Shill-Proof Auctions*

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Abstract

In a single-item auction, a duplicitous seller may masquerade as one or more bidders in order to manipulate the clearing price. This paper characterizes auction formats that are *shill-proof*: a profit-maximizing seller has no incentive to submit any shill bids. We distinguish between *strong* shill-proofness, in which a seller with full knowledge of bidders' valuations can never profit from shilling, and *weak* shill-proofness, which requires only that the expected equilibrium profit from shilling is nonpositive. The Dutch auction (with suitable reserve) is the unique optimal and strongly shill-proof auction. Moreover, the Dutch auction (with no reserve) is the unique prior-independent auction that is both efficient and weakly shill-proof. While there are a multiplicity of strategy-proof, weakly shill-proof, and optimal auctions; any optimal auction can satisfy only two properties in the set $\{\text{static}, \text{strategy-proof}, \text{weakly shill-proof}\}$.

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1 Introduction

1.1 Shill Bidding in Auctions

Shill bidding in practice. Auction theory typically assumes that an auction is carried out as described (by the seller or a third party) and focuses solely on the bidders' incentives. Reality is often different. For example, while major auction houses like Christie's or Sotheby's may appear to be carrying out textbook English (ascending) auctions, a degree of skullduggery is often afoot. According to a *New York Times* article from 2000:

> Some tricks of the trade, like an auctioneer's drumming up excitement by acknowledging nonexistent bids only he hears and potential buyers who bid with nearly imperceptible secret signals, have been around for decades. Making up bids, for instance, is known as "bidding off the chandelier" from an era when the grand auction rooms were adorned with ornate lighting.[1]

The practice continues to this day: Christie's Conditions of Sale for their flagship New York location, in a section titled "Auctioneer's Discretion," states (among other things) that "The auctioneer can... move the bidding backwards or forwards in any way he or she may decide. . . ."[2]

Such chandelier bids or *shill* bids—bids submitted by the seller in order to manipulate the final selling price—appear to be particularly common in online auctions. For example, eBay has long gone out of its way to emphasize that shill bidding is forbidden and will be punished:

> We want to maintain a fair marketplace for all our users, and as such, shill bidding is prohibited on eBay... eBay has a number of systems in place to detect and monitor bidding patterns and practices. If we identify any malicious behavior, we'll take steps to prevent it.[3]

According to many eBay users, however, shill bidding remains rampant. Here's a sample quote from the eBay discussion forums:

> The Sellers post a Buy Now price 3–4 times the actual cost of the item. Then they place the item on an auction at $0.01. This to get as many views as possible. The shill comes in shortly after the auction starts and . . . is there to prevent the item from being sold below their profit margin.[4]

Chen et al. (2020) find that nearly 10% of all bidders on eBay Motors are shill bidders.

[1] See *Genteel Auction Houses Turning Aggressive*, New York Times, April 24, 2000.

[2] See https://www.christies.com/help/buying-guide-important-information/conditions-of-sale.

[3] See https://www.ebay.com/help/policies/selling-policies/selling-practices-policy/shill-bidding-policy?id=4353.

[4] See https://community.ebay.com/t5/Buying/My-experience-with-Shill-bidders/td-p/30402514.

Shill-proof auctions. Much of auction theory to date encourages truthful bidding through careful auction design, while punting on challenges like seller deviations and collusion via appeal to ummodeled concepts such as "rule of law." Anecdotes about eBay and other online platforms suggest that such methods are only partially effective at deterring such deviations. Thus, it makes sense to ask: To what extent can these deviations instead be disincentivized through an auction's design?

The goal of this paper is to understand which auction formats are "shill-proof" in the sense that a seller cannot profit through the submission of shill bids. For private-value auctions, which is our focus here, the reader might well wonder why shill bids matter at all—assuming that the choice of reserve price doesn't affect participation (as it does in the eBay example), isn't a shill bid the same thing as a reserve price?

The answer depends on when the seller has an opportunity to shill and the information available to them at that time. For example, consider an English auction in which the seller also participates as a shill bidder. Suppose the valuations of the (real) bidders are drawn i.i.d. from a regular distribution F and that the opening bid of the auction is set optimally (for revenue), to the monopoly price ρ^* of F. As the auction proceeds, with the offered price p starting at F and increasing from there (in increments of ϵ, say), the seller can shill bid at any time. Suppose that the only additional information known to the seller at a given round of the auction is that the remaining bidders are willing to pay at least p. Then, the seller asks themself: "now that I know how many bidders are willing to pay at least p, do I want to shill and reset the reserve price to $p+\epsilon$?" It turns out that, under our assumption that F is regular, the answer is "no," and an expected revenue–maximizing seller will never shill.[5] (This doesn't necessarily mean that the auctioneers at Christie's are acting suboptimally, as bidders' valuations in art auctions might be strongly positively correlated (Milgrom and Weber, 1982).)

Now suppose that the seller has full knowledge of bidders' realized valuations. In this scenario, the seller will certainly in some cases want to shill in an English auction to push the price up to just below the highest of the bidders' valuations. Lest this informational assumption—that the seller knows the full valuation profile—seem impossibly demanding, consider the Dutch (descending) auction (with an arbitrary reserve price). Here, any shill bid by the seller terminates the auction immediately, leaving the seller holding the item and earning zero revenue.

We map out a theory of "shill-proof" auctions, focusing on the following basic questions:

- Which auction formats are "strongly shill-proof" in the sense of the Dutch auction, i.e., with shill bidding being unprofitable even with full knowledge of bidders' realized valuations?
- Which auction formats are "weakly shill-proof" in the sense of the English auction (with bidders' valuations drawn i.i.d. from a regular distribution and an optimally chosen reserve price), i.e., with shill bidding being unprofitable in expectation at equilibrium?
- To what extent are strong and weak shill-proofness compatible with other desirable

[5] Auction theory experts will now immediately recognize that the English auction with an optimally chosen reserve price is *not* generally shill-proof in this sense when the valuation distribution is not regular.

properties such as optimality, efficiency, strategy-proofness, and sealed-bid implementation?

1.2 Overview of Results

Iterative auction formats like Dutch and English auctions play a central role in our theory, and accordingly we study (real and shill) bidding in the extensive-form game that is induced by a choice of auction format, relying on a framework for extensive-form auction analysis developed by Li (2017) and Akbarpour and Li (2020). We consider single-item auctions with N bidders. A subset of these are shill bidders, which we model as bidders with zero private value for the item and with utility equal to the seller's revenue.[6] We assume that the private valuations of the non-shill bidders are drawn i.i.d. from a known distribution that is regular (see Definition 2.2). An auction is then *weakly shill-proof* (Definition 2.4) if there exists an equilibrium of the induced extensive-form game in which the shill bidders never shill (i.e., bid their true private value of 0). An auction is *strongly shill-proof* (Definition 2.5) if, moreover, shill bidders' equilibrium strategies are dominant strategies. Except when we restrict attention to static auctions (as in the fourth result below), we focus on *public* auctions (Definition 2.6), meaning auctions in which every bidder's action is publicly observable. This is arguably the most natural model for the analysis of typical iterative auctions such as Dutch and English auctions.

Next, we summarize the main results of this paper; see also Figure 1.

Strongly shill-proof auctions. Our first result (Theorem 3.5) is a uniqueness result for strongly shill-proof auctions: the Dutch auction (with consistent tie-breaking and reserve price equal to the valuation distribution's monopoly price) is strongly shill-proof and optimal (i.e., maximizes the seller's expected revenue), and it is the *only* such auction in the public setting. In particular, strongly shill-proof optimal auctions must be iterative (with a possibly large number of rounds) and cannot be strategy-proof. The rough intuition for the proof of this result is that: (i) for any auction format other than a Dutch auction, there exists a history in which some bidder i can indicate that her value is strictly larger than the reserve price without the auction ending immediately; (ii) optimality in tandem with the public setting then implies that this information effectively induces the auction to revise its reserve price upward; and thus (iii) there exist valuations for the bidders such that, if bidder i happens to be a shill bidder, shilling will increase the seller's revenue.

Weakly shill-proof and efficient auctions. Our remaining results concern the richer design space of weakly shill-proof auctions (which contains, at the least, both Dutch and English auctions). We start by investigating *efficient* (and weakly shill-proof) public auctions in which, at equilibrium, the item is always awarded to the (real) bidder with the highest valuation. One example of such an auction is a Dutch auction with a reserve price of 0.

[6] The prior literature has sometimes modeled shill bidding via an unknown number of bidders, with some subset of the bidders who end up participating in the auction being shills. Our framework is essentially equivalent: we can take N to be large and require 0 to be in the support of the valuation distribution; and a bidder with value 0 is equivalent (in terms of outcomes) to a bidder not arriving.

Interestingly, English auctions are not examples: with a non-zero reserve, such an auction is not efficient; with a zero reserve, it is not weakly shill-proof (as shill bidders are motivated to push the price up to the monopoly price). But the Dutch auction is not the only efficient and weakly shill-proof auction: beginning with an English auction at the monopoly price and then, should there be no takers, concluding with a Dutch auction (with no reserve) is another example. (In fact, this auction format closely resembles the Honolulu–Sydney fish auction documented by Hafalir et al. (2023).) That this "hybrid" auction format concludes with a Dutch auction is no accident: we prove (in Theorem 3.9) that every robustly weakly shill-proof and efficient auction must conclude with a Dutch auction when all bidders' valuations fall below the monopoly price ρ^* of the distribution. (Here "robustly" means that the auction specification should depend only on the monopoly price ρ^* of the valuation distribution, and not on its more fine-grained details.) In particular, no strategy-proof auction can be robustly weakly shill-proof and efficient. Compared to Theorem 3.5, the key challenge in the proof of Theorem 3.9 is that shill bidders must now respond to expectations over real bidders' valuations as opposed to conditioning on the valuations themselves. Nevertheless, we can prove that, should an auction deviate from a Dutch auction by giving some bidder an opportunity to signal that their valuation is greater than 0 (without ending the auction), in the event that all bidders' valuations fall below the given monopoly price ρ^*, there exists a regular valuation distribution with monopoly price ρ^* for which the auction is not weakly shill-proof (Corollary 3.11). It follows that the Dutch auction is the unique prior-independent auction (in the sense of Dhangwatnotai et al. (2015), with no dependence whatsoever on the valuation distribution) that is both efficient and weakly shill-proof.

Weakly shill-proof and strategy-proof optimal auctions. The previous two results imply that strategy-proof and public auctions cannot be both strongly shill-proof and optimal, nor can they be (robustly) weakly shill-proof and efficient. The English auction (with an optimally chosen reserve price) is, as we've noted, weakly shill-proof, optimal, and strategy-proof. Is it the unique such auction? Does this combination of properties require a potentially large number of rounds? Our next result (Theorem 4.7) shows that, in general, the answer is no: with a valuation distribution that satisfies the monotone hazard rate condition (Definition 4.5), interrupting an English auction (with an optimally chosen reserve price) after a sufficiently large number of rounds and closing it out with a second-price auction among the remaining bidders is also weakly shill-proof, optimal, and strategy-proof. For certain valuation distributions, switching from an English auction to this alternative format leads to a dramatic reduction in the number of auction rounds that is required in the worst case. (The next result implies that the number of rounds cannot be reduced to 1.) The technical challenge here is in proving that weak shill-proofness obtains, in the public setting, even during the closing second-price auction.[7]

Shill-proof and strategy-proof static auctions. Our last main result (Theorem 5.2), unlike the first three, focuses on *static* or *one-shot* auctions, meaning auction formats that

[7] In the public setting, a "second-price auction" is implemented by having bidders publicly announce their bids one by one in some order, and thus shill bidders may have the opportunity to submit a bid after learning some of the bids by the real bidders.

	Static	Not Static
Strategy-Proof	Impossible (Theorem 5.2)	Ascending, Screening Auction (Theorem 4.7)
Not Strategy-Proof	First-Price Auction (Example 2.7)	Dutch Auction (Theorem 3.5)

(a) Weakly shill-proof and optimal auctions

	Static	Not Static
Strategy-Proof	Impossible (Theorem 5.2)	Not Robustly (Theorem 3.9)
Not Strategy-Proof	Not Robustly (Theorem 3.9)	Dutch Auction (Robustly Unique, Theorem 3.9)

(b) Weakly shill-proof and efficient auctions

	Static	Not Static
Strategy-Proof	Impossible (Theorem 3.5)	Impossible (Theorem 3.5)
Not Strategy-Proof	Impossible (Theorem 3.5)	Dutch Auction (Unique, Theorem 3.5)

(c) Strongly shill-proof and optimal auctions

	Static	Not Static
Strategy-Proof	Impossible (Theorem 3.9)	Impossible (Theorem 3.9)
Not Strategy-Proof	Impossible (Theorem 3.9)	Dutch Auction (Unique, Theorem 3.9)

(d) Strongly shill-proof and efficient auctions

Figure 1: Summary of results. Characterization of single-item auction formats that are strongly or weakly shill-proof, along with other properties such as optimality, efficiency, strategy-proofness, and sealed-bid implementations.

induce extensive-form games in which each bidder moves exactly once. Accordingly, we also consider for this result a more general model of the information available to a real bidder, in the form of an "experiment" that outputs a garbling of the previous bidders' actions. At one extreme point of this model, the experiment reports nothing and thus, from a real bidder's perspective, the auction is equivalent to a sealed-bid auction. The other extreme, in which the experiment reports all the previous bidders' actions, corresponds to the public setting. (Shill bidders continue to be perfectly informed, no matter what the experiment.) Here, in contrast to the prior result, we prove that, for every choice of experiment, no one-shot auction can simultaneously be weakly shill-proof, optimal, and satisfy even a very weak strategy-proofness condition (see Definition 5.1). Thus, an optimal auction can satisfy two and only two of the properties in the set $\{\text{static}, \text{strategy-proof}, \text{weakly shill-proof}\}$.[8] The rough idea of the proof is to consider some bidder i other than the last one to move. Weak shill-proofness implies that i's payment (conditioned on winning) should be independent of what later bidders do. Strategyproofness dictates that i's payment (conditioned on winning) should be independent of her own bid. But then, for any valuation profile in which bidder i's valuation is higher than every bidder who moved before her, bidder i is better off submitting the maximum-possible bid (to maximize her probability of winning) than bidding her true value (and thus the auction cannot be strategy-proof).

1.3 Related Work

While the idea and practice of shill bidding by a seller have long been well known, the auction theory literature on the topic is surprisingly thin. Chakraborty and Kosmopoulou (2004) consider common value auctions and focus on technological barriers (as opposed to auction formats) that can mitigate shill bidding. Lamy (2009) studies shill bidding specifically in English auctions in which bidders' valuations are affiliated in the sense of Milgrom and We-

[8]Assuming a regular valuation distribution and a corresponding optimal reserve price, a second-price auction is optimal, static, and strategy-proof; a first-price auction is optimal, static, and weakly shill-proof (see also Example 2.7); and an English auction is optimal, strategy-proof, and weakly shill-proof.

ber (1982), and proves that shill bidding effectively cancels out the effects of affiliation in equilibrium due to real bidders conditioning on bids being fake (see also Izmalkov (2004)). Porter and Shoham (2005) consider a model similar to public second-price auctions, motivated by "cheating" by online platforms that can announce a manipulated auction outcome subsequent to collecting all of the bidders' bids. More recently, a number of works (e.g., Basu et al. (2023); Chung and Shi (2023); Lavi et al. (2022); Roughgarden (2021)) have considered shill bidding in the context of blockchain transaction fee mechanism design, with an emphasis on knapsack auctions that are strategy-proof, shill-proof, and robust to various forms of collusion. Finally, Ausubel and Milgrom (2006) and Day and Milgrom (2008) consider shill bids by *bidders* in a multi-item auction, who are looking to exploit complementarities to lower their payments in VCG-type mechanisms—as opposed to shill bids by a seller looking to increase revenue, as is the case of this paper.

Our theory of shill-proof auctions is similar in spirit to the theory of credible mechanisms developed by Akbarpour and Li (2020), and leverages their framework for extensive-form auction analysis. That said, our shill-proofness concepts are generally incomparable to the Akbarpour and Li (2020) credibility concept. However, in the case of one-shot auctions (Section 5), we prove in the Appendix that, for every experiment, strong shill-proofness implies credibility, and credibility implies weak shill-proofness. Moreover, for one-shot auctions, weak shill-proofness characterizes credibility in the public setting (Proposition D.8), while strong shill-proofness characterizes it under the null experiment (Proposition D.9). More recent research on credible mechanisms, usually with a focus on evading the impossibility results of Akbarpour and Li (2020) under extra assumptions (such as adding cryptographic tools), includes the work of Essaidi et al. (2022), Ferreira and Weinberg (2020), and Chitra et al. (2023).

More distantly related papers include that of Haupt and Hitzig (2021), who prove a uniqueness result for the Dutch auction under contextual privacy constraints; and Mackenzie (2020) and Mackenzie and Zhou (2022), who, like us, prove revelation principles for extensive-form games in a public setting.

2 Model

In this paper, we consider extensive-form, single item auctions. An extensive-form game G is a tuple of possible histories H, and, for each history $h \in H$, functions mapping h to:

(i) a player taking an action, $P(h)$;

(ii) a set of possible actions $A(h)$;

(iii) an information set,[9] $\mathcal{I}(h)$; and

(iv) the most recent action taken, $\mathcal{A}(h)$.

[9]The information sets induce a partition over all possible histories. Players' actions can only condition on $\mathcal{I}(h)$, not on h.

As further notation, we denote the starting history of the game by $h_\varnothing$ and the set of terminal histories as Z; we say $h' \prec h$ if h' precedes h, i.e., there exists a sequence of actions that lead from h' to h.

We restrict attention to single item auctions, which means that for every terminal history $z \in Z$, we can associate an allocation and transfer vector: $z = (x, t)$, with $\sum_{i=1}^N x_i \leqslant 1$ and $x_i \in \{0, 1\}$ for all i. As abuse of notation, we will use $x(z), t(z)$ to mean the vectors (x, t) associated with the terminal history z. We also assume perfect recall and finite depth. (Definition A.1 in the Appendix gives a more formal and through description of extensive form games.) Note that bidders' values for the item are not built into the extensive-form game G. Instead, a strategy is a function of both the information set of a bidder and her value for the item.

2.1 Bidders – Real and Shill

In the auction, there is a set of potential bidders B, with $|B| = N$, who might participate. Of these potential bidders, a set of real bidders R actually participates. Each bidder $i \in B$ has an independent probability p of participating, $\mathbb{P}[i \in R] = p$.[10] The other bidders, $S = B \backslash R$, are shill bidders whose incentives are completely aligned with the seller/auctioneer's, in the sense that their objective is to maximize the transfer paid by real bidders. Importantly, the auction G cannot directly condition on the realization of R, i.e., the shill bidders are indistinguishable from real bidders during the auction. Each real bidder $i \in R$ has value v_i for the item being sold where $v_i \sim F$ independently for each i. We assume F is discrete, with support $\mathcal{V}$ consisting of the ordered atoms $0 = v^1 < v^2 < \ldots < v^M$, and we define $f(v^k) = \mathbb{P}_{w \sim F}\left[w = v^k\right]$ to be the pmf of the distribution. As notation, for each shill bidder $i \in S$, we assign $v_i = 0$ and let $v = (v_1, \ldots, v_N)$. As we discuss in the next paragraph, the choice of values for shill bidders does not affect their incentives, and by supposing that their values are 0, we can define efficiency and optimality (revenue maximizing) in terms of only v instead of v and R.[11] Observe that given how v is generated, we are in the standard, single-item independent private values (IPV) setting.

Each real bidder has quasi-linear utility: for $i \in R$,

$$u_i(z) = x_i(z)v_i - t_i(z).$$

Real bidders have no information on who else is a real bidder.[12]

Shill bidders know the set of shill bidders and a shill bidder's interests are completely aligned with those of the auctioneer, i.e., their utility is defined by the sum of real bidders' transfers: for $i \in S, u_i(z) = \sum_{j \in R} t_j(z)$.

[10] This randomness does not play much of a role in our analysis—we impose it only so that the overarching structure of our model has bidders with ex ante independent private values.

[11] Here, we assume that the seller has 0 value for the item. Furthermore, when considering optimal auctions, we naturally assume the seller only cares about raising revenue from real bidders.

[12] We do not explicitly consider the possibility that real bidders may update about which bidders may be shills over the course of the auction because under our shill-proofness conditions, shill bidders will never make nontrivial bids in equilibrium. (But out of equilibrium/in practice, of course, we might imagine that bidders who make unexpectedly high bids could be likely to be shills.)

Our equilibrium concept is pure-strategy Perfect Bayesian Equilibrium; a formal definition of the auction equilibrium (G, σ) can be found in Definition A.2. We write $\sigma(v; R)$ for the strategy profile when the value profile is v and the realized set of real bidders is R.

2.2 Auction Environment

Throughout the paper, we focus on auction equilibria that are ex-post monotone and individually rational: The auction equilibrium (G, σ) is **monotone** if, for all i, j, v_{-j}, and $v_j > v'_j$,

$$t_i(\sigma(v; B)) > 0 \implies t_i(\sigma(v; B)) \geqslant t_i\left(\sigma(v'_j, v_{-j}; B)\right),$$

and is **individually rational** (IR) if, for all v and $i \in B$,

$$x_i(\sigma(v; B))v_i - t_i(\sigma(v; B)) \geqslant 0.$$

Because the value distribution is discrete, we must consider what to do if multiple bidders have the same (highest) value. We assume throughout the paper the notion of orderliness introduced by Akbarpour and Li (2020): there exists a fixed priority order—independent of values—over which bidder wins an item if there is a tie.[13]

Definition 2.1. An auction equilibrium (G, σ) is **orderly** if there exists a total ordering $\rhd$ over (v_i, i) with the following properties:

(i) for all $v, i, j; (v_i, i) \rhd (v_j, j) \implies x_j(\sigma(v; B)) = 0$;

(ii) for all $v, i, j; v_i > v_j \implies (v_i, i) \rhd (v_j, j)$;

(iii) for all i, j, if there exists m such that $(v^m, i) \rhd (v^m, j)$, then for all k, $(v^k, i) \rhd (v^k, j)$.

In order to give the optimal auction a well-behaved allocation rule, we suppose the value distribution is regular (with bidders' private values drawn i.i.d.). In Section 5, we relax the regularity assumption. We take our definition of a discrete regular distribution from Elkind (2007):

Definition 2.2. A distribution F is **regular**, if for all k, the virtual value

$$\varphi^k = v^k - (v^{k+1} - v^k)\frac{1 - F(v^k)}{f(v^k)}$$

is non-decreasing.

With regular value distributions, a reserve price ρ^* is optimal if and only if for all $v^k \geqslant \rho^*$, $\varphi^k \geqslant 0$ and for all $v^k < \rho^*$, $\varphi^k < 0$.[14]

[13] For example, if ties are broken lexicographically, then the auction is orderly.

[14] There are multiple optimal reserves in our setting, in general, due to the discrete nature of the distribution.

Remark 2.3. The direct allocation rule in an orderly, optimal auction is

$$\tilde{x}_i^*(v) = \mathbb{1}\left\{v_i \geqslant \rho^*, (v_i, i) = \max_{\triangleright}\{(v_j, j)\}_{j\in B}\right\},$$

and in an orderly, efficient auction the direct allocation rule is

$$\tilde{x}_i^E(v) = \mathbb{1}\left\{(v_i, i) = \max_{\triangleright}\{(v_j, j)\}_{j\in B}\right\}.$$

Note that these allocation rules are a function of the priority order $\triangleright$.

2.3 Shill-Proofness

Next, we define our key shill-proofness desiderata. We are interested in auction equilibria in which shill bidders do not shill. Formally, this corresponds to requiring that shill bidders always act like real bidders who have value 0 for the item—since real bidders who have value 0 will never enter non-trivial bids in equilibrium, requiring shill bidders to have the same actions in equilibrium in effect means that shilling does not occur.

Definition 2.4. An auction equilibrium (G, σ) is **weakly shill-proof** if σ is invariant to the realization of S, i.e., for all v and $S, S' \subseteq \{i : v_i = 0\}$,

$$\sigma\left(v; B\backslash S\right) = \sigma\left(v; B\backslash S'\right).$$

Note that Definition 2.4 is a statement about an equilibrium of an auction—it is possible (although we have not found an example of this) that an auction may have both shill-proof equilibria and non-shill-proof equilibria.

We can also strengthen our notion of shill-proofness from not shilling being an equilibrium strategy to being a dominant strategy:

Definition 2.5. An auction equilibrium (G, σ) is **strongly shill-proof** if it is weakly shill-proof and a dominant strategy profile for shill bidders, i.e., for all σ', S, and v_{-S},

$$\sum_{j\in R} t_j(\sigma(0, v_{-S}; R)) \geqslant \sum_{j\in R} t_j\left(\sigma'_S(0, v_{-S}; R), \sigma_{-S}(0, v_{-S}; R)\right).$$

Strong shill-proofness is obviously preferable to weak shill-proofness (all else equal), especially if there are concerns about a seller somehow acquiring information about real bidders' valuations beyond what is encoded by the prior. As we'll see, however, the design space of weak shill-proof auctions is meaningfully larger than that of strong shill-proof auctions.

2.4 Bidders' Information Sets and Public Auctions

Before we begin our analysis of shill-proofness in extensive form auctions, the last restriction we place is on the information sets bidders have over the course of the auction. We assume that every action is public—when choosing an action, a bidder observes all previously taken actions. Public auctions are common in practice; from open air fish markets, to auctions on eBay, participants often can see every action other bidders take before choosing what to do.[15]

Definition 2.6. An auction equilibrium (G, σ) is **public** if the information set at any history is all previous actions taken. Formally, for any history h, $\mathcal{I}(h) = \{h\}$.

Even though the information set is a singelton set—and so a bidder knows exactly where in the game tree she is—our assumption differs from assuming that bidders have *perfect* information, because we have assumed that only actions are encoded in the tree whereas private values affect future strategy. Also, our definition of public differs from the repeated games literature because it rules out stage games (in particular, no two players move simultaneously). This assumption provides a separation between what is implementable in direct mechanisms a la Myerson (1981) and what is implementable in an extensive form game. The following example illustrates how to simulate a sealed-bid, first-price auction in the public setting, and shows that the resulting auction is not strongly shill-proof.

Example 2.7. Consider the sealed-bid, first-price (pay-as-bid) optimal auction. This auction allocates the item to the highest bidder and the equilibrium of the game has bids of[16]

$$b_i^1(v^m) = v^m - \sum_{k:v^k<v^m} (v^{k+1} - v^k) \frac{\left(F(v^k)\right)^{i-1} \left(F(v^{k-1})\right)^{N-i-1}}{\left(F(v^m)\right)^{i-1} \left(F(v^{m-1})\right)^{N-i-1}},$$

and so the direct transfer rule is $\tilde{t}_i^1(v) = \tilde{x}_i^*(v) \cdot b_i^1(v_i)$, where $\tilde{x}^*$ is the optimal allocation rule from Remark 2.3. The naïve implementation of allocation and transfer rule $(\tilde{x}^*, \tilde{t}^1)$ in a public auction would be to query each bidder sequentially on what her value is and then have the payment rule be $\tilde{t}^1$, but $\tilde{t}^1$ is not the direct transfer rule of any equilibrium of this game. Indeed, consider the last bidder who takes a move, and label that bidder N. If $v_N > \max_{i<N} \{b_i\}$, then the only possible equilibrium bid—and therefore the transfer function—is $\max_{i<N} \{b_i\}$.

If we modify the direct transfer rule to represent the bid that each bidder submits in equilibrium in this sequential form (as one could solve for inductively), the auction would be weakly shill-proof by regularity. In particular, while a shill bid can force later bidders to pay a higher price, the probability that no one will want to pay that higher price outweighs the benefit by regularity. However, such an auction is not strongly shill-proof because, given full knowledge of real bidders' valuations, a shill bidder will be incentivized to bid just below the highest valuation of a subsequent bidder.

While a naïve extensive form implementation of pay-as-bid auctions is not strongly shill-proof, it is in fact the case that only auctions with direct transfer rule $\tilde{t}^1$ are strongly shill-proof. Our main result, that the Dutch auction is the only strongly shill-proof auction, can also be framed as showing that the only extensive form auction that implements the first-price direct transfer rule is the Dutch auction.

[15] Non-examples of public auctions include the FCC spectrum auctions, where bidders typically only learn information on other bidders' actions in rounds (see Milgrom and Segal (2017) for more information). We explore shill-proofness with different information assumptions in auctions where each bidder takes a single action in Section 5, and leave a more general exploration to future work.

[16] See Lemma B.1 in the Appendix for proof that this is the correct form.

2.5 Revelation Principle

In order to make progress in understanding shill-proof auction formats, the following revelation principle will be helpful: for every public auction equilibrium (G, σ), there exists a direct auction that can be summarized by a direct allocation rule $\tilde{x}$, a direct transfer rule $\tilde{t}$, a menu rule μ, and a starting player ξ_0.[17] The first input to the menu rule μ is a set V of valuation profiles of the form $V_1 \times V_2 \times \cdots \times V_N$ with $V_i \subseteq \mathcal{V}$ for all i—intuitively, the valuation profiles that are, in equilibrium, consistent with a particular history. The second input is a player ξ who is to move next. The output of the rule is a collection $\left\{\left(W_\ell, \vec{\xi_\ell}\right)\right\}_{\ell \in \{1,\ldots,L\}}$, where the W_ℓ's are a partition of V_ξ (from which player ξ will choose one, according to its valuation, the equilibrium strategy σ determines the partition) and $\vec{\xi_\ell}$ indicates the next player to move should player ξ choose W_ℓ. If $\vec{\xi_\ell} = \varnothing$, then the game ends should choice ℓ be selected by the bidder ξ. For a typical iterative auction, one generally has $\ell = 2$ with the two sets corresponding to types above and below some value, respectively. Or, for a one-shot auction, the W_ℓ's are generally singletons, with one per type in V_i.

We show that for any implementable outcome $(\tilde{x}, \tilde{t})$ of the auction, one can always find a menu rule that is "informative"—the set of possible outcomes differs across partition selections[18]—that also implements the same outcome. So, without loss of generality, we restrict menu rules in this way and then describe a public auction equilibrium by $(\tilde{x}, \tilde{t}, \mu, \xi_0)$. (See Lemma A.6 in the Appendix for a more formal treatment.) We refer to $(\tilde{x}, \tilde{t}, \mu, \xi_0)$ as an *auction* when convenient. As is always the case with direct mechanisms, the auction encompasses both the game form and the equilibrium, i.e., by appealing to the revelation principle we have implicitly selected the equilibrium already. Thus, we can speak of an auction being shill-proof.

Finally, as notation for later sections, for a set $V = V_1 \times V_2 \times \cdots \times V_N$ of valuation profiles, define $\overline{V}_i = \max\{v_i : v_i \in V_i\}$ to be the maximum possible value of bidder i; $\overline{V}_{-i}$ similarly to be the maximum possible value of bidders $j \neq i$; and $\overline{V} = \max_i \left\{\overline{V}_i\right\}$. We define $\underline{V}_i, \underline{V}_{-i}$, and $\underline{V}$ as the corresponding values for minima instead of maxima.

3 Dutch Auctions

3.1 Strongly Shill-Proof Auctions

In this section, we present our results showing that the Dutch auction is uniquely suited for preventing shill bidding. The Dutch auction is defined as the auction which begins by offering each bidder i the item at $b_i^1(v^M)$, the first price transfer rule from Example 2.7, and then if no bidder chooses to buy the item at that price, the item is offered for $b_i^1(v^{M-1})$ and so on until either a bidder has chosen to buy the item or the price to be offered drops below $b_i^1(\rho^*)$, where ρ^* denotes an optimal reserve price. We consider only orderly auctions and therefore, at each price level, bidders are offered the opportunity to buy the item in priority order. Formally:

[17] This revelation principle is similar to those found in, for example, Ashlagi and Gonczarowski (2018); Mackenzie (2020); Mackenzie and Zhou (2022).

[18] This notion of informativeness is very similar to the pruned condition from Akbarpour and Li (2020).

Definition 3.1. The **Dutch auction with reserve price** ρ^* is defined by the optimal allocation rule $\tilde{x}^*$, first-price transfer rule $\tilde{t}^1$, initial player $(\cdot, \xi_0) = \max_{\triangleright} \{(0, i)\}$, and menu

$$\mu^D(V, \xi) = \left\{ \left(W_{\mathsf{L}}, \vec{\xi}_{\mathsf{L}} \right), \left(W_{\mathsf{H}}, \vec{\xi}_{\mathsf{H}} \right) \right\},$$

$$\text{where } W_{\mathsf{H}} = \left\{ \overline{V}_{\xi} \right\}, W_{\mathsf{L}} = V_{\xi} \setminus \left\{ \overline{V}_{\xi} \right\}, \vec{\xi}_{\mathsf{H}} = \varnothing, \text{ and}$$

$$\vec{\xi}_{\mathsf{L}} = \begin{cases} \left(\cdot, \tilde{\xi} \right) = \max_{\triangleright} \left\{ \left(\overline{V}_i, i \right) : i \neq \xi \right\} & \exists i \neq \xi \text{ such that } |V_i| > 1 \text{ and } \overline{V}_i \geqslant \rho^* \\ \varnothing & \text{otherwise} \end{cases}.$$

We now give two technical conditions in order to simplify the presentation of the class of strongly shill-proof auctions. The first condition, *testing*, can be thought of as the auctioneer querying bidders in reverse priority order on whether they would be interested in participating in the auction. Whether the lowest priority bidder has value equal to ρ^* or to a value strictly below ρ^* is equivalent for the decision-making of other bidders: every other bidder either has a value below ρ^* and so will never win, or has a value at least ρ^* and will never lose to a bid of at most ρ^* by the lowest-priority bidder. So, the auctioneer can query the lowest priority bidder at any time on whether her value is weakly above the reserve price without affecting incentives. The set of strongly shill-proof auctions without the testing condition is the set of Dutch Auctions, but at any point in the auction, the auctioneer can also query the lowest priority bidder on whether her value is weakly above the reserve price. When (or if) such queries takes place has no relevance on the outcome of the auction and so it is convenient for the presentation of the results to simply fix such queries to occur at the start of the auction and not at some later point.

Definition 3.2. An auction $(\tilde{x}, \tilde{t}, \mu, \xi_0)$ is **testing** if $(\cdot, \xi_0) = \min_{\triangleright} \{(0, i)\}$ and

$$\max_i \{\underline{V}_i\} < \rho^* \implies \mu(V, \xi) = \left\{ \left(\{w : w < \rho^*\}, \vec{\xi}_1 \right), \left(\{w : w \geqslant \rho^*\}, \vec{\xi}_2 \right) \right\},$$

$$\text{where } \vec{\xi}_1 = \begin{cases} \left(\cdot, \tilde{\xi} \right) = \min_{\triangleright} \{(0, i) : i \neq \xi\} & \overline{V}_{-\xi} \geqslant \rho^* \\ \varnothing & \text{otherwise} \end{cases}$$

and $\vec{\xi}_2$ is arbitrary.

The second condition, *pooling*, states that if the menu rule has a partition that can rule out all possible values below some threshold, that threshold must be at or above ρ^*. In particular, when a bidder has a value below ρ^*, she must play the same action no matter what her value is. This can be thought of as different actions having ε cost difference and bidders who ex-ante have 0 probability of winning (those with values below the reserve) always select the cheapest action. This condition rules out variants of the Dutch auction where there is an extra action that only some bidders with value below the optimal reserve take.

Definition 3.3. An auction $(\tilde{x}, \tilde{t}, \mu, \xi_0)$ is **pooling** if for all V and ξ, there exists

$$(W, \cdot) \in \mu(V, \xi) \text{ such that } \{w : w < \rho^*\} \cap V_{\xi} \subseteq W.$$

The testing and pooling conditions are without loss for strongly shill-proof auctions in the sense that we can always find an auction that satisfies these conditions and has the same outcome for all value profiles. So, without loss of generality, when we consider strongly shill-proof auctions, we assume that the auction is testing and pooling.

Proposition 3.4. *For any public, strongly shill-proof, and optimal auction* $(\tilde{x}, \tilde{t}, \mu, \xi_0)$*, there exists a public, strongly shill-proof, optimal auction* $(\tilde{x}', \tilde{t}', \mu', \xi_0')$ *that is testing, pooling, and for all* v*,* $\left(\tilde{x}, \tilde{t}\right)(v) = \left(\tilde{x}', \tilde{t}'\right)(v)$*.*

Let us note that an orderly, optimal auction will never end when the only discernible information is that $\max_i \{\underline{V}_i\} \geqslant \rho^*$ and so the auction will not end immediately after completing testing unless all bidders have value below ρ^*. We then define the testing Dutch auction with reserve price ρ^* to be the Dutch auction with reserve price ρ^* and a testing step at the beginning.[19] We are now in a position to present our main result:

Theorem 3.5. *A public and optimal auction is strongly shill-proof if and only if it is the testing Dutch auction with reserve price* ρ^**.*

The intuition behind this result is that shill bidding can ex-interim "increase the reserve price:" shill bidding can be used to imply that no one with value below some new, higher threshold will win the auction. The fact that the auction is public and optimal implies that with a higher reserve price, the transfer from the winner must be higher.[20]

The Dutch auction is strongly shill-proof because any shill bid immediately ends the auction and in that case there would be no transfers from other bidders. To see the uniqueness result, we note that the key property of the testing Dutch auction is that any bid immediately ends the auction. Indeed, for all other auction formats, there exists at least one history such that a bidder can indicate her value is strictly greater than 0 without the auction ending immediately. The testing, pooling, and orderliness conditions together imply that in such an auction, it is possible to ex-interim increase the effective reserve price. We can then conclude that such an auction is not strongly shill-proof because we can consider a valuation vector that generates such a history and have the bidder who can indicate her value is greater than 0 be a shill bidder.

3.2 Weakly Shill-Proof and Efficient Auctions

To dive further into analyzing when a Dutch auction is needed, we now turn to discussing efficient auctions. While the literature has primarily focused on optimal auctions, efficient auctions are important to consider in many cases. One example is two-sided marketplaces, where the auctioneer/market designer and the seller are different entities and may have different objectives. The designer may be interested in allocating goods efficiently while sellers are trying maximize revenue. Our next result shows that in order for an auction to be *weakly* shill-proof and efficient robustly to the prior on the value distribution, part of

[19] $\vec{\xi}_2$ is defined as the initial player in the standard Dutch auction: $\left(\cdot, \vec{\xi}_2\right) = \max_{\triangleright} \left\{\left(\overline{V}_i, i\right)\right\}$.

[20] If the auction were not public, then real bidders' beliefs might not change ex-interim and so their transfers might not change either. For example, in a sealed first-price auction, a shill bidder's actions have no effect on the transfers of other bidders.

its game tree must be a Dutch auction. In particular, the auction must be a semi-Dutch auction:

Definition 3.6. An efficient auction $(\tilde{x}, \tilde{t}, \mu, \xi_0)$ is a **semi-Dutch auction with cutoff** ρ^* if for any v such that $\max_i \{v_i\} < \rho^*$:

(i) $\check{V} = \{w : w < \rho^*\}^N$ is reached and

(ii) $\mu(V, \xi) = \mu^D(V, \xi)$ for any player ξ and possible values $V \subseteq \check{V}$ where μ^D is the Dutch auction menu rule from Definition 3.1.[21]

By *robust*, we mean that if the auction is not a semi-Dutch auction, then we can find a value distribution such that the auction is not weakly shill-proof. In the other sections of the paper, it has been equivalent to think of the auction as a function of v in value space ($v^k \in \mathcal{V}$) or in index space ($k \in \{1, \ldots, M\}$). However, for this robustness theorem, we work in index space, but to standardize notation, we maintain that ρ^* is in value space. More formally, the auction $(\tilde{x}, \tilde{t}, \mu, \xi_0)$ is parameterized by the optimal reserve ρ^*, the number of atoms below the reserve $\underline{M}$ and the number of atoms weakly above the reserve $\overline{M}$.[22]

Before we formally present our robustness result, we build intuition with an example to show how for an auction that is not semi-Dutch, we can find a distribution such that the auction is not weakly shill-proof.

In order to simplify exposition of the example, let us formally define a discrete approximation of a continuous distribution as follows:

Definition 3.7. Let F be a discrete distribution with ordered atoms $0 = v^1 < \ldots < v^M$ and $\mathcal{F}$ be a continuous distribution with p.d.f. f. If $Y_{\mathcal{F}} \sim \mathcal{F}$, then F is a **discrete approximation** of $\mathcal{F}$ when $Y_F \sim F$ is defined as

$$Y_F = \begin{cases} v^1 & Y_{\mathcal{F}} \leqslant v^1 \\ v^k & Y_{\mathcal{F}} \in (v^{k-1}, v^k] \\ v^M & Y_{\mathcal{F}} > v^{M-1} \end{cases}. \tag{2}$$

For such a distribution F, let $\overline{\Delta} = \max_k \left\{v^k - v^{k-1}\right\}$. Note that Definition 3.7 induces the following p.m.f. f for F:

$$f(v^k) = \begin{cases} \mathcal{F}(v^1) & k = 1 \\ \mathcal{F}(v^k) - \mathcal{F}(v^{k-1}) & k \in \{2, \ldots, M-1\} \\ 1 - \mathcal{F}(v^{M-1}) & k = M \end{cases}.$$

As convention, let F^{-1} be the left pseudo-inverse: $F^{-1}(x) = \max\left\{v^k : x \geqslant F(v^k)\right\}$.

[21] Technically, $\mu^D(V, \xi)$ is defined only for optimal auctions; however, it is well-defined for efficient auctions if we redefine $\vec{\xi}_{\mathsf{L}}$ as

$$\vec{\xi}_{\mathsf{L}} = \begin{cases} \left(\cdot, \tilde{\xi}\right) = \max_{\triangleright} \left\{\left(\overline{V}_i, i\right) : i \neq \xi\right\} & \exists i \neq \xi \text{ such that } \overline{V}_i > 0 \\ \varnothing & \text{otherwise} \end{cases}. \tag{1}$$

[22] $\underline{M} + \overline{M} = M$.

Example 3.8. Let $\mathcal{F}(x) = 1-e^{-0.1x}$ correspond to the exponential distribution with rate $\lambda = 0.1$. Let F_1, F_2 be discrete approximations of $\mathcal{F}$ with atoms $\{0, 5, 9, 14, 20\}$ and $\{0, 3, 7, 14, 20\}$, respectively. It can be verified that both these distributions are regular and have optimal reserve $\rho^* = 14$. Consider a variant of the efficient Dutch auction (see Equation (1)), with the modification that, if all bidders have indicated values less than 20, then the auction queries bidders from lowest-to highest-priority as to whether their value is at least 9. If no one indicates that their value is at least 9, then the Dutch auction continues. If at least one person does indicate that their value is at least 9, then bidders are queried from lowest- to highest-priority as to whether their value is 14, and the transfer is 14 if at least two people have value 14, and 9 if only one person does. It can be verified that if the value distribution is F_1, the auction just described is weakly shill-proof, but if the value distribution is F_2, then the auction is not weakly shill-proof. When the value distribution is F_2, in expectation, a shill bidder will want to report that her value is 9. In fact, Lemma B.5 (see appendix) implies that if the value distribution is F_2, then the auction in this example must be a semi-Dutch auction with cutoff at least 14.

As we can see in this example, even when the underlying continuous distribution is the same, simply selecting different atoms can make a weakly shill-proof auction format no longer weakly shill-proof; we show that for any auction that is not a semi-Dutch auction, we can always find such a distribution. In particular, we can construct a distribution such that for the given auction format, it is in expectation profitable to shill for some history and therefore the auction would not be weakly shill-proof under that value distribution.

Theorem 3.9. *For every public and efficient auction that is not a semi-Dutch auction with cutoff ρ^*, there exists a regular value distribution with optimal reserve ρ^* under which the auction is not weakly shill-proof.*

The key step in the proof of Theorem 3.9 resembles that in the proof of Theorem 3.5—in any non-Dutch auction, shill bidders can ex-interim "raise the reserve price" by changing their actions. However, given that we are interested in weak shill-proofness instead of strong shill-proofness, we have to examine shill bidders' incentives when we take expectations over real bidders' values instead of conditioning directly on their values. Regularity implies that above ρ^*, shill bidders do not have an incentive to shill bid in auction formats such as the English auction (see Section 1.1). However, below ρ^*, we can always find a regular distribution such that the ex-interim expected value of raising the reserve price is always positive. So, below ρ^*, the auction must look like the Dutch auction; the class of all such auctions is precisely all semi-Dutch auctions with cutoff ρ^*.

In the Appendix, we construct the claimed sub-class of regular distributions (see Definition B.3). Informally, the atoms of the value distribution have to be far enough apart so that raising the reserve price a single "level" generates a large amount of additional revenue. The following example presents a real-world setting that roughly fits the premises of Theorem 3.9 where a semi-Dutch auction (that is not a Dutch auction) is used:

Example 3.10. We describe the Honolulu-Sydney fish auction documented by Hafalir et al. (2023), which blends elements of the Dutch and English auctions: The auction begins at some intermediate price and if anyone bids, then the price ascends like in the English auction. If no one bids, the price descends until someone bids like in the Dutch auction. Once someone bids, other bidders can counter-bid and raise the price once more. However, in practice there is little counter-bidding.[23] There exist related formats such as that used in the Istanbul flower market that are the same except counter-bidding is not allowed. In that case, it is simply an English auction above the starting price and a Dutch auction below.

The Honolulu-Sydney auction plausibly fits the technical assumptions made in Theorem 3.9: The auctions are public, as they take place in person and all bidders can see other bidders' actions. Market participants are interested in efficient outcomes because the goods are perishable and there are positive disposal costs for the sellers. Speed is of the essence—goods need to be allocated quickly and so the price must rise or fall quickly. Fish markets also tend to have free entry of bidders, which means that shill bidding by the seller is possible. We do not mean to imply that the Honolulu-Sydney auction was instituted precisely because it is shill-proof, but we highlight it as further evidence that in markets where it is difficult to monitor shill bidding, shill-proof mechanisms often arise.

As a final observation in this section, let us note that if instead of allowing the auction format to treat bids above and below the optimal reserve differently, we instead required the auction format to treat all bids identically, then the only public, efficient, and weakly shill-proof auction is the Dutch auction.

Corollary 3.11. *For any public and efficient auction that is not a Dutch auction, there exists a regular value distribution under which the auction is not weakly shill-proof.*

Proof. Observe that a semi-Dutch auction with cutoff $\rho^* = v^M$ is simply a Dutch auction. Then, apply Theorem 3.9 for a regular distribution with optimal reserve $\rho^* = v^M$. $\square$

4 Weakly Shill-Proof and Strategy-Proof Auctions

We have shown that the only optimal auction in which it is a dominant strategy for shill bidders not to shill (strong shill-proofness) and an equilibrium for real bidders is the Dutch auction. We now investigate the reverse question: what optimal auctions have a dominant strategy for real bidders (strategy-proofness) and an equilibrium for shill bidders not to shill (weak shill-proofness)? Before we explore that question, let us formally define strategy-proofness in our public auction setting:

Definition 4.1. An auction $(\tilde{x}, \tilde{t}, \mu, \xi_0)$ is **strategy-proof** if it is a dominant strategy for real bidders to report their values truthfully: for all i, v and v_i',

$$\tilde{x}(v)v_i - \tilde{t}(v) \geqslant \tilde{x}(v_i', v_{-i})v_i - \tilde{t}(v_i', v_{-i}).$$

As we are focused on (augmented) direct games in this paper, we define our notion of strategy-proofness for direct games, not for all extensive-form games. In extensive form games, Definition 4.1 is equivalent to ex-post incentive compatibility, but is a weaker notion than extensive-form strategy-proofness because actions do not have to dominate off-path

[23]On the theoretical side, in an IPV setting, there exists an equilibrium where there is no counter-bidding.

actions. All strategy-proof, optimal auctions must have the second-price transfer rule (Akbarpour and Li, 2020, Proposition 8):

$$\tilde{t}^2(v) = \tilde{x}_i^*(v) \cdot \max\{\rho^*, \text{second-highest value in } \{v_1, \ldots, v_N\}\}.$$

Note that if a shill bidder knew the valuations of all other bidders, then shill bidding would turn a second-price auction into a first price auction, which bounds the expected profit for a shill bidder from shilling. So, in order to find a strategy proof and weakly shill-proof auction, we must find a menu rule that implements a second-price auction where the expected gain from shill bidding is sufficiently small at shill bidders' information sets. The information sets of real bidders are irrelevant when we focus on strategy-proof auctions (as in this section), but for concreteness and continuity we continue to assume that the auction is public. As discussed in Section 1.1, for regular value distributions, one weakly shill-proof, strategy-proof, and optimal auction is the English auction. We formalize the English auction in our framework as follows:

Definition 4.2. The **English auction with reserve price** ρ^* is defined as the auction with the optimal allocation rule $\tilde{x}^*$, second-price transfer rule $\tilde{t}^2$, initial player $(\cdot, \xi_0) = \min_{\triangleright}\{(0,i)\}$, and menu

$$\begin{aligned}
&\mu^E(V,\xi) = \left\{\left(W_{\mathsf{L}}, \vec{\xi}_{\mathsf{L}}\right), \left(W_{\mathsf{H}}, \vec{\xi}_{\mathsf{H}}\right)\right\}, \\
&\quad \text{where } W_{\mathsf{L}} = \{v \in V_\xi : v < \rho^*\} \cup \left\{\underline{V}_\xi\right\}, W_{\mathsf{H}} = V_\xi \backslash W_{\mathsf{L}}, \\
&\qquad \vec{\xi}_{\mathsf{L}} = \vec{\xi}_{\mathsf{H}} = \begin{cases} \left(\cdot, \tilde{\xi}\right) = \min_{\triangleright}\left\{(\underline{V}_i, i) : i \neq \xi, \overline{V}_i = v^M, |V_i| > 1\right\} & \overline{V}_{-\xi} = v^M \\ \varnothing & \text{otherwise} \end{cases}.
\end{aligned}$$

Remark 4.3. The English auction with reserve price ρ^* is weakly shill-proof, strategy-proof, and optimal.

The English auction is not the only strategy-proof and weakly shill-proof auction. While the English auction is used frequently, one drawback is that it is "slow"—each bidder can be queried on their willingness-to-pay on the order of M times (specifically $M - \min\left\{k : v^k \geqslant \rho^*\right\}$ times). To explore if there are weakly shill-proof and strategy-proof auctions that require fewer rounds of communication, we introduce a natural "compression" of the English auction. The ascending, screening auction, comprises the following two phases:

1. An English auction is run from ρ^* to some v^Y.
2. If necessary, a second-price auction is then run among players who have not dropped out before the value level of v^Y.

Definition 4.4. The **ascending, screening auction** with screen level v^Y is defined by the optimal allocation rule $\tilde{x}^*$, second-price transfer rule $\tilde{t}^2$, initial player $(\cdot, \xi_0) = \min_{\triangleright}\left\{\left(\overline{V}_i, i\right)\right\}$, and menu

$$\mu(V,\xi) = \begin{cases} \mu^E(V,\xi) & \exists i \text{ such that } \underline{V}_i < v^Y \text{ and } \overline{V}_i = v^M \\ \left\{\left(\left\{v^k\right\}, \vec{\xi}^k\right)\right\}_{k\in\{Y,Y+1,\ldots,M\}} & \text{otherwise} \end{cases},$$

where $\left(\cdot, \vec{\xi}^Y\right) = \left(\cdot, \vec{\xi}^{Y+1}\right) = \ldots = \left(\cdot, \vec{\xi}^{M-1}\right) = \max_{\triangleright} \left\{(0,i) : |V_i| > 1, \underline{V}_i = v^Y\right\}$ and $\vec{\xi}^M = \varnothing$.

This auction reduces the maximum number of times each bidder is queried by

$$Y - \min\left\{k : v^k \geqslant \rho^*\right\} - 1$$

rounds, as compared to the English auction. Each bidder is queried at most

$$\left|\left\{k : \rho^* \leqslant v^k \leqslant v^Y\right\}\right| + 1$$

times. Because the transfer rule is $\tilde{t}^2$, the ascending, screening auction is strategy-proof and optimal.

We use the ascending, screening auction format to explore how fast a weakly shill-proof, strategy-proof and optimal auction can be, as a function of the underlying value distribution. Our next result shows that when the value distribution has a "thin-enough" right tail, if we ascend for long enough, we can find a screening level such that the ascending, screening auction is weakly shill-proof. In particular, we restrict to the family of monotone hazard rate (MHR) distributions.

Definition 4.5. Let F be a discrete approximation of $\mathcal{F}$. The distribution F is a **monotone hazard rate (MHR) distribution** if $\frac{f(w^k)}{1-F(w^k)}$ is monotonically increasing in k and $h(x) = \frac{f(x)}{1-\mathcal{F}(x)}$ is monotonically increasing in x.

Remark 4.6. Every MHR distribution is regular.

Theorem 4.7. *If the value distribution is a discrete MHR distribution F, then for all*

$$v^Y \geqslant F^{-1}\left(F\left(\rho^*\right) + \max_{1\leqslant n<N}\left\{\left(\max\left\{1 - \frac{\rho^*}{\rho^* + 2\overline{\Delta}}\left(\frac{f(\rho^*)}{1-F(\rho^*)}\right)^n, 0\right\}\right)^{1/n}\right\}\right), \tag{3}$$

the ascending, screening auction with screening level v^Y is a weakly shill-proof, strategy-proof, and optimal auction.

The intuition driving how large v^Y needs to be in order for the ascending, screening auction to be weakly shill-proof is that the less the mass of the value distribution is in the right tail, the less that can be extracted in expectation from shill bidding and the more likely it is that a shill bidder will win the item if she shill bids. Let us observe that Equation (3) is independent of the number of atoms M in the distribution. So, consider a sequence of value distributions $\{F_m\}$ each of which is a discrete MHR distribution and such that for all m, $\overline{\Delta}_m$, ρ^*_m, and $\frac{f_m(\rho^*)}{1-F_m(\rho^*)}$ are the same, and $\operatorname{supp} F_m = \left\{0, \overline{\Delta}, \ldots, m\overline{\Delta}\right\}$. Then, $Y/m \to 0$ as $m \to \infty$, i.e., the worst-case number of queries in the ascending, screening auction is an arbitrarily small fraction of the worst-case for the English auction.

Proof Sketch. The ascending, screening auction is orderly and optimal by construction; to see that it is strategy-proof, note that the English auction phase and the second-price phase both induce the same (strategy-proof) allocation and transfer rule. To prove that the auction is weakly shill-proof, we show that the maximum possible expected amount that can be extracted by shill bidding is less than the expected loss due to the possibility that the shill bidder wins the item. MHR distributions have thin tails, so the maximum expected additional transfer from the winner by shill bidding (i.e., the difference between the conditional first and second moments) is weakly decreasing in v^Y. In addition, the probability a shill bidder wins the item due to shilling is strictly increasing in v^Y. So, for sufficiently large v^Y, the auction is weakly shill-proof.

Several factors govern how large v^Y must be to disincentivize shill bidding. First, it must be a function of the difference between the first and second moments of the distribution. Since every MHR distribution is stochastically dominated by an exponential distribution, this difference can be bounded above by a function of only ρ^*. Second, the jump in price between rounds (i.e., $\overline{\Delta}$) must also be considered. Third, the threshold for v^Y is a function of the probability that shill bidding will result in the shill bidder winning the auction. This probability can be expressed as a function of the probability that the highest value among real bidders is exactly some value v^k conditional on being weakly above said value, i.e., the hazard rate at v^k. Since the value distribution is a MHR distribution, for any value $v^k \geqslant \rho^*$, we can bound this probability from below by $\frac{f(\rho^*)}{1-F(\rho^*)}$. Finally, it must be the case that $v^Y \geqslant \rho^*$, as otherwise there is always an incentive to shill up to price ρ^*. We show in the Appendix that v^Y satisfying Equation (3) is a sufficient condition to maintain the auction being weakly shill-proof. $\square$

5 One-Shot Auctions

Theorem 4.7 shows, by "compressing" an English auction, that there exists a weakly shill-proof and strategy-proof auction in which, for some value distributions, bidders take far fewer actions than in an English auction. We now show that such compression has its limits, and more generally that there is no weakly shill-proof, strategy-proof, and optimal auction in which each bidder takes a single action. This impossibility result holds even after relaxing the assumptions that F is regular and that the auction is public, and after weakening our concept of strategy-proofness.

5.1 Set-up

Let us begin by defining a **one-shot auction**. An auction is considered one-shot when each bidder takes precisely one action in the auction (under all possible histories). More formally, for any $h_N \in Z$, let $h_\varnothing \prec h_1 \prec \ldots h_{N-1} \prec h_N$ be the sequence of histories to reach h_N. Then, for all $i \in B$, there exists a unique $n \leqslant N$ such that $i = P(h_n)$. Without loss, we label the bidders $1, \ldots, N$, in the order that they move and label the action taken by bidder i as a_i. Note that this labelling need not be the same for different histories as the bidder ordering can be endogenous to actions taken. We assume that shill bidders observe all past actions, i.e., for $i \in S$, we assume that i observes $(a_1, \ldots, a_{i-1})$. If $i \in R$, we assume that the information i has when taking an action is a signal $s_i \in \mathcal{S}_i$.

This signal is generated via a deterministic function $\psi_i : \left(\bigtimes_{j<i} A_j\right) \to \mathcal{S}_i$ called an **experiment**.[24] For notational convenience, we assume that ψ_i is surjective for all i. We can think of the experiment as a garbling of the previous bidders' actions—the experiment can pool together multiple actions from previous bidders to a single signal and so a signal is not always perfectly informative of previous actions. We can recover the public setting with a fully informative experiment, meaning setting ψ to the identity function, $\psi = \mathrm{Id}$. We can capture classical static game settings via an uninformative experiment that always return the same output, $\psi = \varnothing$. In this case, a bidder's ex-ante and ex-interim knowledge of other bidders' valuations is the same. One can imagine other experiments, such as a bidder receiving a signal of whether bidders before them took "high" or "low" actions. We use $\psi_i^{-1}(s_i)$ to denote the set of v_{-i} that are possible from the perspective of bidder i given its signal. Note that any game, it is equivalent to consider information sets or experiments; we consider experiments for exposition purposes only.

A revelation principle holds in this setting: for any one-shot auction, we can define the direct allocation and transfer rules as $\tilde{x}(v)$ and $\tilde{t}(v)$, respectively, with the appropriate incentive compatibility and individual rationality constraints (see Lemma D.3 in the Appendix). We also present the appropriate IC constraints for shill bidders when we are interested in weakly and strongly shill-proof auctions (see Lemmatta D.5 and D.6, respectively, in the Appendix).

5.2 One-Shot, Strategy-Proof Auctions Cannot Be Shill-Proof

To finish defining all the terms necessary for our main result of this section, we weaken our notion of strategy-proofness in the one-shot auction setting to be strategy-proof for at least a single bidder:[25]

Definition 5.1. A one-shot auction is **mildly strategy-proof** if for the associated direct mechanism, there exists a real bidder $i < N$ such that truthfulness is an ex-interim dominant strategy conditional on the realization of her signal: there exists bidder $i < N$, such that for all i, v_i, v_i', s_i and $v_{-i}, v'_{-i} \in \psi_i^{-1}(s_i)$,

$$\tilde{x}_i(v) \cdot v_i - \tilde{t}_i(v) \geqslant \tilde{x}_i(v') \cdot v_i - \tilde{t}_i(v').$$

Theorem 5.2. *There exists no one-shot, optimal auction that is mildly strategy-proof and weakly shill-proof.*

Proof Sketch. Consider any real bidder $i < N$. By weak shill-proofness, the transfer from bidder i, conditional on winning (or losing) the auction, is invariant to the values of bidders who take actions after her (Lemma D.10 in the Appendix). If this were not the case, then

[24] Abusing notation, we also sometimes take $\psi_i : \mathcal{V}^{i-1} \to \mathcal{S}_i$, i.e., the experiment maps values to signals instead of actions.

[25] We exclude the last bidder who takes an action from our definition because a take-it-or-leave-it offer to that bidder can be optimal and strategy-proof. Theorem 5.2 would still hold if we instead defined mild strategy-proofness to mean strategy-proof for at least two bidders.

if every bidder $j > i$ is a shill bidder, the shill bidders would report the values that would maximize the transfer from the winning bidder. By mild strategy-proofness, the transfer from bidder i, conditional on winning the auction, is invariant to her value (Lemma D.11 in the Appendix). This is because if there were multiple winning reports with different transfer amounts, only the smallest transfer amount would make truthful reporting of the value a dominant strategy. So, in every one-shot, optimal auction, the transfer from the winning bidder i, can depend only on the values reported by bidders before i. But, this means that if a bidder has positive utility for winning the item (as would be the case if $v_i > v_j$ for all $j < i$), then she should report v^M to maximize the probability of winning (without changing the transfer paid upon winning). Thus, the auction must treat bidder i as if she reported v^M, which violates the allocation rule of an optimal auction. $\square$

5.3 Shill-Proofness vs. Credibility

Another notion of "cheating" by the auctioneer is that of *(in)credibility* introduced by Akbarpour and Li (2020). An auction is credible if a revenue-maximizing auctioneer has no incentive to lie about what other players are doing. The information environment in Akbarpour and Li (2020) is the opposite of the public auctions considered here—whereas we assume that a bidder directly observes all the actions taken by other bidders, Akbarpour and Li (2020) assumed that a bidder's knowledge of other bidders' actions is controlled entirely by the auctioneer.

In the Appendix, we define a generalization of credibility for one-shot auctions that allows for exogenous experiments (such as the public and private experiments mentioned above, or anything "in between") as well as additional communication from the auctioneer. We prove that, for every signal structure, strong shill-proofness implies credibility, and credibility implies weak shill-proofness. We also show that when bidders are perfectly informed about the actions of bidders who move before them, an auction is credible if and only if it is weakly shill-proof (Proposition D.8 in the Appendix), but when bidders have no information about the moves of other bidders, an auction is credible if and only if it is strongly shill-proof (Proposition D.9 in the Appendix).

We now present an example that highlights two points. First, that even in the class of one-shot auctions, there are auctions that are weakly shill-proof, but not credible. Second, this example shows there exists a non-trivial family of one-shot, weakly shill-proof auctions.[26]

Example 5.3. Suppose that bidders have no information about the actions of bidders before them ($\psi = \varnothing$), the value distribution is regular with optimal reserve ρ^*, and the tie-breaking priority order goes in reverse order (bidder N wins ties, followed by bidder $N-1$, and so on). Consider an auction in which a bidder submits a bid and if she wins, she pays the average of her bid and the highest bid that preceded her. Formally, treating action a_i as a bid, the allocation and transfer rules are

$$x_i(a) = \mathbb{1}\left[i = \max\left\{\hat{i} : \hat{i} \in \operatorname{argmax}_j a_j\right\}, a_i \geqslant \rho^*\right] \text{ and } t_i(a) = x_i(a) \cdot \frac{a_i + \max_{j<i}\{a_j\}}{2}.$$

[26]The family of one-shot and mildly strategy-proof auctions is likewise also non-trivial. Thus, it is the combination of (weak) shill-proofness and (mild) strategy-proofness that drives the impossibility result for one-shot auctions in Theorem 5.2.

This auction is not credible because the auctioneer has an incentive to misreport to a bidder the bids that preceded her.[27] In particular, the auctioneer will misreport to the winning bidder i that an earlier bidder $j < i$ also bid a_i. However, the auction is weakly shill-proof. Shill bidders cannot manipulate transfer prices for bidders who precede them. And, while a shill bidder can manipulate the transfers from subsequent bidders, regularity implies that this is not a profitable deviation. The first- and second-price auctions both have the same optimal reserve and regularity disincentivizes shill bidding, which implies that the mixture of a first- and second-price transfer rule also disincentivizes shill bidding.

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A Model (Section 2) Appendix

The following definition for an extensive form auction is taken[28] from Li (2017):

Definition A.1. An **extensive form auction** G is defined as the tuple $(H, \prec, A, \mathcal{A}, P, \{\mathcal{I}_i\}_{i \in B}, (x, t))$ such that:

(i) H is a set of histories, along with a binary relation $\prec$ on H that represents precedence. In addition:

  (a) $\prec$ forms a partial order and $(H, \prec)$ forms an arborescence.

  (b) There exists an initial history $h_\varnothing = h$ such that there does not exists h' where $h' \prec h$.

  (c) The set of terminal histories is $Z \equiv \{h : \neg \exists h \text{ such that } h \prec h'\}$.

  (d) The set of immediate successors to h is $\text{succ}(h)$.

(ii) A is the set of possible actions.

(iii) $\mathcal{A} : H \backslash h_\varnothing \to A$ maps histories to the most recent action taken to reach it. In addition:

  (a) For all h, $\mathcal{A}(h)$ is one-to-one on $\text{succ}(h)$.

  (b) The set of actions available at h is

$$A(h) \equiv \bigcup_{h' \in \text{succ}(h)} \mathcal{A}(h').$$

(iv) $P : H \backslash Z \to B$ is the player function for any given non-terminal history.

(v) $\mathcal{I}_i$ is a partition of $\{h : P(h) = i\}$ such that:

  (a) $A(h) = A(h')$ when h and h' are in the same cell of the partition, and

  (b) $A(h) \cap A(h') = \varnothing$ when h and h' are not in the same cell of the partition.

(vi) For every $z \in Z$, $z = (x, t)$, such that $\sum_{i=1}^{N} x_i \leqslant 1$.

Definition A.2. Consider any set of real bidders R and tuple (G, σ). We restrict the set of potential deviations for shill bidders to

$$\Sigma_S = \{\sigma'_S : \forall v_{-S}, \exists v_S \text{ such that } (\sigma'_S, \sigma_{-S}(v_{-S})) = \sigma(v_S, v_{-S}; R = B)\}.$$

Then, the tuple (G, σ) is an **auction equilibrium** if for all $i \in R$ and deviating strategies σ'_i,

$$\mathbb{E}_{v'_{-i}, \tilde{R}} \left[u_i \left(\sigma \left(v_i, v'_{-i}; \tilde{R} \right) \right) \right] \geqslant \mathbb{E}_{v'_{-i}, \tilde{R}} \left[u_i \left(\sigma'_i \left(v_i, v'_{-i}; \tilde{R} \right), \sigma_{-i} \left(v_i, v'_{-i}; \tilde{R} \right) \right) \right],$$

and for all $\sigma'_S \in \Sigma_S$,

$$\mathbb{E}_{v'_{-S}} \left[u_i \left(\sigma \left(0, v'_{-S}; R \right) \right) \right] \geqslant \mathbb{E}_{v'_{-S}} \left[u_i \left(\sigma'_S \left(0, v'_{-S}; R \right), \sigma_{-S} \left(0, v'_{-S}; R \right) \right) \right].$$

[28] We modify the definition to remove notation we do not use and to make it specific to auctions.

In Definition A.2, we are restricting shill bidders to acting "as-if" they are real bidders by restricting their actions to those of real bidders with some valuation profile. This restriction allows us to move to a direct mechanism where shill-proofness is defined as it being an equilibrium (ex-interim for weak shill-proofness, ex-post for strong shill-proofness) for all shill bidders to report 0.

Note that if we were to enlarge the set Σ_S to be the set $\hat{\Sigma}_S$ of all strategy profiles that are undetectable as coming from shill bidders—strategy profiles such that for every possible information set reached in equilibrium by a real bidder, there is a positive probability such information set would be reached in equilibrium if all bidders were real bidders—are main results would not change. For Theorems 3.5, 3.9 and 4.7, the public assumption immediately implies that $\Sigma_S = \hat{\Sigma}_S$. For Theorem 5.2, we know that $\Sigma_S \subset \hat{\Sigma}_S$ and our impossibility result must still hold if the set of possible deviations by shill bidders is larger; thus, the theorem still holds.

Lemma A.3. *An optimal auction (G, σ) is **winner-paying**: For all i and v,*

$$x_i\left(\sigma(v; B)\right) = 0 \implies t_i\left(\sigma(v; B)\right) = 0.$$

Proof. By the ex-post IR constraint, when $x_i\left(\sigma(v; B)\right) = 0$, we have $t_i\left(\sigma(v; B)\right) \leqslant 0$. It then follows from the optimality that $t_i\left(\sigma(v; B)\right) = 0$. To see this, note that for bidder $j \neq i$, equilibrium constraints on bidder j slacken when moving from $t_i < 0$ to $t_i = 0$ and so her play will remain the same. Meanwhile from bidder i, the transfer strictly increases moving from $t_i < 0$ to $t_i = 0$. ☐

Before we state our revelation principle in this context, we recall (with slight notational modifications) a definition and result from Akbarpour and Li (2020) that will be helpful in the proof.

Definition A.4 (Akbarpour and Li (2020), Definition 2)**.** A game equilibrium (G, σ) is **pruned** if, for any history h:

(i) There exists v such that $h \preceq z(\sigma(v; B))$.

(ii) If $h \notin Z$, then $|\mathrm{succ}(h)| \geqslant 2$.

(iii) If $h \notin Z$, then for $i = P(h)$, there exists v_i,v_i', and v_{-i} such that

  (a) $h \prec z(\sigma(v; B))$,

  (b) $h \prec z(\sigma(v_i', v_{-i}; B))$, and

  (c) $(x, t)\left(\sigma(v; B)\right) \neq (x, t)\left(\sigma(v_i', v_{-i}; B)\right)$.

Lemma A.5 (Akbarpour and Li (2020), Proposition 1)**.** *If (G, σ) is a game equilibrium, then there exists a game equilibrium (G', σ') that is pruned and for all v, $(x, t)\left(\sigma(v; B)\right) \neq (x', t')\left(\sigma'(v; B)\right)$.*

Lemma A.5 means that it is without loss with respect to outcomes to only consider game equilibria such that all outcome-equivalent histories are grouped into a single history and every single possible history in G is visited with positive probability under σ.

Lemma A.6 (Augmented Revelation Principle)**.** *For every public, game equilibrium* (G, σ) *there exists a* ***direct, public auction*** $(\tilde{x}, \tilde{t}, \mu, \xi_0)$ *that meets the following conditions:*

(i) There exists a direct mechanism $(\tilde{x}, \tilde{t})$ *such that for all* v*,*

$$\tilde{x}(v) = x(\sigma(v; B)), \tilde{t}(v) = t(\sigma(v; B)).$$

(ii) There exists a choice menu rule μ *that is a function of the potential values* $V = V_1 \times \cdots \times V_N$ *and bidder* ξ*. This rule has an output of* $L \geqslant 2$ *choices characterized as* $\left\{\left(W_\ell, \vec{\xi_\ell}\right)\right\}_{\ell \in \{1,\ldots,L\}}$ *where:*

(a) $\{W_\ell\}_{\ell \in L}$ *forms a partition of* V_ξ*,* $\vec{\xi_\ell} \in (B \cup \{\varnothing\}) \setminus \{\xi\}$*, and* $\vec{\xi} = \varnothing$ *signifies the game has ended.*

(b) For any ℓ *such that* $\vec{\xi_\ell} \neq \varnothing$*, let* $\hat{V}^\ell = (V_1, \ldots, V_{\xi-1}, W_\ell, V_{\xi+1}, \ldots, V_N)$*. Then, for any such* ℓ*, there exists* $v_{\xi_\ell}, v'_{\xi_\ell} \in \hat{V}^\ell_{\xi_\ell}, v_{-\xi_\ell} \in \hat{V}^\ell_{-\xi_\ell}$ *such that*

$$(\tilde{x}, \tilde{t})\left((v_{\xi_\ell}, v_{-\xi_\ell})\right) \neq (\tilde{x}, \tilde{t})\left((v'_{\xi_\ell}, v_{-\xi_\ell})\right).$$

If $v_\xi \in W_\ell$*, then the next player in the game is* $\vec{\xi_\ell}$ *and the menu presented to her is* $\mu(\hat{V}^\ell, \vec{\xi_\ell})$*.*

(c) If ℓ *is such that* $\vec{\xi_\ell} = \varnothing$*, then for all* $v, v' \in \hat{V}_\ell$*,*

$$\left(\tilde{x}, \tilde{t}\right)(v) = \left(\tilde{x}, \tilde{t}\right)(v').$$

(d) The first player to take an action is ξ_0*, who is presented the menu* $\mu(\mathcal{V}^N, \xi_0)$*.*

Proof. To see that condition i is true, we can simply iterate over all possible v and define $(\tilde{x}, \tilde{t})$ as the outcome of $\sigma(v; B)$ in G.

To prove condition ii, we first observe that by Definition A.2, shill bidders must act "as-if" they were real bidders and that we have restricted to pure strategies. Thus, we can always label actions as classes $\left(W_\ell, \vec{\xi_\ell}\right)$ of a partition of the remaining possible values for the current player ξ and satisfy condition iia. Condition iic follows from the fact that G is well defined (with each terminal history associated with a single outcome). Condition iid is simply mapping the first player in G to ξ_0 and the auctioneer has no information on bidders' values yet. The fact that $L \geqslant 2$ is equivalent to conditions i and ii of Definition A.4, and condition iii of Definition A.4 is equivalent to condition iib here. We can then apply Lemma A.5 to find a game that satisfies these properties. □

B Dutch Auctions (Section 3) Appendix

B.1 Strongly Shill-Proof Auctions (Section 3.1) Appendix

Given our augmented direct mechanism $(\tilde{x}, \tilde{t}, \mu, \xi_0)$, we define $X_i(v_i; V)$ and $T_i(v_i; V)$ to be the ex-interim quantity and transfer rules, respectively, when bidder i has value v_i and the set of potential values for all bidders is $V = V_1 \times \cdots \times V_N$.

Lemma B.1. *For every public, optimal, weakly shill-proof auction $(\tilde{x}, \tilde{t}, \mu, \xi_0)$, the ex-interim transfer rule for bidder i is*

$$T_i(v_i; V) = X_i(v_i; V)v_i - \sum_{m: v^{j_m} < v_i} \left[X_i(v^{j_m}; V) \cdot (v^{j_{m+1}} - v^{j_m})\right],$$

where $\{v^{j_m}\}_m$ are the ordered atoms of V_i.

Proof. To prove that T has the claimed form, we will consider a specific non-truthful reporting: if a bidder has value v^m, she commits to mis-reporting (selecting partitions) $v^{m'}$ for the rest of the game. We now follow the proof of Theorem 1 of Elkind (2007). Since our direct mechanism is truthful, we must have

$$X_i(v^{j_m}; V)v^{j_m} - T_i(v^{j_m}; V) \geqslant X_i(v^{j_{m-1}}; V)v^{j_m} - T_i(v^{j_{m-1}}; V),$$
$$X_i(v^{j_{m-1}}; V)v^{j_{m-1}} - T_i(v^{j_{m-1}}; V) \geqslant X_i(v^{j_m}; V)v^{j_{m-1}} - T_i(v^{j_m}; V).$$

Defining U_i to be the ex-interim utility for bidder i, the preceding expressions become:

$$U_i(v^{j_m}; V) \geqslant U_i(v^{j_{m-1}}; V) + (v^{j_m} - v^{j_{m-1}})X_i(v^{j_{m-1}}; V),$$
$$U_i(v^{j_{m-1}}; V) \geqslant U_i(v^{j_m}; V) - (v^{j_m} - v^{j_{m-1}})X_i(v^{j_m}; V).$$

Thus, we have

$$(v^{j_m} - v^{j_{m-1}})X_i(v^{j_{m-1}}; V) \leqslant U_i(v^{j_m}; V) - U_i(v^{j_{m-1}}; V) \leqslant (v^{j_m} - v^{j_{m-1}})X_i(v^{j_m}; V).$$

Hence, any IC mechanism is such that

$$U_i(v^{j_m}; V) = U_i(v^{j_1}; V) + \sum_{k=2}^{m}(v^{j_m} - v^{j_{m-1}})\tilde{X}_i(v^{j_m}; V)$$
$$\text{where } \tilde{X}_i(v^{j_m}; V) \in \left[X_i(v^{j_{m-1}}; V), X_i(v^{j_m}; V)\right].$$

Therefore, we have

$$T_i(v^{j_m}; V) = X_i(v^{j_m}; V)v^{j_m} - U_i(v^{j_1}; V) - \sum_{k=2}^{m}(v^{j_m} - v^{j_{m-1}})\tilde{X}_i(v^{j_m}; V). \tag{4}$$

By the ex-post IR condition, we have $U_i(v^{j_1}; V) \geqslant 0$ for all V. So, solving for the optimal transfer rule from Equation (4), we have

$$T_i^*(v^{j_m}; V) = \max_{U_i, \tilde{X}} \left[X_i(v^{j_m}; V)v^{j_m} - U_i(v^{j_1}; V) - \sum_{k=2}^{m}(v^{j_m} - v^{j_{m-1}})\tilde{X}_i(v^{j_m}; V)\right]$$
$$\text{such that } U_i(v^{j_1}; V) \geqslant 0 \text{ and } \tilde{X}_i(v^{j_m}; V) \in \left[X_i(v^{j_{m-1}}; V), X_i(v^{j_m}; V)\right]$$

The solution to this maximization is

$$U_i(v^{j_1}; V) = 0, \tilde{X}_i(v^{j_m}; V) = X_i(v^{j_{m-1}}; V).$$

Thus, Equation (4) becomes

$$T_i(v_i; V) = X_i(v_i; V)v_i - \sum_{m: v^{j_m} < v_i} \left[X_i(v^{j_m}; V) \cdot (v^{j_{m+1}} - v^{j_m})\right]. \qquad \square$$

For any value choice $(W, \cdot) \in \mu(\cdot, \cdot)$, let us define $\underline{W} = \min_{w \in W} \{w\}$ and $\overline{W} = \max_{w \in W} \{w\}$, respectively.

Lemma B.2 (Extended Pay-as-Bid)**.** *Consider a strongly shill-proof, public, optimal auction $(\tilde{x}, \tilde{t}, \mu, \xi_0)$. Fix V, ξ and consider any $(W, \vec{\xi}) \in \mu(V, \xi)$. If there exists $v, v' \in V$ such that $v_\xi, v'_\xi \in W$ and $\mu(V, \xi)$ is the last action ξ takes, then,*

$$\tilde{x}_\xi(v) = \tilde{x}_\xi(v') = 1 \implies \tilde{t}_\xi(v) = \tilde{t}_\xi(v') = \frac{T_\xi(\underline{W}; V)}{X_\xi(\underline{W}; V)},$$

i.e., transfers are constant conditional on allocation and are pinned down by the ex-interim outcome functions from the lowest type in the partition.

Proof. First we show if $\tilde{x}_\xi(v) = \tilde{x}_\xi(v')$, then $\tilde{t}_\xi(v) = \tilde{t}_\xi(v')$. Consider any v, v' such that V, ξ is reached and $\tilde{t}_\xi(v) \neq \tilde{t}_\xi(v')$ for $v_\xi, v'_\xi \in W$. Since the auction cannot distinguish between values in the same choice set, we suppose that $v_\xi = v'_\xi$. Now by Lemma A.3, ξ can only have two different transfers if that player wins the item under the allocation. Then, take $R = \{\xi\}$ and by monotonicity, $\tilde{t}_\xi(v) > \tilde{t}_\xi(v') \geqslant \tilde{t}(v'_\xi; 0)$ and thus shilling increases revenue. To conclude the proof and show that $\tilde{t}_\xi(v) = \tilde{t}_\xi(v') = \frac{T_\xi(\underline{W};V)}{X_\xi(\underline{W};V)}$, we note that ξ wins no matter what her value is in W and then apply Lemma A.3 to observe that $T_\xi(W; V) = \tilde{t}_\xi(v) \cdot X_\xi(W; V)$. $\square$

Proof of Proposition 3.4.

Consider any public, optimal and strongly shill-proof auction $(\tilde{x}, \tilde{t}, \mu, \xi_0)$. Let us observe that for any optimal auction, for all i, v_{-i} and $v_i, v'_i < \rho^*$, we have

$$\tilde{x}^*(v) = \tilde{x}^*(v'_i, v_{-i}).$$

Applying Lemma B.2, we then obtain that $\tilde{t}(v) = \tilde{t}(v'_i, v_{-i})$. Furthermore, by Lemma B.1, for all i, v_i, and V, V' such that $\{v^m \geqslant \rho^*\}^N \cap V = \{v^m \geqslant \rho^*\}^N \cap V'$, it is the case $X_i(v_i; V) = X_i(v_i; V')$ and $T_i(v_i; V) = T_i(v_i; V')$. Therefore, incentives for real and shill bidders are equivalent for such V, V'.

To make an auction testing, it must satisfy the implication in Definition 3.2—so in particular, the first menu(s) offered must satisfy the implication. Let $\mu^T(V, \xi)$ be defined as the consequent menu rule:

$$\mu^T(V, \xi) = \left\{ \left(\{w : w < \rho^*\}, \vec{\xi}_1 \right), \left(\{w : w \geqslant \rho^*\}, \vec{\xi}_2 \right) \right\},$$
$$\text{where } \vec{\xi}_1 = \begin{cases} \left(\cdot, \tilde{\xi}\right) = \min_\rhd \{(0, i) : i \neq \xi\} & \overline{V}_{-\xi} \geqslant \rho^* \\ \varnothing & \text{otherwise} \end{cases}$$
$$\text{and } \vec{\xi}_2 \text{ is arbitrary.}$$

Thus, to construct a testing auction that has the same outcomes as $(\tilde{x}, \tilde{t}, \mu, \xi_0)$, we simply prepend the required menus offered by setting $\xi'_0 = \min_\rhd \{(0, i)\}$ and menu rule to be:

$$\mu'(V, \xi) = \begin{cases} \mu^T(V, \xi) & \max_i \{\underline{V}_i\} < \rho^* \\ \mu(V, \xi) & \text{otherwise} \end{cases}.$$

Then, as argued in the preceding paragraph, the outcome and incentives will not change. We select $\vec{\xi}_2$ from Definition 3.2 as ξ_0.

To make an auction pooling, for any menu offered that does not satisfy Definition 3.3, define μ' such that any $w < \rho^*$ are all placed in a single menu choice. Once again, as argued above, the outcome and incentives will not change.

Proof of Theorem 3.5.

Consider the testing Dutch auction. We first show that it is a well-defined, i.e., that the stopping rule allows for the auction to be orderly and optimal.[29] We then show that it is strongly shill-proof. Finally, we show that there are no other public, testing, orderly, optimal auctions.

The Testing Dutch Auction is Orderly and Optimal. The testing Dutch auction quantity and transfer rule are orderly and optimal (as well as ex-post IR and monotone). Indeed, by construction, the next player $\vec{\xi}$ is always the player with the potentially highest value (including for tie-breaking). So, if that player indicates that she is of the highest possible type, the outcome (allocation and transfer) is fully determined and the auction ends. The auction ends once there are no players who could have values weakly greater than ρ^*.[30]

The Testing Dutch Auction is Strongly Shill-Proof. We now prove that a testing Dutch auction is strongly shill-proof. Seeking a contradiction, we suppose that it is not. There are two cases to consider: the first case is that a shill bidder selects the high partition in the testing stage and then always selects the low partition, i.e., misreports ρ^*; the second case is that the shill bidder selects the high partition in the testing stage and then selects the high partition at some later stage during the Dutch auction. In the first case, because the auction is testing and orderly, we know that outcomes for other bidders do not change as a function of whether the shill bidder has value 0 or ρ^*; therefore, by Lemma B.2, this deviation does not increase seller revenue. In the second case, there must exist some $S, \xi \in S$ and V such that $\{\overline{V}_\xi\}$ is selected from the menu $\mu(V, \xi)$. But by construction, this means that the auction immediately ends and the good is allocated to the shill bidder who misreported. By Lemma A.3, the revenue from this deviation is 0, which must be weakly less than any other possible transfer.

Uniqueness. Now, we consider a strongly shill-proof, optimal auction. We prove that it must be a Dutch auction. First consider $\tilde{\mu}$ such that there exists V, ξ such that

$$\tilde{\mu}(V, \xi) = \left\{ \left(W_{\mathsf{L}}, \tilde{\vec{\xi}}_{\mathsf{L}} \right), \left(W_{\mathsf{H}}, \tilde{\vec{\xi}}_{\mathsf{H}} \right) \right\},$$

where $\tilde{\vec{\xi}}_{\mathsf{L}} \neq \vec{\xi}_{\mathsf{L}}$. Now, observe that when the first bidder at any given level is not selected in lexicographic order, then the auction cannot end immediately. This is true even if the

[29] The auction is public and testing by definition.

[30] Recall that the testing step determines if at least one player has a valuation of at least the reserve.

first bidder chooses $\left\{\overline{V}_P\right\}$ by Definition 2.1. So, $\tilde{\vec{\xi}}_{\mathsf{H}} \neq \varnothing$. Note that WLOG, we can assume that bidder b_1 is called first, followed by bidder b_2 with higher priority than b_1, (potentially) followed by the remaining bidders. There must exist such a b_2 because otherwise the auction calls players in the same order as the Dutch auction, which we assumed was not the case. Let us consider the case where $R = B\backslash\{b_1\}, v_{b_2} = v^m$ and for all bidders $i \notin \{b_1, b_2\}, v_i = v^{m-2}$ for some m. Taking the expression from Lemma B.1 and dividing both sides by X_{b_2}, we get

$$\frac{T_{b_2}(v^m; V)}{X_{b_2}(v^m; V)} = v^m - \sum_{k: v^{j_k} < v^m} \frac{X_{b_2}\left(v^{j_{k-1}}; V\right)}{X_{b_2}\left(v^m; V\right)} \cdot \left(v^{j_k} - v^{j_{k-1}}\right) < v^m.$$

(Note that there must be at least one such k in the summation because otherwise $V_{b_2} = \{v^m\}$ and b_2 would not take an action.)

The last choice b_2 makes is to select a partition W such that $\underline{W} \geqslant v^{m-2}$. We can therefore apply Lemma B.2 to conclude that the transfer if b_1 does not shill and report 0 must be $\frac{T_{b_2}(\underline{W}; V)}{X_{b_2}(\underline{W}; V)} \leqslant \frac{T_{b_2}(v^m; V)}{X_{b_2}(v^m; V)} < v^m$. If bidder b_1 instead decides to act as if she has value v^m, then bidder b_2 will win and the revenue will be v^m and so the auction will not be strongly shill-proof. This argument also applies if only the first $n < N$ bidders are chosen in order because of the orderliness assumption. If $\tilde{\vec{\xi}}_{\mathsf{L}} = \vec{\xi}_{\mathsf{L}}$, then $\tilde{\vec{\xi}}_{\mathsf{H}} = \vec{\xi}_{\mathsf{H}}$ because the outcome is fully resolved once a bidder selects the high partition.

Next, consider arbitrary $\tilde{\mu}$ such that there exists V, ξ, where

$$\tilde{\mu}(V, \xi) \neq \mu^D(V, \xi).$$

By the previous argument, $\tilde{\mu}$ must differ from μ^D in the value partitions offered not just in the next player distribution. Observe that by the Definition 3.3, we know that there exists $\tilde{W}$ such that $\{w : w < \rho^*\} \subset \tilde{W}$. Then, because there must be at least two choices in the menu, there exists $(W, \xi) \in \mu(V, \xi)$ such that $\rho^* \leqslant \underline{W}_\ell < \overline{V}_\xi$. It is without loss to suppose that $\underline{W}_\ell \leqslant \overline{V}_{-\xi}$. To see this, note that for all $\ell, V, \xi, (W_\ell, \cdot) \in \mu^D(V, \xi)$, it is the case that $\underline{W}_\ell \leqslant \overline{V}_{-\xi}$ and so the first time that $\tilde{\mu}$ differs from μ^D, it is the case that $\underline{W}_\ell \leqslant \overline{V}_{-\xi}$.

Now, consider the case where there exists m^* such that $\rho^* \leqslant \underline{W}_\ell \leqslant v^{m^*} < \overline{V}_{-\xi}$. Then suppose there exists bidder $i \in R$ such that $v_i \geqslant v^{m^*+1}$; take bidder $\xi \in S$ to shill v^{m^*}; and for $k \notin \{i, \xi\}$, take $v_k = \underline{V}_k < v^{m^*}$. By the testing property of the auction, we can suppose i is such that $(v^{m^*}, \xi) \rhd (\rho^*, i)$. Therefore, by Lemma B.1, observe that for the last action i takes, her ex-interim transfer must be higher when shill ξ reports v^{m^*} than when she reports 0. Thus by Lemma B.2, when $v^{m^*} < \overline{V}_{-\xi}$, there exists a valuation vector v such that a shill bidder would want to deviate away from reporting 0—and therefore such an auction is not strongly shill-proof.

Finally, consider the case where $\rho^* \leqslant \underline{W}_\ell = \overline{V}_{-\xi}$. Since this is our last case, we can without loss suppose that V has been generated via a Dutch auction so far and therefore that ξ is such that for all j, $(\overline{V}_j, j) \rhd (\overline{V}_j, \xi)$, i.e., the current player has the lowest tie-breaking priority. Supposing $j \in R$ and $v_j = \overline{V}_{-\xi}$, take bidder $\xi \in S$ to report $\overline{V}_{-\xi}$; and for $k \notin \{i, \xi\}$, take $v_k = \underline{V}_k < \overline{V}_{-\xi}$. As noted, $(\overline{V}_j, j) \rhd (\overline{V}_j, \xi)$ and so bidder j is allocated the item and not shill bidder ξ. Therefore, by the same argument as above, shill bidding will increase revenue.

B.2 Weakly Shill-Proof and Efficient Auctions (Section 3.2) Appendix

In order to build towards a proof that if an auction is not a semi-Dutch auction, we can find a regular distribution such that the auction is not weakly shill-proof, we will prove that for a certain class of value distributions, every *weakly* shill-proof and efficient auction must have part of its game tree be a Dutch auction. Formally, we assume that the value distribution is sparse:

Definition B.3. A regular distribution F is **sparse** if for all $k < \rho^*$,

$$v^k \cdot \left(1 + \frac{f(v^k)}{f(v^{k+1})}\right) < v^{k+1}. \tag{5}$$

A distribution is sparse if the atoms are sufficiently far apart. Sparsity can also be a reasonable assumption if the auctioneer has preferences for the auction to be completed quickly, or otherwise finds it costly to distinguish between values that are close to each other.

Sparsity is helpful in restricting the set of auctions that are weakly shill-proof because it means that imposing a binding reserve price drives revenue up by a large amount—which means there is an incentive to shill in order to in effect introduce an endogenous binding reserve. Note that if Equation (5) holds for some k, then $\varphi^k < 0$—yet the converse, that $\varphi^k < 0$ implies Equation (5), does not necessarily hold.

Lemma B.4. *Consider an efficient auction and suppose F is regular and sparse. Let R, V such that $V_i = \{v : v \in [\underline{V}_i, \overline{V}_i]\}$ for all $i \in R$, and consider $(\underline{W}, j)$ such that $\underline{W} < \rho^*, j \notin R$ and for all $i \in R, (\overline{V}_i, i) \rhd (\underline{W}, j)$. Then, for all $\gamma < \underline{W}$,*

$$\mathbb{E}\left[\sum_{i \in R} \tilde{t}_i(v) \mid V = (V_{-j}, \{\gamma\})\right] < \mathbb{E}\left[\sum_{i \in R} \tilde{t}_i(v) \mid V = (V_{-j}, \{\underline{W}\})\right].$$

Thus, the following shilling strategy is profitable compared to always reporting 0: if there exists (V, ξ, W) such that $(W, \cdot) \in \mu(V, \xi)$ and $\underline{W} \in (0, \rho^*)$, then select W. Otherwise, select the partition containing 0.

Proof. As we showed in the proof of Lemma B.1, for all $i \in R$ and V, the incentive compatibility constraint implies Equation (4):

$$T_i(v^{j_m}; V) = X_i(v^{j_m}; V)v^{j_m} - \sum_{k<m} \left[\tilde{X}_i^k(V) \cdot (v^{j_{k+1}} - v^{j_k})\right] - U_i(v^{j_1}; V),$$
$$\text{where } \tilde{X}_i^k(V) \in \left[X_i(v^{j_k}; V), X_i(v^{j_{k+1}}; V)\right].$$

Then, for all $i \in R$ and V,

$$\begin{aligned}
\mathbb{E}\left[\tilde{t}_i(v) \mid V\right] &= \mathbb{E}\left[T_i(v_i; V)\right] \\
&= \frac{1}{F(\overline{V}_i)} \sum_m f(v^{j_m}) T_i(v^{j_m}; V) - U_i(v^{j_1}; V) \\
&= \frac{1}{F(\overline{V}_i)} \sum_m f(v^{j_m}) \left(X_i(v^{j_m}; V) v^{j_m} - \sum_{k<m} \left[\tilde{X}_i^k(V) \cdot (v^{j_{k+1}} - v^{j_k}) \right] \right) - U_i(v^{j_1}; V) \\
&= \frac{1}{F(\overline{V}_i)} \left[\sum_m f(v^{j_m}) X_i(v^{j_m}; V) v^{j_m} - \sum_m \sum_{k<m} f(v^{j_m}) \left[\tilde{X}_i^k(V) \cdot (v^{j_{k+1}} - v^{j_k}) \right] \right] - U_i(v^{j_1}; V) \\
&= \frac{1}{F(\overline{V}_i)} \sum_m \left[v^{j_m} X_i(v^{j_m}; V) - (v^{j_{m+1}} - v^{j_m}) \frac{F(\overline{V}_i) - F(v^{j_m})}{f(v^{j_m})} \tilde{X}_i^m(V) \right] f(v^{j_m}) - U_i(v^{j_1}; V).
\end{aligned}$$

By sparsity, for all $m < \rho^*$, we have

$$v^m - (v^{m+1} - v^m) \frac{f(v^{m+1})}{f(v^m)} < 0.$$

Applying Definition 2.1 to an efficient allocation rule $\tilde{x}$, we know that for $(v^m, i) \rhd (\gamma, j)$ and $(v^m, i) \rhd (\gamma', j)$, we can define

$$X_i(v^m; V_{-j}) \equiv X_i(v^m; V_{-j}, \{\gamma\}) = X_i(v^m; V_{-j}, \{\gamma'\}).$$

Note that $\underline{W} \leqslant \min_i \{\overline{V}_i\}$ by assumption and therefore, for $\underline{W} \in (\gamma, \rho^*)$,

$$\begin{aligned}
&\mathbb{E}\left[\tilde{t}_i(v) \mid V = (V_{-j}, \{\gamma\})\right] - \mathbb{E}\left[\tilde{t}_i(v) \mid V = (V_{-j}, \{\underline{W}\})\right] \\
&= \frac{1}{F(\overline{V}_i)} \sum_{m: \gamma \leqslant v^{j_m} < \underline{W}} \left[v^{j_m} X_i(v^{j_m}; V) - (v^{j_{m+1}} - v^{j_m}) \frac{F(\overline{V}_i) - F(v^{j_m})}{f(v^{j_m})} \tilde{X}_i^m(V) \right] f(v^{j_m}) \\
&\leqslant \frac{1}{F(\overline{V}_i)} \sum_{m: \gamma \leqslant v^{j_m} < \underline{W}} \left[v^{j_m} - (v^{j_m+1} - v^{j_m}) \frac{f(v^{j_m+1})}{f(v^{j_m})} \right] f(v^{j_m}) X_i(v^{j_m}; V_{-j}) \\
&< 0
\end{aligned}$$

where the final inequality comes from sparsity. And so,

$$\mathbb{E}\left[\sum_{i \in R} \tilde{t}_i(v) \mid V = (V_{-j}, \{\gamma\}) \right] < \mathbb{E}\left[\sum_{i \in R} \tilde{t}_i(v) \mid V = (V_{-j}, \{\underline{W}\}) \right],$$

as claimed in the lemma. Thus, committing to misreport as $\underline{W}$ is strictly beneficial compared to any strategy that can only report $\gamma < \underline{W}$. $\square$

Lemma B.5. *If F is regular and sparse, then every public, weakly shill-proof, and efficient auction is a semi-Dutch auction with cutoff ρ^*.*

Proof. Suppose F is regular and sparse. Consider an arbitrary weakly shill-proof and efficient auction, $(\tilde{x}, \tilde{t}, \mu, \xi_0)$, and consider any v such that $\max_i \{v_i\} < \rho^*$.

Case 1: First, we prove that, for any player ξ and set of possible values V such that there exists $(W,\cdot) \in \mu(V,\xi)$ where $0 < \underline{W} < \rho^*$, it is the case that $V \subseteq \check{V}$. Towards contradiction, suppose that there exists a (ξ, V, W) such that $V \nsubseteq \check{V}$, $(W,\cdot) \in \mu(V,\xi)$, and $\underline{W} \in (0,\rho^*)$. Let us suppose $\xi \in S$.

If such a (ξ, V, W) exists, then there exists $\left(\tilde{\xi}, \tilde{V}, \tilde{W}\right)$ such that $\tilde{V} \nsubseteq \check{V}$, $\left(\tilde{W},\cdot\right) \in \mu\left(\tilde{V},\tilde{\xi}\right)$, $\underline{\tilde{W}} \in (0,\rho^*)$, and for all i, $\underline{\tilde{V}}_i = 0$ or $\underline{\tilde{V}}_i \geqslant \rho^*$ and therefore it is without loss to assume (ξ, V, W) is such that for all i, $\underline{V}_i = 0$ or $\underline{V}_i \geqslant \rho^*$. To see this is true, if there exists (ξ, V, W) with i such that $\underline{V}_i \in (0,\rho^*)$, let us label that set as V^K and let $V^0 \supset V^1 \supset \ldots \supset V^K$ be the sequence of possible value sets up to this possible value set. Let the players called along the path be $\xi^0, \xi^1, \ldots, \xi^K$ and the value partition selected by player k to be W^k. Note that $V^0 = \mathcal{V}^N$ is such that for all i, $\underline{\tilde{V}}_i = 0$ or $\underline{\tilde{V}}_i \geqslant \rho^*$. So, the set $\mathcal{K} = \left\{k < K : \underline{W}^k \in (0,\rho^*)\right\}$ must be non-empty and therefore $k^* = \min_{k\in\mathcal{K}}\{k\}$ is well-defined. If k is such that $\underline{W}^k \notin (0,\rho^*)$ and for all i, $\underline{V}_i^k \notin (0,\rho^*)$, then it must be the case that for all i, $\underline{V}_i^{k+1} \notin (0,\rho^*)$. Since k^* is the first time in the game that a player selects a partition with $\underline{W}^k \in (0,\rho^*)$, it must be the case that for all i, $\underline{V}_i^k = 0$ or $\underline{\tilde{V}}_i \geqslant \rho^*$. Since $V^{k^*} \supset V^K$, $V^{k^*} \nsubseteq \check{V}$. Thus, $(\xi^{k^*}, V^{k^*}, W^k)$ is such that $V^{k^*} \nsubseteq \check{V}$, $\left(W^{k^*},\cdot\right) \in \mu\left(V^{k^*},\xi^{k^*}\right)$, $\underline{W}^{k^*} \in (0,\rho^*)$, and for all i, $\underline{V}_i^{k^*} = 0$ or $\underline{V}_i^{k^*} \geqslant \rho^*$.

So, in order to have $\underline{W} \in (0,\rho^*)$, it must be the case that $0 \in V_\xi$. Thus it is possible for ξ to be a shill bidder while having so far only selected partitions that contain 0. Let $S = \left\{i : \overline{V}_i < \rho^*\right\} \cup \{\xi\}$. By assumption that $V \nsubseteq \check{V}$, there must exist i such that $\overline{V}_i \geqslant \rho^*$ and thus $R \neq \varnothing$. By assumption that $\underline{W} \in (0,\rho^*)$, it is therefore the case that for all $i \in R$, $\overline{V}_i > \underline{W}$. By Lemma B.4, this would contradict the hypothesis that the auction is weakly shill-proof and so we must have $V \subseteq \check{V}$ when there exists $(W,\cdot) \in \mu(V,\xi)$ such that $0 < \underline{W} < \rho^*$.

Case 2: We now prove that for any player ξ and set of possible values $V \subseteq \check{V}$, it is the case that $\mu(V,\xi) = \mu^D(V,\xi)$. Consider any option $\left(W,\vec{\xi}\right) \in \mu(V,\xi)$. Observe that by Lemma B.4, it is not the case that $0 < \underline{W} < \overline{V}_{-\xi}$. So, $\underline{W} \geqslant \overline{V}_{-\xi}$. Since this is the case for all V, it must therefore be true that $\underline{W} = \overline{V}_\xi$. This is because if $\overline{V}_\xi > \underline{V} \geqslant \overline{V}_{-\xi}$, then there must have existed some earlier menu $\left(\tilde{W},\cdot\right) \in \mu\left(\tilde{V},\tilde{\xi}\right)$ for which $\underline{\tilde{W}} < \overline{\tilde{V}}_{\tilde{\xi}}$.

So far we have proven that

$$\mu(V,\xi) = \left\{\left(W_{\mathsf{L}}, \tilde{\vec{\xi}}_{\mathsf{L}}\right), \left(W_{\mathsf{H}}, \tilde{\vec{\xi}}_{\mathsf{H}}\right)\right\}.$$

To complete the proof, we have to prove that $\tilde{\vec{\xi}}_{\mathsf{L}} = \vec{\xi}_{\mathsf{L}}, \tilde{\vec{\xi}}_{\mathsf{H}} = \vec{\xi}_{\mathsf{H}}$. Seeking a contradiction, suppose $\tilde{\vec{\xi}}_{\mathsf{L}} \neq \vec{\xi}_{\mathsf{L}}$ or $\tilde{\vec{\xi}}_{\mathsf{H}} \neq \vec{\xi}_{\mathsf{H}}$. If $\tilde{\vec{\xi}}_{\mathsf{H}} \neq \vec{\xi}_{\mathsf{H}}$, then, by Lemma A.6, Condition (ii), there exists i such that $\overline{V}_i = \overline{V}_\xi$, $\left(\overline{V}_i, i\right) \rhd \left(\overline{V}_\xi, \xi\right)$. We can let $R = \{i\}$ and then apply Lemma B.4 to contradict the hypothesis that the auction is weakly shill-proof. If $\tilde{\vec{\xi}}_{\mathsf{L}} \neq \vec{\xi}_{\mathsf{L}}$, then, as argued in the proof of Theorem 3.5, the menu presented to $\tilde{\vec{\xi}}_{\mathsf{L}}$ must not have the auction end immediately, no matter what partition $\tilde{\vec{\xi}}_{\mathsf{L}}$ selects. Thus, our previous argument for the case where $\tilde{\vec{\xi}}_{\mathsf{H}} \neq \vec{\xi}_{\mathsf{H}}$ applies, and we can conclude that $\mu(V,\xi) = \mu^D(V,\xi)$. $\square$

Proof of Theorem 3.9

The statement follows as a corollary of Lemma B.5. Consider any optimal reserve ρ^*, $\underline{M}$ atoms below the optimal reserve, and $\overline{M}$ atoms above the optimal reserve. We construct a sparse (and regular) distribution $\tilde{F}$ has optimal reserve ρ^*, $\underline{M}$ atoms below the optimal reserve, and $\overline{M}$ atoms above the optimal reserve. Specifically, we take $\tilde{F}$ to be a discrete approximation of an exponential distribution below the reserve and has the same number of atoms below the same optimal reserve. To begin, we assume that the atoms of $\tilde{F}$ are equally spaced, i.e., $v^{k+1} - v^k = \delta$ for all k such that $ks \leqslant \rho^*$ (we assume the atom directly above the reserve is also part of this distribution). The associated pmf is then

$$\tilde{f}(v^k) = \int_{v^{k-1}}^{v^k} \lambda e^{-\lambda x} dx = e^{-\lambda(k-2)\delta} - e^{-\lambda(k-1)\delta}.$$

Note that

$$\tilde{\varphi}^k = v^k - (v^{k+1} - v^k)\frac{1 - \tilde{F}(v^k)}{\tilde{f}(v^k)} = (k-1)\delta - \delta\frac{e^{-\lambda(k-1)\delta}}{e^{-\lambda(k-2)\delta} - e^{-\lambda(k-1)\delta}},$$

and so

$$\tilde{\varphi}^{k+1} - \tilde{\varphi}^k = k\delta - (k-1)\delta = \delta > 0;$$

hence, $\tilde{F}$ is regular.

In order for ρ^* to be an optimal reserve of $\tilde{F}$, it must be the case that for k^* such that $\tilde{\varphi}^{k^*} \geqslant 0$ and $\tilde{\varphi}^{k^*-1} < 0$, it is also the case that $\rho^* \in ((k^*-1)\delta, k^*\delta]$. Such a k^* must be equal to $\lceil \check{k} \rceil$, where

$$\check{k}\delta - \frac{\delta}{e^{\lambda\delta} - 1} = 0.$$

Thus, $\underline{M} - 1 = k^* = \lceil \frac{1}{e^{\lambda\delta}-1} \rceil$.

In order for $\tilde{F}$ to be sparse, it must satisfy Equation (5), which here simplifies to

$$(k-1)\delta \cdot \left(1 + \frac{e^{-\lambda(k-2)\delta} - e^{-\lambda(k-1)\delta}}{e^{-\lambda(k-1)\delta} - e^{-\lambda(k)\delta}}\right) < k\delta \implies \frac{e^{2\lambda\delta} - 1}{e^{\lambda\delta} - 1} < \frac{k}{k-1}.$$

So, $\tilde{F}$ is sparse if

$$e^{2\lambda\delta} < \frac{2 + (e^{\lambda\delta} - 1)(2(k^* - \check{k}) - 1)}{1 - ((k^* - \check{k}) - 1)(e^{\lambda\delta} - 1)}. \tag{6}$$

Selecting (λ, δ) such that $\underline{M} - 1 = \hat{k} = k^*$, Equation (6) is satisfied.

Finally, to finish constructing $\tilde{F}$, we simply select atoms $v^{\underline{M}+2}, \ldots, v^{\underline{M}+\overline{M}}$ and respective probability weights subject to the conditions that

$$\tilde{\varphi}^k \text{ is non-decreasing, and} \tag{7}$$

$$\sum_{k=\underline{M}+2}^{\underline{M}+\overline{M}} \tilde{f}(v^k) = e^{-\lambda(\underline{M}+1)\delta}. \tag{8}$$

The system of constraints (7)–(8) has at most $\overline{M}$ constraints and $2(\overline{M} - 1)$ free variables, so the system can be satisfied.

Thus, we have constructed a regular and sparse $\tilde{F}$ that has the required values of ρ^*,$\underline{M}$, and $\overline{M}$. Then, we can apply Lemma B.5 to conclude the proof.

C Weakly Shill-Proof and Strategy-Proof Auctions (Section 4) Appendix

Proof of Theorem 4.7.

The Ascending, Screening Auction is Orderly and Optimal. We first prove that the auction is well-defined while being orderly and optimal. The transfer and allocation function are orderly and optimal, so we only have to show that the menu rule can induce this outcome function. Let us examine the English auction phase first. The auction ends if and only if $\overline{V} < v^M$. When that occurs, the auction has determined v_i for all i given $v_i \geqslant \rho^*$. Thus, the outcome is fully determined. In the second-price auction phase, each value weakly greater than v^Y is determined precisely (and there are at least two players with values weakly greater than v^Y) and so the outcome rule is determined.

The Ascending, Screening Auction is Strategy-Proof. Observe from Definition 4.1, that the definition of strategy-proofness is a function solely of the direct mechanism $\tilde{x}, \tilde{t}$ and not of the menu rule. Both the English auction phase and the second-price auction use the same transfer function: the second highest value is the winner's transfer. The entire auction has the optimal allocation rule and so the ascending, screening auction has the same direct mechanism as the second-price auction which is strategy-proof Vickrey (1961).

The Ascending, Screening Auction is Weakly Shill-Proof. By assumption that the screen level is at least ρ^*, any potential shill bidder will be asked to play at least once in the English auction before being able to play in the second-price auction. If the optimal shill bid is 0 in the first round of the English auction, then the auction is weakly shill-proof because once a bidder reports 0, she "drops out" of the auction and does not take another action.

Observe that given the form of the transfer rule, the maximum amount that a bidder i with value v_i would have to pay is v_i. Thus, the maximum possible gain in revenue from a shill bidder deviating is at most the difference between the first and second moment of v. Next, note that MHR distributions are regular. Regularity implies that if a shill bidder reports a non-zero value in the English auction stage and the auction concludes before reaching the second-price stage, the expected gain must be weakly less than 0. So, when considering the expected gain of misreporting, we can think of the expected gain from manipulating outcomes in the English auction component as at most 0 and can focus on manipulating outcomes in the second-price stage. Therefore, the total gains from misreporting as a shill bidder must be bounded above by the probability that a shill bidder is able to manipulate the outcome of the second-price auction multiplied by the expected difference between the first and second moments of the value distribution conditional on reaching the second-price auction stage.

Let $\mathcal{F}$ be the continuous distribution for which F is a discrete approximation. For an exponential distribution with rate λ, the expected difference between the first and second moments of T independent draws is $\frac{1}{\lambda}$. The exponential distribution, with its constant hazard rate, has the thickest right tail of any MHR distribution and so has the largest expected difference between its first and second moments (see proof of Theorem 5.1 in Bahrani et al.

(2024)). In particular, since we are only interested in value draws above the reserve $\rho^*_{\mathcal{F}}$ and $\mathcal{F}$ has a non-decreasing hazard rate, we can take the rate $\lambda = h(\rho^*_{\mathcal{F}}) = \frac{1}{\rho^*_{\mathcal{F}}}$ and conclude that the maximum difference between the first and second moments of $\mathcal{F}$ must be bounded above by $\rho^*_{\mathcal{F}}$. Recall that $h(\rho^*_{\mathcal{F}}) = \frac{1}{\rho^*_{\mathcal{F}}}$ because $\mathcal{F}$ is regular and $\rho^*_{\mathcal{F}}$ is the optimal reserve of $\mathcal{F}$. Examining Equation (2), we can see that our discrete approximation pools draws from a continuous distribution upwards to atoms and so, if the absolute difference between two samples of the continuous distribution is κ, the absolute difference between the discrete approximation samples would be at most $\kappa + \overline{\Delta}$. Thus, the maximum possible expected difference between the first and second moments of F conditional on being above the reserve is at most $\rho^*_{\mathcal{F}} + \overline{\Delta}$. Further note that this also implies that $\left|\rho^* - \rho^*_{\mathcal{F}}\right| \leqslant \overline{\Delta}$.

Suppose bidder i is a shill bidder and it is the first time she is taking an action. Then, under the rules of the auction, she has not indicated that her value is greater than ρ^* yet. For any real bidder $j \neq i$, there are two cases: either she has indicated her value is weakly greater than ρ^* or she has not yet taken an action. In the first case, the probability $v_j < v^Y$ is $F(v^Y)$. In the second case, the probability $v_j < v^Y$ is $F(v^Y) - F(\rho^*)$. So, the probability is at least $F(v^Y) - F(\rho^*)$. So, if $K \leqslant N$ real bidders have not dropped out yet (i.e., indicated that their value is less than ρ^*), then the probability that the auction would continue to the second-price auction is at most $1 - \left(F\left(v^Y\right) - F\left(\rho^*\right)\right)^K$. Therefore, the maximum expected gain for a shill bidder from misreporting in her first action of the English auction phase when K bidders have not dropped is at most

$$\left(1 - \left(F\left(v^Y\right) - F\left(\rho^*\right)\right)^K\right)\left(\rho^*_{\mathcal{F}} + \overline{\Delta}\right). \tag{9}$$

We now turn to bounding the loss from reporting a non-zero value as a shill bidder. If shill bidder i misreports her value as v^m at some point in the English auction phase and then she wins the item without taking another action, then the transfer the seller would have received had i not misreported is at least $\max\{\rho^*, v^{m-1}\} \geqslant \rho^*$, assuming at least one real bidder has value weakly above the reserve. To bound the probability that a real bidder j would have won the item if not for shill bidder i's misreport, we can consider the probability that bidder j has indicated her value is at least $\underline{V}_j \geqslant \rho^*$. By Definition 4.5, the hazard rate of $\mathcal{F}$ is non-decreasing. So,

$$\mathbb{P}\left[v_j \leqslant v^m\right] \geqslant \frac{\sum_{\left\{k:\underline{V}_j \leqslant v^k < v^m\right\}} f(v^k)}{1 - F(\underline{V}_j)} \geqslant \frac{f(\underline{V}_j)}{1 - F(\underline{V}_j)} \geqslant \frac{f(\rho^*)}{1 - F(\rho^*)}.$$

Combining the preceding inequality with our hypothesis that K bidders have not dropped out yet, the expected loss for a shill bidder of misreporting is at least

$$\rho^* \cdot \left(\frac{f(\rho^*)}{1 - F(\rho^*)}\right)^K. \tag{10}$$

We conclude the proof by showing that v^Y satisfying Equation (3) implies that the expected revenue loss from misreporting as a shill is weakly larger than the expected gain.

Beginning with Equation (3), we can see that for all $K < N$,

$$
\begin{aligned}
v^S &\geqslant F^{-1}\left(F\left(\rho^*\right) + \max_{1\leqslant n<N}\left\{\left(\max\left\{1-\frac{\rho^*}{\rho^*+2\overline{\Delta}}\left(\frac{f(\rho^*)}{1-F(\rho^*)}\right)^n, 0\right\}\right)^{1/n}\right\}\right)\\
&\geqslant F^{-1}\left(F\left(\rho^*\right) + \left(\max\left\{1-\frac{\rho^*}{\rho^*+2\overline{\Delta}}\left(\frac{f(\rho^*)}{1-F(\rho^*)}\right)^K, 0\right\}\right)^{1/K}\right)\\
&\geqslant F^{-1}\left(F\left(\rho^*\right) + \left(\max\left\{1-\frac{\rho^*}{\rho^*_{\mathcal{F}}+\overline{\Delta}}\left(\frac{f(\rho^*)}{1-F(\rho^*)}\right)^K, 0\right\}\right)^{1/K}\right).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
(F(v^S)-F(\rho^*))^K &\geqslant \max\left\{1-\frac{\rho^*}{\rho^*_{\mathcal{F}}+\overline{\Delta}}\left(\frac{f(\rho^*)}{1-F(\rho^*)}\right)^K, 0\right\}\\
&\geqslant 1-\frac{\rho^*}{\rho^*_{\mathcal{F}}+\overline{\Delta}}\left(\frac{f(\rho^*)}{1-F(\rho^*)}\right)^K.
\end{aligned}
$$

Therefore,

$$
\rho^* \cdot \left(\frac{f(\rho^*)}{1-F(\rho^*)}\right)^K \geqslant \left(1-(F(v^Y)-F(\rho^*))^K\right)\left(\rho^*_{\mathcal{F}}+\overline{\Delta}\right).
$$

The left-hand side of the preceding equation corresponds to Equation (10), the lower bound on the expected loss from misreporting as a shill bidder, and the right-hand side corresponds to Equation (9), the upper bound on the expected gain from misreporting and thus we have shown that it is equilibrium not to shill when v^Y is sufficiently high.

D One-Shot Auctions (Section 5) Appendix

Definition D.1. Fix a one-shot, optimal auction and a set of real bidders R. The set of safe deviations to report to $i \in B$ is

$$
\begin{aligned}
&\mathcal{A}_i(v_{j\leqslant i}) =\\
&\quad \left\{a : \exists \tilde{v}_{-i} \text{ such that } \left[j<i, v_j \neq 0 \implies \tilde{v}_j = v_j\right] \text{ and } a = \left(\sigma_i(v_i;\psi_i(\tilde{v}_{j<i})), \sigma_{-i}\left(v_i, \tilde{v}_{-i}; B\right)\right)\right\}.
\end{aligned}
$$

The total set of **safe deviations** is

$$
\mathcal{A}(v) = \left\{\left\{a^{\rightsquigarrow i}\right\} : a^{\rightsquigarrow i} \in \mathcal{A}_i(v_{j\leqslant i}) \text{ and } \sum_{i=1}^{N} x_i\left(a^{\rightsquigarrow i}\right) \leqslant 1\right\}.
$$

Definition D.1 allows the auctioneer to report any value she chooses when a bidder's declared valuation is 0. Note that if we take the canonical setting where there are no exogenous signals, the above assumption is without loss.

Definition D.2. A one-shot, optimal auction is **credible** if for all v and $\{a^{\rightsquigarrow i}\} \in \mathcal{A}(v)$, we have

$$\sum_i t_i(a^{\rightsquigarrow i}) \leqslant \sum_i t_i(\sigma(v; B)).$$

Lemma D.3. *For any one-shot, optimal auction, there exist unique*

$$\tilde{x} : \mathcal{V}^N \to \{0,1\}^N \quad \textit{and} \quad \tilde{t} : \mathcal{V}^N \to \mathbb{R}^N$$

such that:

(i) (Correspondence) For all $v \in \mathcal{V}^N$,

$$\tilde{x}(v) = x(\sigma(v; B)) \text{ and } \tilde{t}(v) = t\left(\sigma(v; B)\right). \tag{11}$$

(ii) (Individual Rationality) For all $i \in B$ and v,

$$\tilde{x}_i(v)v_i - \tilde{t}_i(v) \geqslant 0 \tag{12}$$

(iii) (Incentive Compatibility) For all R, $i \in R$, v_i, and v_i',

$$\begin{aligned}\mathbb{E}_{v_{-i},\tilde{R}} &\left[\tilde{x}_i\left(\sigma(v; \tilde{R})\right) v_i - \tilde{t}_i(\sigma(v; \tilde{R})) \mid v_i, s_i = \psi(v_{j<i})\right] \\ &\geqslant \mathbb{E}_{v_{-i},\tilde{R}}\left[\tilde{x}_i\left(\sigma(v_i', v_{-i}; \tilde{R})\right) v_i - \tilde{t}_i\left(\sigma(v_i', v_{-i}; \tilde{R})\right) \mid v_i, s_i = \psi(v_{j<i})\right].\end{aligned} \tag{13}$$

Proof. To begin, let us note that we can define $(\tilde{x}, \tilde{t})$ as claimed point-wise from $\sigma(v; B)$. Next, the IR constraint (Equation (12)) follows immediately from the ex-post IR condition and our construction of $(\tilde{x}, \tilde{t})$. Finally, Equation (13) comes from Definition A.2 and recalling that we restrict shill bidders to actions that could have been taken by real bidders. ☐

Lemma D.4. *Suppose a one-shot, optimal auction is weakly shill-proof, but not strongly shill-proof. Then, there exist R, v_R, and v_{-R} such that*

$$\sum_{k \in R} \tilde{t}_k(v_R, v_{-R}) > \sum_{k \in R} \tilde{t}_k(v_R, 0). \tag{14}$$

Proof. Suppose that (G, σ) is weakly shill-proof, but not strongly shill-proof. Because (G, σ) is weakly shill-proof, for all v and R, R', we can define $\hat{\sigma}(v) \equiv \sigma(v; R) = \sigma(v; R')$. Since (G, σ) is not strongly shill-proof, $\hat{\sigma}$ must not be a dominant strategy for the shill bidders. So, for some realization of R and v_R there exists a profitable deviation for the shill bidders; examining the set of possible deviations Σ_S in Definition A.2, we see that any profitable deviating actions induces a profitable misreport v_S in the direct mechanism. Therefore, we have found R, v_R, and v_{-R} such that Equation (14) is satisfied. ☐

Lemma D.5. *If a one-shot, optimal auction is weakly shill-proof, then for all R, $v_{j<\min S}$,*[31] *and $\{v_i\}_{i \in S}$,*

$$\mathbb{E}_v\left[\sum_{k \in R} \tilde{t}_k\left(\{v_i\}_{i \in S}, \{v_i\}_{i \notin S}\right) \mid v_{j<\min S}\right] \leqslant \mathbb{E}_v\left[\sum_{k \in R} \tilde{t}_k\left(0, \{v_i\}_{i \notin S}\right) \mid v_{j<\min S}\right].$$

[31] This is condensed notation for the set $\{v_j : j < \min_{i \in S}\{i\}\}$ where $S = B \backslash R$ is the set of shill bidders.

Proof. Towards contradiction, suppose there exists R, $v_{j<\min S}$, and $\{v_i\}_{i\in S}$ such that

$$\mathbb{E}_v\left[\sum_{k\in R}\tilde{t}_k\left(\{v_i\}_{i\in S}, \{v_i\}_{i\notin S}\right) \mid v_{j<\min S}\right] > \mathbb{E}_v\left[\sum_{k\in R}\tilde{t}_k\left(0, \{v_i\}_{i\notin S}\right) \mid v_{j<\min S}\right].$$

We now prove that the deviation by the coalition S where they report $\{v_i\}_{i\in S}$ is profitable and therefore that the auction is not weakly shill-proof. By assumption, a shill bidder observes actions by all bidders who take actions before her. So, $\{v_i\}_{i\in S}$ can condition on $v_{j<\min S}$ when making decisions. Then, the strategy by S of committing to report $\{v_i\}_{i\in S}$ regardless of what other bidders play after $\min_{i\in S}\{i\}$ must be strictly profitable compared to always reporting 0. Thus, we have found a strategy that does strictly better than always reporting 0: When the values before $\min_{i\in S}\{i\}$ are reported as $v_{j<\min S}$, report $\{v_i\}_{i\in S}$. Otherwise, report 0. This strategy in the direct game immediately translates to a profitable deviation in the auction by Definition A.2 and Lemma D.3 and thus the equilibrium is not weakly shill-proof. $\square$

Lemma D.6. *If a one-shot, optimal auction is strongly shill-proof, then for all R, $i \notin R$, v_i, and v_{-i},*

$$\sum_{k\in R}\tilde{t}_k(v_i, v_{-i}) \leqslant \sum_{k\in R}\tilde{t}_k(0, v_{-i}).$$

Proof. Towards contradiction, suppose that there exists R, $i \notin R$, v_i, and v_{-i} such that

$$\sum_{k\in R}\tilde{t}_k(v_i, v_{-i}) > \sum_{k\in R}\tilde{t}_k(0, v_{-i}).$$

That means in the direct game, reporting 0 is not a dominant strategy for shill bidders. This implies, from Equation (11), that there exists a deviation in the auction such that for some value vectors, the auctioneer raises more revenue. Therefore, the auction is not strongly shill-proof. $\square$

Lemma D.7. *For a one-shot, optimal auction, define the augmented (direct) inverse $\breve{\psi}_i^{-1}$ as*

$$\breve{\psi}_i^{-1}(v) = \{0\} \cup \psi_i^{-1}(v_{j<i}).$$

Then for

$$\mathcal{V}(v) = \left\{\{v^{\rightsquigarrow i}\} : v^{\rightsquigarrow i} \in \breve{\psi}_i^{-1}(v), \sum_i \tilde{x}_i(v^{\rightsquigarrow i}) \leqslant 1\right\},.$$

the auction is credible if and only if for all v, and $\{v^{\rightsquigarrow i}\} \in \mathcal{V}(v)$,

$$\sum_i \tilde{t}_i(v^{\rightsquigarrow i}) \leqslant \sum_i \tilde{t}_i(v).$$

Proof. Apply Lemma D.3, specifically the unique mapping between $(\tilde{x}, \tilde{t})$ and (x, t) to Definitions D.1 and D.2 to see that the lemma holds. $\square$

In a one-shot auction, we define $\psi = \mathrm{Id}$ to mean that $\psi_i(a_{j<i}) = a_{j<i}$, i.e., that the signals reveal the actions of previous bidders (and also the signals of previous bidders). We define $\psi = \varnothing$ to mean the opposite, $\psi_i(a_{j<i}) = \varnothing$.

Proposition D.8. *Consider an one-shot, optimal auction. If the auction is credible, then it is weakly shill-proof. Moreover, if the auction has experiments $\psi = \mathrm{Id}$ and is weakly shill-proof, then it is credible.*

Proof.

Weak Shill-Proofness → Credibility. Suppose the auction is not credible and $\psi = \mathrm{Id}$. Then, combining Lemma D.7 with the ex-post IR condition, there exist $v, \{v^{\rightsquigarrow i}\} \in \mathcal{V}(v), k^*$ such that $\tilde{t}_{k^*}(v^{\rightsquigarrow k^*}) > \tilde{t}_{k^*}(v)$. Applying the winner-paying property, it is the case that for all $j \neq k^*$, $\tilde{t}_j(v^{\rightsquigarrow k^*}) = 0$. Since $\psi = \mathrm{Id}$, for all $j \leqslant k$, it is the case that $v_j^{\rightsquigarrow k^*} = v_j$ or $v_j = 0$. Let $R = \{1, \ldots, k^*\}$. Then,

$$\begin{aligned}\sum_{i \in R} \tilde{t}_i(v^{\rightsquigarrow k^*}) &= \tilde{t}_{k^*}(v^{\rightsquigarrow k^*}) \\ &\geqslant \tilde{t}_{k^*}\left(v_1, \ldots, v_{k^*}, v_{k^*+1}^{\rightsquigarrow k^*}, \ldots, v_N^{\rightsquigarrow k^*}\right) \\ &\geqslant \tilde{t}_{k^*}\left(v_1, \ldots, v_{k^*}, 0, \ldots, 0\right) \\ &= \sum_{i \in R} \tilde{t}_i(v_R, 0).\end{aligned}$$

Thus, we can apply Lemma D.5 to conclude that the auction is not weakly shill-proof.

Credibility → Weak Shill-Proofness. Suppose the auction is credible. Towards contradiction, suppose the auction is not weakly shill-proof. So, there exists R, v such that $\sigma(v; R) \neq \sigma(v; B)$. In particular, this means that shill bidders have, in expectation, a profitable deviation relative to acting as real bidders with valuation 0. If this is true in expectation, there must then exist $v = (v_R, 0)$ and $\tilde{v}_{-R}$ such that

$$\sum_{i \in R} \tilde{t}_i((v_R, \tilde{v}_{-R})) > \sum_{i \in R} \tilde{t}_i((v_R, 0)).$$

Now, let us consider the messaging deviation

$$\left\{v^{\rightsquigarrow i}\right\}_{i \in B} = \begin{cases} (v_R, \tilde{v}_{-R}) & i \in R \\ (v_R, 0) & \text{otherwise} \end{cases}.$$

By the definition of credibility, the auctioneer can report any value to other bidders when the value reported to him is 0 and bidders with value 0 are told the other bidders' true reports. Therefore, $\{v^{\rightsquigarrow i}\} \in \mathcal{V}((v_R, 0))$ and

$$\sum_i \tilde{t}_i(v^{\rightsquigarrow i}) = \sum_{i \in R} \tilde{t}_i((v_R, \tilde{v}_{-R})) + \sum_{i \notin R} \tilde{t}_i((v_R, 0)) > \sum_i \tilde{t}_i((v_R, 0)).$$

This contradicts Lemma D.7, and so we see that the auction must be weakly shill-proof. $\square$

Proposition D.9. *Consider an one-shot, optimal auction. If the auction is strongly shill-proof, then it is credible. If the auction has experiments $\psi = \varnothing$ and is credible, then it is strongly shill-proof.*

Proof. **Credibility → Strong Shill-Proofness.** Suppose that the auction is not strongly shill-proof and $\psi = \varnothing$. There are two cases to consider: either the auction is not weakly shill-proof or it is. If the auction is not weakly shill-proof, then we can apply Proposition D.8 to conclude the auction is not credible. If the auction is weakly shill-proof, but not strongly shill-proof, then by Lemma D.4,

$$\text{there exists } R, k^* \in R, v_R, \text{ and } v_{-R} \text{ such that } \tilde{t}_{k^*}(v) > \tilde{t}_{k^*}(v_R, 0).$$

Thus, we can construct the following profitable auctioneer reporting deviation:

$$\left\{v^{\rightsquigarrow i}\right\}_{i \in B} = \begin{cases} (v_R, v_{-R}) & i = k^* \\ (v_R, 0) & \text{otherwise} \end{cases}.$$

Since $\psi = \varnothing$, we know that $\{\tilde{v}^{\rightsquigarrow i}\} \in \mathcal{V}(v_R, 0)$. The total transfers is then

$$\sum_i \tilde{t}_i\left(v^{\rightsquigarrow i}\right) = \tilde{t}_{k^*}(v) + \sum_{i \neq k^*} \tilde{t}_i\left(v_R, 0\right) > \sum_i \tilde{t}_i\left(v_R, 0\right);$$

hence, by Lemma D.7, we see that the auction is not credible.

Strong Shill-Proofness → Credibility. Suppose that the auction is not credible. Then, combining Lemma D.7 with the ex-post IR condition, there exists $v, \{v^{\rightsquigarrow i}\} \in \mathcal{V}(v), k^*$ such that $\tilde{t}_{k^*}(v^{\rightsquigarrow k^*}) > \tilde{t}_{k^*}(v)$. Recall, by the definition of ψ^{-1}, that $v_{k^*}^{\rightsquigarrow k^*} = v_{k^*}$. Suppose $R = \{k^*\}$. Then,

$$\sum_{i \in R} \tilde{t}_i(v^{\rightsquigarrow k^*}) = \tilde{t}_{k^*}(v^{\rightsquigarrow k^*}) > \tilde{t}_{k^*}(v) \geqslant \tilde{t}_{k^*}(v_{k^*}, 0) = \sum_{i \in R} \tilde{t}_{k^*}(v_{k^*}, 0).$$

Therefore, by Lemma D.6, the auction is not strongly shill-proof. ☐

Now, when discussing one-shot, optimal auctions, we focus on the direct mechanisms associated to weakly shill-proof auctions and so we will refer to an auction as $(\tilde{x}, \tilde{t}, \psi)$.

Lemma D.10. *Suppose a one-shot, optimal auction $(\tilde{x}, \tilde{t}, \psi)$ is weakly shill-proof. Then, for all i, v, and $v'_{j>i}$,*

$$\tilde{x}_i\left(v\right) = \tilde{x}_i\left(v_{j \leqslant i}, v'_{j>i}\right) \implies \tilde{t}_i\left(v\right) = \tilde{t}_i\left(v_{j \leqslant i}, v'_{j>i}\right).$$

Proof. Towards contradiction, suppose there exists i, v, and $v'_{j>i}$, such that

$$\tilde{x}_i\left(v\right) = \tilde{x}_i\left(v_{j \leqslant i}, v'_{j>i}\right), \text{but } \tilde{t}_i\left(v; s\right) > \tilde{t}_i\left(v_{j \leqslant i}, v'_{j>i}; s\right).$$

Because the auction is winner-paying, it must then be the case that $\tilde{x}_i\left(v\right) = \tilde{x}_i\left(v_{j \leqslant i}, v'_{j>i}\right) = 1$. Let $R = \{1, \ldots, i\}$. Then,

$$\begin{aligned} \mathbb{E}_v\left[\sum_{k \in R} \tilde{t}_k(\{v_i\}_{i \in S}, \{v_i\}_{i \notin S}) \mid v_{j \leqslant \min S}\right] &= \tilde{t}_i\left(v\right) \\ &> \tilde{t}_i\left(v_{j \leqslant i}, v'_{j>i}\right) \\ &\geqslant \tilde{t}_i\left(v_{j \leqslant i}, 0\right). \end{aligned}$$

This violates Lemma D.5, and so we have reached a contradiction. ☐

Lemma D.11. *Suppose a one-shot, optimal auction $(\tilde{x}, \tilde{t}, \psi)$ is mildly strategy-proof and weakly shill-proof. Then, there exists $i < N$ such that for all v_i, v_i', and $v_{-i} \in \psi_i^{-1}(v_{j<i})$,*

$$\tilde{x}_i(v) = \tilde{x}_i(v_i', v_{-i}) \implies \tilde{t}_i(v) = \tilde{t}_i(v_i', v_{-i}).$$

Proof. Let $i < N, R \ni i, v_i, v_i', v_{-i} \in \psi_i^{-1}(v_{j<i})$ such that $\tilde{x}_i(v) = \tilde{x}_i(v_i', v_{-i})$. WLOG, suppose $v_i > v_i'$. By monotonicity, $\tilde{t}(v) \geqslant \tilde{t}(v_i', v_{-i})$. Towards contradiction, suppose $\tilde{t}(v) > \tilde{t}(v_i', v_{-i})$. By the winner-paying property, $\tilde{t}(v) > \tilde{t}(v_i', v_{-i})$ implies that $\tilde{x}_i(v) = 1$—and hence $\tilde{x}_i(v_i', v_{-i}) = 1$ as well because $\tilde{x}_i(v_i', v_{-i}) = \tilde{x}_i(v)$. However, note that $\tilde{t}(v) > \tilde{t}(v_i', v_{-i})$ would mean that the utility of reporting v_i' would be higher than truthful reporting under true value v_i which would violate the mildly strategy-proofness and thus $\tilde{t}(v_i', v_{-i}) = \tilde{t}(v_i', v_{-i})$. □

Proof of Theorem 5.2.

Towards contradiction, suppose such an auction did exist. Fix $i < N, s$ and suppose $v_i < v^M$. Combining Lemmatta D.10 and D.11, we can see that for all v_i' and $v_{-i}, v_{-i}' \in \psi_i^{-1}(v_{j<i})$,

$$\tilde{x}_i(v) = \tilde{x}_i(v') \implies \tilde{t}_i(v) = \tilde{t}_i(v').$$

So, for all v, v' such that $\tilde{x}_i(v) = \tilde{x}_i(v') = 1$, we can define $\tilde{t}_i^* \equiv \tilde{t}_i(v) = \tilde{t}_i(v')$.

Monotonicity implies that when $\tilde{x}_i(v) = 1$, it must also be the case that $\tilde{x}_i(v^M, v_{-i}) = 1$. So, applying the winner-paying property (and suppressing that the expectation is conditioned on $s_i = \psi_i(v_{j<i})$), we have

$$\begin{aligned} &\mathbb{E}_{v_{-i}}\left[\tilde{x}_i\left(v^M, v_{-i}\right) v_i - \tilde{t}_i\left(v^M, v_{-i}\right)\right] \\ &\qquad = \mathbb{E}_{v_{-i}}\left[v_i - \tilde{t}_i^* \mid \tilde{x}_i(v) = 1\right] + \mathbb{E}_{v_{-i}}\left[v_i - \tilde{t}_i^* \mid \tilde{x}_i(v) = 0, \tilde{x}_i\left(v^M, v_{-i}\right) = 1\right] \end{aligned} \tag{15}$$

and

$$\mathbb{E}_{v_{-i}}\left[\tilde{x}_i\left(v\right) v_i - \tilde{t}_i\left(v\right)\right] = \mathbb{E}_{v_{-i}}\left[v_i - \tilde{t}_i^* \mid \tilde{x}_i(v) = 1\right]. \tag{16}$$

Taking the difference between Equation (15) and Equation (16), we see that

$$\begin{aligned} &\mathbb{E}_{v_{-i}}\left[\tilde{x}_i\left(v^M, v_{-i}\right) v_i - \tilde{t}_i\left(v^M, v_{-i}\right)\right] - \mathbb{E}_{v_{-i}}\left[\tilde{x}_i\left(v\right) v_i - \tilde{t}_i\left(v\right)\right] \\ &\qquad = \mathbb{E}_{v_{-i}}\left[v_i - \tilde{t}_i^* \mid \tilde{x}_i(v) = 0, \tilde{x}_i\left(v^M, v_{-i}\right) = 1\right] \end{aligned} \tag{17}$$

Now, $\tilde{x}$ is monotone because the auction is optimal, and by assumption $\tilde{t}$ is monotone. If there exists v^m such that $\mathbb{P}[\tilde{x}_i(v^m, v_{-i}) = 1] < \mathbb{P}[\tilde{x}_i(v^M, v_{-i}) = 1]$, then for such m,

$$\begin{aligned} \mathbb{E}_{v_{-i}}\left[\tilde{x}_i(v^m, v_{-i})v^m - \tilde{t}_i(v^m, v_{-i})\right] &\geqslant \mathbb{E}_{v_{-i}}\left[\tilde{x}_i(v^{m-1}, v_{-i})v^m - \tilde{t}_i(v^{m-1}, v_{-i})\right] \\ &> \mathbb{E}_{v_{-i}}\left[\tilde{x}_i(v^{m-1}, v_{-i})v^{m-1} - \tilde{t}_i(v^{m-1}, v_{-i})\right] \\ &\geqslant 0, \end{aligned}$$

where the last inequality comes from the IR condition. Thus, the IR constraint does not bind for $v_i = v^m$. Since the good has to be allocated to the highest type, for all $i < N$, there exists v_{-i} such that $\tilde{x}_i(v) = 0$ and $\tilde{x}_i(v^M, v_{-i}) = 1$. Thus,

$$\mathbb{E}_{v_{-i}}\left[v_i - \tilde{t}_i^* \mid \tilde{x}_i(v) = 0, \tilde{x}_i\left(v^M, v_{-i}\right) = 1\right] > 0. \tag{18}$$

Combining Equations (17) and (18), we see that

$$\mathbb{E}_{v_{-i}}\left[\tilde{x}_i\left(v^M, v_{-i}\right) v_i - \tilde{t}_i\left(v^M, v_{-i}\right)\right] > \mathbb{E}_{v_{-i}}\left[\tilde{x}_i\left(v\right) v_i - \tilde{t}_i\left(v\right)\right].$$

We then apply the weak shill-proofness condition to simplify Equation (13) to

$$\mathbb{E}_{v_{-i}}\left[\tilde{x}_i\left(v\right) v_i - \tilde{t}_i(v)\right] \geqslant \mathbb{E}_{v_{-i}}\left[\tilde{x}_i\left(v_i', v_{-i}\right) v_i - \tilde{t}_i\left(v_i', v_{-i}\right)\right].$$

Taking $v_i' = v^M$, we have therefore shown that the IC constraint from Lemma D.3 is violated—and thus have reached a contradiction.