University of Toronto

Physics 407: Computational Physics Final Project

A Brief Numerical Inspection of Solitons and their Similarities in Fluid Dynamics and Nonlinear Optics

Interacting Solitons

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Abstract

Solitons are "solitary-wave pules" that arrive in the solutions to several physical systems [6].
The situations of particular interest for this work will be their appearance in solutions to the Korteweg de-Vries Equation (K-dW), extending the results to non-linear Schrodinger Equations (NLSE) which appear in optical systems. The appearance in solitons in the solutions to both of these equations have import physical ramifications ranging from the field of nonlinear optics to fluid dynamics.
This report will detail some significant historical progress in realizing numerical solutions and conclude with some novel observations about the effect of non-linearity parameters on soliton stability.

1 Introduction

Solitons were discovered in 1834 by J. Scott Russell, and published in his 1844 'Report on Waves' [2]. The report details chasing a soliton on horseback in the Edinburgh-Glasgow Canal and his subsequent experimentation to reproduce the phenomenon witnessed.
Soliton solutions arise during nonlinear wave equation solutions where the nonlinear effects and dispersive and/or dissipative effects cancel, creating stability of the wave that would not seem intuitive.
From this phenomenon, soliton research has grown from a single Scotsman dropping rocks in a tub into an entire-sub-field of applied partial differential equations and mathematical physics. At the heart of the excitement of solition solutions is the fact that "these nonlinear waves can interact strongly and then continue thereafter almost as if there had been no interaction at all". [2]. As the author's current interests and research with Dr. Aephraim Steinberg relate to the fields of nonlinear optics and quantum information science, this report will trace the history of solitons solutions and attempt to draw conclusions of the importance of soliton research within the twenty-first century.

2 Physical Systems

2.1 Korteweg de-Vries Equation

The equation under inspection here is as follows

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

Kd-W equations describe a variety of physical systems [3] such as the one-dimensional, long-time asymptotic behavior of small, but finite amplitude shallow-water waves, hydro-magnetic waves in a cold plasma, ion-acoustic waves, and acoustic waves in an anharmonic crystal.
Accurate and insightful solutions to the equation therefore provide clarity to several sub-fields of physics. The subsequent sections offer a thorough study of the associated systems through the original proposed solution and subsequent evolution into a modern sophisticated approach.

2.1.1 Original Leapfrog Method

The earliest numerical solution [6] follows the approach of traditional numerical PDE solutions such as the Lax-Welshoff method or other leapfrog methods. It begins with a discretized temporal and spatial mesh denoted by the temporal and spatial intervals of k and $h = \frac{1}{N}$ respectively. The solution is restricted by a periodic interval $0 \leq x < 2$. and setting $u_i j = u^j_{i+2N}$. The equations therefore approximates $u^j_i = u(ih, jk)$ by the following.

$$\begin{aligned} u_i^{j+1} = u_i^{j-1} - \frac{1}{3}(k/h)\left(u^j_{i+1} + u^j_i + u^j_{i-1}\right)\left(u^j_{i+1} - u^j_{i-1}\right) \\ - \left(\delta^2 k/h^3\right)\left(u^j_{i+2} - 2u^j_{i+1} + 2u^j_{i-1} - u^j_{i-2}\right) \end{aligned} \tag{2}$$

By setting an initial condition of $u(x,0) = 8\,\mathrm{sech}^2(2(x+8)) + 2\,\mathrm{sech}^2((x+1))$ and $\delta = 0.022$, the numerical solutions are as follows

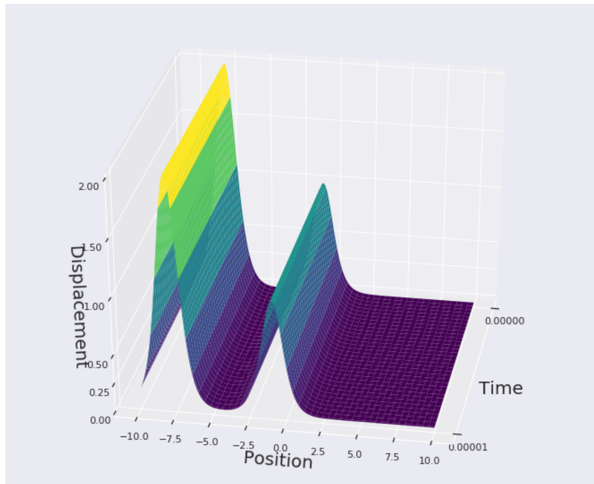

Figure 1: Spatial and Time Plot for Single Soliton

2.1.2 Split-Step Pseudo-Spectral Method

The more sophisticated solution used in contemporary numerical analysis is the spectral method. The term refers to decomposing the solution into its spectra using the Fourier Transform and then recovering the solution. [1] Consider writing the Fourier Transform KdV equation of the form

$$\partial_t \hat{u} + 3ik\widehat{(u^2)} - ik^3\hat{u} = 0 \tag{3}$$

where in split step method solve the parts simultaneously. Consider the two parts

$$\partial_t \hat{u} = ik^3\hat{u} \tag{4}$$

and

$$\partial_t \hat{u} = -3ik\widehat{(u^2)} \tag{5}$$

Then using the Forward Euler method and a Fourier Transform, consider the two equations as the solution

$$\hat{u}_1(k, t + \Delta t) = \quad \hat{u}(k, t)\mathrm{e}^{\mathrm{i}k^3\Delta t} \tag{6}$$

$$\hat{u}(k, t + \Delta t) = \hat{u}_1(k, t + \Delta t) - 3ik\Delta t \left(\mathcal{F}\left(\left(\mathcal{F}^{-1}\left[\hat{u}_1(k, t + \Delta t)\right]\right)^2\right)\right) \tag{7}$$

From the citation, significant adaptations were made to provided code converted from MATLAB to Python resulting in Figure 2. The same initial position was given, but this time, there was an added velocity to each soliton of $v_1 = 6$ and $v_2 = -3$. These are in arbitrary units based on the spatial and temporal mesh.

Note that in order to provide a velocity to the code used here, it was necessary to repeat the initial step with a time velocity, although the method itself is robust.Therefore, the more interesting computation falls in subsequent sections.

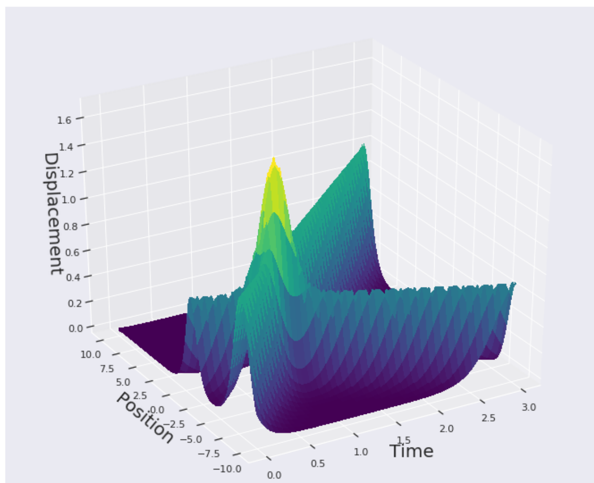

Figure 2: Spatial and Time Plot for Single Soliton

2.2 Optical Systems and NLS Equations

Non-Linear Schrodinger Equations (NLSE) describe some of the fundamental behavior of non-linear optics or other classical field theory problems. [4]. Within these equations, soliton solutions appear often. Their physical significance varies, but the stability of solitons describes several unique propagation of systems, whether temporal or spatial, or representing the existence or lack of energy, sometimes referred to as "bright" or "dark" solitons within the field of optics. . Note that here is where the real computational project begins. The KdV served as an educational playground in which to understand soliton solutions. However with analytic solutions available and no literature to discuss KdV solutions with interesting features, even as simple as velocity, there is little science to be done. Now that they have been discussed though, without further ado, the NLSE that represents optical pulses in media with saturation is as follows

$$i\frac{\partial\psi}{\partial t}+\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}+\frac{|\psi|^2\psi}{1+S|\psi|^2}=0 \tag{8}$$

A solution involving solitons show up in the form

$$\psi(x,t)=\frac{2\sqrt{2}e^{\sqrt{2}x}}{1+\left(\frac{3}{2}-2S\right)e^{2\sqrt{2}x}}e^{it+iv} \tag{9}$$

2.2.1 Numerical Solutions

The spectral method introduced in Section 2.1.1 applies to the NLSE as well. Here we will consider the split-step system for a NLSE in the context of optical pulses. Here, again using the same source for solution. Here the nonlinear solution is [5]

$$\psi(x,t)=\psi\left(x,t_0\right)e^{\frac{|\psi|^2}{1+S|\psi|^2}t} \tag{10}$$

and the linear solution using a Fourier Transform is

$$\psi_{j,k+1}=2\tau i\left(0.5\psi_{xx}+A_{j,k}\psi_{j,k}\right)+\psi_{j,k-1} \tag{11}$$

The results for two colliding soliton solutions of the same magnitude is provided below.

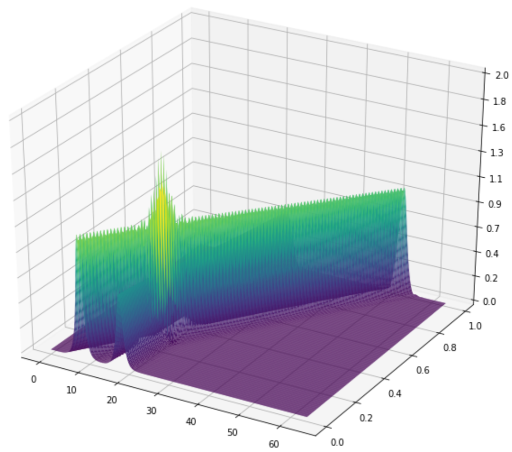

Figure 3: Spatial and Time Plot for Two Soliton Solution

For this figure, the citation recommended the settings of mesh points $N = 512$ and a simulation time of $T = 1$. The spatial length is 64 and the time step $\tau = 0.01$ This satisfies the conclusion of the stability analysis that [5]

$$\tau < \frac{h^2}{2} \tag{12}$$

3 Reflection on Accuracy

The advantage of solutions that stem from current literature is that accuracy can be most closely determined by the resemblance of solutions to the solutions provided by literature. In order to provide a more meaningful contribution, we examine an accuracy check particularly related to the application of soliton methods to physical system. The reference for the NLSE solution, perhaps the most interesting problem here, provided qualitative assessment for the susceptibility of the solution to physical parameters. Below we provide a quantitative extension, using conservation of normalization as reference points, as this would be insightful for realistic problems.

3.1 Error Estimates

The stability condition given in the literature for the explicit finite-difference method was

$$kh^3/\left|2 - h^2U\right| \tag{13}$$

however, even if that is satisfied, running the code for a long enough period of time $(0.01)s$ provides significant errors and artifacts in the solution.

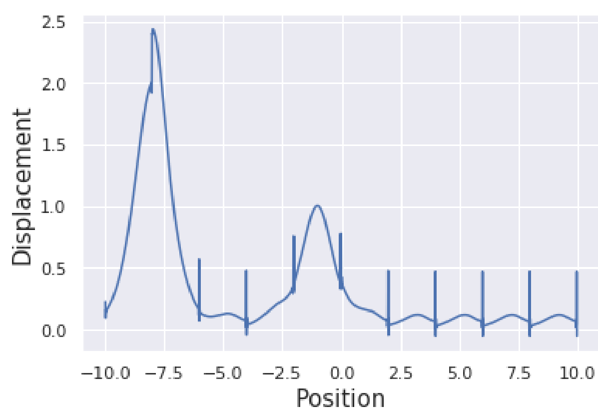

Figure 4: Plot illustrating the statistical artifacts within the code

3.2 Parameter Susceptibility

We now turn our attention to the most meaningful part of this report, some quantitative analysis about the stability of the two soliton solution given a different nonlinear parameter, S. As one of the most interesting effects of true soliton solutions is their ability to preserve their form despite interacting, the best quantitative approach to this is to calculate the normalization before and after interaction and seeing how much it is disrupted. The results are as follows. Consider the normalization constant

$$N = \int_{-\infty}^{\infty} |\psi|^2 dx \tag{14}$$

which is calculated using the composite trapezoidal method.

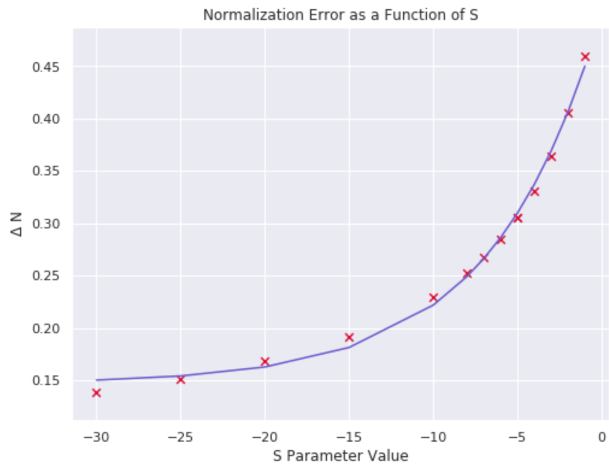

Figure 5: A line of best fit, $y = 0.35e^{0.16x} + 0.15$ showing the relationship of accuracy to non-linearity parameter

here we see that a larger negative value of S tends to actually lead to more accurate solutions, where the solitons retain their form better. S has many physical interpretations, but it is related to the non-linearity of the equation, which makes this an interesting phenomenon, particularly because the relationship can be fit with an analytic line. The author does not have good intuition for why this might be, but since solitons form due to the balance of non-linearity and dispersion, perhaps this is a physical phenomenon rather than a mere numerical artifact.

4 Conclusions

Solitons remain a fascinating intersection of mathematics and physics, to a degree largely beyond the comprehension of the author. What exists here is proof of concept of several approximations of solitons solutions and the remarkable power of the numerical methods used to understand them. Perhaps the most significant conclusion then is that the work remains ongoing. With this work as a springboard, these solutions could be used to grapple genuine physical problems and perhaps gain more genuine insight. The most immediate of these would be to understand the physical significance of the S parameter in optical systems and do thorough tests of how the preservation of soliton solutions is related to this along with velocity.

4.1 Observations

Since this project is regarded as entirely theoretical solutions, the most insightful comments here to make are the computational ones. The split-step spectral methods are considerably quicker in computational time and more robust to different parameters or extrema of conditions. If further work were to explore physical systems, this method would be considerably more recommended.

4.2 Potential Future Studies

There are two separate avenues of further work that would be beneficial to explore. The first is further computational improvements that could be made. In particular, the split-step spectral method currently uses and Euler-Forward Step solution. It could be extended via a 4^{th} order Runge Kutta solution that would provide additional accuracy. These sorts of improvements are generally meaningless until the systems become more complicated such as a physical system. Perhaps these can be best summarized by the current literature as higher order terms of the equations, more accurate dissipation, discreteness effects that come from the underlying lattice. The extreme complications that physical systems offer motivation to continue an undergraduate career.

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