1 Vector Fields and the Tangent Bundle

1.1 Vector Fields

Definition 1.1 (Vector Fields)**.**
A vector field X on $V \subseteq \mathcal{M}$ is any rule of assigning a tangent vector $X(p) = X_p \in T_p\mathcal{M}$ for all $p \in V$.

Definition 1.2 (Components of a Vector Field)**.**
Let X be a vector field on $V \subseteq \mathcal{M}$, $\dim \mathcal{M} = m$. For any chart $\phi : U \to \phi(U) \in \Phi$ with induced coordinates $x^1, \dots, x^m$ and any $p \in V \cap U$, the decomposition $X(p) = X^j(p) \left(\partial_j\right)_p$ is unique, and therefore we write

$$X = X^j \partial_j$$

on $V \cap U$, and $X^j : V \cap U \to \mathbb{R}$ are called the components of X on $V \cap U$.

Definition 1.3 (Smoothness of Vector Field)**.**
A vector field X on $V \subseteq \mathcal{M}$ is C^k near $p \in V$ iff $\exists\ \phi \in \Phi$ s.t. $p \in U_\phi$ and all the components X^j induced by ϕ are C^k. That is,

$$X^j \circ \phi^{-1} \text{ are } C^k \text{ on } \phi(V \cap U).$$

1.2 Tangent Bundle

1.2.1 Definition

Definition 1.4 (Tangent Bundle)**.**
The tangent bundle $T\mathcal{M}$ is

$$T\mathcal{M} := \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}.$$

Remark.
Why not $T\mathcal{M} := \left\{ T_p\mathcal{M} \;\middle|\; p \in \mathcal{M} \right\}$? ♍

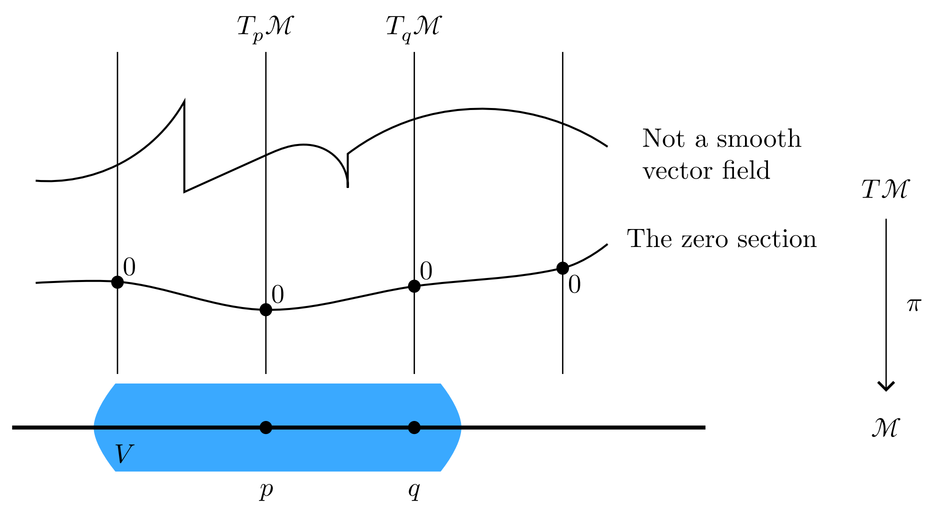
1.2.2 Projection

Definition 1.5 (Canonical Projection)**.**
The canonical projection is the map

$$\begin{aligned}\pi : T\mathcal{M} &\to \mathcal{M}\\ T_p\mathcal{M} &\mapsto p.\end{aligned}$$

Definition 1.6 (Alternative Definition of Vector Fields)**.**
A tangent vector field X on $V \subseteq \mathcal{M}$ is a map $X : V \to T\mathcal{M}$ s.t. $(\pi \circ X)(p) = p$ for all $p \in V$.

Remark.
Let's see why vector fields are often called a "cross-section" of a tangent bundle.



♍

1.2.3 Topological and Manifold Structure

Definition 1.7 (Smooth Structure on Tangent Bundle)**.**
Let a manifold $\mathcal{M}$ with dimension m and atlas Φ. Consider a chart $\phi_j \in \Phi : U_j \to \mathbb{R}^m$. We define a chart $\tilde{\phi}_j$ for the tangent bundle $T\mathcal{M}$ accordingly,

$$\begin{aligned} \tilde{\phi}_j : \tilde{\pi}(U_j) \subseteq T\mathcal{M} &\to \mathbb{R}^{2m} \\ (p, v^i \partial_i) &\mapsto (\phi_j(p), v^1, \dots, v^m), \end{aligned}$$

where $\tilde{\pi}$ is the inverse set map of the canonical projection function Definition 1.5. We thus see the tangent bundle $T\mathcal{M}$ is itself a manifold of dimension $2m$.
Also, we define the topology of the tangent bundle by

$$A \subseteq T\mathcal{M} \text{ is open iff } \tilde{\phi}_j(A \cap \tilde{\pi}(U_j)) \text{ is open.}$$

Theorem 1.1.
The smooth structure given by Definition 1.7 is unique in the sense of,

1. $\pi : T\mathcal{M} \to \mathcal{M}$ is C^∞,
2. For all open sets $V \subseteq \mathcal{M}$ and any vector field X on V, X is $C^\infty \iff X : V \to T\mathcal{M}$ is C^∞.

Remark.
At first thought, one may think of $T\mathcal{M} \simeq \mathcal{M} \times \mathbb{R}^m$. This is not the case, as one can consider the Moebius strip. The tangent bundle is only locally isomorphic to $U \times \mathbb{R}^m$.
If indeed $T\mathcal{M} \simeq \mathcal{M} \times \mathbb{R}^m$, then the bundle is called trivial tangent bundle, and the manifold is called parallelizable. ♍

1.3 Integral Curves and Local Flows

1.3.1 Integral Curves

Definition 1.8 (Integral Curves)**.**
Let $\mathcal{M}$ be a manifold, and X be a vector field on $V \subseteq \mathcal{M}$. If one single curve $\sigma : (-\epsilon, \epsilon) \to V$, $\epsilon > 0$ satisfies

$$\begin{aligned} \sigma(0) &= p \in V \\ X_{\sigma(t)} &= [\sigma] \ \forall t \in (-\epsilon, \epsilon) \end{aligned}$$

then σ is called an integral curve of the vector field X through the point p.

Theorem 1.2 (Differential Equations of Integral Curve)**.**
The components X^μ of X determine the integral curve σ by the following ODE with boundary conditions,

$$\begin{aligned} X^\mu(\sigma(t)) &= \frac{d}{dt} x^\mu(\sigma(t)) \\ x^\mu(\sigma(0)) &= x^\mu(p), \mu = 1, 2, \dots, m. \end{aligned}$$

1.3.2 Local Flows

Definition 1.9 (Local 1D Family of Local Diffeomorphisms)**.**
A local, 1D family of local diffeomorphisms at $p \in \mathcal{M}$ is made up of a family of diffeomorphisms $\{ \sigma_t : U \to \mathcal{M} \mid t \in (-\epsilon, \epsilon) \}$ with $\epsilon > 0$, $U \subseteq \mathcal{M}$ an open set s.t.

1. Every σ_t is a smooth function in t and p.
2. $\forall t, s \in \mathbb{R}$ and $|t|, |s|, |t + s| < \epsilon$, and $\forall p \in U$ s.t. $\sigma_t(p), \sigma_s(p), \sigma_{t+s}(p) \in U$, we have
$$\sigma_s(\sigma_t(p)) = \sigma_{s+t}(p).$$
3. $\sigma_0(p) = p$.

Remark.

The first "local" refers to the parameter t, which is limited to $(-\epsilon, \epsilon)$. The second "local" refers to the spatial limitation to U. You can view $\phi_t(q)$ as a curve that brings $t \in (-\epsilon, \epsilon)$ to $\phi_t(q) \in \mathcal{M}$. ♍

Definition 1.10.

Consider a family of local diffeomorphisms given by Definition 1.9 ϕ_t. We denote the following for convenience, if its meaning is clear from context,

$$\sigma(p,t) := \sigma_t(p),$$
$$\sigma_p(t) := \sigma_t(p).$$

Definition 1.11 (Local Flow)**.**

Let X be a vector field on $V \subseteq \mathcal{M}$, $p \in V$. Consider a family of diffeomorphisms Definition 1.9 σ on V. Let the corresponding curve family $\sigma_p(t)$ satisfy,

$$\begin{aligned} \sigma_p(0) &= p && \forall p \in V, \\ [\sigma_p] = X_{\sigma_p(t)} &= \sigma_{p*}\left(\frac{d}{dt}\right)_t && \forall t \in (-\epsilon, \epsilon), p \in V. \end{aligned}$$

Then we say the curve family $\sigma_p(t)$ is the local flow for X on V. We may abbreviate and just say σ is the local flow for X.

Theorem 1.3.

Local flows always exist and are unique. Thus for a vector field X defined on V, we may denote its local flow of over an open set U by σ^X.

1.4 Lie Derivative of Vector Fields

1.4.1 Lie Bracket

Definition 1.12 (Lie Derivative of Functions)**.**
We can view a vector field as a function $X : C^\infty \to C^\infty$, given by

$$(Xf)(p) := X_p f = \frac{\partial}{\partial t}(f \circ \sigma^X(p,t))\Big|_{t=0}.$$

We can interpret this as: How much the function f changes beginning at p, along the flow lines of X?

Definition 1.13 (Composition of Vector Fields)**.**
We can view $X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, and so does Y. Therefore, we define

$$(X \circ Y)(f) := X(Yf).$$

Theorem 1.4 (Flow Interpretation)**.**

$$X_p(Yf) = \frac{\partial^2}{\partial s \partial t}\left(f\left(\sigma^Y\left(\sigma^X(p,t),s\right)\right)\right)\Big|_{(s,t)=(0,0)}.$$

We can interpret $X(Yf)$ as: How much the function f changes beginning at the point p, first following the flow lines of X, then following the flow lines of Y?

Definition 1.14 (Lie Bracket/Vector Field Commutator)**.**
We define the Lie Bracket of two vector fields X, Y to be

$$[X,Y] := X \circ Y - Y \circ X.$$

Remark.
Lie Bracket Definition 1.14 is a vector field, while the expression $X \circ Y$ is not, because it contains second differential terms. See the following proof. ♍

Theorem 1.5 (Lie Bracket Components)**.**

$$[X, Y]^\mu = (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu).$$

Proof. Given $X = X^\mu \partial_\mu, Y = Y^\nu \partial_\nu$, we try to write the component of $X \circ Y$.

$$X \circ Y(f) = X^\mu \partial_\mu \left(Y^\nu \partial_\nu f\right).$$

However, notice that

$$\begin{aligned}
&Y^\nu := Y x^\nu \in C^\infty(\mathcal{M}); \\
&\partial_\nu : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), \\
&\implies \partial_\nu f \in C^\infty(\mathcal{M}).
\end{aligned}$$

So we need to use the Leibniz property of ∂_μ in order to evaluate the second term. Doing this for $X \circ Y(f)$ and $Y \circ X(f)$, we have

$$\begin{aligned}
X \circ Y(f) &= X^\mu \big((\partial_\mu Y^\nu)(\partial_\nu f) + Y^\nu \partial_\mu \partial_\nu f\big). \\
Y \circ X(f) &= Y^\nu \big((\partial_\nu X^\mu)(\partial_\mu f) + X^\mu \partial_\nu \partial_\mu f\big).
\end{aligned}$$

So if $\partial_\mu \partial_\nu f = \partial_\nu \partial_\mu f$, then by subtracting, we can cancel the second order terms, and we are done. We prove so now.

$$\begin{aligned}
(\partial_\mu \partial_\nu f)(p) &= \frac{\partial}{\partial u^\mu}\left((\partial_\nu f) \circ \phi^{-1}\right)\Big|_{\phi(p)} \\
&= \frac{\partial}{\partial u^\mu}\Big((\partial_\nu)_{\phi^{-1}(u)} f\Big)\Big|_{\phi(p)} \\
&= \frac{\partial}{\partial u^\mu}\left(\frac{\partial}{\partial u^\nu}(f \circ \phi^{-1})\Big|_u\right)\Bigg|_{\phi(p)} \\
&= \frac{\partial}{\partial u^\nu}\left(\frac{\partial}{\partial u^\mu}(f \circ \phi^{-1})\Big|_u\right)\Bigg|_{\phi(p)} \\
&= (\partial_\nu \partial_\mu f)(p).
\end{aligned}$$

▨

Theorem 1.6 (Properties of Lie Brackets)**.**

$$\begin{cases} [X,Y] = -[Y,X] & \text{(antisymmetry)} \\ \sum_{\text{cyc}}[X,[Y,Z]] = 0. & \text{(Jacobi Identity)} \end{cases}$$

1.4.2 Lie Derivative of Vector Fields

Definition 1.15 (Lie Derivative)**.**
Let X, Y be two vector fields defined on $U, V \subseteq \mathcal{M}$, respectively. Then the lie derivative of Y along X is defined as

$$(\mathcal{L}_X Y)_p := \lim_{t\to 0} \frac{\sigma^X_{-t*}(Y_{\sigma^X_t(p)}) - Y_p}{t},$$

where the latter is often abbreviated as

$$\frac{d}{dt}(\sigma^X_{-t*}(Y_{\sigma^X_t(p)}))\bigg|_{t=0} = \lim_{t\to 0} \frac{\sigma^X_{-t*}(Y_{\sigma^X_t(p)}) - Y_p}{t}.$$

Theorem 1.7 (Lie Bracket and Lie Derivative)**.**

$$\mathcal{L}_X Y = [X,Y].$$

That is,

$$\frac{d}{dt}(\sigma^X_{-t*}(Y_{\sigma^X_t(p)}))\bigg|_{t=0} = X \circ Y - Y \circ X.$$

Proof. We start with

$$\begin{aligned} \frac{d}{dt}(\sigma^X_{-t*}(Y_{\sigma^X_t(p)}))\bigg|_{t=0} f &= \lim_{t\to 0} \frac{\sigma^X_{-t*}(Y_{\sigma^X_t(p)f}) - Y_p f}{t} \\ &= \lim_{t\to 0} \frac{Y_{\sigma^X_t(p)}(f \circ \sigma^X_{-t}) - Y_{\sigma^X_0(p)}(f \circ \sigma^X_0)}{t}. \end{aligned}$$

It motivates us to consider the function $H(r,s) = Y_{\sigma^X_r(p)}(f \circ \sigma^X_{-s})$. Then we see

$$\begin{aligned} \frac{d}{dt}(\sigma^X_{-t*}(Y_{\sigma^X_t(p)}))\bigg|_{t=0} f = \frac{d}{dt} H(t,t)\bigg|_{t=0} \\ = \frac{\partial H}{\partial r}(0,0) + \frac{\partial H}{\partial s}(0,0). \end{aligned}$$

(r part: moving along X)

$$\begin{aligned}
\frac{\partial H}{\partial r}(0,0) &= \frac{\partial}{\partial r} Y_{\sigma_r^X(p)}(f \circ \underbrace{\sigma_0^X}_{\mathrm{id}})\Big|_{r=0} \\
&= \frac{\partial}{\partial r}(Yf)(\sigma_r^X(p))\Big|_{r=0} \\
&= X_p(Yf).
\end{aligned}$$

(s part: moving along Y)

$$\begin{aligned}
\frac{\partial H}{\partial s}(0,0) &= \frac{\partial}{\partial s} Y_p(f \circ \sigma_{-s}^X)\Big|_{s=0} \\
&= Y_p \, \frac{\partial}{\partial s}(f \circ \sigma_{-s}^X)\Big|_{s=0} \\
&= -Y_p(Xf),
\end{aligned}$$

And therefore completes the proof. ▨