1 Integration of Differential Forms

1.1 Partition of Unity

Definition 1.1 (Support)**.**
Let X be a topological space, and $f : X \to \mathbb{R}$. Then the support of f is defined as

$$\operatorname{supp} f := \{ x \in X \mid f(x) \neq 0 \}.$$

Theorem 1.1 (Partition of Unity)**.**
Let $\mathcal{M}$ be a C^∞ manifold with dimension m with atlas Φ.
Let $\Phi = \left\{ \phi_j \mid \phi_j : V_j \to \phi_j(V_j), j \in J \right\}$.
Then it is possible to construct a set of C^∞ functions $\rho_j, j \in J$ s.t.

$$1 = \sum_{j \in J} \rho_j,\ \operatorname{supp} \rho_j \subseteq V_j$$

1.2 Orientation

1.2.1 Definition

Definition 1.2 (Compatible Coordinate Charts)**.**
Given a manifold $\mathcal{M}$, its two coordinate charts are called compatible (have the same orientation) if,

$$\det J > 0.$$

Where J is the Jacobian matrix **??**.
If the manifold has a maximal compatible atlas, then we say the manifold is orientable, and we may call its corresponding orientation positive and denote the atlas Φ_+.

Theorem 1.2.
A manifold has either no orientation (any atlas is not compatible) or two orientations.

Theorem 1.3 (Orientability and Existence of Forms of Highest Degree)**.**
A manifold is orientable iff there exists a nowhere vanishing differential form of the highest degree.

1.2.2 Positively Oriented Boundary

Definition 1.3 (Positively Oriented Boundary)**.**
Let $\mathcal{M}$ be a orientable C^∞ manifold with dimension m, positively oriented by compatible atlas Φ_+. Define coordinate charts on $\partial\mathcal{M}$ from Φ as follows,

$$\phi^{\partial\mathcal{M}} : U_\phi \cap \partial\mathcal{M} \to \mathbb{R}^{m-1},$$

Then $\Phi_+^{\partial\mathcal{M}} := \left\{ \phi^{\partial\mathcal{M}} \mid \phi \in \Phi_+ \right\}$ determines an orientation on $\partial\mathcal{M}$, called the positive orientation.

1.3 Pseudoforms

Definition 1.4 (Pseudoforms)**.**
A C^∞ pseudo k-form $\tilde{\omega} \in \tilde{\Lambda}^k(\mathcal{M})$ consists of a family of differential k-forms $\omega_\phi \in \Lambda^k(\phi(U_\phi))$, $\phi \in \Phi$, $\phi : U_\phi \to V_\phi = \phi(U_\phi) \subseteq (-\infty, 0] \times \mathbb{R}^{m-1}$, with an additional requirement that

$$\omega_{\phi'}\big|_{\phi'(U_\phi \cap U_{\phi'})} = (\operatorname{sgn}\det J)\left(\phi \circ \phi'^{-1}\right)^* \left(\omega_\phi\big|_{\phi(U_\phi \cap U_{\phi'})}\right), \ \forall \phi, \phi' \in \Phi,$$

where $\operatorname{sgn}\det J$ denotes the sign of Jacobian determinant.

Remark.
The transformation rule differ from usual forms only in the choice of sign.
One cannot pullback a pseudoform. ♍

1.4 Integration of Forms of Highest Degree

Definition 1.5 (Integration)**.**
Let $\mathcal{M}$ be a paracompact C^∞ manifold of dimension m. Choose a C^∞ partition of unity $\rho_j, j \in J$ of $\mathcal{M}$ s.t. $\operatorname{supp}\rho_j \subseteq U_{\phi_j} := U_j$.
Let a pseudo-m-form $\tilde{\omega} \in \tilde{\Lambda}^m(\mathcal{M})$ has local expression $\tilde{\omega}_{\phi_j} = f_j(x)dx_j^1 \wedge \cdots \wedge dx_j^m$, we say

$$\int_{\mathcal{M}} \tilde{\omega} = \sum_{j \in J} \int_{(-\infty,0] \times \mathbb{R}^{m-1}} (\rho_j \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m$$

if the finite sum exists and has the same value for all choices of ρ_j and ϕ_j.

Remark.
The following theorem reveals why we integrate pseudoforms, not usual forms.
♍

Theorem 1.4 (Criterion of Existence of Integral)**.**
If $\operatorname{supp}\tilde{\omega}$ is compact, then $\int_{\mathcal{M}} \tilde{\omega}$ exists.

Proof. Let two sets of coordinate charts be

$$\begin{aligned} \phi_j &: U_j \to V_j, j \in J \\ \phi'_k &: U'_k \to V'_k, k \in K. \end{aligned}$$

And cooresponding partition of unity be ρ_j, ρ'_k.

(The goal) Show

$$\begin{aligned} &\sum_{j \in J} \int (\rho_j \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m \\ &= \sum_{k \in K} \int (\rho'_k \circ {\phi'_k}^{-1})(x') f'_k(x') d{x'}^1 \dots d{x'}^m \end{aligned}$$

(Split using ρ'_k)

$$\begin{aligned} \int_{\mathcal{M}} \omega &= \sum_{j \in J} \int (\rho_j \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m \\ &= \sum_{j \in J} \int \sum_{k \in K} (\rho'_k \circ \phi_j^{-1})(x)(\rho_j \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m. \end{aligned}$$

Since the sum is finite, and $\operatorname{supp}\omega$ is compact, and therefore the integral is not improper; thus, there can be no limit or Fubini problems on exchanging sums and integrals. So

$$\int_{\mathcal{M}} \omega = \sum_{j\in J}\sum_{k\in K}\int (\rho_j\rho'_k \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m$$

(Change of variables) First fix j, k.

$$\begin{aligned}
&\int (\rho_j\rho'_k \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m \\
=&\int (\rho_j\rho'_k \circ \phi_j^{-1})(\phi_j \circ {\phi'_k}^{-1}(x')) f_j(\phi_j \circ {\phi'_k}^{-1}(x')) \left|\det\left(\frac{\partial x}{\partial x'}\right)\right| d{x'}^1 \dots d{x'}^m \\
=&\int (\rho_j\rho'_k \circ {\phi'_k}^{-1}(x')) f_j(\phi_j \circ {\phi'_k}^{-1}(x')) \left|\det\left(\frac{\partial x}{\partial x'}\right)\right| d{x'}^1 \dots d{x'}^m
\end{aligned}$$

(Use pullback requirement) From Definition 1.4,

$$\omega_{\phi'_k} = (\operatorname{sgn}\det J)(\phi_j \circ {\phi'_k}^{-1})^* \omega_{\phi_j},$$

we see

$$\begin{aligned}
&f'_j(x') dx^1 \wedge \dots \wedge dx^m \\
=&(\operatorname{sgn}\det J) f_j(\phi_j \circ {\phi'_k}^{-1}(x')) \left(\frac{\partial x^1}{\partial {x'}^{\ell_1}} d{x'}^{\ell_1}\right) \wedge \dots \wedge \left(\frac{\partial x^m}{\partial {x'}^{\ell_m}} d{x'}^{\ell_m}\right) \\
=&(\operatorname{sgn}\det J) \sum_{\sigma\in S_m} f_j(\phi_j \circ {\phi'_k}^{-1}(x'))(\operatorname{sgn}\sigma) \frac{\partial x^1}{\partial {x'}^{\sigma(1)}} \dots \frac{\partial x^m}{\partial {x'}^{\sigma(m)}} d{x'}^1 \wedge \dots \wedge d{x'}^m \\
=&(\operatorname{sgn}\det J) f_j(\phi_j \circ {\phi'_k}^{-1}(x')) \det\left(\frac{\partial x}{\partial x'}\right) d{x'}^1 \wedge \dots \wedge d{x'}^m \\
=&f_j(\phi_j \circ {\phi'_k}^{-1}(x')) \left|\det\left(\frac{\partial x}{\partial x'}\right)\right| d{x'}^1 \wedge \dots \wedge d{x'}^m
\end{aligned}$$

Therefore, the integral

$$\begin{aligned}
&\int (\rho_j\rho'_k \circ {\phi'_k}^{-1}(x')) f_j(\phi_j \circ {\phi'_k}^{-1}(x')) \left|\det\left(\frac{\partial x}{\partial x'}\right)\right| d{x'}^1 \dots d{x'}^m \\
=&\int (\rho_j\rho'_k \circ {\phi'_k}^{-1}(x')) f'_j(x') d{x'}^1 \dots d{x'}^m
\end{aligned}$$

(Closing) By moving the sum wrt $j \in J$ into the integral and using the property of partition of unity, the proof is completed. ▨

1.5 Integration of Forms of Lower Degree

1.5.1 Definition

Definition 1.6 (Integration of Lower Degree Forms).
Let $\mathcal{Z}$ be an oriented C^∞ manifold of dimension d, $f : \mathcal{Z} \to \mathcal{M}$ be a C^∞ map to a C^∞ manifold $\mathcal{M}$ of dimension m.
Let $\omega \in \Lambda^d(\mathcal{M})$, we define

$$\int_{\mathcal{Z}} \omega := \int_{\mathcal{Z}} f^*\omega$$

using the positive orientation of $\mathcal{Z}$, if it latter exists and the pullback function f is clear from context.

1.5.2 Stoke's theorem

Theorem 1.5 (Stoke's Theorem).
If $\mathcal{M}$ is an oriented C^∞ manifold of dimension m and $\omega \in \Lambda^{m-1}(\mathcal{M})$, then

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega := \int_{\partial\mathcal{M}} i^*\omega,$$

where $i : \partial\mathcal{M} \to \mathcal{M}$ is just the immersion map, $i : p \mapsto p$.

Proof. (Partition Using Charts) Choose a C^∞ partition of unity Theorem 1.1 $\rho_j, j \in J$ s.t. $\operatorname{supp}\rho_j$ are compact and $\operatorname{supp}\rho_j \subseteq U_{\phi_j} := U_j$.

Now $\omega = \sum_{j\in J} \rho_j\omega$ is a finite sum. So it suffices to show that, if $\eta \in \Lambda^{m-1}(\mathcal{M})$, $\operatorname{supp}\eta$ compact and $\operatorname{supp}\eta \subseteq U_\phi$ then $\int_{\mathcal{M}} d\eta = \int_{\partial\mathcal{M}} \eta$, and apply $\eta = \rho_j\omega$ for all $j \in J$.

(The integral) Suppose the coordinates of ϕ is labeled $x^1, \dots, x^m$. Locally, let

$$\eta = \sum_{\ell=1}^{m} f_\ell dx^1 \wedge \cdots \cancel{dx^\ell} \cdots \wedge dx^m,$$

where $f_\ell : (-\infty, 0] \times \mathbb{R}^{m-1} \to \mathbb{R}$, and it is C^∞. Then, also locally,

$$\begin{aligned} d\eta &= \sum_{\ell=1}^{m} \frac{\partial f_\ell}{\partial x^\ell} dx^\ell \wedge dx^1 \wedge \cdots \cancel{dx^\ell} \cdots \wedge dx^m \\ &= \left(\sum_{\ell=1}^{m} (-1)^{\ell-1} \frac{\partial f_\ell}{\partial x^\ell} \right) dx^1 \wedge \cdots \wedge dx^m. \end{aligned}$$

Then by Definition 1.5,

$$\int_{\mathcal{M}} d\eta = \int_{(-\infty,0]\times\mathbb{R}^{m-1}} \left(\sum_{\ell=1}^{m}(-1)^{\ell-1}\frac{\partial f_\ell}{\partial x^\ell}\right) dx^1 \dots dx^m.$$

(To be precise, we need to choose another partition of unity ρ_j' to do intergration. But we can just choose it to cover all of $\operatorname{supp}\eta$ and don't care all other parts, so that doesn't matter too much.)

Choose a rectangular region R s.t.

$$\operatorname{supp}\eta \subseteq [a_1, 0] \times \cdots \times [a_m, b_m]$$

and define

$$R_\ell := [a_1, 0] \times \dots \cancel{[a_\ell, b_\ell]} \cdots \times [a_m, b_m].$$

($\ell = 2, \dots, m$) In this case, by Fubini and FTC,

$$\begin{aligned}
&\int_R \frac{\partial f_\ell}{\partial x^\ell} dx^1 \dots dx^m \\
=&\int_{R_\ell} \left(\int_{a_\ell}^{b_\ell} \frac{\partial f_\ell}{\partial x^\ell} dx^\ell\right) dx^1 \dots \cancel{dx^\ell} \dots dx^m \\
=&\int_{R_\ell} (\underbrace{f_\ell(x^1, \dots, b_\ell, \dots, x^m)}_{0} - \underbrace{f_\ell(x^1, \dots, a_\ell, \dots, x^m)}_{0}) dx^1 \dots \cancel{dx^\ell} \dots dx^m \\
=&0,
\end{aligned}$$

since $\operatorname{supp}\eta \subseteq R$, so on the boundary $f = 0$.

($\ell = 1$) Now the integral has only one term left.

$$\begin{aligned}
\int_{\mathcal{M}} d\eta &= \int_R \frac{\partial f_1}{\partial x^1} dx^1 \dots dx^m \\
&= \int_{R_1} \left(\int_{a_1}^{0} \frac{\partial f_1}{\partial x^1} dx^1\right) dx^2 \dots dx^m \\
&= \int_{R_\ell} (f_1(0, x^2, \dots, x^m) - \underbrace{f_1(a_1, x^2, \dots, x^m)}_{0}) dx^2 \dots dx^m \\
&= \int_{\mathbb{R}^{m-1}} (f_1 \circ i) dx^2 \dots dx^m \\
&= \int_{\partial\mathcal{M}} i^*\eta.
\end{aligned}$$

▨