MAT223: Group Report 2

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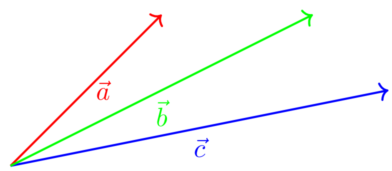
Group report 2 iterates upon the concepts introduced in Group Report 1. Hence it is advised to read Group Report 1 before Group Report 2.

Contents

1 Linear Independence and Dependence of a Set of Vectors

|In Group Report 1, we delved into the theory of vector spaces, which we were interested in defining $\mathbb{R}^3$. As a result, we devised three linearly independent vectors, which span would represent $\mathbb{R}^3$ itself. Therefore in this matter, we are interested in the distinction between linear independence and dependence among a set of vectors. In the theory of vector spaces, one knows a set of vectors is linearly independent if no nontrivial linear combinations of vectors are equal to the zero vector. The negation of linear independence is linear dependence. We can define linear dependence as a set of vectors in which one or more vectors are linear combinations of another.

To familiarize ourselves with linear dependence, we shall view a mental model that illustrates a linear-dependent set of vectors in $\mathbb{R}^2$:



$$I = \left\{\vec{a}, \vec{b}, \vec{c}\right\}$$

|In this case, this set contains a redundant vector which makes this set of vectors linearly dependent. A redundant vector means the entire set can be expressed as linear combinations of other vectors. So, how does one make this set linearly independent? The answer is simple: remove one of the three vectors in this set and receive a set of linearly independent vectors.

Thus,

1.0.1 The Geometric Definition of Linearly Dependent and Independent

We say the vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ are linearly independent if for at least one i,

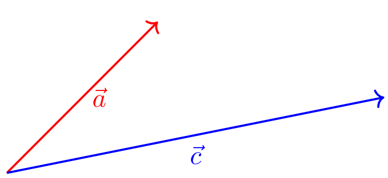
$$\vec{v}_i \in span\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_{i-1}, \vec{v}_{i+1}, ..., \vec{v}_n\}$$

Otherwise, they are called linearly independent.

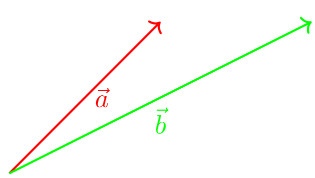
|Here are illustrations of the same vectors, now as linearly independent sets of vectors in $\mathbb{R}^2$:

$$I = \left\{\vec{a}, \vec{b}, \vec{c}\right\}$$

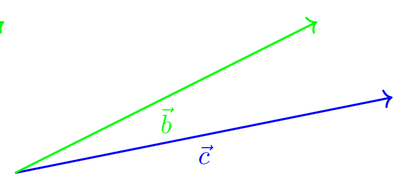
$$A = \{\vec{a}\}, B = \{\vec{b}\}, C = \{\vec{c}\} \implies A \cup B \cup C = I$$



$I \cap (A \cup C)$



$I \cap (A \cup B)$



$I \cap (B \cup C)$

|In a more mathematically rigorous approach, we may determine whether a set of vectors is linearly independent or linearly dependent. Then, we can look at the trivial and non-trivial solutions of

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + ... + \alpha_n\vec{v}_n = \vec{0}\ ,\ \textit{where}\ \alpha_1, \alpha_2, ..., \alpha_n\ \textit{are scalars.}$$

Therefore,

1.0.2 Definition of Trivial Linear Combination

The linear combination $\alpha_1\vec{v}_1 + ... + \alpha_n\vec{v}_n$ is called trivial if $\alpha_1 = ... = \alpha_n = 0$. If at least one $\alpha_i \neq 0$, the linear combination is called non-trivial.

The set is linearly dependent if the linear combination contains a non-zero scalar. Otherwise, if all scalars equal zero, the set of vectors is linearly independent.

1.1 Subspaces in $\mathbb{R}^3$

|A vector space is a collection of vectors that can be combined and multiplied by scalars in a linearly consistent manner. For instance, the set of all 3-dimensional vectors with real-number coordinates is a vector space, where their components define vector addition and scalar multiplication. Now, what is a subspace? First, we may ponder a subspace as a subset of a vector space. This subset of vectors, in turn, has properties of a vector space. In other words, a subspace is a smaller vector space contained in a larger one.

1.1.1 Definition of Subspace

Subspace. A non-empty set $V \subset \mathbb{R}^n$ is called a subspace if for all $\vec{u}, \vec{v} \in V$ and all scalars k we have: (1.) $\vec{u} + \vec{v} \in V$; and (2.) $k\vec{u} \in V$

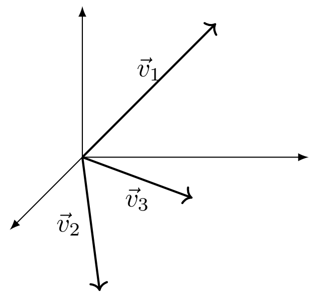
|Subspaces are equivalent to spans, and this duality permits us to determine whether a set of vectors are linearly independent or dependent. Let's say, S = $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a subset of $\mathbb{R}^3$. Then, using the abovementioned methods, we may determine whether the set S is linearly independent or dependent.

1.1.2 Linear Independence of a Subspace

$$S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \implies \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = \vec{0}$$

|To give an example. let's choose arbitrary vectors for our set S, that happens to be linearly independent.

$$S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$



|To uncover if a subspace is linearly independent, we must express the subspace as a matrix. Then employ reduced row echelon form to determine if there is a trivial solution to the linear combination of the set that composes this subspace.

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \right\} \to \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -2 & 2 & 4 \end{bmatrix}$$

|Utilizing the methods in (Group report 1 section 1.2), we find the trivial solution to linear combination of vectors.

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -2 & 2 & 4 \end{bmatrix} R_2 = R_2 - R_1 \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ -2 & 2 & 4 \end{bmatrix} R_3 = R_3 + 2R_2 \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 4 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 4 & 10 \end{bmatrix} R_2 = -\tfrac{R_2}{2} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 4 & 10 \end{bmatrix} R_1 = R_1 - R_2 \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 4 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 4 & 10 \end{bmatrix} R_3 = R_3 - 4R_2 \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{bmatrix} R_3 = \tfrac{R_3}{6} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} R_1 = R_1 - 2R_3 \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} R_2 = R_2 - R_3 \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This reduced row echelon indicates that these arbitrary vectors are linearly independent. Since:

$$\alpha_1 = \alpha_2 = \alpha_3 = 0$$

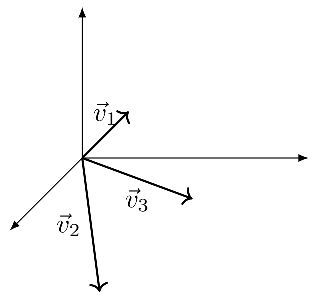
1.1.3 Linear Dependence of a Subspace

|For an example of a subspace in $\mathbb{R}^3$, whose vectors are linearly dependent, we must choose a set of vectors that produce a non-trivial solution to the linear combination whose output produces the zero vector:

$$S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \implies \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = \vec{0}$$

|Let us choose three vectors that comprise our linearly dependent subspace:

$$S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \rightarrow \vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \vec{v}_2 = \begin{bmatrix}1\\-1\\2\end{bmatrix} \vec{v}_3 = \begin{bmatrix}3\\1\\4\end{bmatrix}$$



|Much like linear independence, we must express the set that composes such a subspace in a matrix to find if a subspace is linearly dependent. Then utilize reduced row reduction to determine if there is a non-trivial solution.

$$S = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}3\\1\\4\end{bmatrix} \right\} \rightarrow \begin{bmatrix}1 & 1 & 3\\1 & -1 & 1\\1 & 2 & 4\end{bmatrix}$$

|Utilizing the methods in (Group report 1 section 1.2), we find the non-trivial solution to linear combination of vectors.

$$\begin{bmatrix}1 & 1 & 3\\1 & -1 & 1\\1 & 2 & 4\end{bmatrix} R_2 = R_2 - R_1 \rightarrow \begin{bmatrix}1 & 1 & 3\\0 & -2 & -2\\1 & 2 & 4\end{bmatrix} R_3 = R_3 - R_1 \rightarrow \begin{bmatrix}1 & 1 & 3\\0 & -2 & -2\\0 & 1 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 3\\0 & -2 & -2\\0 & 1 & 1\end{bmatrix} R_2 = -\frac{R_2}{2} \rightarrow \begin{bmatrix}1 & 1 & 3\\0 & 1 & 1\\0 & 1 & 1\end{bmatrix} R_1 = R_1 - R_2 \rightarrow \begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 1 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 1 & 1\end{bmatrix} R_3 = R_3 - R_2 \rightarrow \begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$$

This reduced row echelon indicates that these arbitrary vectors are linearly dependent due to the formation of non-trivial solution:

$$\alpha_1 + 2\alpha_3 = 0$$
$$\alpha_2 + \alpha_3 = 0$$

1.1.4 Methods of determining if something is linearly independent or dependent

|As a footnote, the reader may have been pondering what means would be for us to find instantly if a set of vectors is linearly dependent or linearly independent. With the starting linear combination, I recommend

choosing non-trivial or trivial solutions, which implies linear independence or dependence, then derive what vectors composed the trivial or non-trivial solutions.

2 Projections onto Subspaces

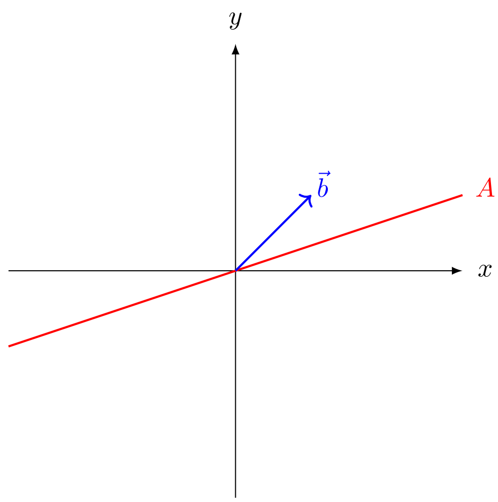
|A projection onto a subspace is a linear transformation that maps a vector onto a subspace, constructing the nearest vector in that subspace to the initial vector.

Thus,

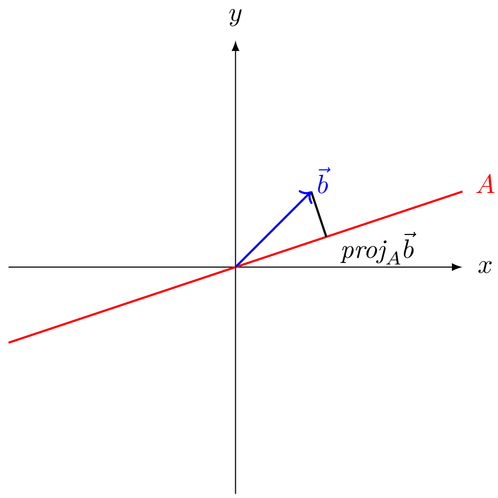
2.0.1 Definition of Projection

Let $A \subset \mathbb{R}^n$ be a set. The projection of a vector $\vec{b} \in \mathbb{R}^n$ onto A, written $proj_A\vec{b}$, is the closest point in A to $\vec{b}$.

|To understand this concept more thoroughly, let us have a mathematical exploration of what a projection is, which may help us develop a greater understanding of this concept in linear algebra. Here we have two mathematical objects in $\mathbb{R}^2$, a vector $\vec{b}$ and a one dimentional subspace in $\mathbb{R}^2$ we denote as A.



|The closest point from $\vec{b}$ to line (A) would be considered $proj_A\vec{b}$. When we draw this new line, we notice that it is normal to the subspace.



|With the illustration above, one may notice that we inserted a line in black that represents the projection of $\vec{b}$ onto set A. We may derive a means of calculating such a projection in this particular scenario. We can think of the line as an error between set A and $\vec{b}$. Therefore let us express this error mathematically as:

$$\vec{e} = \vec{b} - proj_A\vec{b}$$

|Since the projection is a point in the one-dimensional subspace, we may express it as a multiple of a direction that may span that given subspace.

$$proj_A\vec{b} = \alpha\vec{a} \text{ , } where\ \alpha\ is\ some\ scalar$$

|Now rearrange, for the equation to be equal to zero.

$$\vec{e} = \vec{b} - proj_A\vec{b}\ and\ proj_A\vec{b} = \alpha\vec{a} \implies \vec{a}\cdot\vec{e} = 0 \implies \vec{a}^\intercal\vec{e} = 0 \implies \vec{a}^\intercal(\vec{b} - proj_A\vec{b}) = 0 \implies \vec{a}^\intercal(\vec{b} - \alpha\vec{a}) = 0$$

|The equation equal to zero offers us a step closer to finding the value of α, giving us the vector that provides the projection. Now rearrange this equation for α.

$$\vec{a}^\intercal(\vec{b} - \alpha\vec{a}) = 0 \implies \alpha\vec{a}^\intercal\vec{a} - \vec{a}^\intercal\vec{b} = 0 \implies \alpha\vec{a}^\intercal\vec{a} = \vec{a}^\intercal\vec{b}$$

$$\alpha\vec{a}^\intercal\vec{a} = \vec{a}^\intercal\vec{b} \implies \frac{\alpha\vec{a}^\intercal\vec{a}}{\vec{a}^\intercal\vec{a}} = \frac{\vec{a}^\intercal\vec{b}}{\vec{a}^\intercal\vec{a}} \implies \alpha = \frac{\vec{a}^\intercal\vec{b}}{\vec{a}^\intercal\vec{a}}$$

Since we have an expression for projection, we find a means of calculating projection of subspaces by rearranging the expressions we have derived:

$$\alpha = \frac{\vec{a}^\intercal\vec{b}}{\vec{a}^\intercal\vec{a}} \qquad proj_A\vec{b} = \alpha\vec{a}$$

$$proj_A\vec{b} = \frac{\vec{a}^\intercal\vec{b}}{\vec{a}^\intercal\vec{a}}\vec{a} \implies proj_A\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a}$$

|Now with some understanding of the mathematical machinery behind projections. How does one break a projection mathematically? Before that, I would like to look at a more intuitive argument for the functioning of projections.

2.1 Shadows

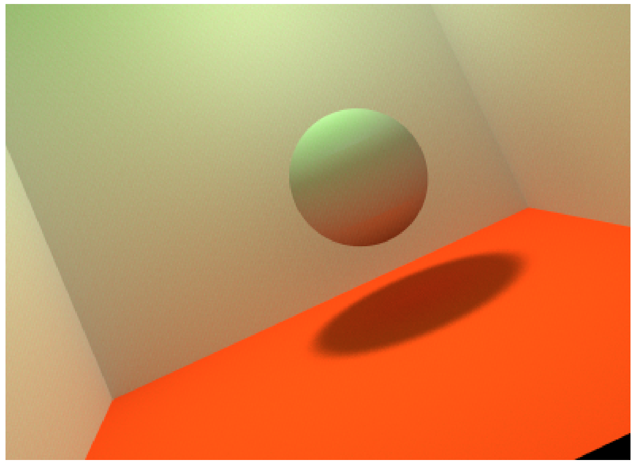


Illustration: Behaviour of path traced light in $\mathbb{R}^3$ that depicts $\vec{b}$ as a sphere and set A in red as a two-dimensional subspace.

|With this illustration above, we get a glimpse that the projection of a vector onto a subspace can be thought of as the shadow of the vector on the subspace. For this shadow to accurately depict the projection, we must place a light that shines uniformly orthogonal to A. This fact offers us insight into this particular projection, which is that it has a unique solution. The solution is unique since only one shadow is cast.

2.1.1 A Non-Existent Shadow

In this duality, conceptually breaking down a projection is far more approachable. This would occur when there is no set for the object to cast its shadow. However, the concept of projection being a shadow is troublesome since whenever an object becomes an element of a two-dimensional subspace, it will no longer cast a shadow onto that subspace. The projection of such a vector is an element of a subspace onto that same subspace will reveal the vector we initially projected.

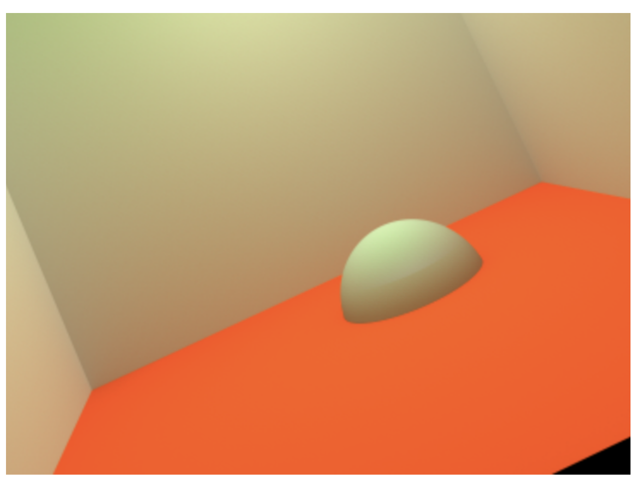


Illustration: Behaviour of path traced light in $\mathbb{R}^3$ that depicts $\vec{b}$ as a sphere and set A in red as a two-dimensional subspace. When a subset of the sphere is an element of the subspace.

2.1.2 Contradiction to The Duality of Projections and Shadows

|Let us contrast this with an example in $\mathbb{R}^2$ that demonstrates this contradiction:

$$\textit{Choose a direction vector } \vec{d} \textit{ that is not equal the zero vector in } \mathbb{R}^2.$$
$$\textit{This direction vector will span one-dimensional subspace set } B.\ B = span\left\{\vec{d}\right\}$$
$$\textit{Then, choose another vector } \vec{c} \textit{ that is colinear to our direction vector. } \vec{c} = \left\{k\vec{d} : k \neq 1\right\}$$
$$\textit{Consider an arbitrary direction vector: } \quad \vec{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\textit{Consider scalar multiple of direction vector: } k = 2 \ \ldots \ \textit{Then, } \vec{c} = 2\vec{d} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$\textit{We may compute the result of projecting set } B \textit{ onto } \vec{c} \textit{ with the formula we derived for projections.}$$

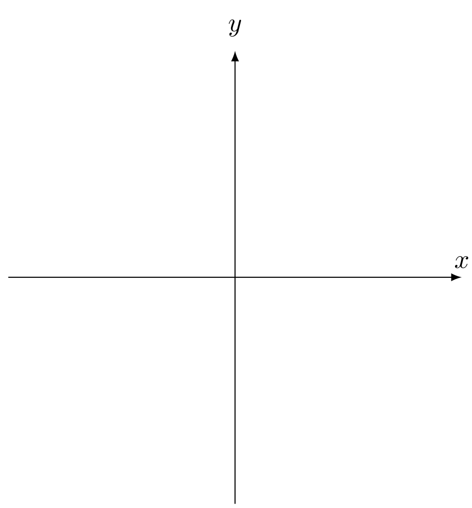
$$proj_B\vec{c} = \frac{\vec{d}\cdot\vec{c}}{\vec{d}\cdot\vec{d}}\vec{d} = 2\vec{d} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \vec{c}$$

Q.E.D.

|Mathematically, the projection exists in the arbitrary one-dimensional subspace, while our mental model of a projection being a shadow says the projection cannot exist. Therefore, exceptions exist to our mental model of a projection being a shadow.

2.2 A Non-Existent Projection onto an Arbitrary Subspace

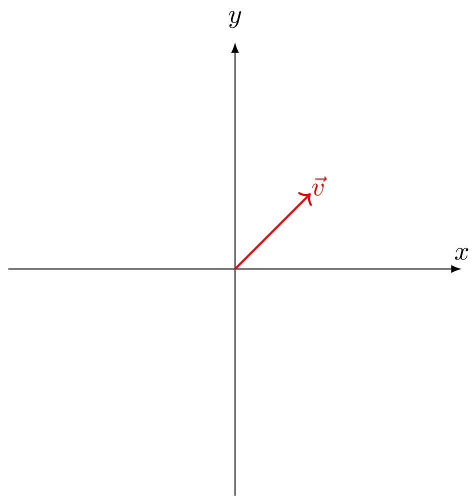
|In section 2.1, we proposed we could construct a projection that did not exist. The assertion predicates the notion that if a set contained no vectors, there would be nothing for another given vector, to project itself onto the set. Thus we need a mathematical object that relates to no elements in a set. A set with no elements is called an empty set $\emptyset$.



$\emptyset$ *in* $\mathbb{R}^2$

|Let us make a non-existent projection, for this we shall refer to the empty set $\emptyset$ as set X, and let us choose

an arbitrary vector $\vec{v}$ in $\mathbb{R}^2$. Now if we were to project the vector $\vec{v}$ onto set X, the projection would not exist since there is no closest vector in set X to $\vec{v}$.



$$X = \emptyset \textit{ and } \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \implies \textit{proj}_X \vec{v} \textit{ D.N.E.}$$

$$Q.E.D.$$

3 A Means of Determining a Non-Standard Basis for a Given Subspace

|In the previous Group Report, we outlined the concepts of a basis. In that case, we used three standard basis vectors, in which, if we were to take the span of these vectors, it would be equivalent to the set of all vectors in $\mathbb{R}^3$.

3.0.1 Definition of Vector Basis

The vector basis of a vector space is a fundamental concept in linear algebra. It provides a way to understand a vector space's structure by breaking it down into linearly independent vectors, which span composes such a subspace. To be more precise, a vector basis for X is a subset $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ of the vectors in X that satisfies two properties:

Linear independence: No vector in $\mathcal{V}$ may be expressed as a linear combination of the other vectors in $\mathcal{V}$. Therefore, for any scalars, $\alpha_1, \alpha_2, \ldots, \alpha_n$, the equation $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ holds if and only if $c_1 = c_2 = \cdots = c_n = 0$.

Tracing the Subspace: This suggests that every vector in X can be written as a linear combination of the vectors in $\mathcal{V}$. In other words, for any vector $x \in X$, there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$.

The first property assures that the vectors in $\mathcal{V}$ are linearly independent and not redundant because they cannot be expressed as a linear combination of each other. The second property ensures that the vectors in $\mathcal{V}$ are sufficient to describe every vector in X. These two properties define the vectors in $\mathcal{V}$ as the fundamental structure of X.

|How does one differ from a standard basis vector? We know that a standard basis vector is a standard basis, an orthonormal vector basis in which each basis vector has a single nonzero entry with value 1. Therefore the basis vectors in $\mathbb{R}^3$ are:

$$\mathbb{R}^3 = span\left\{\hat{\imath}, \hat{\jmath}, \hat{k}\right\} = span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

|To receive non-standard basis vectors, let us choose vectors that are not equal to $\left\{\hat{\imath}, \hat{\jmath}, \hat{k}\right\}$. With this in mind, let us choose three linearly independent vectors in $\mathbb{R}^3$. From section one, we learned that a set of vectors are linearly independent if their solution is trivial. Therefore from the vectors we choose, we may verify if they are linearly independent.

Therefore let us call our set of non-standard basis vectors $\mathcal{B}$.

$$Let,\ \mathcal{B} = \left\{\vec{b}_1, \vec{b}_2, \vec{b}_3\right\} = \left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\0\end{bmatrix}\right\}$$

|Now let us verify if this set is linearly independent by determining if there is a trivial solution to this set of vectors:

$$\begin{bmatrix}1&4&7\\2&5&8\\3&6&0\end{bmatrix} R_2 = R_2 - R_1 \rightarrow \begin{bmatrix}1&4&7\\0&-3&-6\\3&6&0\end{bmatrix} R_3 = R_3 + 3R_1 \rightarrow \begin{bmatrix}1&4&7\\0&-2&-2\\0&-6&-21\end{bmatrix}$$

$$\begin{bmatrix}1&4&7\\0&-2&-2\\0&-6&-21\end{bmatrix} R_2 = \tfrac{-R_2}{3} \rightarrow \begin{bmatrix}1&4&7\\0&1&2\\0&-6&-21\end{bmatrix} R_1 = R_1 - 4R_2 \rightarrow \begin{bmatrix}1&0&-1\\0&1&2\\0&-6&-21\end{bmatrix}$$

$$\begin{bmatrix}1&0&-1\\0&1&2\\0&-6&-21\end{bmatrix} R_3 = R_3 + 6R_2 \rightarrow \begin{bmatrix}1&0&-1\\0&1&2\\0&0&-9\end{bmatrix} R_3 = \tfrac{-R_3}{9} \rightarrow \begin{bmatrix}1&0&-1\\0&1&2\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&-1\\0&1&2\\0&0&1\end{bmatrix} R_1 = R_1 + R_3 \rightarrow \begin{bmatrix}1&0&0\\0&1&2\\0&0&1\end{bmatrix} R_2 = R_2 - 2R_3 \rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

This reduced row echelon indicates that the vectors in this set $\mathcal{B}$ are linearly independent. Since:

$$\alpha_1 = \alpha_2 = \alpha_3 = 0$$

|From this, we can conclude that the set satisfies the first property. We must verify if it can span the subspace $\mathbb{R}^3$. In addition, Finding a unique set of coefficients $\alpha_1, \alpha_2, ..., \alpha_n$ for each vector in the subspace is unnecessary. As long as some coefficients work, the set of vectors is considered to span the subspace. Therefore let us find these coefficients:

$$(x, y, z) = \hat{\imath}(1, 2, 3) + \hat{\jmath}(4, 5, 6) + \hat{k}(7, 8, 0)$$

$$(x, y, z) = ((\hat{\imath} + 4\hat{\jmath} + 7\hat{k}), (2\hat{\imath} + 5\hat{\jmath} + 8\hat{k}), (3\hat{\imath} + 6\hat{\jmath}))$$

|Now we established their relation to the standard basis vectors, which we know spans the subspace of $\mathbb{R}^3$. This fact is beneficial since the difference between these non-standard basis vectors is only the coefficients of $\hat{i}, \hat{j}, \hat{k}$.

$$x = \hat{i} + 4\hat{j} + 7\hat{k}$$

$$y = 2\hat{i} + 5\hat{j} + 8\hat{k}$$

$$z = 3\hat{i} + 6\hat{j}$$

4 Subspaces

4.0.1 Definition of Subspaces

4.0.2 Proving Whether Something Is a Subspace

5 Proving P is a Subspace Using The Definition of Subspace

|In section two, when deriving the projection of a vector onto a set of vectors, we accidentally came across the normal form. Although this form can describe many things, in section two, it was intended to represent a one-dimensional subspace. Now in section five, we will prove that a plane expressed in normal form is a subspace. Continuing derivation of normal form from, section 2:

$$\vec{a} \cdot \vec{e} = 0 \implies \vec{a}^{\intercal}\vec{e} = 0 \implies \vec{a}^{\intercal}(\vec{b} - proj_A\vec{b}) = 0 \implies \vec{a}^{\intercal}(\vec{b} - \alpha\vec{a}) = 0$$

$$\vec{a}^{\intercal}(\vec{b} - \alpha\vec{a}) = 0 \implies \vec{a} \cdot (\vec{b} - \vec{c}) = 0$$

5.0.1 Normal Form of a Line

A line $l \subset \mathbb{R}^2$ is expressed in normal form if there exist vectors $\vec{a}$ and $\vec{c}$ so that l is the solution set to the equation:

$$\vec{a} \cdot (\vec{b} - \vec{c}) = 0$$

The equation $\vec{a} \cdot (\vec{b} - \vec{c}) = 0$ is called normal form of l.

|Now we have derived the normal form, let us use this form of describing subspaces to prove that a plane is a subspace in $\mathbb{R}^3$. Let P be the plane given by:

$$P = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) = 0$$

|Prove P is a subspace using the definition of subspace.

$$\implies \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) - \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) = 0$$

$$\implies 2x + y + x - 0 - 0 - 0 = 0$$

$$\implies 2x + y + x = 0$$

|For P in the subspace, there are three main points to satisfy for all $\vec{u}, \vec{v} \in P$ and all scalars k:

1. $\vec{0}$ is in P (non-emptiness, additive identity)

2. $\vec{u} + \vec{v} \in P$ (closed with respect to vector addition)

3. $k\vec{u} \in P$ (closed with respect to scalar multiplication)

· First point:

P does not contain the zero vector because if x, y, z equal 0

$\implies 2x + y + z = 0$

$\implies 2 \cdot 0 + 0 + 0 = 0$

$\implies 0 = 0$

· Second point:

We will take arbitrary points $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ which lie on P consider $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$ and we will use $2x + y + z = 0$ By substituting:

$2(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = 0$

$(2x_1 + y_1 + z_1) + (2x_2 + y_2 + z_2) = 0$

Which proves closed under addition.

· Third Point:

Consider $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ on P. If we multiply $k \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} kx_1 \\ ky_1 \\ kz_1 \end{bmatrix}$

By plugging it in:

$2kx_1 + ky_1 + kz_1 = 0$

$k(2x_1 + y_1 + z_1) = 0$

We know, $2x_1 + y_1 + z_1 = 0$ so $k \cdot 0 = 0$ which proves closed under scalar mulitiplication.

Q.E.D.