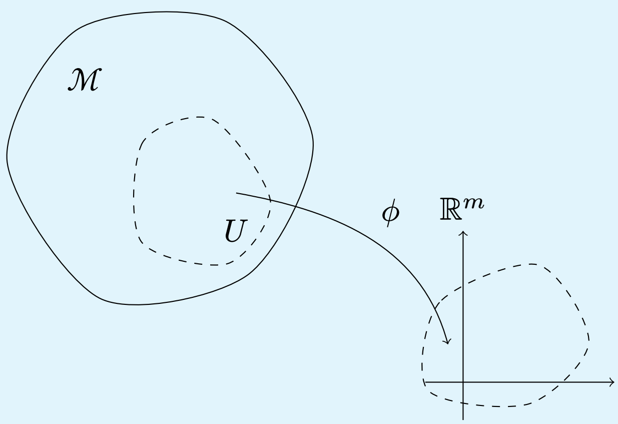
1 Differentiable Manifolds

1.1 Definition

1.1.1 Coordinate Charts

Definition 1.1 (Coordinate Charts)**.**
An m-dimensional, $m \neq \infty$ coordinate chart on a topological space $\mathcal{M}$ is a pair

$$(U, \phi) \begin{cases} U \subseteq \mathcal{M}, U \text{ open} \\ \phi : U \to \mathbb{R}^m, \phi \text{ homeomorphism} \end{cases}$$

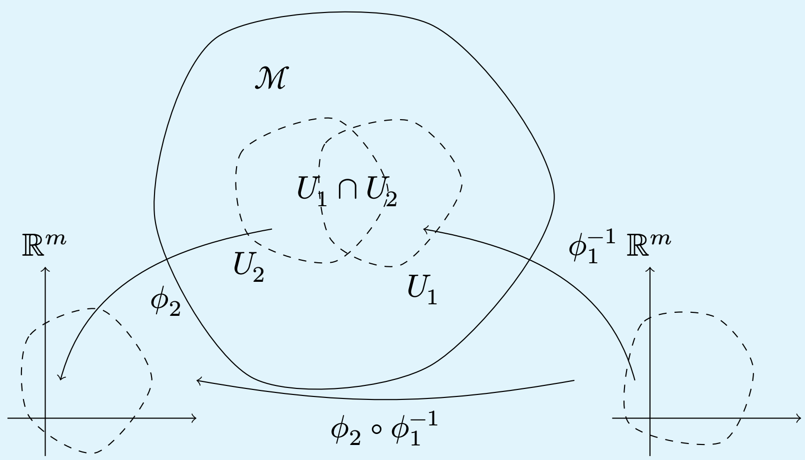


Remark.
If $U = \mathcal{M}$, then we say the coordinate chart ϕ is globally defined; if not, then it is locally defined. Few manifolds have globally defined property. ♍

Remark.
The basic method of studying manifolds is to analyze it in the familiar Euclidean space via coordinate charts. ♍

Definition 1.2 (Overlap Function).
Let $(U_1, \phi_1), (U_2, \phi_2)$ be a pair of m-dimensional coordinate charts with $U_1 \cap U_2 \neq \varnothing$. Then the overlap function is defined as

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \subseteq \mathbb{R}^m \to \phi_2(U_1 \cap U_2) \subseteq \mathbb{R}^m.$$



Definition 1.3 (Atlas).
An m-dimensional atlas on $\mathcal{M}$ is a family of m-dimensional coordinate charts $(U_i, \phi_i), i \in I$ s.t.

1. $\mathcal{M} = \bigcup_{i \in I} U_i$.
2. Each overlap function $\phi_j \circ \phi_i^{-1}, i, j \in I$ is C^∞.

Definition 1.4 (Differentiable Manifolds).
An m-dimensional differentiable manifold is a topological space $\mathcal{M}$ equipped with an atlas.

Remark.
We didn't define a differentiable manifold by regulating the differentiability of the coordinate charts themselves. That's because differentiation is not defined on a manifold, so we need to rely on Euclidean spaces. ♍

1.2 Dimension of a Manifold

Remark.
Consider a manifold that consists of a rod attached to a disk. The dimension is not same everywhere. We give a criterion on how to describe such a scenario. ♍

Theorem 1.1 (Invariance of Domain)**.**
For all $A, B \subseteq S^n$, if $\exists\ f : A \to B$ homeomorphic and $B \in \tau_{S^n}$, then $A \in \tau_{S^n}$ too.

Remark.
Theorem 1.1 is an early theorem in algebraic topology. ♍

Corollary 1.1.1 (Dimension is Well-defined)**.**
Given $U \in \tau_{\mathbb{R}^n}, U' \in \tau_{\mathbb{R}^{n'}}$, and if $\exists\ f : U \to U'$ homeomorphic, then $n = n'$.

Proof. If $n = n'$, it is trivially true.

If $n < n'$, embed $\mathbb{R}^n$ to $\mathbb{R}^{n'}$ by $f : \vec{x} \mapsto (\vec{x}, \vec{0})$. Via stereographic projection, we can map homeomorphically

$$\phi : U \mapsto V \subseteq S^{n'},$$
$$\phi' : U' \mapsto V' \subseteq S^{n'}.$$

Since the compositions are also homeomorphic, we see V and V' are homeomorphic. However, V' is not an open subset of $\mathbb{R}^{n'}$ because of the 0's, contradicting Theorem 1.1. ※ ▨

Remark.
Since the definition of a differentiable manifold requires every overlap function to be diffeomorphic, if $U \cap U' \neq \varnothing$, their dimensions must be equal via the above corollary. We can bypass this by demanding $U \cap U' = \varnothing$, as in the rod and disk case. ♍

Corollary 1.1.2.

If $g : V \to \mathbb{R}^n$ is a continuous injection and $V \in \tau_{\mathbb{R}^n}$, then $g(V)$ is homeomorphic to V, and $g(V) \in \tau_{\mathbb{R}^n}$.

Proof. On $g(V)$, g is surjective and therefore a homeomorphism. Use stereographic projection and the result is obvious. ▨

1.3 Coordinate Functions

Definition 1.5 (Coordinate Functions)**.**

The coordinate functions are the (Euclidean) components of coordinate.

$$\begin{aligned} \phi : U &\to \mathbb{R}^m & p &\mapsto \phi(p), \\ \phi^\mu : U &\to \mathbb{R} & \text{s.t. } \phi(p) &= \begin{pmatrix} \phi^1(p) \\ \vdots \\ \phi^m(p) \end{pmatrix}. \end{aligned}$$

An alternative notation is

$$x^\mu := \phi^\mu.$$

Remark.

There are (Euclidean) projection functions,

$$u^\mu : \mathbb{R}^m \to \mathbb{R}.$$

But I think mention it will cause a lot of confusion. Just remember in the future when we say $\frac{\partial}{\partial u^\mu}$, we are referring to the Euclidean partial derivative wrt the μ-th component. ♍

Definition 1.6 (Jacobian Matrix of Coordinate Transformation)**.**

Let $\mathcal{M}$ be a C^∞ manifold of $\dim \mathcal{M} = m$. Choose two coordinate functions

$$\phi : U_\phi \to V_\phi, p \mapsto (x^1(p), \ldots, x^m(p)),$$
$$\psi : U_\psi \to V_\psi, p \mapsto (y^1(p), \ldots, y^m(p)).$$

Define the Jacobian matrix of coordinate transformation to be

$$J := \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial y^m}{\partial x^1} & \cdots & \frac{\partial y^m}{\partial x^m} \end{pmatrix}$$

In short,

$$J^\nu{}_\mu := \frac{\partial y^\nu}{\partial x^\mu},$$
$$(J^{-1})^\nu{}_\mu := \frac{\partial x^\nu}{\partial y^\mu},$$

where $\frac{\partial y^\nu}{\partial x^\mu} := \frac{\partial (y^\nu \circ \phi^{-1})}{\partial x^\mu}$.

1.4 Manifold With Boundary

1.4.1 Generalized Coordinate Charts

Definition 1.7 (Generalized Coordinate Charts)**.**
A generalized coordinate chart allows chart that

$$\phi : U \to \phi(U) \subseteq (-\infty, 0] \times \mathbb{R}^{n-1},$$

U is open, and ϕ is homeomorphic.

Remark.
Essentially, this allows a chart to map to "half planes". In this case, even if a set $\phi(U)$ contains $\{0\} \times \mathbb{R}^{n-1}$ and therefore not open in the Euclidean topology of $\mathbb{R}^n$, it is still considered open in the product topology of $(-\infty, 0] \times \mathbb{R}^{n-1}$. ♍

Definition 1.8 (Manifold With Boundary)**.**
A manifold with boundary is a manifold whose atlas consists of generalized coordinate charts.

Definition 1.9 (Boundary Points of a Manifold)**.**
For all $p \in \mathcal{M}$ is a boundary point of a manifold with boundary $\mathcal{M}$ if $\exists\ \phi_\alpha \in \Phi$ atlas s.t. $\phi^1_\alpha(p) = 0$.
The set of all boundary points of $\mathcal{M}$ is denoted $\partial\mathcal{M}$.

1.4.2 Boundary is Well-defined

Remark.
A natural question regarding Definition 1.9 is that, the definition only asks for existence, but it does not guarantee the existence of

$$\exists\ \phi_\alpha, \phi_\beta \text{ s.t. } \phi^1_\alpha(p) = 0, \phi^1_\beta(p) \neq 0.$$

We resolve this in the following. ♍

Theorem 1.2.
Suppose U, U' are open sets in the product topology $(-\infty, 0] \times \mathbb{R}^{n-1}$, and $\exists\, f : U \to U'$ homeomorphic. Then

$$f(U \cap (\{0\} \times \mathbb{R}^{n-1})) = U' \cap (\{0\} \times \mathbb{R}^{n-1}).$$

Proof. We show instead that

$$f(U \cap ((-\infty, 0) \times \mathbb{R}^{n-1})) = U' \cap ((-\infty, 0) \times \mathbb{R}^{n-1}).$$

Via Corollary 1.1.2, we see $f(U \cap ((-\infty, 0) \times \mathbb{R}^{n-1}))$ must be an open set in the Euclidean topology of $\mathbb{R}^n$. Therefore, it cannot contain $\{0\} \times \mathbb{R}^{n-1}$. ▨

Corollary 1.2.1 (Boundary is Well-defined)**.**
$\forall p \in \mathcal{M}$, if $\exists\ \phi^1_\alpha(p) = 0$, then $\forall \phi \in \Phi$ atlas, $\phi^1(p) = 0$.

Proof. Make use of the fact that $\phi \circ \phi_\alpha^{-1}$ is a homeomorphism. ▨

1.5 Submanifolds

1.5.1 Definition

Definition 1.10 (Local Submanifold)**.**
$A \subseteq \mathcal{M}$ is a submanifold of codimension r around $p \in A \subseteq \mathcal{M}$ if there exists a chart $\phi : U \to \phi(U) \in \Phi$ atlas s.t. $p \in U$ and

$$\phi(U \cap A) = \phi(U) \cap (\mathbb{R}^{m-r} \times \{\underbrace{0, \dots, 0}_{r \text{ zeroes}}\}).$$

Definition 1.11 (Submanifold)**.**
If A is a C^k local submanifold of $\mathcal{M}$ of codimension r around every $p \in \mathcal{M}$, then we say A is a submanifold of $\mathcal{M}$ of codimension r.

1.5.2 Dimension of Submanifolds

Definition 1.12 (Regular Value).
Let $\mathcal{M}, \mathcal{N}$ be C^∞ manifolds, and $f : \mathcal{M} \to \mathcal{N}$ be C^∞. We say $q \in \mathcal{N}$ is a regular value of f if $\forall p \in \tilde{f}(q)$,

$$\begin{cases} \operatorname{rank} f_* : T_p\mathcal{M} \to T_{f(p)}\mathcal{N} = \dim \mathcal{N} \\ \operatorname{rank} f_* : T_p\partial\mathcal{M} \to T_{f(p)}\mathcal{N} = \dim \mathcal{N} \quad \text{if } p \in \partial\mathcal{M}. \end{cases}$$

Theorem 1.3 (Regular Value and Submanifolds - With Boundary).
Let $f : \mathcal{M} \to \mathcal{N}$ be C^∞. If $q \in \mathcal{N}$ is a regular value of f, then $\tilde{f}(q) = \varnothing$ or a C^∞ submanifold of $\mathcal{M}$ of codimension $\dim \mathcal{N}$. Also, $\partial\left(\tilde{f}(q)\right) = \tilde{f}(q) \cap \partial\mathcal{M}$.

Theorem 1.4 (Equation and Submanifolds).
Let $f : \mathcal{M} \to \mathcal{N}$ be C^∞. If $\partial\mathcal{M} = \varnothing$ and $\operatorname{rank} f_* = r$ for all $p \in \mathcal{M}$, then for all $q \in \mathcal{N}$, $\tilde{f}(q) = \varnothing$ or a C^∞ submanifold of $\mathcal{M}$ of codimension r.

1.5.3 Dimension of Submanifolds of $GL_n(\mathbb{R})$

Theorem 1.5 (Dimension of Special Linear Group $SL_n(\mathbb{R})$).
$SL_n(\mathbb{R})$ is a submanifold of $GL_n(\mathbb{R})$ codimension 1; $\dim SL_n(\mathbb{R}) = n^2 - 1$.

Proof. Consider $f = \det : \mathcal{M} = GL_n(\mathbb{R}) \to \mathbb{R}$. We want to show 1 is a regular value of $f = \det$.

(Elementary argument) One can simply write f_* in local coordinates and work out its rank. First choose a curve $A(t) : (-\epsilon, \epsilon) \to \mathcal{M}$ s.t. $A(0) = A_0$. Also define $\sigma(t) = \det A(t) : (-\epsilon, \epsilon) \to \mathbb{R}$. Then $\det_*([A]) \in T_{\det A_0}\mathbb{R} \simeq \mathbb{R}$. Referring to **??**, and identify $(\partial_1)_{\det A_0} \in T_{\det A_0}\mathbb{R}$ with 1, we see

$$\det{}_*([A]) = \left.\frac{\partial \det A}{\partial x^{ij}}\right|_{A=A_0} A'_{ij}(0) = (\operatorname{adj}(A_0))_{ji} A'_{ij}(0).$$

Clearly it didn't vanish everywhere, so it has rank 1. So 1 is indeed a regular value of f, and so by Theorem 1.3, the statement is true.

(Lie argument) Consider the following diagram, which leverages the Lie group property of $SL_n(\mathbb{R})$.

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "f = \det"] \arrow[d, "\ell_{A_0}"'] & \mathbb{R} \arrow[d, "\ell_{\det A_0}"] \\
\mathcal{M} \arrow[r, "f = \det"] & \mathbb{R}
\end{tikzcd}$$

So we know

$$f = \ell^{-1}_{\det A_0} \circ f \circ \ell_{A_0}.$$

We know that both $\ell_{A_0} : A \mapsto A_0 A$ and $\ell_{\det A_0} : a \mapsto a \det A_0$ are linear and thus C^∞ diffeomorphisms. Therefore,

$$\begin{aligned}
\operatorname{rank} f_{*A_0} &= \operatorname{rank}(\ell^{-1}_{\det A_0^{-1}} \circ f \circ \ell_{A_0^{-1}})_{*A_0} \\
&= \operatorname{rank}(\ell^{-1}_{\det A_0^{-1}} \circ f)_{*I} \\
&= \operatorname{rank}(f_{*I}).
\end{aligned}$$

Then one can use the elementary argument again, but this time with significantly less complexity. ▨

Theorem 1.6 (Dimension of Orthogonal Group $O_n(\mathbb{R})$)**.**
The orthogonal group $O_n(\mathbb{R}) = \left\{ A \in M_n(\mathbb{R}) \;\middle|\; AA^\top = I_n \right\}$ is a C^∞ submanifold of $GL_n(\mathbb{R})$ of codimension $\frac{n(n+1)}{2}$, and thus $\dim O_n(\mathbb{R}) = \frac{n(n-1)}{2}$.

Proof. Consider the following diagram,

$$\begin{tikzcd}
\boxed{A} & \mathcal{M} \arrow[r, "f = AA^\top"] \arrow[d, "\ell_{A_0}"'] & M_n(\mathbb{R}) \arrow[d, "\ell_{A_0} \circ r_{A_0^\top}"] & \boxed{AA^\top} \\
\boxed{A_0 A} & \mathcal{M} \arrow[r, "f = AA^\top"] & M_n(\mathbb{R}) & \boxed{\begin{array}{l} A_0 A A^\top A_0^\top \\ = (A_0 A)(A_0 A)^\top \end{array}}
\end{tikzcd}$$

where the red words are determined such that the diagram commutes. ▨