1 Integration of Differential Forms

1.1 Partition of Unity

Definition 1.1 (Support)**.**
Let X be a topological space, and $f : X \to \mathbb{R}$. Then the support of f is defined as

$$\operatorname{supp} f := \{ x \in X \mid f(x) \neq 0 \} .$$

Theorem 1.1 (Partition of Unity)**.**
Let $\mathcal{M}$ be a C^∞ manifold with dimension m with atlas Φ.
Let $\Phi = \{ \phi_j \mid \phi_j : V_j \to \phi_j(V_j), j \in J \}$.
Then it is possible to construct a set of C^∞ functions $\rho_j, j \in J$ s.t.

$$1 = \sum_{j \in J} \rho_j, \ \operatorname{supp} \rho_j \subseteq V_j$$

1.2 Orientation

1.2.1 Definition

Definition 1.2 (Compatible Coordinate Charts)**.**
Given a manifold $\mathcal{M}$, its two coordinate charts are called compatible (have the same orientation) if,

$$\det J > 0.$$

Where J is the Jacobian matrix **??**.
If the manifold has a maximal compatible atlas, then we say the manifold is orientable, and we may call its corresponding orientation positive and denote the atlas Φ_+.

Theorem 1.2.
A manifold has either no orientation (any atlas is not compatible) or two orientations.

Theorem 1.3 (Orientability and Existence of Forms of Highest Degree).
A manifold is orientable iff there exists a nowhere vanishing differential form of the highest degree.

1.2.2 Positively Oriented Boundary

Definition 1.3 (Positively Oriented Boundary).
Let $\mathcal{M}$ be a orientable C^∞ manifold with dimension m, positively oriented by compatible atlas Φ_+. Define coordinate charts on $\partial\mathcal{M}$ from Φ as follows,

$$\phi^{\partial\mathcal{M}} : U_\phi \cap \partial\mathcal{M} \to \mathbb{R}^{m-1},$$

Then $\Phi_+^{\partial\mathcal{M}} := \left\{ \phi^{\partial\mathcal{M}} \;\middle|\; \phi \in \Phi_+ \right\}$ determines an orientation on $\partial\mathcal{M}$, called the positive orientation.

1.3 Pseudoforms (Densities)

1.3.1 Pseudoscalars

Definition 1.4 (Space of Pseudoscalars).
Given a manifold $\mathcal{M}$ and a point p on it. Choose an arbitrary orientation $o_p = \Phi_+$ of $T_p\mathcal{M}$. On the set $\mathbb{R} \times \{o_p, -o_p\}$, define an equivalence relation,

$$\equiv : (c, o_p) \equiv (-c, -o_p).$$

We denote $\tilde{P}_p := \mathbb{R} \times \{o_p, -o_p\}/\equiv$, and call it the space of pseudoscalars. Under a choice of local orientation o, we often denote its element $(c, o) \equiv (-c, -o)$ by $\tilde{c}$.

Theorem 1.4 (Pseudoscalar Vector Space).
The space $\tilde{P}_p$ forms a vector space under the operations,

$$\begin{aligned} &(a, o) + (b, o) = (a + b, o), \\ &c(a, o) = (ca, o). \end{aligned}$$

Specifically, $\dim \tilde{P}_p = 1$, since $\forall \tilde{c} \in \tilde{P}_p$, $\exists\, c \in \mathbb{R}$ s.t. $\tilde{c} = c\tilde{1}$.

Theorem 1.5 (Pseudoscalar Vector Bundle)**.**
On the space $\tilde{P} = \bigcup_{p\in\mathcal{M}} \tilde{P}_p$, define the canonical projection $\pi : \tilde{P} \to \mathcal{M}$ by $\tilde{P}_p \mapsto p$.
For any chart $\phi : U \to \mathbb{R}^m \in \Phi$ of the underlying manifold $\mathcal{M}$, it induces an orientation on $T_p\mathcal{M}$, denoted by o_ϕ. We define the local trivialization by

$$\begin{aligned} \tilde{\phi} : \tilde{\pi}(U) &\to U \times \mathbb{R} \\ (p, (c, o_\phi)) &\mapsto (p, c). \end{aligned}$$

And this makes pseudoscalars a vector bundle on any manifold $\mathcal{M}$.

Theorem 1.6 (Pseudoscalars Under Coordinate Transformation)**.**
Given two charts $\phi : U \to \mathbb{R}^m, \psi : V \to \mathbb{R}^m \in \Phi$ of a manifold $\mathcal{M}$, define local trivialization of the pseudoscalar bundle $\tilde{\phi}, \tilde{\psi}$ according to Theorem 1.5. If we throw away the point part and retain only the scalar part, we have

$$\tilde{\phi}((p, (c, o_\phi))) = (\operatorname{sgn}\det J)\tilde{\psi}((p, (c, o_\phi)))$$

Proof. If $o_\phi = o_\psi$, that is, $\det J > 0$ and their orientations agree, then $(c, o_\phi) \equiv (c, o_\psi) \mapsto c$.

But if $o_\phi = -o_\psi$, that is, $\det J < 0$ and their orientations disagree, then $(c, o_\phi) \equiv (-c, o_\psi)$. The former is mapped to c, the latter is mapped to $-c$.

This shows that pseudoscalars "flip sign" under charts of different orientation. ▨

1.3.2 Definition

Definition 1.5 (Pseudoforms)**.**
The space of pseudoforms $\tilde{\Lambda}^k(\mathcal{M})$ is defined by the space of differential forms tensored with (twisted by) the pseudoscalar bundle,

$$\tilde{\Lambda}^k(\mathcal{M}) = \bigwedge^k T^*\mathcal{M} \otimes \tilde{P}(\mathcal{M}).$$

This means, pseudoforms are defined locally as

$$\tilde{\omega} = \tilde{1}\omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

1.3.3 Pseudoforms and Forms on Orientable Manifolds

Definition 1.6 (Pseudoforms to Forms)**.**
For a pseudoform $\tilde{\omega} \in \tilde{\Lambda}^k(\mathcal{M})$ on an orientable manifold $\mathcal{M}$ of degree m positively oriented by the atlas Φ_+, the positive orientation is chosen in a continuous manner. Then $\tilde{P}(\mathcal{M})$ is isomorphic to $\mathbb{R}$ by the positive orientation in a continuous manner, so we can say

$$\begin{aligned} o_+ : \tilde{\Lambda}^k(\mathcal{M}) &\to \Lambda^k(\mathcal{M}) \\ \tilde{1}\omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} &\mapsto \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \end{aligned}$$

defines a smooth differential form on $\mathcal{M}$.

1.4 Integration of Forms of Highest Degree

Definition 1.7 (Integration)**.**
Let $\mathcal{M}$ be a paracompact C^∞ manifold of dimension m. Choose a C^∞ partition of unity $\rho_j, j \in J$ of $\mathcal{M}$ s.t. $\operatorname{supp} \rho_j \subseteq U_{\phi_j} := U_j$.
Let a pseudo-m-form $\tilde{\omega} \in \tilde{\Lambda}^m(\mathcal{M})$ has local expression $\tilde{\omega}_{\phi_j} = \tilde{1} f_j(x) dx_j^1 \wedge \dots \wedge dx_j^m$, we say

$$\int_{\mathcal{M}} \tilde{\omega} = \sum_{j \in J} \int_{(-\infty,0] \times \mathbb{R}^{m-1}} \tilde{\phi}_j(\tilde{1})(\rho_j \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m$$

if the finite sum exists and has the same value for all choices of ρ_j and ϕ_j.

Remark.
The following theorem reveals why we integrate pseudoforms, not usual forms.
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Theorem 1.7 (Criterion of Existence of Integral)**.**
If $\operatorname{supp} \tilde{\omega}$ is compact, then $\int_{\mathcal{M}} \tilde{\omega}$ exists.

Proof. Let two sets of coordinate charts be

$$\begin{aligned} \phi_j &: U_j \to V_j, j \in J \\ \phi'_k &: U'_k \to V'_k, k \in K. \end{aligned}$$

And cooresponding partition of unity be ρ_j, ρ'_k.

(The goal) Show

$$\begin{aligned} &\sum_{j \in J} \int \tilde{\phi}_j(\tilde{1})(\rho_j \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m \\ =&\sum_{k \in K} \int \tilde{\phi}'_k(\tilde{1})(\rho'_k \circ {\phi'_k}^{-1})(x') f'_k(x') d{x'}^1 \dots d{x'}^m \end{aligned}$$

(Split using ρ'_k)

$$\begin{aligned} \int_{\mathcal{M}} \omega &= \sum_{j \in J} \int \tilde{\phi}_j(\tilde{1})(\rho_j \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m \\ &= \sum_{j \in J} \int \sum_{k \in K} \tilde{\phi}_j(\tilde{1})(\rho'_k \circ \phi_j^{-1})(x)(\rho_j \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m. \end{aligned}$$

Since the sum is finite, and $\operatorname{supp} \omega$ is compact, and therefore the integral is not improper; thus, there can be no limit or Fubini problems on exchanging sums and integrals. So

$$\int_{\mathcal{M}} \omega = \sum_{j \in J} \sum_{k \in K} \int \tilde{\phi}_j(\tilde{1})(\rho_j \rho'_k \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m$$

(Change of variables) First fix j, k.

$$\begin{aligned} &\int \tilde{\phi}_j(\tilde{1})(\rho_j \rho'_k \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m \\ =&\int \tilde{\phi}_j(\tilde{1})(\rho_j \rho'_k \circ \phi_j^{-1})(\phi_j \circ {\phi'_k}^{-1}(x')) f_j(\phi_j \circ {\phi'_k}^{-1}(x')) \left| \det \left(\frac{\partial x}{\partial x'} \right) \right| d{x'}^1 \dots d{x'}^m \\ =&\int \tilde{\phi}_j(\tilde{1})(\rho_j \rho'_k \circ {\phi'_k}^{-1}(x')) f_j(\phi_j \circ {\phi'_k}^{-1}(x')) \left| \det \left(\frac{\partial x}{\partial x'} \right) \right| d{x'}^1 \dots d{x'}^m \end{aligned}$$

(Pseudoscalar) From Theorem 1.6,

$$\tilde{\phi}_j(\tilde{1}) = (\operatorname{sgn}\det J)\tilde{\phi}'_k(\tilde{1})$$

we see

$$\begin{aligned}
&\tilde{\phi}'_k(\tilde{1}) f'_j(x') dx^1 \wedge \cdots \wedge dx^m \\
=&(\operatorname{sgn}\det J)\tilde{\phi}_j(\tilde{1}) f_j(\phi_j \circ {\phi'_k}^{-1}(x')) \left(\frac{\partial x^1}{\partial {x'}^{\ell_1}} d{x'}^{\ell_1}\right) \wedge \cdots \wedge \left(\frac{\partial x^m}{\partial {x'}^{\ell_m}} d{x'}^{\ell_m}\right) \\
=&(\operatorname{sgn}\det J) \sum_{\sigma \in S_m} \tilde{\phi}_j(\tilde{1}) f_j(\phi_j \circ {\phi'_k}^{-1}(x'))(\operatorname{sgn}\sigma) \frac{\partial x^1}{\partial {x'}^{\sigma(1)}} \dots \frac{\partial x^m}{\partial {x'}^{\sigma(m)}} d{x'}^1 \wedge \cdots \wedge d{x'}^m \\
=&(\operatorname{sgn}\det J)\tilde{\phi}_j(\tilde{1}) f_j(\phi_j \circ {\phi'_k}^{-1}(x')) \det\left(\frac{\partial x}{\partial x'}\right) d{x'}^1 \wedge \cdots \wedge d{x'}^m \\
=&\tilde{\phi}_j(\tilde{1}) f_j(\phi_j \circ {\phi'_k}^{-1}(x')) \left|\det\left(\frac{\partial x}{\partial x'}\right)\right| d{x'}^1 \wedge \cdots \wedge d{x'}^m
\end{aligned}$$

Therefore, the integral

$$\begin{aligned}
&\int (\rho_j \rho'_k \circ {\phi'_k}^{-1}(x'))\tilde{\phi}_j(\tilde{1}) f_j(\phi_j \circ {\phi'_k}^{-1}(x')) \left|\det\left(\frac{\partial x}{\partial x'}\right)\right| d{x'}^1 \dots d{x'}^m \\
=&\int (\rho_j \rho'_k \circ {\phi'_k}^{-1}(x'))\tilde{\phi}'_k(\tilde{1}) f'_k(x') d{x'}^1 \dots d{x'}^m
\end{aligned}$$

(Closing) By moving the sum wrt $j \in J$ into the integral and using the property of partition of unity, the proof is completed. ▨

1.5 Integration of Forms of Lower Degree

1.5.1 Definition

Definition 1.8 (Integration of Lower Degree Forms).
Let $\mathcal{Z}$ be an oriented C^∞ manifold of dimension d, $f : \mathcal{Z} \to \mathcal{M}$ be a C^∞ map to a C^∞ manifold $\mathcal{M}$ of dimension m.
Let $\omega \in \Lambda^d(\mathcal{M})$, we define

$$\int_{\mathcal{Z}} \omega := \int_{\mathcal{Z}} f^* \omega$$

using the positive orientation of $\mathcal{Z}$, if it exists and the pullback function f is clear from context.

Remark.
Let's look at an example. Choose $\mathcal{M} = \mathbb{R}^2$, $\mathcal{Z} = [-1, 1]$, $\omega = dx^2$, and $f : \mathcal{Z} \to \mathcal{M}$ defined trivially by $p \mapsto (0, p)$.
If we choose the positive orientation by setting $\mathrm{id} \in \Phi_+$, then we see

$$\begin{aligned} \eta &:= f^*\omega = dx^2 \\ \tilde{\eta} &= dx^2 \\ \int_{\mathcal{Z}} \tilde{\eta} &= 1. \end{aligned}$$

If we choose the positive orientation by setting $-\mathrm{id} \in \Phi_+$, then

$$\begin{aligned} \eta' &:= f^*\omega = dx^2 \\ \tilde{\eta}' &= -dx^2 \\ \int_{\mathcal{Z}} \tilde{\eta}' &= -1. \end{aligned}$$

♍

1.5.2 Stoke's theorem

Theorem 1.8 (Stoke's Theorem)**.**
If $\mathcal{M}$ is an oriented C^∞ manifold of dimension m and $\omega \in \Lambda^{m-1}(\mathcal{M})$, then

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega := \int_{\partial\mathcal{M}} i^*\omega,$$

where $i : \partial\mathcal{M} \to \mathcal{M}$ is just the immersion map, $i : p \mapsto p$.

Proof. (Partition Using Charts) Choose a C^∞ partition of unity Theorem 1.1 $\rho_j, j \in J$ s.t. $\operatorname{supp} \rho_j$ are compact and $\operatorname{supp} \rho_j \subseteq U_{\phi_j} := U_j$.

Now $\omega = \sum_{j\in J} \rho_j\omega$ is a finite sum. So it suffices to show that, if $\eta \in \Lambda^{m-1}(\mathcal{M})$, $\operatorname{supp} \eta$ compact and $\operatorname{supp} \eta \subseteq U_\phi$ then $\int_{\mathcal{M}} d\eta = \int_{\partial\mathcal{M}} \eta$, and apply $\eta = \rho_j\omega$ for all $j \in J$.

(The integral) Suppose the coordinates of ϕ is labeled $x^1, \dots, x^m$. Locally, let

$$\eta = \sum_{\ell=1}^{m} f_\ell dx^1 \wedge \cdots \cancel{dx^\ell} \cdots \wedge dx^m,$$

where $f_\ell : (-\infty, 0] \times \mathbb{R}^{m-1} \to \mathbb{R}$, and it is C^∞. Then, also locally,

$$\begin{aligned} d\eta &= \sum_{\ell=1}^{m} \frac{\partial f_\ell}{\partial x^\ell} dx^\ell \wedge dx^1 \wedge \cdots \cancel{dx^\ell} \cdots \wedge dx^m \\ &= \left(\sum_{\ell=1}^{m} (-1)^{\ell-1} \frac{\partial f_\ell}{\partial x^\ell} \right) dx^1 \wedge \cdots \wedge dx^m. \end{aligned}$$

Then by Definition 1.7,

$$\int_{\mathcal{M}} d\eta = \int_{(-\infty,0]\times\mathbb{R}^{m-1}} \left(\sum_{\ell=1}^{m} (-1)^{\ell-1} \frac{\partial f_\ell}{\partial x^\ell} \right) dx^1 \dots dx^m.$$

(To be precise, we need to choose another partition of unity ρ_j' to do intergration. But we can just choose it to cover all of $\operatorname{supp} \eta$ and don't care all other parts, so that doesn't matter too much.)

Choose a rectangular region R s.t.

$$\operatorname{supp} \eta \subseteq [a_1, 0] \times \cdots \times [a_m, b_m]$$

and define

$$R_\ell \coloneqq [a_1, 0] \times \dots \cancel{[a_\ell, b_\ell]} \cdots \times [a_m, b_m].$$

($\ell = 2, \dots, m$) In this case, by Fubini and FTC,

$$\begin{aligned} &\int_R \frac{\partial f_\ell}{\partial x^\ell} dx^1 \dots dx^m \\ =&\int_{R_\ell} \left(\int_{a_\ell}^{b_\ell} \frac{\partial f_\ell}{\partial x^\ell} dx^\ell \right) dx^1 \dots \cancel{dx^\ell} \dots dx^m \\ =&\int_{R_\ell} (\underbrace{f_\ell(x^1, \dots, b_\ell, \dots, x^m)}_{0} - \underbrace{f_\ell(x^1, \dots, a_\ell, \dots, x^m)}_{0}) dx^1 \dots \cancel{dx^\ell} \dots dx^m \\ =&0, \end{aligned}$$

since $\operatorname{supp} \eta \subseteq R$, so on the boundary $f = 0$.

($\ell = 1$) Now the integral has only one term left.

$$\begin{aligned}
\int_{\mathcal{M}} d\eta &= \int_R \frac{\partial f_1}{\partial x^1} dx^1 \ldots dx^m \\
&= \int_{R_1} \left(\int_{a_1}^{0} \frac{\partial f_1}{\partial x^1} dx^1 \right) dx^2 \ldots dx^m \\
&= \int_{R_\ell} (f_1(0, x^2, \ldots, x^m) - \underbrace{f_1(a_1, x^2, \ldots, x^m)}_{0}) dx^2 \ldots dx^m \\
&= \int_{\mathbb{R}^{m-1}} (f_1 \circ i) dx^2 \ldots dx^m \\
&= \int_{\partial \mathcal{M}} i^* \eta.
\end{aligned}$$

▨