

1 Differentiable Manifolds

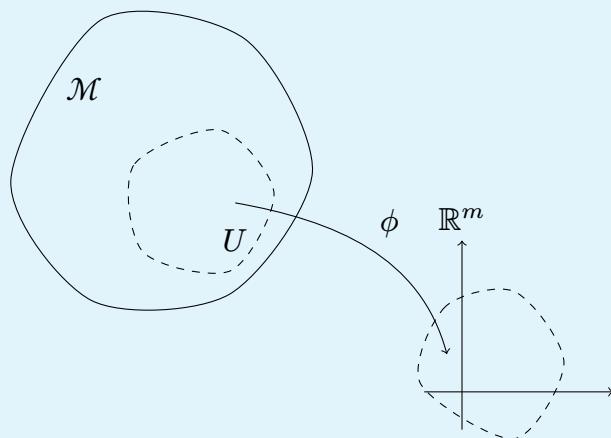
1.1 Definition

1.1.1 Coordinate Charts

Definition 1.1 (Coordinate Charts).

An m -dimensional, $m \neq \infty$ coordinate chart on a topological space \mathcal{M} is a pair

$$(U, \phi) \begin{cases} U \subseteq \mathcal{M}, U \text{ open} \\ \phi : U \rightarrow \mathbb{R}^m, \phi \text{ homeomorphism} \end{cases}$$



↗ **Remark.**

If $U = \mathcal{M}$, then we say the coordinate chart ϕ is globally defined; if not, then it is locally defined. Few manifolds have globally defined property. ℳ

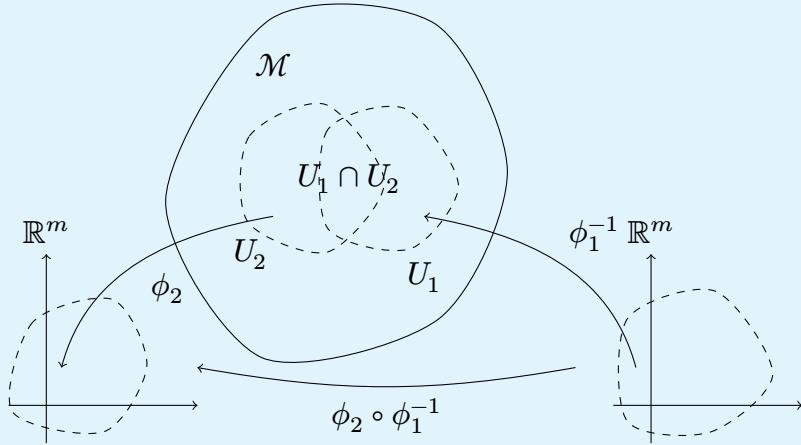
↗ **Remark.**

The basic method of studying manifolds is to analyze it in the familiar Euclidean space via coordinate charts. ℳ

Definition 1.2 (Overlap Function).

Let $(U_1, \phi_1), (U_2, \phi_2)$ be a pair of m -dimensional coordinate charts with $U_1 \cap U_2 \neq \emptyset$. Then the overlap function is defined as

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \subseteq \mathbb{R}^m \rightarrow \phi_2(U_1 \cap U_2) \subseteq \mathbb{R}^m.$$



Definition 1.3 (Atlas).

An m -dimensional atlas on \mathcal{M} is a family of m -dimensional coordinate charts $(U_i, \phi_i), i \in I$ s.t.

1. $\mathcal{M} = \bigcup_{i \in I} U_i$.
2. Each overlap function $\phi_j \circ \phi_i^{-1}, i, j \in I$ is C^∞ .

Definition 1.4 (Differentiable Manifolds).

An m -dimensional differentiable manifold is a topological space \mathcal{M} equipped with an atlas.

Remark.

We didn't define a differentiable manifold by regulating the differentiability of the coordinate charts themselves. That's because differentiation is not defined on a manifold, so we need to rely on Euclidean spaces. ℳ

1.2 Dimension of a Manifold

↗ **Remark.**

Consider a manifold that consists of a rod attached to a disk. The dimension is not same everywhere. We give a criterion on how to describe such a scenario.



Theorem 1.1 (Invariance of Domain).

For all $A, B \subseteq S^n$, if $\exists f : A \rightarrow B$ homeomorphic and $B \in \tau_{S^n}$, then $A \in \tau_{S^n}$ too.

↗ **Remark.**

Theorem 1.1 is an early theorem in algebraic topology.



Corollary 1.1.1 (Dimension is Well-defined).

Given $U \in \tau_{\mathbb{R}^n}, U' \in \tau_{\mathbb{R}^{n'}}$, and if $\exists f : U \rightarrow U'$ homeomorphic, then $n = n'$.

Proof. If $n = n'$, it is trivially true.

If $n < n'$, embed \mathbb{R}^n to $\mathbb{R}^{n'}$ by $f : \vec{x} \mapsto (\vec{x}, \vec{0})$. Via stereographic projection, we can map homeomorphically

$$\begin{aligned}\phi : U &\mapsto V \subseteq S^{n'}, \\ \phi' : U' &\mapsto V' \subseteq S^{n'}.\end{aligned}$$

Since the compositions are also homeomorphic, we see V and V' are homeomorphic. However, V' is not an open subset of $\mathbb{R}^{n'}$ because of the 0 's, contradicting Theorem 1.1. \ast



↗ **Remark.**

Since the definition of a differentiable manifold requires every overlap function to be diffeomorphic, if $U \cap U' \neq \emptyset$, their dimensions must be equal via the above corollary. We can bypass this by demanding $U \cap U' = \emptyset$, as in the rod and disk case.



Corollary 1.1.2.

If $g : V \rightarrow \mathbb{R}^n$ is a continuous injection and $V \in \tau_{\mathbb{R}^n}$, then $g(V)$ is homeomorphic to V , and $g(V) \in \tau_{\mathbb{R}^n}$.

Proof. On $g(V)$, g is surjective and therefore a homeomorphism. Use stereographic projection and the result is obvious. \blacksquare

1.3 Coordinate Functions

Definition 1.5 (Coordinate Functions).

The coordinate functions are the (Euclidean) components of coordinate.

$$\begin{aligned}\phi : U &\rightarrow \mathbb{R}^m & p &\mapsto \phi(p), \\ \phi^\mu : U &\rightarrow \mathbb{R} & \text{s.t. } \phi(p) = \begin{pmatrix} \phi^1(p) \\ \vdots \\ \phi^m(p) \end{pmatrix}.\end{aligned}$$

An alternative notation is

$$x^\mu := \phi^\mu.$$

↗ Remark.

There are (Euclidean) projection functions,

$$u^\mu : \mathbb{R}^m \rightarrow \mathbb{R}.$$

But I think mention it will cause a lot of confusion. Just remember in the future when we say $\frac{\partial}{\partial u^\mu}$, we are referring to the Euclidean partial derivative wrt the μ -th component. \mathfrak{M}

Definition 1.6 (Jacobian Matrix of Coordinate Transformation).

Let \mathcal{M} be a C^∞ manifold of $\dim \mathcal{M} = m$. Choose two coordinate functions

$$\begin{aligned}\phi : U_\phi &\rightarrow V_\phi, p \mapsto (x^1(p), \dots, x^m(p)), \\ \psi : U_\psi &\rightarrow V_\psi, p \mapsto (y^1(p), \dots, y^m(p)).\end{aligned}$$

Define the Jacobian matrix of coordinate transformation to be

$$J := \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \dots & \frac{\partial y^1}{\partial x^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial y^m}{\partial x^1} & \dots & \frac{\partial y^m}{\partial x^m} \end{pmatrix}$$

In short,

$$\begin{aligned} J^\nu{}_\mu &:= \frac{\partial y^\nu}{\partial x^\mu}, \\ (J^{-1})^\nu{}_\mu &:= \frac{\partial x^\nu}{\partial y^\mu}, \end{aligned}$$

where $\frac{\partial y^\nu}{\partial x^\mu} := \frac{\partial(y^\nu \circ \phi^{-1})}{\partial x^\mu}$.

1.4 Manifold With Boundary

1.4.1 Generalized Coordinate Charts

Definition 1.7 (Generalized Coordinate Charts).

A generalized coordinate chart allows chart that

$$\phi : U \rightarrow \phi(U) \subseteq (-\infty, 0] \times \mathbb{R}^{n-1},$$

U is open, and ϕ is homeomorphic.

↗ **Remark.**

Essentially, this allows a chart to map to "half planes". In this case, even if a set $\phi(U)$ contains $\{0\} \times \mathbb{R}^{n-1}$ and therefore not open in the Euclidean topology of \mathbb{R}^n , it is still considered open in the product topology of $(-\infty, 0] \times \mathbb{R}^{n-1}$. \mathfrak{M}

Definition 1.8 (Manifold With Boundary).

A manifold with boundary is a manifold whose atlas consists of generalized coordinate charts.

Definition 1.9 (Boundary Points of a Manifold).

For all $p \in \mathcal{M}$ is a boundary point of a manifold with boundary \mathcal{M} if $\exists \phi_\alpha \in \Phi$ atlas s.t. $\phi_\alpha^1(p) = 0$.

The set of all boundary points of \mathcal{M} is denoted $\partial \mathcal{M}$.

1.4.2 Boundary is Well-defined

↗ **Remark.**

A natural question regarding **Definition 1.9** is that, the definition only asks for existence, but it does not guarantee the existence of

$$\exists \phi_\alpha, \phi_\beta \text{ s.t. } \phi_\alpha^1(p) = 0, \phi_\beta^1(p) \neq 0.$$

We resolve this in the following. \mathfrak{M}

Theorem 1.2.

Suppose U, U' are open sets in the product topology $(-\infty, 0] \times \mathbb{R}^{n-1}$, and $\exists f : U \rightarrow U'$ homeomorphic. Then

$$f(U \cap (\{0\} \times \mathbb{R}^{n-1})) = U' \cap (\{0\} \times \mathbb{R}^{n-1}).$$

Proof. We show instead that

$$f(U \cap ((-\infty, 0) \times \mathbb{R}^{n-1})) = U' \cap ((-\infty, 0) \times \mathbb{R}^{n-1}).$$

Via [Corollary 1.1.2](#), we see $f(U \cap ((-\infty, 0) \times \mathbb{R}^{n-1}))$ must be an open set in the Euclidean topology of \mathbb{R}^n . Therefore, it cannot contain $\{0\} \times \mathbb{R}^{n-1}$. \blacksquare

Corollary 1.2.1 (Boundary is Well-defined).

$\forall p \in \mathcal{M}$, if $\exists \phi_\alpha^1(p) = 0$, then $\forall \phi \in \Phi$ atlas, $\phi^1(p) = 0$.

Proof. Make use of the fact that $\phi \circ \phi_\alpha^{-1}$ is a homeomorphism. \blacksquare

1.5 Submanifolds

1.5.1 Definition

Definition 1.10 (Local Submanifold).

$A \subseteq \mathcal{M}$ is a submanifold of codimension r around $p \in A \subseteq \mathcal{M}$ if there exists a chart $\phi : U \rightarrow \phi(U) \in \Phi$ atlas s.t. $p \in U$ and

$$\phi(U \cap A) = \phi(U) \cap (\mathbb{R}^{m-r} \times \underbrace{\{0, \dots, 0\}}_{r \text{ zeroes}}).$$

Definition 1.11 (Submanifold).

If A is a C^k local submanifold of \mathcal{M} of codimension r around every $p \in \mathcal{M}$, then we say A is a submanifold of \mathcal{M} of codimension r .

2 Tangent Spaces

↗ **Remark.**

The definition of manifold do not require the entity to be embeded in a higher dimensional space. Therefore, the traditional view of tangency is not valid here.

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2.1 Curves and Vectors

Definition 2.1 (Curve).

A curve on \mathcal{M} is a C^∞ map,

$$\sigma : (-\epsilon, \epsilon) \rightarrow \mathcal{M}.$$

Definition 2.2 (Curve Tangency).

Two curves σ_1, σ_2 are tangent at $p \in \mathcal{M}$ if

1. $\sigma_1(0) = \sigma_2(0) = p$.
2. $\frac{d}{dt}(x^i \circ \sigma_1(t))|_{t=0} = \frac{d}{dt}(x^i \circ \sigma_2(t))|_{t=0}, \quad 1 \leq i \leq m$.

↗ **Remark.**

Written more compactly,

$$\frac{d}{dt}(\phi \circ \sigma_1)|_{t=0} = \frac{d}{dt}(\phi \circ \sigma_2)|_{t=0}$$

ℳ

Definition 2.3 (Tangent Vectors).

A tangent vector at $p \in \mathcal{M}$ is an equivalence class of curves where the equivalence relation is that they are tangent. It will be denoted as

$$v = [\sigma].$$

Definition 2.4 (Tangent Space).

The tangent space $T_p\mathcal{M}$ at point p is the set of all tangent vectors at point p .

Definition 2.5 (Tangent Bundle).

The tangent bundle $T\mathcal{M}$ is

$$T\mathcal{M} := \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}.$$

2.2 Curves and Derivation

Definition 2.6 (Directional Derivative).

For any $f : \mathcal{M} \rightarrow \mathbb{R}$ s.t. $f \in C^\infty$, we define

$$v(f) := \frac{d}{dt}(f \circ \sigma(t)) \Big|_{t=0},$$

where $v = [\sigma]$.

Theorem 2.1.

The definition **Definition 2.6** is well-defined. That is, $v(f)$ is independent of the curve σ chosen as well as $v = [\sigma]$.

Proof. Let $v_1 = [\sigma_1] = [\sigma_2] = v_2$. Then

$$v_1(f) = \frac{d}{dt}(f \circ \sigma_1) \Big|_{t=0},$$

$$v_2(f) = \frac{d}{dt}(f \circ \sigma_2) \Big|_{t=0},$$

$$\frac{d}{dt}(\phi \circ \sigma_1) \Big|_{t=0} = \frac{d}{dt}(\phi \circ \sigma_2) \Big|_{t=0}.$$

Then

$$\begin{aligned}
v_1(f) &= \frac{d}{dt} \left(\underbrace{(f \circ \phi^{-1})}_{\mathbb{R} \leftarrow \mathbb{R}^m} \circ \underbrace{(\phi \circ \sigma_1)}_{\mathbb{R}^m \leftarrow \mathbb{R}} \right) \Big|_{t=0} \\
&= (f \circ \phi^{-1})' \circ (\phi \circ \sigma_1) \cdot (\phi \circ \sigma_1)' \Big|_{t=0} \\
&= (f \circ \phi^{-1})' \circ (\phi \circ \sigma_2) \cdot (\phi \circ \sigma_2)' \Big|_{t=0} \\
&= v_2(f),
\end{aligned}$$

since $\phi \circ \sigma_1(0) = \phi \circ \sigma_2(0) = \phi(p)$, and $(\phi \circ \sigma_1)' = (\phi \circ \sigma_2)'$ by equivalence. \blacksquare

2.3 Isomorphism with Euclidean Spaces

Definition 2.7 ("Straight Lines").

Choose a coordinate chart $\phi \in \Phi$ atlas near p . For all $v \in \mathbb{R}^m$, we define

$$\gamma_v^\phi(t) := \phi^{-1}(\phi(p) + tv).$$

In simple words, it is such a curve on manifold that it is a straight line on maps.

Theorem 2.2 (Isomorphism with Straight Lines).

Let $p \in \mathcal{M}$, $p \in \phi \in \Phi$. For any curve γ passing through p , $\exists! v \in \mathbb{R}^m$ s.t. γ is tangent to γ_v^ϕ , where v can be explicitly given by $(\phi \circ \gamma)'(0)$.

In other words, the map

$$\begin{aligned}
\ell_p^\phi : \mathbb{R}^m &\rightarrow T_p \mathcal{M} \\
v &\mapsto [\gamma_v^\phi]
\end{aligned}$$

is a bijection with inverse

$$\begin{aligned}
(\ell_p^\phi)^{-1} : T_p \mathcal{M} &\rightarrow \mathbb{R}^m \\
[\gamma] &\mapsto (\phi \circ \gamma)'(0).
\end{aligned}$$

Proof. It is almost by definition that γ is tangent to $(\phi \circ \gamma)'(0)$.

Suppose v_1, v_2 satisfies γ is tangent to $\gamma_{v_1}^\phi, \gamma_{v_2}^\phi$. Then they are tangent too. So

$$(\phi \circ \gamma_{v_1}^\phi)'(0) = (\phi \circ \gamma_{v_2}^\phi)'(0).$$

Therefore, $v_1 = v_2$. \blacksquare

Definition 2.8 (Alternative Definition for Addition).

For $v_1, v_2 \in T_p \mathcal{M}$, we define addition via the isomorphism [Theorem 2.2](#),

$$v_1 + v_2 := (\ell_p^\phi)^{-1}(\ell_p^\phi(v_1) + \ell_p^\phi(v_2)).$$

Definition 2.9 (Alternative Definition of Basis Tangent Vectors).

$$(\partial_\mu)_p := \ell_p^\phi(e^\mu).$$

Theorem 2.3.

$$(\partial_\mu)_p(x^\nu) = \delta_\mu^\nu.$$

Proof. By [Definition 2.9](#),

$$(\partial_\mu)_p := \ell_p^\phi(e^\mu) = \phi^{-1}(\phi(p) + te^\mu),$$

where e^μ is the column vector with μ -th component 1, others 0. Then

$$\begin{aligned} (\partial_\mu)_p(x^\nu) &= (x^\nu \circ \phi^{-1}(\phi(p) + te^\mu))'(0) \\ &= (x^\nu \circ \phi^{-1})'(\phi(p))e^\mu. \end{aligned}$$

But since x^ν has only one component, $(x^\nu \circ \phi^{-1})'$ is a row vector with ν -th component 1, others 0. So the result follows immediately. \blacksquare

Theorem 2.4 (Linear Independence of Basis Tangent Vectors).

The basis tangent vectors $(\partial_\mu)_p, 1 \leq \mu \leq \dim \mathcal{M}$ are linear independent.

Proof. Suppose $a^\mu (\partial_\mu)_p = 0$. Then

$$a^\mu (\partial_\mu)_p(x^\nu) = a^\mu \delta_\mu^\nu = 0(x^\nu) = 0.$$

So $a^\nu = 0$. \blacksquare

Theorem 2.5 (Coordinate Expansion of Tangent Vectors).

For all $v \in T_p \mathcal{M}$, we have

$$v = v^\mu (\partial_\mu)_p,$$

where Einstein notation was used, and $v^\mu = v(x^\mu)$.

↗ **Remark.**

A natural question is that whether two charts behave "the same" if $U_1 \cap U_2 \neq \emptyset$. Under this perspective, the criterion is clear: we need only to check whether a straight line is still a straight line in another chart, which is true indeed. It also produces the coordinate transformation formula for free. See below. \mathfrak{M}

Theorem 2.6 (Coordinate Transformation of Straight Lines).

Choose $\phi, \psi \in \Phi$, $\phi : U_1 \rightarrow \mathbb{R}^m$, $\psi : U_2 \rightarrow \mathbb{R}^m$ s.t. $U_1 \cap U_2 \neq \emptyset$ and $p \in U_1 \cap U_2$. Let the corresponding straight line isomorphisms ℓ_p^ϕ, ℓ_p^ψ . Then $(\ell_p^\psi)^{-1} \circ \ell_p^\phi$ is a linear isomorphism.

Let the local coordinates induced by ϕ be x^1, \dots, x^m , ψ be y^1, \dots, y^m , then $(\ell_p^\psi)^{-1} \circ \ell_p^\phi$ can be expressed in terms of the Jacobian **Definition 1.6** $J^\nu_\mu := \frac{\partial y^\nu}{\partial x^\mu} \Big|_{\phi(p)}$, namely,

$$(\ell_p^\psi)^{-1} \circ \ell_p^\phi(v) = Jv.$$

Proof.

$$\begin{aligned} (\ell_p^\psi)^{-1} \circ \ell_p^\phi(v) &= (\ell_p^\psi)^{-1}(\phi^{-1}(\phi(p) + vt)) \\ &= ((\psi \circ \phi^{-1})(\phi(p) + vt))'(0) \\ &= Jv \end{aligned}$$

◻

Definition 2.10 (Contravariancy and Covariancy).

Let \mathcal{M} be a m -dimensional C^∞ manifold. Choose two coordinate charts

$$\begin{aligned} \phi : U_\phi \rightarrow V_\phi, p \mapsto (x^1(p), \dots, x^m(p)), \\ \psi : U_\psi \rightarrow V_\psi, p \mapsto (y^1(p), \dots, y^m(p)). \end{aligned}$$

and the corresponding Jacobian matrix **Definition 1.6** $J^\nu_\mu := \frac{\partial y^\nu}{\partial x^\mu} \Big|_{\phi(p)}$. We

define covariancy to be anything that transforms like

$$(\text{new})_\nu = (\text{old})_\mu (J^{-1})^\mu{}_\nu.$$

Corollary 2.6.1 (Contravariancy of Tangent Vectors).

The basis vectors transform covariantly, whereas the components of vectors transform contravariantly. That is, choose two overlapping coordinate charts (x^1, \dots, x^m) and (y^1, \dots, y^m) and define their Jacobian matrix [Definition 1.6](#)

$$J^\nu{}_\mu := \frac{\partial y^\nu}{\partial x^\mu} \Big|_{\phi(p)},$$

$$\begin{cases} \left(\frac{\partial}{\partial y^\nu} \right) = \left(\frac{\partial}{\partial x^\mu} \right) (J^{-1})^\mu{}_\nu & (\text{covariant}) \\ v^{\nu'} = J^{\nu'}{}_\mu v^\mu & (\text{contravariant}) \end{cases}$$

2.4 Pushforward

2.4.1 Definition and Linearity

↗ Remark.

The pushforward $h_* : T_p \mathcal{M} \rightarrow T_{h(p)} \mathcal{N}$ of a specific function $h : \mathcal{M} \rightarrow \mathcal{N}$ can be thought of as local linearization of the function.

ℳ

Definition 2.11 (Pushforward).

Given a function $h : \mathcal{M} \rightarrow \mathcal{N}$ and $v \in T_p \mathcal{M}$, then we define the pushforward $h_* : T_p \mathcal{M} \rightarrow T_{h(p)} \mathcal{N}$ by

$$h_*(v) := [h \circ \sigma], \quad v = [\sigma].$$

Theorem 2.7.

The pushforward operation [Definition 2.11](#) is well-defined. That is, $h_*(v_1) = h_*(v_2)$ if $v_1 = [\sigma_1] = [\sigma_2] = v_2$.

Theorem 2.8 (Algebraic Definition of Pushforward).

The definition of pushforward [Definition 2.11](#) is equivalent to the following: let $h : \mathcal{M} \rightarrow \mathcal{N}$, $h_* : D_p \mathcal{M} \rightarrow D_{h(p)} \mathcal{N}$ is defined by,

$$(h_* v)(f) := v(f \circ h).$$

Proof. (\rightarrow)

$$\begin{aligned} h_*(v)(f) &= [h \circ \sigma](f) = \frac{d}{dt}(f \circ h \circ \sigma(t)) \Big|_{t=0} \\ &= \frac{d}{dt}((f \circ h) \circ \sigma(t)) \Big|_{t=0} \\ &:= v(f \circ h). \end{aligned}$$

(\leftarrow) This direction is similar. □

Theorem 2.9 (Linearity of Pushforward).

The pushforward map $h_* : T_p \mathcal{M} \rightarrow T_{h(p)} \mathcal{N}$ is linear.

$$h_*(rv_1 + v_2) = rh_*(v_1) + h_*(v_2).$$

Proof. (Using [Definition 2.11](#)) Let $p \in (U, \phi) \subseteq \mathcal{M}$, and $h(p) \in (V, \psi) \subseteq \mathcal{N}$. Choose ϕ s.t. $\phi(p) = 0$. It is obvious that $h_*(rv_1 + v_2)(0) = (rh_*(v_1) + h_*(v_2))(0) = h(p)$.

Consider

$$\begin{aligned} \frac{d}{dt} \underbrace{(\psi \circ h_*(rv_1 + v_2))}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathbb{R}} \Big|_{t=0} &= \frac{d}{dt} \left(\underbrace{\psi \circ h \circ (\phi^{-1})}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathcal{M} \leftarrow \mathbb{R}^m} \circ \underbrace{(r\phi \circ \sigma_1 + \phi \circ \sigma_2)}_{\mathbb{R}^m \leftarrow \mathcal{M} \leftarrow \mathbb{R}} \right) \Big|_{t=0} \\ &= (\psi \circ h \circ \phi^{-1})' \circ (r\phi \circ \sigma_1 + \phi \circ \sigma_2) \cdot (r\phi \circ \sigma_1 + \phi \circ \sigma_2)' \Big|_{t=0} \\ &= (\psi \circ h \circ \phi^{-1})'(0) \cdot ((r\phi \circ \sigma_1)' + (\phi \circ \sigma_2)') \Big|_{t=0}. \end{aligned}$$

And

$$\frac{d}{dt} \left(\underbrace{\psi}_{\mathbb{R}^n \leftarrow} \circ \underbrace{(rh_*(v_1) + h_*(v_2))}_{\mathcal{N} \leftarrow \mathbb{R}} \right) \Big|_{t=0} = \frac{d}{dt} \underbrace{(r\psi \circ h \circ \sigma_1 + \psi \circ h \circ \sigma_2)}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathcal{M} \leftarrow \mathbb{R}} \Big|_{t=0}$$

$$\begin{aligned}
&= (\underbrace{r\psi \circ h \circ \phi^{-1}}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathcal{M} \leftarrow \mathbb{R}^m} \circ \underbrace{\phi \circ \sigma_1}_{\mathbb{R}^m \leftarrow \mathcal{M} \leftarrow \mathbb{R}} + \psi \circ h \circ \phi^{-1} \circ \phi \circ \sigma_2)'|_{t=0} \\
&= (r(\psi \circ h \circ \phi^{-1})' \circ (\phi \circ \sigma_1) \cdot (\phi \circ \sigma_1)')|_{t=0} \\
&\quad + ((\psi \circ h \circ \phi^{-1})' \circ (\phi \circ \sigma_2) \cdot (\phi \circ \sigma_2)')|_{t=0} \\
&= (\psi \circ h \circ \phi^{-1})'(0) \cdot (r(\phi \circ \sigma_1)' + (\phi \circ \sigma_2))|_{t=0}.
\end{aligned}$$

So we see the two are equal.

(Using [Theorem 2.8](#))

$$\begin{aligned}
(h_*(rv_1 + v_2))(f) &= (rv_1 + v_2)(f \circ h) \\
&= rv_1(f \circ h) + v_2(f \circ h) \\
&= r(h_*v_1)f + (h_*v_2)f.
\end{aligned}$$

(Using straight line isomorphism: [Definition 2.8](#))

$$\begin{array}{ccc}
T_p \mathcal{M} & \xrightarrow{h_*} & T_{h(p)} \mathcal{N} \\
\ell_p^\phi \uparrow & & \downarrow (\ell_{h(p)}^\psi)^{-1} \\
\mathbb{R}^m & \longrightarrow & \mathbb{R}^n
\end{array}$$

Since ℓ_p^ϕ and $(\ell_{h(p)}^\psi)^{-1}$ are linear, to prove h_* is linear, we need only to show $(\ell_{h(p)}^\psi)^{-1} \circ h_* \circ \ell_p^\phi$ is linear.

$$\begin{aligned}
&(\ell_{h(p)}^\psi)^{-1} \circ h_* \circ \ell_p^\phi(v) \\
&= (\ell_{h(p)}^\psi)^{-1} \circ h_*([\phi^{-1}(\phi(p) + tv)]) \\
&= (\ell_{h(p)}^\psi)^{-1}([h \circ \phi^{-1}(\phi(p) + tv)]) \\
&= (\psi \circ h \circ \phi^{-1}(\phi(p) + tv))'(0) \\
&= (\psi \circ h \circ \phi^{-1})(\phi(p))v.
\end{aligned}$$

□

Theorem 2.10 (Associativity of Pushforwards).

Given manifolds $\mathcal{M}, \mathcal{N}, \mathcal{P}$ and $h : \mathcal{M} \rightarrow \mathcal{N}, k : \mathcal{N} \rightarrow \mathcal{P}$, then

$$(k \circ h)_* = k_* \circ h_*.$$

2.4.2 Jacobian

Theorem 2.11 (Local Representative of Pushforward).

Let $\dim \mathcal{M} = m, \dim \mathcal{N} = n, h : \mathcal{M} \rightarrow \mathcal{N}, \{x^1, \dots, x^m\}$ be the local coordinates of \mathcal{M} around p , and $\{y^1, \dots, y^n\}$ be the local coordinates of \mathcal{N} around $h(p)$. Then

$$h_* v = \sum_{\mu=1}^m \sum_{\nu=1}^n (\partial_\nu)_{h(p)} \frac{\partial h^\nu}{\partial x^\mu} \Big|_p v^\mu,$$

where $J^\nu_\mu := \frac{\partial h^\nu}{\partial x^\mu} \Big|_p := (\partial_\mu)_p (y^\nu \circ h)$ is the Jacobian matrix.

Proof. First expand v in terms of local coordinates and use linearity,

$$h_* v = h_*(v^\mu (\partial_\mu)_p) = v^\mu h_*((\partial_\mu)_p).$$

Expand the result in local coordinates of \mathcal{N} ,

$$h_*((\partial_\mu)_p) = \left(h_* (\partial_\mu)_p \right)^\nu (\partial_\nu)_{h(p)}.$$

Using [Theorem 2.8](#),

$$\begin{aligned} \left(h_* (\partial_\mu)_p \right)^\nu &= \left(h_* (\partial_\mu)_p \right) \circ y^\nu \\ &= (\partial_\mu)_p (y^\nu \circ h) \\ &:= (\partial_\mu)_p h^\nu. \end{aligned}$$

So,

$$h_*((\partial_\mu)_p) = (\partial_\mu)_p h^\nu (\partial_\nu)_{h(p)}.$$

And,

$$h_* v = v^\mu (\partial_\mu)_p h^\nu (\partial_\nu)_{h(p)}.$$



Theorem 2.12 (Using Curve to Pushforward).

Given $c : (-\epsilon, \epsilon) \rightarrow \mathcal{M}$ a curve, and choose the coordinate chart of \mathbb{R} to be the identity, then

$$c_* \left(\frac{d}{dt} \right)_0 = [c] \in T_p \mathcal{M}.$$

Proof. First we clarify what is $(\frac{d}{dt})_0$. Since on the trivial manifold \mathbb{R} there is only one coordinate, namely t , we need not specify the number. Also, considering our functions are scalar valued $f : \mathcal{M} \rightarrow \mathbb{R}$, this motivates us to write "total differential".

For all $f \in C^\infty$,

$$c_* \left(\frac{d}{dt} \right)_0 f = \left(\frac{d}{dt} \right)_0 (f \circ c).$$

Since the coordinate chart is the identity,

$$\begin{aligned} \left(\frac{d}{dt} \right)_0 (f \circ c) &= \frac{d}{dt} (f \circ c \circ I) \Big|_{I(t)=0} \\ &= \frac{d}{dt} (f \circ c) \Big|_{t=0} \\ &= [c]f. \end{aligned}$$

□

3 Formal Differential Form

↗ Remark.

In this section, we follow a local, coordinate approach. We focus on the requirements that makes form a form. We will postpone the realization of differential forms. ℳ

3.1 Euclidean Spaces

Definition 3.1 (Formal Differential Form on \mathbb{R}^m).

A formal differential k -form on \mathbb{R}^m is composed of m^k functions, arranged in the form

$$\omega = \sum_{1 \leq i_1, \dots, i_k \leq m} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

We require,

1. All the functions $\omega_{i_1 \dots i_k}$ are $(-\infty, 0] \times \mathbb{R}^{m-1} \rightarrow (-\infty, 0] \times \mathbb{R}^{m-1}$ and C^∞ .
2. The operation $+$ is commutative and associative, just like the usual addition.
3. The operation \wedge is associative and distributes over $+$, resembling the usual multiplication.
4. \wedge also has anticommutativity,

$$\begin{cases} dx^i \wedge dx^j = -dx^j \wedge dx^i & \forall i \neq j \\ dx^i \wedge dx^i = 0 \end{cases}$$

Definition 3.2 (Equality of Formal Differential k-forms).

Let two formal k -forms ω, η be

$$\begin{aligned} \omega &= \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \\ \eta &= \eta_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \end{aligned}$$

We say they are "equal", denoted $\omega \equiv \eta$, iff

$$\sum_{\sigma \in S_k} \omega_{\sigma(i_1) \dots \sigma(i_k)} = \sum_{\sigma \in S_k} \eta_{\sigma(i_1) \dots \sigma(i_k)} \quad \forall 1 \leq i_1 < \dots < i_k \leq m.$$

Remark. 1. For a formal differential k -form $\omega = dx^1 \wedge dx^2$, $\omega_{12} = 1$, but $\omega_{21} = 0$. The m^k components in the definition just presents a general form, so that it includes the scenario $\omega' = dx^1 \wedge dx^2 - dx^2 \wedge dx^1$. If you insist on arranging indices even in the definition, then ω' would not satisfy the definition, which is weird.

2. For demonstration of equality, consider the following example in \mathbb{R}^3 ,

$$\begin{aligned}\omega &= dx^1 \wedge dx^2 + dx^2 \wedge dx^3 \\ \eta &= -dx^2 \wedge dx^1 + dx^2 \wedge dx^3.\end{aligned}$$

Choose $i_1 = 1, i_2 = 2$, and

$$S_3 = \left\{ \underbrace{\text{id}, (1 2 3), (1 3 2)}_{\text{even}}, \underbrace{(1 2), (2 3), (1 3)}_{\text{odd}} \right\}.$$

Then, in order,

$$\begin{aligned}&\sum_{\sigma \in S_n} (\text{sgn } \sigma) \omega \\ &= \color{red}{\omega_{12}} + \color{blue}{\omega_{23}} + \color{green}{\omega_{31}} - \color{red}{\omega_{21}} - \color{green}{\omega_{13}} - \color{blue}{\omega_{32}} \\ &= \color{red}{1} + \color{blue}{1} + \color{green}{0} - \color{red}{0} - \color{green}{0} - \color{blue}{0}.\end{aligned}$$

Notice how all permutations of a given component (paired in color) appears exactly once in this relation, and the sign is fixed accordingly by $\text{sgn } \sigma$.

3. Viewed this way, we see in the language of formal differential forms,

$$\begin{aligned}\omega &= dx^1 \wedge dx^2 \\ \eta &= -dx^2 \wedge dx^1 \\ \nu &= dx^1 \wedge dx^2 - dx^2 \wedge dx^1,\end{aligned}$$

$\omega \equiv \eta$, since $\omega_{12} - \omega_{21} = 1 = \eta_{12} - \eta_{21}$. But $\omega \not\equiv \nu$, since $\nu_{12} - \nu_{21} = 2$.

ℳ

3.2 Operations

Definition 3.3 (Exterior Product).

Let two formal k -forms be $\omega = \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$, $\eta = \eta_{j_1 \dots j_l} dx^{j_1} \wedge \dots \wedge dx^{j_l}$. Then

$$\omega \wedge \eta := \omega_{i_1 \dots i_k} \eta_{j_1 \dots j_l} dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}.$$

Which is equivalent [Definition 3.2](#) to,

$$\sum_{1 \leq s_1 < \dots < s_{k+l} \leq m} \left(\sum_S (\text{sgn } \sigma) \omega_{i_1 \dots i_k} \eta_{j_1 \dots j_l} \right) dx^{s_1} \wedge \dots \wedge dx^{s_{k+l}}.$$

Where S is all the combinations of $S_1 = \{i_1, \dots, i_k, j_1, \dots, j_l\}$ that is equal to $S_2 = \{s_1, \dots, s_{k+l}\}$, and the order does not matter. σ is the function $S_1 \rightarrow S_2$.

Remark.

Example. Let $\dim \mathcal{M} = 6$, and

$$\begin{aligned} \omega &= \omega_{12} dx^1 \wedge dx^2 + \omega_{21} dx^2 \wedge dx^1 \\ \eta &= \eta_{456} dx^4 \wedge dx^5 \wedge dx^6. \end{aligned}$$

Then

$$\omega \wedge \eta = (\omega_{12} \eta_{456} - \omega_{21} \eta_{456}) dx^{1,2,4,5,6}$$

ℳ

Definition 3.4 (Pullback).

Let $f : U \subseteq \mathbb{R}^m \rightarrow V \subseteq (-\infty, 0] \times \mathbb{R}^{n-1}$ is C^∞ . Choose local coordinates on U to be $x = (x^1, \dots, x^m)$, on V to be $y = (y^1, \dots, y^n)$. Define $f^\nu := y^\nu \circ f$.

Let $\omega = \omega_{j_1 \dots j_k} dy^{j_1} \wedge \dots \wedge dy^{j_k}$ be a formal k -form on U . Then

$$f^* \omega := (\omega_{j_1 \dots j_k} \circ f) \frac{\partial f^{j_1}}{\partial x^{i_1}} \dots \frac{\partial f^{j_k}}{\partial x^{i_k}} dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

- ↗ **Remark.**
1. The motivation is just somehow $\omega \circ f$. Chain the component function with f , and express the (resulting) coordinates as $dy^\nu = \frac{\partial f^\nu}{\partial x^\mu} dx^\mu$.
 2. It is possible that $U \subseteq V$. Pulling back onto a subset is essentially a "limitation" on ω .
 3. It is also possible that $U \subseteq \partial V$, i.e. U is the boundary of V . $f : p \mapsto (0, p)$ is just the immersion map in that case.

ℳ

Definition 3.5 (Exterior Differentiation).

Let a formal k -form be $\omega = \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$. Then

$$d\omega := \left(\frac{\partial \omega_{i_1 \dots i_k}}{\partial x^{i_0}} dx^{i_0} \right) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

- ↗ **Remark.**

Let $U \subseteq \mathbb{R}^3$.

1. Consider a formal 0-form, i.e. $f \in C^\infty$. Then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \rightarrow \nabla f.$$

2. Consider a formal 1-form $\omega = Pdx + Qdy + Rdz$. Then

$$\begin{aligned} d\omega &= \left(\frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz \right) \wedge dx + \left(\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial z} dz \right) \wedge dy + \left(\frac{\partial R}{\partial x} dx + \frac{\partial R}{\partial y} dy \right) \wedge dz \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left(\frac{\partial P}{\partial x} - \frac{\partial R}{\partial z} \right) dx \wedge dz + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy \\ &\rightarrow \text{curl } \omega. \end{aligned}$$

3. Consider a formal 2-form $\eta = Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy$. Then

$$\begin{aligned} d\eta &= \frac{\partial A}{\partial x} dx \wedge dy \wedge dz + \frac{\partial B}{\partial y} dy \wedge dz \wedge dx + \frac{\partial C}{\partial z} dz \wedge dx \wedge dy \\ &= \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) dx \wedge dy \wedge dz \\ &\rightarrow \operatorname{div} \eta. \end{aligned}$$

4. In usual vector calculus terms, we say ∇ produces a vector, div produces a scalar, and curl produces a vector. This r -form to $(m - r)$ -form correspondance is provided by the Hodge star operation.

\mathfrak{M}

3.3 Differential Form as Equivalence Classes

3.3.1 Operations are Well-Defined

Theorem 3.1.

If $\omega \equiv \omega'$, $\eta \equiv \eta'$ are formal k -forms on \mathbb{R}^m , then

$$\omega \wedge \eta \equiv \omega' \wedge \eta'.$$

Theorem 3.2.

If $\omega \equiv \omega'$ are formal k -forms on \mathbb{R}^m , and $f \in C^\infty(\mathbb{R}^m)$, then

$$f^* \omega \equiv f^* \omega'.$$

Theorem 3.3.

If $\omega \equiv \omega'$ are formal k -forms on \mathbb{R}^m , then

$$d\omega \equiv d\omega'.$$

3.3.2 Equivalence Classes

Definition 3.6 (Euclidean Differential k-form).

Denote the set of all formal k -forms on \mathbb{R}^m be $A^k(\mathbb{R}^m)$. Then the set of all differential k -forms on \mathbb{R}^m is defined to be

$$\Lambda^k(\mathbb{R}^m) := A^k(\mathbb{R}^m)/\equiv.$$

3.4 Properties of Operations

Theorem 3.4 (Properties of Exterior Product).

Let $\omega \in \Lambda^k(\mathbb{R}^m)$, $\eta \in \Lambda^l(\mathbb{R}^m)$. Then

$$\begin{aligned} (\omega_1 + \omega_2) \wedge \eta &= \omega_1 \wedge \eta + \omega_2 \wedge \eta \\ \omega \wedge \eta &= (-1)^{kl} \eta \wedge \omega. \end{aligned}$$

Theorem 3.5 (Properties of Pullback).

Let $f \in C^\infty$, $\omega \in \Lambda^k(\mathbb{R}^m)$. Then

$$\begin{aligned} f^*(\omega_1 + \omega_2) &= f^*\omega_1 + f^*\omega_2 \\ f^*(\omega_1 \wedge \omega_2) &= (f^*\omega_1) \wedge (f^*\omega_2) \\ g^*(f^*\eta) &= (f \circ g)^*\eta. \end{aligned}$$

Theorem 3.6 (Properties of Exterior Differentiation).

Let $f \in C^\infty$, $\omega \in \Lambda^k(\mathbb{R}^m)$. Then

$$\begin{aligned} f^*(d\omega) &= d(f^*\omega) \\ d(\omega \wedge \eta) &= (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta) \\ d(d\omega) &= 0. \end{aligned}$$

3.5 Differential Forms on Manifolds

3.5.1 Requirements of Manifold Forms

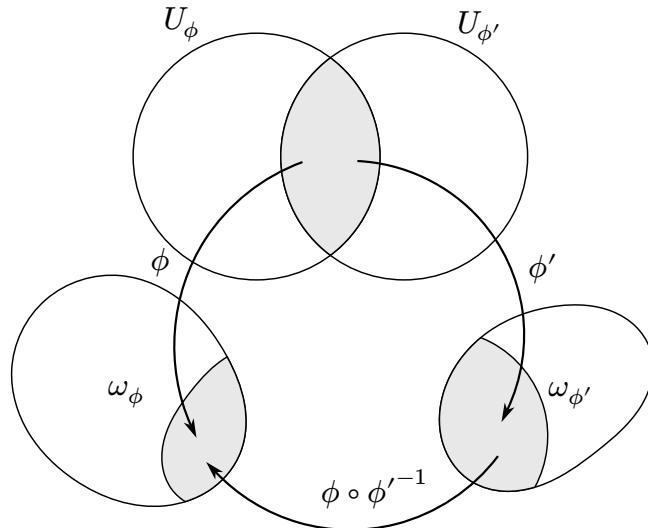
Definition 3.7 (Differential Form on a Manifold).

A C^∞ differential k -form ω on manifold \mathcal{M} , $\omega \in \Lambda^k(\mathcal{M})$, consists of a family of differential k -forms $\omega_\phi \in \Lambda^k(\phi(U_\phi))$, $\phi \in \Phi$, $\phi : U_\phi \rightarrow V_\phi = \phi(U_\phi) \subseteq (-\infty, 0] \times \mathbb{R}^{m-1}$, with an additional requirement that

$$(\phi \circ \phi'^{-1})^* (\omega_\phi|_{\phi(U_\phi \cap U_{\phi'})}) = \omega_{\phi'}|_{\phi'(U_\phi \cap U_{\phi'})}, \quad \forall \phi, \phi' \in \Phi.$$

ω_ϕ is called the local expression of ω on U_ϕ via ϕ .

↗ **Remark.**



The motivation is that, if two differential forms describe the same set, they should "agree" on that portion of manifold. The "agreement" is done by pull-back using the overlap function. ℳ

Theorem 3.7 (Covariancy of Differential Forms).

Choose two overlapping charts x^1, \dots, x^m and y^1, \dots, y^m and define their Jacobian matrix **Definition 1.6** $J^\nu_\mu := \frac{\partial y^\nu}{\partial x^\mu} \Big|_{\phi(p)}$, combining **Definition 3.4** and **Definition 3.7**, we have

$$\begin{cases} dy^\nu = J^\nu_\mu dx^\mu & (\text{covariant}) \\ \omega_{\nu'} = \omega_\mu (J^{-1})^\mu_\nu & (\text{contravariant}) \end{cases}$$

3.5.2 Operations on Manifold Forms

Definition 3.8 (Addition).

Given two forms $\omega, \eta \in \Lambda^k(\mathcal{M})$, $m = \dim \mathcal{M}$, define their sum to be $\omega + \eta$, whose chart components are given by

$$(\omega + \eta)_\phi := \omega_\phi + \eta_\phi.$$

They satisfy **Definition 3.7** thanks to the linearity of pullback.

Definition 3.9 (Exterior Product).

Given two forms $\omega, \eta \in \Lambda^k(\mathcal{M})$, $m = \dim \mathcal{M}$, define their exterior product to be $\omega \wedge \eta$, whose chart components are given by

$$(\omega \wedge \eta)_\phi := \omega_\phi \wedge \eta_\phi.$$

They satisfy **Definition 3.7** because pullback commutes with exterior product.

Definition 3.10 (Exterior Differentiation).

Given a form $\omega \in \Lambda^k(\mathcal{M})$, $m = \dim \mathcal{M}$, define its exterior derivative to be $d\omega$, whose chart components are given by

$$(d\omega)_\phi := d(\omega_\phi).$$

They satisfy **Definition 3.7** because pullback commutes with exterior product.

Definition 3.11 (Pullback).

Given a form $\omega \in \Lambda^k(\mathcal{N})$ and a function $f : \mathcal{M} \rightarrow \mathcal{N}$, $m = \dim \mathcal{M}, n = \dim \mathcal{N}$. Choose coordinate functions $\phi_i : U_i \rightarrow \phi_i(U_i)$ on \mathcal{M} , and $\psi_j : V_j \rightarrow \psi_j(V_j)$ on \mathcal{N} .

To define $(f^*\omega)_{\phi_1}$, choose any V_1, \dots, V_s s.t. $U_1 \subseteq \bigcup_{j=1}^s f^{-1}(V_j)$. Then

$$(f^*\omega)_{\phi_1} := \sum_j f^* \left(\omega_{\psi_j} \Big|_{\phi_1(f(U_1) \cap V_j)} \right)$$

They satisfy [Definition 3.7](#).

4 Integration of Differential Forms

4.1 Partition of Unity

Definition 4.1 (Support).

Let X be a topological space, and $f : X \rightarrow \mathbb{R}$. Then the support of f is defined as

$$\text{supp } f := \{x \in X \mid f(x) \neq 0\}.$$

Theorem 4.1 (Partition of Unity).

Let \mathcal{M} be a C^∞ manifold with dimension m with atlas Φ .

Let $\Phi = \{\phi_j \mid \phi_j : V_j \rightarrow \phi_j(V_j), j \in J\}$.

Then it is possible to construct a set of C^∞ functions $\rho_j, j \in J$ s.t.

$$1 = \sum_{j \in J} \rho_j, \quad \text{supp } \rho_j \subseteq V_j$$

4.2 Orientation

4.2.1 Definition

Definition 4.2 (Compatible Coordinate Charts).

Given a manifold \mathcal{M} , its two coordinate charts are called compatible (have the same orientation) if,

$$\det J > 0.$$

Where J is the Jacobian matrix [Definition 1.6](#).

The collection of all compatible charts of a specific orientation are denoted $\tilde{\Phi}$.

Theorem 4.2.

A manifold has either no orientation (any atlas is not compatible) or two orientations.

4.2.2 Positively Oriented Boundary

Definition 4.3 (Positively Oriented Boundary).

Let \mathcal{M} be a orientable C^∞ manifold with dimension m , with compatible atlas $\tilde{\Phi}$. Define coordinate charts on $\partial\mathcal{M}$ from Φ as follows,

$$\phi^{\partial\mathcal{M}} : U_\phi \cap \partial\mathcal{M} \rightarrow \mathbb{R}^{m-1},$$

Then $\tilde{\Phi}^{\partial\mathcal{M}} := \{ \phi^{\partial\mathcal{M}} \mid \phi \in \tilde{\Phi} \}$ determines an orientation on $\partial\mathcal{M}$, called the positive orientation.

4.3 Integration of Forms of Highest Degree

Definition 4.4 (Integration).

Let \mathcal{M} be a paracompact C^∞ manifold of dimension m , oriented by compatible atlas $\tilde{\Phi} = \{ \phi_j \mid j \in J \}$. Choose a C^∞ partition of unity $\rho_j, j \in J$ of \mathcal{M} s.t. $\text{supp } \rho_j \subseteq U_{\phi_j} := U_j$.

Let an m -form $\omega \in \Lambda^m(\mathcal{M})$ has local expression $\omega_{\phi_j} = f_j(x)dx_j^1 \wedge \cdots \wedge dx_j^m$, we say

$$\int_{\mathcal{M}} \omega = \sum_{j \in J} \int_{(-\infty, 0] \times \mathbb{R}^{m-1}} (\rho_j \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m$$

if the finite sum exists and has the same value for all choices of ρ_j and ϕ_j .

↗ **Remark.**

The following theorem reveals why orientation is a significant factor in the context of integration. ℳ

Theorem 4.3 (Criterion of Existence of Integral).

If $\text{supp } \omega$ is compact, then $\int_{\mathcal{M}} \omega$ exists.

Proof. Let two sets of coordinate charts be

$$\begin{aligned} \phi_j &: U_j \rightarrow V_j, j \in J \\ \phi'_k &: U'_k \rightarrow V'_k, k \in K. \end{aligned}$$

And cooresponding partition of unity be ρ_j, ρ'_k .

(The goal) Show

$$\begin{aligned} & \sum_{j \in J} \int (\rho_j \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m \\ &= \sum_{k \in K} \int (\rho'_k \circ \phi_k'^{-1})(x') f'_k(x') dx'^1 \dots dx'^m \end{aligned}$$

(Split using ρ'_k)

$$\begin{aligned} \int_{\mathcal{M}} \omega &= \sum_{j \in J} \int (\rho_j \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m \\ &= \sum_{j \in J} \int \sum_{k \in K} (\rho'_k \circ \phi_j^{-1})(x) (\rho_j \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m. \end{aligned}$$

Since the sum is finite, and $\text{supp } \omega$ is compact, and therefore the integral is not improper; thus, there can be no limit or Fubini problems on exchanging sums and integrals. So

$$\int_{\mathcal{M}} \omega = \sum_{j \in J} \sum_{k \in K} \int (\rho_j \rho'_k \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m$$

(Change of variables) First fix j, k .

$$\begin{aligned} & \int (\rho_j \rho'_k \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m \\ &= \int (\rho_j \rho'_k \circ \phi_j^{-1})(\phi_j \circ \phi_k'^{-1}(x')) f_j(\phi_j \circ \phi_k'^{-1}(x')) \left| \det \left(\frac{\partial x}{\partial x'} \right) \right| dx'^1 \dots dx'^m \\ &= \int (\rho_j \rho'_k \circ \phi_k'^{-1}(x')) f_j(\phi_j \circ \phi_k'^{-1}(x')) \left| \det \left(\frac{\partial x}{\partial x'} \right) \right| dx'^1 \dots dx'^m \end{aligned}$$

(Use pullback requirement) From [Definition 3.7](#),

$$(\phi_j \circ \phi_k'^{-1})^* \omega_{\phi_j} = \omega_{\phi_k'},$$

we see

$$\begin{aligned} & f'_j(x') dx'^1 \wedge \dots \wedge dx'^m \\ &= f_j(\phi_j \circ \phi_k'^{-1}(x')) \left(\frac{\partial x^1}{\partial x'^{\ell_1}} dx'^{\ell_1} \right) \wedge \dots \wedge \left(\frac{\partial x^m}{\partial x'^{\ell_m}} dx'^{\ell_m} \right) \\ &= \sum_{\sigma \in S_m} f_j(\phi_j \circ \phi_k'^{-1}(x')) (\text{sgn } \sigma) \frac{\partial x^1}{\partial x'^{\sigma(1)}} \dots \frac{\partial x^m}{\partial x'^{\sigma(m)}} dx'^1 \wedge \dots \wedge dx'^m \\ &= f_j(\phi_j \circ \phi_k'^{-1}(x')) \det \left(\frac{\partial x}{\partial x'} \right) dx'^1 \wedge \dots \wedge dx'^m \end{aligned}$$

(Orientation) Within a compatible atlas, however, $|\det(\frac{\partial x}{\partial x'})| = \det(\frac{\partial x}{\partial x'})$. Therefore, the integral

$$\begin{aligned} & \int (\rho_j \rho_k' \circ \phi_k'^{-1}(x')) f_j(\phi_j \circ \phi_k'^{-1}(x')) \left| \det \left(\frac{\partial x}{\partial x'} \right) \right| dx'^1 \dots dx'^m \\ &= \int (\rho_j \rho_k' \circ \phi_k'^{-1}(x')) f'_j(x') dx'^1 \dots dx'^m \end{aligned}$$

(Closing) By moving the sum wrt $j \in J$ into the integral and using the property of partition of unity, the proof is completed. \blacksquare

4.4 Integration of Forms of Lower Degree

4.4.1 Definition

Definition 4.5 (Integration of Lower Degree Forms).

Let \mathcal{Z} be an oriented C^∞ manifold of dimension d , $f : \mathcal{Z} \rightarrow \mathcal{M}$ be a C^∞ map to a C^∞ manifold \mathcal{M} of dimension m .

Let $\omega \in \Lambda^d(\mathcal{M})$, we define

$$\int_{\mathcal{Z}} \omega := \int_{\mathcal{Z}} f^* \omega$$

if the latter exists and the pullback function f is clear from context.

4.4.2 Stoke's theorem

Theorem 4.4 (Stoke's Theorem).

If \mathcal{M} is an oriented C^∞ manifold of dimension m and $\omega \in \Lambda^{m-1}(\mathcal{M})$, then

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega := \int_{\partial\mathcal{M}} i^* \omega,$$

where $i : \partial\mathcal{M} \rightarrow \mathcal{M}$ is just the immersion map, $i : p \mapsto p$.

Proof. (Partition Using Charts) Choose a C^∞ partition of unity [Theorem 4.1](#) $\rho_j, j \in J$ s.t. $\text{supp } \rho_j$ are compact and $\text{supp } \rho_j \subseteq U_{\phi_j} := U_j$.

Now $\omega = \sum_{j \in J} \rho_j \omega$ is a finite sum. So it suffices to show that, if $\eta \in \Lambda^{m-1}(\mathcal{M})$, $\text{supp } \eta$ compact and $\text{supp } \eta \subseteq U_\phi$ then $\int_{\mathcal{M}} d\eta = \int_{\partial\mathcal{M}} \eta$, and apply $\eta = \rho_j \omega$ for all $j \in J$.

(The integral) Suppose the coordinates of ϕ is labeled x^1, \dots, x^m . Locally, let

$$\eta = \sum_{\ell=1}^m f_\ell dx^1 \wedge \cdots \not{dx^\ell} \cdots \wedge dx^m,$$

where $f_\ell : (-\infty, 0] \times \mathbb{R}^{m-1} \rightarrow \mathbb{R}$, and it is C^∞ . Then, also locally,

$$\begin{aligned} d\eta &= \sum_{\ell=1}^m \frac{\partial f_\ell}{\partial x^\ell} dx^\ell \wedge dx^1 \wedge \cdots \not{dx^\ell} \cdots \wedge dx^m \\ &= \left(\sum_{\ell=1}^m (-1)^{\ell-1} \frac{\partial f_\ell}{\partial x^\ell} \right) dx^1 \wedge \cdots \wedge dx^m. \end{aligned}$$

Then by [Definition 4.4](#),

$$\int_{\mathcal{M}} d\eta = \int_{(-\infty, 0] \times \mathbb{R}^{m-1}} \left(\sum_{\ell=1}^m (-1)^{\ell-1} \frac{\partial f_\ell}{\partial x^\ell} \right) dx^1 \cdots dx^m.$$

(To be precise, we need to choose another partition of unity ρ'_j to do intergration. But we can just choose it to cover all of $\text{supp } \eta$ and don't care all other parts, so that doesn't matter too much.)

Choose a rectangular region R s.t.

$$\text{supp } \eta \subseteq [a_1, 0] \times \cdots \times [a_m, b_m]$$

and define

$$R_\ell := [a_1, 0] \times \cdots \not{[a_\ell, b_\ell]} \cdots \times [a_m, b_m].$$

($\ell = 2, \dots, m$) In this case, by Fubini and FTC,

$$\begin{aligned} &\int_R \frac{\partial f_\ell}{\partial x^\ell} dx^1 \cdots dx^m \\ &= \int_{R_\ell} \left(\int_{a_\ell}^{b_\ell} \frac{\partial f_\ell}{\partial x^\ell} dx^\ell \right) dx^1 \cdots \not{dx^\ell} \cdots dx^m \\ &= \int_{R_\ell} (\underbrace{f_\ell(x^1, \dots, b_\ell, \dots, x^m)}_0 - \underbrace{f_\ell(x^1, \dots, a_\ell, \dots, x^m)}_0) dx^1 \cdots \not{dx^\ell} \cdots dx^m \\ &= 0, \end{aligned}$$

since $\text{supp } \eta \subseteq R$, so on the boundary $f = 0$.

$(\ell = 1)$ Now the integral has only one term left.

$$\begin{aligned}
 \int_{\mathcal{M}} d\eta &= \int_R \frac{\partial f_1}{\partial x^1} dx^1 \dots dx^m \\
 &= \int_{R_1} \left(\int_{a_1}^0 \frac{\partial f_1}{\partial x^1} dx^1 \right) dx^2 \dots dx^m \\
 &= \int_{R_\ell} (f_1(0, x^2, \dots, x^m) - \underbrace{f_1(a_1, x^2, \dots, x^m)}_0) dx^2 \dots dx^m \\
 &= \int_{\mathbb{R}^{m-1}} (f_1 \circ i) dx^2 \dots dx^m \\
 &= \int_{\partial \mathcal{M}} i^* \eta.
 \end{aligned}$$

■

5 Vector Fields

5.1 Definition

Definition 5.1 (Vector Fields).

A vector field X on \mathcal{M} is a smooth assignment of a tangent vector $X_p \in T_p\mathcal{M} \forall p \in \mathcal{M}$.

"Smooth" assignment is defined to be that the Lie derivative **Definition 5.2** is smooth.

Definition 5.2 (Lie Derivative).

The Lie-derivative of function f with respect to vector field X is defined as

$$\mathcal{L}_X f := Xf,$$

and at a specific point $p \in \mathcal{M}$,

$$\mathcal{L}_X f(p) := Xf(p) := X_p f.$$

Theorem 5.1 (Properties of Lie Derivative).

The Lie derivative has the following properties,

1. $X(rf + g) = rXf + Xg$
2. $X(fg) = fXg + gXf.$

Theorem 5.2 (Component of Vector Field).

Given a chart (U, ϕ) on \mathcal{M} , we can write

$$X_U = X_U x^\mu \partial_\mu.$$

When the context is clear or for convenience, we write

$$X = X x^\mu \partial_\mu := X^\mu \partial_\mu.$$

Proof. We know

$$(Xf)(p) = X_p f = X_p x^\mu (\partial_\mu)_p f = (Xx^\mu)(p) (\partial_\mu)_p f.$$

□

↗ **Remark.**

∂_μ is a vector field that assigns each point $p \in \mathcal{M}$ with the vector $(\partial_\mu)_p \in T_p \mathcal{M}$.

ℳ

Theorem 5.3 (Contravariancy of Vector Fields).

Given two coordinate charts (U, ϕ) and (U', ϕ') s.t. $U \cap U' = S \neq \emptyset$. On S ,

$$X^{\nu'} = \sum_{\mu=1}^m X^\mu \frac{\partial x'^\nu}{\partial x^\mu}.$$

Analogous to ??.

5.2 Lie Bracket

Definition 5.3 (Composition of Vector Fields).

We can view $X : C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$, and so does Y . Therefore, we define

$$(X \circ Y)(f) := X(Yf).$$

Definition 5.4 (Lie Bracket (Commutator)).

We define the Lie Bracket of two vector fields X, Y to be

$$[X, Y] := X \circ Y - Y \circ X.$$

In particular,

$$[X, Y](f) = \mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f)$$

↗ **Remark.**

Lie Bracket **Definition 5.4** is a vector field, while the expression $X \circ Y$ is not, because it contains second differential terms. See the following proof.

Theorem 5.4 (Lie Bracket Components).

$$[X, Y]^\mu = (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu).$$

Proof. Given $X = X^\mu \partial_\mu, Y = Y^\nu \partial_\nu$, we try to write the component of $X \circ Y$.

$$X \circ Y(f) = X^\mu \partial_\mu (Y^\nu \partial_\nu f).$$

However, notice that

$$\begin{aligned} Y^\nu &:= Yx^\nu \in C^\infty(\mathcal{M}); \\ \partial_\nu : C^\infty(\mathcal{M}) &\rightarrow C^\infty(\mathcal{M}), \\ \implies \partial_\nu f &\in C^\infty(\mathcal{M}). \end{aligned}$$

So we need to use the Leibniz property of ∂_μ ?? in order to evaluate the second term. Doing this for $X \circ Y(f)$ and $Y \circ X(f)$, we have

$$\begin{aligned} X \circ Y(f) &= X^\mu ((\partial_\mu Y^\nu)(\partial_\nu f) + Y^\nu \partial_\mu \partial_\nu f). \\ Y \circ X(f) &= Y^\nu ((\partial_\nu X^\mu)(\partial_\mu f) + X^\mu \partial_\nu \partial_\mu f). \end{aligned}$$

So if $\partial_\mu \partial_\nu f = \partial_\nu \partial_\mu f$, then by subtracting, we can cancel the second order terms, and we are done. We prove so now.

$$\begin{aligned} (\partial_\mu \partial_\nu f)(p) &= \frac{\partial}{\partial u^\mu} ((\partial_\nu f) \circ \phi^{-1})|_{\phi(p)} \\ &= \frac{\partial}{\partial u^\mu} \left((\partial_\nu)_{\phi^{-1}(u)} f \right)|_{\phi(p)} \\ &= \frac{\partial}{\partial u^\mu} \left(\frac{\partial}{\partial u^\nu} (f \circ \phi^{-1})|_u \right)|_{\phi(p)} \\ &= \frac{\partial}{\partial u^\nu} \left(\frac{\partial}{\partial u^\mu} (f \circ \phi^{-1})|_u \right)|_{\phi(p)} \\ &= (\partial_\nu \partial_\mu f)(p). \end{aligned}$$

◻

Theorem 5.5 (Properties of Lie Brackets).

1. $[X, Y] = -[Y, X]$ (antisymmetry)

2. $\sum_{\text{cyc}} [X, [Y, Z]] = 0$. (Jacobi Identity)

5.3 Integral Curves and Flows

Definition 5.5 (Integral Curve).

Let X be a vector field on \mathcal{M} , $p \in \mathcal{M}$. Then an integral curve of X through p is a curve $\sigma : (-\epsilon, \epsilon) \rightarrow \mathcal{M}$ s.t.

$$\begin{aligned}\sigma(0) &= p, \\ \sigma_* \left(\frac{d}{dt} \right)_t &= X_{\sigma(t)}.\end{aligned}$$

↗ **Remark.**

Qualitatively, using [Theorem 2.12](#), this pushforward is just $[\sigma] \in T_{\sigma(t)}\mathcal{M}$. Therefore, the second condition is saying in some sense that the curve is tangent to the vector field on the manifold. For quantitative description, see below.

ℳ

Definition 5.6 (Differential Equations of Integral Curve).

The components X^μ of X determine the integral curve σ by the following ODE with boundary conditions,

$$\begin{aligned}X^\mu(\sigma(t)) &= \frac{d}{dt}x^\mu(\sigma(t)) \\ x^\mu(\sigma(0)) &= x^\mu(p), \mu = 1, 2, \dots, m.\end{aligned}$$

5.3.1 One-parameter Family of Diffeomorphisms

Definition 5.7 (Local 1D Family of Local Diffeomorphisms).

A local, 1D family of local diffeomorphisms at $p \in \mathcal{M}$ is made up of (1) an open neighborhood U of p , (2) $\epsilon > 0$ (3) a family of diffeomorphisms $\{\phi_t \mid |t| < \epsilon\}$, $\phi_t : U \rightarrow \mathcal{M}$ s.t.

1. Every ϕ_t is a smooth function in t and q .
 2. $\forall t, s \in \mathbb{R}$ and $|t|, |s|, |t+s| < \epsilon$, and $\forall q \in U$ s.t. $\phi_t(q), \phi_s(q), \phi_{t+s}(q) \in U$, we have
- $$\phi_s(\phi_t(q)) = \phi_{s+t}(q).$$
3. $\phi_0(q) = q$.

☞ **Remark.**

The first "local" refers to the parameter t , which is limited to $(-\epsilon, \epsilon)$. The second "local" refers to the spatial limitation to U . You can view $\phi_t(q)$ as a curve that brings $t \in (-\epsilon, \epsilon)$ to $\phi_t(q) \in \mathcal{M}$.

ℳ

Definition 5.8 (Induced Vector Field).

By taking tangents to the curve family [Definition 5.7](#), we have the induced vector field X^ϕ given by

$$X_q^\phi(f) := \frac{d}{dt}(f(\phi_t(q))) \Big|_{t=0}$$

Theorem 5.6.

The curve family $t \mapsto \phi_t(q)$ is the integral curve of the induced vector field [Definition 5.8](#) X_q^ϕ .

Proof.

$$\begin{aligned} X_{\phi_s(q)}^\phi &= \frac{d}{dt}(f \circ \phi_t \circ \phi_s(q)) \Big|_{t=0} \\ &= \frac{d}{dt}(f \circ \phi_{t+s}(q)) \Big|_{t=0}. \end{aligned}$$

Let $u = t + s$. Then

$$\begin{aligned} X_{\phi_s(q)}^\phi &= \frac{d}{du}(f \circ \phi_u(q)) \Big|_{u=s} \\ &= \phi_{q*} \left(\frac{d}{dt} \right)_s f. \end{aligned}$$

■

5.3.2 Local Flows

Definition 5.9 (Local Flow).

Let X be a vector field on open $U \subseteq \mathcal{M}$, and $p \in U$. A local flow at p is a local one-parameter family of local diffeomorphisms [Definition 5.7](#) defined on some open $V \subseteq U$ s.t. $p \in V$ and the induced vector field [Definition 5.8](#) is X .

↗ **Remark.**

Local flows always exist and are unique. In contrast, global flows (which means $t \in \mathbb{R}$ instead of a restricted interval) may not exist.

ℳ

5.3.3 Lie Derivative

Theorem 5.7 (Interpretation of Lie Bracket).

If X, Y are two vector fields on \mathcal{M} , and define the following quantity, which can be interpreted as the change of Y when following the integral curves of X , as

$$\frac{d}{dt}(\phi_{-t*}^X(Y)) \Big|_{t=0} := \lim_{\epsilon \rightarrow 0} \frac{\phi_{-\epsilon*}^X(Y_{\phi_\epsilon^X(p)}) - Y_p}{\epsilon}.$$

Then,

$$\frac{d}{dt}(\phi_{-t*}^X(Y)) \Big|_{t=0} = [X, Y].$$

6 Cotangent Spaces

6.1 Cotangent Vectors

Definition 6.1 (Cotangent Spaces).

The cotangent space $T_p^*\mathcal{M}$ at $p \in \mathcal{M}$ is the set of all linear functions $f : T_p\mathcal{M} \rightarrow \mathbb{R}$.

Its member is called a cotangent vector.

$$\dim T_p^*\mathcal{M} = \dim T_p\mathcal{M}.$$

Definition 6.2 (One-Form).

A one-form on \mathcal{M} is a smooth assignment of cotangent vectors $\omega : p \mapsto \omega_p$.

It may be understood as a covector field.

Definition 6.3 (Basis Cotangent Vectors).

The basis cotangent vectors is chosen to be the dual basis of the basis tangent vectors ??,

$$(dx^\mu)_p((\partial_\nu)_p) = \delta^\mu_\nu.$$

Theorem 6.1 (Coordinate Expression of Cotangent Vectors).

Any $f \in T_p^*\mathcal{M}$ can be expanded as

$$f = f_\mu (dx^\mu)_p.$$

Any one-form ω can be expressed as

$$\omega = \omega_\mu dx^\mu.$$

6.2 Pullback

6.2.1 Definition

Definition 6.4 (Pullback).

Given a function and its pushforward, we define pullback to be the dual of pushforward, i.e.,

$$\begin{array}{ccc} h : & \mathcal{M} & \rightarrow & \mathcal{N}, \\ h_* : & T_p \mathcal{M} & \rightarrow & T_{h(p)} \mathcal{N}, \\ h^* : & T_{h(p)}^* \mathcal{N} & \rightarrow & T_p^* \mathcal{M}, \end{array}$$

s.t. given $f \in T_{h(p)}^* \mathcal{N}$ and $v \in T_p \mathcal{M}$,

$$(h^* f)(v) := f(h_* v).$$

↗ **Remark.**

Note especially on the direction of original function and its induced pullback. This is crucial to the covariancy of one-forms. ℳ

Theorem 6.2.

Given ω a one-form on \mathcal{N} , and a function $h : \mathcal{M} \rightarrow \mathcal{N}$, the pullback $h^* \omega$ is defined as

$$(h^* \omega)(v)_p = \omega(h_* v)_{h(p)}.$$

Theorem 6.3 (Associativity of Pullbacks).

Analogous to [Theorem 2.10](#), given manifolds $\mathcal{M}, \mathcal{N}, \mathcal{P}$ and $h : \mathcal{M} \rightarrow \mathcal{N}$, $k : \mathcal{N} \rightarrow \mathcal{P}$, then

$$(k \circ h)^* = k^* \circ h^*.$$

6.2.2 Jacobian

Theorem 6.4 (Local Representative of Pullback).

Let $\dim \mathcal{M} = m, \dim \mathcal{N} = n, h : \mathcal{M} \rightarrow \mathcal{N}, \{x^1, \dots, x^m\}$ be the local coordinates of \mathcal{M} around p , and $\{y^1, \dots, y^n\}$ be the local coordinates of \mathcal{N} around $h(p)$.

Then

$$h^*\omega = \sum_{\mu=1}^m \sum_{\nu=1}^n \omega_\nu \left. \frac{\partial h^\nu}{\partial x^\mu} \right|_p (dx^\mu)_p,$$

where $J^\nu_\mu := \left. \frac{\partial h^\nu}{\partial x^\mu} \right|_p := (\partial_\mu)_p (y^\nu \circ h)$ is the Jacobian matrix.

Proof. We know by [Definition 6.4](#),

$$(h^*\omega)_\mu(p) = h^*\omega(\partial_\mu) = \omega(h_*\partial_\mu).$$

Expand it in local coordinates of \mathcal{N} ,

$$(h^*\omega)_\mu(p) = \omega_\nu dy^\nu(h_*\partial_\mu).$$

Via similar procedure in [Theorem 2.11](#), we arrive at

$$(h^*\omega)_\mu(p) = \omega_\nu \frac{\partial h^\nu}{\partial x^\mu}.$$



6.3 Transformation Properties

Theorem 6.5 (Covariancy and Contravariancy).

Given two coordinate charts (U, ϕ) and (U', ϕ') s.t. $U \cap U' = S \neq \emptyset$, then on S ,

$$X^{\nu'} = \sum_{\mu=1}^m \frac{\partial x'^{\nu}}{\partial x^{\mu}} X^{\mu},$$

$$\omega_{\nu'} = \sum_{\mu=1}^m \omega_{\mu} \frac{\partial x^{\mu}}{\partial x'^{\nu}}.$$

If Jacobian matrix is given,

$$J^{\nu'}_{\mu} := \left. \frac{\partial x'^{\nu}}{\partial x^{\mu}} \right|_p := (\partial_{\mu})_p x'^{\nu},$$

$$(J^{-1})^{\mu}_{\nu'} := \left. \frac{\partial x^{\mu}}{\partial x'^{\nu}} \right|_p := (\partial_{\nu'})_p x^{\mu},$$

then,

$$X^{\nu'} = J^{\nu'}_{\mu} X^{\mu}, \quad (\text{contravariant})$$

$$\omega_{\nu'} = \omega_{\mu} (J^{-1})^{\mu}_{\nu'}. \quad (\text{covariant})$$

Proof. The contravariant part is proved in ???. Now we turn to the covariant part.

Let $h = \text{id} : (U, \phi) \subseteq \mathcal{M} \rightarrow (U', \phi') \subseteq \mathcal{M}$, consider its pullback.

$$(\text{id}^* \omega)_p = \omega_{\nu'} \frac{\partial x'^{\nu}}{\partial x^{\mu}} dx^{\mu}.$$

Then,

$$\omega_{\mu} = \omega_{\nu'} \frac{\partial x'^{\nu}}{\partial x^{\mu}}.$$

Inverting the matrix equation above, we get the desired result. ◻

7 Tensors

Definition 7.1 (Tensors).

If $\dim \mathcal{M} \neq \infty$, the tensors of type (r, s) $T_p^{r,s}\mathcal{M}$ are all the linear functions

$$f : \bigtimes^r T_p^*\mathcal{M} \times \bigtimes^s T_p\mathcal{M} \rightarrow \mathbb{R}.$$

I.e., it eats r covectors and s vectors.

Theorem 7.1 (Dimensions of General Tensor Space).

The dimension of $T_p^{r,s}\mathcal{M}$ is $m^r m^s$. In particular, a basis for the space is,

$$\bigotimes_{1 \leq \mu_1 \cdots \mu_r \leq m} (\partial_{\mu_i})_p \otimes \bigotimes_{1 \leq \nu_1 \cdots \nu_s \leq m} (dx^{\nu_i})_p$$

↗ **Remark.**

For a detailed proof, see Hoffman.



8 n-Forms

8.1 Definition

Definition 8.1 (n-Forms).

An n-form is a tensor field of type $(0, n)$ that is totally skew-symmetric (or alternating, or totally antisymmetric), i.e.,

$$\omega(X_1, X_2, \dots, X_n) = (\text{sgn } \sigma)\omega(X_{\sigma(1)}, \dots, X_{\sigma(n)}), \quad \forall \sigma \in S_n.$$

The set of all n-forms on \mathcal{M} is denoted as $\Lambda^n(\mathcal{M})$.

The set of all forms is $\Lambda(\mathcal{M}) = \bigoplus_{n=0}^{\dim \mathcal{M}} \Lambda^n(\mathcal{M})$.

Conventionally, we classify functions as 0-forms.

8.2 The Exterior Product

Definition 8.2 (Exterior Product).

Given $\omega_1 \in \Lambda^{n_1}(\mathcal{M}), \omega_2 \in \Lambda^{n_2}(\mathcal{M})$, their exterior product is a $(n_1 + n_2)$ -form given by,

$$\omega_1 \wedge \omega_2 := \frac{1}{n_1!n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\text{sgn } \sigma)(\omega_1 \otimes \omega_2)_{\sigma}.$$

Written explicitly,

$$(\omega_1 \wedge \omega_2)(X_1, \dots, X_{n_1+n_2}) := \frac{1}{n_1!n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\text{sgn } \sigma)(\omega_1 \otimes \omega_2)(X_{\sigma(1)}, \dots, X_{\sigma(n_1+n_2)})$$

↗ **Remark.**

I'll take the alternating property and associativity of the exterior product for granted. For a detailed proof, see Hoffman. ℳ

Theorem 8.1 (Commutativity with Pullback).

Given $h : \mathcal{M} \rightarrow \mathcal{N}$ and $\alpha, \beta \in \Lambda(\mathcal{N})$, then

$$h^*(\alpha \wedge \beta) = (h^*\alpha) \wedge (h^*\beta).$$

↗ **Remark.**

For a "generalized" pullback, we have,

$$(h^*(\alpha))(X_1, \dots, X_{n_1}) = \alpha(h_*X_1, \dots, h_*X_{n_1}).$$

ℳ

Proof.

$$\begin{aligned} & (h^*\alpha) \wedge (h^*\beta) \\ &= \frac{1}{n_1!n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\text{sgn } \sigma) \alpha \otimes \beta(h_*X_{\sigma(1)}, \dots, h_*X_{\sigma(n_1+n_2)}). \\ &= \frac{1}{n_1!n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\text{sgn } \sigma) h^* \left(\alpha \otimes \beta(X_{\sigma(1)}, \dots, X_{\sigma(n_1+n_2)}) \right). \\ &= h^* \left(\frac{1}{n_1!n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\text{sgn } \sigma) \alpha \otimes \beta(X_{\sigma(1)}, \dots, X_{\sigma(n_1+n_2)}) \right). \\ &= h^*(\alpha \wedge \beta). \end{aligned}$$

◻

Theorem 8.2 (Skew-Symmetry).

The exterior product makes $\Lambda(\mathcal{M})$ a graded algebra with skew-symmetry given by

$$\omega_1 \wedge \omega_2 = (-1)^{n_1 n_2} \omega_2 \wedge \omega_1.$$

Proof. In the definition of exterior product, first fix $\sigma = \sigma_0$ to consider only one term.

When we switch ω_1 and ω_2 , we are essentially doing

$$\begin{aligned} & (\omega_2 \otimes \omega_1)(X_{\sigma_0(1)}, \dots, X_{\sigma_0(n_2)}, \underbrace{X_{\sigma_0(n_2+1)}, \dots, X_{\sigma_0(n_1+n_2)}}) \\ &= (\omega_1 \otimes \omega_2)(\underbrace{X_{\sigma_0(n_2+1)}, \dots, X_{\sigma_0(n_1+n_2)}}, X_{\sigma_0(1)}, \dots, X_{\sigma_0(n_2)}). \end{aligned}$$

Now,

$$\underbrace{1, 2, \dots, n_2}_{\downarrow \text{ } n_2 \text{ times}}, \underbrace{n_2 + 1, \dots, n_1 + n_2}_{\downarrow \text{ } (n_1 - 1)n_2 \text{ times}}$$

$$\underbrace{n_2 + 1, 1, 2, \dots, n_2}_{\downarrow \text{ } (n_1 - 1)n_2 \text{ times}}, \dots, n_1 + n_2$$

$$\underbrace{n_2 + 1, \dots, n_1 + n_2, 1, 2, \dots, n_2}_{\downarrow \text{ } (n_1 - 1)n_2 \text{ times}}$$

So $n_1 n_2$ transposes can achieve the desired effect. Therefore, every term in the summation is multiplied by $(-1)^{n_1 n_2}$, and we get the desired result. \blacksquare

Theorem 8.3 (Dimension of n-Forms).

Let $\dim \mathcal{M} = m$. If $1 \leq n \leq m$, then $\Lambda^n(\mathcal{M}) = \binom{m}{n}$. If $n > m$, then $\Lambda^n(\mathcal{M}) = 0$.

Moreover, a basis for $\Lambda^n(\mathcal{M})_p$ is given by,

$$(dx^{\mu_1})_p \wedge (dx^{\mu_2})_p \wedge \cdots \wedge (dx^{\mu_n})_p, \quad 1 \leq \mu_1 \leq \cdots \leq \mu_n \leq m.$$

Remark.

The proof is quite a pleasure to read (and to think of). Please see Hoffman. \mathfrak{M}

8.3 The Exterior Derivative

Definition 8.3 (Exterior Derivative).

Let ω be an n -form on \mathcal{M} , $1 \leq n < \dim \mathcal{M}$. Then the exterior derivative $d\omega$ is a $(n+1)$ -form. Let $d\omega(\mathbf{X}) = d\omega(X_1, \dots, X_{n+1})$, then

$$\begin{aligned} d\omega(\mathbf{X}) := & \sum_{i=1}^{n+1} (-1)^{i+1} \mathcal{L}_{X_i} (\omega(\mathbf{X} \setminus \{X_i\})) \\ & + \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], \mathbf{X} \setminus \{X_i, X_j\}). \end{aligned}$$

If $\omega \in \Lambda^{\dim \mathcal{M}}(\mathcal{M})$, we define $d\omega = 0$.

Theorem 8.4.

In particular for a 0-form $f \in C^\infty(\mathcal{M})$,

$$df(X) := \mathcal{L}_X f.$$

In coordinates,

$$df = (\partial_\mu f)(dx^\mu).$$

Theorem 8.5.

In particular for a 1-form ω ,

$$d\omega(X, Y) = \mathcal{L}_X(\omega(Y)) - \mathcal{L}_Y(\omega(X)) - \omega([X, Y]).$$

Theorem 8.6 (Coordinate Expansion for Exterior Derivative).

In local coordinates, if $\omega = \omega_{\mu_1 \mu_2 \dots \mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n}$, then

$$d\omega = \partial_\nu \omega_{\mu_1 \mu_2 \dots \mu_n} dx^\nu \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n}$$

Theorem 8.7 (Exterior Derivative and Product).

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d\omega_2.$$

Theorem 8.8 (Exterior Derivative and Pullback).

Given $h : \mathcal{M} \rightarrow \mathcal{N}$, ω an n-form on \mathcal{N} , then

$$d(h^*\omega) = h^*(d\omega).$$

Theorem 8.9 (Functional Linearity of Exterior Derivative).

Let ω be a 1-form on \mathcal{M} . Then $d\omega$ satisfies,

$$d\omega(fX, Y) = f d\omega(X, Y), \quad \forall f \in C^\infty(\mathcal{M}),$$

where fX is a vector field that gives $(fX)(p) = f(p)X_p$.

Proof. By [Definition 8.3](#),

$$d\omega(fX, Y) = \mathcal{L}_{fX}(\omega(Y)) - \mathcal{L}_Y(\omega(fX)) - \omega([fX, Y]).$$

We break it down term by term. Firstly,

$$(\mathcal{L}_{fX}(\omega(Y)))(p) = f(p)X_p(\omega(Y)) = f(p)(\mathcal{L}_X(\omega(Y)))(p).$$

So

$$\mathcal{L}_{fX}(\omega(Y)) = f \cdot \mathcal{L}_X(\omega(Y)).$$

Secondly, we tackle $\mathcal{L}_Y(\omega(fX))$. In particular,

$$\omega(fX)(p) = \omega_p(f(p)X_p) = f(p)\omega_p(X_p) = f(p)(\omega(X))(p).$$

Therefore,

$$\mathcal{L}_Y(\omega(fX)) = \mathcal{L}_Y(f \cdot \omega(X)) = (\mathcal{L}_Y f)\omega(X) + f \cdot \mathcal{L}_Y(\omega(X)).$$

Thirdly,

$$\omega([fX, Y]) = \omega((fX) \circ Y - Y \circ (fX)).$$

In particular,

$$((Y \circ (fX))(g))(p) = Y_p((fX)(g)) = Y_p(f \cdot Xg) = (Y_p f)((Xg)(p)) + f(p) \cdot Y_p(Xg).$$

So,

$$Y \circ (fX) = (\mathcal{L}_Y f)X + f \cdot Y \circ X.$$

Substituting back,

$$\begin{aligned}\omega([fX, Y]) &= \omega(f \cdot X \circ Y - (\mathcal{L}_Y f)X - f \cdot Y \circ X) \\ &= \omega(f[X, Y] - (\mathcal{L}_Y f)X) \\ &= f\omega([X, Y]) - (\mathcal{L}_Y f)\omega(X).\end{aligned}$$

Finally,

$$\begin{aligned}d\omega(fX, Y) &= \mathcal{L}_{fX}(\omega(Y)) - \mathcal{L}_Y(\omega(fX)) - \omega([fX, Y]) \\ &= f \cdot \mathcal{L}_X(\omega(Y)) - (\mathcal{L}_Y f)\omega(X) - f \cdot \mathcal{L}_Y(\omega(X)) - f\omega([X, Y]) + (\mathcal{L}_Y f)\omega(X) \\ &= f(\mathcal{L}_X(\omega(Y)) - \mathcal{L}_Y(\omega(X)) - \omega([X, Y])) \\ &= fd\omega(X, Y).\end{aligned}$$

◻

Corollary 8.9.1 (Local Nature of Exterior Derivative).

When ω is fixed, the value of $d\omega$ depends only on the local values of vector fields.

$$d\omega(X, Y)(p) = X^\mu(p)Y^\nu(p)d\omega(\partial_\mu, \partial_\nu)(p).$$

◻

Proof. Write $X = X^\mu \partial_\mu$, noting that $X^\mu \in C^\infty(\mathcal{M})$, and use [Theorem 8.9](#).

8.4 DeRham Cohomology

Theorem 8.10 (Twice Exterior Differential).

For all $\omega \in \Lambda^n(\mathcal{M})$, $1 \leq n \leq \dim M$, we have

$$d^2\omega = 0.$$

↗ **Remark.**

This means

$$\text{Im}(d : \Lambda^{n-1}(\mathcal{M}) \rightarrow \Lambda^n(\mathcal{M})) \subseteq \text{Ker}(d : \Lambda^n(\mathcal{M}) \rightarrow \Lambda^{n+1}(\mathcal{M})).$$

This type of structure is called a differential complex, and is common in many structures. ℳ

Definition 8.4 (Closed Form).

An n -form ω is closed if $d\omega = 0$. The set of all closed n -forms is denoted $Z^n(\mathcal{M})$.

Definition 8.5 (Exact Form).

An n -form ω is exact if $\omega = d\beta$ for some $(n-1)$ -form β . The set of all exact n -forms is denoted $B^n(\mathcal{M})$.

↗ **Remark.**

It is guaranteed that $B^n(\mathcal{M}) \subseteq Z^n(\mathcal{M})$, that is, exactness implies closure. But how much closed form is not exact is the study of cohomology theory. ℳ

Theorem 8.11 (Poincare's Lemma).

On Euclidean space \mathbb{R}^m ,

$$B^n(\mathcal{M}) = Z^n(\mathcal{M}), \quad \forall n > 0.$$

Definition 8.6 (DeRham Cohomology Groups).

The DeRham cohomology groups $H^n(\mathcal{M})$, $0 \leq n \leq \dim \mathcal{M}$ are the quotient

spaces

$$H^n(\mathcal{M}) := Z^n(\mathcal{M})/B^n(\mathcal{M}).$$

↗ **Remark.**

Recall the definition of quotient groups that $H^n(\mathcal{M})$ consists of elements of form $z + B^n(\mathcal{M}), z \in Z^n(\mathcal{M})$.

If all closed forms are exact, $Z^n(\mathcal{M}) \subseteq B^n(\mathcal{M})$, then $H^n(\mathcal{M}) \cong \{0\}$. ℳ

Theorem 8.12 (Criterion of Exact ODE).

On the Euclidean space \mathbb{R}^2 , given a 1-form $\omega = \omega_1 dx^1 + \omega_2 dx^2$. Then

$$\omega \in B^1(\mathbb{R}^2) \iff \partial_2 \omega_1 = \partial_1 \omega_2.$$

↗ **Remark.**

This is an important theorem to me, for it connects the "exactness of differential forms" to the familiar notion of "exactness of differential equations".

It also provides the first hints that we are actually integrating forms, and that exterior differentiation of a 0-form resembles gradient in usual vector calculus terms. ℳ

Proof. Via Poincare lemma [Theorem 8.11](#), on \mathbb{R}^2 , exactness is equivalent to closure. So we need only to determine the condition that $d\omega = 0$. Using [Theorem 8.6](#),

$$\begin{aligned} d\omega &= \partial_\nu \omega_{\mu_1} dx^\nu \wedge dx^{\mu_1} \\ &= \partial_2 \omega_1 dx^2 \wedge dx^1 + \partial_1 \omega_2 dx^1 \wedge dx^2 \\ &= (\partial_2 \omega_1 - \partial_1 \omega_2) dx^2 \wedge dx^1. \end{aligned}$$

