

# 1 Differentiable Manifolds

## 1.1 Definition

### 1.1.1 Coordinate Charts

#### Definition 1.1 (Coordinate Charts).

An  $m$ -dimensional,  $m \neq \infty$  coordinate chart on a topological space  $\mathcal{M}$  is a pair

$$(U, \phi) \begin{cases} U \subseteq \mathcal{M}, U \text{ open} \\ \phi : U \rightarrow \mathbb{R}^m, \phi \text{ homeomorphism} \end{cases}$$

#### ✍ Remark.

If  $U = \mathcal{M}$ , then we say the coordinate chart  $\phi$  is globally defined; if not, then it is locally defined. Few manifolds have globally defined property.

□

#### Definition 1.2 (Overlap Function).

Let  $(U_1, \phi_1), (U_2, \phi_2)$  be a pair of  $m$ -dimensional coordinate charts with  $U_1 \cap U_2 \neq \emptyset$ . Then the overlap function is defined as

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \subseteq \mathbb{R}^m \rightarrow \phi_2(U_1 \cap U_2) \subseteq \mathbb{R}^m.$$

#### Definition 1.3 (Atlas).

An  $m$ -dimensional atlas on  $\mathcal{M}$  is a family of  $m$ -dimensional coordinate charts  $(U_i, \phi_i), i \in I$  s.t.

1.  $\mathcal{M} = \bigcup_{i \in I} U_i$ .
2. Each overlap function  $\phi_j \circ \phi_i^{-1}, i, j \in I$  is  $C^\infty$ .

### Definition 1.4 (Differentiable Manifolds).

An  $m$ -dimensional differentiable manifold is a topological space  $\mathcal{M}$  equipped with an atlas.

#### ✍ Remark.

We didn't define a differentiable manifold by regulating the differentiability of the coordinate charts themselves. That's because differentiation is not defined on a manifold, so we need to rely on Euclidean spaces.  $\mathfrak{M}$

## 1.2 Dimension of a Manifold

#### ✍ Remark.

Consider a manifold that consists of a rod attached to a disk. The dimension is not same everywhere. We give a criterion on how to describe such a scenario.  $\mathfrak{M}$

### Theorem 1.1 (Invariance of Domain).

For all  $A, B \subseteq S^n$ , if  $\exists f : A \rightarrow B$  homeomorphic and  $B \in \tau_{S^n}$ , then  $A \in \tau_{S^n}$  too.

#### ✍ Remark.

**Theorem 1.1** is an early theorem in algebraic topology.  $\mathfrak{M}$

### Corollary 1.1.1 (Dimension is Well-defined).


Given  $U \in \tau_{\mathbb{R}^n}, U' \in \tau_{\mathbb{R}^{n'}}$ , and if  $\exists f : U \rightarrow U'$  homeomorphic, then  $n = n'$ .

*Proof.* If  $n = n'$ , it is trivially true.


If  $n < n'$ , embed  $\mathbb{R}^n$  to  $\mathbb{R}^{n'}$  by  $f : \vec{x} \mapsto (\vec{x}, \vec{0})$ . Via stereographic projection, we

can map homeomorphically

$$\begin{aligned}\phi : U &\mapsto V \subseteq S^{n'}, \\ \phi' : U' &\mapsto V' \subseteq S^{n'}.\end{aligned}$$

Since the compositions are also homeomorphic, we see  $V$  and  $V'$  are homeomorphic. However,  $V'$  is not an open subset of  $R^{n'}$  because of the 0's, contradicting **Theorem 1.1**.  $\times$  

### **Remark.**

Since the definition of a differentiable manifold requires every overlap function to be diffeomorphic, if  $U \cap U' \neq \emptyset$ , their dimensions must be equal via the above corollary. We can bypass this by demanding  $U \cap U' = \emptyset$ , as in the rod and disk case. 

### **Corollary 1.1.2.**

If  $g : V \rightarrow \mathbb{R}^n$  is a continuous injection and  $V \in \tau_{\mathbb{R}^n}$ , then  $g(V)$  is homeomorphic to  $V$ , and  $g(V) \in \tau_{\mathbb{R}^n}$ .

*Proof.* On  $g(V)$ ,  $g$  is surjective and therefore a homeomorphism. Use stereographic projection and the result is obvious. 

## 1.3 Coordinate Functions

### **Definition 1.5 (Coordinate Functions).**

The coordinate functions are the (Euclidean) components of coordinate.

$$\begin{aligned}\phi : U &\rightarrow \mathbb{R}^m & p &\mapsto \phi(p), \\ \phi^\mu : U &\rightarrow \mathbb{R} & \text{s.t. } \phi(p) &= \begin{pmatrix} \phi^1(p) \\ \vdots \\ \phi^m(p) \end{pmatrix}.\end{aligned}$$

An alternative notation is

$$x^\mu := \phi^\mu.$$

✍ **Remark.**

There are (Euclidean) projection functions,

$$u^\mu : \mathbb{R}^n \rightarrow \mathbb{R}.$$

But I think mention it will cause a lot of confusion. Just remember in the future when we say  $\frac{\partial}{\partial u^\mu}$ , we are referring to the Euclidean partial derivative wrt the  $\mu$ -th component. ✍

## 1.4 Manifold With Boundary

### 1.4.1 Generalized Coordinate Charts

**Definition 1.6 (Generalized Coordinate Charts).**

A generalized coordinate chart allows chart that

$$\phi : U \rightarrow \phi(U) \subseteq (-\infty, 0] \times \mathbb{R}^{n-1},$$

$U$  is open, and  $\phi$  is homeomorphic.

✍ **Remark.**

Essentially, this allows a chart to map to "half planes". In this case, even if a set  $\phi(U)$  contains  $\{0\} \times \mathbb{R}^{n-1}$  and therefore not open in the Euclidean topology of  $\mathbb{R}^n$ , it is still considered open in the product topology of  $(-\infty, 0] \times \mathbb{R}^{n-1}$ . ✍

**Definition 1.7 (Manifold With Boundary).**

A manifold with boundary is a manifold whose atlas consists of generalized coordinate charts.

**Definition 1.8 (Boundary Points of a Manifold).**

For all  $p \in \mathcal{M}$  is a boundary point of a manifold with boundary  $\mathcal{M}$  if  $\exists \phi_\alpha \in \Phi$  atlas s.t.  $\phi_\alpha^1(p) = 0$ .

The set of all boundary points of  $\mathcal{M}$  is denoted  $\partial\mathcal{M}$ .

**1.4.2 Boundary is Well-defined****✍ Remark.**

A natural question regarding **Definition 1.8** is that, the definition only asks for existence, but it does not guarantee the existence of

$$\exists \phi_\alpha, \phi_\beta \text{ s.t. } \phi_\alpha^1(p) = 0, \phi_\beta^1(p) \neq 0.$$

We resolve this in the following.

□

**Theorem 1.2.**

Suppose  $U, U'$  are open sets in the product topology  $(-\infty, 0] \times \mathbb{R}^{n-1}$ , and  $\exists f : U \rightarrow U'$  homeomorphic. Then

$$f(U \cap (\{0\} \times \mathbb{R}^{n-1})) = U' \cap (\{0\} \times \mathbb{R}^{n-1}).$$

*Proof.* We show instead that

$$f(U \cap ((-\infty, 0) \times \mathbb{R}^{n-1})) = U' \cap ((-\infty, 0) \times \mathbb{R}^{n-1}).$$

Via **Corollary 1.1.2**, we see  $f(U \cap ((-\infty, 0) \times \mathbb{R}^{n-1}))$  must be an open set in the Euclidean topology of  $\mathbb{R}^n$ . Therefore, it cannot contain  $\{0\} \times \mathbb{R}^{n-1}$ . ▣

**Corollary 1.2.1 (Boundary is Well-defined).**

$\forall p \in \mathcal{M}$ , if  $\exists \phi_\alpha^1(p) = 0$ , then  $\forall \phi \in \Phi$  atlas,  $\phi^1(p) = 0$ .

*Proof.* Make use of the fact that  $\phi \circ \phi_\alpha^{-1}$  is a homeomorphism. ▣

## 2 Tangent Spaces

### 2.1 The Curve Formulation of Tangent Spaces

✈ Remark.

The definition of manifold do not require the entity to be embedded in a higher dimensional space. Therefore, the traditional view of tangency is not valid here.

□

✈ Remark.

The curve formulation remains valid in the infinite-dimensional case, while the algebraic formulation is not. However, in the finite-dimensional case, they are isomorphic.

□

#### 2.1.1 Curves and Vectors

**Definition 2.1 (Curve).**

A curve on  $\mathcal{M}$  is a  $C^\infty$  map,

$$\sigma : (-\epsilon, \epsilon) \rightarrow \mathcal{M}.$$

**Definition 2.2 (Curve Tangency).**

Two curves  $\sigma_1, \sigma_2$  are tangent at  $p \in \mathcal{M}$  if

1.  $\sigma_1(0) = \sigma_2(0) = p$ .
2.  $\frac{d}{dt}(x^i \circ \sigma_1(t))\big|_{t=0} = \frac{d}{dt}(x^i \circ \sigma_2(t))\big|_{t=0}, \quad 1 \leq i \leq m.$

✈ Remark.

Written more compactly,

$$\left. \frac{d}{dt}(\phi \circ \sigma_1) \right|_{t=0} = \left. \frac{d}{dt}(\phi \circ \sigma_2) \right|_{t=0}$$

□

**Definition 2.3 (Tangent Vectors).**

A tangent vector at  $p \in \mathcal{M}$  is an equivalence class of curves where the equivalence relation is that they are tangent. It will be denoted as

$$v = [\sigma].$$

**Definition 2.4 (Tangent Space).**

The tangent space  $T_p\mathcal{M}$  at point  $p$  is the set of all tangent vectors at point  $p$ .

**Definition 2.5 (Tangent Bundle).**

The tangent bundle  $T\mathcal{M}$  is

$$T\mathcal{M} := \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}.$$

### 2.1.2 Addition and Scalar Multiplication

**Definition 2.6 (Addition and Scalar Multiplication).**

Let  $v_1 = [\sigma_1], v_2 = [\sigma_2] \in T_p\mathcal{M}$ , and  $r \in \mathbb{R}$ . Then define

$$\begin{aligned} v_1 + v_2 &:= [\phi^{-1} \circ (\phi \circ \sigma_1 + \phi \circ \sigma_2)], \\ rv_1 &:= [\phi^{-1} \circ (r\phi \circ \sigma_1)]. \end{aligned}$$

**Theorem 2.1.**

The definition **Definition 2.6** is well-defined. That is, they are independent of the choice of chart  $(U, \phi)$  and  $\sigma_1, \sigma_2$  as long as  $v_1 = [\sigma_1]$  and  $v_2 = [\sigma_2]$ .

Therefore,  $T_p \mathcal{M}$  is a real vector space.

*Proof.* Let  $v_1 = [\sigma_1] = v'_1 := [\tau_1], v_2 = [\sigma_2] = v'_2 := [\tau_2]$ . First check (1) of **Definition 2.2**,

$$\begin{aligned} (rv_1 + v_2)(0) &= (\phi^{-1} \circ (r\phi \circ \sigma_1(0) + \phi \circ \sigma_2(0))) \\ &= (\phi^{-1} \circ (r\phi \circ \tau_1(0) + \phi \circ \tau_2(0))) \\ &= (rv'_1 + v'_2)(0), \end{aligned}$$

since  $\phi \circ \sigma_1(0) = \phi \circ \tau_1(0) = \phi(p)$  by equivalence, and the same for  $\sigma_2$ .

Now consider

$$\begin{aligned} \left. \frac{d}{dt}(\phi \circ (rv_1 + v_2)) \right|_{t=0} &= \left. \frac{d}{dt}(r\phi \circ \sigma_1 + \phi \circ \sigma_2) \right|_{t=0} \\ &= r \left. \frac{d}{dt}(\phi \circ \sigma_1) \right|_{t=0} + \left. \frac{d}{dt}(\phi \circ \sigma_2) \right|_{t=0} \\ &= r \left. \frac{d}{dt}(\phi \circ \tau_1) \right|_{t=0} + \left. \frac{d}{dt}(\phi \circ \tau_2) \right|_{t=0} \\ &= \left. \frac{d}{dt}(\phi \circ (rv'_1 + v'_2)) \right|_{t=0}, \end{aligned}$$

since  $\left. \frac{d}{dt}(\phi \circ \sigma_1) \right|_{t=0} = \left. \frac{d}{dt}(\phi \circ \tau_1) \right|_{t=0}$  by equivalence, and the same for  $\sigma_2$ . ▀

**2.1.3 Curves and Derivation****Definition 2.7 (Directional Derivative).**

For any  $f : \mathcal{M} \rightarrow \mathbb{R}$  s.t.  $f \in C^\infty$ , we define

$$v(f) := \left. \frac{d}{dt}(f \circ \sigma(t)) \right|_{t=0},$$

where  $v = [\sigma]$ .

**Theorem 2.2.**

The definition **Definition 2.7** is well-defined. That is,  $v(f)$  is independent of the curve  $\sigma$  chosen as well as  $v = [\sigma]$ .



*Proof.* Let  $v_1 = [\sigma_1] = [\sigma_2] = v_2$ . Then

$$\begin{aligned} v_1(f) &= \left. \frac{d}{dt}(f \circ \sigma_1) \right|_{t=0}, \\ v_2(f) &= \left. \frac{d}{dt}(f \circ \sigma_2) \right|_{t=0}, \\ \frac{d}{dt}(\phi \circ \sigma_1) \Big|_{t=0} &= \frac{d}{dt}(\phi \circ \sigma_2) \Big|_{t=0}. \end{aligned}$$

Then

$$\begin{aligned} v_1(f) &= \left. \frac{d}{dt} \left( \underbrace{(f \circ \phi^{-1})}_{\mathbb{R} \leftarrow \mathbb{R}^m} \circ \underbrace{(\phi \circ \sigma_1)}_{\mathbb{R}^m \leftarrow \mathbb{R}} \right) \right|_{t=0} \\ &= (f \circ \phi^{-1})' \circ (\phi \circ \sigma_1) \cdot (\phi \circ \sigma_1)' \Big|_{t=0} \\ &= (f \circ \phi^{-1})' \circ (\phi \circ \sigma_2) \cdot (\phi \circ \sigma_2)' \Big|_{t=0} \\ &= v_2(f), \end{aligned}$$

since  $\phi \circ \sigma_1(0) = \phi \circ \sigma_2(0) = \phi(p)$ , and  $(\phi \circ \sigma_1)' = (\phi \circ \sigma_2)'$  by equivalence. ▣

## 2.2 The Algebraic Formulation of Tangent Spaces

### 2.2.1 The Space of Derivations

#### Definition 2.8 (Derivation).

A derivation at  $p \in \mathcal{M}$  is a map  $v : C^\infty(\mathcal{M}) \rightarrow \mathbb{R}$  s.t.

1.  $v(rf + g) = rv(f) + v(g)$ , (Linear)
2.  $v(fg) = f(p)v(g) + g(p)v(f)$ , (Leibniz)

where  $f, g \in C^\infty$ .

#### Definition 2.9 (Tangent Space (Algebraic)).

The space of all derivations at  $p \in \mathcal{M}$  is denoted  $D_p\mathcal{M}$ .

**Definition 2.10** (Addition and Scalar Multiplication).

Given  $v_1, v_2 \in D_p \mathcal{M}$ , define

$$\begin{aligned}(v_1 + v_2)(f) &:= v_1(f) + v_2(f) \\ (rv)(f) &:= rv(f).\end{aligned}$$

**Theorem 2.3.**

$D_p \mathcal{M}$  is a real vector space.

**2.2.2 The Basis Tangent Vectors****Definition 2.11** (Basis Tangent Vectors).

We define the basis tangent vectors via derivations by

$$(\partial_\mu)_p f := \frac{\partial}{\partial u^\mu} (f \circ \phi^{-1}(\vec{u})) \Big|_{\vec{u}=\phi(p)}, \quad 1 \leq \mu \leq \dim \mathcal{M}.$$

where  $u \in \mathbb{R}^m$ ,  $f : \mathcal{M} \rightarrow \mathbb{R}$ ,  $f \in C^\infty$ . For the use of  $u^\mu$ , see [Definition 1.5](#).

**Theorem 2.4.**

$$(\partial_\mu)_p x^\nu = \delta^\nu_\mu.$$

*Proof.* Although a simple exercise, it was a good chance to explain the sophisticated notation.

$$(\partial_\mu)_p x^\mu = \frac{\partial}{\partial u^\mu} \left( x^\mu \circ \phi^{-1} \begin{pmatrix} u^1 \\ \vdots \\ u^m \end{pmatrix} \right) \Big|_{\phi(p)}.$$

The coordinate  $u \in \mathbb{R}^m$  was brought to  $\mathcal{M}$  and projected to  $\mathbb{R}^m$  again and taken out the  $\mu$ -th component. So

$$= \frac{\partial}{\partial u^\mu} (u^\mu) \Big|_{\phi(p)} = 1.$$

▣

**Theorem 2.5 (Linear Independence of Basis Tangent Vectors).**

The basis tangent vectors  $(\partial_\mu)_p, 1 \leq \mu \leq \dim \mathcal{M}$  are linear independent.

*Proof.* Suppose  $a^\mu (\partial_\mu)_p = 0$ . Then

$$a^\mu (\partial_\mu)_p (x^\nu) = a^\mu \delta^\nu_\mu = 0(x^\nu) = 0.$$

So  $a^\mu = 0$ . ▣

**Theorem 2.6 (Coordinate Expansion of Tangent Vectors).**

For all  $v \in D_p \mathcal{M}$ , we have

$$v = v^\mu (\partial_\mu)_p,$$

where Einstein notation was used, and  $v^\mu = v(x^\mu)$ .

**✍ Remark.**

The proof was sophisticated and did not teach me much. ☞

## 2.3 Isomorphism of Curves and Derivations

**Theorem 2.7 (Isomorphism of Curves and Derivations).**

Similar to **Definition 2.7**, we define the linear map  $\iota : T_p \mathcal{M} \rightarrow D_p \mathcal{M}$  acting on  $v = [\sigma] \in T_p \mathcal{M}$  by

$$\iota(v)(f) := \left. \frac{d}{dt}(f \circ \sigma(t)) \right|_{t=0}.$$

Then  $\iota$  is a linear isomorphism. Note that  $\text{RHS} \in D_p \mathcal{M}$ .

*Proof.* (linearity) Choose  $\phi$  s.t.  $\phi(p) = 0$ .

$$\begin{aligned}
\iota(rv_1 + v_2)(f) &= \frac{d}{dt}(f \circ \phi^{-1} \circ (r\phi \circ \sigma_1 + \phi \circ \sigma_2)) \Big|_{t=0} \\
&= ((f \circ \phi^{-1})' \circ (r\phi \circ \sigma_1 + \phi \circ \sigma_2) \cdot (r\phi \circ \sigma_1 + \phi \circ \sigma_2)') \Big|_{t=0} \\
&= ((f \circ \phi^{-1})'(0) \cdot ((r\phi \circ \sigma_1)' + (\phi \circ \sigma_2)')) \Big|_{t=0} \\
&= (r(f \circ \phi^{-1})' \circ (\phi \circ \sigma_1) \cdot (\phi \circ \sigma_1)' + (f \circ \phi^{-1})' \circ (\phi \circ \sigma_2) \cdot (\phi \circ \sigma_2)') \Big|_{t=0} \\
&= r\iota(v_1) + \iota(v_2)(f).
\end{aligned}$$

(surjectivity) To show surjectivity, we need to construct a curve for all  $v' \in D_p\mathcal{M}$  s.t.  $\iota(v) = v'$ .

Let  $v' \in D_p\mathcal{M}$  and construct  $\sigma : (-\epsilon, \epsilon) \rightarrow \mathcal{M}$  s.t.

$$\begin{aligned}
\sigma(0) &= p, \\
v^\mu &= v(x^\mu) = \frac{d}{dt}(x^\mu \circ \sigma(t)) \Big|_{t=0}.
\end{aligned}$$

Then

$$v(f) = v^\mu (\partial_\mu)_p f = \frac{d}{dt}(x^\mu \circ \sigma(t)) \Big|_{t=0} (\partial_\mu)_p f.$$

Also,

$$\begin{aligned}
\frac{d}{dt}(f \circ \sigma(t)) \Big|_{t=0} &= \frac{d}{dt}(f \circ \phi^{-1} \circ \phi \circ \sigma(t)) \Big|_{t=0} \\
&= \sum_{\mu=1}^m \frac{\partial}{\partial u^\mu}(f \circ \phi^{-1}) \Big|_{\phi(p)} \frac{d}{dt}(u^\mu \circ \phi \circ \sigma) \Big|_{t=0} \quad (\text{component-wise}) \\
&= \sum_{\mu=1}^m (\partial_\mu)_p f \frac{d}{dt}(x^\mu \circ \sigma) \Big|_{t=0} \\
&= v(f).
\end{aligned}$$

Thus completing the proof. ▣

### ✍ Remark.

It's tempting to use the powerful theorem that surjectivity of linear transformations is equivalent to injectivity. However, the requirement of that theorem is that **the dimensions of the two spaces are equal**. At this point, we don't know the dimension of  $T_p\mathcal{M}$ , so we cannot do this. ▮

## 2.4 Pushforward

### 2.4.1 Definition and Linearity

✍ **Remark.**

The pushforward  $h_* : T_p\mathcal{M} \rightarrow T_{h(p)}\mathcal{N}$  of a specific function  $h : \mathcal{M} \rightarrow \mathcal{N}$  can be thought of as local linearization of the function.

□

**Definition 2.12 (Pushforward).**

Given a function  $h : \mathcal{M} \rightarrow \mathcal{N}$  and  $v \in T_p\mathcal{M}$ , then we define the pushforward  $h_* : T_p\mathcal{M} \rightarrow T_{h(p)}\mathcal{N}$  by

$$h_*(v) := [h \circ \sigma], \quad v = [\sigma].$$

**Theorem 2.8.**

The pushforward operation **Definition 2.12** is well-defined. That is,  $h_*(v_1) = h_*(v_2)$  if  $v_1 = [\sigma_1] = [\sigma_2] = v_2$ .

**Theorem 2.9 (Algebraic Definition of Pushforward).**

The definition of pushforward **Definition 2.12** is equivalent to the following: let  $h : \mathcal{M} \rightarrow \mathcal{N}$ ,  $h_* : D_p\mathcal{M} \rightarrow D_{h(p)}\mathcal{M}$  is defined by,

$$(h_*v)(f) := v(f \circ h).$$

*Proof.* ( $\rightarrow$ )

$$\begin{aligned} h_*(v)(f) &= [h \circ \sigma](f) = \left. \frac{d}{dt}(f \circ h \circ \sigma(t)) \right|_{t=0} \\ &= \left. \frac{d}{dt}((f \circ h) \circ \sigma(t)) \right|_{t=0} \\ &:= v(f \circ h). \end{aligned}$$

( $\leftarrow$ ) This direction is similar. □

**Theorem 2.10 (Linearity of Pushforward).**

The pushforward map  $h_* : T_p \mathcal{M} \rightarrow T_{h(p)} \mathcal{N}$  is linear.

$$h_*(rv_1 + v_2) = rh_*(v_1) + h_*(v_2).$$

*Proof.* (Using [Definition 2.12](#)) Let  $p \in (U, \phi) \subseteq \mathcal{M}$ , and  $h(p) \in (V, \psi) \subseteq \mathcal{N}$ . Choose  $\phi$  s.t.  $\phi(p) = 0$ . It is obvious that  $h_*(rv_1 + v_2)(0) = (rh_*(v_1) + h_*(v_2))(0) = h(p)$ .

Consider

$$\begin{aligned} \left. \frac{d}{dt} \underbrace{(\psi \circ h_*(rv_1 + v_2))}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathbb{R}} \right|_{t=0} &= \left. \frac{d}{dt} \left( \underbrace{\psi \circ h \circ (\phi^{-1})}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathcal{M} \leftarrow \mathbb{R}^m} \circ \underbrace{(r\phi \circ \sigma_1 + \phi \circ \sigma_2)}_{\mathbb{R}^m \leftarrow \mathcal{M} \leftarrow \mathbb{R}} \right) \right|_{t=0} \\ &= (\psi \circ h \circ \phi^{-1})' \circ (r\phi \circ \sigma_1 + \phi \circ \sigma_2) \cdot (r\phi \circ \sigma_1 + \phi \circ \sigma_2)' \Big|_{t=0} \\ &= (\psi \circ h \circ \phi^{-1})'(0) \cdot ((r\phi \circ \sigma_1)' + (\phi \circ \sigma_2)') \Big|_{t=0}. \end{aligned}$$

And

$$\begin{aligned} \left. \frac{d}{dt} \left( \underbrace{\psi \circ (rh_*(v_1) + h_*(v_2))}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathbb{R}} \right) \right|_{t=0} &= \left. \frac{d}{dt} \underbrace{(r\psi \circ h \circ \sigma_1 + \psi \circ h \circ \sigma_2)}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathcal{M} \leftarrow \mathbb{R}} \right|_{t=0} \\ &= \left( \underbrace{r\psi \circ h \circ \phi^{-1}}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathcal{M} \leftarrow \mathbb{R}^m} \circ \underbrace{\phi \circ \sigma_1}_{\mathbb{R}^m \leftarrow \mathcal{M} \leftarrow \mathbb{R}} + \psi \circ h \circ \phi^{-1} \circ \phi \circ \sigma_2 \right)' \Big|_{t=0} \\ &= (r(\psi \circ h \circ \phi^{-1})' \circ (\phi \circ \sigma_1) \cdot (\phi \circ \sigma_1)') \Big|_{t=0} \\ &\quad + ((\psi \circ h \circ \phi^{-1})' \circ (\phi \circ \sigma_2) \cdot (\phi \circ \sigma_2)') \Big|_{t=0} \\ &= (\psi \circ h \circ \phi^{-1})'(0) \cdot (r(\phi \circ \sigma_1)' + (\phi \circ \sigma_2)') \Big|_{t=0}. \end{aligned}$$

So we see the two are equal.

(Using [Theorem 2.9](#))

$$\begin{aligned} (h_*(rv_1 + v_2))(f) &= (rv_1 + v_2)(f \circ h) \\ &= rv_1(f \circ h) + v_2(f \circ h) \\ &= r(h_*v_1)f + (h_*v_2)f. \end{aligned}$$

▣

**Theorem 2.11 (Associativity of Pushforwards).**

Given manifolds  $\mathcal{M}, \mathcal{N}, \mathcal{P}$  and  $h : \mathcal{M} \rightarrow \mathcal{N}$ ,  $k : \mathcal{N} \rightarrow \mathcal{P}$ , then

$$(k \circ h)_* = k_* \circ h_*.$$

### 2.4.2 Jacobian

#### Theorem 2.12 (Local Representative of Pushforward).

Let  $\dim \mathcal{M} = m$ ,  $\dim \mathcal{N} = n$ ,  $h : \mathcal{M} \rightarrow \mathcal{N}$ ,  $\{x^1, \dots, x^m\}$  be the local coordinates of  $\mathcal{M}$  around  $p$ , and  $\{y^1, \dots, y^n\}$  be the local coordinates of  $\mathcal{N}$  around  $h(p)$ . Then

$$h_* v = \sum_{\mu=1}^m \sum_{\nu=1}^n (\partial_\nu)_{h(p)} \left. \frac{\partial h^\nu}{\partial x^\mu} \right|_p v^\mu,$$

where  $J^\nu_\mu := \left. \frac{\partial h^\nu}{\partial x^\mu} \right|_p := (\partial_\mu)_p (y^\nu \circ h)$  is the Jacobian matrix.

*Proof.* First expand  $v$  in terms of local coordinates and use linearity,

$$h_* v = h_* (v^\mu (\partial_\mu)_p) = v^\mu h_* ((\partial_\mu)_p).$$

Expand the result in local coordinates of  $\mathcal{N}$ ,

$$h_* ((\partial_\mu)_p) = \left( h_* (\partial_\mu)_p \right)^\nu (\partial_\nu)_{h(p)}.$$

Using Theorem 2.9,

$$\begin{aligned} \left( h_* (\partial_\mu)_p \right)^\nu &= \left( h_* (\partial_\mu)_p \right) \circ y^\nu \\ &= (\partial_\mu)_p (y^\nu \circ h) \\ &:= (\partial_\mu)_p h^\nu. \end{aligned}$$

So,

$$h_* ((\partial_\mu)_p) = (\partial_\mu)_p h^\nu (\partial_\nu)_{h(p)}.$$

And,

$$h_* v = v^\mu (\partial_\mu)_p h^\nu (\partial_\nu)_{h(p)}.$$

▣

#### Theorem 2.13 (Using Curve to Pushforward).

Given  $c : (-\epsilon, \epsilon) \rightarrow \mathcal{M}$  a curve, and choose the coordinate chart of  $\mathbb{R}$  to be the identity, then

$$c_* \left( \frac{d}{dt} \right)_0 = [c] \in T_p \mathcal{M}.$$

*Proof.* First we clarify what is  $\left(\frac{d}{dt}\right)_0$ . Since on the trivial manifold  $\mathbb{R}$  there is only one coordinate, namely  $t$ , we need not specify the number. Also, considering our functions are scalar valued  $f : \mathcal{M} \rightarrow \mathbb{R}$ , this motivates us to write "total differential".

For all  $f \in C^\infty$ ,

$$c_* \left( \frac{d}{dt} \right)_0 f = \left( \frac{d}{dt} \right)_0 (f \circ c).$$

Since the coordinate chart is the identity,

$$\begin{aligned} \left( \frac{d}{dt} \right)_0 (f \circ c) &= \frac{d}{dt} (f \circ c \circ I) \Big|_{I(t)=0} \\ &= \frac{d}{dt} (f \circ c) \Big|_{t=0} \\ &= [c]f. \end{aligned}$$

▣

**Theorem 2.14 (Contravariancy of Tangent Vectors).**

The components of tangent vectors are contravariant, i.e., given two coordinate charts  $(U, \phi)$  and  $(U', \phi')$  s.t.  $U \cap U' = S \neq \emptyset$ , then on  $S$ ,

$$v'^\nu = \sum_{\mu=1}^m v^\mu \frac{\partial x'^\nu}{\partial x^\mu}.$$

*Proof.* Given that we have the local representative of pushforward at hand, consider the identity pushforward  $\text{id}_* : T_p \mathcal{M} \rightarrow T_p \mathcal{M}$ ,

$$\text{id}_* v = \sum_{\mu=1}^m \sum_{\nu=1}^m v^\mu \frac{\partial x'^\nu}{\partial x^\mu} \Big|_p (\partial_{\nu'})_p.$$

We see immediately that the result holds.

▣



## 3 Vector Fields

### 3.1 Definition

#### Definition 3.1 (Vector Fields).

A vector field  $X$  on  $\mathcal{M}$  is a smooth assignment of a tangent vector  $X_p \in T_p\mathcal{M} \ \forall p \in \mathcal{M}$ .

”Smooth” assignment is defined to be that the Lie derivative [Definition 3.2](#) is smooth.

#### Definition 3.2 (Lie Derivative).

The Lie-derivative of function  $f$  with respect to vector field  $X$  is defined as

$$\mathcal{L}_X f := Xf,$$

and at a specific point  $p \in \mathcal{M}$ ,

$$\mathcal{L}_X f(p) := Xf(p) := X_p f.$$

#### Theorem 3.1 (Properties of Lie Derivative).

The Lie derivative has the following properties,

1.  $X(rf + g) = rXf + Xg$
2.  $X(fg) = fXg + gXf$ .

#### Theorem 3.2 (Component of Vector Field).

Given a chart  $(U, \phi)$  on  $\mathcal{M}$ , we can write

$$X_U = X_U x^\mu \partial_\mu.$$

When the context is clear or **for convenience**, we write

$$X = X x^\mu \partial_\mu := X^\mu \partial_\mu.$$

*Proof.* We know

$$(Xf)(p) = X_p f = X_p x^\mu (\partial_\mu)_p f = (X x^\mu)(p) (\partial_\mu)_p f.$$

▣

✂ **Remark.**

$\partial_\mu$  is a vector field that assigns each point  $p \in \mathcal{M}$  with the vector  $(\partial_\mu)_p \in T_p \mathcal{M}$ .

⌘

**Theorem 3.3 (Contravariancy of Vector Fields).**

Given two coordinate charts  $(U, \phi)$  and  $(U', \phi')$  s.t.  $U \cap U' = S \neq \emptyset$ . On  $S$ ,

$$X^{\nu'} = \sum_{\mu=1}^m X^\mu \frac{\partial x'^\nu}{\partial x^\mu}.$$

Analogous to **Theorem 2.14**.

## 3.2 Lie Bracket

**Definition 3.3 (Composition of Vector Fields).**

We can view  $X : C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$ , and so does  $Y$ . Therefore, we define

$$(X \circ Y)(f) := X(Yf).$$

**Definition 3.4 (Lie Bracket (Commutator)).**

We define the Lie Bracket of two vector fields  $X, Y$  to be

$$[X, Y] := X \circ Y - Y \circ X.$$

In particular,

$$[X, Y](f) = \mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f)$$

✂ Remark.

Lie Bracket **Definition 3.4** is a vector field, while the expression  $X \circ Y$  is not, because it contains second differential terms. See the following proof.

□

**Theorem 3.4 (Lie Bracket Components).**

$$[X, Y]^\mu = (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu).$$

*Proof.* Given  $X = X^\mu \partial_\mu$ ,  $Y = Y^\nu \partial_\nu$ , we try to write the component of  $X \circ Y$ .

$$X \circ Y(f) = X^\mu \partial_\mu (Y^\nu \partial_\nu f).$$

However, notice that

$$\begin{aligned} Y^\nu &:= Y x^\nu \in C^\infty(\mathcal{M}); \\ \partial_\nu &: C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M}), \\ \implies \partial_\nu f &\in C^\infty(\mathcal{M}). \end{aligned}$$

So we need to use the Leibniz property of  $\partial_\mu$  **Definition 2.8** in order to evaluate the second term. Doing this for  $X \circ Y(f)$  and  $Y \circ X(f)$ , we have

$$\begin{aligned} X \circ Y(f) &= X^\mu ((\partial_\mu Y^\nu)(\partial_\nu f) + Y^\nu \partial_\mu \partial_\nu f). \\ Y \circ X(f) &= Y^\nu ((\partial_\nu X^\mu)(\partial_\mu f) + X^\mu \partial_\nu \partial_\mu f). \end{aligned}$$

So if  $\partial_\mu \partial_\nu f = \partial_\nu \partial_\mu f$ , then by subtracting, we can cancel the second order terms, and we are done. We prove so now.

$$\begin{aligned} (\partial_\mu \partial_\nu f)(p) &= \frac{\partial}{\partial u^\mu} ((\partial_\nu f) \circ \phi^{-1}) \Big|_{\phi(p)} \\ &= \frac{\partial}{\partial u^\mu} \left( (\partial_\nu)_{\phi^{-1}(u)} f \right) \Big|_{\phi(p)} \\ &= \frac{\partial}{\partial u^\mu} \left( \frac{\partial}{\partial u^\nu} (f \circ \phi^{-1}) \Big|_u \right) \Big|_{\phi(p)} \\ &= \frac{\partial}{\partial u^\nu} \left( \frac{\partial}{\partial u^\mu} (f \circ \phi^{-1}) \Big|_u \right) \Big|_{\phi(p)} \\ &= (\partial_\nu \partial_\mu f)(p). \end{aligned}$$

□

### Theorem 3.5 (Properties of Lie Brackets).

1.  $[X, Y] = -[Y, X]$  (antisymmetry)
2.  $\sum_{\text{cyc}} [X, [Y, Z]] = 0$ . (Jacobi Identity)

## 3.3 Integral Curves and Flows

### Definition 3.5 (Integral Curve).

Let  $X$  be a vector field on  $\mathcal{M}$ ,  $p \in \mathcal{M}$ . Then an integral curve of  $X$  through  $p$  is a curve  $\sigma : (-\epsilon, \epsilon) \rightarrow \mathcal{M}$  s.t.

$$\begin{aligned}\sigma(0) &= p, \\ \sigma_* \left( \frac{d}{dt} \right)_t &= X_{\sigma(t)}.\end{aligned}$$

### ✍ Remark.

Qualitatively, using **Theorem 2.13**, this pushforward is just  $[\sigma] \in T_{\sigma(t)}\mathcal{M}$ . Therefore, the second condition is saying in some sense that the curve is tangent to the vector field on the manifold. For quantitative description, see below.

□

### Definition 3.6 (Differential Equations of Integral Curve).

The components  $X^\mu$  of  $X$  determine the integral curve  $\sigma$  by the following ODE with boundary conditions,

$$\begin{aligned}X^\mu(\sigma(t)) &= \frac{d}{dt}x^\mu(\sigma(t)) \\ x^\mu(\sigma(0)) &= x^\mu(p), \mu = 1, 2, \dots, m.\end{aligned}$$

### 3.3.1 One-parameter Family of Diffeomorphisms

#### Definition 3.7 (Local 1D Family of Local Diffeomorphisms).

A local, 1D family of local diffeomorphisms at  $p \in \mathcal{M}$  is made up of (1) an open neighborhood  $U$  of  $p$ , (2)  $\epsilon > 0$  (3) a family of diffeomorphisms  $\{ \phi_t \mid |t| < \epsilon \}$ ,  $\phi_t : U \rightarrow \mathcal{M}$  s.t.

1. Every  $\phi_t$  is a smooth function in  $t$  and  $q$ .
2.  $\forall t, s \in \mathbb{R}$  and  $|t|, |s|, |t+s| < \epsilon$ , and  $\forall q \in U$  s.t.  $\phi_t(q), \phi_s(q), \phi_{t+s}(q) \in U$ , we have

$$\phi_s(\phi_t(q)) = \phi_{s+t}(q).$$

3.  $\phi_0(q) = q$ .

#### ✍ Remark.

The first "local" refers to the parameter  $t$ , which is limited to  $(-\epsilon, \epsilon)$ . The second "local" refers to the spatial limitation to  $U$ . You can view  $\phi_t(q)$  as a curve that brings  $t \in (-\epsilon, \epsilon)$  to  $\phi_t(q) \in \mathcal{M}$ .

□

#### Definition 3.8 (Induced Vector Field).

By taking tangents to the curve family [Definition 3.7](#), we have the induced vector field  $X^\phi$  given by

$$X_q^\phi(f) := \left. \frac{d}{dt}(f(\phi_t(q))) \right|_{t=0}$$

#### Theorem 3.6.

The curve family  $t \mapsto \phi_t(q)$  is the integral curve of the induced vector field [Definition 3.8](#)  $X_q^\phi$ .

*Proof.*

$$\begin{aligned} X_{\phi_s(q)}^\phi &= \left. \frac{d}{dt} (f \circ \phi_t \circ \phi_s(q)) \right|_{t=0} \\ &= \left. \frac{d}{dt} (f \circ \phi_{t+s}(q)) \right|_{t=0}. \end{aligned}$$

Let  $u = t + s$ . Then

$$\begin{aligned} X_{\phi_s(q)}^\phi &= \left. \frac{d}{du} (f \circ \phi_u(q)) \right|_{u=s} \\ &= \phi_{q*} \left( \left. \frac{d}{dt} \right|_s \right) f. \end{aligned}$$

▢

### 3.3.2 Local Flows

#### Definition 3.9 (Local Flow).

Let  $X$  be a vector field on open  $U \subseteq \mathcal{M}$ , and  $p \in U$ . A local flow at  $p$  is a local one-parameter family of local diffeomorphisms [Definition 3.7](#) defined on some open  $V \subseteq U$  s.t.  $p \in V$  and the induced vector field [Definition 3.8](#) is  $X$ .

#### ✍ Remark.

Local flows always exist and are unique. In contrast, global flows (which means  $t \in \mathbb{R}$  instead of a restricted interval) may not exist.

⌘

### 3.3.3 Lie Derivative

#### Theorem 3.7 (Interpretation of Lie Bracket).

If  $X, Y$  are two vector fields on  $\mathcal{M}$ , and define the following quantity, which can be interpreted as the change of  $Y$  when following the integral curves of  $X$ , as

$$\left. \frac{d}{dt} (\phi_{-t*}^X(Y)) \right|_{t=0} := \lim_{\epsilon \rightarrow 0} \frac{\phi_{-\epsilon*}^X(Y_{\phi_\epsilon^X(p)}) - Y_p}{\epsilon}.$$

Then,

$$\left. \frac{d}{dt} (\phi_{-t*}^X(Y)) \right|_{t=0} = [X, Y].$$

## 4 Cotangent Spaces

### 4.1 Cotangent Vectors

#### Definition 4.1 (Cotangent Spaces).

The cotangent space  $T_p^*\mathcal{M}$  at  $p \in \mathcal{M}$  is the set of all linear functions  $f : T_p\mathcal{M} \rightarrow \mathbb{R}$ .

Its member is called a cotangent vector.

$$\dim T_p^*\mathcal{M} = \dim T_p\mathcal{M}.$$

#### Definition 4.2 (One-Form).

A one-form on  $\mathcal{M}$  is a smooth assignment of cotangent vectors  $\omega : p \mapsto \omega_p$ .

It may be understood as a covector field.

#### Definition 4.3 (Basis Cotangent Vectors).

The basis cotangent vectors is chosen to be the dual basis of the basis tangent vectors [Definition 2.11](#),

$$(dx^\mu)_p((\partial_\nu)_p) = \delta^\mu_\nu.$$

#### Theorem 4.1 (Coordinate Expression of Cotangent Vectors).

Any  $f \in T_p^*\mathcal{M}$  can be expanded as

$$f = f_\mu (dx^\mu)_p.$$

Any one-form  $\omega$  can be expressed as

$$\omega = \omega_\mu dx^\mu.$$

## 4.2 Pullback

### 4.2.1 Definition

#### Definition 4.4 (Pullback).

Given a function and its pushforward, we define pullback to be the dual of pushforward, i.e.,

$$\begin{aligned} h : \mathcal{M} &\rightarrow \mathcal{N}, \\ h_* : T_p \mathcal{M} &\rightarrow T_{h(p)} \mathcal{N}, \\ h^* : T_{h(p)}^* \mathcal{N} &\rightarrow T_p^* \mathcal{M}, \end{aligned}$$

s.t. given  $f \in T_{h(p)}^* \mathcal{N}$  and  $v \in T_p \mathcal{M}$ ,

$$(h^* f)(v) := f(h_* v).$$

#### ✍ Remark.

Note especially on the direction of original function and its induced pullback. This is crucial to the covariancy of one-forms. □

#### Theorem 4.2.

Given  $\omega$  a one-form on  $\mathcal{N}$ , and a function  $h : \mathcal{M} \rightarrow \mathcal{N}$ , the pullback  $h^* \omega$  is defined as

$$(h^* \omega)(v)_p = \omega(h_* v)_{h(p)}.$$

#### Theorem 4.3 (Associativity of Pullbacks).

Analogous to [Theorem 2.11](#), given manifolds  $\mathcal{M}, \mathcal{N}, \mathcal{P}$  and  $h : \mathcal{M} \rightarrow \mathcal{N}$ ,  $k : \mathcal{N} \rightarrow \mathcal{P}$ , then

$$(k \circ h)^* = k^* \circ h^*.$$



### 4.2.2 Jacobian

**Theorem 4.4 (Local Representative of Pullback).**

Let  $\dim \mathcal{M} = m$ ,  $\dim \mathcal{N} = n$ ,  $h : \mathcal{M} \rightarrow \mathcal{N}$ ,  $\{x^1, \dots, x^m\}$  be the local coordinates of  $\mathcal{M}$  around  $p$ , and  $\{y^1, \dots, y^n\}$  be the local coordinates of  $\mathcal{N}$  around  $h(p)$ . Then

$$h^*\omega = \sum_{\mu=1}^m \sum_{\nu=1}^n \omega_\nu \frac{\partial h^\nu}{\partial x^\mu} \Big|_p (dx^\mu)_p,$$

where  $J^\nu_\mu := \frac{\partial h^\nu}{\partial x^\mu} \Big|_p := (\partial_\mu)_p (y^\nu \circ h)$  is the Jacobian matrix.

*Proof.* We know by [Definition 4.4](#),

$$(h^*\omega)_\mu(p) = h^*\omega(\partial_\mu) = \omega(h_*\partial_\mu).$$

Expand it in local coordinates of  $\mathcal{N}$ ,

$$(h^*\omega)_\mu(p) = \omega_\nu dy^\nu(h_*\partial_\mu).$$

Via similar procedure in [Theorem 2.12](#), we arrive at

$$(h^*\omega)_\mu(p) = \omega_\nu \frac{\partial h^\nu}{\partial x^\mu}.$$

▣

### 4.3 Transformation Properties

#### Theorem 4.5 (Covariancy and Contravariancy).

Given two coordinate charts  $(U, \phi)$  and  $(U', \phi')$  s.t.  $U \cap U' = S \neq \emptyset$ , then on  $S$ ,

$$X^{\nu'} = \sum_{\mu=1}^m \frac{\partial x'^{\nu}}{\partial x^{\mu}} X^{\mu},$$

$$\omega_{\nu'} = \sum_{\mu=1}^m \omega_{\mu} \frac{\partial x^{\mu}}{\partial x'^{\nu}}.$$

If Jacobian matrix is given,

$$J^{\nu'}_{\mu} := \left. \frac{\partial x'^{\nu}}{\partial x^{\mu}} \right|_p := (\partial_{\mu})_p x'^{\nu},$$

$$(J^{-1})^{\mu}_{\nu'} := \left. \frac{\partial x^{\mu}}{\partial x'^{\nu}} \right|_p := (\partial_{\nu'})_p x^{\mu},$$

then,

$$X^{\nu'} = J^{\nu'}_{\mu} X^{\mu}, \quad (\text{contravariant})$$

$$\omega_{\nu'} = \omega_{\mu} (J^{-1})^{\mu}_{\nu'}. \quad (\text{covariant})$$

*Proof.* The contravariant part is proved in [Theorem 2.14](#). Now we turn to the covariant part.

Let  $h = \text{id} : (U, \phi) \subseteq \mathcal{M} \rightarrow (U', \phi') \subseteq \mathcal{M}$ , consider its pullback.

$$(\text{id}^* \omega)_p = \omega_{\nu'} \frac{\partial x'^{\nu}}{\partial x^{\mu}} dx^{\mu}.$$

Then,

$$\omega_{\mu} = \omega_{\nu'} \frac{\partial x'^{\nu}}{\partial x^{\mu}}.$$

Inverting the matrix equation above, we get the desired result. ▣

## 5 Tensors

### Definition 5.1 (Tensors).

If  $\dim \mathcal{M} \neq \infty$ , the tensors of type  $(r, s)$   $T_p^{r,s} \mathcal{M}$  are all the linear functions

$$f : \bigotimes^r T_p^* \mathcal{M} \times \bigotimes^s T_p \mathcal{M} \rightarrow \mathbb{R}.$$

I.e., it eats  $r$  covectors and  $s$  vectors.

### Theorem 5.1 (Dimensions of General Tensor Space).

The dimension of  $T_p^{r,s} \mathcal{M}$  is  $m^r m^s$ . In particular, a basis for the space is,

$$\bigotimes_{1 \leq \mu_1 \dots \mu_r \leq m} \left( \partial_{\mu_i} \right)_p \otimes \bigotimes_{1 \leq \nu_1 \dots \nu_s \leq m} (dx^{\nu_i})_p$$

### ✍ Remark.

For a detailed proof, see Hoffman.

□

## 6 n-Forms

### 6.1 Definition

#### Definition 6.1 (n-Forms).

An n-form is a tensor field of type  $(0, n)$  that is totally skew-symmetric (or alternating, or totally antisymmetric), i.e.,

$$\omega(X_1, X_2, \dots, X_n) = (\text{sgn } \sigma) \omega(X_{\sigma(1)}, \dots, X_{\sigma(n)}), \quad \forall \sigma \in S_n.$$

The set of all n-forms on  $\mathcal{M}$  is denoted as  $\Lambda^n(\mathcal{M})$ .

The set of all forms is  $\Lambda(\mathcal{M}) = \bigoplus_{n=0}^{\dim \mathcal{M}} \Lambda^n(\mathcal{M})$ .

Conventionally, we classify functions as 0-forms.

### 6.2 The Exterior Product

#### Definition 6.2 (Exterior Product).

Given  $\omega_1 \in \Lambda^{n_1}(\mathcal{M})$ ,  $\omega_2 \in \Lambda^{n_2}(\mathcal{M})$ , their exterior product is a  $(n_1 + n_2)$ -form given by,

$$\omega_1 \wedge \omega_2 := \frac{1}{n_1! n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\text{sgn } \sigma) (\omega_1 \otimes \omega_2)_\sigma.$$

Written explicitly,

$$\begin{aligned} (\omega_1 \wedge \omega_2)(X_1, \dots, X_{n_1+n_2}) &:= \\ &\frac{1}{n_1! n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\text{sgn } \sigma) (\omega_1 \otimes \omega_2)(X_{\sigma(1)}, \dots, X_{\sigma(n_1+n_2)}) \end{aligned}$$

#### ✍ Remark.

I'll take the alternating property and associativity of the exterior product for granted. For a detailed proof, see Hoffman. ⌘

**Theorem 6.1 (Commutativity with Pullback).**

Given  $h : \mathcal{M} \rightarrow \mathcal{N}$  and  $\alpha, \beta \in \Lambda(\mathcal{N})$ , then

$$h^*(\alpha \wedge \beta) = (h^*\alpha) \wedge (h^*\beta).$$

**Remark.**

For a "generalized" pullback, we have,

$$(h^*(\alpha))(X_1, \dots, X_{n_1}) = \alpha(h_*X_1, \dots, h_*X_{n_1}).$$

□

*Proof.*

$$\begin{aligned} & (h^*\alpha) \wedge (h^*\beta) \\ &= \frac{1}{n_1!n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\text{sgn } \sigma) \alpha \otimes \beta(h_*X_{\sigma(1)}, \dots, h_*X_{\sigma(n_1+n_2)}). \\ &= \frac{1}{n_1!n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\text{sgn } \sigma) h^* \left( \alpha \otimes \beta(X_{\sigma(1)}, \dots, X_{\sigma(n_1+n_2)}) \right). \\ &= h^* \left( \frac{1}{n_1!n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\text{sgn } \sigma) \alpha \otimes \beta(X_{\sigma(1)}, \dots, X_{\sigma(n_1+n_2)}) \right). \\ &= h^*(\alpha \wedge \beta). \end{aligned}$$

□

**Theorem 6.2 (Skew-Symmetry).**

The exterior product makes  $\Lambda(\mathcal{M})$  a graded algebra with skew-symmetry given by

$$\omega_1 \wedge \omega_2 = (-1)^{n_1 n_2} \omega_2 \wedge \omega_1.$$

*Proof.* In the definition of exterior product, first fix  $\sigma = \sigma_0$  to consider only one term.

When we switch  $\omega_1$  and  $\omega_2$ , we are essentially doing

$$\begin{aligned} & (\omega_2 \otimes \omega_1)(X_{\sigma_0(1)}, \dots, X_{\sigma_0(n_2)}, \underbrace{X_{\sigma_0(n_2+1)}, \dots, X_{\sigma_0(n_1+n_2)}}_{\text{...}}) \\ &= (\omega_1 \otimes \omega_2)(\underbrace{X_{\sigma_0(n_2+1)}, \dots, X_{\sigma_0(n_1+n_2)}}_{\text{...}}, X_{\sigma_0(1)}, \dots, X_{\sigma_0(n_2)}). \end{aligned}$$

Now,

$$\underbrace{1, 2, \dots, n_2}_{\text{...}}, \underbrace{n_2 + 1, \dots, n_1 + n_2}_{\text{...}}$$

$\downarrow$   $n_2$  times

$$\underbrace{n_2 + 1, \dots, n_1 + n_2}_{\text{...}}, \underbrace{1, 2, \dots, n_2}_{\text{...}}$$

$\downarrow$   $(n_1 - 1)n_2$  times

$$\underbrace{n_2 + 1, \dots, n_1 + n_2}_{\text{...}}, \underbrace{1, 2, \dots, n_2}_{\text{...}}$$

So  $n_1 n_2$  transposes can achieve the desired effect. Therefore, every term in the summation is multiplied by  $(-1)^{n_1 n_2}$ , and we get the desired result.  $\square$

### **Theorem 6.3 (Dimension of n-Forms).**

Let  $\dim \mathcal{M} = m$ . If  $1 \leq n \leq m$ , then  $\Lambda^n(\mathcal{M}) = \binom{m}{n}$ . If  $n > m$ , then  $\Lambda^n(\mathcal{M}) = 0$ .

Moreover, a basis for  $\Lambda^n(\mathcal{M})_p$  is given by,

$$(dx^{\mu_1})_p \wedge (dx^{\mu_2})_p \wedge \dots \wedge (dx^{\mu_n})_p, \quad 1 \leq \mu_1 \leq \dots \leq \mu_n \leq m.$$

### **✍ Remark.**

The proof is quite a pleasure to read (and to think of). Please see Hoffman.  $\square$

### 6.3 The Exterior Derivative

#### Definition 6.3 (Exterior Derivative).

Let  $\omega$  be an  $n$ -form on  $\mathcal{M}$ ,  $1 \leq n < \dim \mathcal{M}$ . Then the exterior derivative  $d\omega$  is a  $(n+1)$ -form. Let  $d\omega(\mathbf{X}) = d\omega(X_1, \dots, X_{n+1})$ , then

$$\begin{aligned} d\omega(\mathbf{X}) := & \sum_{i=1}^{n+1} (-1)^{i+1} \mathcal{L}_{X_i}(\omega(\mathbf{X} \setminus \{X_i\})) \\ & + \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], \mathbf{X} \setminus \{X_i, X_j\}). \end{aligned}$$

If  $\omega \in \Lambda^{\dim \mathcal{M}}(\mathcal{M})$ , we define  $d\omega = 0$ .

#### Theorem 6.4.

In particular for a 0-form  $f \in C^\infty(\mathcal{M})$ ,

$$df(X) := \mathcal{L}_X f.$$

In coordinates,

$$df = (\partial_\mu f)(dx^\mu).$$

#### Theorem 6.5.

In particular for a 1-form  $\omega$ ,

$$d\omega(X, Y) = \mathcal{L}_X(\omega(Y)) - \mathcal{L}_Y(\omega(X)) - \omega([X, Y]).$$

#### Theorem 6.6 (Coordinate Expansion for Exterior Derivative).

In local coordinates, if  $\omega = \omega_{\mu_1 \mu_2 \dots \mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n}$ , then

$$d\omega = \partial_\nu \omega_{\mu_1 \mu_2 \dots \mu_n} dx^\nu \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n}$$

**Theorem 6.7** (Exterior Derivative and Product).

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d\omega_2.$$

**Theorem 6.8** (Exterior Derivative and Pullback).

Given  $h : \mathcal{M} \rightarrow \mathcal{N}$ ,  $\omega$  an  $n$ -form on  $\mathcal{N}$ , then

$$d(h^*\omega) = h^*(d\omega).$$

**Theorem 6.9** (Functional Linearity of Exterior Derivative).

Let  $\omega$  be a 1-form on  $\mathcal{M}$ . Then  $d\omega$  satisfies,

$$d\omega(fX, Y) = fd\omega(X, Y), \quad \forall f \in C^\infty(\mathcal{M}),$$

where  $fX$  is a vector field that gives  $(fX)(p) = f(p)X_p$ .

*Proof.* By **Definition 6.3**,

$$d\omega(fX, Y) = \mathcal{L}_{fX}(\omega(Y)) - \mathcal{L}_Y(\omega(fX)) - \omega([fX, Y]).$$

We break it down term by term. Firstly,

$$(\mathcal{L}_{fX}(\omega(Y)))(p) = f(p)X_p(\omega(Y)) = f(p)(\mathcal{L}_X(\omega(Y)))(p).$$

So

$$\mathcal{L}_{fX}(\omega(Y)) = f \cdot \mathcal{L}_X(\omega(Y)).$$

Secondly, we tackle  $\mathcal{L}_Y(\omega(fX))$ . In particular,

$$\omega(fX)(p) = \omega_p(f(p)X_p) = f(p)\omega_p(X_p) = f(p)(\omega(X))(p).$$

Therefore,

$$\mathcal{L}_Y(\omega(fX)) = \mathcal{L}_Y(f \cdot \omega(X)) = (\mathcal{L}_Y f)\omega(X) + f \cdot \mathcal{L}_Y(\omega(X)).$$

Thirdly,

$$\omega([fX, Y]) = \omega((fX) \circ Y - Y \circ (fX)).$$



In particular,

$$((Y \circ (fX))(g))(p) = Y_p((fX)(g)) = Y_p(f \cdot Xg) = (Y_p f)((Xg)(p)) + f(p) \cdot Y_p(Xg).$$

So,

$$Y \circ (fX) = (\mathcal{L}_Y f)X + f \cdot Y \circ X.$$

Substituting back,

$$\begin{aligned} \omega([fX, Y]) &= \omega(f \cdot X \circ Y - (\mathcal{L}_Y f)X - f \cdot Y \circ X) \\ &= \omega(f[X, Y] - (\mathcal{L}_Y f)X) \\ &= f\omega([X, Y]) - (\mathcal{L}_Y f)\omega(X). \end{aligned}$$

Finally,

$$\begin{aligned} d\omega(fX, Y) &= \mathcal{L}_{fX}(\omega(Y)) - \mathcal{L}_Y(\omega(fX)) - \omega([fX, Y]) \\ &= f \cdot \mathcal{L}_X(\omega(Y)) - (\mathcal{L}_Y f)\omega(X) - f \cdot \mathcal{L}_Y(\omega(X)) - f\omega([X, Y]) + (\mathcal{L}_Y f)\omega(X) \\ &= f(\mathcal{L}_X(\omega(Y)) - \mathcal{L}_Y(\omega(X)) - \omega([X, Y])) \\ &= fd\omega(X, Y). \end{aligned}$$

▣

**Corollary 6.9.1 (Local Nature of Exterior Derivative).**

When  $\omega$  is fixed, the value of  $d\omega$  depends only on the local values of vector fields.

$$d\omega(X, Y)(p) = X^\mu(p)Y^\nu(p)d\omega(\partial_\mu, \partial_\nu)(p).$$

*Proof.* Write  $X = X^\mu \partial_\mu$ , noting that  $X^\mu \in C^\infty(\mathcal{M})$ , and use **Theorem 6.9**. ▣

## 6.4 DeRham Cohomology

### Theorem 6.10 (Twice Exterior Differential).

For all  $\omega \in \Lambda^n(\mathcal{M})$ ,  $1 \leq n \leq \dim M$ , we have

$$d^2\omega = 0.$$

### ✍ Remark.

This means

$$\text{Im}(d : \Lambda^{n-1}(\mathcal{M}) \rightarrow \Lambda^n(\mathcal{M})) \subseteq \text{Ker}(d : \Lambda^n(\mathcal{M}) \rightarrow \Lambda^{n+1}(\mathcal{M})).$$

This type of structure is called a differential complex, and is common in many structures. ✍

### Definition 6.4 (Closed Form).

An  $n$ -form  $\omega$  is closed if  $d\omega = 0$ . The set of all closed  $n$ -forms is denoted  $Z^n(\mathcal{M})$ .

### Definition 6.5 (Exact Form).

An  $n$ -form  $\omega$  is exact if  $\omega = d\beta$  for some  $(n-1)$ -form  $\beta$ . The set of all exact  $n$ -forms is denoted  $B^n(\mathcal{M})$ .

### ✍ Remark.

It is guaranteed that  $B^n(\mathcal{M}) \subseteq Z^n(\mathcal{M})$ , that is, exactness implies closure. But how much closed form is not exact is the study of cohomology theory. ✍

### Theorem 6.11 (Poincare's Lemma).

On Euclidean space  $\mathbb{R}^m$ ,

$$B^n(\mathcal{M}) = Z^n(\mathcal{M}), \quad \forall n > 0.$$

**Definition 6.6 (DeRham Cohomology Groups).**

The DeRham cohomology groups  $H^n(\mathcal{M}), 0 \leq n \leq \dim \mathcal{M}$  are the quotient spaces

$$H^n(\mathcal{M}) := Z^n(\mathcal{M})/B^n(\mathcal{M}).$$

**✍ Remark.**

Recall the definition of quotient groups that  $H^n(\mathcal{M})$  consists of elements of form  $z + B^n(\mathcal{M}), z \in Z^n(\mathcal{M})$ .

If all closed forms are exact,  $Z^n(\mathcal{M}) \subseteq B^n(\mathcal{M})$ , then  $H^n(\mathcal{M}) \cong \{0\}$ . □

**Theorem 6.12 (Criterion of Exact ODE).**

On the Euclidean space  $\mathbb{R}^2$ , given a 1-form  $\omega = \omega_1 dx^1 + \omega_2 dx^2$ . Then

$$\omega \in B^1(\mathbb{R}^2) \iff \partial_2 \omega_1 = \partial_1 \omega_2.$$

**✍ Remark.**

This is an important theorem to me, for it connects the "exactness of differential forms" to the familiar notion of "exactness of differential equations".

It also provides the first hints that we are actually integrating forms, and that exterior differentiation of a 0-form resembles gradient in usual vector calculus terms. □

*Proof.* Via Poincare lemma **Theorem 6.11**, on  $\mathbb{R}^2$ , exactness is equivalent to closure. So we need only to determine the condition that  $d\omega = 0$ . Using **Theorem 6.6**,

$$\begin{aligned} d\omega &= \partial_\nu \omega_{\mu_1} dx^\nu \wedge dx^{\mu_1} \\ &= \partial_2 \omega_1 dx^2 \wedge dx^1 + \partial_1 \omega_2 dx^1 \wedge dx^2 \\ &= (\partial_2 \omega_1 - \partial_1 \omega_2) dx^2 \wedge dx^1. \end{aligned}$$

