1 Tensors

Definition 1.1 (Tensors)**.**
If $\dim \mathcal{M} \neq \infty$, the tensors of type (r, s) $T_p^{r,s}\mathcal{M}$ are all the linear functions

$$f : \bigtimes^{r} T_p^*\mathcal{M} \times \bigtimes^{s} T_p\mathcal{M} \to \mathbb{R}.$$

I.e., it eats r covectors and s vectors.

Theorem 1.1 (Dimensions of General Tensor Space)**.**
The dimension of $T_p^{r,s}\mathcal{M}$ is $m^r m^s$. In particular, a basis for the space is,

$$\bigotimes_{1\leq\mu_1\cdots\mu_r\leq m} \left(\partial_{\mu_i}\right)_p \otimes \bigotimes_{1\leq\nu_1\cdots\nu_s\leq m} (dx^{\nu_i})_p$$

Remark.
For a detailed proof, see Hoffman. ♍

Theorem 1.2 (Transformation Properties of Tensor Fields)**.**
Given a manifold $\mathcal{M}$ of dimension m, choose two overlapping charts $\phi : U \to V$ with local coordinates $x^1, \dots, x^m$, $\phi' : U' \to V'$ with ${x'}^1, \dots, {x'}^m$. A tensor field T with local representation on U given by

$$T^{\mu_1\cdots\mu_r}{}_{\nu_1\dots\nu_s}\,\partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_r} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_s}$$

transforms like

$$T^{\mu'_1\cdots\mu'_r}{}_{\nu'_1\dots\nu'_s} = T^{\mu_1\cdots\mu_r}{}_{\nu_1\dots\nu_s}\frac{\partial {x'}^{\mu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial {x'}^{\mu'_r}}{\partial x^{\mu_r}}\frac{\partial x^{\nu_1}}{\partial {x'}^{\nu'_1}}\cdots\frac{\partial x^{\nu_1}}{\partial {x'}^{\nu'_s}}$$