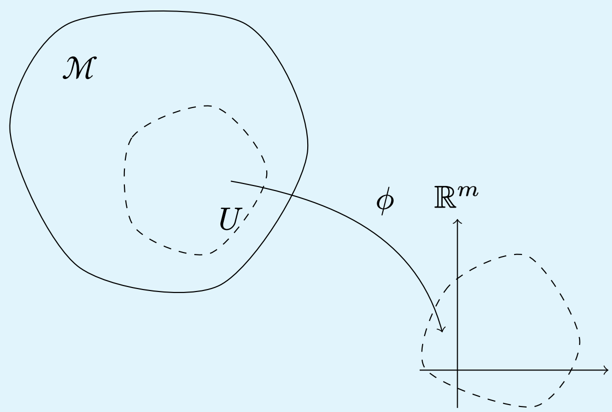
1 Differentiable Manifolds

1.1 Definition

1.1.1 Coordinate Charts

Definition 1.1 (Coordinate Charts).
An m-dimensional, $m \neq \infty$ coordinate chart on a topological space $\mathcal{M}$ is a pair

$$(U, \phi) \begin{cases} U \subseteq \mathcal{M}, U \text{ open} \\ \phi : U \to \mathbb{R}^m, \phi \text{ homeomorphism} \end{cases}$$

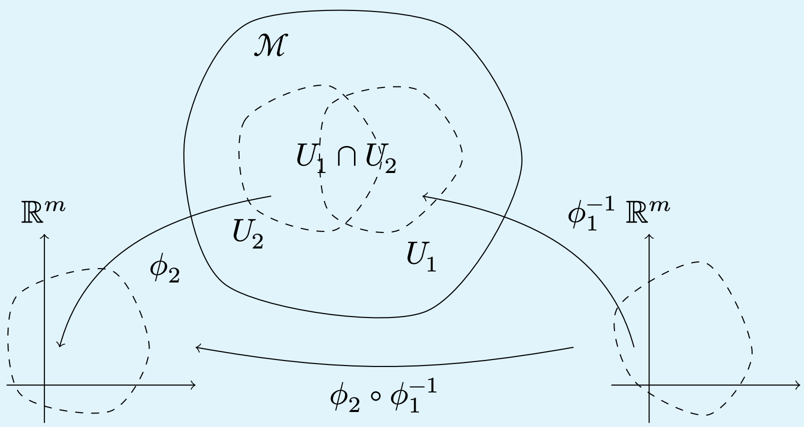


Remark.
If $U = \mathcal{M}$, then we say the coordinate chart ϕ is globally defined; if not, then it is locally defined. Few manifolds have globally defined property.

♍

Definition 1.2 (Overlap Function)**.**
Let $(U_1, \phi_1), (U_2, \phi_2)$ be a pair of m-dimensional coordinate charts with $U_1 \cap U_2 \neq \varnothing$. Then the overlap function is defined as

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \subseteq \mathbb{R}^m \to \phi_2(U_1 \cap U_2) \subseteq \mathbb{R}^m.$$



Definition 1.3 (Atlas)**.**
An m-dimensional atlas on $\mathcal{M}$ is a family of m-dimensional coordinate charts $(U_i, \phi_i), i \in I$ s.t.

1. $\mathcal{M} = \bigcup_{i \in I} U_i$.
2. Each overlap function $\phi_j \circ \phi_i^{-1}, i, j \in I$ is C^∞.

Definition 1.4 (Differentiable Manifolds)**.**
An m-dimensional differentiable manifold is a topological space $\mathcal{M}$ equipped with an atlas.

Remark.
We didn't define a differentiable manifold by regulating the differentiability of the coordinate charts themselves. That's because differentiation is not defined on a manifold, so we need to rely on Euclidean spaces. ♍

1.2 Dimension of a Manifold

Remark.
Consider a manifold that consists of a rod attached to a disk. The dimension is not same everywhere. We give a criterion on how to describe such a scenario. ♍

Theorem 1.1 (Invariance of Domain)**.**
For all $A, B \subseteq S^n$, if $\exists\ f : A \to B$ homeomorphic and $B \in \tau_{S^n}$, then $A \in \tau_{S^n}$ too.

Remark.
Theorem 1.1 is an early theorem in algebraic topology. ♍

Corollary 1.1.1 (Dimension is Well-defined)**.**
Given $U \in \tau_{\mathbb{R}^n}, U' \in \tau_{\mathbb{R}^{n'}}$, and if $\exists\ f : U \to U'$ homeomorphic, then $n = n'$.

Proof. If $n = n'$, it is trivially true.

If $n < n'$, embed $\mathbb{R}^n$ to $\mathbb{R}^{n'}$ by $f : \vec{x} \mapsto (\vec{x}, \vec{0})$. Via stereographic projection, we can map homeomorphically

$$\begin{aligned} \phi &: U \mapsto V \subseteq S^{n'}, \\ \phi' &: U' \mapsto V' \subseteq S^{n'}. \end{aligned}$$

Since the compositions are also homeomorphic, we see V and V' are homeomorphic. However, V' is not an open subset of $R^{n'}$ because of the 0's, contradicting Theorem 1.1. ※ ▨

Remark.
Since the definition of a differentiable manifold requires every overlap function to be diffeomorphic, if $U \cap U' \neq \varnothing$, their dimensions must be equal via the above corollary. We can bypass this by demanding $U \cap U' = \varnothing$, as in the rod and disk case. ♍

Corollary 1.1.2.
If $g : V \to \mathbb{R}^n$ is a continuous injection and $V \in \tau_{\mathbb{R}^n}$, then $g(V)$ is homeomorphic to V, and $g(V) \in \tau_{\mathbb{R}^n}$.

Proof. On $g(V)$, g is surjective and therefore a homeomorphism. Use stereographic projection and the result is obvious. ▨

1.3 Coordinate Functions

Definition 1.5 (Coordinate Functions)**.**
The coordinate functions are the (Euclidean) components of coordinate.

$$\begin{aligned} \phi &: U \to \mathbb{R}^m & p &\mapsto \phi(p), \\ \phi^\mu &: U \to \mathbb{R} & \text{s.t. } \phi(p) &= \begin{pmatrix} \phi^1(p) \\ \vdots \\ \phi^m(p) \end{pmatrix}. \end{aligned}$$

An alternative notation is

$$x^\mu := \phi^\mu.$$

Remark.
There are (Euclidean) projection functions,

$$u^\mu : \mathbb{R}^m \to \mathbb{R}.$$

But I think mention it will cause a lot of confusion. Just remember in the future when we say $\frac{\partial}{\partial u^\mu}$, we are referring to the Euclidean partial derivative wrt the μ-th component. ♍

1.4 Manifold With Boundary

1.4.1 Generalized Coordinate Charts

Definition 1.6 (Generalized Coordinate Charts).
A generalized coordinate chart allows chart that

$$\phi : U \to \phi(U) \subseteq (-\infty, 0] \times \mathbb{R}^{n-1},$$

U is open, and ϕ is homeomorphic.

Remark.
Essentially, this allows a chart to map to "half planes". In this case, even if a set $\phi(U)$ contains $\{0\} \times \mathbb{R}^{n-1}$ and therefore not open in the Euclidean topology of $\mathbb{R}^n$, it is still considered open in the product topology of $(-\infty, 0] \times \mathbb{R}^{n-1}$. ♍

Definition 1.7 (Manifold With Boundary).
A manifold with boundary is a manifold whose atlas consists of generalized coordinate charts.

Definition 1.8 (Boundary Points of a Manifold).
For all $p \in \mathcal{M}$ is a boundary point of a manifold with boundary $\mathcal{M}$ if $\exists\ \phi_\alpha \in \Phi$ atlas s.t. $\phi_\alpha^1(p) = 0$.

The set of all boundary points of $\mathcal{M}$ is denoted $\partial\mathcal{M}$.

1.4.2 Boundary is Well-defined

Remark.
A natural question regarding Definition 1.8 is that, the definition only asks for existence, but it does not guarantee the existence of

$$\exists\ \phi_\alpha, \phi_\beta \text{ s.t. } \phi_\alpha^1(p) = 0, \phi_\beta^1(p) \neq 0.$$

We resolve this in the following. ♍

Theorem 1.2.
Suppose U, U' are open sets in the product topology $(-\infty, 0] \times \mathbb{R}^{n-1}$, and $\exists\ f : U \to U'$ homeomorphic. Then

$$f(U \cap (\{0\} \times \mathbb{R}^{n-1})) = U' \cap (\{0\} \times \mathbb{R}^{n-1}).$$

Proof. We show instead that

$$f(U \cap ((-\infty, 0) \times \mathbb{R}^{n-1})) = U' \cap ((-\infty, 0) \times \mathbb{R}^{n-1}).$$

Via Corollary 1.1.2, we see $f(U \cap ((-\infty, 0) \times \mathbb{R}^{n-1}))$ must be an open set in the Euclidean topology of $\mathbb{R}^n$. Therefore, it cannot contain $\{0\} \times \mathbb{R}^{n-1}$. ▨

Corollary 1.2.1 (Boundary is Well-defined)**.**
$\forall p \in \mathcal{M}$, if $\exists\ \phi_\alpha^1(p) = 0$, then $\forall \phi \in \Phi$ atlas, $\phi^1(p) = 0$.

Proof. Make use of the fact that $\phi \circ \phi_\alpha^{-1}$ is a homeomorphism. ▨

2 Tangent Spaces

2.1 The Curve Formulation of Tangent Spaces

Remark.
The definition of manifold do not require the entity to be embeded in a higher dimensional space. Therefore, the traditional view of tangency is not valid here.

♍

Remark.
The curve formulation remains valid in the infinite-dimensional case, while the algebraic formulation is not. However, in the finite-dimensional case, they are isomorphic.

♍

2.1.1 Curves and Vectors

Definition 2.1 (Curve)**.**
A curve on $\mathcal{M}$ is a C^∞ map,

$$\sigma : (-\epsilon, \epsilon) \to \mathcal{M}.$$

Definition 2.2 (Curve Tangency)**.**
Two curves σ_1, σ_2 are tangent at $p \in \mathcal{M}$ if

1. $\sigma_1(0) = \sigma_2(0) = p$.
2. $\frac{d}{dt}(x^i \circ \sigma_1(t))\big|_{t=0} = \frac{d}{dt}(x^i \circ \sigma_2(t))\big|_{t=0}\,,\ 1 \leq i \leq m$.

Remark.
Written more compactly,

$$\frac{d}{dt}(\phi \circ \sigma_1)\Big|_{t=0} = \frac{d}{dt}(\phi \circ \sigma_2)\Big|_{t=0}$$

♍

Definition 2.3 (Tangent Vectors)**.**
A tangent vector at $p \in \mathcal{M}$ is an equivalence class of curves where the equivalence relation is that they are tangent. It will be denoted as

$$v = [\sigma].$$

Definition 2.4 (Tangent Space)**.**
The tangent space $T_p\mathcal{M}$ at point p is the set of all tangent vectors at point p.

Definition 2.5 (Tangent Bundle)**.**
The tangent bundle $T\mathcal{M}$ is

$$T\mathcal{M} := \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}.$$

2.1.2 Addition and Scalar Multiplication

Definition 2.6 (Addition and Scalar Multiplication)**.**
Let $v_1 = [\sigma_1], v_2 = [\sigma_2] \in T_p\mathcal{M}$, and $r \in \mathbb{R}$. Then define

$$\begin{aligned} v_1 + v_2 &:= [\phi^{-1} \circ (\phi \circ \sigma_1 + \phi \circ \sigma_2)], \\ rv_1 &:= [\phi^{-1} \circ (r\phi \circ \sigma_1)]. \end{aligned}$$

Theorem 2.1.
The definition Definition 2.6 is well-defined. That is, they are independent of the choice of chart (U, ϕ) and σ_1, σ_2 as long as $v_1 = [\sigma_1]$ and $v_2 = [\sigma_2]$.

Therefore, $T_p\mathcal{M}$ is a real vector space.

Proof. Let $v_1 = [\sigma_1] = v_1' := [\tau_1], v_2 = [\sigma_2] = v_2' := [\tau_2]$. First check (1) of Definition 2.2,

$$\begin{aligned}(rv_1 + v_2)(0) &= (\phi^{-1} \circ (r\phi \circ \sigma_1(0) + \phi \circ \sigma_2(0))) \\ &= (\phi^{-1} \circ (r\phi \circ \tau_1(0) + \phi \circ \tau_2(0))) \\ &= (rv_1' + v_2')(0),\end{aligned}$$

since $\phi \circ \sigma_1(0) = \phi \circ \tau_1(0) = \phi(p)$ by equivalence, and the same for σ_2.

Now consider

$$\begin{aligned}\frac{d}{dt}(\phi \circ (rv_1 + v_2))\Big|_{t=0} &= \frac{d}{dt}(r\phi \circ \sigma_1 + \phi \circ \sigma_2)\Big|_{t=0} \\ &= r\,\frac{d}{dt}(\phi \circ \sigma_1)\Big|_{t=0} + \frac{d}{dt}(\phi \circ \sigma_2)\Big|_{t=0} \\ &= r\,\frac{d}{dt}(\phi \circ \tau_1)\Big|_{t=0} + \frac{d}{dt}(\phi \circ \tau_2)\Big|_{t=0} \\ &= \frac{d}{dt}(\phi \circ (rv_1' + v_2'))\Big|_{t=0},\end{aligned}$$

since $\frac{d}{dt}(\phi \circ \sigma_1)\big|_{t=0} = \frac{d}{dt}(\phi \circ \tau_1)\big|_{t=0}$ by equivalence, and the same for σ_2. ▨

2.1.3 Curves and Derivation

Definition 2.7 (Directional Derivative)**.**
For any $f : \mathcal{M} \to \mathbb{R}$ s.t. $f \in C^\infty$, we define

$$v(f) := \frac{d}{dt}(f \circ \sigma(t))\Big|_{t=0},$$

where $v = [\sigma]$.

Theorem 2.2.
The definition Definition 2.7 is well-defined. That is, $v(f)$ is independent of the curve σ chosen as well as $v = [\sigma]$.

Proof. Let $v_1 = [\sigma_1] = [\sigma_2] = v_2$. Then

$$v_1(f) = \frac{d}{dt}(f \circ \sigma_1)\Big|_{t=0},$$
$$v_2(f) = \frac{d}{dt}(f \circ \sigma_2)\Big|_{t=0},$$

$$\frac{d}{dt}(\phi \circ \sigma_1)\Big|_{t=0} = \frac{d}{dt}(\phi \circ \sigma_2)\Big|_{t=0}.$$

Then

$$\begin{aligned} v_1(f) &= \frac{d}{dt}(\underbrace{(f \circ \phi^{-1})}_{\mathbb{R} \leftarrow \mathbb{R}^m} \circ \underbrace{(\phi \circ \sigma_1)}_{\mathbb{R}^m \leftarrow \mathbb{R}})\Bigg|_{t=0} \\ &= (f \circ \phi^{-1})' \circ (\phi \circ \sigma_1) \cdot (\phi \circ \sigma_1)'\big|_{t=0} \\ &= (f \circ \phi^{-1})' \circ (\phi \circ \sigma_2) \cdot (\phi \circ \sigma_2)'\big|_{t=0} \\ &= v_2(f), \end{aligned}$$

since $\phi \circ \sigma_1(0) = \phi \circ \sigma_2(0) = \phi(p)$, and $(\phi \circ \sigma_1)' = (\phi \circ \sigma_2)'$ by equivalence. ▨

2.2 The Algebraic Formulation of Tangent Spaces

2.2.1 The Space of Derivations

Definition 2.8 (Derivation)**.**
A derivation at $p \in \mathcal{M}$ is a map $v : C^\infty(\mathcal{M}) \to \mathbb{R}$ s.t.

1. $v(rf + g) = rv(f) + v(g)$, (Linear)
2. $v(fg) = f(p)v(g) + g(p)v(f)$, (Leibniz)

where $f, g \in C^\infty$.

Definition 2.9 (Tangent Space (Algebraic))**.**
The space of all derivations at $p \in \mathcal{M}$ is denoted $D_p\mathcal{M}$.

Definition 2.10 (Addition and Scalar Multiplication)**.**
Given $v_1, v_2 \in D_p\mathcal{M}$, define

$$(v_1 + v_2)(f) := v_1(f) + v_2(f)$$
$$(rv)(f) := rv(f).$$

Theorem 2.3.
$D_p\mathcal{M}$ is a real vector space.

2.2.2 The Basis Tangent Vectors

Definition 2.11 (Basis Tangent Vectors)**.**
We define the basis tangent vectors via derivations by

$$\left(\partial_\mu\right)_p f := \frac{\partial}{\partial u^\mu}(f \circ \phi^{-1}(\vec{u}))\Big|_{\vec{u}=\phi(p)}, 1 \leq \mu \leq \dim \mathcal{M}.$$

where $u \in \mathbb{R}^m$, $f : \mathcal{M} \to \mathbb{R}$, $f \in C^\infty$. For the use of u^μ, see Definition 1.5.

Theorem 2.4.

$$\left(\partial_\mu\right)_p x^\nu = \delta^\nu{}_\mu.$$

Proof. Although a simple exercise, it was a good chance to explain the sophisticated notation.

$$\left(\partial_\mu\right)_p x^\mu = \frac{\partial}{\partial u^\mu}\left(x^\mu \circ \phi^{-1}\begin{pmatrix} u^1 \\ \vdots \\ u^m \end{pmatrix}\right)\Bigg|_{\phi(p)}.$$

The coordinate $u \in \mathbb{R}^m$ was brought to $\mathcal{M}$ and projected to $\mathbb{R}^m$ again and taken out the μ-th component. So

$$= \frac{\partial}{\partial u^\mu}(u^\mu)\Big|_{\phi(p)} = 1.$$

▨

Theorem 2.5 (Linear Independence of Basis Tangent Vectors).
The basis tangent vectors $\left(\partial_\mu\right)_p, 1 \leq \mu \leq \dim \mathcal{M}$ are linear independent.

Proof. Suppose $a^\mu \left(\partial_\mu\right)_p = 0$. Then

$$a^\mu \left(\partial_\mu\right)_p (x^\nu) = a^\mu \delta^\nu{}_\mu = 0(x^\nu) = 0.$$

So $a^\mu = 0$. ▨

Theorem 2.6 (Coordinate Expansion of Tangent Vectors).
For all $v \in D_p\mathcal{M}$, we have

$$v = v^\mu \left(\partial_\mu\right)_p,$$

where Einstein notation was used, and $v^\mu = v(x^\mu)$.

Remark.
The proof was sophisticated and did not teach me much.

♍

2.3 Isomorphism of Curves and Derivations

Theorem 2.7 (Isomorphism of Curves and Derivations).
Similar to Definition 2.7, we define the linear map $\iota : T_p\mathcal{M} \to D_p\mathcal{M}$ acting on $v = [\sigma] \in T_p\mathcal{M}$ by

$$\iota(v)(f) := \frac{d}{dt}(f \circ \sigma(t))\bigg|_{t=0}.$$

Then ι is a linear isomorphism. Note that RHS $\in D_p\mathcal{M}$.

Proof. (linearity) Choose ϕ s.t. $\phi(p) = 0$.

$$
\begin{aligned}
\iota(rv_1 + v_2)(f) &= \frac{d}{dt}(f \circ \phi^{-1} \circ (r\phi \circ \sigma_1 + \phi \circ \sigma_2))\Big|_{t=0} \\
&= ((f \circ \phi^{-1})' \circ (r\phi \circ \sigma_1 + \phi \circ \sigma_2) \cdot (r\phi \circ \sigma_1 + \phi \circ \sigma_2)')\big|_{t=0} \\
&= ((f \circ \phi^{-1})'(0) \cdot ((r\phi \circ \sigma_1)' + (\phi \circ \sigma_2)'))\big|_{t=0} \\
&= (r(f \circ \phi^{-1})' \circ (\phi \circ \sigma_1) \cdot (\phi \circ \sigma_1)' + (f \circ \phi^{-1})' \circ (\phi \circ \sigma_2) \cdot (\phi \circ \sigma_2)')\big|_{t=0} \\
&= r\iota(v_1) + \iota(v_2)(f).
\end{aligned}
$$

(surjectivity) To show surjectivity, we need to construct a curve for all $v' \in D_p\mathcal{M}$ s.t. $\iota(v) = v'$.

Let $v' \in D_p\mathcal{M}$ and construct $\sigma : (-\epsilon, \epsilon) \to \mathcal{M}$ s.t.

$$
\begin{aligned}
\sigma(0) &= p, \\
v^\mu = v(x^\mu) &= \frac{d}{dt}(x^\mu \circ \sigma(t))\Big|_{t=0}.
\end{aligned}
$$

Then

$$
v(f) = v^\mu \left(\partial_\mu\right)_p f = \frac{d}{dt}(x^\mu \circ \sigma(t))\Big|_{t=0} \left(\partial_\mu\right)_p f.
$$

Also,

$$
\begin{aligned}
\frac{d}{dt}(f \circ \sigma(t))\Big|_{t=0} &= \frac{d}{dt}(f \circ \phi^{-1} \circ \phi \circ \sigma(t))\Big|_{t=0} \\
&= \sum_{\mu=1}^{m} \frac{\partial}{\partial u^\mu}(f \circ \phi^{-1})\Big|_{\phi(p)} \frac{d}{dt}(u^\mu \circ \phi \circ \sigma)\Big|_{t=0} \quad \text{(component-wise)} \\
&= \sum_{\mu=1}^{m} \left(\partial_\mu\right)_p f \, \frac{d}{dt}(x^\mu \circ \sigma)\Big|_{t=0} \\
&= v(f).
\end{aligned}
$$

Thus completing the proof. ▨

> ↗ **Remark.**
> It's tempting to use the powerful theorem that surjectivity of linear transformations is equivalent to injectivity. However, the requirement of that theorem is that **the dimensions of the two spaces are equal**. At this point, we don't know the dimension of $T_p\mathcal{M}$, so we cannot do this. ♍

2.4 Pushforward

2.4.1 Definition and Linearity

Remark.
The pushforward $h_* : T_p\mathcal{M} \to T_{h(p)}\mathcal{N}$ of a specific function $h : \mathcal{M} \to \mathcal{N}$ can be thought of as local linearization of the function.

♍

Definition 2.12 (Pushforward)**.**
Given a function $h : \mathcal{M} \to \mathcal{N}$ and $v \in T_p\mathcal{M}$, then we define the pushforward $h_* : T_p\mathcal{M} \to T_{h(p)}\mathcal{N}$ by

$$h_*(v) := [h \circ \sigma],\ \ v = [\sigma].$$

Theorem 2.8.
The pushforward operation Definition 2.12 is well-defined. That is, $h_*(v_1) = h_*(v_2)$ if $v_1 = [\sigma_1] = [\sigma_2] = v_2$.

Theorem 2.9 (Algebraic Definition of Pushforward)**.**
The definition of pushforward Definition 2.12 is equivalent to the following: let $h : \mathcal{M} \to \mathcal{N}$, $h_* : D_p\mathcal{M} \to D_{h(p)}\mathcal{M}$ is defined by,

$$(h_*v)(f) := v(f \circ h).$$

Proof. $(\rightarrow)$

$$\begin{aligned} h_*(v)(f) = [h \circ \sigma](f) &= \frac{d}{dt}(f \circ h \circ \sigma(t))\bigg|_{t=0} \\ &= \frac{d}{dt}((f \circ h) \circ \sigma(t))\bigg|_{t=0} \\ &:= v(f \circ h). \end{aligned}$$

$(\leftarrow)$ This direction is similar. ▨

Theorem 2.10 (Linearity of Pushforward)**.**
The pushforward map $h_* : T_p\mathcal{M} \to T_{h(p)}\mathcal{N}$ is linear.

$$h_*(rv_1 + v_2) = rh_*(v_1) + h_*(v_2).$$

Proof. (Using Definition 2.12) Let $p \in (U, \phi) \subseteq \mathcal{M}$, and $h(p) \in (V, \psi) \subseteq \mathcal{N}$. Choose ϕ s.t. $\phi(p) = 0$. It is obvious that $h_*(rv_1 + v_2)(0) = (rh_*(v_1) + h_*(v_2))(0) = h(p)$.

Consider

$$\begin{aligned}
\frac{d}{dt}\underbrace{(\psi \circ h_*(rv_1 + v_2))}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathbb{R}}\Bigg|_{t=0} &= \frac{d}{dt}(\underbrace{\psi \circ h \circ (\phi^{-1}}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathcal{M} \leftarrow \mathbb{R}^m} \circ \underbrace{(r\phi \circ \sigma_1 + \phi \circ \sigma_2)}_{\mathbb{R}^m \leftarrow \mathcal{M} \leftarrow \mathbb{R}}))\Bigg|_{t=0} \\
&= (\psi \circ h \circ \phi^{-1})' \circ (r\phi \circ \sigma_1 + \phi \circ \sigma_2) \cdot (r\phi \circ \sigma_1 + \phi \circ \sigma_2)'\big|_{t=0} \\
&= (\psi \circ h \circ \phi^{-1})'(0) \cdot ((r\phi \circ \sigma_1)' + (\phi \circ \sigma_2)')\big|_{t=0}.
\end{aligned}$$

And

$$\begin{aligned}
\frac{d}{dt}(\underbrace{\psi}_{\mathbb{R}^n \leftarrow} \circ \underbrace{(rh_*(v_1) + h_*(v_2))}_{\mathcal{N} \leftarrow \mathbb{R}})\Bigg|_{t=0} &= \frac{d}{dt}\underbrace{(r\psi \circ h \circ \sigma_1 + \psi \circ h \circ \sigma_2)}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathcal{M} \leftarrow \mathbb{R}}\Bigg|_{t=0} \\
=&(\underbrace{r\psi \circ h \circ \phi^{-1}}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathcal{M} \leftarrow \mathbb{R}^m} \circ \underbrace{\phi \circ \sigma_1}_{\mathbb{R}^m \leftarrow \mathcal{M} \leftarrow \mathbb{R}} + \psi \circ h \circ \phi^{-1} \circ \phi \circ \sigma_2)'|_{t=0} \\
=&(r(\psi \circ h \circ \phi^{-1})' \circ (\phi \circ \sigma_1) \cdot (\phi \circ \sigma_1)')|_{t=0} \\
&+ ((\psi \circ h \circ \phi^{-1})' \circ (\phi \circ \sigma_2) \cdot (\phi \circ \sigma_2)')|_{t=0} \\
=&(\psi \circ h \circ \phi^{-1})'(0) \cdot (r(\phi \circ \sigma_1)' + (\phi \circ \sigma_2)')|_{t=0}.
\end{aligned}$$

So we see the two are equal.

(Using Theorem 2.9)

$$\begin{aligned}
(h_*(rv_1 + v_2))(f) &= (rv_1 + v_2)(f \circ h) \\
&= rv_1(f \circ h) + v_2(f \circ h) \\
&= r(h_* v_1)f + (h_* v_2)f.
\end{aligned}$$

▨

Theorem 2.11 (Associativity of Pushforwards)**.**
Given manifolds $\mathcal{M}, \mathcal{N}, \mathcal{P}$ and $h : \mathcal{M} \to \mathcal{N}$, $k : \mathcal{N} \to \mathcal{P}$, then

$$(k \circ h)_* = k_* \circ h_*.$$

2.4.2 Jacobian

Theorem 2.12 (Local Representative of Pushforward)**.**
Let $\dim \mathcal{M} = m, \dim \mathcal{N} = n$, $h : \mathcal{M} \to \mathcal{N}$, $\{x^1, \dots, x^m\}$ be the local coordinates of $\mathcal{M}$ around p, and $\{y^1, \dots, y^n\}$ be the local coordinates of $\mathcal{N}$ around $h(p)$. Then

$$h_* v = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} (\partial_\nu)_{h(p)} \left.\frac{\partial h^\nu}{\partial x^\mu}\right|_p v^\mu,$$

where $J^\nu{}_\mu := \left.\frac{\partial h^\nu}{\partial x^\mu}\right|_p := \left(\partial_\mu\right)_p (y^\nu \circ h)$ is the Jacobian matrix.

Proof. First expand v in terms of local coordinates and use linearity,

$$h_* v = h_*(v^\mu \left(\partial_\mu\right)_p) = v^\mu h_*(\left(\partial_\mu\right)_p).$$

Expand the result in local coordinates of $\mathcal{N}$,

$$h_*(\left(\partial_\mu\right)_p) = \left(h_* \left(\partial_\mu\right)_p\right)^\nu (\partial_\nu)_{h(p)}.$$

Using Theorem 2.9,

$$\begin{aligned}\left(h_* \left(\partial_\mu\right)_p\right)^\nu &= \left(h_* \left(\partial_\mu\right)_p\right) \circ y^\nu \\ &= \left(\partial_\mu\right)_p (y^\nu \circ h) \\ &:= \left(\partial_\mu\right)_p h^\nu.\end{aligned}$$

So,

$$h_*(\left(\partial_\mu\right)_p) = \left(\partial_\mu\right)_p h^\nu (\partial_\nu)_{h(p)}.$$

And,

$$h_* v = v^\mu \left(\partial_\mu\right)_p h^\nu (\partial_\nu)_{h(p)}.$$

▨

Theorem 2.13 (Using Curve to Pushforward)**.**
Given $c : (-\epsilon, \epsilon) \to \mathcal{M}$ a curve, and choose the coordinate chart of $\mathbb{R}$ to be the identity, then

$$c_* \left(\frac{d}{dt}\right)_0 = [c] \in T_p\mathcal{M}.$$

Proof. First we clarify what is $\left(\frac{d}{dt}\right)_0$. Since on the trivial manifold $\mathbb{R}$ there is only one coordinate, namely t, we need not specify the number. Also, considering our functions are scalar valued $f : \mathcal{M} \to \mathbb{R}$, this motivates us to write "total differential".

For all $f \in C^\infty$,

$$c_* \left(\frac{d}{dt}\right)_0 f = \left(\frac{d}{dt}\right)_0 (f \circ c).$$

Since the coordinate chart is the identity,

$$\begin{aligned} \left(\frac{d}{dt}\right)_0 (f \circ c) &= \frac{d}{dt}(f \circ c \circ I)\Big|_{I(t)=0} \\ &= \frac{d}{dt}(f \circ c)\Big|_{t=0} \\ &= [c]f. \end{aligned}$$

▨

Theorem 2.14 (Contravariancy of Tangent Vectors)**.**
The components of tangent vectors are contravariant, i.e., given two coordinate charts (U, ϕ) and (U', ϕ') s.t. $U \cap U' = S \neq \varnothing$, then on S,

$$v'^\nu = \sum_{\mu=1}^{m} v^\mu \frac{\partial x'^\nu}{\partial x^\mu}.$$

Proof. Given that we have the local representative of pushforward at hand, consider the identity pushforward $\mathrm{id}_* : T_p\mathcal{M} \to T_p\mathcal{M}$,

$$\mathrm{id}_* \, v = \sum_{\mu=1}^{m} \sum_{\nu=1}^{m} v^\mu \left. \frac{\partial x'^\nu}{\partial x^\mu} \right|_p \left(\partial_{\nu'}\right)_p.$$

We see immediately that the result holds. ▨

3 Vector Fields

3.1 Definition

Definition 3.1 (Vector Fields)**.**
A vector field X on $\mathcal{M}$ is a smooth assignment of a tangent vector $X_p \in T_p\mathcal{M}$ $\forall p \in \mathcal{M}$.

"Smooth" assignment is defined to be that the Lie derivative Definition 3.2 is smooth.

Definition 3.2 (Lie Derivative)**.**
The Lie-derivative of function f with respect to vector field X is defined as

$$\mathcal{L}_X f := Xf,$$

and at a specific point $p \in \mathcal{M}$,

$$\mathcal{L}_X f(p) := Xf(p) := X_p f.$$

Theorem 3.1 (Properties of Lie Derivative)**.**
The Lie derivative has the following properties,

1. $X(rf + g) = rXf + Xg$
2. $X(fg) = fXg + gXf$.

Theorem 3.2 (Component of Vector Field)**.**
Given a chart (U, ϕ) on $\mathcal{M}$, we can write

$$X_U = X_U x^\mu \partial_\mu.$$

When the context is clear or **for convenience**, we write

$$X = Xx^\mu \partial_\mu := X^\mu \partial_\mu.$$

Proof. We know

$$(Xf)(p) = X_p f = X_p x^\mu \left(\partial_\mu\right)_p f = (Xx^\mu)(p) \left(\partial_\mu\right)_p f.$$

Remark.
∂_μ is a vector field that assigns each point $p \in \mathcal{M}$ with the vector $\left(\partial_\mu\right)_p \in T_p\mathcal{M}$.

Theorem 3.3 (Contravariancy of Vector Fields)**.**
Given two coordinate charts (U, ϕ) and (U', ϕ') s.t. $U \cap U' = S \neq \varnothing$. On S,

$$X^{\nu'} = \sum_{\mu=1}^{m} X^\mu \frac{\partial x'^\nu}{\partial x^\mu}.$$

Analogous to Theorem 2.14.

3.2 Lie Bracket

Definition 3.3 (Composition of Vector Fields)**.**
We can view $X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, and so does Y. Therefore, we define

$$(X \circ Y)(f) := X(Yf).$$

Definition 3.4 (Lie Bracket (Commutator))**.**
We define the Lie Bracket of two vector fields X, Y to be

$$[X, Y] := X \circ Y - Y \circ X.$$

In particular,

$$[X, Y](f) = \mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f)$$

Remark.
Lie Bracket Definition 3.4 is a vector field, while the expression $X \circ Y$ is not, because it contains second differential terms. See the following proof.

Theorem 3.4 (Lie Bracket Components)**.**

$$[X, Y]^\mu = (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu).$$

Proof. Given $X = X^\mu \partial_\mu, Y = Y^\nu \partial_\nu$, we try to write the component of $X \circ Y$.

$$X \circ Y(f) = X^\mu \partial_\mu \left(Y^\nu \partial_\nu f\right).$$

However, notice that

$$\begin{aligned} &Y^\nu := Y x^\nu \in C^\infty(\mathcal{M}); \\ &\partial_\nu : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), \\ &\implies \partial_\nu f \in C^\infty(\mathcal{M}). \end{aligned}$$

So we need to use the Leibniz property of ∂_μ Definition 2.8 in order to evaluate the second term. Doing this for $X \circ Y(f)$ and $Y \circ X(f)$, we have

$$\begin{aligned} X \circ Y(f) &= X^\mu \left((\partial_\mu Y^\nu)(\partial_\nu f) + Y^\nu \partial_\mu \partial_\nu f\right). \\ Y \circ X(f) &= Y^\nu \left((\partial_\nu X^\mu)(\partial_\mu f) + X^\mu \partial_\nu \partial_\mu f\right). \end{aligned}$$

So if $\partial_\mu \partial_\nu f = \partial_\nu \partial_\mu f$, then by subtracting, we can cancel the second order terms, and we are done. We prove so now.

$$\begin{aligned} (\partial_\mu \partial_\nu f)(p) &= \frac{\partial}{\partial u^\mu} \left((\partial_\nu f) \circ \phi^{-1}\right)\Big|_{\phi(p)} \\ &= \frac{\partial}{\partial u^\mu} \left((\partial_\nu)_{\phi^{-1}(u)} f\right)\Big|_{\phi(p)} \\ &= \frac{\partial}{\partial u^\mu} \left(\frac{\partial}{\partial u^\nu} (f \circ \phi^{-1})\Big|_u\right)\Bigg|_{\phi(p)} \\ &= \frac{\partial}{\partial u^\nu} \left(\frac{\partial}{\partial u^\mu} (f \circ \phi^{-1})\Big|_u\right)\Bigg|_{\phi(p)} \\ &= (\partial_\nu \partial_\mu f)(p). \end{aligned}$$

Theorem 3.5 (Properties of Lie Brackets)**.**

1. $[X, Y] = -[Y, X]$ (antisymmetry)
2. $\sum_{\text{cyc}}[X, [Y, Z]] = 0.$ (Jacobi Identity)

3.3 Integral Curves and Flows

Definition 3.5 (Intergral Curve)**.**
Let X be a vector field on $\mathcal{M}$, $p \in \mathcal{M}$. Then an integral curve of X through p is a curve $\sigma : (-\epsilon, \epsilon) \to \mathcal{M}$ s.t.

$$\sigma(0) = p,$$
$$\sigma_* \left(\frac{d}{dt}\right)_t = X_{\sigma(t)}.$$

Remark.
Qualitatively, using Theorem 2.13, this pushforward is just $[\sigma] \in T_{\sigma(t)}\mathcal{M}$. Therefore, the second condition is saying in some sense that the curve is tangent to the vector field on the manifold. For quantitative description, see below.

♍

Definition 3.6 (Differential Equations of Integral Curve)**.**
The components X^μ of X determine the integral curve σ by the following ODE with boundary conditions,

$$X^\mu(\sigma(t)) = \frac{d}{dt}x^\mu(\sigma(t))$$
$$x^\mu(\sigma(0)) = x^\mu(p), \mu = 1, 2, \dots, m.$$

3.3.1 One-parameter Family of Diffeomorphisms

Definition 3.7 (Local 1D Family of Local Diffeomorphisms)**.**
A local, 1D family of local diffeomorphisms at $p \in \mathcal{M}$ is made up of (1) an open neighborhood U of p, (2) $\epsilon > 0$ (3) a family of diffeomorphisms $\{ \phi_t \mid |t| < \epsilon \}$, $\phi_t : U \to \mathcal{M}$ s.t.

1. Every ϕ_t is a smooth function in t and q.
2. $\forall t, s \in \mathbb{R}$ and $|t|, |s|, |t+s| < \epsilon$, and $\forall q \in U$ s.t. $\phi_t(q), \phi_s(q), \phi_{t+s}(q) \in U$, we have
$$\phi_s(\phi_t(q)) = \phi_{s+t}(q).$$
3. $\phi_0(q) = q$.

Remark.
The first "local" refers to the parameter t, which is limited to $(-\epsilon, \epsilon)$. The second "local" refers to the spatial limitation to U. You can view $\phi_t(q)$ as a curve that brings $t \in (-\epsilon, \epsilon)$ to $\phi_t(q) \in \mathcal{M}$.

♍

Definition 3.8 (Induced Vector Field)**.**
By taking tangents to the curve family Definition 3.7, we have the induced vector field X^ϕ given by

$$X_q^\phi(f) := \frac{d}{dt}(f(\phi_t(q)))\Big|_{t=0}$$

Theorem 3.6.
The curve family $t \mapsto \phi_t(q)$ is the integral curve of the induced vector field Definition 3.8 X_q^ϕ.

Proof.
$$
\begin{aligned}
X^{\phi}_{\phi_s(q)} &= \frac{d}{dt}(f\circ\phi_t\circ\phi_s(q))\bigg|_{t=0} \\
&= \frac{d}{dt}(f\circ\phi_{t+s}(q))\bigg|_{t=0}.
\end{aligned}
$$
Let $u = t+s$. Then
$$
\begin{aligned}
X^{\phi}_{\phi_s(q)} &= \frac{d}{du}(f\circ\phi_u(q))\bigg|_{u=s}. \\
&= \phi_{q*}\left(\frac{d}{dt}\right)_s f.
\end{aligned}
$$
▨

3.3.2 Local Flows

Definition 3.9 (Local Flow)**.**
Let X be a vector field on open $U \subseteq \mathcal{M}$, and $p \in U$. A local flow at p is a local one-parameter family of local diffeomorphisms Definition 3.7 defined on some open $V \subseteq U$ s.t. $p \in V$ and the induced vector field Definition 3.8 is X.

Remark.
Local flows always exist and are unique. In contrast, global flows (which means $t \in \mathbb{R}$ instead of a restricted interval) may not exist.

♍

3.3.3 Lie Derivative

Theorem 3.7 (Interpretation of Lie Bracket)**.**
If X, Y are two vector fields on $\mathcal{M}$, and define the following quantity, which can be interpreted as the change of Y when following the integral curves of X, as
$$
\frac{d}{dt}(\phi^X_{-t*}(Y))\bigg|_{t=0} := \lim_{\epsilon\to 0}\frac{\phi^X_{-\epsilon*}(Y_{\phi^X_\epsilon(p)}) - Y_p}{\epsilon}.
$$
Then,
$$
\frac{d}{dt}(\phi^X_{-t*}(Y))\bigg|_{t=0} = [X, Y].
$$

4 Cotangent Spaces

4.1 Cotangent Vectors

Definition 4.1 (Cotangent Spaces)**.**
The cotangent space $T_p^*\mathcal{M}$ at $p \in \mathcal{M}$ is the set of all linear functions $f : T_p\mathcal{M} \to \mathbb{R}$.
Its member is called a cotangent vector.
$\dim T_p^*\mathcal{M} = \dim T_p\mathcal{M}$.

Definition 4.2 (One-Form)**.**
A one-form on $\mathcal{M}$ is a smooth assignment of cotangent vectors $\omega : p \mapsto \omega_p$.
It may be understood as a covector field.

Definition 4.3 (Basis Cotangent Vectors)**.**
The basis cotangent vectors is chosen to be the dual basis of the basis tangent vectors Definition 2.11,

$$(dx^\mu)_p((\partial_\nu)_p) = \delta^\mu{}_\nu.$$

Theorem 4.1 (Coordinate Expression of Cotangent Vectors)**.**
Any $f \in T_p^*\mathcal{M}$ can be expanded as

$$f = f_\mu (dx^\mu)_p.$$

Any one-form ω can be expressed as

$$\omega = \omega_\mu dx^\mu.$$

4.2 Pullback

4.2.1 Definition

Definition 4.4 (Pullback)**.**
Given a function and its pushforward, we define pullback to be the dual of pushforward, i.e.,

$$\begin{aligned} h : \quad & \mathcal{M} & \to \quad & \mathcal{N}, \\ h_* : \quad & T_p\mathcal{M} & \to \quad & T_{h(p)}\mathcal{N}, \\ h^* : \quad & T^*_{h(p)}\mathcal{N} & \to \quad & T^*_p\mathcal{M}, \end{aligned}$$

s.t. given $f \in T^*_{h(p)}\mathcal{N}$ and $v \in T_p\mathcal{M}$,

$$(h^* f)(v) := f(h_* v).$$

Remark.
Note especially on the direction of original function and its induced pullback. This is crucial to the covariancy of one-forms. ♍

Theorem 4.2.
Given ω a one-form on $\mathcal{N}$, and a function $h : \mathcal{M} \to \mathcal{N}$, the pullback $h^*\omega$ is defined as

$$(h^*\omega)(v)_p = \omega(h_* v)_{h(p)}.$$

Theorem 4.3 (Associativity of Pullbacks)**.**
Analogous to Theorem 2.11, given manifolds $\mathcal{M}, \mathcal{N}, \mathcal{P}$ and $h : \mathcal{M} \to \mathcal{N}$, $k : \mathcal{N} \to \mathcal{P}$, then

$$(k \circ h)^* = k^* \circ h^*.$$

4.2.2 Jacobian

Theorem 4.4 (Local Representative of Pullback)**.**
Let $\dim \mathcal{M} = m, \dim \mathcal{N} = n$, $h : \mathcal{M} \to \mathcal{N}$, $\{x^1, \dots, x^m\}$ be the local coordinates of $\mathcal{M}$ around p, and $\{y^1, \dots, y^n\}$ be the local coordinates of $\mathcal{N}$ around $h(p)$. Then

$$h^*\omega = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} \omega_\nu \left. \frac{\partial h^\nu}{\partial x^\mu} \right|_p (dx^\mu)_p,$$

where $J^\nu{}_\mu := \left. \frac{\partial h^\nu}{\partial x^\mu} \right|_p := \left(\partial_\mu \right)_p (y^\nu \circ h)$ is the Jacobian matrix.

Proof. We know by Definition 4.4,

$$(h^*\omega)_\mu(p) = h^*\omega(\partial_\mu) = \omega(h_*\partial_\mu).$$

Expand it in local coordinates of $\mathcal{N}$,

$$(h^*\omega)_\mu(p) = \omega_\nu dy^\nu(h_*\partial_\mu).$$

Via similar procedure in Theorem 2.12, we arrive at

$$(h^*\omega)_\mu(p) = \omega_\nu \frac{\partial h^\nu}{\partial x^\mu}.$$

▨

4.3 Transformation Properties

Theorem 4.5 (Covariancy and Contravariancy)**.**
Given two coordinate charts (U, ϕ) and (U', ϕ') s.t. $U \cap U' = S \neq \varnothing$, then on S,

$$X^{\nu'} = \sum_{\mu=1}^{m} \frac{\partial x'^{\nu}}{\partial x^{\mu}} X^{\mu},$$
$$\omega_{\nu'} = \sum_{\mu=1}^{m} \omega_{\mu} \frac{\partial x^{\mu}}{\partial x'^{\nu}}.$$

If Jacobian matrix is given,

$$J^{\nu'}{}_{\mu} := \left.\frac{\partial x'^{\nu}}{\partial x^{\mu}}\right|_{p} := \left(\partial_{\mu}\right)_{p} x'^{\nu},$$
$$(J^{-1})^{\mu}{}_{\nu'} := \left.\frac{\partial x^{\mu}}{\partial x'^{\nu}}\right|_{p} := \left(\partial_{\nu'}\right)_{p} x^{\mu},$$

then,

$$X^{\nu'} = J^{\nu'}{}_{\mu} X^{\mu}, \qquad \text{(contravariant)}$$
$$\omega_{\nu'} = \omega_{\mu} (J^{-1})^{\mu}{}_{\nu'}. \qquad \text{(covariant)}$$

Proof. The contravariant part is proved in Theorem 2.14. Now we turn to the covariant part.

Let $h = \mathrm{id} : (U, \phi) \subseteq \mathcal{M} \to (U', \phi') \subseteq \mathcal{M}$, consider its pullback.

$$(\mathrm{id}^* \omega)_p = \omega_{\nu'} \frac{\partial x'^{\nu}}{\partial x^{\mu}} dx^{\mu}.$$

Then,

$$\omega_{\mu} = \omega_{\nu'} \frac{\partial x'^{\nu}}{\partial x^{\mu}}.$$

Inverting the matrix equation above, we get the desired result. ▨

5 Tensors

Definition 5.1 (Tensors)**.**
If $\dim \mathcal{M} \neq \infty$, the tensors of type (r, s) $T_p^{r,s}\mathcal{M}$ are all the linear functions

$$f : \bigtimes^{r} T_p^*\mathcal{M} \times \bigtimes^{s} T_p\mathcal{M} \to \mathbb{R}.$$

I.e., it eats r covectors and s vectors.

Theorem 5.1 (Dimensions of General Tensor Space)**.**
The dimension of $T_p^{r,s}\mathcal{M}$ is $m^r m^s$. In particular, a basis for the space is,

$$\bigotimes_{1 \leq \mu_1 \cdots \mu_r \leq m} \left(\partial_{\mu_i}\right)_p \otimes \bigotimes_{1 \leq \nu_1 \cdots \nu_s \leq m} (dx^{\nu_i})_p$$

Remark.
For a detailed proof, see Hoffman. ♍

6 n-Forms

6.1 Definition

Definition 6.1 (n-Forms)**.**
An n-form is a tensor field of type $(0, n)$ that is totally skew-symmetric (or alternating, or totally antisymmetric), i.e.,

$$\omega(X_1, X_2, \dots, X_n) = (\operatorname{sgn}\sigma)\omega(X_{\sigma(1)}, \dots, X_{\sigma(n)}), \ \forall\sigma \in S_n.$$

The set of all n-forms on $\mathcal{M}$ is denoted as $\Lambda^n(\mathcal{M})$.
The set of all forms is $\Lambda(\mathcal{M}) = \bigoplus_{n=0}^{\dim\mathcal{M}} \Lambda^n(\mathcal{M})$.
Conventionally, we classify functions as 0-forms.

6.2 The Exterior Product

Definition 6.2 (Exterior Product)**.**
Given $\omega_1 \in \Lambda^{n_1}(\mathcal{M}), \omega_2 \in \Lambda^{n_2}(\mathcal{M})$, their exterior product is a $(n_1 + n_2)$-form given by,

$$\omega_1 \wedge \omega_2 := \frac{1}{n_1!n_2!} \sum_{\sigma\in S_{n_1+n_2}} (\operatorname{sgn}\sigma)(\omega_1 \otimes \omega_2)_\sigma.$$

Written explicitly,

$$\begin{aligned}(\omega_1 \wedge \omega_2)(X_1, \dots, X_{n_1+n_2}) :=&\\ &\frac{1}{n_1!n_2!} \sum_{\sigma\in S_{n_1+n_2}} (\operatorname{sgn}\sigma)(\omega_1 \otimes \omega_2)(X_{\sigma(1)}, \dots, X_{\sigma(n_1+n_2)})\end{aligned}$$

Remark.
I'll take the alternating property and associativity of the exterior product for granted. For a detailed proof, see Hoffman. ♍

Theorem 6.1 (Commutativity with Pullback).
Given $h : \mathcal{M} \to \mathcal{N}$ and $\alpha, \beta \in \Lambda(\mathcal{N})$, then

$$h^*(\alpha \wedge \beta) = (h^*\alpha) \wedge (h^*\beta).$$

Remark.
For a "generalized" pullback, we have,

$$(h^*(\alpha))(X_1, \dots, X_{n_1}) = \alpha(h_* X_1, \dots, h_* X_{n_1}).$$

♍

Proof.

$$\begin{aligned}
(h^*\alpha)\wedge(h^*\beta) & \\
&= \frac{1}{n_1! n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\operatorname{sgn}\sigma) \alpha \otimes \beta(h_* X_{\sigma(1)}, \dots, h_* X_{\sigma(n_1+n_2)}). \\
&= \frac{1}{n_1! n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\operatorname{sgn}\sigma) h^* \left(\alpha \otimes \beta(X_{\sigma(1)}, \dots, X_{\sigma(n_1+n_2)})\right). \\
&= h^* \left(\frac{1}{n_1! n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\operatorname{sgn}\sigma) \alpha \otimes \beta(X_{\sigma(1)}, \dots, X_{\sigma(n_1+n_2)}) \right). \\
&= h^*(\alpha \wedge \beta).
\end{aligned}$$

▨

Theorem 6.2 (Skew-Symmetry).
The exterior product makes $\Lambda(\mathcal{M})$ a graded algebra with skew-symmetry given by

$$\omega_1 \wedge \omega_2 = (-1)^{n_1 n_2} \omega_2 \wedge \omega_1.$$

Proof. In the definition of exterior product, first fix $\sigma = \sigma_0$ to consider only one term.

When we switch ω_1 and ω_2, we are essentially doing

$$\begin{aligned}&(\omega_2 \otimes \omega_1)(X_{\sigma_0(1)}, \dots, X_{\sigma_0(n_2)}, \underbrace{X_{\sigma_0(n_2+1)}, \dots, X_{\sigma_0(n_1+n_2)}})\\ =&(\omega_1 \otimes \omega_2)(\underbrace{X_{\sigma_0(n_2+1)}, \dots, X_{\sigma_0(n_1+n_2)}}, X_{\sigma_0(1)}, \dots, X_{\sigma_0(n_2)}).\end{aligned}$$

Now,

$$\underbrace{1, 2, \dots, n_2}, \underbrace{n_2+1}, \dots, n_1+n_2$$

$$\downarrow\ n_2 \text{ times}$$

$$\underbrace{n_2+1}, \underbrace{1, 2, \dots, n_2}, \dots, n_1+n_2$$

$$\downarrow\ (n_1-1)n_2 \text{ times}$$

$$\underbrace{n_2+1, \dots, n_1+n_2}, \underbrace{1, 2, \dots, n_2}$$

So $n_1 n_2$ transposes can achieve the desired effect. Therefore, every term in the summation is multiplied by $(-1)^{n_1 n_2}$, and we get the desired result. ▨

Theorem 6.3 (Dimension of n-Forms)**.**
Let $\dim \mathcal{M} = m$. If $1 \leq n \leq m$, then $\Lambda^n(\mathcal{M}) = \binom{m}{n}$. If $n > m$, then $\Lambda^n(\mathcal{M}) = 0$.
Moreover, a basis for $\Lambda^n(\mathcal{M})_p$ is given by,

$$(dx^{\mu_1})_p \wedge (dx^{\mu_2})_p \wedge \cdots \wedge (dx^{\mu_n})_p,\ 1 \leq \mu_1 \leq \cdots \leq \mu_n \leq m.$$

Remark.
The proof is quite a pleasure to read (and to think of). Please see Hoffman. ♍

6.3 The Exterior Derivative

Definition 6.3 (Exterior Derivative)**.**
Let ω be an n-form on $\mathcal{M}$, $1 \leq n < \dim \mathcal{M}$. Then the exterior derivative $d\omega$ is a $(n+1)$-form. Let $d\omega(\mathbf{X}) = d\omega(X_1, \ldots, X_{n+1})$, then

$$\begin{aligned} d\omega(\mathbf{X}) := &\sum_{i=1}^{n+1} (-1)^{i+1} \mathcal{L}_{X_i}(\omega(\mathbf{X} \backslash \{X_i\})) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], \mathbf{X} \backslash \{X_i, X_j\}). \end{aligned}$$

If $\omega \in \Lambda^{\dim \mathcal{M}}(\mathcal{M})$, we define $d\omega = 0$.

Theorem 6.4.
In particular for a 0-form $f \in C^\infty(\mathcal{M})$,

$$df(X) := \mathcal{L}_X f.$$

In coordinates,

$$df = (\partial_\mu f)(dx^\mu).$$

Theorem 6.5.
In particular for a 1-form ω,

$$d\omega(X, Y) = \mathcal{L}_X(\omega(Y)) - \mathcal{L}_Y(\omega(X)) - \omega([X, Y]).$$

Theorem 6.6 (Coordinate Expansion for Exterior Derivative)**.**
In local coordinates, if $\omega = \omega_{\mu_1 \mu_2 \cdots \mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n}$, then

$$d\omega = \partial_\nu \omega_{\mu_1 \mu_2 \cdots \mu_n} dx^\nu \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n}$$

Theorem 6.7 (Exterior Derivative and Product)**.**

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d\omega_2.$$

Theorem 6.8 (Exterior Derivative and Pullback)**.**
Given $h : \mathcal{M} \to \mathcal{N}$, ω an n-form on $\mathcal{N}$, then

$$d(h^* \omega) = h^*(d\omega).$$

Theorem 6.9 (Functional Linearity of Exterior Derivative)**.**
Let ω be a 1-form on $\mathcal{M}$. Then $d\omega$ satisfies,

$$d\omega(fX, Y) = f d\omega(X, Y), \ \forall f \in C^\infty(\mathcal{M}),$$

where fX is a vector field that gives $(fX)(p) = f(p)X_p$.

Proof. By Definition 6.3,

$$d\omega(fX, Y) = \mathcal{L}_{fX}(\omega(Y)) - \mathcal{L}_Y(\omega(fX)) - \omega([fX, Y]).$$

We break it down term by term. Firstly,

$$(\mathcal{L}_{fX}(\omega(Y)))(p) = f(p)X_p(\omega(Y)) = f(p)(\mathcal{L}_X(\omega(Y)))(p).$$

So

$$\mathcal{L}_{fX}(\omega(Y)) = f \cdot \mathcal{L}_X(\omega(Y)).$$

Secondly, we tackle $\mathcal{L}_Y(\omega(fX))$. In particular,

$$\omega(fX)(p) = \omega_p(f(p)X_p) = f(p)\omega_p(X_p) = f(p)(\omega(X))(p).$$

Therefore,

$$\mathcal{L}_Y(\omega(fX)) = \mathcal{L}_Y(f \cdot \omega(X)) = (\mathcal{L}_Y f)\omega(X) + f \cdot \mathcal{L}_Y(\omega(X)).$$

Thirdly,

$$\omega([fX, Y]) = \omega((fX) \circ Y - Y \circ (fX)).$$

In particular,

$$((Y \circ (fX))(g))(p) = Y_p((fX)(g)) = Y_p(f \cdot Xg) = (Y_p f)((Xg)(p)) + f(p) \cdot Y_p(Xg).$$

So,

$$Y \circ (fX) = (\mathcal{L}_Y f)X + f \cdot Y \circ X.$$

Substituting back,

$$\begin{aligned} \omega([fX, Y]) &= \omega(f \cdot X \circ Y - (\mathcal{L}_Y f)X - f \cdot Y \circ X) \\ &= \omega(f[X, Y] - (\mathcal{L}_Y f)X) \\ &= f\omega([X, Y]) - (\mathcal{L}_Y f)\omega(X). \end{aligned}$$

Finally,

$$\begin{aligned} d\omega(fX, Y) &= \mathcal{L}_{fX}(\omega(Y)) - \mathcal{L}_Y(\omega(fX)) - \omega([fX, Y]) \\ &= f \cdot \mathcal{L}_X(\omega(Y)) - (\mathcal{L}_Y f)\omega(X) - f \cdot \mathcal{L}_Y(\omega(X)) - f\omega([X, Y]) + (\mathcal{L}_Y f)\omega(X) \\ &= f(\mathcal{L}_X(\omega(Y)) - \mathcal{L}_Y(\omega(X)) - \omega([X, Y])) \\ &= f d\omega(X, Y). \end{aligned}$$

▨

Corollary 6.9.1 (Local Nature of Exterior Derivative)**.**
When ω is fixed, the value of $d\omega$ depends only on the local values of vector fields.

$$d\omega(X, Y)(p) = X^\mu(p)Y^\nu(p)d\omega(\partial_\mu, \partial_\nu)(p).$$

Proof. Write $X = X^\mu \partial_\mu$, noting that $X^\mu \in C^\infty(\mathcal{M})$, and use Theorem 6.9. ▨

6.4 DeRham Cohomology

Theorem 6.10 (Twice Exterior Differential)**.**
For all $\omega \in \Lambda^n(\mathcal{M}), 1 \leq n \leq \dim M$, we have

$$d^2\omega = 0.$$

Remark.
This means

$$\operatorname{Im}(d : \Lambda^{n-1}(\mathcal{M}) \to \Lambda^n(\mathcal{M})) \subseteq \operatorname{Ker}(d : \Lambda^n(\mathcal{M}) \to \Lambda^{n+1}(\mathcal{M})).$$

This type of structure is called a differential complex, and is common in many structures. ♍

Definition 6.4 (Closed Form)**.**
An n-form ω is closed if $d\omega = 0$. The set of all closed n-forms is denoted $Z^n(\mathcal{M})$.

Definition 6.5 (Exact Form)**.**
An n-form ω is exact if $\omega = d\beta$ for some $(n-1)$-form β. The set of all exact n-forms is denoted $B^n(\mathcal{M})$.

Remark.
It is guaranteed that $B^n(\mathcal{M}) \subseteq Z^n(\mathcal{M})$, that is, exactness implies closure. But how much closed form is not exact is the study of cohomology theory. ♍

Theorem 6.11 (Poincare's Lemma)**.**
On Euclidean space $\mathbb{R}^m$,

$$B^n(\mathcal{M}) = Z^n(\mathcal{M}),\ \forall n > 0.$$

Definition 6.6 (DeRham Cohomology Groups).
The DeRham cohomology groups $H^n(\mathcal{M}), 0 \leq n \leq \dim \mathcal{M}$ are the quotient spaces

$$H^n(\mathcal{M}) := Z^n(\mathcal{M})/B^n(\mathcal{M}).$$

Remark.
Recall the definition of quotient groups that $H^n(\mathcal{M})$ consists of elements of form $z + B^n(\mathcal{M}), z \in Z^n(\mathcal{M})$.
If all closed forms are exact, $Z^n(\mathcal{M}) \subseteq B^n(\mathcal{M})$, then $H^n(\mathcal{M}) \cong \{0\}$. ♍

Theorem 6.12 (Criterion of Exact ODE).
On the Euclidean space $\mathbb{R}^2$, given a 1-form $\omega = \omega_1 dx^1 + \omega_2 dx^2$. Then

$$\omega \in B^1(\mathbb{R}^2) \iff \partial_2 \omega_1 = \partial_1 \omega_2.$$

Remark.
This is an important theorem to me, for it connects the "exactness of differential forms" to the familiar notion of "exactness of differential equations".
It also provides the first hints that we are actually integrating forms, and that exterior differentiation of a 0-form resembles gradient in usual vector calculus terms. ♍

Proof. Via Poincare lemma Theorem 6.11, on $\mathbb{R}^2$, exactness is equivalent to closure. So we need only to determine the condition that $d\omega = 0$. Using Theorem 6.6,

$$\begin{aligned} d\omega &= \partial_\nu \omega_{\mu_1} dx^\nu \wedge dx^{\mu_1} \\ &= \partial_2 \omega_1 dx^2 \wedge dx^1 + \partial_1 \omega_2 dx^1 \wedge dx^2 \\ &= (\partial_2 \omega_1 - \partial_1 \omega_2)\, dx^2 \wedge dx^1. \end{aligned}$$

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