1 Vector Fields and the Tangent Bundle

1.1 Vector Fields

Definition 1.1 (Vector Fields)**.**
A vector field X on $V \subseteq \mathcal{M}$ is any rule of assigning a tangent vector $X(p) = X_p \in T_p\mathcal{M}$ for all $p \in V$.

Definition 1.2 (Components of a Vector Field)**.**
Let X be a vector field on $V \subseteq \mathcal{M}$, $\dim \mathcal{M} = m$. For any chart $\phi : U \to \phi(U) \in \Phi$ with induced coordinates $x^1, \dots, x^m$ and any $p \in V \cap U$, the decomposition $X(p) = X^j(p) \left(\partial_j\right)_p$ is unique, and therefore we write

$$X = X^j \partial_j$$

on $V \cap U$, and $X^j : V \cap U \to \mathbb{R}$ are called the components of X on $V \cap U$.

Definition 1.3 (Smoothness of Vector Field)**.**
A vector field X on $V \subseteq \mathcal{M}$ is C^k near $p \in V$ iff $\exists\, \phi \in \Phi$ s.t. $p \in U_\phi$ and all the components X^j induced by ϕ are C^k. That is,

$$X^j \circ \phi^{-1} \text{ are } C^k \text{ on } \phi(V \cap U).$$

1.2 Tangent Bundle

1.2.1 Definition

Definition 1.4 (Tangent Bundle)**.**
The tangent bundle $T\mathcal{M}$ is

$$T\mathcal{M} := \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}.$$

Remark.
Why not $T\mathcal{M} := \left\{ T_p\mathcal{M} \;\middle|\; p \in \mathcal{M} \right\}$? ♍

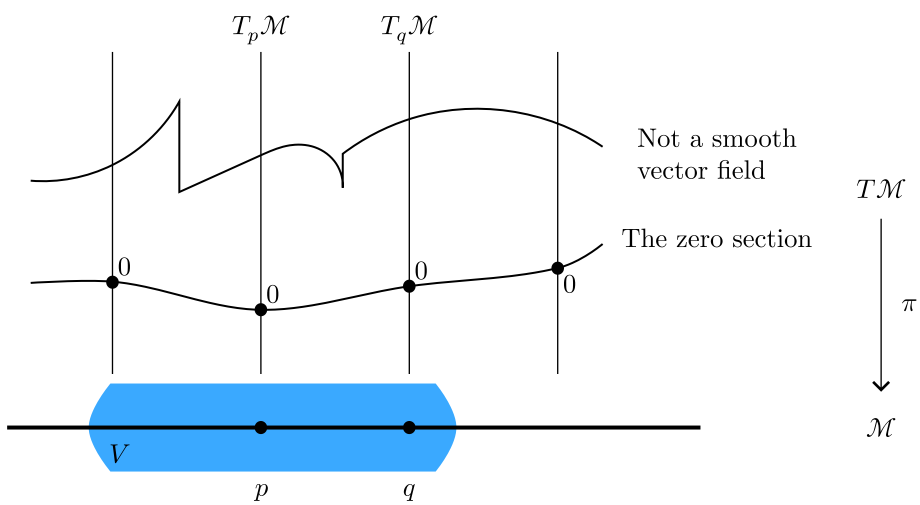
1.2.2 Projection

Definition 1.5 (Canonical Projection)**.**
The canonical projection is the map

$$\begin{aligned} \pi : T\mathcal{M} &\to \mathcal{M} \\ T_p\mathcal{M} &\mapsto p. \end{aligned}$$

Definition 1.6 (Alternative Definition of Vector Fields)**.**
A tangent vector field X on $V \subseteq \mathcal{M}$ is a map $X : V \to T\mathcal{M}$ s.t. $(\pi \circ X)(p) = p$ for all $p \in V$.

Remark.
Let's see why vector fields are often called a "cross-section" of a tangent bundle.



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1.2.3 Topological and Manifold Structure

Definition 1.7 (Smooth Structure on Tangent Bundle)**.**
Let a manifold $\mathcal{M}$ with dimension m and atlas Φ. Consider a chart $\phi_j \in \Phi$:

$U_j \to \mathbb{R}^m$. We define a chart $\tilde{\phi}_j$ for the tangent bundle $T\mathcal{M}$ accordingly,

$$\begin{aligned}\tilde{\phi}_j : \tilde{\pi}(U_j) \subseteq T\mathcal{M} &\to \mathbb{R}^{2m} \\ (p, v^i \partial_i) &\mapsto (\phi_j(p), v^1, \dots, v^m),\end{aligned}$$

where $\tilde{\pi}$ is the inverse set map of the canonical projection function Definition 1.5. We thus see the tangent bundle $T\mathcal{M}$ is itself a manifold of dimension $2m$.

Also, we define the topology of the tangent bundle by

$$A \subseteq T\mathcal{M} \text{ is open iff } \tilde{\phi}_j(A \cap \tilde{\pi}(U_j)) \text{ is open.}$$

Theorem 1.1.

The smooth structure given by Definition 1.7 is unique in the sense of,

1. $\pi : T\mathcal{M} \to \mathcal{M}$ is C^∞,
2. For all open sets $V \subseteq \mathcal{M}$ and any vector field X on V, X is $C^\infty \iff X : V \to T\mathcal{M}$ is C^∞.

Remark.

At first thought, one may think of $T\mathcal{M} \simeq \mathcal{M} \times \mathbb{R}^m$. This is not the case, as one can consider the Moebius strip. The tangent bundle is only locally isomorphic to $U \times \mathbb{R}^m$.

If indeed $T\mathcal{M} \simeq \mathcal{M} \times \mathbb{R}^m$, then the bundle is called trivial tangent bundle, and the manifold is called parallelizable. ♍

Definition 1.8 (Lie Derivative)**.**

The Lie-derivative of function f with respect to vector field X is defined as

$$\mathcal{L}_X f := Xf,$$

and at a specific point $p \in \mathcal{M}$,

$$\mathcal{L}_X f(p) := Xf(p) := X_p f.$$

Theorem 1.2 (Properties of Lie Derivative)**.**
The Lie derivative has the following properties,

1. $X(rf + g) = rXf + Xg$
2. $X(fg) = fXg + gXf$.

Proof. We know

$$(Xf)(p) = X_p f = X_p x^\mu \left(\partial_\mu\right)_p f = (Xx^\mu)(p) \left(\partial_\mu\right)_p f.$$

▨

Remark.
∂_μ is a vector field that assigns each point $p \in \mathcal{M}$ with the vector $\left(\partial_\mu\right)_p \in T_p\mathcal{M}$.

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Theorem 1.3 (Contravariancy of Vector Fields)**.**
Given two coordinate charts (U, ϕ) and (U', ϕ') s.t. $U \cap U' = S \neq \varnothing$. On S,

$$X^{\nu'} = \sum_{\mu=1}^{m} X^\mu \frac{\partial x'^\nu}{\partial x^\mu}.$$

Analogous to **??**.

1.3 Lie Bracket

Definition 1.9 (Composition of Vector Fields)**.**
We can view $X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, and so does Y. Therefore, we define

$$(X \circ Y)(f) := X(Yf).$$

Definition 1.10 (Lie Bracket (Commutator))**.**
We define the Lie Bracket of two vector fields X, Y to be

$$[X, Y] := X \circ Y - Y \circ X.$$

In particular,

$$[X,Y](f) = \mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f)$$

Remark.

Lie Bracket Definition 1.10 is a vector field, while the expression $X \circ Y$ is not, because it contains second differential terms. See the following proof.

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Theorem 1.4 (Lie Bracket Components)**.**

$$[X,Y]^\mu = (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu).$$

Proof. Given $X = X^\mu \partial_\mu, Y = Y^\nu \partial_\nu$, we try to write the component of $X \circ Y$.

$$X \circ Y(f) = X^\mu \partial_\mu \left(Y^\nu \partial_\nu f\right).$$

However, notice that

$$\begin{aligned} Y^\nu &:= Y x^\nu \in C^\infty(\mathcal{M}); \\ \partial_\nu &: C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), \\ &\implies \partial_\nu f \in C^\infty(\mathcal{M}). \end{aligned}$$

So we need to use the Leibniz property of ∂_μ **??** in order to evaluate the second term. Doing this for $X \circ Y(f)$ and $Y \circ X(f)$, we have

$$\begin{aligned} X \circ Y(f) &= X^\mu \left((\partial_\mu Y^\nu)(\partial_\nu f) + Y^\nu \partial_\mu \partial_\nu f\right). \\ Y \circ X(f) &= Y^\nu \left((\partial_\nu X^\mu)(\partial_\mu f) + X^\mu \partial_\nu \partial_\mu f\right). \end{aligned}$$

So if $\partial_\mu \partial_\nu f = \partial_\nu \partial_\mu f$, then by subtracting, we can cancel the second order terms, and we are done. We prove so now.

$$\begin{aligned} (\partial_\mu \partial_\nu f)(p) &= \frac{\partial}{\partial u^\mu} \left((\partial_\nu f) \circ \phi^{-1}\right)\Big|_{\phi(p)} \\ &= \frac{\partial}{\partial u^\mu} \left((\partial_\nu)_{\phi^{-1}(u)} f\right)\Big|_{\phi(p)} \\ &= \frac{\partial}{\partial u^\mu} \left(\frac{\partial}{\partial u^\nu}(f \circ \phi^{-1})\Big|_u\right)\Bigg|_{\phi(p)} \\ &= \frac{\partial}{\partial u^\nu} \left(\frac{\partial}{\partial u^\mu}(f \circ \phi^{-1})\Big|_u\right)\Bigg|_{\phi(p)} \\ &= (\partial_\nu \partial_\mu f)(p). \end{aligned}$$

▨

Theorem 1.5 (Properties of Lie Brackets).

1. $[X,Y] = -[Y,X]$ (antisymmetry)
2. $\sum_{\text{cyc}}[X,[Y,Z]] = 0.$ (Jacobi Identity)

1.4 Integral Curves and Flows

Definition 1.11 (Intergral Curve).
Let X be a vector field on $\mathcal{M}$, $p \in \mathcal{M}$. Then an integral curve of X through p is a curve $\sigma : (-\epsilon, \epsilon) \to \mathcal{M}$ s.t.

$$\sigma(0) = p,$$
$$\sigma_* \left(\frac{d}{dt}\right)_t = X_{\sigma(t)}.$$

Remark.
Qualitatively, using **??**, this pushforward is just $[\sigma] \in T_{\sigma(t)}\mathcal{M}$. Therefore, the second condition is saying in some sense that the curve is tangent to the vector field on the manifold. For quantitative description, see below.

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Definition 1.12 (Differential Equations of Integral Curve).
The components X^μ of X determine the integral curve σ by the following ODE with boundary conditions,

$$X^\mu(\sigma(t)) = \frac{d}{dt}x^\mu(\sigma(t))$$
$$x^\mu(\sigma(0)) = x^\mu(p), \mu = 1, 2, \ldots, m.$$

1.4.1 One-parameter Family of Diffeomorphisms

Definition 1.13 (Local 1D Family of Local Diffeomorphisms)**.**
A local, 1D family of local diffeomorphisms at $p \in \mathcal{M}$ is made up of (1) an open neighborhood U of p, (2) $\epsilon > 0$ (3) a family of diffeomorphisms $\{ \phi_t \mid |t| < \epsilon \}$, $\phi_t : U \to \mathcal{M}$ s.t.

1. Every ϕ_t is a smooth function in t and q.
2. $\forall t, s \in \mathbb{R}$ and $|t|, |s|, |t+s| < \epsilon$, and $\forall q \in U$ s.t. $\phi_t(q), \phi_s(q), \phi_{t+s}(q) \in U$, we have
$$\phi_s(\phi_t(q)) = \phi_{s+t}(q).$$
3. $\phi_0(q) = q$.

Remark.
The first "local" refers to the parameter t, which is limited to $(-\epsilon, \epsilon)$. The second "local" refers to the spatial limitation to U. You can view $\phi_t(q)$ as a curve that brings $t \in (-\epsilon, \epsilon)$ to $\phi_t(q) \in \mathcal{M}$.

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Definition 1.14 (Induced Vector Field)**.**
By taking tangents to the curve family Definition 1.13, we have the induced vector field X^ϕ given by

$$X^\phi_q(f) := \frac{d}{dt}(f(\phi_t(q)))\Big|_{t=0}$$

Theorem 1.6.
The curve family $t \mapsto \phi_t(q)$ is the integral curve of the induced vector field Definition 1.14 X^ϕ_q.

Proof.
$$\begin{aligned} X^\phi_{\phi_s(q)} &= \frac{d}{dt}(f \circ \phi_t \circ \phi_s(q))\Big|_{t=0} \\ &= \frac{d}{dt}(f \circ \phi_{t+s}(q))\Big|_{t=0}. \end{aligned}$$

Let $u = t + s$. Then

$$X^{\phi}_{\phi_s(q)} = \frac{d}{du}(f \circ \phi_u(q))\Big|_{u=s}.$$
$$= \phi_{q*}\left(\frac{d}{dt}\right)_s f.$$

1.4.2 Local Flows

Definition 1.15 (Local Flow)**.**
Let X be a vector field on open $U \subseteq \mathcal{M}$, and $p \in U$. A local flow at p is a local one-parameter family of local diffeomorphisms Definition 1.13 defined on some open $V \subseteq U$ s.t. $p \in V$ and the induced vector field Definition 1.14 is X.

Remark.
Local flows always exist and are unique. In contrast, global flows (which means $t \in \mathbb{R}$ instead of a restricted interval) may not exist.

1.4.3 Lie Derivative

Theorem 1.7 (Interpretation of Lie Bracket)**.**
If X, Y are two vector fields on $\mathcal{M}$, and define the following quantity, which can be interpreted as the change of Y when following the integral curves of X, as

$$\frac{d}{dt}(\phi^X_{-t*}(Y))\Big|_{t=0} := \lim_{\epsilon \to 0} \frac{\phi^X_{-\epsilon*}(Y_{\phi^X_\epsilon(p)}) - Y_p}{\epsilon}.$$

Then,

$$\frac{d}{dt}(\phi^X_{-t*}(Y))\Big|_{t=0} = [X, Y].$$