FEB22009-21: Financial Case Study
Specialisation: Quantitative Finance & Econometrics
"They don't realise I'm down": Realised GARCH for modelling market downturns

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1 Introduction

In 1973, The Hollies sang "They don't realise I'm down" in a song also containing the line "I, I saw the light". Investors certainly realised that markets were down during much of the Covid crisis, but in the end they saw the light, with stock markets re-bouncing and reaching all-time highs. During this strong recovery, the VIX declined from its peak to more usual levels (see Figure 1).

The 'VIX' is short-hand for the volatility index provided by the Chicago Board Options Exchange (CBOE). It is constructed from option prices and intends to measure the expectation among investors of the (annualised) standard deviation of the S&P500 return over the next month. For a more detailed explanation of the VIX, click here. The annualisation simply means that the (monthly) standard deviation is multiplied by $\sqrt{12}$ to get an 'annual' figure. Its level of around 80 at the height of the Corona crisis meant that investors were expecting the standard deviation of the next month's return to be $80/\sqrt{12} = 23\%$. Typically, the standard deviation of S&P500 returns, on an *annual* basis, is around 20%. The fact that more volatility was expected in a single month than normally would be expected in an entire year illustrates the magnitude of the Covid crisis. The VIX is currently hovering around 30 or 40 (i.e., well above its long-term average of around 20) due to the war in Ukraine, worries about higher interest rates (making stocks less attractive) and inflation.

Typically, when the VIX goes up, the S&P500 goes down. Figure 2 displays daily changes in the VIX on the horizontal axis versus S&P500 close-to-close log returns on the vertical axis. Clearly, the index goes down when the VIX goes up, and vice versa. Unfortunately for investors, the relation is contemporaneous, i.e., there is no lead-lag relationship. For any given day, movements of the VIX and the index are uncertain, but they will probably move in opposite directions. This effect is known as the 'leverage effect', so coined by Fischer Black, who supposed that a decline in the market valuation of a particular stock increased the company's 'leverage ratio', thereby making it more risky hence more volatile. In this interpretation, the drop in the stock price is leading, while the increase in volatility is lagging.

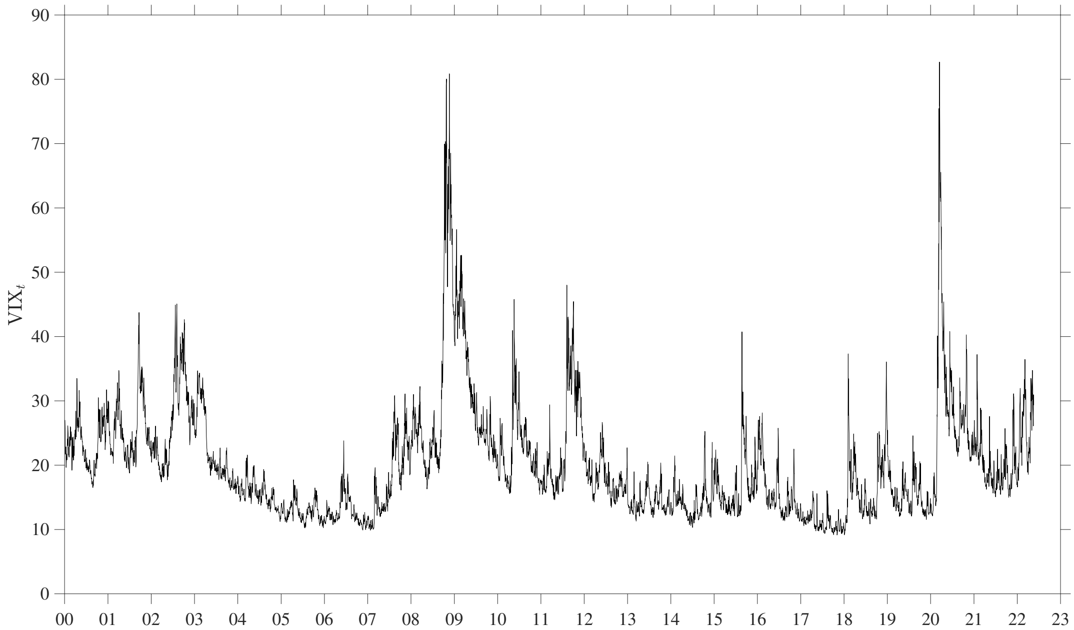
Black's explanation of the leverage effect is seemingly contradicted by our Figure 2, which suggests the relation between the S&P500 and the VIX is contemporaneous. Perhaps the increase in volatility is leading, while the drop in the market is lagging, while still occurring on the same day. Indeed, the so-called 'volatility feedback effect' supposes that increased volatility causes the market to drop, as risk averse investors exit the market. As a result, stocks are sold cheaply, and those who buy them are rewarded with a risk premium. In the aftermath of the Covid crisis, this effect may have played a large role. Indeed, van der Kroft et al. (2020) predicted the recovery before it happened, predicting that stocks would recover as the VIX slid down to more normal levels. Classic volatility models (discussed in more detail below) maintain the original leverage effect as proposed by Fischer Black, taking negative stock market returns to be the cause of increased volatility, rather than the other way around.

The VIX is a 'forward-looking' volatility measure, in the sense that it captures investors' expectations regarding the magnitude of changes in the S&P500 in the next month. This is in contrast with 'realised variance' measures, which are based directly on (historical) movements of the index itself. Roughly speaking, realised variance is a measure of variability obtained by summing the squares of intra-day returns. For example, the Oxford library of realised variance publishes daily measures of the realised variance of the S&P500 index based on intra-day returns at the five minute frequency.[1] Realised variance is essentially a 'backward-looking' volatility measure in the sense that it can only be computed in hindsight.

Given the crucial role of volatility in financial markets and portfolio management, volatility modelling started as early as 1982 (see Engle, 1982). Researchers have since proposed a variety of models, such as the

[1] For the Oxford library, click here. In this assignment, you will use the variable rv5_ss.

Figure 1: The VIX from 4 January 2000 until 20 May 2022. The data are from from Yahoo Finance (https://finance.yahoo.com/quote/%5EVIX/).



autoregressive conditional heteroskedasticity (ARCH) and generalized ARCH (GARCH) models of Bollerslev (1986), the latter of which will be used in this assignment. These models account for the fact that volatility is persistent but time-varying. That is, there are prolonged periods of large swings in the market, as well as extensive periods of stability. This phenomenon is known as 'volatility clustering'. While volatility changes over time, it does so gradually, such that periods of high volatility cluster together. While the high-volatility cluster of the Covid crisis seems to have passed, investors are currently anxious for other reasons.

To estimate volatility models, researchers have historically had access to daily returns, such as those available from Yahoo Finance.[2] Nowadays, researchers additionally have access to realised variance measures based on high-frequency data, as well as market expectations of volatility given by the VIX. Given this abundance of new, high-quality data, we should perhaps reconsider the validity of 'classic' volatility models, such as GARCH. GARCH models use a single, noisy proxy to measure volatility: the daily squared return. If the market is extremely volatile during the day, but the closing price equals the closing price of the previous day, a GARCH model would 'think' that the daily volatility was zero. To overcome this issue, Hansen et al. (2011) introduce Realised GARCH models that incorporate realised variance of high-frequency intra-day returns. Realised measures are more reliable, as they take into account all intermediate market movements. A separate strand of the literature proposes to model the realised variance directly, i.e. without a GARCH structure. The heterogeneous autoregressive realised volatility (HAR-RV) of Corsi (2009) is an important example of this approach, and will be used as a benchmark model in this assignment.

These considerations lead to the following research questions:

1. Can the volatility of the S&P500 index over a given period be better predicted using classic GARCH models or more recent models relying on realised variance measures such as the Realised GARCH model of Hansen et al. (2011)?

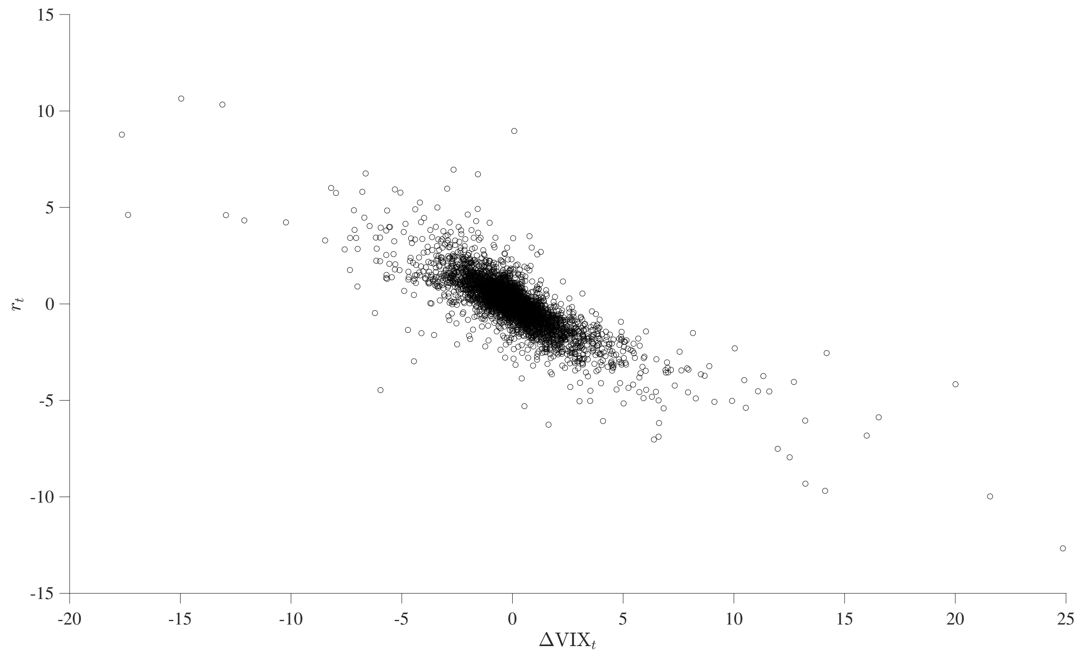
2. Can any of these models beat the HAR-RV model of Corsi (2009)?

3. Can any of these models beat market expectations of volatility, as provided by the VIX?

The (actual) variance of the S&P500 index over some period, which we hope to predict, can be approximated either by summing squared close-to-close log returns, or by summing daily realised variances, obtained from the Oxford library. As such, we have two possible 'target variables' that we hope to predict, using two advanced

[2]For example, see https://finance.yahoo.com/quote/%5EGSPC/ for the S&P500. Daily returns are also included in the Oxford realised variance library, click here.

Figure 2: Changes in the VIX from 4 January 2000 until 20 May 2022 (horizontally) versus close-to-close returns on the S&P500 (vertically).



models (GARCH and Realised GARCH) and two simple benchmark models (HAR-RV and the VIX). We are interested in making predictions over a horizon of one trading day, five trading days (corresponding to one calendar week), or 21 trading days (corresponding to one calendar month). To compare the quality of different forecasts, you may consider the robust loss function proposed in equation (24) of Patton (2011), important special cases of which are the mean-squared error (MSE) and so-called 'QLIKE' loss functions.[3] The purpose of these loss functions is to compare our predictions against the target and assign some 'loss value' when they are different. In sum, you will need to compute (at least) 4 models $\times$ 2 targets $\times$ 3 horizons $\times$ 2 loss functions $= 48$ loss values, report these in a meaningful way and draw economic conclusions.

To answer these research questions, you are provided with an Excel sheet titled `data.xlsx`, which contains four series related to the S&P500 index: (i) date, (ii) 100$\times$ close-to-close log return on the index, (iii) the realised variance (variable name: rv5_ss), (iv) the closing level of the VIX. For your convenience, these data have been pre-loaded in the Matlab file `data.mat`. The Matlab file `demo.m` that you have been provided with plots the data, and contains an implementation of the simplest possible GARCH model.

Sections 2, 3 and 4 provide background information that should help you understand the models and how to work with them. Section 5 provides a concrete roadmap with step-by-step suggestions for how to answer the research questions.

2 Classic volatility modelling: GARCH

Let the r_t denote the close-to-close log return on the index, that is $r_t = 100 \times \log(y_t/y_{t-1})$, where y_t is the level of the index at the end of day t. These returns have already been constructed for you in the data set provided. Many models for time-varying volatility assume that r_t is generated by a model like this:

$$r_t \;=\; \mu + \sigma_t\,\varepsilon_t, \qquad 1 \le t \le n, \tag{2.1}$$

where n denotes the sample size, and the $\varepsilon_t's$ are independently and identically distributed (i.i.d.) 'shocks' with $\mathrm{E}[\varepsilon_t] = 0$, $\mathrm{V}[\varepsilon_t] = 1$. Throughout this assignment, we will assume the distribution of ε_t is normal; concisely, $\varepsilon_t \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0,1)$. People often use the word 'volatility' to mean either σ_t or σ_t^2, depending on the context. If μ and σ_t are known, then the variance of returns is $\mathrm{V}[r_t] = \sigma_t^2$. The phenomenon of 'volatility clustering' implies that σ_t is time-varying (as indicated by the subscript) but also persistent, which means that $\sigma_{t+1} \approx \sigma_t$. In other words, σ_t is changing over time, but not too much.

[3]For alternative loss functions that could also be considered, see section 3 of Hansen and Lunde (2005).

Classic GARCH models assume that our estimate of σ_t is based on the information up to (and including) time $t-1$. Hence σ_t is based on the information set $\mathcal{I}_{t-1}$, which is essentially short-hand for the returns contained in the set $\{r_1, \ldots, r_{t-1}\}$. Hence, σ_t is really a one-step-ahead prediction (made at time $t-1$) of the volatility at time t. A rule for constructing σ_t is called a 'filter'. Any filter allows us to compute the implied shock, or implied residual, ε_t, as follows:

$$r_t \;=\; \mu \,+\, \sigma_t\, \varepsilon_t, \qquad \Leftrightarrow \qquad \varepsilon_t \;=\; (r_t - \mu)\,/\,\sigma_t. \tag{2.2}$$

The next subsection discusses the classic filter introduced by Bollerslev (1986), which has been cited over 30,000 times. Our treatment below covers only the basics; for more detail, you are encouraged to read Chapter 7 of Franses et al. (2014).

2.1 GARCH

The industry workhorse for filtering volatility is the generalized autoregressive conditional heteroskedasticity (GARCH) model of order $(1,1)$ introduced by Bollerslev (1986), which reads

$$\begin{aligned} \sigma^2_{t+1} \;&=\; \omega \,+\, \alpha\,(r_t - \mu)^2 \,+\, \beta\,\sigma^2_t, \\ &=\; \omega \,+\, \alpha\,\sigma^2_t\,\varepsilon^2_t \,+\, \beta\,\sigma^2_t, \end{aligned} \tag{2.3}$$

where the second line follows from (2.2). Here, $\omega, \alpha, \beta \geq 0$ are nonnegative parameters, ensuring $\sigma^2_t \geq 0$ for all t. Everything on the right-hand side is evaluated at time t, such that σ^2_{t+1} can be computed at time step t. The GARCH equation ensures that volatility changes gradually over time, such that σ^2_{t+1} will be close to σ^2_t. Note that the unconditional variance of r_t is given by

$$\mathrm{E}[(r_t - \mu)^2] \;=\; \mathrm{E}[\sigma^2_t\,\varepsilon^2_t] \;=\; \mathrm{E}[\sigma^2_t]\,\mathrm{E}[\varepsilon^2_t] \;=\; \sigma^2,$$

where $\mathrm{E}[\sigma^2_t] = \sigma^2$ denotes the unconditional expectation of σ^2_t. By taking expectations on both sides of the GARCH equation (2.3), it follows that $\mathrm{E}[\sigma^2_{t+1}] = \omega + \alpha\,\mathrm{E}[\sigma^2_t]\,\mathrm{E}[\varepsilon^2_t] + \beta\,\mathrm{E}[\sigma^2_t]$. In the steady state (for large t), we expect $\mathrm{E}[\sigma^2_{t+1}] = \mathrm{E}[\sigma^2_t] = \sigma^2$, which would imply

$$\sigma^2 = \frac{\omega}{1 - \alpha - \beta}, \tag{2.4}$$

as long as $\alpha + \beta < 1$. Using $\omega = (1 - \alpha - \beta)\sigma^2$, GARCH equation (2.3) can equivalently be written as

$$\begin{aligned} \sigma^2_{t+1} \;&=\; (1 - \alpha - \beta)\,\sigma^2 + \alpha\,\sigma^2_t\,\varepsilon^2_t + \beta\,\sigma^2_t, \\ &=\; \sigma^2 + \alpha\,\sigma^2_t(\varepsilon^2_t - 1) + (\alpha + \beta)\,(\sigma^2_t - \sigma^2), \end{aligned} \tag{2.5}$$

which is called the 'innovations form', as the term $\varepsilon^2_t - 1$ has expectation zero. This simple reformulation demonstrates that the parameter combination $0 \leq \alpha + \beta < 1$ controls how quickly or slowly σ^2_t mean-reverts to σ^2, while the parameter $\alpha \geq 0$ controls how sensitive the filter is to each individual observation r_t. It is common to set $\sigma^2_1 = \sigma^2$, i.e. start off at the unconditional mean.

2.2 Asymmetric GARCH

According to Black's 'leverage effect', positive shocks ε_t affect tomorrow's volatility differently than negative shocks. To model this phenomenon, we consider the following asymmetric GARCH model by Glosten et al. (1993):

$$\begin{aligned} \sigma^2_{t+1} \;&=\; \omega \,+\, (\alpha_1\,\mathbb{1}_{\varepsilon_t<0} \,+\, \alpha_2\,\mathbb{1}_{\varepsilon_t\geq 0})\;\sigma^2_t\,\varepsilon^2_t \,+\, \beta\,\sigma^2_t, \qquad (2.6) \\ &=\; \omega + \alpha_1\left[\varepsilon^2_t\,\mathbb{1}_{\varepsilon_t<0} - \frac{1}{2}\right]\sigma^2_t + \alpha_2\left[\varepsilon^2_t\,\mathbb{1}_{\varepsilon_t\geq 0} - \frac{1}{2}\right]\sigma^2_t + (\alpha + \beta)\,\sigma^2_t, \\ &=\; \sigma^2 + \alpha_1\left[\varepsilon^2_t\,\mathbb{1}_{\varepsilon_t<0} - \frac{1}{2}\right]\sigma^2_t + \alpha_2\left[\varepsilon^2_t\,\mathbb{1}_{\varepsilon_t\geq 0} - \frac{1}{2}\right]\sigma^2_t + (\alpha + \beta)\,(\sigma^2_t - \sigma^2), \end{aligned}$$

where $\alpha := (\alpha_1 + \alpha_2)/2$, and the indicator function $\mathbb{1}_A$ equals one if the event A happens and zero if it does not. Here, $\omega, \alpha_2, \alpha_1, \beta \geq 0$ are parameters. The coefficients α_1 and α_2 describe the sensitivity of volatility with respect to negative and positive shocks, respectively. Note that both terms in square brackets have expectation zero. The last line can be viewed as a definition of σ^2, implying $\sigma^2 := \omega/(1 - \beta - \alpha)$ as before, except now $\alpha := (\alpha_1 + \alpha_2)/2$. Hence σ^2 is still the unconditional mean of σ^2_t as long as $\alpha + \beta < 1$. It is common to set $\sigma^2_1 = \sigma^2$. As you should verify for yourself, classic (symmetric) GARCH is recovered if $\alpha_1 = \alpha_2$.

2.3 Estimation of constant parameters

This section discusses the estimation of GARCH models using maximum likelihood (ML). The return r_t is distributed according to the probability density function (p.d.f.) $f(r_t|\mathcal{I}_{t-1})$:

$$f(r_t|\mathcal{I}_{t-1}) \;=\; \frac{1}{\sigma_t}\, p\left(\frac{r_t-\mu}{\sigma_t}\right), \tag{2.7}$$

where $p(\cdot)$ is the p.d.f. of the shock ε_t, which we assume to be Gaussian. To estimate the constant parameters, consider the GARCH filter (2.6), in which case the vector of constant parameters reads $\boldsymbol{\theta} = (\mu, \omega, \alpha_1, \alpha_2, \beta)'$. The likelihood function $L(\boldsymbol{\theta})$ is the joint p.d.f. of the returns, that is

$$L(\boldsymbol{\theta}) = f(r_1, r_2, \ldots, r_n; \boldsymbol{\theta}) = \prod_{t=1}^{n} f(r_t|\mathcal{I}_{t-1}; \boldsymbol{\theta}).$$

ML estimation involves finding the parameter vector $\boldsymbol{\theta}$ that maximises (the logarithm of) this function, that is

$$\begin{aligned}
\widehat{\boldsymbol{\theta}}_{\mathrm{ML}} = \operatorname{argmax}_{\boldsymbol{\theta}}\ \log L(\boldsymbol{\theta}) \;&=\; \operatorname{argmax}_{\boldsymbol{\theta}} \sum_{t=1}^{n} \log f(r_t|\mathcal{I}_{t-1}; \boldsymbol{\theta}),\\
&=\; \operatorname{argmax}_{\boldsymbol{\theta}} \sum_{t=1}^{n} \left[-\log(\sigma_t) + \log p\left(\frac{r_t-\mu}{\sigma_t}\right)\right],\\
&=\; \operatorname{argmax}_{\boldsymbol{\theta}} \sum_{t=1}^{n} \frac{1}{2}\left[-\log(2\pi) - \log\left(\sigma_t^2\right) - \frac{(r_t-\mu)^2}{\sigma_t^2}\right].
\end{aligned} \tag{2.8}$$

In general, no closed-form or analytic solutions are available, but 'standard' numerical optimisation procedures can be used to obtain $\widehat{\boldsymbol{\theta}}_{\mathrm{ML}}$. We have provided you with a very basic Matlab demo code, which estimates the simplest GARCH(1,1) model given by equations (2.2), (2.3) and (2.8). This demo code uses the Matlab function `fmincon` to minimise the negative log-likelihood function. Because Matlab likes to minimise rather than maximise, we work with the negative log likelihood. You may use this demo code as your starting point. However, note that no further coding help will be available from the supervisors.

2.4 Predictions

Having investigated how the GARCH-filtered quantity σ_t^2 changes from day to day, we are also interested in d-day-ahead volatility forecasts. Because the terms in square brackets in equation (2.6) have expectation zero for any t, it follows that the d-day ahead prediction of the variance is given by

$$\mathrm{E}_t[\sigma_{t+d}^2] = \sigma^2 + (\alpha+\beta)^{d-1}\,(\sigma_{t+1}^2 - \sigma^2), \quad d \geq 1, \tag{2.9}$$

where E_t is an expectation conditional on the knowledge at time t. You should check for yourself that this equation holds for $d = 1$ or $d = 2$. This equation says that the d-day-ahead forecast mean-reverts to the unconditional variance σ^2 at the geometric rate $\alpha + \beta < 1$. To obtain the variance not for a particular day but for *the sum of days* up to (and including) some day d, we compute the predicted variance (PV) for the next d days as follows:

$$\begin{aligned}
\mathrm{PV}_{t,d} := \mathrm{E}_t\left[\sum_{i=1}^{d} r_{t+i}^2\right] \;&=\; \sum_{i=1}^{d} \mathrm{E}_t\left[(\mu + \sigma_{t+i}\varepsilon_{t+i})^2\right] = d\mu^2 + \sum_{i=1}^{d} \mathrm{E}_t[\sigma_{t+i}^2],\\
&=\; d\mu^2 + \sum_{i=1}^{d}\left[\sigma^2 + (\alpha+\beta)^{i-1}\,(\sigma_{t+1}^2 - \sigma^2)\right],\\
&=\; d(\mu^2+\sigma^2) + \frac{1-(\alpha+\beta)^d}{1-\alpha-\beta}(\sigma_{t+1}^2 - \sigma^2),
\end{aligned} \tag{2.10}$$

where $\alpha = (\alpha_1 + \alpha_2)/2$ allows for asymmetry (i.e., the leverage effect). The first equality uses (2.2), while the second equality uses the fact that the shocks ε_t are i.i.d. with mean zero and unit variance. The third equality uses (2.9) above, while the last line uses the fact $\sum_{i=1}^{d} z^{i-1} = (1-z^d)/(1-z)$ for $0 \leq z < 1$ with $z = \alpha + \beta$.

The variable $\mathrm{PV}_{t,d}$ can be viewed as being a 'pseudo out-of-sample' prediction if the whole data set is used to estimate the constant parameters $(\mu, \omega, \alpha_1, \alpha_2, \beta, \nu)$. Conversely, prediction (2.10) can be viewed as a genuine out-of-sample prediction if the constant parameters are estimated using only data that would have been available at time t. More details can be found in Chapter 7 of Franses et al. (2014).

3 A modern extension of GARCH: Realised GARCH

Classic GARCH uses a single, noisy proxy (the squared close-to-close return) to estimate the current level of volatility. The availability of high-frequency data has led to the development of the Realised GARCH model of Hansen et al. (2011), as discussed next.

3.1 Realised GARCH

The Realised GARCH model introduced by Hansen et al. (2011) is defined by the following equations:

$$
\begin{aligned}
r_t &= \mu + \sigma_t \varepsilon_t, && (3.1)\\
\sigma_{t+1}^2 &= \omega + \alpha (r_t - \mu)^2 + \beta \sigma_t^2 + \gamma x_t, && (3.2)\\
x_t &= \xi + \varphi \sigma_t^2 + \tau(\varepsilon_t) + u_t, && (3.3)
\end{aligned}
$$

where x_t is the realised variance on day t, which is the variable rv5_ss$_t$ in our case. The first and second equations are exactly as before, except for the additional term γx_t in the second equation. Everything on the right-hand side is evaluated at time t, such that σ_{t+1}^2 can be computed at t. We impose $\omega, \alpha, \beta, \gamma \geq 0$ to ensure $\sigma_t^2 \geq 0$ for all t. As before, we have $\varepsilon_t \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0,1)$.

The third equation is known as the 'measurement equation', because the observed realised variance x_t can be interpreted as a 'measurement' of σ_t^2, where $u_t \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma_u^2)$ is the 'measurement noise'. Here, $\tau(\varepsilon_t)$ is the leverage function, which captures the dependence between returns and future volatility. We follow Hansen et al. (2011) in taking $\tau(\varepsilon_t) = \tau_1 \varepsilon_t + \tau_2(\varepsilon_t^2 - 1)$, which can generate an asymmetric response in volatility to return shocks. Because σ_t^2 is the variance of close-to-close returns, while x_t is the realised variance constructed from five-minute returns between the market open and close, we expect $\varphi < 1$.

The ten parameters of the model are $(\mu, \omega, \alpha, \beta, \gamma, \xi, \varphi, \tau_1, \tau_2, \sigma_u)$. The Realised GARCH framework contains as a special case the classic GARCH model, which can be obtained by setting $\gamma = 0$.

3.2 Estimation of constant parameters

The vector of constant parameters in the realized GARCH model consists of $\boldsymbol{\theta} = (\mu, \omega, \alpha, \beta, \gamma, \xi, \varphi, \tau_1, \tau_2, \sigma_u)'$. Similarly to the classic GARCH, these parameters can be estimated using maximum likelihood. The main difference, however, is that now the likelihood function depends not only on the return r_t, but also on the realised variance x_t. That is, we consider the joint p.d.f. $f(r_t, x_t|\mathcal{I}_{t-1})$, such that the likelihood function is

$$L(\boldsymbol{\theta}) = f(\{r_t, x_t\}_{t=1}^n; \boldsymbol{\theta}) = \prod_{t=1}^n f(r_t, x_t|\mathcal{I}_{t-1}; \boldsymbol{\theta}).$$

To write the log-likelihood function, we can factorize the joint p.d.f. $f(r_t, x_t|\mathcal{I}_{t-1})$ into the marginal p.d.f. of r_t and the conditional p.d.f. of x_t given r_t:

$$f(r_t, x_t|\mathcal{I}_{t-1}; \boldsymbol{\theta}) = f(r_t|\mathcal{I}_{t-1}; \boldsymbol{\theta}) f(x_t|r_t, \mathcal{I}_{t-1}; \boldsymbol{\theta}).$$

The ML estimator for the parameter vector $\boldsymbol{\theta}$ then reads

$$
\begin{aligned}
\widehat{\boldsymbol{\theta}}_{\text{ML}} &= \operatorname{argmax}_{\boldsymbol{\theta}} \sum_{t=1}^n \log f(r_t, x_t|\mathcal{I}_{t-1}; \boldsymbol{\theta}),\\
&= \operatorname{argmax}_{\boldsymbol{\theta}} \sum_{t=1}^n \left[\log f(r_t|\mathcal{I}_{t-1}; \boldsymbol{\theta}) + \log f(x_t|r_t, \mathcal{I}_{t-1}; \boldsymbol{\theta})\right],\\
&= \operatorname{argmax}_{\boldsymbol{\theta}} \underbrace{\sum_{t=1}^n \frac{1}{2}\left[-\log(2\pi) - \log\left(\sigma_t^2\right) - \frac{(r_t-\mu)^2}{\sigma_t^2}\right]}_{l(r)} + \underbrace{\sum_{t=1}^n \frac{1}{2}\left[-\log(2\pi) - \log\left(\sigma_u^2\right) - \frac{u_t^2}{\sigma_u^2}\right]}_{l(x|r)}. \qquad (3.4)
\end{aligned}
$$

The first term in (3.4) is the partial log-likelihood of the returns, $l(r)$, for which the calculation is analogous to that for the classic GARCH model. Importantly, this term depends *only* on the parameters $(\mu, \omega, \alpha, \beta, \gamma)$ because σ_t^2 depends on these parameters via equation (3.2). The second term, $l(x|r)$, is obtained by noting that, given $(r_t, \sigma_t, \varepsilon_t)$, the term $u_t = x_t - \xi - \varphi\sigma_t^2 - \tau_1\varepsilon_t - \tau_2(\varepsilon_t^2 - 1)$ follows a normal distribution with mean zero and variance σ_u^2. Hence the second term depends directly on the parameters $(\xi, \varphi, \tau_1, \tau_2, \sigma_u)$, and indirectly on the parameters $(\mu, \omega, \alpha, \beta, \gamma)$, which affect σ_t^2 and ε_t^2 for all t. Before attempting the ten-dimensional optimisation problem (3.4), a pragmatic approach is to optimise the first term with respect to $(\mu, \omega, \alpha, \beta, \gamma)$. Based on these parameters, we can compute σ_t^2 and ε_t^2 for all t. Then we can regress x_t on a constant, σ_t^2, ε_t and $\varepsilon_t^2 - 1$ to obtain reasonable starting values for $\xi, \varphi, \tau_1, \tau_2$. Having obtained these starting values, you can attempt the full optimisation problem (3.4). The partial log-likelihood $l(r)$ evaluated at the optimal parameters can be compared against the log-likelihood for GARCH.

3.3 Predictions

As in classic GARCH, the variance σ^2_{t+1} is known at time t, while σ^2_{t+2} is random at time t. For $d \geq 2$, we can compute the d-day-ahead expectation of the daily variance as follows:

$$
\begin{aligned}
\mathrm{E}_t[\sigma^2_{t+d}] \;&=\; \omega \,+\, \alpha\,\mathrm{E}_t[(r_{t+d-1}-\mu)^2] \,+\, \beta\,\mathrm{E}_t[\sigma^2_{t+d-1}] \,+\, \gamma\,\mathrm{E}_t[x_{t+d-1}], \\
&=\; \omega \,+\, \alpha\,\mathrm{E}_t[\sigma^2_{t+d-1}\,\varepsilon^2_{t+d-1}] \,+\, \beta\,\mathrm{E}_t[\sigma^2_{t+d-1}] \,+\, \gamma\,(\xi + \varphi \mathrm{E}_t[\sigma^2_{t+d-1}]), \\
&=\; \omega \,+ \gamma\,\xi \;+\, (\alpha+\beta+\gamma\varphi)\,\mathrm{E}_t[\sigma^2_{t+d-1}], \qquad d \geq 2,
\end{aligned} \tag{3.5}
$$

Assuming $\alpha + \beta + \gamma\,\varphi < 1$, we can define

$$
\sigma^2 := \frac{\omega + \gamma\,\xi}{1-\alpha-\beta-\gamma\,\varphi}, \tag{3.6}
$$

which is a generalisation of equation (2.4). We use definition (3.6) to rewrite equation (3.5) as

$$
\mathrm{E}_t[\sigma^2_{t+d}] = \sigma^2 + (\alpha+\beta+\gamma\varphi)\big(\mathrm{E}_t[\sigma^2_{t+d-1}] - \sigma^2\big), \qquad d \geq 2.
$$

Applying this formula for $d = 2$ and $d = 3$, we obtain

$$
\begin{aligned}
&\mathrm{E}_t[\sigma^2_{t+1}] = \sigma^2_{t+1}, && d = 1, \\
&\mathrm{E}_t[\sigma^2_{t+2}] = \sigma^2 + (\alpha+\beta+\gamma\varphi)(\sigma^2_{t+1} - \sigma^2), && d = 2, \\
&\mathrm{E}_t[\sigma^2_{t+3}] = \sigma^2 + (\alpha+\beta+\gamma\varphi)(\mathrm{E}_t[\sigma^2_{t+2}] - \sigma^2) = \sigma^2 + (\alpha+\beta+\gamma\varphi)^2(\sigma^2_{t+1} - \sigma^2), && d = 3.
\end{aligned}
$$

These relations can be collectively summarised as

$$
\mathrm{E}_t[\sigma^2_{t+d}] = \sigma^2 + (\alpha+\beta+\gamma\varphi)^{d-1}\big(\sigma^2_{t+1} - \sigma^2\big), \qquad d \geq 1,
$$

which holds for any $d \geq 1$ and generalises equation (2.9) for classic GARCH. By the same arguments as before, the cumulative d-day-ahead predicted variance (PV) equals

$$
\begin{aligned}
\mathrm{PV}_{t,d} := \mathrm{E}_t\left[\sum_{i=1}^{d} r^2_{t+i}\right] \;&=\; \sum_{i=1}^{d} \mathrm{E}_t\left[(\mu + \sigma_{t+i}\,\varepsilon_{t+i})^2\right] = d\mu^2 + \sum_{i=1}^{d} \mathrm{E}_t[\sigma^2_{t+i}], \\
&=\; d\mu^2 + \sum_{i=1}^{d}\left[\sigma^2 + (\alpha+\beta+\gamma\,\varphi)^{i-1}\,(\sigma^2_{t+1} - \sigma^2)\right], \\
&=\; d(\mu^2+\sigma^2) + \frac{1-(\alpha+\beta+\gamma\,\varphi)^d}{1-\alpha-\beta-\gamma\,\varphi}(\sigma^2_{t+1} - \sigma^2),
\end{aligned} \tag{3.7}
$$

which is a direct generalisation of (2.10), where σ^2 is now given by equation (3.6).

4 Evaluating model forecasts

To evaluate the accuracy of GARCH predictions and Realised GARCH predictions, we consider two 'target variables' as follows:

1. **Target variable 1: Target variance (TV) computed from daily returns**. Since we have access to daily returns, we can compute the target variance over the next d days when viewed from day t as follows:

$$
\mathrm{TV}_{t,d} = \sum_{i=1}^{d} r^2_{t+i}, \tag{4.1}
$$

   where r_{t+i} is the close-to-close log return on day $t+i$. While the (forward-looking) target variable $\mathrm{TV}_{t,d}$ is not available on day t, it represents our *target* in the sense that we would like to predict it on day t.

2. **Target variable 2: Target variance (TV) computed from intra-day returns**. Alternatively, we may use the Oxford library of realised variances, which contains daily realised-variance measures obtained from intra-day returns. While this measure excludes overnight returns, experience suggests that around seventy percent of the total (daily and overnight) variance is realised during the day. For this reason, we

'scale up' the Oxford realised variance by a factor $1/0.7 \approx 1.4$.[4] The target variance over a d-day horizon can thus be defined as

$$\mathrm{TV}_{t,d} = 1.4 \times \sum_{i=1}^{d} \mathrm{rv5\_ss}_{t+i}, \tag{4.2}$$

where $\mathrm{rv5\_ss}_{t+i}$ is realised variance of intra-day returns on day $t+i$. While the precise definition rv5_ss can be found here, for the purpose of this assignment you may simply take this variable as given.

The advantage of target variable 1, which uses close-to-close log returns, is that it includes both daily and overnight volatility. The disadvantage is that much information is potentially ignored by using only closing prices. The advantage of target variable 2, which uses intra-day returns, is that it accurately estimates the volatility during the opening hours of the market. The disadvantage is that overnight volatility is only artificially included via the *ad hoc* multiplicative factor of 1.4.

We now have two models (GARCH and Realised GARCH) and two target variables. Moreover, we are interested in three horizons, namely $d = 1$, $d = 5$ and $d = 21$ trading days, which correspond to roughly to one calendar day, one calendar week and one calendar month, respectively. As our benchmark models, we consider the VIX and a simple autoregressive model for realised volatility, as discussed next.

4.1 Benchmark model 1: The VIX

The VIX intends to predict the (annualised) standard deviation of the return of the S&P500 index in the next month. Based on the VIX, the predicted variance at time t over a d-day horizon is

$$\mathrm{PV}_{t,d} = d/250 \times \mathrm{VIX}_t^2. \tag{4.3}$$

As the VIX squared can be interpreted as an annualised variance, we multiply by $d/250$ to account for the d-day horizon. The factor of 250 originates from the fact that there are roughly 250 days per year on which the market is open.

4.2 Benchmark model 2: The HAR-RV model

Corsi (2009) proposes an heterogeneous autoregressive realised variance (HAR-RV) model that directly models the target variable that we are interested in. He models the target variable as a linear function of several one-period lagged realised variances, which are measured at different (i.e., heterogeneous) frequencies, explaining the name of the model.

Specifically, the realised variances in the HAR-RV model correspond to the daily, weekly and monthly frequencies. The daily realised variance, denoted $\mathrm{RV}_t^{(d)}$, is simply $\mathrm{rv5\_ss}_t$. The weekly and monthly realised variances, denoted $\mathrm{RV}_t^{(w)}$ and $\mathrm{RV}_t^{(m)}$, respectively, equal the *average* realised variance over these periods, i.e.

$$\text{weekly:} \qquad \mathrm{RV}_t^{(w)} = \frac{1}{5}\left(\mathrm{RV}_t^{(d)} + \mathrm{RV}_{t-1}^{(d)} + \ldots + \mathrm{RV}_{t-4}^{(d)}\right),$$

$$\text{monthly:} \qquad \mathrm{RV}_t^{(m)} = \frac{1}{21}\left(\mathrm{RV}_t^{(d)} + \mathrm{RV}_{t-1}^{(d)} + \ldots + \mathrm{RV}_{t-20}^{(d)}\right).$$

The HAR-RV model is implemented by regressing via ordinary least squares (OLS) the target variable $\mathrm{TV}_{t,d}$ onto the three realised-variance components as follows:

$$\mathrm{TV}_{t,d} = c_d + \beta_d^{(d)}\mathrm{RV}_t^{(d)} + \beta_d^{(w)}\mathrm{RV}_t^{(w)} + \beta_d^{(m)}\mathrm{RV}_t^{(m)} + e_t, \tag{4.4}$$

where the error e_t satisfies the appropriate assumptions. The OLS output (estimates of $c_d, \beta_d^{(d)}, \beta_d^{(w)}, \beta_d^{(m)}$) will differ for different d-day-ahead prediction horizons, as indicated by the subscripts. If we use data before date t to estimate the constant parameters, we can compute the out-of-sample predicted variance (PV) over a d-day horizon as follows:

$$\mathrm{PV}_{t,d} = \hat{c}_d + \hat{\beta}_d^{(d)}\mathrm{RV}_t^{(d)} + \hat{\beta}_d^{(w)}\mathrm{RV}_t^{(w)} + \hat{\beta}_d^{(m)}\mathrm{RV}_t^{(m)}, \tag{4.5}$$

where hats denote OLS estimates based on data that would have been available at time t.

In sum, you now have four predicted variances (PVs), given by equations (2.10), (3.7), (4.3), and (4.5), two target variances (TVs), given in equations (4.1) and (4.2), and three horizons ($d = 1, 5, 21$). For any given loss function (e.g. MSE or QLIKE), this allows you compute $4 \times 2 \times 3 = 24$ loss values.

[4] This is consistent with equation (1) in Hansen and Lunde (2005), because the factor 1.4 brings the average of target variable 2 in line with that of target variable 1.

5 Roadmap

This section contains a logical sequence of modelling steps that you may consider. Note, however, that this is *not* how you should write your report. It is up to you to decide which analyses to perform. When you have completed (some or all of) the modelling steps presented below, you will have to decide which results to report and which to leave out.

1. **In-sample analysis.** Estimate four models by maximum likelihood (ML) using the entire data set:

   (a) Symmetric GARCH (SGARCH), i.e. set $\alpha = \alpha_1 = \alpha_2$. You can use the demo code for this purpose

   (b) Asymmetric GARCH (AGARCH), i.e. allow $\alpha_1 \neq \alpha_2$

   (c) Symmetric Realised GARCH (SRGARCH), i.e. set $\tau(\varepsilon_t) = \tau_2(\varepsilon_t^2 - 1)$

   (d) Asymmetric Realised GARCH (ARGARCH), i.e. set $\tau(\varepsilon_t) = \tau_1\varepsilon_t + \tau_2(\varepsilon_t^2 - 1)$

   In a single table, report parameter estimates for all models and corresponding (possibly partial) log likelihoods. For parameter estimates, report 3 decimal places. For log likelihoods, one decimal place is enough. You may include additional statistics if you want.

2. Of the two GARCH models, which do you think is best? Of the two Realised GARCH models, which do you think is best? Use statistical arguments. From now on, consider only your favourite GARCH model and your favourite Realised GARCH model.

3. For your two favourite models, discuss the difference in parameter estimates. As much as possible, draw economic (not econometric) conclusions.

4. **Out-of-sample analysis.** Use only the first x% of the data to estimate the constant parameters, e.g. 50%. Alternatively, you may also use an expanding window or moving window to estimate the constant parameters. In the out-of-sample period, construct volatility predictions for $d = 1$, $d = 5$ and $d = 21$ days ahead, for the HAR-RV, VIX and your favourite (realised) GARCH models, leading to $4 \times 3 = 12$ series of predictions.

5. Compare your 12 series of predictions against both possible targets using the MSE and QLIKE robust loss functions of Patton (2011). This means you have 4 models (GARCH, Realised GARCH, VIX, HAR-RV), 3 horizons, 2 targets, and at least 2 loss functions, leading to at least $4 \times 3 \times 2 \times 2 = 48$ numbers. Organise your results in such a way that they can be easily read and interpreted. Pretty tables are important; it will be detrimental to your grade if assessors have trouble understanding them.

6. **Bonus question**: In equation (3.2), can you think of ways to decompose the term involving α into two parts involving α_1 (for negative returns) and α_2 (for positive returns). Similarly, can you decompose the term involving γ into two parts involving γ_1 (for negative returns) and γ_2 (for positive returns)? When this change is made, equation (3.7) will have to be adjusted (you need to re-do the derivation). For the purpose of making predictions, is this additional effort worth the trouble?

7. **Bonus question:** Does any model *significantly* outperform the others in making predictions?

8. **Conclusions.** In the conclusion of your report, you may address the provocative question raised by Hansen and Lunde (2005): "Does anything beat a GARCH(1,1)?". They conclude that the answer is essentially "no" as long as you allow for asymmetric GARCH as in equation (2.6). Do you agree with their assessment?

9. *Note for the final version:* Your reference list should be carefully edited. Please see the APA reference style used at the end of this case study: each reference has a journal name with appropriate capitalisation, as well as volume, issue and page numbers. Book titles are capitalised differently from journal titles. The year goes in brackets after the authors. Authors have initials, not full names. References are listed alphabetically, not numerically. Hyperlinks are useful but not mandatory.

6 Econometric software

You may use any software package, such as EVIEWS, Matlab, R, etc. To encourage you to write your own code, a basic demo code for estimating a simple GARCH(1,1) model is provided in Matlab. No additional coding help will be available from the supervisors.

7 Proposal

The research proposal should be no more than two pages and is due by 20:00 on Tuesday 31 May. Given the short time-frame of this case study, we suggest that the research proposal should contain:

1. Output for steps 1 through 3 of the Roadmap in Section 5.
2. Intended output for the remaining steps in the Roadmap, e.g. tables. These tables may still be *empty*, i.e. contain no numbers, but the rows and columns should be clearly labelled, so that supervisors get a sense of the output you intend to produce. The table should enable the reader to make interesting comparisons. Ask yourself: is it meaningful to compare horizons, loss functions, targets or models? Depending on the answer, some table lay-outs are more appropriate than others.

Hint: Use the proposal to sketch your intended output, *not* to summarise the case; the supervisors know the case already. You will receive verbal feedback on your proposal on Wednesday 1 June. Proposals that do not contain the minimum requirements will receive minimal feedback.

8 Second meeting

A second meeting with one of the supervisors will take place on Wednesday 8 June. You will be given the opportunity to ask a minimum of three and a maximum of five questions.

9 Final report

The deadline for the final report is Friday 17 June 18:00. Late reports will not be graded. Reports are to be written in Dutch or English for FEB22009 students and in English for FEB22009X, FEB22009Q and FEB22009S students. For the word limit and lay-out, see section 4 in the Course Information Document.

Appendices are allowed but should not contain important information; we probably won't read them. The focus of your report should be on presenting your findings and explaining them in econometric and/or economic terms. The report is *not* meant to summarise the case; e.g. you need not replicate all formulas or derivations. Since GARCH is a special case of Realised GARCH, you may present only the latter, noting simply that the former is a special case corresponding to $\gamma = 0$. Reports should be in line with common rules and practice in academic publishing as taught in the skills course. You are strongly encouraged to use LaTeX.

10 Summary of deadlines

1. Proposal: 20:00, Tuesday 31 May. You will receive verbal feedback on Wednesday 1 June.
2. Second meeting with one of the supervisors on Wednesday 8 June.
3. Final report (version 1): 23:59, Monday 13 June
4. Peer review: 11:59 (i.e., noon!), Wednesday 15 June
5. Final report (version 2): 18:00, Friday 17 June

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