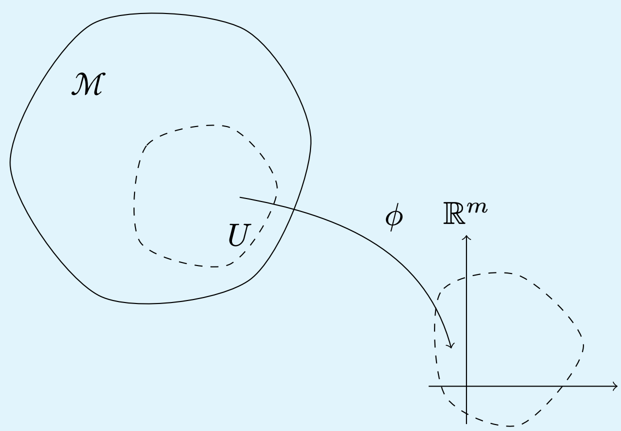
# 1 Differentiable Manifolds

## 1.1 Definition

### 1.1.1 Coordinate Charts

**Definition 1.1** (Coordinate Charts)**.**
An $m$-dimensional, $m \neq \infty$ coordinate chart on a topological space $\mathcal{M}$ is a pair

$$(U, \phi) \begin{cases} U \subseteq \mathcal{M}, U \text{ open} \\ \phi : U \to \mathbb{R}^m, \phi \text{ homeomorphism} \end{cases}$$

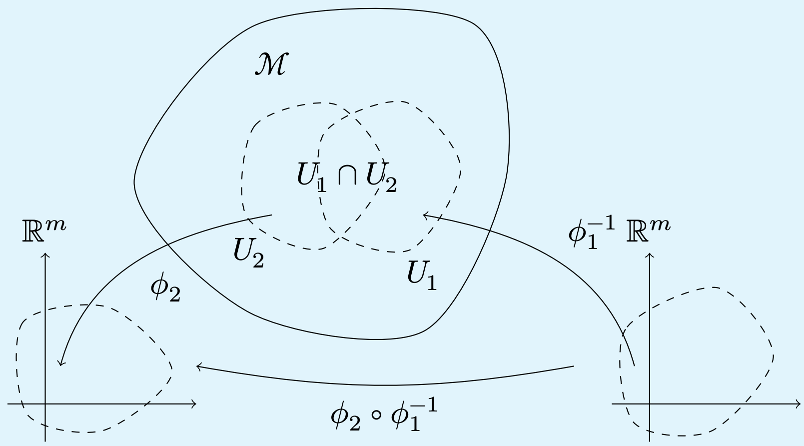



**Remark.**
If $U = \mathcal{M}$, then we say the coordinate chart $\phi$ is globally defined; if not, then it is locally defined. Few manifolds have globally defined property. ♍

**Remark.**
The basic method of studying manifolds is to analyze it in the familiar Euclidean space via coordinate charts. ♍

**Definition 1.2** (Overlap Function).
Let $(U_1, \phi_1), (U_2, \phi_2)$ be a pair of $m$-dimensional coordinate charts with $U_1 \cap U_2 \neq \varnothing$. Then the overlap function is defined as

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \subseteq \mathbb{R}^m \to \phi_2(U_1 \cap U_2) \subseteq \mathbb{R}^m.$$




**Definition 1.3** (Atlas).
An $m$-dimensional atlas on $\mathcal{M}$ is a family of $m$-dimensional coordinate charts $(U_i, \phi_i), i \in I$ s.t.

1. $\mathcal{M} = \bigcup_{i \in I} U_i$.
2. Each overlap function $\phi_j \circ \phi_i^{-1}, i, j \in I$ is $C^\infty$.

**Definition 1.4** (Differentiable Manifolds).
An $m$-dimensional differentiable manifold is a topological space $\mathcal{M}$ equipped with an atlas.

**Remark.**
We didn't define a differentiable manifold by regulating the differentiability of the coordinate charts themselves. That's because differentiation is not defined on a manifold, so we need to rely on Euclidean spaces. ♍

## 1.2 Dimension of a Manifold

**Remark.**
Consider a manifold that consists of a rod attached to a disk. The dimension is not same everywhere. We give a criterion on how to describe such a scenario. ♍

**Theorem 1.1** (Invariance of Domain)**.**
For all $A, B \subseteq S^n$, if $\exists\ f : A \to B$ homeomorphic and $B \in \tau_{S^n}$, then $A \in \tau_{S^n}$ too.

**Remark.**
Theorem 1.1 is an early theorem in algebraic topology. ♍

**Corollary 1.1.1** (Dimension is Well-defined)**.**
Given $U \in \tau_{\mathbb{R}^n}, U' \in \tau_{\mathbb{R}^{n'}}$, and if $\exists\ f : U \to U'$ homeomorphic, then $n = n'$.

*Proof.* If $n = n'$, it is trivially true.

If $n < n'$, embed $\mathbb{R}^n$ to $\mathbb{R}^{n'}$ by $f : \vec{x} \mapsto (\vec{x}, \vec{0})$. Via stereographic projection, we can map homeomorphically

$$\phi : U \mapsto V \subseteq S^{n'},$$
$$\phi' : U' \mapsto V' \subseteq S^{n'}.$$

Since the compositions are also homeomorphic, we see $V$ and $V'$ are homeomorphic. However, $V'$ is not an open subset of $\mathbb{R}^{n'}$ because of the 0's, contradicting Theorem 1.1. ※ ▨

**Remark.**
Since the definition of a differentiable manifold requires every overlap function to be diffeomorphic, if $U \cap U' \neq \varnothing$, their dimensions must be equal via the above corollary. We can bypass this by demanding $U \cap U' = \varnothing$, as in the rod and disk case. ♍

**Corollary 1.1.2.**
If $g : V \to \mathbb{R}^n$ is a continuous injection and $V \in \tau_{\mathbb{R}^n}$, then $g(V)$ is homeomorphic to $V$, and $g(V) \in \tau_{\mathbb{R}^n}$.

*Proof.* On $g(V)$, $g$ is surjective and therefore a homeomorphism. Use stereographic projection and the result is obvious. ▨

## 1.3 Coordinate Functions

**Definition 1.5** (Coordinate Functions)**.**
The coordinate functions are the (Euclidean) components of coordinate.

$$\begin{aligned} \phi : U &\to \mathbb{R}^m & p &\mapsto \phi(p), \\ \phi^\mu : U &\to \mathbb{R} & \text{s.t. } \phi(p) &= \begin{pmatrix} \phi^1(p) \\ \vdots \\ \phi^m(p) \end{pmatrix}. \end{aligned}$$

An alternative notation is

$$x^\mu := \phi^\mu.$$

**Remark.**
There are (Euclidean) projection functions,

$$u^\mu : \mathbb{R}^m \to \mathbb{R}.$$

But I think mention it will cause a lot of confusion. Just remember in the future when we say $\frac{\partial}{\partial u^\mu}$, we are referring to the Euclidean partial derivative wrt the $\mu$-th component. ♍

**Definition 1.6** (Jacobian Matrix of Coordinate Transformation)**.**
Let $\mathcal{M}$ be a $C^\infty$ manifold of $\dim \mathcal{M} = m$. Choose two coordinate functions

$$\phi : U_\phi \to V_\phi, p \mapsto (x^1(p), \dots, x^m(p)),$$
$$\psi : U_\psi \to V_\psi, p \mapsto (y^1(p), \dots, y^m(p)).$$

Define the Jacobian matrix of coordinate transformation to be

$$J := \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial y^m}{\partial x^1} & \cdots & \frac{\partial y^m}{\partial x^m} \end{pmatrix}$$

In short,

$$J^{\nu}{}_{\mu} := \frac{\partial y^{\nu}}{\partial x^{\mu}},$$

$$(J^{-1})^{\nu}{}_{\mu} := \frac{\partial x^{\nu}}{\partial y^{\mu}},$$

where $\frac{\partial y^{\nu}}{\partial x^{\mu}} := \frac{\partial (y^{\nu} \circ \phi^{-1})}{\partial x^{\mu}}$.

## 1.4 Manifold With Boundary

### 1.4.1 Generalized Coordinate Charts

**Definition 1.7** (Generalized Coordinate Charts).
A generalized coordinate chart allows chart that

$$\phi : U \to \phi(U) \subseteq (-\infty, 0] \times \mathbb{R}^{n-1},$$

$U$ is open, and $\phi$ is homeomorphic.

**Remark.**
Essentially, this allows a chart to map to "half planes". In this case, even if a set $\phi(U)$ contains $\{0\} \times \mathbb{R}^{n-1}$ and therefore not open in the Euclidean topology of $\mathbb{R}^n$, it is still considered open in the product topology of $(-\infty, 0] \times \mathbb{R}^{n-1}$. ♍

**Definition 1.8** (Manifold With Boundary).
A manifold with boundary is a manifold whose atlas consists of generalized coordinate charts.

**Definition 1.9** (Boundary Points of a Manifold).
For all $p \in \mathcal{M}$ is a boundary point of a manifold with boundary $\mathcal{M}$ if $\exists\ \phi_\alpha \in \Phi$ atlas s.t. $\phi_\alpha^1(p) = 0$.
The set of all boundary points of $\mathcal{M}$ is denoted $\partial\mathcal{M}$.

### 1.4.2 Boundary is Well-defined

**Remark.**
A natural question regarding Definition 1.9 is that, the definition only asks for existence, but it does not guarantee the existence of

$$\exists\ \phi_\alpha, \phi_\beta \text{ s.t. } \phi_\alpha^1(p) = 0, \phi_\beta^1(p) \neq 0.$$

We resolve this in the following. ♍

**Theorem 1.2.**
Suppose $U, U'$ are open sets in the product topology $(-\infty, 0] \times \mathbb{R}^{n-1}$, and $\exists\, f : U \to U'$ homeomorphic. Then

$$f(U \cap (\{0\} \times \mathbb{R}^{n-1})) = U' \cap (\{0\} \times \mathbb{R}^{n-1}).$$

*Proof.* We show instead that

$$f(U \cap ((-\infty, 0) \times \mathbb{R}^{n-1})) = U' \cap ((-\infty, 0) \times \mathbb{R}^{n-1}).$$

Via Corollary 1.1.2, we see $f(U \cap ((-\infty, 0) \times \mathbb{R}^{n-1}))$ must be an open set in the Euclidean topology of $\mathbb{R}^n$. Therefore, it cannot contain $\{0\} \times \mathbb{R}^{n-1}$. ▨

**Corollary 1.2.1** (Boundary is Well-defined).
$\forall p \in \mathcal{M}$, if $\exists\, \phi^1_\alpha(p) = 0$, then $\forall \phi \in \Phi$ atlas, $\phi^1(p) = 0$.

*Proof.* Make use of the fact that $\phi \circ \phi_\alpha^{-1}$ is a homeomorphism. ▨

## 1.5 Submanifolds

### 1.5.1 Definition

**Definition 1.10** (Local Submanifold).
$A \subseteq \mathcal{M}$ is a submanifold of codimension $r$ around $p \in A \subseteq \mathcal{M}$ if there exists a chart $\phi : U \to \phi(U) \in \Phi$ atlas s.t. $p \in U$ and

$$\phi(U \cap A) = \phi(U) \cap (\mathbb{R}^{m-r} \times \{\underbrace{0, \dots, 0}_{r \text{ zeroes}}\}).$$

**Definition 1.11** (Submanifold).
If $A$ is a $C^k$ local submanifold of $\mathcal{M}$ of codimension $r$ around every $p \in \mathcal{M}$, then we say $A$ is a submanifold of $\mathcal{M}$ of codimension $r$.

### 1.5.2 Dimension of Submanifolds

**Definition 1.12** (Regular Value).
Let $\mathcal{M}, \mathcal{N}$ be $C^\infty$ manifolds, and $f : \mathcal{M} \to \mathcal{N}$ be $C^\infty$. We say $q \in \mathcal{N}$ is a regular value of $f$ if $\forall p \in \tilde{f}(q)$,

$$\begin{cases} \operatorname{rank} f_* : T_p\mathcal{M} \to T_{f(p)}\mathcal{N} = \dim \mathcal{N} \\ \operatorname{rank} f_* : T_p\partial\mathcal{M} \to T_{f(p)}\mathcal{N} = \dim \mathcal{N} & \text{if } p \in \partial\mathcal{M}. \end{cases}$$

**Theorem 1.3** (Regular Value and Submanifolds - With Boundary).
Let $f : \mathcal{M} \to \mathcal{N}$ be $C^\infty$. If $q \in \mathcal{N}$ is a regular value of $f$, then $\tilde{f}(q) = \varnothing$ or a $C^\infty$ submanifold of $\mathcal{M}$ of codimension $\dim \mathcal{N}$. Also, $\partial\left(\tilde{f}(q)\right) = \tilde{f}(q) \cap \partial\mathcal{M}$.

**Theorem 1.4** (Equation and Submanifolds).
Let $f : \mathcal{M} \to \mathcal{N}$ be $C^\infty$. If $\partial\mathcal{M} = \varnothing$ and $\operatorname{rank} f_* = r$ for all $p \in \mathcal{M}$, then for all $q \in \mathcal{N}$, $\tilde{f}(q) = \varnothing$ or a $C^\infty$ submanifold of $\mathcal{M}$ of codimension $r$.

### 1.5.3 Dimension of Submanifolds of $GL_n(\mathbb{R})$

**Theorem 1.5** (Dimension of Special Linear Group $SL_n(\mathbb{R})$).
$SL_n(\mathbb{R})$ is a submanifold of $GL_n(\mathbb{R})$ codimension 1; $\dim SL_n(\mathbb{R}) = n^2 - 1$.

*Proof.* Consider $f = \det : \mathcal{M} = GL_n(\mathbb{R}) \to \mathbb{R}$. We want to show 1 is a regular value of $f = \det$.

(Elementary argument) One can simply write $f_*$ in local coordinates and work out its rank. First choose a curve $A(t) : (-\epsilon, \epsilon) \to \mathcal{M}$ s.t. $A(0) = A_0$. Also define $\sigma(t) = \det A(t) : (-\epsilon, \epsilon) \to \mathbb{R}$. Then $\det_*([A]) \in T_{\det A_0}\mathbb{R} \simeq \mathbb{R}$. Referring to Theorem 2.11, and identify $(\partial_1)_{\det A_0} \in T_{\det A_0}\mathbb{R}$ with 1, we see

$$\det{}_*([A]) = \left.\frac{\partial \det A}{\partial x^{ij}}\right|_{A=A_0} A'_{ij}(0) = (\operatorname{adj}(A_0))_{ji} A'_{ij}(0).$$

Clearly it didn't vanish everywhere, so it has rank 1. So 1 is indeed a regular value of $f$, and so by Theorem 1.3, the statement is true.

(Lie argument) Consider the following diagram, which leverages the Lie group property of $SL_n(\mathbb{R})$.

$$
\begin{tikzcd}
\mathcal{M} \arrow[r, "f = \det"] \arrow[d, "\ell_{A_0}"'] & \mathbb{R} \arrow[d, "\ell_{\det A_0}"] \\
\mathcal{M} \arrow[r, "f = \det"] & \mathbb{R}
\end{tikzcd}
$$

So we know

$$f = \ell^{-1}_{\det A_0} \circ f \circ \ell_{A_0}.$$

We know that both $\ell_{A_0} : A \mapsto A_0 A$ and $\ell_{\det A_0} : a \mapsto a \det A_0$ are linear and thus $C^\infty$ diffeomorphisms. Therefore,

$$
\begin{aligned}
\operatorname{rank} f_{*A_0} &= \operatorname{rank}(\ell^{-1}_{\det A_0^{-1}} \circ f \circ \ell_{A_0^{-1}})_{*A_0} \\
&= \operatorname{rank}(\ell^{-1}_{\det A_0^{-1}} \circ f)_{*I} \\
&= \operatorname{rank}(f_{*I}).
\end{aligned}
$$

Then one can use the elementary argument again, but this time with significantly less complexity. ▨

**Theorem 1.6** (Dimension of Orthogonal Group $O_n(\mathbb{R})$)**.**
The orthogonal group $O_n(\mathbb{R}) = \left\{ A \in M_n(\mathbb{R}) \mid AA^\top = I_n \right\}$ is a $C^\infty$ submanifold of $GL_n(\mathbb{R})$ of codimension $\frac{n(n+1)}{2}$, and thus $\dim O_n(\mathbb{R}) = \frac{n(n-1)}{2}$.

*Proof.* Consider the following diagram,

$$
\begin{tikzcd}
\boxed{A} & \mathcal{M} \arrow[r, "f = AA^\top"] \arrow[d, "\ell_{A_0}"'] & M_n(\mathbb{R}) \arrow[d, "\ell_{A_0} \circ r_{A_0^\top}"] & \boxed{AA^\top} \\
\boxed{A_0 A} & \mathcal{M} \arrow[r, "f = AA^\top"] & M_n(\mathbb{R}) & \boxed{\begin{array}{l} A_0 A A^\top A_0^\top \\ = (A_0 A)(A_0 A)^\top \end{array}}
\end{tikzcd}
$$

where the red words are determined such that the diagram commutes. ▨

# 2 Tangent Spaces

**Remark.**
The definition of manifold do not require the entity to be embeded in a higher dimensional space. Therefore, the traditional view of tangency is not valid here.

♍

## 2.1 Curves and Vectors

**Definition 2.1** (Curve)**.**
A curve on $\mathcal{M}$ is a $C^\infty$ map,

$$\sigma : (-\epsilon, \epsilon) \to \mathcal{M}.$$

**Definition 2.2** (Curve Tangency)**.**
Two curves $\sigma_1, \sigma_2$ are tangent at $p \in \mathcal{M}$ if

1. $\sigma_1(0) = \sigma_2(0) = p$.
2. $\frac{d}{dt}(x^i \circ \sigma_1(t))\big|_{t=0} = \frac{d}{dt}(x^i \circ \sigma_2(t))\big|_{t=0}, \; 1 \leq i \leq m$ .

**Remark.**
Written more compactly,

$$\frac{d}{dt}(\phi \circ \sigma_1)\bigg|_{t=0} = \frac{d}{dt}(\phi \circ \sigma_2)\bigg|_{t=0}$$

♍

**Definition 2.3** (Tangent Vectors)**.**
A tangent vector at $p \in \mathcal{M}$ is an equivalence class of curves where the equivalence relation is that they are tangent. It will be denoted as

$$v = [\sigma].$$

**Definition 2.4** (Tangent Space).
The tangent space $T_p\mathcal{M}$ at point $p$ is the set of all tangent vectors at point $p$.

## 2.2 Curves and Derivation

**Definition 2.5** (Directional Derivative).
For any $f : \mathcal{M} \to \mathbb{R}$ s.t. $f \in C^\infty$, we define

$$v(f) := \frac{d}{dt}(f \circ \sigma(t))\Big|_{t=0},$$

where $v = [\sigma]$.

**Theorem 2.1.**
The definition Definition 2.5 is well-defined. That is, $v(f)$ is independent of the curve $\sigma$ chosen as well as $v = [\sigma]$.

*Proof.* Let $v_1 = [\sigma_1] = [\sigma_2] = v_2$. Then

$$v_1(f) = \frac{d}{dt}(f \circ \sigma_1)\Big|_{t=0},$$
$$v_2(f) = \frac{d}{dt}(f \circ \sigma_2)\Big|_{t=0},$$

$$\frac{d}{dt}(\phi \circ \sigma_1)\Big|_{t=0} = \frac{d}{dt}(\phi \circ \sigma_2)\Big|_{t=0}.$$

Then

$$\begin{aligned}
v_1(f) &= \frac{d}{dt}(\underbrace{(f \circ \phi^{-1})}_{\mathbb{R} \leftarrow \mathbb{R}^m} \circ \underbrace{(\phi \circ \sigma_1)}_{\mathbb{R}^m \leftarrow \mathbb{R}})\Bigg|_{t=0} \\
&= (f \circ \phi^{-1})' \circ (\phi \circ \sigma_1) \cdot (\phi \circ \sigma_1)'\big|_{t=0} \\
&= (f \circ \phi^{-1})' \circ (\phi \circ \sigma_2) \cdot (\phi \circ \sigma_2)'\big|_{t=0} \\
&= v_2(f),
\end{aligned}$$

since $\phi \circ \sigma_1(0) = \phi \circ \sigma_2(0) = \phi(p)$, and $(\phi \circ \sigma_1)' = (\phi \circ \sigma_2)'$ by equivalence. ▨

## 2.3 Isomorphism with Euclidean Spaces

**Definition 2.6** ("Straight Lines")**.**
Choose a coordinate chart $\phi \in \Phi$ atlas near $p$. For all $v \in \mathbb{R}^m$, we define

$$\gamma_v^\phi(t) := \phi^{-1}(\phi(p) + tv).$$

In simple words, it is such a curve on manifold that it is a straight line on maps.

**Theorem 2.2** (Isomorphism with Straight Lines)**.**
Let $p \in \mathcal{M}$, $p \in \phi \in \Phi$. For any curve $\gamma$ passing through $p$, $\exists!\ v \in \mathbb{R}^m$ s.t. $\gamma$ is tangent to $\gamma_v^\phi$, where $v$ can be explicitly given by $(\phi \circ \gamma)'(0)$.
In other words, the map

$$\begin{aligned} \ell_p^\phi :&\mathbb{R}^m \to T_p\mathcal{M} \\ &v \mapsto [\gamma_v^\phi] \end{aligned}$$

is a bijection with inverse

$$\begin{aligned} (\ell_p^\phi)^{-1} :&T_p\mathcal{M} \to \mathbb{R}^m \\ &[\gamma] \mapsto (\phi \circ \gamma)'(0). \end{aligned}$$

*Proof.* It is almost by definition that $\gamma$ is tangent to $(\phi \circ \gamma)'(0)$.

Suppose $v_1, v_2$ satisfies $\gamma$ is tangent to $\gamma_{v_1}^\phi, \gamma_{v_2}^\phi$. Then they are tangent too. So

$$(\phi \circ \gamma_{v_1}^\phi)'(0) = (\phi \circ \gamma_{v_2}^\phi)'(0).$$

Therefore, $v_1 = v_2$. ▨

**Definition 2.7** (Alternative Definition for Addition)**.**
For $v_1, v_2 \in T_p\mathcal{M}$, we define addition via the isomorphism Theorem 2.2,

$$v_1 + v_2 := (\ell_p^\phi)^{-1}(\ell_p^\phi(v_1) + \ell_p^\phi(v_2)).$$

**Definition 2.8** (Alternative Definition of Basis Tangent Vectors)**.**

$$\left(\partial_\mu\right)_p := \ell_p^\phi(e^\mu).$$

**Theorem 2.3.**

$$\left(\partial_\mu\right)_p (x^\nu) = \delta_\mu{}^\nu.$$

*Proof.* By Definition 2.8,

$$\left(\partial_\mu\right)_p := \ell_p^\phi(e^\mu) = \phi^{-1}(\phi(p) + te^\mu),$$

where $e^\mu$ is the column vector with $\mu$-th component 1, others 0. Then

$$\begin{aligned}\left(\partial_\mu\right)_p (x^\nu) &= (x^\nu \circ \phi^{-1}(\phi(p) + te^\mu))'(0) \\ &= (x^\nu \circ \phi^{-1})'(\phi(p))e^\mu.\end{aligned}$$

But since $x^\nu$ has only one component, $(x^\nu \circ \phi^{-1})'$ is a row vector with $\nu$-th component 1, others 0. So the result follows immediately. ▨

**Theorem 2.4** (Linear Independence of Basis Tangent Vectors)**.**
The basis tangent vectors $\left(\partial_\mu\right)_p, 1 \leq \mu \leq \dim \mathcal{M}$ are linear independent.

*Proof.* Suppose $a^\mu \left(\partial_\mu\right)_p = 0$. Then

$$a^\mu \left(\partial_\mu\right)_p (x^\nu) = a^\mu \delta^\nu{}_\mu = 0(x^\nu) = 0.$$

So $a^\nu = 0$. ▨

**Theorem 2.5** (Coordinate Expansion of Tangent Vectors)**.**
For all $v \in T_p\mathcal{M}$, we have

$$v = v^\mu \left(\partial_\mu\right)_p,$$

where Einstein notation was used, and $v^\mu = v(x^\mu)$.

**Remark.**
A natural question is that whether two charts behave "the same" if $U_1 \cap U_2 \neq \varnothing$. Under this perspective, the criterion is clear: we need only to check whether a straight line is still a straight line in another chart, which is true indeed. It also produces the coordinate transformation formula for free. See below. ♍

**Theorem 2.6** (Coordinate Transformation of Straight Lines)**.**
Choose $\phi, \psi \in \Phi$, $\phi : U_1 \to \mathbb{R}^m$, $\psi : U_2 \to \mathbb{R}^m$ s.t. $U_1 \cap U_2 \neq \varnothing$ and $p \in U_1 \cap U_2$. Let the corresponding straight line isomorphisms $\ell_p^\phi, \ell_p^\psi$. Then $(\ell_p^\psi)^{-1} \circ \ell_p^\phi$ is a linear isomorphism.
Let the local coordinates induced by $\phi$ be $x^1, \dots, x^m$, $\psi$ be $y^1, \dots, y^m$, then $(\ell_p^\psi)^{-1} \circ \ell_p^\phi$ can be expressed in terms of the Jacobian Definition 1.6 $J^\nu{}_\mu := \frac{\partial y^\nu}{\partial x^\mu}\Big|_{\phi(p)}$, namely,

$$(\ell_p^\psi)^{-1} \circ \ell_p^\phi(v) = Jv.$$

*Proof.*

$$\begin{aligned}
(\ell_p^\psi)^{-1} \circ \ell_p^\phi(v) &= (\ell_p^\psi)^{-1}(\phi^{-1}(\phi(p) + vt)) \\
&= ((\psi \circ \phi^{-1})(\phi(p) + vt))'(0) \\
&= Jv
\end{aligned}$$

▨

**Definition 2.9** (Contravariancy and Covariancy)**.**
Let $\mathcal{M}$ be a $m$-dimensional $C^\infty$ manifold. Choose two coordinate charts

$$\begin{aligned}
\phi : U_\phi \to V_\phi,\ p \mapsto (x^1(p), \dots, x^m(p)), \\
\psi : U_\psi \to V_\psi,\ p \mapsto (y^1(p), \dots, y^m(p)).
\end{aligned}$$

and the corresponding Jacobian matrix Definition 1.6 $J^\nu{}_\mu := \frac{\partial y^\nu}{\partial x^\mu}\Big|_{\phi(p)}$. We define covariancy to be anything that transforms like

$$(\text{new})_\nu = (\text{old})_\mu (J^{-1})^\mu{}_\nu.$$

**Corollary 2.6.1** (Contravariancy of Tangent Vectors)**.**
The basis vectors transform covariantly, whereas the components of vectors

transform contravariantly. That is, choose two overlapping coordinate charts $(x^1, \dots, x^m)$ and $(y^1, \dots, y^m)$ and define their Jacobian matrix Definition 1.6 $J^{\nu}{}_{\mu} := \frac{\partial y^{\nu}}{\partial x^{\mu}}\Big|_{\phi(p)}$,

$$\begin{cases} \left(\frac{\partial}{\partial y^{\nu}}\right) = \left(\frac{\partial}{\partial x^{\mu}}\right) (J^{-1})^{\mu}{}_{\nu} & \text{(covariant)} \\ v^{\nu'} = J^{\nu'}{}_{\mu} v^{\mu} & \text{(contravariant)} \end{cases}$$

## 2.4 Pushforward

### 2.4.1 Definition and Linearity

**Remark.**
The pushforward $h_* : T_p\mathcal{M} \to T_{h(p)}\mathcal{N}$ of a specific function $h : \mathcal{M} \to \mathcal{N}$ can be thought of as local linearization of the function.

♍

**Definition 2.10** (Pushforward).
Given a function $h : \mathcal{M} \to \mathcal{N}$ and $v \in T_p\mathcal{M}$, then we define the pushforward $h_* : T_p\mathcal{M} \to T_{h(p)}\mathcal{N}$ by

$$h_*(v) := [h \circ \sigma], \; v = [\sigma].$$

**Theorem 2.7.**
The pushforward operation Definition 2.10 is well-defined. That is, $h_*(v_1) = h_*(v_2)$ if $v_1 = [\sigma_1] = [\sigma_2] = v_2$.

**Theorem 2.8** (Algebraic Definition of Pushforward).
The definition of pushforward Definition 2.10 is equivalent to the following: let $h : \mathcal{M} \to \mathcal{N}$, $h_* : D_p\mathcal{M} \to D_{h(p)}\mathcal{M}$ is defined by,

$$(h_* v)(f) := v(f \circ h).$$

*Proof.* ($\to$)

$$\begin{aligned} h_*(v)(f) = [h \circ \sigma](f) &= \frac{d}{dt}(f \circ h \circ \sigma(t))\Big|_{t=0} \\ &= \frac{d}{dt}((f \circ h) \circ \sigma(t))\Big|_{t=0} \\ &:= v(f \circ h). \end{aligned}$$

($\leftarrow$) This direction is similar. ▨

**Theorem 2.9** (Linearity of Pushforward)**.**
The pushforward map $h_* : T_p\mathcal{M} \to T_{h(p)}\mathcal{N}$ is linear.

$$h_*(rv_1 + v_2) = rh_*(v_1) + h_*(v_2).$$

*Proof.* (Using Definition 2.10) Let $p \in (U, \phi) \subseteq \mathcal{M}$, and $h(p) \in (V, \psi) \subseteq \mathcal{N}$. Choose $\phi$ s.t. $\phi(p) = 0$. It is obvious that $h_*(rv_1 + v_2)(0) = (rh_*(v_1) + h_*(v_2))(0) = h(p)$.

Consider

$$\begin{aligned} \frac{d}{dt}\underbrace{(\psi \circ h_*(rv_1 + v_2))}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathbb{R}}\Bigg|_{t=0} &= \frac{d}{dt}(\underbrace{\psi \circ h \circ (\phi^{-1}}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathcal{M} \leftarrow \mathbb{R}^m} \circ \underbrace{(r\phi \circ \sigma_1 + \phi \circ \sigma_2)}_{\mathbb{R}^m \leftarrow \mathcal{M} \leftarrow \mathbb{R}}))\Bigg|_{t=0} \\ &= (\psi \circ h \circ \phi^{-1})' \circ (r\phi \circ \sigma_1 + \phi \circ \sigma_2) \cdot (r\phi \circ \sigma_1 + \phi \circ \sigma_2)'\big|_{t=0} \\ &= (\psi \circ h \circ \phi^{-1})'(0) \cdot ((r\phi \circ \sigma_1)' + (\phi \circ \sigma_2)')\big|_{t=0}. \end{aligned}$$

And

$$\begin{aligned} \frac{d}{dt}(\underbrace{\psi}_{\mathbb{R}^n \leftarrow} \circ \underbrace{(rh_*(v_1) + h_*(v_2))}_{\mathcal{N} \leftarrow \mathbb{R}})\Bigg|_{t=0} &= \frac{d}{dt}\underbrace{(r\psi \circ h \circ \sigma_1 + \psi \circ h \circ \sigma_2)}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathcal{M} \leftarrow \mathbb{R}}\Bigg|_{t=0} \\ =&(\underbrace{r\psi \circ h \circ \phi^{-1}}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathcal{M} \leftarrow \mathbb{R}^m} \circ \underbrace{\phi \circ \sigma_1}_{\mathbb{R}^m \leftarrow \mathcal{M} \leftarrow \mathbb{R}} + \psi \circ h \circ \phi^{-1} \circ \phi \circ \sigma_2)'|_{t=0} \\ =&(r(\psi \circ h \circ \phi^{-1})' \circ (\phi \circ \sigma_1) \cdot (\phi \circ \sigma_1)')|_{t=0} \\ &+ ((\psi \circ h \circ \phi^{-1})' \circ (\phi \circ \sigma_2) \cdot (\phi \circ \sigma_2)')|_{t=0} \\ =&(\psi \circ h \circ \phi^{-1})'(0) \cdot (r(\phi \circ \sigma_1)' + (\phi \circ \sigma_2)')|_{t=0}. \end{aligned}$$

So we see the two are equal.

(Using Theorem 2.8)

$$\begin{aligned} (h_*(rv_1 + v_2))(f) &= (rv_1 + v_2)(f \circ h) \\ &= rv_1(f \circ h) + v_2(f \circ h) \\ &= r(h_*v_1)f + (h_*v_2)f. \end{aligned}$$

(Using straight line isomorphism: Definition 2.7)

$$\begin{array}{ccc} T_p\mathcal{M} & \xrightarrow{h_*} & T_{h(p)\mathcal{N}} \\ \ell_p^\phi \uparrow & & \downarrow (\ell_{h(p)}^\psi)^{-1} \\ \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \end{array}$$

Since $\ell_p^\phi$ and $(\ell_{h(p)}^\psi)^{-1}$ are linear, to prove $h_*$ is linear, we need only to show $(\ell_{h(p)}^\psi)^{-1} \circ h_* \circ \ell_p^\phi$ is linear.

$$\begin{aligned} & (\ell_{h(p)}^\psi)^{-1} \circ h_* \circ \ell_p^\phi(v) \\ =& (\ell_{h(p)}^\psi)^{-1} \circ h_*([\phi^{-1}(\phi(p)+tv)]) \\ =& (\ell_{h(p)}^\psi)^{-1}([h \circ \phi^{-1}(\phi(p)+tv)]) \\ =& (\psi \circ h \circ \phi^{-1}(\phi(p)+tv))'(0) \\ =& (\psi \circ h \circ \phi^{-1})'(\phi(p))v. \end{aligned}$$

**Theorem 2.10** (Associativity of Pushforwards).
Given manifolds $\mathcal{M}, \mathcal{N}, \mathcal{P}$ and $h : \mathcal{M} \to \mathcal{N}$, $k : \mathcal{N} \to \mathcal{P}$, then

$$(k \circ h)_* = k_* \circ h_*.$$

#### 2.4.2 Jacobian

**Theorem 2.11** (Local Representative of Pushforward).
Let $\dim \mathcal{M} = m, \dim \mathcal{N} = n$, $h : \mathcal{M} \to \mathcal{N}$, $\{x^1, \dots, x^m\}$ be the local coordinates of $\mathcal{M}$ around $p$, and $\{y^1, \dots, y^n\}$ be the local coordinates of $\mathcal{N}$ around $h(p)$. Then

$$h_* v = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} (\partial_\nu)_{h(p)} \left.\frac{\partial h^\nu}{\partial x^\mu}\right|_p v^\mu,$$

where ${J^\nu}_\mu := \left.\frac{\partial h^\nu}{\partial x^\mu}\right|_p := \left(\partial_\mu\right)_p (y^\nu \circ h)$ is the Jacobian matrix.

*Proof.* First expand $v$ in terms of local coordinates and use linearity,

$$h_*v = h_*(v^\mu \left(\partial_\mu\right)_p) = v^\mu h_*(\left(\partial_\mu\right)_p).$$

Expand the result in local coordinates of $\mathcal{N}$,

$$h_*(\left(\partial_\mu\right)_p) = \left(h_* \left(\partial_\mu\right)_p\right)^\nu \left(\partial_\nu\right)_{h(p)}.$$

Using Theorem 2.8,

$$\begin{aligned}\left(h_* \left(\partial_\mu\right)_p\right)^\nu &= \left(h_* \left(\partial_\mu\right)_p\right) \circ y^\nu \\ &= \left(\partial_\mu\right)_p (y^\nu \circ h) \\ &:= \left(\partial_\mu\right)_p h^\nu.\end{aligned}$$

So,

$$h_*(\left(\partial_\mu\right)_p) = \left(\partial_\mu\right)_p h^\nu \left(\partial_\nu\right)_{h(p)}.$$

And,

$$h_*v = v^\mu \left(\partial_\mu\right)_p h^\nu \left(\partial_\nu\right)_{h(p)}.$$

▨

**Theorem 2.12** (Using Curve to Pushforward)**.**
Given $c : (-\epsilon, \epsilon) \to \mathcal{M}$ a curve, and choose the coordinate chart of $\mathbb{R}$ to be the identity, then

$$c_* \left(\frac{d}{dt}\right)_0 = [c] \in T_p\mathcal{M}.$$

*Proof.* First we clarify what is $\left(\frac{d}{dt}\right)_0$. Since on the trivial manifold $\mathbb{R}$ there is only one coordinate, namely $t$, we need not specify the number. Also, considering our functions are scalar valued $f : \mathcal{M} \to \mathbb{R}$, this motivates us to write "total differential".

For all $f \in C^\infty$,

$$c_* \left(\frac{d}{dt}\right)_0 f = \left(\frac{d}{dt}\right)_0 (f \circ c).$$

Since the coordinate chart is the identity,

$$\begin{aligned}\left(\frac{d}{dt}\right)_0 (f \circ c) &= \frac{d}{dt}(f \circ c \circ I)\Big|_{I(t)=0} \\ &= \frac{d}{dt}(f \circ c)\Big|_{t=0} \\ &= [c]f.\end{aligned}$$

▨

# 3 Vector Fields and the Tangent Bundle

## 3.1 Vector Fields

**Definition 3.1** (Vector Fields).
A vector field $X$ on $V \subseteq \mathcal{M}$ is any rule of assigning a tangent vector $X(p) = X_p \in T_p\mathcal{M}$ for all $p \in V$.

**Definition 3.2** (Components of a Vector Field).
Let $X$ be a vector field on $V \subseteq \mathcal{M}$, $\dim \mathcal{M} = m$. For any chart $\phi : U \to \phi(U) \in \Phi$ with induced coordinates $x^1, \ldots, x^m$ and any $p \in V \cap U$, the decomposition $X(p) = X^j(p) \left(\partial_j\right)_p$ is unique, and therefore we write

$$X = X^j \partial_j$$

on $V \cap U$, and $X^j : V \cap U \to \mathbb{R}$ are called the components of $X$ on $V \cap U$.

**Definition 3.3** (Smoothness of a Vector Field).
A vector field $X$ on $V \subseteq \mathcal{M}$ is $C^k$ near $p \in V$ iff $\exists\ \phi \in \Phi$ s.t. $p \in U_\phi$ and all the components $X^j$ induced by $\phi$ are $C^k$. That is,

$$X^j \circ \phi^{-1} \text{ are } C^k \text{ on } \phi(V \cap U).$$

## 3.2 Tangent Bundle

### 3.2.1 Definition

**Definition 3.4** (Tangent Bundle).
The tangent bundle $T\mathcal{M}$ is

$$T\mathcal{M} := \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}.$$

**Remark.**
Why not $T\mathcal{M} := \left\{ T_p\mathcal{M} \mid p \in \mathcal{M} \right\}$? ♍

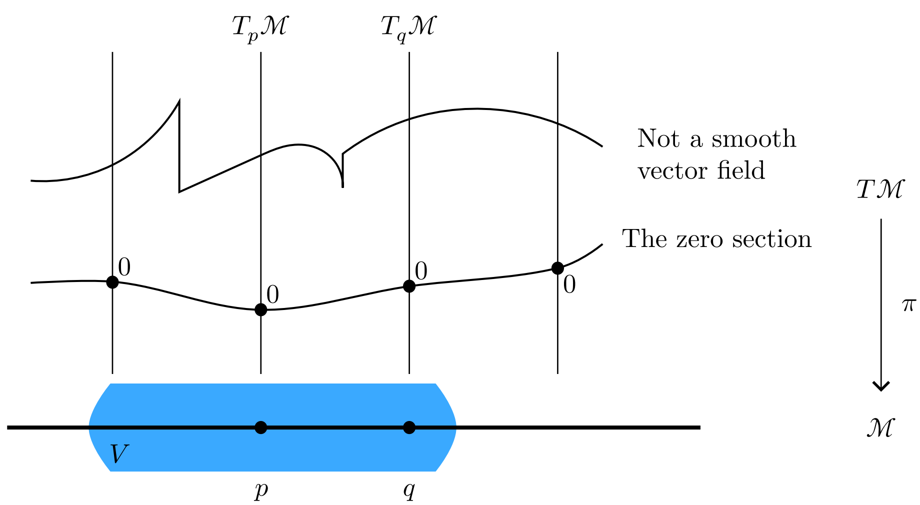
### 3.2.2 Projection

**Definition 3.5** (Canonical Projection)**.**
The canonical projection is the map

$$\begin{aligned} \pi : T\mathcal{M} &\to \mathcal{M} \\ T_p\mathcal{M} &\mapsto p. \end{aligned}$$

**Definition 3.6** (Alternative Definition of Vector Fields)**.**
A tangent vector field $X$ on $V \subseteq \mathcal{M}$ is a map $X : V \to T\mathcal{M}$ s.t. $(\pi \circ X)(p) = p$ for all $p \in V$.

**Remark.**
Let's see why vector fields are often called a "cross-section" of a tangent bundle.




♍

### 3.2.3 Topological and Manifold Structure

**Definition 3.7** (Smooth Structure on Tangent Bundle)**.**
Let a manifold $\mathcal{M}$ with dimension $m$ and atlas $\Phi$. Consider a chart $\phi_j \in \Phi : U_j \to \mathbb{R}^m$. We define a chart $\tilde{\phi}_j$ for the tangent bundle $T\mathcal{M}$ accordingly,

$$\begin{aligned}\tilde{\phi}_j : \tilde{\pi}(U_j) \subseteq T\mathcal{M} &\to \mathbb{R}^{2m} \\ (p, v^i \partial_i) &\mapsto (\phi_j(p), v^1, \dots, v^m),\end{aligned}$$

where $\tilde{\pi}$ is the inverse set map of the canonical projection function Definition 3.5. We thus see the tangent bundle $T\mathcal{M}$ is itself a manifold of dimension $2m$.
Also, we define the topology of the tangent bundle by

$$A \subseteq T\mathcal{M} \text{ is open iff } \tilde{\phi}_j(A \cap \tilde{\pi}(U_j)) \text{ is open.}$$

**Theorem 3.1.**
The smooth structure given by Definition 3.7 is unique in the sense of,

1. $\pi : T\mathcal{M} \to \mathcal{M}$ is $C^\infty$,
2. For all open sets $V \subseteq \mathcal{M}$ and any vector field $X$ on $V$, $X$ is $C^\infty \iff X : V \to T\mathcal{M}$ is $C^\infty$.

**Remark.**
At first thought, one may think of $T\mathcal{M} \simeq \mathcal{M} \times \mathbb{R}^m$. This is not the case, as one can consider the Moebius strip. The tangent bundle is only locally isomorphic to $U \times \mathbb{R}^m$.
If indeed $T\mathcal{M} \simeq \mathcal{M} \times \mathbb{R}^m$, then the bundle is called trivial tangent bundle, and the manifold is called parallelizable. ♍

## 3.3 Integral Curves and Local Flows

### 3.3.1 Integral Curves

**Definition 3.8** (Integral Curves)**.**
Let $\mathcal{M}$ be a manifold, and $X$ be a vector field on $V \subseteq \mathcal{M}$. If one single curve $\sigma : (-\epsilon, \epsilon) \to V$, $\epsilon > 0$ satisfies

$$\sigma(0) = p \in V$$
$$X_{\sigma(t)} = [\sigma] \; \forall t \in (-\epsilon, \epsilon)$$

then $\sigma$ is called an integral curve of the vector field $X$ through the point $p$.

**Theorem 3.2** (Differential Equations of Integral Curve)**.**
The components $X^\mu$ of $X$ determine the integral curve $\sigma$ by the following ODE with boundary conditions,

$$X^\mu(\sigma(t)) = \frac{d}{dt} x^\mu(\sigma(t))$$
$$x^\mu(\sigma(0)) = x^\mu(p), \mu = 1, 2, \dots, m.$$

### 3.3.2 Local Flows

**Definition 3.9** (Local 1D Family of Local Diffeomorphisms)**.**
A local, 1D family of local diffeomorphisms at $p \in \mathcal{M}$ is made up of a family of diffeomorphisms $\{ \sigma_t : U \to \mathcal{M} \mid t \in (-\epsilon, \epsilon) \}$ with $\epsilon > 0$, $U \subseteq \mathcal{M}$ an open set s.t.

1. Every $\sigma_t$ is a smooth function in $t$ and $p$.
2. $\forall t, s \in \mathbb{R}$ and $|t|, |s|, |t+s| < \epsilon$, and $\forall p \in U$ s.t. $\sigma_t(p), \sigma_s(p), \sigma_{t+s}(p) \in U$, we have
$$\sigma_s(\sigma_t(p)) = \sigma_{s+t}(p).$$
3. $\sigma_0(p) = p$.

**Remark.**
The first "local" refers to the parameter $t$, which is limited to $(-\epsilon, \epsilon)$. The second "local" refers to the spatial limitation to $U$. You can view $\phi_t(q)$ as a curve that brings $t \in (-\epsilon, \epsilon)$ to $\phi_t(q) \in \mathcal{M}$. ♍

**Definition 3.10.**
Consider a family of local diffeomorphisms given by Definition 3.9 $\phi_t$. We denote the following for convenience, if its meaning is clear from context,

$$\sigma(p,t) := \sigma_t(p),$$
$$\sigma_p(t) := \sigma_t(p).$$

**Definition 3.11** (Local Flow)**.**
Let $X$ be a vector field on $V \subseteq \mathcal{M}$, $p \in V$. Consider a family of diffeomorphisms Definition 3.9 $\sigma$ on $V$. Let the corresponding curve family $\sigma_p(t)$ satisfy,

$$\begin{aligned} \sigma_p(0) &= p && \forall p \in V, \\ [\sigma_p] &= X_{\sigma_p(t)} = \sigma_{p*}\left(\frac{d}{dt}\right)_t && \forall t \in (-\epsilon, \epsilon), p \in V. \end{aligned}$$

Then we say the curve family $\sigma_p(t)$ is the local flow for $X$ on $V$. We may abbreviate and just say $\sigma$ is the local flow for $X$.

**Theorem 3.3.**
Local flows always exist and are unique. Thus for a vector field $X$ defined on $V$, we may denote its local flow of over an open set $U$ by $\sigma^X$.

## 3.4 Lie Derivative of Vector Fields

### 3.4.1 Lie Bracket

**Definition 3.12** (Lie Derivative of Functions).
We can view a vector field as a function $X : C^\infty \to C^\infty$, given by

$$(Xf)(p) := X_p f = \frac{\partial}{\partial t}(f \circ \sigma^X(p,t))\Big|_{t=0}.$$

We can interpret this as: How much the function $f$ changes beginning at $p$, along the flow lines of $X$?

**Definition 3.13** (Composition of Vector Fields).
We can view $X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, and so does $Y$. Therefore, we define

$$(X \circ Y)(f) := X(Yf).$$

**Theorem 3.4** (Flow Interpretation).

$$X_p(Yf) = \frac{\partial^2}{\partial s \partial t}\left(f\left(\sigma^Y\left(\sigma^X(p,t),s\right)\right)\right)\Big|_{(s,t)=(0,0)}.$$

We can interpret $X(Yf)$ as: How much the function $f$ changes beginning at the point $p$, first following the flow lines of $X$, then following the flow lines of $Y$?

**Definition 3.14** (Lie Bracket/Vector Field Commutator).
We define the Lie Bracket of two vector fields $X, Y$ to be

$$[X,Y] := X \circ Y - Y \circ X.$$

**Remark.**
Lie Bracket Definition 3.14 is a vector field, while the expression $X \circ Y$ is not, because it contains second differential terms. See the following proof. ♍

**Theorem 3.5** (Lie Bracket Components)**.**

$$[X,Y]^\mu = (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu).$$

*Proof.* Given $X = X^\mu \partial_\mu, Y = Y^\nu \partial_\nu$, we try to write the component of $X \circ Y$.

$$X \circ Y(f) = X^\mu \partial_\mu \left(Y^\nu \partial_\nu f\right).$$

However, notice that

$$\begin{aligned}
&Y^\nu := Y x^\nu \in C^\infty(\mathcal{M});\\
&\partial_\nu : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}),\\
&\implies \partial_\nu f \in C^\infty(\mathcal{M}).
\end{aligned}$$

So we need to use the Leibniz property of $\partial_\mu$ in order to evaluate the second term. Doing this for $X \circ Y(f)$ and $Y \circ X(f)$, we have

$$\begin{aligned}
X \circ Y(f) &= X^\mu \big((\partial_\mu Y^\nu)(\partial_\nu f) + Y^\nu \partial_\mu \partial_\nu f\big).\\
Y \circ X(f) &= Y^\nu \big((\partial_\nu X^\mu)(\partial_\mu f) + X^\mu \partial_\nu \partial_\mu f\big).
\end{aligned}$$

So if $\partial_\mu \partial_\nu f = \partial_\nu \partial_\mu f$, then by subtracting, we can cancel the second order terms, and we are done. We prove so now.

$$\begin{aligned}
(\partial_\mu \partial_\nu f)(p) &= \frac{\partial}{\partial u^\mu}\left((\partial_\nu f) \circ \phi^{-1}\right)\Big|_{\phi(p)}\\
&= \frac{\partial}{\partial u^\mu}\Big((\partial_\nu)_{\phi^{-1}(u)} f\Big)\Big|_{\phi(p)}\\
&= \frac{\partial}{\partial u^\mu}\left(\frac{\partial}{\partial u^\nu}(f \circ \phi^{-1})\Big|_u\right)\Bigg|_{\phi(p)}\\
&= \frac{\partial}{\partial u^\nu}\left(\frac{\partial}{\partial u^\mu}(f \circ \phi^{-1})\Big|_u\right)\Bigg|_{\phi(p)}\\
&= (\partial_\nu \partial_\mu f)(p).
\end{aligned}$$

▨

**Theorem 3.6** (Properties of Lie Brackets).

$$\begin{cases} [X,Y] = -[Y,X] & \text{(antisymmetry)} \\ \sum_{\text{cyc}}[X,[Y,Z]] = 0. & \text{(Jacobi Identity)} \end{cases}$$

### 3.4.2 Lie Derivative of Vector Fields

**Definition 3.15** (Lie Derivative).
Let $X, Y$ be two vector fields defined on $U, V \subseteq \mathcal{M}$, respectively. Then the lie derivative of $Y$ along $X$ is defined as

$$(\mathcal{L}_X Y)_p := \lim_{t\to 0} \frac{\sigma^X_{-t*}(Y_{\sigma^X_t(p)}) - Y_p}{t},$$

where the latter is often abbreviated as

$$\left.\frac{d}{dt}(\sigma^X_{-t*}(Y_{\sigma^X_t(p)}))\right|_{t=0} = \lim_{t\to 0} \frac{\sigma^X_{-t*}(Y_{\sigma^X_t(p)}) - Y_p}{t}.$$

**Remark.**
It is common to interpret the Lie derivative of $Y$ along $X$ to be: How much do $Y$ change when one moves along the flow lines of $X$?
However, one cannot directly compare the two vectors $Y_{\sigma^X_t(p)} \in T_{\sigma^X_t(p)}\mathcal{M}$ and $Y_p \in T_p\mathcal{M}$. Therefore, the most intuitive way is to "pull it back", i.e., push it forward along the flow lines of $X$ by negative time. ♍

**Theorem 3.7** (Lie Bracket and Lie Derivative).

$$\mathcal{L}_X Y = [X,Y].$$

That is,

$$\left.\frac{d}{dt}(\sigma^X_{-t*}(Y_{\sigma^X_t(p)}))\right|_{t=0} = X \circ Y - Y \circ X.$$

*Proof.* We start with

$$\begin{aligned}
\frac{d}{dt}(\sigma^X_{-t*}(Y_{\sigma^X_t(p)}))\bigg|_{t=0} f &= \lim_{t\to 0}\frac{\sigma^X_{-t*}(Y_{\sigma^X_t(p)f}) - Y_p f}{t} \\
&= \lim_{t\to 0}\frac{Y_{\sigma^X_t(p)}(f\circ\sigma^X_{-t}) - Y_{\sigma^X_0(p)}(f\circ\sigma^X_0)}{t}.
\end{aligned}$$

It motivates us to consider the function $H(r,s) = Y_{\sigma^X_r(p)}(f\circ\sigma^X_{-s})$. Then we see

$$\begin{aligned}
\frac{d}{dt}(\sigma^X_{-t*}(Y_{\sigma^X_t(p)}))\bigg|_{t=0} f &= \frac{d}{dt}H(t,t)\bigg|_{t=0} \\
&= \frac{\partial H}{\partial r}(0,0) + \frac{\partial H}{\partial s}(0,0).
\end{aligned}$$

($r$ part: moving along $X$)

$$\begin{aligned}
\frac{\partial H}{\partial r}(0,0) &= \frac{\partial}{\partial r}Y_{\sigma^X_r(p)}(f\circ\underbrace{\sigma^X_0}_{\mathrm{id}})\bigg|_{r=0} \\
&= \frac{\partial}{\partial r}(Yf)(\sigma^X_r(p))\bigg|_{r=0} \\
&= X_p(Yf).
\end{aligned}$$

($s$ part: moving along $Y$)

$$\begin{aligned}
\frac{\partial H}{\partial s}(0,0) &= \frac{\partial}{\partial s}Y_p(f\circ\sigma^X_{-s})\bigg|_{s=0} \\
&= Y_p\,\frac{\partial}{\partial s}(f\circ\sigma^X_{-s})\bigg|_{s=0} \\
&= -Y_p(Xf),
\end{aligned}$$

and therefore completes the proof. ▨

# 4 Formal Differential Form

**Remark.**
In this section, we follow a local, coordinate approach. We focus on the requirements that makes form a form. We will postpone the realization of differential forms. ♍

## 4.1 Euclidean Spaces

**Definition 4.1** (Formal Differential Form on $\mathbb{R}^m$).
A formal differential $k$-form on $\mathbb{R}^m$ is composed of $m^k$ functions, arranged in the form

$$\omega = \sum_{1 \le i_1, \dots, i_k \le m} \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

We require,

1. All the functions $\omega_{i_1 \cdots i_k}$ are $(-\infty, 0] \times \mathbb{R}^{m-1} \to (-\infty, 0] \times \mathbb{R}^{m-1}$ and $C^\infty$.
2. The operation $+$ is commutative and associative, just like the usual addition.
3. The operation $\wedge$ is associative and distributes over $+$, resembling the usual multiplication.
4. $\wedge$ also has anticommutativity,

$$\begin{cases} dx^i \wedge dx^j = -dx^j \wedge dx^i & \forall i \neq j \\ dx^i \wedge dx^i = 0 \end{cases}$$

**Definition 4.2** (Equality of Formal Differential k-forms).
Let two formal $k$-forms $\omega, \eta$ be

$$\omega = \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$
$$\eta = \eta_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

We say they are "equal", denoted $\omega \equiv \eta$, iff

$$\sum_{\sigma \in S_k} \omega_{\sigma(i_1)\dots\sigma(i_k)} = \sum_{\sigma \in S_k} \eta_{\sigma(i_1)\dots\sigma(i_k)} \ \forall 1 \le i_1 < \dots < i_k \le m.$$

**Remark.** 1. For a formal differential $k$-form $\omega = dx^1 \wedge dx^2$, $\omega_{12} = 1$, but $\omega_{21} = 0$. The $m^k$ components in the definition just presents a general form, so that it includes the scenario $\omega' = dx^1 \wedge dx^2 - dx^2 \wedge dx^1$. If you insist on arranging indices even in the definition, then $\omega'$ would not satisfy the definition, which is wierd.

2. For demonstration of equality, consider the following example in $\mathbb{R}^3$,

$$\omega = dx^1 \wedge dx^2 + dx^2 \wedge dx^3$$
$$\eta = -dx^2 \wedge dx^1 + dx^2 \wedge dx^3.$$

Choose $i_1 = 1, i_2 = 2$, and

$$S_3 = \{\underbrace{\mathrm{id}, (1\ 2\ 3), (1\ 3\ 2)}_{\text{even}}, \underbrace{(1\ 2), (2\ 3), (1\ 3)}_{\text{odd}}\}.$$

Then, in order,

$$\begin{aligned} &\sum_{\sigma \in S_n} (\operatorname{sgn} \sigma)\omega \\ =&\omega_{12} + \omega_{23} + \omega_{31} - \omega_{21} - \omega_{13} - \omega_{32} \\ =&1 + 1 + 0 - 0 - 0 - 0. \end{aligned}$$

Notice how all permutations of a given component (paired in color) appears exactly once in this relation, and the sign is fixed accordingly by $\operatorname{sgn} \sigma$.

3. Viewed this way, we see in the language of formal differential forms,

$$\omega = dx^1 \wedge dx^2$$
$$\eta = -dx^2 \wedge dx^1$$
$$\nu = dx^1 \wedge dx^2 - dx^2 \wedge dx^1,$$

$\omega \equiv \eta$, since $\omega_{12} - \omega_{21} = 1 = \eta_{12} - \eta_{21}$. But $\omega \not\equiv \nu$, since $\nu_{12} - \nu_{21} = 2$.

♍

## 4.2 Operations

**Definition 4.3** (Exterior Product)**.**
Let two formal $k$-forms be $\omega = \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, $\eta = \eta_{j_1 \cdots j_l} dx^{j_1} \wedge \cdots \wedge dx^{j_l}$. Then

$$\omega \wedge \eta := \omega_{i_1 \cdots i_k} \eta_{j_1 \cdots j_l} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l}.$$

Which is equivalent Definition 4.2 to,

$$\sum_{1 \le s_1 < \cdots < s_{k+l} \le m} \left( \sum_S (\operatorname{sgn} \sigma) \omega_{i_1 \cdots i_k} \eta_{j_1 \cdots j_l} \right) dx^{s_1} \wedge \cdots \wedge dx^{s_{k+l}}.$$

Where $S$ is all the combinations of $S_1 = \{i_1, \ldots, i_k, j_1, \ldots, j_l\}$ that is equal to $S_2 = \{s_1, \ldots, s_{k+l}\}$, and the order does not matter. $\sigma$ is the function $S_1 \to S_2$.

**Remark.**
Example. Let $\dim \mathcal{M} = 6$, and

$$\begin{aligned} \omega &= \omega_{12} dx^1 \wedge dx^2 + \omega_{21} dx^2 \wedge dx^1 \\ \eta &= \eta_{456} dx^4 \wedge dx^5 \wedge dx^6. \end{aligned}$$

Then

$$\omega \wedge \eta = (\omega_{12}\eta_{456} - \omega_{21}\eta_{456}) dx^{1,2,4,5,6}$$

♍

**Definition 4.4** (Pullback)**.**
Let $f : U \subseteq \mathbb{R}^m \to V \subseteq (-\infty, 0] \times \mathbb{R}^{n-1}$ is $C^\infty$. Choose local coordinates on $U$ to be $x = (x^1, \ldots, x^m)$, on $V$ to be $y = (y^1, \ldots, y^n)$. Define $f^\nu := y^\nu \circ f$.
Let $\omega = \omega_{j_1 \cdots j_k} dy^{j_1} \wedge \cdots \wedge dy^{j_k}$ be a formal $k$-form on $U$. Then

$$f^*\omega := (\omega_{j_1 \cdots j_k} \circ f) \frac{\partial f^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{j_k}}{\partial x^{i_k}} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

**Remark.** 1. The motivation is just somehow $\omega \circ f$. Chain the component function with $f$, and express the (resulting) coordinates as $dy^\nu = \frac{\partial f^\nu}{\partial x^\mu} dx^\mu$.

2. It is possible that $U \subseteq V$. Pulling back onto a subset is essentially a "limitation" on $\omega$.

3. It is also possible that $U \subseteq \partial V$, i.e. $U$ is the boundary of $V$. $f : p \mapsto (0, p)$ is just the immersion map in that case.

♍

**Definition 4.5** (Exterior Differentiation).
Let a formal $k$-form be $\omega = \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Then

$$d\omega := \left( \frac{\partial \omega_{i_1 \ldots i_k}}{\partial x^{i_0}} dx^{i_0} \right) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

**Remark.**
Let $U \subseteq \mathbb{R}^3$.

1. Consider a formal 0-form, i.e. $f \in C^\infty$. Then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \to \nabla f.$$

2. Consider a formal 1-form $\omega = Pdx + Qdy + Rdz$. Then

$$\begin{aligned} d\omega &= \left( \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz \right) \wedge dx + \left( \frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial z} dz \right) \wedge dy + \left( \frac{\partial R}{\partial x} dx + \frac{\partial R}{\partial y} dy \right) \wedge dz \\ &= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) dx \wedge dz + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy \\ &\to \operatorname{curl} \omega. \end{aligned}$$

3. Consider a formal 2-form $\eta = Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy$. Then

$$\begin{aligned}
d\eta &= \frac{\partial A}{\partial x} dx \wedge dy \wedge dz + \frac{\partial B}{\partial y} dy \wedge dz \wedge dx + \frac{\partial C}{\partial z} dz \wedge dx \wedge dy \\
&= \left( \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) dx \wedge dy \wedge dz \\
&\to \operatorname{div} \eta.
\end{aligned}$$

4. In usual vector calculus terms, we say $\nabla$ produces a vector, div produces a scalar, and curl produces a vector. This $r$-form to $(m-r)$-form correspondance is provided by the Hodge star operation.

♍

## 4.3 Differential Form as Equivalence Classes

### 4.3.1 Operations are Well-Defined

**Theorem 4.1.**
If $\omega \equiv \omega'$, $\eta \equiv \eta'$ are formal $k$-forms on $\mathbb{R}^m$, then

$$\omega \wedge \eta \equiv \omega' \wedge \eta'.$$

**Theorem 4.2.**
If $\omega \equiv \omega'$ are formal $k$-forms on $\mathbb{R}^m$, and $f \in C^\infty(\mathbb{R}^m)$, then

$$f^*\omega \equiv f^*\omega'.$$

**Theorem 4.3.**
If $\omega \equiv \omega'$ are formal $k$-forms on $\mathbb{R}^m$, then

$$d\omega \equiv d\omega'.$$

### 4.3.2 Equivalence Classes

**Definition 4.6** (Euclidean Differential k-form).
Denote the set of all formal $k$-forms on $\mathbb{R}^m$ be $A^k(\mathbb{R}^m)$. Then the set of all differential $k$-forms on $\mathbb{R}^m$ is defined to be

$$\Lambda^k(\mathbb{R}^m) := A^k(\mathbb{R}^m)/\equiv.$$

## 4.4 Properties of Operations

**Theorem 4.4** (Properties of Exterior Product).
Let $\omega \in \Lambda^k(\mathbb{R}^m)$, $\eta \in \Lambda^l(\mathbb{R}^m)$. Then

$$\begin{aligned}(\omega_1 + \omega_2) \wedge \eta &= \omega_1 \wedge \eta + \omega_2 \wedge \eta \\ \omega \wedge \eta &= (-1)^{kl} \eta \wedge \omega.\end{aligned}$$

**Theorem 4.5** (Properties of Pullback).
Let $f \in C^\infty$, $\omega \in \Lambda^k(\mathbb{R}^m)$. Then

$$\begin{aligned}f^*(\omega_1 + \omega_2) &= f^*\omega_1 + f^*\omega_2 \\ f^*(\omega_1 \wedge \omega_2) &= (f^*\omega_1) \wedge (f^*\omega_2) \\ g^*(f^*\eta) &= (f \circ g)^*\eta.\end{aligned}$$

**Theorem 4.6** (Properties of Exterior Differentiation).
Let $f \in C^\infty$, $\omega \in \Lambda^k(\mathbb{R}^m)$. Then

$$\begin{aligned}f^*(d\omega) &= d(f^*\omega) \\ d(\omega \wedge \eta) &= (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta) \\ d(d\omega) &= 0.\end{aligned}$$

## 4.5 Differential Forms on Manifolds

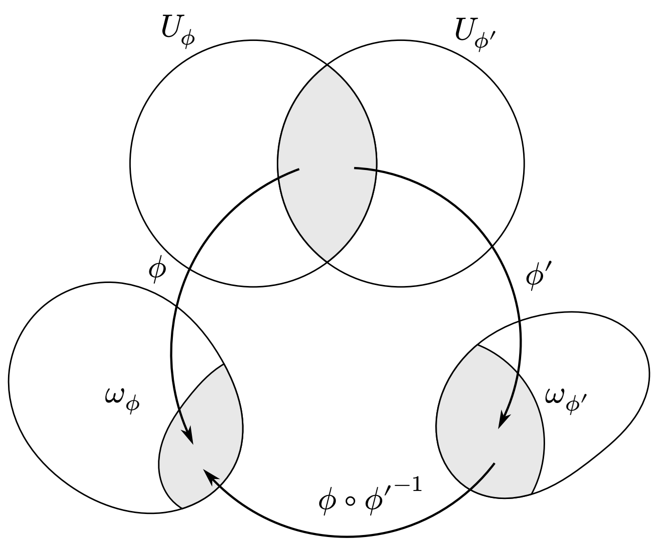
### 4.5.1 Requirements of Manifold Forms

**Definition 4.7** (Differential Form on a Manifold).
A $C^\infty$ differential $k$-form $\omega$ on manifold $\mathcal{M}$, $\omega \in \Lambda^k(\mathcal{M})$, consists of a family of differential $k$-forms $\omega_\phi \in \Lambda^k(\phi(U_\phi))$, $\phi \in \Phi$, $\phi : U_\phi \to V_\phi = \phi(U_\phi) \subseteq (-\infty, 0] \times \mathbb{R}^{m-1}$, with an additional requirement that

$$\omega_{\phi'}\big|_{\phi'(U_\phi \cap U_{\phi'})} = \left(\phi \circ \phi'^{-1}\right)^* \left(\omega_\phi\big|_{\phi(U_\phi \cap U_{\phi'})}\right),\ \forall \phi, \phi' \in \Phi.$$

$\omega_\phi$ is called the local expression of $\omega$ on $U_\phi$ via $\phi$.

**Remark.**




The motivation is that, if two differential forms describe the same set, they should "agree" on that portion of manifold. The "agreement" is done by pull-back using the overlap function.

**Theorem 4.7** (Covariancy of Differential Forms).
Choose two overlapping charts $x^1, \dots, x^m$ and $y^1, \dots, y^m$ and define their Jacobian matrix Definition 1.6 $J^\nu{}_\mu := \frac{\partial y^\nu}{\partial x^\mu}\Big|_{\phi(p)}$, combining Definition 4.4 and Definition 4.7, we have

$$\begin{cases} dy^\nu = J^\nu{}_\mu \, dx^\mu & \text{(covariant)} \\ \omega_{\nu'} = \omega_\mu (J^{-1})^\mu{}_\nu & \text{(contravariant)} \end{cases}$$

### 4.5.2 Operations on Manifold Forms

**Definition 4.8** (Addition).
Given two forms $\omega, \eta \in \Lambda^k(\mathcal{M})$, $m = \dim \mathcal{M}$, define their sum to be $\omega + \eta$, whose chart components are given by

$$(\omega + \eta)_\phi := \omega_\phi + \eta_\phi.$$

They satisfy Definition 4.7 thanks to the linearity of pullback.

**Definition 4.9** (Exterior Product).
Given two forms $\omega, \eta \in \Lambda^k(\mathcal{M})$, $m = \dim \mathcal{M}$, define their exterior product to be $\omega \wedge \eta$, whose chart components are given by

$$(\omega \wedge \eta)_\phi := \omega_\phi \wedge \eta_\phi.$$

They satisfy Definition 4.7 because pullback commutes with exterior product.

**Definition 4.10** (Exterior Differentiation).
Given a form $\omega \in \Lambda^k(\mathcal{M})$, $m = \dim \mathcal{M}$, define its exterior derivative to be $d\omega$, whose chart components are given by

$$(d\omega)_\phi := d(\omega_\phi).$$

They satisfy Definition 4.7 because pullback commutes with exterior product.

**Definition 4.11** (Pullback)**.**
Given a form $\omega \in \Lambda^k(\mathcal{N})$ and a function $f : \mathcal{M} \to \mathcal{N}$, $m = \dim \mathcal{M}, n = \dim \mathcal{N}$.
Choose coordinate functions $\phi_i : U_i \to \phi_i(U_i)$ on $\mathcal{M}$, and $\psi_j : V_j \to \psi_j(V_j)$ on $\mathcal{N}$.
To define $(f^*\omega)_{\phi_1}$, choose any $V_1, \dots, V_s$ s.t. $U_1 \subseteq \bigcup_{j=1}^{s} \tilde{f}(V_j)$. Then

$$(f^*\omega)_{\phi_i} := \sum_j (\phi_i \circ f \circ \psi_j^{-1})^* \left( \omega_{\psi_j}\Big|_{\phi(f(U_i) \cap V_j)} \right)$$

They satisfy Definition 4.7.

# 5 Realization of Differential Forms

## 5.1 Cotangent Spaces

**Definition 5.1** (Cotangent Spaces)**.**
The cotangent space $T_p^*\mathcal{M}$ at $p \in \mathcal{M}$ is the set of all linear functions $f : T_p\mathcal{M} \to \mathbb{R}$.
Its member is called a cotangent vector.
$\dim T_p^*\mathcal{M} = \dim T_p\mathcal{M}$.

**Definition 5.2** (One-Form)**.**
A one-form on $\mathcal{M}$ is a smooth assignment of cotangent vectors $\omega : p \mapsto \omega_p$.
It may be understood as a covector field.

**Definition 5.3** (Basis Cotangent Vectors)**.**
The basis cotangent vectors is chosen to be the dual basis of the basis tangent vectors Definition 2.8,

$$(dx^\mu)_p((\partial_\nu)_p) = \delta^\mu{}_\nu.$$

**Theorem 5.1** (Coordinate Expression of Cotangent Vectors)**.**
Any $f \in T_p^*\mathcal{M}$ can be expanded as

$$f = f_\mu (dx^\mu)_p.$$

Any one-form $\omega$ can be expressed as

$$\omega = \omega_\mu dx^\mu.$$

**Theorem 5.2** (Pullback as Dual of Pushforward).
Given two manifolds $\mathcal{M}$ with dimension $m$ and $\mathcal{N}$ with dimension $n$ and a $C^\infty$ function $f : \mathcal{M} \to \mathcal{N}$, the pullback of a one-form is the dual of pushforward. That is,

$$(h^*\omega)_p(v) := \omega_p(h_* v).$$

**Definition 5.4** (n-Forms).
An n-form is a tensor field of type $(0, n)$ that is totally skew-symmetric (or alternating, or totally antisymmetric), i.e.,

$$\omega(X_1, X_2, \dots, X_n) = (\operatorname{sgn} \sigma)\omega(X_{\sigma(1)}, \dots, X_{\sigma(n)}), \ \forall \sigma \in S_n.$$

The set of all n-forms on $\mathcal{M}$ is denoted as $\Lambda^n(\mathcal{M})$.
The set of all forms is $\Lambda(\mathcal{M}) = \bigoplus_{n=0}^{\dim \mathcal{M}} \Lambda^n(\mathcal{M})$.
Conventionally, we classify functions as 0-forms.

## 5.2 The Exterior Product

**Definition 5.5** (Exterior Product).
Given $\omega_1 \in \Lambda^{n_1}(\mathcal{M}), \omega_2 \in \Lambda^{n_2}(\mathcal{M})$, their exterior product is a $(n_1 + n_2)$-form given by,

$$\omega_1 \wedge \omega_2 := \frac{1}{n_1! n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\operatorname{sgn} \sigma)(\omega_1 \otimes \omega_2)_\sigma.$$

Written explicitly,

$$(\omega_1 \wedge \omega_2)(X_1, \dots, X_{n_1+n_2}) := \frac{1}{n_1! n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\operatorname{sgn} \sigma)(\omega_1 \otimes \omega_2)(X_{\sigma(1)}, \dots, X_{\sigma(n_1+n_2)})$$

**Remark.**
I'll take the alternating property and associativity of the exterior product for granted. For a detailed proof, see Hoffman. ♍

**Theorem 5.3** (Commutativity with Pullback)**.**
Given $h : \mathcal{M} \to \mathcal{N}$ and $\alpha, \beta \in \Lambda(\mathcal{N})$, then

$$h^*(\alpha \wedge \beta) = (h^*\alpha) \wedge (h^*\beta).$$

**Remark.**
For a "generalized" pullback, we have,

$$(h^*(\alpha))(X_1, \dots, X_{n_1}) = \alpha(h_* X_1, \dots, h_* X_{n_1}).$$

♍

*Proof.*

$$\begin{aligned}
(h^*\alpha)\wedge&(h^*\beta) \\
&= \frac{1}{n_1! n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\operatorname{sgn}\sigma) \alpha \otimes \beta(h_* X_{\sigma(1)}, \dots, h_* X_{\sigma(n_1+n_2)}). \\
&= \frac{1}{n_1! n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\operatorname{sgn}\sigma) h^* \left(\alpha \otimes \beta(X_{\sigma(1)}, \dots, X_{\sigma(n_1+n_2)})\right). \\
&= h^* \left( \frac{1}{n_1! n_2!} \sum_{\sigma \in S_{n_1+n_2}} (\operatorname{sgn}\sigma) \alpha \otimes \beta(X_{\sigma(1)}, \dots, X_{\sigma(n_1+n_2)}) \right). \\
&= h^*(\alpha \wedge \beta).
\end{aligned}$$

▨

**Theorem 5.4** (Skew-Symmetry)**.**
The exterior product makes $\Lambda(\mathcal{M})$ a graded algebra with skew-symmetry given by

$$\omega_1 \wedge \omega_2 = (-1)^{n_1 n_2} \omega_2 \wedge \omega_1.$$

*Proof.* In the definition of exterior product, first fix $\sigma = \sigma_0$ to consider only one term.

When we switch $\omega_1$ and $\omega_2$, we are essentially doing

$$\begin{aligned}
&(\omega_2 \otimes \omega_1)(X_{\sigma_0(1)}, \dots, X_{\sigma_0(n_2)}, \underbrace{X_{\sigma_0(n_2+1)}, \dots, X_{\sigma_0(n_1+n_2)}}) \\
=&(\omega_1 \otimes \omega_2)(\underbrace{X_{\sigma_0(n_2+1)}, \dots, X_{\sigma_0(n_1+n_2)}}, X_{\sigma_0(1)}, \dots, X_{\sigma_0(n_2)}).
\end{aligned}$$

Now,

$$\underbrace{1, 2, \ldots, n_2}, \underbrace{n_2 + 1}, \ldots, n_1 + n_2$$

$$\downarrow \; n_2 \text{ times}$$

$$\underbrace{n_2 + 1}, \underbrace{1, 2, \ldots, n_2}, \ldots, n_1 + n_2$$

$$\downarrow \; (n_1 - 1)n_2 \text{ times}$$

$$\underbrace{n_2 + 1, \ldots, n_1 + n_2}, \underbrace{1, 2, \ldots, n_2}$$

So $n_1 n_2$ transposes can achieve the desired effect. Therefore, every term in the summation is multiplied by $(-1)^{n_1 n_2}$, and we get the desired result. ▨

**Theorem 5.5** (Dimension of n-Forms)**.**
Let $\dim \mathcal{M} = m$. If $1 \leq n \leq m$, then $\Lambda^n(\mathcal{M}) = \binom{m}{n}$. If $n > m$, then $\Lambda^n(\mathcal{M}) = 0$.
Moreover, a basis for $\Lambda^n(\mathcal{M})_p$ is given by,

$$(dx^{\mu_1})_p \wedge (dx^{\mu_2})_p \wedge \cdots \wedge (dx^{\mu_n})_p, \; 1 \leq \mu_1 \leq \cdots \leq \mu_n \leq m.$$

**Remark.**
The proof is quite a pleasure to read (and to think of). Please see Hoffman. ♍

## 5.3 The Exterior Derivative

**Definition 5.6** (Exterior Derivative)**.**
Let a $k$-form be $\omega = \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$. Then

$$d\omega := \left( \frac{\partial \omega_{i_1 \dots i_k}}{\partial x^{i_0}} dx^{i_0} \right) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

If $\omega \in \Lambda^{\dim \mathcal{M}}(\mathcal{M})$, we define $d\omega = 0$.

**Theorem 5.6** (Exterior Derivative and Product)**.**

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d\omega_2.$$

**Theorem 5.7** (Exterior Derivative and Pullback)**.**
Given $h : \mathcal{M} \to \mathcal{N}$, $\omega$ an n-form on $\mathcal{N}$, then

$$d(h^* \omega) = h^*(d\omega).$$

**Theorem 5.8** (Functional Linearity of Exterior Derivative)**.**
Let $\omega$ be a 1-form on $\mathcal{M}$. Then $d\omega$ satisfies,

$$d\omega(fX, Y) = f d\omega(X, Y), \ \forall f \in C^\infty(\mathcal{M}),$$

where $fX$ is a vector field that gives $(fX)(p) = f(p)X_p$.

*Proof.* By Definition 5.6,

$$d\omega(fX, Y) = \mathcal{L}_{fX}(\omega(Y)) - \mathcal{L}_Y(\omega(fX)) - \omega([fX, Y]).$$

We break it down term by term. Firstly,

$$(\mathcal{L}_{fX}(\omega(Y)))(p) = f(p)X_p(\omega(Y)) = f(p)(\mathcal{L}_X(\omega(Y)))(p).$$

So

$$\mathcal{L}_{fX}(\omega(Y)) = f \cdot \mathcal{L}_X(\omega(Y)).$$

Secondly, we tackle $\mathcal{L}_Y(\omega(fX))$. In particular,

$$\omega(fX)(p) = \omega_p(f(p)X_p) = f(p)\omega_p(X_p) = f(p)(\omega(X))(p).$$

Therefore,

$$\mathcal{L}_Y(\omega(fX)) = \mathcal{L}_Y(f \cdot \omega(X)) = (\mathcal{L}_Y f)\omega(X) + f \cdot \mathcal{L}_Y(\omega(X)).$$

Thirdly,

$$\omega([fX, Y]) = \omega((fX) \circ Y - Y \circ (fX)).$$

In particular,

$$((Y \circ (fX))(g))(p) = Y_p((fX)(g)) = Y_p(f \cdot Xg) = (Y_p f)((Xg)(p)) + f(p) \cdot Y_p(Xg).$$

So,

$$Y \circ (fX) = (\mathcal{L}_Y f)X + f \cdot Y \circ X.$$

Substituting back,

$$\begin{aligned}
\omega([fX, Y]) &= \omega(f \cdot X \circ Y - (\mathcal{L}_Y f)X - f \cdot Y \circ X) \\
&= \omega(f[X, Y] - (\mathcal{L}_Y f)X) \\
&= f\omega([X, Y]) - (\mathcal{L}_Y f)\omega(X).
\end{aligned}$$

Finally,

$$\begin{aligned}
d\omega(fX, Y) &= \mathcal{L}_{fX}(\omega(Y)) - \mathcal{L}_Y(\omega(fX)) - \omega([fX, Y]) \\
&= f \cdot \mathcal{L}_X(\omega(Y)) - (\mathcal{L}_Y f)\omega(X) - f \cdot \mathcal{L}_Y(\omega(X)) - f\omega([X, Y]) + (\mathcal{L}_Y f)\omega(X) \\
&= f(\mathcal{L}_X(\omega(Y)) - \mathcal{L}_Y(\omega(X)) - \omega([X, Y])) \\
&= f d\omega(X, Y).
\end{aligned}$$

▨

**Corollary 5.8.1** (Local Nature of Exterior Derivative)**.**
When $\omega$ is fixed, the value of $d\omega$ depends only on the local values of vector fields.

$$d\omega(X, Y)(p) = X^\mu(p)Y^\nu(p)d\omega(\partial_\mu, \partial_\nu)(p).$$

*Proof.* Write $X = X^\mu\partial_\mu$, noting that $X^\mu \in C^\infty(\mathcal{M})$, and use Theorem 5.8. ▨

## 5.4 DeRham Cohomology

**Theorem 5.9** (Twice Exterior Differential)**.**
For all $\omega \in \Lambda^n(\mathcal{M}), 1 \leq n \leq \dim M$, we have

$$d^2\omega = 0.$$

**Remark.**
This means

$$\operatorname{Im}(d : \Lambda^{n-1}(\mathcal{M}) \to \Lambda^n(\mathcal{M})) \subseteq \operatorname{Ker}(d : \Lambda^n(\mathcal{M}) \to \Lambda^{n+1}(\mathcal{M})).$$

This type of structure is called a differential complex, and is common in many structures. ♍

**Definition 5.7** (Closed Form)**.**
An n-form $\omega$ is closed if $d\omega = 0$. The set of all closed n-forms is denoted $Z^n(\mathcal{M})$.

**Definition 5.8** (Exact Form)**.**
An n-form $\omega$ is exact if $\omega = d\beta$ for some $(n-1)$-form $\beta$. The set of all exact n-forms is denoted $B^n(\mathcal{M})$.

**Remark.**
It is guaranteed that $B^n(\mathcal{M}) \subseteq Z^n(\mathcal{M})$, that is, exactness implies closure. But how much closed form is not exact is the study of cohomology theory. ♍

**Theorem 5.10** (Poincare's Lemma)**.**
On Euclidean space $\mathbb{R}^m$,

$$B^n(\mathcal{M}) = Z^n(\mathcal{M}),\ \forall n > 0.$$

**Definition 5.9** (DeRham Cohomology Groups)**.**
The DeRham cohomology groups $H^n(\mathcal{M}), 0 \leq n \leq \dim \mathcal{M}$ are the quotient

spaces

$$H^n(\mathcal{M}) := Z^n(\mathcal{M})/B^n(\mathcal{M}).$$

**Remark.**
Recall the definition of quotient groups that $H^n(\mathcal{M})$ consists of elements of form $z + B^n(\mathcal{M}), z \in Z^n(\mathcal{M})$.
If all closed forms are exact, $Z^n(\mathcal{M}) \subseteq B^n(\mathcal{M})$, then $H^n(\mathcal{M}) \simeq \{0\}$. ♍

**Theorem 5.11** (Criterion of Exact ODE)**.**
On the Euclidean space $\mathbb{R}^2$, given a 1-form $\omega = \omega_1 dx^1 + \omega_2 dx^2$. Then

$$\omega \in B^1(\mathbb{R}^2) \iff \partial_2\omega_1 = \partial_1\omega_2.$$

**Remark.**
This is an important theorem to me, for it connects the "exactness of differential forms" to the familiar notion of "exactness of differential equations".
It also provides the first hints that we are actually integrating forms, and that exterior differentiation of a 0-form resembles gradient in usual vector calculus terms. ♍

*Proof.* Via Poincare lemma Theorem 5.10, on $\mathbb{R}^2$, exactness is equivalent to closure. So we need only to determine the condition that $d\omega = 0$. Using Definition 5.6,

$$\begin{aligned} d\omega &= \partial_\nu \omega_{\mu_1} dx^\nu \wedge dx^{\mu_1} \\ &= \partial_2\omega_1 dx^2 \wedge dx^1 + \partial_1\omega_2 dx^1 \wedge dx^2 \\ &= (\partial_2\omega_1 - \partial_1\omega_2)\, dx^2 \wedge dx^1. \end{aligned}$$

▨

# 6 Tensors

**Definition 6.1** (Tensors)**.**
If $\dim \mathcal{M} \neq \infty$, the tensors of type $(r, s)$ $T_p^{r,s}\mathcal{M}$ are all the linear functions

$$f : \bigtimes^{r} T_p^*\mathcal{M} \times \bigtimes^{s} T_p\mathcal{M} \to \mathbb{R}.$$

I.e., it eats $r$ covectors and $s$ vectors.

**Theorem 6.1** (Dimensions of General Tensor Space)**.**
The dimension of $T_p^{r,s}\mathcal{M}$ is $m^r m^s$. In particular, a basis for the space is,

$$\bigotimes_{1\leq\mu_1\cdots\mu_r\leq m} \left(\partial_{\mu_i}\right)_p \otimes \bigotimes_{1\leq\nu_1\cdots\nu_s\leq m} (dx^{\nu_i})_p$$

**Remark.**
For a detailed proof, see Hoffman. ♍

**Theorem 6.2** (Transformation Properties of Tensor Fields)**.**
Given a manifold $\mathcal{M}$ of dimension $m$, choose two overlapping charts $\phi : U \to V$ with local coordinates $x^1, \dots, x^m$, $\phi' : U' \to V'$ with $x'^1, \dots, x'^m$. A tensor field $T$ of type $(r, s)$ with local representation on $U$ given by

$$T^{\mu_1\cdots\mu_r}{}_{\nu_1\dots\nu_s}\,\partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_r} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_s}$$

transforms like

$$T^{\mu'_1\cdots\mu'_r}{}_{\nu'_1\dots\nu'_s} = T^{\mu_1\cdots\mu_r}{}_{\nu_1\dots\nu_s}\frac{\partial x'^{\mu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial x'^{\mu'_r}}{\partial x^{\mu_r}}\frac{\partial x^{\nu_1}}{\partial x'^{\nu'_1}}\cdots\frac{\partial x^{\nu_1}}{\partial x'^{\nu'_s}}$$

# 7 Integration of Differential Forms

## 7.1 Partition of Unity

**Definition 7.1** (Support)**.**
Let $X$ be a topological space, and $f : X \to \mathbb{R}$. Then the support of $f$ is defined as

$$\operatorname{supp} f := \{\, x \in X \mid f(x) \neq 0 \,\} .$$

**Theorem 7.1** (Partition of Unity)**.**
Let $\mathcal{M}$ be a $C^\infty$ manifold with dimension $m$ with atlas $\Phi$.
Let $\Phi = \left\{ \phi_j \;\middle|\; \phi_j : V_j \to \phi_j(V_j), j \in J \right\}$.
Then it is possible to construct a set of $C^\infty$ functions $\rho_j, j \in J$ s.t.

$$1 = \sum_{j \in J} \rho_j, \ \operatorname{supp} \rho_j \subseteq V_j$$

## 7.2 Orientation

### 7.2.1 Definition

**Definition 7.2** (Compatible Coordinate Charts)**.**
Given a manifold $\mathcal{M}$, its two coordinate charts are called compatible (have the same orientation) if,

$$\det J > 0.$$

Where $J$ is the Jacobian matrix Definition 1.6.
If the manifold has a maximal compatible atlas, then we say the manifold is orientable, and we may call its corresponding orientation positive and denote the atlas $\Phi_+$.

**Theorem 7.2.**
A manifold has either no orientation (any atlas is not compatible) or two orientations.

**Theorem 7.3** (Orientability and Existence of Forms of Highest Degree).
A manifold is orientable iff there exists a nowhere vanishing differential form of the highest degree.

### 7.2.2 Positively Oriented Boundary

**Definition 7.3** (Positively Oriented Boundary).
Let $\mathcal{M}$ be a orientable $C^\infty$ manifold with dimension $m$, positively oriented by compatible atlas $\Phi_+$. Define coordinate charts on $\partial\mathcal{M}$ from $\Phi$ as follows,

$$\phi^{\partial\mathcal{M}} : U_\phi \cap \partial\mathcal{M} \to \mathbb{R}^{m-1},$$

Then $\Phi_+^{\partial\mathcal{M}} := \left\{ \phi^{\partial\mathcal{M}} \mid \phi \in \Phi_+ \right\}$ determines an orientation on $\partial\mathcal{M}$, called the positive orientation.

## 7.3 Pseudoforms

### 7.3.1 Definition

**Definition 7.4** (Pseudoforms).
A $C^\infty$ pseudo $k$-form $\tilde{\omega} \in \tilde{\Lambda}^k(\mathcal{M})$ consists of a family of differential $k$-forms $\omega_\phi \in \Lambda^k(\phi(U_\phi))$, $\phi \in \Phi$, $\phi : U_\phi \to V_\phi = \phi(U_\phi) \subseteq (-\infty, 0] \times \mathbb{R}^{m-1}$, with an additional requirement that

$$\omega_{\phi'}\big|_{\phi'(U_\phi \cap U_{\phi'})} = (\operatorname{sgn}\det J) \left(\phi \circ {\phi'}^{-1}\right)^* \left(\omega_\phi\big|_{\phi(U_\phi \cap U_{\phi'})}\right), \ \forall \phi, \phi' \in \Phi,$$

where $\operatorname{sgn}\det J$ denotes the sign of Jacobian determinant.

**Remark.**
The transformation rule differ from usual forms only in the choice of sign.
One cannot pullback a pseudoform. ♍

### 7.3.2 Pseudoforms and Forms on Orientable Manifolds

**Definition 7.5** (Pseudoforms to Forms)**.**
For a pseudoform $\tilde{\omega} \in \tilde{\Lambda}^k(\mathcal{M})$ on an orientable manifold $\mathcal{M}$ of degree $m$ positively oriented by the atlas $\Phi_+$, we define its underlying real form by letting them agree on the positive charts, that is,

$$\omega_\phi := o(\phi)\tilde{\omega}_\phi, \ \forall \phi \in \Phi,$$

where

$$o(\phi) := \begin{cases} 1 & \forall \phi \in \Phi_+ \\ -1 & \forall \phi \in \Phi_- \end{cases}$$

**Definition 7.6** (Forms to Pseudoforms)**.**
For a form $\omega \in \Lambda^k(\mathcal{M})$ on an orientable manifold $\mathcal{M}$ of degree $m$, we define its corresponding pseudoform by taking its positive value. That is,

$$\tilde{\omega}_\phi := o(\phi)\omega_\phi, \ \forall \phi \in \Phi,$$

## 7.4 Integration of Forms of Highest Degree

**Definition 7.7** (Integration)**.**
Let $\mathcal{M}$ be a paracompact $C^\infty$ manifold of dimension $m$. Choose a $C^\infty$ partition of unity $\rho_j, j \in J$ of $\mathcal{M}$ s.t. $\operatorname{supp} \rho_j \subseteq U_{\phi_j} := U_j$.
Let a pseudo-$m$-form $\tilde{\omega} \in \tilde{\Lambda}^m(\mathcal{M})$ has local expression $\tilde{\omega}_{\phi_j} = f_j(x) dx_j^1 \wedge \cdots \wedge dx_j^m$, we say

$$\int_{\mathcal{M}} \tilde{\omega} = \sum_{j \in J} \int_{(-\infty,0] \times \mathbb{R}^{m-1}} (\rho_j \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m$$

if the finite sum exists and has the same value for all choices of $\rho_j$ and $\phi_j$.

**Remark.**
The following theorem reveals why we integrate pseudoforms, not usual forms.
♍

**Theorem 7.4** (Criterion of Existence of Integral)**.**
If $\operatorname{supp} \tilde{\omega}$ is compact, then $\int_{\mathcal{M}} \tilde{\omega}$ exists.

*Proof.* Let two sets of coordinate charts be

$$\begin{aligned} \phi_j &: U_j \to V_j, j \in J \\ \phi'_k &: U'_k \to V'_k, k \in K. \end{aligned}$$

And cooresponding partition of unity be $\rho_j, \rho'_k$.

(The goal) Show

$$\begin{aligned} &\sum_{j \in J} \int (\rho_j \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m \\ &= \sum_{k \in K} \int (\rho'_k \circ {\phi'_k}^{-1})(x') f'_k(x') d{x'}^1 \dots d{x'}^m \end{aligned}$$

(Split using $\rho'_k$)

$$\begin{aligned} \int_{\mathcal{M}} \omega &= \sum_{j \in J} \int (\rho_j \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m \\ &= \sum_{j \in J} \int \sum_{k \in K} (\rho'_k \circ \phi_j^{-1})(x) (\rho_j \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m. \end{aligned}$$

Since the sum is finite, and $\operatorname{supp} \omega$ is compact, and therefore the integral is not improper; thus, there can be no limit or Fubini problems on exchanging sums and integrals. So

$$\int_{\mathcal{M}} \omega = \sum_{j \in J} \sum_{k \in K} \int (\rho_j \rho'_k \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m$$

(Change of variables) First fix $j, k$.

$$\begin{aligned} &\int (\rho_j \rho'_k \circ \phi_j^{-1})(x) f_j(x) dx^1 \dots dx^m \\ &= \int (\rho_j \rho'_k \circ \phi_j^{-1})(\phi_j \circ {\phi'_k}^{-1}(x')) f_j(\phi_j \circ {\phi'_k}^{-1}(x')) \left| \det \left( \frac{\partial x}{\partial x'} \right) \right| d{x'}^1 \dots d{x'}^m \\ &= \int (\rho_j \rho'_k \circ {\phi'_k}^{-1}(x')) f_j(\phi_j \circ {\phi'_k}^{-1}(x')) \left| \det \left( \frac{\partial x}{\partial x'} \right) \right| d{x'}^1 \dots d{x'}^m \end{aligned}$$

(Use pullback requirement) From Definition 7.4,

$$\omega_{\phi'_k} = (\operatorname{sgn} \det J)(\phi_j \circ {\phi'_k}^{-1})^* \omega_{\phi_j},$$

we see

$$
\begin{aligned}
& f_j'(x')dx^1 \wedge \cdots \wedge dx^m \\
=& (\operatorname{sgn}\det J) f_j(\phi_j \circ {\phi_k'}^{-1}(x')) \left( \frac{\partial x^1}{\partial {x'}^{\ell_1}} d{x'}^{\ell_1} \right) \wedge \cdots \wedge \left( \frac{\partial x^m}{\partial {x'}^{\ell_m}} d{x'}^{\ell_m} \right) \\
=& (\operatorname{sgn}\det J) \sum_{\sigma \in S_m} f_j(\phi_j \circ {\phi_k'}^{-1}(x'))(\operatorname{sgn}\sigma) \frac{\partial x^1}{\partial {x'}^{\sigma(1)}} \cdots \frac{\partial x^m}{\partial {x'}^{\sigma(m)}} d{x'}^1 \wedge \cdots \wedge d{x'}^m \\
=& (\operatorname{sgn}\det J) f_j(\phi_j \circ {\phi_k'}^{-1}(x')) \det\left( \frac{\partial x}{\partial x'} \right) d{x'}^1 \wedge \cdots \wedge d{x'}^m \\
=& f_j(\phi_j \circ {\phi_k'}^{-1}(x')) \left| \det\left( \frac{\partial x}{\partial x'} \right) \right| d{x'}^1 \wedge \cdots \wedge d{x'}^m
\end{aligned}
$$

Therefore, the integral

$$
\begin{aligned}
& \int (\rho_j \rho_k' \circ {\phi_k'}^{-1}(x')) f_j(\phi_j \circ {\phi_k'}^{-1}(x')) \left| \det\left( \frac{\partial x}{\partial x'} \right) \right| d{x'}^1 \ldots d{x'}^m \\
=& \int (\rho_j \rho_k' \circ {\phi_k'}^{-1}(x')) f_j'(x') d{x'}^1 \ldots d{x'}^m
\end{aligned}
$$

(Closing) By moving the sum wrt $j \in J$ into the integral and using the property of partition of unity, the proof is completed. ▨

## 7.5 Integration of Forms of Lower Degree

### 7.5.1 Definition

**Definition 7.8** (Integration of Lower Degree Forms)**.**
Let $\mathcal{Z}$ be an oriented $C^\infty$ manifold of dimension $d$, $f : \mathcal{Z} \to \mathcal{M}$ be a $C^\infty$ map to a $C^\infty$ manifold $\mathcal{M}$ of dimension $m$.
Let $\omega \in \Lambda^d(\mathcal{M})$, we define

$$
\int_{\mathcal{Z}} \omega := \int_{\mathcal{Z}} f^* \omega
$$

using the positive orientation of $\mathcal{Z}$, if it exists and the pullback function $f$ is clear from context.

**Remark.**
Let's look at an example. Choose $\mathcal{M} = \mathbb{R}^2$, $\mathcal{Z} = [-1, 1]$, $\omega = dx^2$, and $f : \mathcal{Z} \to \mathcal{M}$ defined trivially by $p \mapsto (0, p)$.

If we choose the positive orientation by setting $\mathrm{id} \in \Phi_+$, then we see

$$\begin{aligned} \eta &:= f^*\omega = dx^2 \\ \tilde{\eta} &= dx^2 \\ \int_{\mathcal{Z}} \tilde{\eta} &= 1. \end{aligned}$$

If we choose the positive orientation by setting $-\mathrm{id} \in \Phi_+$, then

$$\begin{aligned} \eta' &:= f^*\omega = dx^2 \\ \tilde{\eta}' &= -dx^2 \\ \int_{\mathcal{Z}} \tilde{\eta}' &= -1. \end{aligned}$$

♍

### 7.5.2 Stoke's theorem

**Theorem 7.5** (Stoke's Theorem)**.**
If $\mathcal{M}$ is an oriented $C^\infty$ manifold of dimension $m$ and $\omega \in \Lambda^{m-1}(\mathcal{M})$, then

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega := \int_{\partial\mathcal{M}} i^*\omega,$$

where $i : \partial\mathcal{M} \to \mathcal{M}$ is just the immersion map, $i : p \mapsto p$.

*Proof.* (Partition Using Charts) Choose a $C^\infty$ partition of unity Theorem 7.1 $\rho_j, j \in J$ s.t. $\operatorname{supp}\rho_j$ are compact and $\operatorname{supp}\rho_j \subseteq U_{\phi_j} := U_j$.

Now $\omega = \sum_{j\in J} \rho_j\omega$ is a finite sum. So it suffices to show that, if $\eta \in \Lambda^{m-1}(\mathcal{M})$, $\operatorname{supp}\eta$ compact and $\operatorname{supp}\eta \subseteq U_\phi$ then $\int_{\mathcal{M}} d\eta = \int_{\partial\mathcal{M}} \eta$, and apply $\eta = \rho_j\omega$ for all $j \in J$.

(The integral) Suppose the coordinates of $\phi$ is labeled $x^1, \dots, x^m$. Locally, let

$$\eta = \sum_{\ell=1}^{m} f_\ell dx^1 \wedge \cdots \cancel{dx^\ell} \cdots \wedge dx^m,$$

where $f_\ell : (-\infty, 0] \times \mathbb{R}^{m-1} \to \mathbb{R}$, and it is $C^\infty$. Then, also locally,

$$
\begin{aligned}
d\eta &= \sum_{\ell=1}^{m} \frac{\partial f_\ell}{\partial x^\ell} dx^\ell \wedge dx^1 \wedge \cdots \cancel{dx^\ell} \cdots \wedge dx^m \\
&= \left( \sum_{\ell=1}^{m} (-1)^{\ell-1} \frac{\partial f_\ell}{\partial x^\ell} \right) dx^1 \wedge \cdots \wedge dx^m.
\end{aligned}
$$

Then by Definition 7.7,

$$
\int_{\mathcal{M}} d\eta = \int_{(-\infty,0]\times\mathbb{R}^{m-1}} \left( \sum_{\ell=1}^{m} (-1)^{\ell-1} \frac{\partial f_\ell}{\partial x^\ell} \right) dx^1 \dots dx^m.
$$

(To be precise, we need to choose another partition of unity $\rho_j'$ to do intergration. But we can just choose it to cover all of $\operatorname{supp} \eta$ and don't care all other parts, so that doesn't matter too much.)

Choose a rectangular region $R$ s.t.

$$
\operatorname{supp} \eta \subseteq [a_1, 0] \times \cdots \times [a_m, b_m]
$$

and define

$$
R_\ell := [a_1, 0] \times \dots \cancel{[a_\ell, b_\ell]} \cdots \times [a_m, b_m].
$$

($\ell = 2, \dots, m$) In this case, by Fubini and FTC,

$$
\begin{aligned}
&\int_R \frac{\partial f_\ell}{\partial x^\ell} dx^1 \dots dx^m \\
=& \int_{R_\ell} \left( \int_{a_\ell}^{b_\ell} \frac{\partial f_\ell}{\partial x^\ell} dx^\ell \right) dx^1 \dots \cancel{dx^\ell} \dots dx^m \\
=& \int_{R_\ell} (\underbrace{f_\ell(x^1, \dots, b_\ell, \dots, x^m)}_{0} - \underbrace{f_\ell(x^1, \dots, a_\ell, \dots, x^m)}_{0}) dx^1 \dots \cancel{dx^\ell} \dots dx^m \\
=& 0,
\end{aligned}
$$

since $\operatorname{supp} \eta \subseteq R$, so on the boundary $f = 0$.

($\ell = 1$) Now the integral has only one term left.

$$\begin{aligned}
\int_{\mathcal{M}} d\eta &= \int_R \frac{\partial f_1}{\partial x^1} dx^1 \dots dx^m \\
&= \int_{R_1} \left( \int_{a_1}^{0} \frac{\partial f_1}{\partial x^1} dx^1 \right) dx^2 \dots dx^m \\
&= \int_{R_\ell} (f_1(0, x^2, \dots, x^m) - \underbrace{f_1(a_1, x^2, \dots, x^m)}_{0}) dx^2 \dots dx^m \\
&= \int_{\mathbb{R}^{m-1}} (f_1 \circ i) dx^2 \dots dx^m \\
&= \int_{\partial \mathcal{M}} i^* \eta.
\end{aligned}$$

▨