Theorem 0.1. *Let M be a compact manifold and let $g\colon M \to \mathbb{R}^k$ be a smooth map, with $k \geq 2n+1$. Then, for every $\varepsilon > 0$ there exists an a smooth embedding $g'\colon M \to \mathbb{R}^k$, such that $\sup_{x\in M} |g(x) = g'(x)| < \varepsilon$.*

Theorem 0.2 (Parametric transversality theorem)**.** *Suppose $F\colon X \times S \to Y$ is smooth, and $Z \subset Y$ is a submanifold. Denote $F_s\colon X \to Y$ by $F_s(\cdot) = F(s, \cdot)$. Suppose that $F \pitchfork Z$. Then for almost all $s \in S$, $F_s \pitchfork V$.*

Lemma 0.3. *If $s \in S$ is a regular value of π then $F_s \pitchfork Z$.*

Proof of Lemma 0.3. Since the claim is local, we work in local coordinates. Let (x, s) be coordinates on $X \times S$, and choose coordinates (u, z) on Y such that z is the coordinate on Z and locally Z is given be $\{(z, u)\colon u = 0\}$ and $F(0, 0) = 0$. In these coordinates we can write

$$F(x, s) = (A(x, s), B(x, s)).$$

$$W = F^{-1}(Z) = \{(x, s)\colon A(x, s) = 0\}.$$

By Example **??**, $T_0W = \left\{(\dot{x}, \dot{s}) \in T_0X \times T_0S\colon \frac{\partial A}{\partial x} \cdot \dot{x} + \frac{\partial A}{\partial s} \cdot \dot{s} = 0\right\}$.

1. The transversality of F to Z means that $A_*(0)$ is onto. Indeed, by our our choice of coordinates, $T_0Z = \{(0, \dot{z})\} \subset T_0Y = \{(\dot{u}, \dot{z})\}$. Additionally, $F_*(T_0X \times S) = \big(A_*(T_0X \times S), B_*(T_0X \times S)\big)$. Therefore, the condition $F_*(T_0X \times S) + T_0Z = T_0Y$ translates to surjectivity of $A_*(0)$. Thus for any $\dot{u}$ there exist $\dot{x}, \dot{s}$ such that $\frac{\partial A}{\partial x} \cdot \dot{x} + \frac{\partial A}{\partial s} \cdot \dot{s} = \dot{u}$.
2. By assumption, $s = 0$ is a regular value of the projection $\pi\big|_W$. Now, $\pi_*\colon T_0W \to T_0S, \quad (\dot{x}, \dot{s}) \mapsto \dot{s}$. Therefore, the surjectivity of $(\pi\big|_W)_*$ means that for any $\dot{s}$ we can find $\dot{x}$ such that $(\dot{x}, \dot{s}) \in T_0W$, that is, such that $\frac{\partial A}{\partial x}\dot{x} + \frac{\partial A}{\partial s}\dot{s} = 0$.

Finally, our goal is to prove the transversality of F_0 to Z. As before, this amounts to requiring that $\dfrac{\partial A}{\partial x}(0)$ is surjective, that is, for any $\dot{u}$ we should find $\dot{x}$ such that $\dfrac{\partial A}{\partial x} \cdot \dot{x} = \dot{u}$. So, let $\dot{u}$. By condition (1), we have $\dot{x}_1, \dot{s}$ such that $\frac{\partial A}{\partial x} \cdot \dot{x}_1 + \frac{\partial A}{\partial s} \cdot \dot{s} = \dot{u}$. Now by condition (2) we have $\dot{x}_2$ such that $\frac{\partial A}{\partial x} \cdot \dot{x}_2 + \frac{\partial A}{\partial s} \cdot \dot{s} = \dot{u}$. Subtracting the two equations, we find that $\frac{\partial A}{\partial x} \cdot (\dot{x}_1 - \dot{x}_2) = \dot{u}$. This proves the surjectivity of $\dfrac{\partial A}{\partial x}(0)$ and hence the lemma. □

Theorem 0.4 (Approximation theorem)**.** *Let $f\colon X \to Y$ be smooth, and let $Z \subset Y$ be a submanifold. Then there is $N \in \mathbb{N}$ and a smooth $F\colon X \times D^N \to Y$ so that $F \pitchfork Z$ and $F(x, 0) = f(x)$.*

Therefore, by the parametric transversality theorem, $F_s \pitchfork Z$ for almost all $s \in S$. In particular, we can find $s_i \operatorname{conv} 0$ such that $F_{s_i} \pitchfork Z$. In a suitably chosen topology on the space of smooth maps $X \to Y$, one obtains $F_{s_i} \operatorname{conv} f$ as $s_i \operatorname{conv} 0$. Moreover, we note that by taking $s \to 0$ we in fact obtain a homotopy F_s such that $F_0 = 0$ and $F_s \pitchfork Z$ for almost any s. Thus Theorem 0.4 states that *any map between manifolds can be approximated by homotopic maps transversal to Z.*

For the proof of Theorem 0.4 we require the notion of a *tubular neighbourhood*: we assume that Y is compact, and consider it embedded in $\mathbb{R}^N$. Let $U_\varepsilon(Y) = \{w \in \mathbb{R}^N\colon d(w, Y) < \varepsilon\}$. Consider the *normal bundle* to Y in $\mathbb{R}^N$: $NY = \{(y, \xi)\colon y \in Y,\ \xi \in T_y\mathbb{R}^N,\ \xi \perp Y_y\mathbb{R}^N\}$, and its ε-disc bundle: $N_\varepsilon Y = \{(y, \xi) \in NY\colon |\xi| < \varepsilon\}$.

Exercise 0.5. Show that for small $\varepsilon > 0$ the map $N_\varepsilon Y \to U_\varepsilon(Y), \quad (y, \xi) \mapsto y + \xi$ is a diffeomorphism.

Exercise 0.6. Let $\pi\colon N_\varepsilon Y \to Y$ be the projection $(y, \xi) \mapsto y$. Prove that π is a submersion.

Next, define $G\colon Y \times D^N \to Y, \quad (y, \varepsilon) \mapsto \pi(y + \varepsilon\xi)$.

Exercise 0.7. Prove that for any $y \in Y$, the map $D^N \to Y, \quad \xi \mapsto G(y, \xi)$ is a submersion.

Proof of Theorem 0.4. Define $F\colon X \times D^N \to Y, \quad F(x, \xi) = G(f(x), \xi) = \pi(f(x) + \varepsilon\xi)$. Note that $F(x, 0) = f(x)$. By Exercise 0.7, $\frac{\partial F}{\partial \xi}(x, \xi) = \frac{\partial G}{\partial \xi}(f(x), \xi)\colon T_\xi D^N \to T_{F(x,\xi)}Y$ is onto for all (x, ξ). In particular, $F \pitchfork Z$. □

Intersection Index First, a bit of terminology: A compact and boundaryless manifold is called *closed.* Define the *intersection index* of f and Z by $I(f, Z) = \sum_{x \in f^{-1}(Z)} (f_*T_xX) \circ Z$.

Lemma 0.8. *Assume that $X = \partial W$ with W oriented and X oriented with the boundary orientation. Let $F\colon X \to Y$ such that $F \pitchfork Z$ and $F\big|_X \pitchfork Z$. Then $I(F\big|_X, Z) = 0$.*

Theorem 0.9 (Classification of 1-manifolds)**.** *Let M be a 1-dimensional compact connected manifold with boundary. Then M is diffeomorphic either to S^1 or $[0, 1]$.*

Lemma 0.10. *Assume that $X = \partial W$ with W oriented and X oriented with the boundary orientation. Let $F\colon X \to Y$ such that $F \pitchfork Z$ and $F\big|_X \pitchfork Z$. Then $I(F\big|_X, Z) = 0$.*

Proof of Lemma 0.10. Denote $f = F\big|_X$ and $\gamma = F^{-1}(Z)$. If $\gamma = \emptyset$ then no problem, so assume $\gamma \neq \emptyset$. Since $\dim W = \dim X + 1$, γ is a 1-dimensional submanifold, and by an exercise from last time, $\partial\gamma = F^{-1}(Z) \cap X$, and by definition, this is just $f^{-1}(Z)$. Now, by Lemma 0.9, γ a finite union of circles and intervals, and hence each component of γ is either boundaryless or has exactly two boundary points, denote them $x_\pm$. For each such point x put $\varepsilon(x) = f_*(T_x X) \circ T_{f(x)} Z$. We will show that for any such pair $\varepsilon(x_+) + \varepsilon(x_-) = 0$. Therefore we can assume wlog that $\gamma \approx [0,1]$, and $\partial\gamma = f^{-1}(Z)$ contains exactly two point. Choose a parametrization $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_-,\ \gamma(0) = x_+,$ and $\dot\gamma(t) \neq 0\ \forall t \in [0,1]$. Note that, since $\dim F_{x_\pm} X + \dim T_{f(x_\pm)} Z = \dim T_{f(x_\pm)} Y$, the transversality of f and Z implies that $f_*(T_{x_\pm}) \cap T_{f(x_\pm)} = \{0\}$. In particular, since $F_*\big|_{T_{x_\pm} X} = f_*$, $F_*(T_{x_\pm}) \cap T_{f(x_\pm)} = \{0\}$. On the other hand, note that since $F(\gamma(t)) \in Z$ for all $t \in [0,1]$, we have $F_*\dot\gamma(t) \in T_{F(\gamma(t))} Z$. In particular, $F_*\dot\gamma(0) \in T_{F(x_-)} Z,\quad F_*\dot\gamma(1) \in T_{F(x_+)} Z$. Therefore, $\dot\gamma(0) \notin T_{x_-} X$ and $\dot\gamma(1) \notin T_{x_+} X$, which allows us to pick a Riemannian metric on W such that $\dot\gamma(0) \perp T_{x_-} X$ and $\dot\gamma(1) \perp T_{x_+} X$. Set $E(t) = \{\dot\gamma(t)\}^\perp$. By definition, $E(0) = T_{x_-} X$ and $E(1) = T_{x_+} X$. Observe that since $F \pitchfork Z$ and $F_*\dot\gamma(t) \in T_{F(\gamma(t))} Z$, $F_*(E(t)) \pitchfork T_{F(\gamma(t))} Z \quad \forall t \in [0,1]$. Thus the function $t \mapsto F_*(E(t)) \circ T_{F(\gamma(t))} Z$ is defined for all $t \in [0,1]$ and continuous. Therefore, it is constant. In particular, $F_*(T_{x_-} X) \circ T_{F(\gamma(0))} Z = F_*(T_{x_+} X) \circ T_{F(\gamma(1))} Z$. However, we note that this function is constant when considered with the *product* orientation. In view of Remark **??**, in terms of the induced orientation on X, $-f_*(T_{x_-} X) \circ T_{f(\gamma(0))} Z = f_*(T_{x_+} X) \circ T_{f(\gamma(1))} Z$. Therefore, $\varepsilon(x_-) + \varepsilon(x_+) = 0$ and the lemma is proved. $\square$

Proof of Theorem **??**. Suppose $f, g\colon X \to Y$ are homotopic and transversal to Z. Put $W = X \times [0,1]$ and let $F\colon W \to Y$ be a homotopy between f and g. By the approximation theorem, we may assume that $F \pitchfork Z$. Now, considered with the boundary orientation, $\partial W = (X \times \{0\}, -or) \sqcup (X \times \{1\}, +or)$. Now by Lemma 0.10 $I(f, Z, -or) + I(g, Z, +or) = 0$, thus with respect to the standard orientation on X, $I(f, Z) = I(g, Z)$. $\square$

Theorem 0.11. *Let W^{n+1}, $V^n = \partial W$ as above, and assume $n \geq 2$. Let $f\colon W \to S^n$ be a smooth map with $\deg f = 0$. Then f extends to $F\colon W \to S^n$.*

Lemma 0.12 (Homotopy extension)**.** *Suppose $f, g\colon V \to S^n$ are homotopic maps, and g extends to W. Then f extends as well.*

Lemma 0.13. *Let M be a smooth manifold, $X \subset N$ a smooth subset and $f\colon X \to \mathbb{R}^d$ a smooth map. Then f extends to a smooth $F\colon M \to \mathbb{R}^d$.*

Lemma 0.14. *Let $A_0, A_1 \in GL(n, \mathbb{R})$ be two matrices such that*

$$\operatorname{sign}\det A_0 = \operatorname{sign}\det A_1.$$

Then there exists a smooth $G\colon D^n \times [0,1] \to \mathbb{R}^n$ such that $G(x,0) = A_0 x,\quad G(x,1) = A_1 x,\quad G(x,t) = 0 \iff x = 0.$

Proof of Theorem 0.11. Let $y \in S^n$ be a regular value of f. If $f^{-1}(y) = \emptyset$, f is not surjective, and hence since the sphere with one point removed is contractible, f is homotopic to a constant map, which clearly extends to W, and hence by Lemma 0.12 f extends to W as well.

Therefore, assume $f^{-1}(y) \neq \emptyset$. In this case, we know that it is a finite collection of points, say $x_1, \ldots, x_m$. Then $\deg(g) = \sum_j \varepsilon(x_j)$, where, as usual, $\varepsilon(x_j) = \begin{cases} +1, & D_{x_j} f\colon T_{x_j} V \to T_y S^n \text{ is orientation preserving,} \\ -1, & D_{x_j} f\colon T_{x_j} V \to T_y S^n \text{ is orientation reversing.} \end{cases}$ Since $\deg f = 0$, $m = 2k$ must be even, and we can assume that $\varepsilon(x_{2j-1}) + \varepsilon(x_{2j}) = 0$ for all j. Choose pair-wise non-intersecting embedded arcs $\gamma_1, \ldots, \gamma_k$ such that γ_j connects x_{2j-1} to x_{2j}. This can be achieved since by the transversality theorem we may assume that each pair is transversal. But since each arc is one-dimensional and by assumption $\dim W \geq 3$, if $\gamma_{2j-1} \pitchfork \gamma_{2j}$ they must be non-intersecting. Additionally, equip W with a Riemannian metric such that $\gamma_j \perp \partial W$ for all j (as we did in the proof of the transversality theorem).

For each j, let U_j be a tubular neighbourhood of γ_j, such that $U_j \simeq D^n \times [0,1]$. Choose local coordinates near $x_j \in W$ which agree with the product orientation on $U_j \simeq D^n \times [0,1]$. Choose coordinates near $y \in S^n$ such that $f(x_j) = 0$ and write in local coordinates $f(q) = \frac{\partial f}{\partial q}(0) \cdot q + O(q^2)$. Now, using a local homotopy near x_j, we can 'kill' the remainder term, so that in local coordinates $f(q) = \frac{\partial f}{\partial q}(0) \cdot q$. Now, the matrix $\frac{\partial f}{\partial q}(0)$ is simply the local coordinate expression for $D_{x_j} f$. Since $\varepsilon(x_{2j-1}) = -\varepsilon(x_{2j})$ we have, in our coordinates, $\operatorname{sign}\det D_{x_{2j-1}} f = \operatorname{sign}\det D_{x_{2j}} f$. The signs are reversed since the coordinates respect the *product* orientation. Now, applying Lemma 0.14 to each U_j we can extend f to $F'\colon V \cup \bigcup_{j=1}^{k} U_j = Z \to S^n$. Note that the last condition in the lemma ensures that $F'(z) = y$ iff $z \in \gamma_j$. Therefore, we have $F'\colon Z \setminus \bigcup_j \gamma_j \to S^n \setminus \{y\} \simeq \mathbb{R}^n$. Now we can use Lemma 0.13 to extend F' to $F\colon W \setminus \bigcup_j \gamma_j \to S^n \setminus \{y\}$ Now we note that by construction, F agrees with F' on a neighbourhood of each γ_j and hence extends f smoothly to W. $\square$

Definition 0.15. The *Euler characteristic* of X is the intersection index $X \circ X$ in TX. It is denoted $\chi(X)$.

Example 0.16. We know that in general $A \circ B = (-1)^{\dim A \dim B} B \circ A$. Therefore, if $\dim X$ is odd, $\chi(X) = 0$.

In general, if X' is sufficiently close to X, then X' is the graph of some vector field. This is since $\pi\big|_X\colon X \to X$ is a diffeomorphism, so for small enough perturbations, $\pi\big|_{X'}\colon X' \to X$ is still a diffeomorphism. Now, assume that X' is the graph of a vector field v. Then $X \cap X'$ is in bijection with the set of zeroes of v. By definition,

$$\chi(X) = X \circ X' = \sum_{x\colon v(x)=0} \operatorname{ind}_x v, \tag{1}$$

where $\operatorname{ind}_x v = T_x x \circ T_x \operatorname{graph} v$.

Proposition 0.17. *In local positively-oriented coordinates $(q_1, \dots, q_n)$ on X, $\mathrm{ind}_q\, v = \mathrm{sign}\det \frac{\partial v}{\partial q}(q)$.*

Proof. Take the natural coordinates $(q_1, \dots, q_n \dot{q}_1, \dots, \dot{q}_n)$. Then $X = \{(q, 0)\}$. Therefore, a positive basis for T_qX is $\left(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_n}\right)$. Additionally, $\mathrm{graph}\, v = \{(q, v(q))\}$. Therefore, a positive basis of $T_q\, \mathrm{graph}\, v$ is $\left(\frac{\partial}{\partial q_1} \oplus \frac{\partial v}{\partial q_1}, \dots, \frac{\partial}{\partial q_n} \oplus \frac{\partial v}{\partial q_n}\right)$. We need to determine whether the union of these bases is positively oriented with respect to the natural orientation on TX. Therefore, we form the $2n \times 2n$ matrix with these vectors as columns, and compute its determinant. We get, in block notation: $\det \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ 0 & \frac{\partial v}{\partial q} \end{pmatrix} = \det \frac{\partial v}{\partial q}$. This proves the claim. $\square$

The Gauss Map Let M be a closed surface embedded in $\mathbb{R}^3$.

Let $v(x)$ define the outward unit normal to M. $|v(x)| = 1$, $v(x) \perp T_xM$.

Definition 0.18. The **Gauss map** $M \to S^2$ is given by $x \mapsto v(x)$.

Theorem 0.19. $\deg g = \frac{1}{2}\chi(M)$

Proof. Choose $v \in S^2$ s.t. both $v, -v$ are regular values of g. Wlog, $v = [0, 0, 1]^t$. For $q \in M$, let $w(q) = v(q) \times v$. $w(q) \in T_qM$.
Note. $w(q) = 0 \Leftrightarrow v(q) = \pm v$

Assume $v(q) = v$. Calculate $\mathrm{ind}_q(w)$:

In local coordinates, we may express M as the set (picture) $z = \varphi(x, y)$. Since $v(q) = v$ (the unit normal is pointing up), $\frac{\partial\varphi}{\partial x}(0) = \frac{\partial\varphi}{\partial y}(0) = 0$.

The parametrization $(x, y) \mapsto (x, y, \varphi(x, y))$ induces $\frac{\partial}{\partial x} \mapsto (1, 0, \frac{\partial\varphi}{\partial x})$ and $\frac{\partial}{\partial y} \mapsto (0, 1, \frac{\partial\varphi}{\partial y})$ on $T\mathbb{R}^2$. Then $v(q) = \frac{n(q)}{|n(q)|}$ $n(q) :=$
$\begin{pmatrix} i & j & k \\ 1 & 0 & \frac{\partial\varphi}{\partial x} \\ 0 & 1 & \frac{\partial\varphi}{\partial y} \end{pmatrix} = \left(\frac{-\partial\varphi}{\partial x}, -\frac{\partial\varphi}{\partial y}, 1\right)$

Now we can compute $w(q)$: $w(q) = \frac{n(q) \times v}{|n(q)|} = \frac{1}{|n(q)|} \begin{vmatrix} i & j & k \\ -\frac{\partial\varphi}{\partial x} & -\frac{\partial\varphi}{\partial y} & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{|n(q)|}\left(-\frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial x}, 0\right)$ In local coordinates

$w(x, y) = \frac{\left(-\frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial x}\right)}{\left[1 + (\frac{\partial\varphi}{\partial x})^2 + (\frac{\partial\varphi}{\partial y})^2\right]^{1/2}}$ To compute $\left.\frac{\partial w}{\partial q}\right|_{q=0}$ we first note that $\left.\frac{\partial\varphi}{\partial y}\right|_{q=0} = \left.\frac{\partial\varphi}{\partial x}\right|_{q=0} = 0$, so $\left.\frac{\partial w}{\partial q}\right|_{q=0} = \frac{1}{|n(q)|}\begin{pmatrix} -\varphi_{yx} & -\varphi_{yy} \\ \varphi_{xx} & \varphi_{xy} \end{pmatrix}$
$\Rightarrow \det\left(\left.\frac{\partial w}{\partial q}\right|_{q=0}\right) = -\varphi_{xy}^2 + \varphi_{xx}\varphi_{yy}$

Calculate the contribution of this point to $\deg(g)$.

$v(v, y) = \frac{n(x,y)}{|n(x,y)|} = \frac{\left(-\frac{\partial\varphi}{\partial x}, -\frac{\partial\varphi}{\partial y}, 1\right)}{|n(x,y)|}$

In x, y-coordinates on M and S^2, we get $v(x, y) = \frac{\left(-\frac{\partial\varphi}{\partial x}, -\frac{\partial\varphi}{\partial y}\right)}{|n(x,y)|}$

Hence (using again that $\left.\frac{\partial\varphi}{\partial y}\right|_{q=0} = \left.\frac{\partial\varphi}{\partial x}\right|_{q=0} = 0$), $\left.\frac{\partial v}{\partial q}\right|_{q=0} = \frac{1}{|n(x,y)|}\begin{pmatrix} -\varphi_x x & -\varphi_{xy} \\ -\varphi_{yx} & -\varphi_{yy} \end{pmatrix} \Rightarrow \det\left(\left.\frac{\partial v}{\partial q}\right|_{q=0}\right) = \varphi_{xx}\varphi_{yy} - \varphi_{xy}^2$

Finally $\chi(M) = \sum_{w(q)=0} \mathrm{ind}_q(w) = \sum_{v(q)=v} \mathrm{ind}_q(v) + \sum_{v(q)=-v} \mathrm{ind}_q(v)$
$= \sum_{v(q)=v} \mathrm{sgn}\left(\det \frac{\partial v}{\partial q}\right) + \sum_{v(q)=-v} \mathrm{sgn}\left(\det \frac{\partial v}{\partial q}\right) = 2\deg(g)$ $\square$

Let X^n be a closed, connected, oriented manifold. Let $\Delta \subset X \times X$-diagonal. Note: Δ is teh graph of the identity map $X \to X$. Assume $f : X \to X$ is smooth.

Definition 0.20. A : **fixed point** of f is a point x s.t. $f(x) = x$. In $X \times X$, the set of fixed points is $\Gamma(f) \cap \Delta$, where $\Gamma(f)$ denotes the graph of f.

Definition 0.21. $L(f) := \Delta \circ \Gamma(f)$

Remark 0.22. $L(f) = L(g)$ if $f \sim g$, another remark#Fix$(f) \geq |L(f)|$ generically.

What does $\Gamma(f) \pitchfork_q \Delta$ mean?

Lemma 0.23. $\Gamma(f) \pitchfork \Delta \Leftrightarrow \det\left(\frac{\partial f}{\partial q}(q) - 1\right) \neq 0$. *Then* $\Delta \circ \Gamma(f) = sgn \det\left(\frac{\partial f}{\partial q} q - 1\right)$.

Proof. Near the fixed point q, we may approximate $f(q)$ by $\frac{\partial f}{\partial q} q$. Locally: 1. $X = \mathbb{R}^n$ 2. $f(x) = Ax$ 3. 0 is a fixed point and $A = \frac{\partial f}{\partial q}(q)$ 4. $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ 5. $\Delta = \mathrm{Span}(e_1 \oplus e_1, \dots, e_n \oplus e_n)$ 6. $\Gamma(f) = \mathrm{Span}(e_1 \oplus Ae_1, \dots, e_n Ae_n)$

$$\Delta \circ \Gamma(f) = \mathrm{sgn}\det(e_1 \oplus e_1, \dots, e_n \oplus e_n, e_1 \oplus Ae_1, \dots, e_n \oplus Ae_n) = \det\begin{pmatrix} I & I \\ I & A \end{pmatrix} = \det\begin{pmatrix} I & 0 \\ I & A - I \end{pmatrix} = \det(A - I)$$

$\square$

Theorem 0.24. $\chi(x) = L(1_X)$

Proof. We need to perturb 1_X. Choose a vector field v on X. We can solve $\dot{x} = v(x)$ to get $f_t(x(0)) = x(t)$. This gives $f_0 = 1_X$ and $f_t \sim f_0$ for any t.

Locally, $f_t(x) = x + tv(x) + O(t^2)$. Assume v has non-degenerate zeroes. Then for any fixed point x, $x + tv(x) = x \Rightarrow v(x) = 0$. So for $|t| \ll 1$, we have that fixed points of f_t are 0's of v. By the Taylor expansion, $\frac{\partial f_t}{\partial x} - \mathrm{id} = t \cdot \frac{\partial v}{\partial x}, t > 0$

$$\Rightarrow \det\left(\frac{\partial f_t}{\partial x} - \mathrm{id}\right) = t^n \left(\frac{\partial v}{\partial x}\right) \Rightarrow L(f_t) = \sum_{f_t(x)=x} \operatorname{sgn}\det\left(\frac{\partial f_t}{\partial x} - 1\right) = \sum_{v(x)=0} \operatorname{sgn}\det\left(\frac{\partial v}{\partial x}\right) = \chi(X)$$

$\square$

3. Let p : E to M^n be a vector bundle, f : N^n to E be a smooth map transversal to all fibers (that is the sets of the form p^{-1}(x)). Show that p$\circ$ f : N to M is a submersion (since M and N have the same dimension, f is a submersion whenever it is an immersion). 5.2. Let S2 in R3 be the unit sphere. Show that $f : S2 \to CP1, (x1, x2, x3) to [1 - x3 : x1 + ix2]$ is a diffeomorphism. 5.3. Calculate the self-intersection number CP1$\circ$CP1 of the projective line inside the projective plane CP2. 5.4. Let p(z) = zm+am-1zm-1+...+a1z+a0be a polynomial with complex coefficients. Consider the corresponding homogeneous polynomial q(z,w) = zm+ am-1zm-1w + ... + a1zwm-1+ a0wm so that p(z) = q(z,1). Find the degree of the map g : CP1? CP1, [z0: z1] ?? [q(z0,z1) : zm 1] .

Homework 4.

2. Let a0,..., an be non-zero complex numbers. Consider a map fa : CP n to CPn [z0 : ... : zn] to [a0z0 : ... : anzn] Find all fixed points of fa. For which a all of them are non-degenerate? For such a, calculate their indices and find the Lefschetz number of fa. Apply this to the calculation of χ(CPn)

3. Consider the torus Tn = Rn/Zn Let A be a n x n matrix with integer . coeffcients. Calculate the degree and the Lefschetz number of a map fA : Tn to Tn, x to Ax mod Zn

4. Let U $\subset$ Rn be a compact domain (that is, a compact n-dimensional submanifold with boundary) with a smooth boundary M. Dene the Gauss map GU : M to Sn-1; x $\rightarrow v(x)$; where v(x) is the outward Euclidean normal to M. 4.1. Let V be a vector field on U which coincides with v on the boundary M and whose zeroes in U are non-degenerate. Show that $\sum_{v(x)=0} indx(v) = deg(GU)$ 4.2. Let U be a tubular neighborhood Show that $\chi(M) = deg(GU)$.

5. Prove that two maps f and g from S1 to S1 are homotopic if and only if they have the same degree.

6. Let Mn be a closed connected oriented manifold, and let f : Sn to M be a map of degree 1. Show that M is simply connected.

homework 5

1. Let v be a vector eld dened in a neighborhood U of 0 in R n so that v(0) = 0 and v(q) != 0 for all q in U -0 (the point 0 is called an isolated zero of v). Choose e ¿ 0 so that the sphere S (n-1)$_e$ of radius e centered at 0 is fully contained in U and put f$_e$: S$_e$ to S(n-1)$_1$ by q to v(q)/$|v(q)|$

Show that there exists I in Z and e ¿ 0 so that deqf$_b$ = I for all 0 ¡ b ¡ e. Show that if v has a non-degenerate zero at 0, I coincides with the usual index: I = ind0(v). Hint: reduce the second part of the problem to the case when v(q) = Aq where A is an invertible matrix n n, and use the polar decomposition. 2. Let M be a connected manifold of dimension ¿ 1. Let x1; :::; xN, y1; :::; yN be two collections of pair-wise distinct points on M. Show that there exists a dieomorphism f : M ! M with yj = f(xj) for all j = 1; :::;N. See Guillemin-Pollack, Chapter 3, beginning of Section 6. Is the same result true for M = S 1 and N = 3? N = 4? 3. Let M be a closed oriented connected manifold with χ(M) = 0. Show that M admits a nowhere vanishing vector eld. Hint: Start with any vector eld v with non-degenerate zeroes. Reduce the problem to the case when all the zeroes of v are contained in a small ball B in M (use Problem 2). Use Problem 1 and a theorem proved in the class (which exactly?) in order to modify v in the interior of B to a desired vector eld without zeroes