(\cos\theta) = \frac{k}{5}\left(8\left(\frac{r}{R}\right)^3 P_3(\cos\theta) - 3\left(\frac{r}{R}\right) P_1(\cos\theta)\right) = \frac{k}{5}\frac{r}{R}\cos\theta\left(4\left(\frac{r}{R}\right)^2 [5\cos^2\theta - 3] - 3\right)}$$

Now let us find the B_ℓ's. Notice that example 3.8 already tells us that $B_\ell = -A_\ell R^{2\ell+1}$ $\rightarrow$ with our bc's this would be $B_\ell = A_\ell R^{2\ell+1}$

So we are left with $B_1 = A_1 R^3 = \frac{-3k}{5R} R^3 = \frac{-3}{5} k R^2$ and $B_3 = \frac{8k}{5R^3} R^7 = \frac{8k}{5} R^4$

We may then write:

$$V_{out}(r,\theta) = \frac{B_1}{r^2} P_1(\cos\theta) + \frac{B_3}{r^4} P_3(\cos\theta) = \frac{-3kR^2}{5r^2} P_1(\cos\theta) + \frac{8kR^4}{5r^4} P_3(\cos\theta) = \frac{k}{5}\left[8\left(\frac{R}{r}\right)^4 P_3(\cos\theta) - 3\left(\frac{R}{r}\right)^2 P_1(\cos\theta)\right].$$

$$\boxed{V_{out}(r,\theta) = \frac{k}{5}\left(\frac{R}{r}\right)^2 \cos\theta\left(4\left(\frac{R}{r}\right)^2 [5\cos^2\theta - 3] - 3\right).}$$

Now we can compute the surface charge density

Where example 3.9 equation 3.83 tells us: $\qquad$ $=R$).

$$\sum_{\ell=0}^{\infty} (2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos\theta) = \frac{1}{\varepsilon_0}\sigma(\theta).$$

$$\Rightarrow \sigma(\theta) = \varepsilon_0 (3A_1 P_1 + 7A_3 R^2 P_3) \leftarrow \text{plugging in}$$

$$= \frac{\varepsilon_0 k}{5R}\left[-9P_1(\cos\theta) + 56 P_3(\cos\theta)\right]$$

$$= \frac{\varepsilon_0 k}{5R}\left(-9\cos\theta + \frac{1}{2} 56 (5\cos^3\theta - 3\cos\theta)\right)$$

$$\boxed{\sigma(\theta) = \frac{\varepsilon_0 k}{5R}\cos\theta(140\cos^2\theta - 93)}$$

Linear Dielectric response.

Recall in semi-classical model of an atom we found induced dipole $\vec{p}_{dipole} = \alpha \vec{E}$, where $\vec{E}$ is applied field and $\alpha = a_0^3$ is "polarizability", a_0 = radius of cloud.

In most materials for "weak" applied fields,

$$\vec{P} = \epsilon_0 \underbrace{\chi_e}_{} \vec{E} \leftarrow \text{tot } \vec{E} \text{ field.}$$

dimensionless susceptibility.

applied field plus induced field from surrounding matter.

Displacement field

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 (1+\chi_e)\vec{E} = \epsilon \vec{E}$$

where permittivity $\epsilon \equiv \epsilon_0 (1+\chi_e)$.

where permittivity

Dielectric Constant (Relative Permittivity)

$$\epsilon_r \equiv \epsilon/\epsilon_0 = 1+\chi_e > 1$$

(For time-varying fields, this is generalized to freq-dep dielectric const)

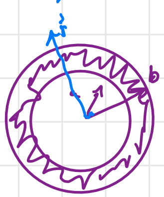
Recall in special situations,
$\vec{D}$ can be determined entirely from ρ_f (free charge distribution)

provided $0 = \vec{\nabla}\times\vec{D} = \vec{\nabla}\times(\epsilon\vec{E})$ Now let $\epsilon = \epsilon(\vec{r})$

$$(\vec{\nabla}\times\vec{D})_i = \epsilon_{ijk}\partial_j(\epsilon E_k) = \epsilon_{ijk}[(\partial_j\epsilon)E_k + \underbrace{\epsilon(\partial_j E_k)}_{\text{neglect since } \vec{\nabla}\times\vec{E}=0.}]$$
$$= (\vec{\nabla}\epsilon\times\vec{E})_i$$

Spatial variations in $\epsilon(\vec{r})$ must be $\parallel \vec{E}$.

Example 1: Conducting sphere with charge Q, surrounded by dielectric shell.



What is the potential of conductor?

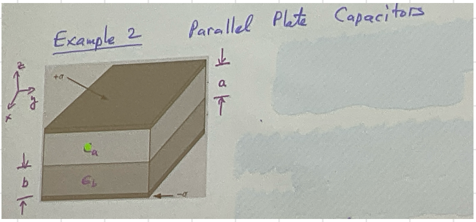
Is $\vec{\nabla}\epsilon \parallel \vec{E}$? (otherwise need σ_b).

for $r > a$ $\vec{D} = \frac{Q\hat{r}}{4\pi r^2} \Rightarrow \vec{E} = \vec{D}/\epsilon = \begin{cases} \frac{Q\hat{r}}{4\pi\epsilon r^2} & \text{for } a<r\le b \\ \frac{Q\hat{r}}{4\pi\epsilon_0 r^2} & \text{for } r>b. \end{cases}$

$$\varphi(a) = -\int_\infty^a \vec{E}\cdot d\vec{l} \quad \text{when } d\vec{l} = dr\hat{r} + \underbrace{r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}}_{\perp \vec{E}}$$

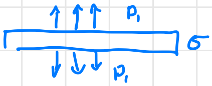
$$\frac{1}{\epsilon_0}\int_a^\infty \frac{dr}{r^2} + \frac{1}{\epsilon}\int_b^a \frac{dr}{r^2} = \frac{1}{\epsilon_0}\left[-\frac{1}{r}\right]_a^b + \frac{1}{\epsilon}\left[-\frac{1}{r}\right]_b^a$$

$$\Rightarrow \varphi(a) = \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} + \frac{1}{\epsilon b}\right).$$



Dielectric constants (relative permittivity), $\varepsilon_a, \varepsilon_b$
thickness a, b.
$\vec{E}$-field is vertical and $\vec{\nabla}\varepsilon$ is $\parallel$ except near edge of parallel capacitor plates.
Approx: neglect edge effects.

Just below top plate



$\nabla \cdot D = \rho_f$.

Gauss's law: $2D_1A = \sigma A$

$$D_1 = \sigma/2$$

Same contribution from lower plate: $D_a = \sigma/2$

$$\vec{D} = \begin{cases} -\sigma\hat{z} & \text{inside cap} \\ 0 & \text{outside} \end{cases}$$

$\vec{E}$ field $\quad \vec{E} = \vec{D}/\epsilon \qquad \epsilon = \epsilon_0 \underbrace{\epsilon_{a/b}}_{\text{dielectric constants.}}$

$$\vec{E} = \frac{-\sigma\hat{z}}{\varepsilon_0}\begin{cases} \frac{1}{\varepsilon_a} & \text{in slab } a \\ \frac{1}{\varepsilon_b} & \text{in slab } b, \end{cases}$$

Polarization: $\vec{P} = \epsilon_0 \chi_e \vec{E}$ where $\chi_e = \varepsilon_{a/b} - 1$

$$= -\sigma\hat{z}\begin{cases} 1 - \frac{1}{\varepsilon_a} & (\text{slab } a) \\ 1 - \frac{1}{\varepsilon_b} & (\text{slab } b). \end{cases}$$

Potential difference:

$$\Delta\varphi \equiv \varphi_+ - \varphi_- = -\int_-^+ \vec{E}\cdot d\vec{l} = \frac{\sigma}{\varepsilon_0}\left(\frac{a}{\varepsilon_a} + \frac{b}{\varepsilon_b}\right)$$

Capacitance: $\quad C = \frac{Q}{\Delta\varphi} = \sigma A \frac{\varepsilon_0}{\sigma}\left(\frac{a}{\varepsilon_a} + \frac{b}{\varepsilon_b}\right)^{-1} = A\varepsilon_0\left(\frac{a}{\varepsilon_a} + \frac{b}{\varepsilon_b}\right)^{-1}$

Note: if $b=0$, this is a capacitor with only ε_a and plates seperated by a.

$$C_a = \frac{A\varepsilon_0\varepsilon_a}{a}$$

Similarly, $C_b = \frac{A\varepsilon_0\varepsilon_b}{b}$

$$\therefore \boxed{\frac{1}{C} = \frac{1}{C_a} + \frac{1}{C_b}}$$ Addition rule for capacitors in series.

Bound charges: $\sigma_{bound} = \vec{p} \cdot \hat{n}$

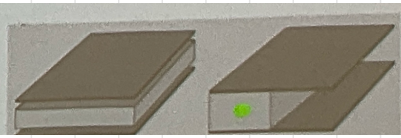
Near top plate: $\vec{p} \cdot \hat{n} = -\sigma\left(1 - \frac{1}{\epsilon_a}\right)$
$\hat{n} = \hat{z}$

Near bottom plate: $\vec{p} \cdot \hat{n} = +\sigma\left(1 - \frac{1}{\epsilon_b}\right)$
$\hat{n} = -\hat{z}$

Interface between ϵ_a and ϵ_b: $\hat{n} = -\hat{z}$ from ϵ_a
$\hat{n} = \hat{z}$ from ϵ_b

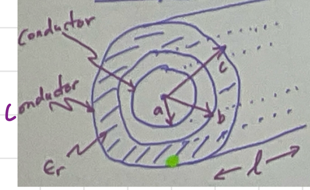
$$\sigma_{bound} = +\sigma\left(1 - \frac{1}{\epsilon_a}\right) - \sigma\left(1 - \frac{1}{\epsilon_b}\right)$$

Exercise: How would you treat the following?



Example 3 Coaxial Cable (Cylindrical Symmetry)

Conductors at $r = a$ and $r = c$



What is the capacitance per unit length.

$C = Q/\Delta\varphi$

Place charge $\begin{cases} +Q \\ -Q \end{cases}$ over same length of $\begin{cases} \text{inner conductor} \\ \text{outer conductor} \end{cases}$

Here both $\vec{\nabla}\epsilon$ and $\vec{E}$ are purely radial so $\nabla \times \vec{D} = 0$

Let r be the distance from the central axis

conductors at $r = a$ and $r = c$.

Dielectrics $b < r < c$

Displacement Field line charge density: $\lambda = Q/l$

Gauss's law $D\, 2\pi r l = Q \Rightarrow D = \frac{\lambda}{2\pi r}$

Electric field $\vec{E} = \vec{D}/\epsilon = \vec{D}/(\epsilon_0 \epsilon_r) = \frac{\lambda}{2\pi\epsilon_0 r} \begin{cases} 1 & a < r < b \\ 1/\epsilon_r & b < r < c \end{cases}$

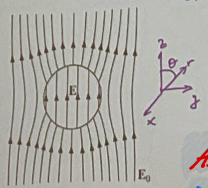
Pot. ential diff

$$\Delta\varphi = -\int_c^a \vec{E} \cdot d\vec{l} = \int_a^c \vec{E} \cdot d\vec{l} = \frac{\lambda}{2\pi\epsilon_0}\left[\int_a^b \frac{dr}{r} + \int_b^c \frac{dr}{\epsilon_r r}\right]$$

$$= \frac{\lambda}{2\pi\epsilon_0}\left[\ln(b/a) + \frac{1}{\epsilon_r}\ln c/b\right]$$

$C = \frac{\lambda l}{\Delta\varphi}$, $\frac{C}{l} = 2\pi\epsilon_0\left[\ln b/a + \frac{1}{\epsilon_r}\ln c/b\right]^{-1}$ Capacitance per unit length.

Example 4 Uniform Uncharged Dielectric Sphere of radius R in External Applied Field $\vec{E}_0$



sphere dielectric constant ϵ_r

$\rho_f = 0 \Rightarrow \vec{\nabla}\cdot\vec{D} = 0$ (Divergence of displacement field).

But $\vec{\nabla}\times\vec{D} \neq 0$ since $\vec{\nabla}\epsilon$ is $\parallel \hat{r}$ whereas $\vec{E}$ is not $\parallel \hat{r}$.

All charges are bound to sphere surface.

$\Rightarrow \nabla^2\varphi = 0$ except on sphere surface.

Treat as boundary value problem.

1. At ∞ $\varphi = -E_0 r\cos\theta$ so that $-\vec{\nabla}\varphi = E_0\hat{z}$.

2. At $r=R$, φ is continuous.

and $\vec{D}\cdot\hat{r}|_{out} - \vec{D}\cdot\hat{r}|_{in} = \sigma_f = 0$ and $\vec{D} = \epsilon\vec{E}$.

$\epsilon_0 \frac{\partial\varphi}{\partial r}\Big|_{out} - \epsilon\frac{\partial\varphi}{\partial r} = 0$, when $\epsilon = \epsilon_0\epsilon_r$.

We showed previously that for $r<R$

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \text{ for some } \{A_l\}$$

whereas for $r>R$

$$\varphi(r,\theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty}\frac{B_l P_l}{r^{l+1}}(\cos\theta) \text{ for some } \{B_l\}$$

$\cos\theta = P_1(\cos\theta)$ and $P_l(\cos\theta)$ are orthogonal functions, φ is continuous at R (must be true for each l),

$$l=1 \quad A_1 R = -E_0 R + B_1/R^2 \quad (1)$$

$$l\neq 1 \quad A_l R^l = B_l/R^{l+1} \quad (2).$$

$$\frac{\partial\varphi}{\partial r} = \begin{cases} \sum_l l A_l r^{l-1}P_l(\cos\theta) & r<R \\ -E_0\cos\theta + \sum_l \frac{-(l+1)B_l}{r^{l+2}} & r>R \end{cases}$$

$$l=1 \quad \epsilon A_1 = \epsilon_0\left[-E_0 - \frac{2B_1}{R^3}\right] \quad (3).$$

$$l\neq 1 \quad \epsilon A_l = -\frac{\epsilon_0 (l+1)B_l}{R^{l+1}} \quad (4).$$

(2) and (4) $\Rightarrow A_l = B_l = 0$ $(l\neq 1)$ (they cannot simultaneously have same and opposite signs).

(1) and (3) two equations and two unknowns

$$\therefore A_1 = \frac{-3E_0}{\epsilon_r+2} \text{ and } B_1 = \left(\frac{\epsilon_r-1}{\epsilon_r+2}\right)R^3E_0$$

For $r<R$ $\varphi(r,\theta) = \frac{-3E_0}{\epsilon_r+2}\underbrace{rP_1(\cos\theta)}_{z}$.

$$\boxed{\vec{E} = -\vec{\nabla}\varphi = \frac{3E_0}{\epsilon_r+2}}$$

Uniform total field within sphere!

Relationship between Microscopic and Macroscopic descriptions

Recall microscopic response of atomic/molecular dipole

$$\vec{p}_{dipole} = \underbrace{\alpha}_{\text{polarizability}} \vec{E} \leftarrow \text{applied field.}$$

Macroscopic description:

Polarization density $\vec{P} \equiv \langle n_p \vec{p} \rangle$ where n_p = number density of microscopic dipoles $\vec{p}$,
and $\langle \; \rangle = \frac{1}{\Delta V} \int_{\Delta V} d^3\vec{r}$ Spatial average over small volume ΔV (but large compared to microscopic scale).

For 'linear dielectrics', we asserted

$$\vec{P} = \epsilon_0 \underbrace{\chi_e}_{\text{susceptibility}} \vec{E} \leftarrow \text{applied field plus induced field = total field.}$$

What is the relationship between "microscopic" and "macroscopic" χ_e?

For a very dilute 'gas' of dipoles, we should expect.

$\epsilon_0 \chi_e = n_p \alpha$ (when field due to other dipoles in medium is negligible)

<u>Instead</u> consider the response from the dense collection of dipoles (e.g. liquid or solid).

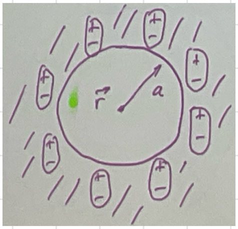
Now total (microscopic) electric field at $\vec{r}$ (within medium).

$$\vec{E}_{tot} = \vec{E}_{applied} + \underbrace{\vec{E}_{local}}_{\text{field due to nearby dipoles (induced by } \vec{E}_{applied}) \sim \frac{1}{r^3}}$$

$$\vec{P} = n_p \alpha (\vec{E}_{appl} + \vec{E}_{local}) \leftarrow \text{depends on } \vec{P} \text{ itself.}$$

Consider a spherical cavity surrounding point $\vec{r}$.

Consider a spherical cavity surrounding point $\vec r$.
(ideal for isotropic medium: liquid or amorphous solid).

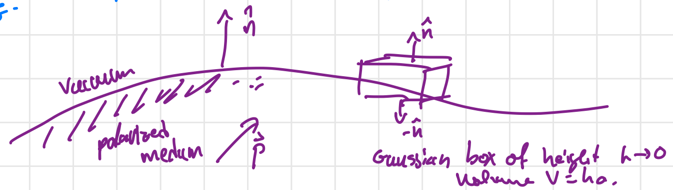


$\vec E_{local}$ arises from surface polarization charge
Recall Polarization charge density.

$$\rho_b = -\vec\nabla\cdot\vec p.$$

Choose "a" to a microscopic scale so that Macroscopic $\vec P$ is uniform around the cavity.

Recall surface charge density.

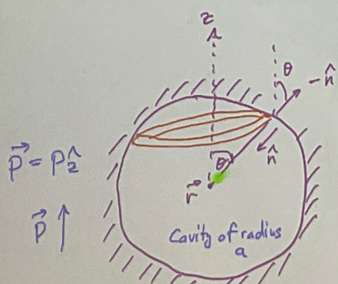


$$A h \rho_b = \int dV \rho_b$$

$$= -\int dV(\vec\nabla\cdot\vec p) = -\oint d\vec s\cdot\vec p = -(-\hat n\cdot\vec p)A$$ (only bottom surface contributes).

$\therefore \boxed{\sigma_b = \hat n\cdot\vec P}$ where $\hat n$ points from the solid to vacuum.

$\vec E_{local}$ at center of cavity is the sum of contributions from rings at angle θ:



Ring area $dA = 2\pi\underbrace{a\sin\theta}_{\text{ring radius.}}\underbrace{(a\,d\theta)}_{\text{ring thickness}}$

All charge in ring is distance "a" from point $\vec r$ of interest.

By symmetry, ring contributes only in z-direction.

z-component of $\vec E$-field due to ring. $dE_z = \dfrac{-dq}{4\pi\epsilon_0 a^2}\overbrace{\cos\theta}^{z\text{-component.}}$ where $dq = \sigma_b\, dA$ and $\sigma_b = -P\cos\theta$

Put it all together: $\vec E_{local} = \vec P\int_0^\pi d\theta \sin\theta\cos^2\theta\,\dfrac{2\pi a^2}{4\pi\epsilon_0 a^2} = \dfrac{\vec P}{2\epsilon_0}\int_{-1}^{1} dx\, x^2$ where $x \equiv \cos\theta$.

$$= \frac{\vec P}{3\epsilon_0}$$

$$\therefore \vec P = n_p\alpha\left[\vec E_{applied} + \frac{\vec P}{3\epsilon_0}\right]$$

Solve for $\vec P$:

$$\vec P = \underbrace{\left[\frac{n_p\alpha}{1-\frac{n_p\alpha}{3\epsilon_0}}\right]}_{\equiv\epsilon_0\chi_e}\vec E_{applied}$$

Dielectric susceptibility (Clausius-Mossotti)

$$\chi_e = \frac{n_p\alpha/\epsilon_0}{1-n_p\alpha/(3\epsilon_0)}$$

Dielectric constant: $\epsilon_r = 1 + \chi_e$

From a macroscopic measurement of ϵ_r can we infer the microscopic polarizability α?

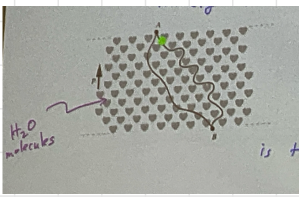
Invert: $\frac{1}{\chi_e} = -\frac{1}{3} + \frac{1}{\chi}$ where $\chi \equiv n_p\alpha/\epsilon$

$$\chi = \frac{1}{\chi_e^{-1} + \frac{1}{3}} = \frac{3\chi_e}{3+\chi_e} = \frac{3(\epsilon_r - 1)}{\epsilon_r + 2}$$

$$\therefore \boxed{\alpha = \frac{3\epsilon_0}{n_p}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)}$$ Lorentz-Lorenz equation.

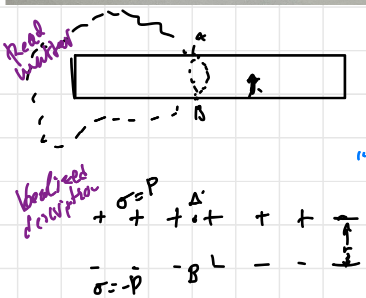
Physical interpretation of Averaged microscopic fields.

In real matter, microscopic fields can be huge and point in random directions.
Near atomic nuclei $\sim 10^{11}$ Volts/meter
In water with polarized H_2O molecules $\sim 10^8$ V/m.
Nevertheless, electrodynamics still holds at large length scales, namely $\vec{\nabla}\times\vec{E} = 0$ for static charges.



$\therefore \int_A^B \vec{E}_m \cdot d\vec{l}$ is independent of path $= \varphi_A - \varphi_B$

Line integral from A to B is the same for outside path where fields are smooth



Our idealized description depicts this as layers of bound surface charge

$$\sigma = \vec{p}\cdot\hat{n} = \pm p \text{ and}$$

"macroscopic electric field" inside $\vec{E} = -\hat{z}\sigma/\epsilon_0$

$$\varphi_A - \varphi_B = \varphi_{A'} - \varphi_{B'} = \frac{\sigma t}{\epsilon_0} = \frac{Pt}{\epsilon_0}$$

Assume slab of thickness "t" has very large area A such that φ_A and φ_B are constant over their respective planes

Spatial Avg of Microscopic Field $\langle \vec{E}_m \rangle = \frac{1}{V}\int d^3r\, \vec{E}_M$

Perform sampling of $\vec{E}_m$ along infinitesimal columns of length t and cross section da.

$$E_z = \frac{1}{V}\sum_i \underbrace{\int_{B_i}^{A_i} \vec{E}_m \cdot d\vec{l}}\, da_i$$ $-\frac{Pt}{\epsilon_0}$ for each i.

$$\therefore E_z = -\frac{1}{V}\frac{P}{\epsilon_0}\underbrace{\sum_i t\, da_i} = -\frac{P}{\epsilon_0}$$

In other words, average of microscopic field $\vec{E}_m$ over volume is equal to $-\vec{P}/\epsilon_0$.

$$\boxed{\langle \vec{E}_m \rangle = -\vec{P}/\epsilon_0}$$

Well defined value even though $\vec{E}_m$ has singularities near point charges.

$\vec{P}$ is the sum over microscopic dipole moments of molecules divided by average volume.

Lecture 15: Magnetic fields and Magnetization in matter

Common types of magnetism:

paramagnetism (alignment with electron spin with $\vec{B}$-field)

Diamagnetism (orbital motion of electrons opposes applied $\vec{B}$)

Ferromagnetism (due to coulomb repulsion between electrons and Pauli exclusion principle).

Antiferromagnetism (due to lowering of quantum mechanical energy of electron hoping between atoms).

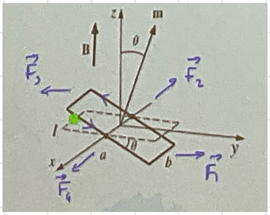
Forces and Torque on Magnetic Dipoles.

General current loop can be broken into smaller loops

internal circuits cancel pairwise

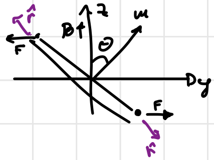
only parameter current survives.

$d\vec{F} = I\,d\vec{l} \times \vec{B}$ (lorentz-force).



$\vec{F}_1 + \vec{F}_3 = 0 \qquad \vec{F}_2 + \vec{F}_4 = 0$ No net force in uniform $\vec{B}$.

Torque $\vec{r} \times \vec{F}_2 = 0 = \vec{r} \times \vec{F}_4$



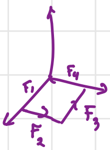
$r \times \vec{F}_1 = r \times \vec{F}_3 = \hat{x}(Ib)B\frac{a}{2}\sin\theta$

Total torque $\vec{\tau} = \hat{x}\underbrace{Iab}_{m} B\sin\theta$ magnetic dipole moment

$$\boxed{\vec{\tau} = \vec{m} \times \vec{B}}$$

What if the mag field was non-uniform?

For a small loop (side length $\epsilon \to 0$): $\vec{F} = \sum_{i=1}^{4} \vec{F}_i$



$$\vec{F}_1 = \hat{x}(I\epsilon) \times \vec{B}(x,0,0)$$
$$\vec{F}_2 = \hat{y}(I\epsilon) \times \vec{B}(x,\epsilon,0)$$
$$\vec{F}_3 = -\hat{x}(I\epsilon) \times \vec{B}(x,\epsilon,0)$$
$$\vec{F}_4 = -\hat{y}(I\epsilon) \times \vec{B}(0,y,0)$$

$$\vec{F}_1 + \vec{F}_3 = I\epsilon\hat{x} \times [\vec{B}(x,0,0) - \vec{B}(x,\epsilon,0)] \simeq -I\epsilon^2 \hat{x} \times \frac{\partial \vec{B}}{\partial y}$$
$$\vec{F}_2 + \vec{F}_4 = I\epsilon\hat{y} \times [\vec{B}(\epsilon,y,0) - \vec{B}(0,y,0)] \simeq \underbrace{I\epsilon^2}_{m} \hat{y} \times \frac{\partial \vec{B}}{\partial x}$$

$$\therefore \vec{F} = m\left[\hat{y} \times \frac{\partial \vec{B}}{\partial x} - \hat{x} \times \frac{\partial \vec{B}}{\partial y}\right]$$

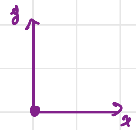
$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 1 & 0 \\ \frac{\partial B_x}{\partial x} & \frac{\partial B_y}{\partial x} & \frac{\partial B_z}{\partial x} \end{vmatrix} - \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & 0 \\ \frac{\partial B_x}{\partial y} & \frac{\partial B_y}{\partial y} & \frac{\partial B_z}{\partial y} \end{vmatrix}$$

$$= \hat{x}\frac{\partial B_z}{\partial x} - \hat{z}\frac{\partial B_x}{\partial x} + \hat{y}\frac{\partial B_z}{\partial y} - \hat{z}\frac{\partial B_y}{\partial y}$$

Combine using $\vec{\nabla} \cdot \vec{B} = 0$ to get $+\hat{z}\frac{\partial B_z}{\partial z}$

$$\therefore \vec{F} \simeq m\vec{\nabla}B_z = \vec{\nabla}(\vec{m} \cdot \vec{B})$$

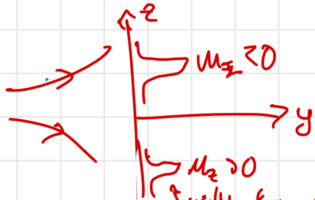
Example: Magnet with field gradient

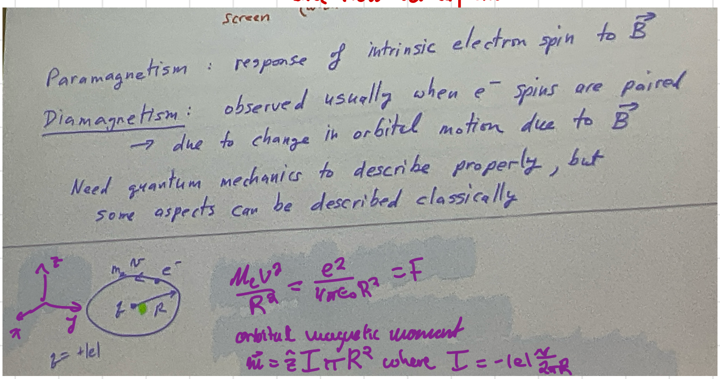


Cross section of magnet

Field gradient in $-\hat{z}$ direction (stern gerlach experiment)

Send beams of electrons in y-direction with various orientations of $\vec{m}$ along center of apparatus





Paramagnetism : response of intrinsic electron spin to $\vec{B}$

Diamagnetism : observed usually when e^- spins are paired
→ due to change in orbital motion due to $\vec{B}$

Need quantum mechanics to describe properly, but
some aspects can be described classically

$$\frac{m_e v^2}{R} = \frac{e^2}{4\pi\epsilon_0 R^2} = F$$

orbital magnetic moment

$\vec{m} = \hat{z} I \pi R^2$ where $I = -|e| \frac{v}{2\pi R}$

$$\vec{m} = -\hat{z}|e| v \frac{R}{2}$$

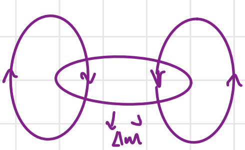
What if we apply a $\vec{B}$ field in z direction? Where $\vec{B} = \Delta B \hat{z}$.

Now $F \to F + \Delta F$ where $\Delta F = |e| v \Delta B$

Change in velocity Δv:

$$2 m_e \frac{v \Delta v}{R} = |e| v \Delta B \qquad \Delta v = \frac{|e| R}{2 m_e} \Delta B$$

$$\Rightarrow \Delta \vec{m} = -\hat{z} |e| \frac{R}{2} \Delta v = -\hat{z} \frac{e^2 R^2}{4 m_e} \Delta B$$



in opposite direction to $\vec{B}$

$\Delta \vec{m}$ creates additional field in $-\hat{z}$ direction to oppose $\vec{B} = \Delta B \hat{z}$.

But magnetic field does no work, where did the energy come from to speed up electron?

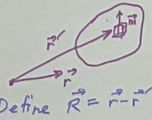
Recall Faraday's law $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\oint_{\text{orbital path}} \vec{E} \cdot d\vec{l} \neq 0$ work done as $\vec{B}$ turned on.

Magnetization width matter.

$\vec{M}(\vec{r}) = \langle n_m \vec{m} \rangle$ $\quad n_m$ = number density of mag dipoles.

$\langle \ \rangle = \frac{1}{\Delta V}\int_{\Delta V} d^3\vec{r}'$ spatial average

Macroscopic magnetic field due to $\vec{M}$.



Define $\vec{R} = \vec{r} - \vec{r}'$

Recall $\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{R}}{R^2}$ $\quad \vec{m} \to \vec{M}\, d^3\vec{r}'$

$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3\vec{r}'\, \frac{\vec{M}\times\hat{R}}{R^2}$

Use identity $\vec{\nabla}'\frac{1}{R} = -\vec{\nabla}\underbrace{\frac{1}{R}}_{4\pi\epsilon_0\varphi} = \underbrace{\frac{\hat{R}}{R^2}}_{4\pi\epsilon_0\vec{E}}$ due to unit charge at $\vec{r}'$.

$\therefore \vec{A} = \frac{\mu_0}{4\pi}\int d^3\vec{r}'\, \vec{M}\times\vec{\nabla}'\frac{1}{R}$

Consider $\vec{\nabla}'\times(\vec{M}/R) = \epsilon_{ijk}\partial_j(M_k/R) = \epsilon_{ijk}\left[\frac{\partial_j M_k}{R} + M_k\,\partial_j\left(\frac{1}{R}\right)\right] = \frac{1}{R}\vec{\nabla}'\times\vec{M} - \underbrace{\vec{\nabla}'\left(\frac{1}{R}\right)\times\vec{M}}_{-\vec{M}\times\vec{\nabla}'\frac{1}{R}}$

$\therefore \vec{A} = \frac{\mu_0}{4\pi}\int d^3\vec{r}'\left[\frac{1}{R}\underbrace{\vec{\nabla}'\times\vec{M}}_{J_b \text{ bound current density.}} - \vec{\nabla}\times'(\vec{M}/R)\right]$

For second term, consider divergence thm: $\int d^3\vec{r}\ \vec{\nabla}\cdot(\vec{v}\times\vec{c}) = \oint_S \underbrace{\vec{v}\times\vec{c}\cdot d\vec{s}}_{ds_i\,\epsilon_{ijk}v_j c_k \,=\, -\epsilon_{kji}c_k v_j ds_i \,=\, -\vec{c}\cdot(\vec{v}\times d\vec{s})}$ for $\vec{c}$ = constant.

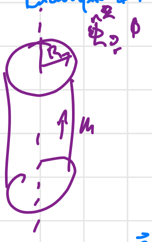
$\partial_i \epsilon_{ijk} v_j c_k = \epsilon_{ijk}(\partial_i v_j)c_k$
$= \epsilon_{kij}c_k(\partial_i v_j)$
$= \vec{c}\cdot\vec{\nabla}\times\vec{v}$

Since this is true for arbitrary $\vec{c}$:

$$\int d^3\vec{r}\,(\vec{\nabla}\times\vec{v}) = -\oint_S \vec{v}\times d\vec{s}$$

$\therefore \vec{A} = \frac{\mu_0}{4\pi}\left[\int d^3\vec{r}\,\frac{\vec{J}_b(\vec{r}')}{R} + \oint_S \frac{1}{R}\underbrace{(\vec{M}\times\hat{n})}_{K_B \text{ surface current density.}} ds\right]$

Example 1: Magnetic field of a long uniformly magnetized cylinder.



for $r<R$, $\vec{J}_b = \vec{\nabla}\times\vec{M} = 0$

$r = R$ $\quad \vec{K}_b = \vec{M}\times\vec{n} = M\hat{z}\times\hat{r} = M\hat{\phi}$.

This is the same as the solenoid.

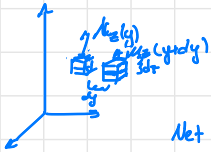
$\vec{B} = 0$ for $r > R$

$\vec{B} = B\hat{z}$ is uniform for $r<R$. $\quad$ Amps law: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$ $\quad BL = \mu_0 |K_b| L = \mu_0 M L$

$\therefore \vec{B} = \mu_0 M\hat{z}$.

In the notes there are more examples that are relevant.

Intuitive Picture of Bound Currents due to Spatially Varying magnetization.



In small volume $dx\,dy\,dz$

magnetic dipole $m_z = M_z\,dx\,dy\,dz = I\,dx\,dy$

Net current in x-direction on surface when adjacent volumes join is

$$[M_z(y+dy) - M_z(y)]\,dz = \frac{\partial M_z}{\partial y}\,dy\,dz$$

contribution to x-component of bound current density $J_{b,x}^{(1)} = \frac{\partial M_z}{\partial y}$

The only contribution to $J_{b,x}$ comes from vertically aligned volumes:

$$m_y = M_y\,dx\,dy\,dz = I\,dx\,dz$$

$$[-M_y(z+dz) + M_y(z)]\,dy = -\frac{\partial M_y}{\partial z}\,dz\,dy$$

$$\therefore J_{b,x}^{(2)} = -\frac{\partial M_y}{\partial y}$$

Total bound charge den $J_{b,x} = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} = (\vec{\nabla}\times\vec{M})_x$

Same argument for each component of $\vec{J}_b = \vec{\nabla}\times\vec{M}$

<u>Bound surface current</u>

dipole moment $m = M a t = I a$

$\therefore k_b = M$ in direction orthogonal to both $\vec{M}$ and $\hat{n}$.

$$\boxed{\vec{k}_b = \vec{M}\times\hat{n}}$$

Total current $\vec{J} = \vec{J}_b + \vec{J}_f$

free current

Total $\vec{J}$ is the source of $\vec{B}$:

$$\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} = \vec{J}_f + \vec{J}_b$$

$$\therefore \vec{\nabla}\times\vec{H} = \vec{J}_f \quad \text{where} \quad \vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M}$$

Amperes law in matter: $\oint \vec{H}\cdot d\vec{l} = I_f$

free current passing through C.

Can we uniquely determine $\vec{H}$ and $\vec{B}$?

It worked for $\vec{B}$, since we knew $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ and $\vec{\nabla}\cdot\vec{B} = 0$.

But now, $\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M} \neq 0$ in general.

In specific (high sym) situations $\vec{\nabla}\cdot\vec{M} = 0$

[Analogous to situations where $\vec{\nabla}\times\vec{D} = \vec{\nabla}\times(\epsilon\vec{E}) = 0$ where we required $\vec{\nabla}\epsilon \parallel \vec{E}$].

Example 1: Infinitely long cylinder with uniform $\vec{\mu}$ of radius R and axial magnetization $\vec{M} = M r \hat{z}$.

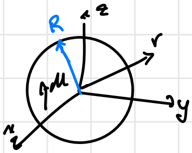
Surface
$\vec{\nabla}\times\vec{M}=0$
$\vec{\nabla}\cdot\vec{M}=0$

Previously, we would first determine $\vec{J}_b = \vec{\nabla}\times\vec{M}$ and $\vec{K}_b = \vec{M}\times\hat{n}$ to find $\vec{B}$.

Here, $\vec{\nabla}\cdot\vec{M}=0$ everywhere, so use $\oint \vec{H}\cdot d\vec{l} = I_{f,enc} = 0$

$$\vec{H} = 0 \Rightarrow \vec{B} = \mu_0\vec{M} = \begin{cases} \mu_0 M r\hat{z} & \text{for } r<R \\ 0 & \text{for } r>R \end{cases}$$

Example 2: Uniformly magnetized sphere.



$$\vec{J}_f = 0 \Rightarrow \vec{\nabla}\times\vec{H} = 0$$

So we could write $\vec{H} = -\vec{\nabla}\varphi_m$ for some fictitious "magnetic potential"

$\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M}$ is zero except on surface of sphere.

This is equivalent to solving poissons equation

$-\nabla^2\varphi_m = \rho_m$ where $\rho_m = -\vec{\nabla}\cdot\vec{M}$ is a fictitious "surface charge".

This has a unique solution provided boundary conditions at $r=0$, $r=R$ and $r\to\infty$ are satisfied

Recall general solution to Laplace's equation $\nabla^2\varphi_m = 0$

$$\varphi_m = \begin{cases} \sum_l A_l r^l P_l(\cos\theta) & r<R \\ \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta) & r>R \end{cases}$$

Boundary conditions at $r=R$

① thin pill box on sphere surface
$\epsilon\to 0$

$$\int_{\Delta V} d^3r\, \vec{\nabla}\cdot\vec{H} = -\int d^3r\, \vec{\nabla}\cdot\vec{M}$$

$d\vec{s} = A\hat{n}$

$$\int_S \vec{H}\cdot d\vec{s} = -\int_S \vec{M}\cdot d\vec{s}$$

$$H^{\perp}_{above} - H^{\perp}_{below} = M^{\perp}_{below} - M^{\perp}_{above} = \vec{M}\cdot\hat{n} = \vec{M}\cdot\hat{r} = M\cos\theta$$

② φ_m is continuous at $r=R$ (otherwise $\vec{H}$ is divergent)

Apply ① and ② to General solution

① $\Rightarrow$ $\sum_l P_l(\cos\theta)\left[\frac{(l+1)B_l}{R^{l+2}} + l A_l R^{l-1}\right] = \underbrace{M\cos\theta}_{P_1(\cos\theta)}$

$\therefore A_l = B_l = 0$ for $l \neq 1$

$$\frac{2B_1}{R^3} + A_1 = M.$$

② $\Rightarrow A_1 R = B_1/R^2$ $\quad\therefore \frac{3B_1}{R^3} = M$, $B_1 = \frac{MR^3}{3}$ and $A = M/3$

For $r<R$, $\varphi_m = \frac{M}{3} r\cos\theta = Mz/3$

$\vec{H} = -\vec{\nabla}\varphi_m = -\frac{M\hat{z}}{3} = -\vec{M}/3$, $\vec{B} = \mu_0(\vec{H}+\vec{M}) = \frac{2}{3}\mu_0\vec{M}$ uniform inside sphere

For $r>R$, $\varphi_m = \frac{B_1}{r^2}\cos\theta = \frac{MR^3}{3}\frac{\cos\theta}{r^2} = \frac{R^3}{3}\frac{\vec{M}\cdot\hat{r}}{r^2}$

Total magnetic moment $\vec{m} = \frac{4}{3}\pi R^3\vec{M}$

$\vec{B} = \underbrace{\mu_0\vec{H}}_{\vec{M}=0\ (r>R)} = \frac{-\mu_0}{4\pi}\vec{\nabla}\left(\frac{\vec{m}\cdot\hat{r}}{r^2}\right)$ same as the field of perfect dipole.

Summary of Magnetic Boundary conditions

As shown before, for any surface current density K.

$$B^{\perp}_{above} - B^{\perp}_{below} = 0 \qquad (\vec{\nabla}\cdot\vec{B} = \vec{0})$$

$$B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K \qquad (\vec{\nabla}\times\vec{B} = \mu_0\vec{J})$$

Generalize $\vec{B}^{\parallel}_{above} - \vec{B}_{below} = \mu_0\vec{K}\times\hat{n}$ (exercise)

$$H^{\perp}_{above} - H^{\perp}_{below} = -(M^{\perp}_{above} - M^{\perp}_{below})$$

$$\vec{H}^{\parallel}_{above} - \vec{H}^{\parallel}_{below} = \mu_0 \vec{K}_f \times \hat{n}$$

Definition of magnetic permeability

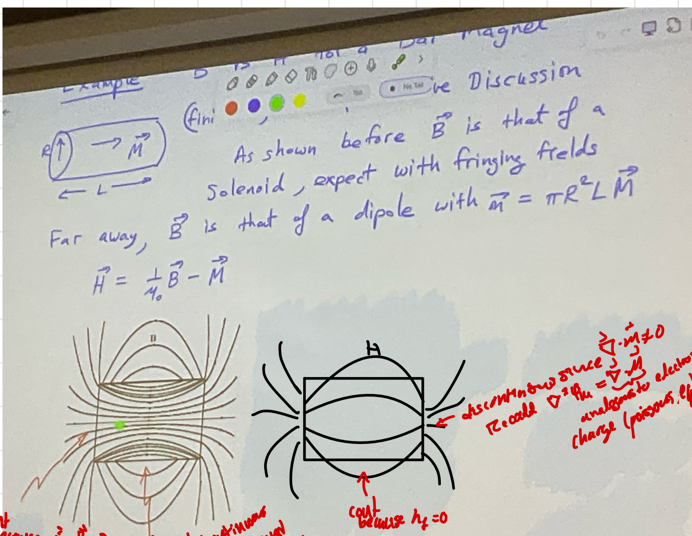
analogous to electric susceptibility $\vec{P} = \epsilon_0 \chi_e \vec{E}$, here we use $\underbrace{\vec{H}}_{\text{related to applied current}}$ instead of $\vec{B}$ ← depends on details of mag field material.

Def'n $\vec{M} = \chi_m \vec{H}$ (when is true, medium is called "linear medium")

Then $\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(1+\chi_m)\vec{H} = \mu\vec{H}$

where $\mu = \mu_0(1+\chi_m)$ is called "permeability"

Note** $\vec{\nabla}\cdot\vec{H} \neq \frac{1}{\mu}\vec{\nabla}\cdot\vec{B}$ since $\mu = \underbrace{\mu(\vec{r})}_{\text{especially at interface between media}}$



Cont because $\vec{\nabla}\cdot\vec{B} = 0$ and no bound surface current $\vec{M}\times\hat{n} = 0$

discontinuous due to bound solenoid current

Energy of Magnetic Dipole in Field $\vec{B}$

Recall force on magnetic dipole $\vec{m}$: $\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$

Energy = Work required to bring dipole in from infinity

$$U = -\int_{\infty}^{\vec{r}} \vec{F}\cdot d\vec{l} = -\int_{\infty}^{\vec{r}} d\vec{l}\cdot\vec{\nabla}(\vec{m}\cdot\vec{B})$$

$$= -\vec{m}\cdot\vec{B}(\vec{r}) + \vec{m}\cdot\underbrace{\vec{B}(\infty)}_{\text{choose to be zero}}$$

$$\therefore \boxed{U = -\vec{m}\cdot\vec{B}}$$

analogous to energy of electric dipole