1 Differentiable Manifolds

1.1 Definition

1.1.1 Coordinate Charts

Definition 1.1 (Coordinate Charts).
An m-dimensional, $m \neq \infty$ coordinate chart on a topological space $\mathcal{M}$ is a pair

$$(U, \phi) \begin{cases} U \subseteq \mathcal{M}, U \text{ open} \\ \phi : U \to \mathbb{R}^m, \phi \text{ homeomorphism} \end{cases}$$

Remark.
If $U = \mathcal{M}$, then we say the coordinate chart ϕ is globally defined; if not, then it is locally defined. Few manifolds have globally defined property.

♍

Definition 1.2 (Overlap Function).
Let $(U_1, \phi_1), (U_2, \phi_2)$ be a pair of m-dimensional coordinate charts with $U_1 \cap U_2 \neq \varnothing$. Then the overlap function is defined as

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \subseteq \mathbb{R}^m \to \phi_2(U_1 \cap U_2) \subseteq \mathbb{R}^m.$$

Definition 1.3 (Atlas).
An m-dimensional atlas on $\mathcal{M}$ is a family of m-dimensional coordinate charts $(U_i, \phi_i), i \in I$ s.t.

1. $\mathcal{M} = \bigcup_{i \in I} U_i$.
2. Each overlap function $\phi_j \circ \phi_i^{-1}, i, j \in I$ is C^∞.

Definition 1.4 (Differentiable Manifolds)**.**
An m-dimensional differentiable manifold is a topological space $\mathcal{M}$ equipped with an atlas.

Remark.
We didn't define a differentiable manifold by regulating the differentiability of the coordinate charts themselves. That's because differentiation is not defined on a manifold, so we need to rely on Euclidean spaces.

♍

Definition 1.5 (Coordinate Functions)**.**
The coordinate functions are the (Euclidean) components of coordinate.

$$\begin{aligned} \phi : U \to \mathbb{R}^m \qquad & \qquad p \mapsto \phi(p), \\ \phi^\mu : U \to \mathbb{R} \qquad & \qquad \text{s.t. } \phi(p) = \begin{pmatrix} \phi^1(p) \\ \vdots \\ \phi^m(p) \end{pmatrix}. \end{aligned}$$

An alternative notation is

$$x^\mu := \phi^\mu.$$

Remark.
There are (Euclidean) projection functions,

$$u^\mu : \mathbb{R}^m \to \mathbb{R}.$$

But I think mention it will cause a lot of confusion. Just remember in the future when we say $\frac{\partial}{\partial u^\mu}$, we are referring to the Euclidean partial derivative wrt the μ-th component.

♍

2 Tangent Spaces

2.1 The Curve Formulation of Tangent Spaces

Remark.
The definition of manifold do not require the entity to be embeded in a higher dimensional space. Therefore, the traditional view of tangency is not valid here.

Remark.
The curve formulation remains valid in the infinite-dimensional case, while the algebraic formulation is not. However, in the finite-dimensional case, they are isomorphic.

2.1.1 Curves and Vectors

Definition 2.1 (Curve)**.**
A curve on $\mathcal{M}$ is a C^∞ map,

$$\sigma : (-\epsilon, \epsilon) \to \mathcal{M}.$$

Definition 2.2 (Curve Tangency)**.**
Two curves σ_1, σ_2 are tangent at $p \in \mathcal{M}$ if

1. $\sigma_1(0) = \sigma_2(0) = p$.
2. $\frac{d}{dt}(x^i \circ \sigma_1(t))\big|_{t=0} = \frac{d}{dt}(x^i \circ \sigma_2(t))\big|_{t=0}, \ 1 \leq i \leq m$.

Remark.

Written more compactly,

$$\frac{d}{dt}(\phi \circ \sigma_1)\bigg|_{t=0} = \frac{d}{dt}(\phi \circ \sigma_2)\bigg|_{t=0}$$

♍

Definition 2.3 (Tangent Vectors)**.**
A tangent vector at $p \in \mathcal{M}$ is an equivalence class of curves where the equivalence relation is that they are tangent. It will be denoted as

$$v = [\sigma].$$

Definition 2.4 (Tangent Space)**.**
The tangent space $T_p\mathcal{M}$ at point p is the set of all tangent vectors at point p.

Definition 2.5 (Tangent Bundle)**.**
The tangent bundle $T\mathcal{M}$ is

$$T\mathcal{M} := \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}.$$

2.1.2 Addition and Scalar Multiplication

Definition 2.6 (Addition and Scalar Multiplication)**.**
Let $v_1 = [\sigma_1], v_2 = [\sigma_2] \in T_p\mathcal{M}$, and $r \in \mathbb{R}$. Then define

$$v_1 + v_2 := [\phi^{-1} \circ (\phi \circ \sigma_1 + \phi \circ \sigma_2)],$$
$$rv_1 := [\phi^{-1} \circ (r\phi \circ \sigma_1)].$$

Theorem 2.1.
The definition 2.6 is well-defined. That is, they are independent of the choice

of chart (U, ϕ) and σ_1, σ_2 as long as $v_1 = [\sigma_1]$ and $v_2 = [\sigma_2]$.

Therefore, $T_p\mathcal{M}$ is a real vector space.

Proof. Let $v_1 = [\sigma_1] = v_1' := [\tau_1], v_2 = [\sigma_2] = v_2' := [\tau_2]$. First check (1) of 2.2,

$$\begin{aligned}(rv_1 + v_2)(0) &= (\phi^{-1} \circ (r\phi \circ \sigma_1(0) + \phi \circ \sigma_2(0))) \\ &= (\phi^{-1} \circ (r\phi \circ \tau_1(0) + \phi \circ \tau_2(0))) \\ &= (rv_1' + v_2')(0),\end{aligned}$$

since $\phi \circ \sigma_1(0) = \phi \circ \tau_1(0) = \phi(p)$ by equivalence, and the same for σ_2.

Now consider

$$\begin{aligned}\frac{d}{dt}(\phi \circ (rv_1 + v_2))\Big|_{t=0} &= \frac{d}{dt}(r\phi \circ \sigma_1 + \phi \circ \sigma_2)\Big|_{t=0} \\ &= r\,\frac{d}{dt}(\phi \circ \sigma_1)\Big|_{t=0} + \frac{d}{dt}(\phi \circ \sigma_2)\Big|_{t=0} \\ &= r\,\frac{d}{dt}(\phi \circ \tau_1)\Big|_{t=0} + \frac{d}{dt}(\phi \circ \tau_2)\Big|_{t=0} \\ &= \frac{d}{dt}(\phi \circ (rv_1' + v_2'))\Big|_{t=0},\end{aligned}$$

since $\frac{d}{dt}(\phi \circ \sigma_1)\big|_{t=0} = \frac{d}{dt}(\phi \circ \tau_1)\big|_{t=0}$ by equivalence, and the same for σ_2. ▨

2.1.3 Curves and Derivation

Definition 2.7 (Directional Derivative)**.**
For any $f : \mathcal{M} \to \mathbb{R}$ s.t. $f \in C^\infty$, we define

$$v(f) := \frac{d}{dt}(f \circ \sigma(t))\Big|_{t=0},$$

where $v = [\sigma]$.

Theorem 2.2.
The definition 2.7 is well-defined. That is, $v(f)$ is independent of the curve σ chosen as well as $v = [\sigma]$.

Proof. Let $v_1 = [\sigma_1] = [\sigma_2] = v_2$. Then

$$v_1(f) = \frac{d}{dt}(f \circ \sigma_1)\Big|_{t=0},$$
$$v_2(f) = \frac{d}{dt}(f \circ \sigma_2)\Big|_{t=0},$$

$$\frac{d}{dt}(\phi \circ \sigma_1)\Big|_{t=0} = \frac{d}{dt}(\phi \circ \sigma_2)\Big|_{t=0}.$$

Then

$$\begin{aligned} v_1(f) &= \frac{d}{dt}(\underbrace{(f \circ \phi^{-1})}_{\mathbb{R} \leftarrow \mathbb{R}^m} \circ \underbrace{(\phi \circ \sigma_1)}_{\mathbb{R}^m \leftarrow \mathbb{R}})\Bigg|_{t=0} \\ &= (f \circ \phi^{-1})' \circ (\phi \circ \sigma_1) \cdot (\phi \circ \sigma_1)'\big|_{t=0} \\ &= (f \circ \phi^{-1})' \circ (\phi \circ \sigma_2) \cdot (\phi \circ \sigma_2)'\big|_{t=0} \\ &= v_2(f), \end{aligned}$$

since $\phi \circ \sigma_1(0) = \phi \circ \sigma_2(0) = \phi(p)$, and $(\phi \circ \sigma_1)' = (\phi \circ \sigma_2)'$ by equivalence. ▨

2.2 The Algebraic Formulation of Tangent Spaces

2.2.1 The Space of Derivations

Definition 2.8 (Derivation)**.**
A derivation at $p \in \mathcal{M}$ is a map $v : C^\infty(\mathcal{M}) \to \mathbb{R}$ s.t.

1. $v(rf + g) = rv(f) + v(g)$, (Linear)
2. $v(fg) = f(p)v(g) + g(p)v(f)$, (Leibniz)

where $f, g \in C^\infty$.

Definition 2.9 (Tangent Space (Algebraic))**.**
The space of all derivations at $p \in \mathcal{M}$ is denoted $D_p\mathcal{M}$.

Definition 2.10 (Addition and Scalar Multiplication)**.**
Given $v_1, v_2 \in D_p\mathcal{M}$, define

$$(v_1 + v_2)(f) := v_1(f) + v_2(f)$$
$$(rv)(f) := rv(f).$$

Theorem 2.3.
$D_p\mathcal{M}$ is a real vector space.

2.2.2 The Basis Tangent Vectors

Definition 2.11 (Basis Tangent Vectors)**.**
We define the basis tangent vectors via derivations by

$$\left(\partial_\mu\right)_p f := \frac{\partial}{\partial u^\mu}(f \circ \phi^{-1}(\vec{u}))\Big|_{\vec{u}=\phi(p)}, 1 \leq \mu \leq \dim \mathcal{M}.$$

where $u \in \mathbb{R}^m$, $f : \mathcal{M} \to \mathbb{R}$, $f \in C^\infty$. For the use of u^μ, see 1.5.

Theorem 2.4.

$$\left(\partial_\mu\right)_p x^\nu = \delta^\nu{}_\mu.$$

Proof. Although a simple exercise, it was a good chance to explain the sophisticated notation.

$$\left(\partial_\mu\right)_p x^\mu = \frac{\partial}{\partial u^\mu}\left(x^\mu \circ \phi^{-1}\begin{pmatrix} u^1 \\ \vdots \\ u^m \end{pmatrix}\right)\Bigg|_{\phi(p)}.$$

The coordinate $u \in \mathbb{R}^m$ was brought to $\mathcal{M}$ and projected to $\mathbb{R}^m$ again and taken out the μ-th component. So

$$= \frac{\partial}{\partial u^\mu}(u^\mu)\Big|_{\phi(p)} = 1.$$

▨

Theorem 2.5 (Linear Independence of Basis Tangent Vectors)**.**
The basis tangent vectors $\left(\partial_\mu\right)_p, 1 \leq \mu \leq \dim \mathcal{M}$ are linear independent.

Proof. Suppose $a^\mu \left(\partial_\mu\right)_p = 0$. Then

$$a^\mu \left(\partial_\mu\right)_p (x^\nu) = a^\mu \delta^\nu{}_\mu = 0(x^\nu) = 0.$$

So $a^\mu = 0$. ▨

Theorem 2.6 (Coordinate Expansion of Tangent Vectors)**.**
For all $v \in D_p\mathcal{M}$, we have

$$v = v^\mu \left(\partial_\mu\right)_p,$$

where Einstein notation was used, and $v^\mu = v(x^\mu)$.

Remark.
The proof was sophisticated and did not teach me much.

♍

2.3 Isomorphism of Curves and Derivations

Theorem 2.7 (Isomorphism of Curves and Derivations)**.**
Similar to 2.7, we define the linear map $\iota : T_p\mathcal{M} \to D_p\mathcal{M}$ acting on $v = [\sigma] \in T_p\mathcal{M}$ by

$$\iota(v)(f) := \frac{d}{dt}(f \circ \sigma(t))\Big|_{t=0}.$$

Then ι is a linear isomorphism. Note that RHS $\in D_p\mathcal{M}$.

Proof. (linearity) Choose ϕ s.t. $\phi(p) = 0$.

$$\begin{aligned}
\iota(rv_1 + v_2)(f) &= \frac{d}{dt}(f \circ \phi^{-1} \circ (r\phi \circ \sigma_1 + \phi \circ \sigma_2))\Big|_{t=0} \\
&= ((f \circ \phi^{-1})' \circ (r\phi \circ \sigma_1 + \phi \circ \sigma_2) \cdot (r\phi \circ \sigma_1 + \phi \circ \sigma_2)')\big|_{t=0} \\
&= ((f \circ \phi^{-1})'(0) \cdot ((r\phi \circ \sigma_1)' + (\phi \circ \sigma_2)'))\big|_{t=0} \\
&= (r(f \circ \phi^{-1})' \circ (\phi \circ \sigma_1) \cdot (\phi \circ \sigma_1)' + (f \circ \phi^{-1})' \circ (\phi \circ \sigma_2) \cdot (\phi \circ \sigma_2)')\big|_{t=0} \\
&= r\iota(v_1) + \iota(v_2)(f).
\end{aligned}$$

(surjectivity) Since ι is linear, surjectivity is equivalent to injectivity and therefore to bijectivity. To show surjectivity, we need to construct a curve for all $v' \in D_p\mathcal{M}$ s.t. $\iota(v) = v'$.

Let $v' \in D_p\mathcal{M}$ and construct $\sigma : (-\epsilon, \epsilon) \to \mathcal{M}$ s.t.

$$\begin{aligned}
\sigma(0) &= p, \\
v^\mu = v(x^\mu) &= \frac{d}{dt}(x^\mu \circ \sigma(t))\Big|_{t=0}.
\end{aligned}$$

Then

$$v(f) = v^\mu \left(\partial_\mu\right)_p f = \frac{d}{dt}(x^\mu \circ \sigma(t))\Big|_{t=0} \left(\partial_\mu\right)_p f.$$

Also,

$$\begin{aligned}
\frac{d}{dt}(f \circ \sigma(t))\Big|_{t=0} &= \frac{d}{dt}(f \circ \phi^{-1} \circ \phi \circ \sigma(t))\Big|_{t=0} \\
&= \sum_{\mu=1}^{m} \frac{\partial}{\partial u^\mu}(f \circ \phi^{-1})\Big|_{\phi(p)} \frac{d}{dt}(u^\mu \circ \phi \circ \sigma)\Big|_{t=0} \quad \text{(component-wise)} \\
&= \sum_{\mu=1}^{m} \left(\partial_\mu\right)_p f \, \frac{d}{dt}(x^\mu \circ \sigma)\Big|_{t=0} \\
&= v(f).
\end{aligned}$$

Thus completing the proof. ▨

2.4 Pushforward

2.4.1 Definition and Linearity

Remark.

The pushforward $h_* : T_p\mathcal{M} \to T_{h(p)}\mathcal{N}$ of a specific function $h : \mathcal{M} \to \mathcal{N}$ can be thought of as local linearization of the function.

♍

Definition 2.12 (Pushforward)**.**

Given a function $h : \mathcal{M} \to \mathcal{N}$ and $v \in T_p\mathcal{M}$, then we define the pushforward $h_* : T_p\mathcal{M} \to T_{h(p)}\mathcal{N}$ by

$$h_*(v) := [h \circ \sigma],\ v = [\sigma].$$

Theorem 2.8.

The pushforward operation 2.12 is well-defined. That is, $h_*(v_1) = h_*(v_2)$ if $v_1 = [\sigma_1] = [\sigma_2] = v_2$.

Theorem 2.9 (Algebraic Definition of Pushforward)**.**

The definition of pushforward 2.12 is equivalent to the following: let $h : \mathcal{M} \to \mathcal{N}$, $h_* : D_p\mathcal{M} \to D_{h(p)}\mathcal{M}$ is defined by,

$$(h_*v)(f) := v(f \circ h).$$

Proof. ($\to$)

$$\begin{aligned} h_*(v)(f) = [h \circ \sigma](f) &= \frac{d}{dt}(f \circ h \circ \sigma(t))\Big|_{t=0} \\ &= \frac{d}{dt}((f \circ h) \circ \sigma(t))\Big|_{t=0} \\ &:= v(f \circ h). \end{aligned}$$

($\leftarrow$) This direction is similar. ▨

Theorem 2.10 (Linearity of Pushforward)**.**

The pushforward map $h_* : T_p\mathcal{M} \to T_{h(p)}\mathcal{N}$ is linear.

$$h_*(rv_1 + v_2) = rh_*(v_1) + h_*(v_2).$$

Proof. (Using 2.12) Let $p \in (U,\phi) \subseteq \mathcal{M}$, and $h(p) \in (V,\psi) \subseteq \mathcal{N}$. Choose ϕ s.t. $\phi(p) = 0$. It is obvious that $h_*(rv_1 + v_2)(0) = (rh_*(v_1) + h_*(v_2))(0) = h(p)$.

Consider

$$\begin{aligned}
\frac{d}{dt}\underbrace{(\psi\circ h_*(rv_1+v_2))}_{\mathbb{R}^n\leftarrow\mathcal{N}\leftarrow\mathbb{R}}\Big|_{t=0} &= \frac{d}{dt}(\underbrace{\psi\circ h\circ(\phi^{-1}}_{\mathbb{R}^n\leftarrow\mathcal{N}\leftarrow\mathcal{M}\leftarrow\mathbb{R}^m}\circ\underbrace{(r\phi\circ\sigma_1+\phi\circ\sigma_2)))}_{\mathbb{R}^m\leftarrow\mathcal{M}\leftarrow\mathbb{R}}\Big|_{t=0} \\
&= (\psi\circ h\circ\phi^{-1})'\circ(r\phi\circ\sigma_1+\phi\circ\sigma_2)\cdot(r\phi\circ\sigma_1+\phi\circ\sigma_2)'\big|_{t=0} \\
&= (\psi\circ h\circ\phi^{-1})'(0)\cdot((r\phi\circ\sigma_1)'+(\phi\circ\sigma_2)')\big|_{t=0}.
\end{aligned}$$

And

$$\begin{aligned}
\frac{d}{dt}(\underbrace{\psi}_{\mathbb{R}^n\leftarrow}\circ\underbrace{(rh_*(v_1)+h_*(v_2)))}_{\mathcal{N}\leftarrow\mathbb{R}}\Big|_{t=0} &= \frac{d}{dt}\underbrace{(r\psi\circ h\circ\sigma_1+\psi\circ h\circ\sigma_2)}_{\mathbb{R}^n\leftarrow\mathcal{N}\leftarrow\mathcal{M}\leftarrow\mathbb{R}}\Big|_{t=0} \\
=&(\underbrace{r\psi\circ h\circ\phi^{-1}}_{\mathbb{R}^n\leftarrow\mathcal{N}\leftarrow\mathcal{M}\leftarrow\mathbb{R}^m}\circ\underbrace{\phi\circ\sigma_1}_{\mathbb{R}^m\leftarrow\mathcal{M}\leftarrow\mathbb{R}}+\psi\circ h\circ\phi^{-1}\circ\phi\circ\sigma_2)'|_{t=0} \\
=&(r(\psi\circ h\circ\phi^{-1})'\circ(\phi\circ\sigma_1)\cdot(\phi\circ\sigma_1)')|_{t=0} \\
&+((\psi\circ h\circ\phi^{-1})'\circ(\phi\circ\sigma_2)\cdot(\phi\circ\sigma_2)')|_{t=0} \\
=&(\psi\circ h\circ\phi^{-1})'(0)\cdot(r(\phi\circ\sigma_1)'+(\phi\circ\sigma_2)')|_{t=0}.
\end{aligned}$$

So we see the two are equal.

(Using 2.9)

$$\begin{aligned}
(h_*(rv_1+v_2))(f) &= (rv_1+v_2)(f\circ h) \\
&= rv_1(f\circ h)+v_2(f\circ h) \\
&= r(h_*v_1)f+(h_*v_2)f.
\end{aligned}$$

▨

Theorem 2.11.
Given manifolds $\mathcal{M},\mathcal{N},\mathcal{P}$ and $h:\mathcal{M}\to\mathcal{N}$, $k:\mathcal{N}\to\mathcal{P}$, then

$$(k\circ h)_* = k_*\circ h_*.$$

2.4.2 Jacobian

Theorem 2.12 (Local Representative of Pushforward)**.**
Let $\dim \mathcal{M} = m, \dim \mathcal{N} = n$, $h : \mathcal{M} \to \mathcal{N}$, $\{x^1, \dots, x^m\}$ be the local coordinates of $\mathcal{M}$ around p, and $\{y^1, \dots, y^n\}$ be the local coordinates of $\mathcal{N}$ around $h(p)$. Then

$$h_* v = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} v^\mu \left. \frac{\partial h^\nu}{\partial x^\mu} \right|_p (\partial_\nu)_{h(p)},$$

where $J_{\nu\mu} := \left. \frac{\partial h^\nu}{\partial x^\mu} \right|_p := \left(\partial_\mu\right)_p (y^\nu \circ h)$ is the inverse Jacobian matrix.

Proof. First expand v in terms of local coordinates and use linearity,

$$h_* v = h_*(v^\mu \left(\partial_\mu\right)_p) = v^\mu h_*(\left(\partial_\mu\right)_p).$$

Expand the result in local coordinates of $\mathcal{N}$,

$$h_*(\left(\partial_\mu\right)_p) = \left(h_* \left(\partial_\mu\right)_p\right)^\nu (\partial_\nu)_{h(p)}.$$

Using 2.9,

$$\begin{aligned} \left(h_* \left(\partial_\mu\right)_p\right)^\nu &= \left(h_* \left(\partial_\mu\right)_p\right) \circ y^\nu \\ &= \left(\partial_\mu\right)_p (y^\nu \circ h) \\ &:= \left(\partial_\mu\right)_p h^\nu. \end{aligned}$$

So,

$$h_*(\left(\partial_\mu\right)_p) = \left(\partial_\mu\right)_p h^\nu (\partial_\nu)_{h(p)}.$$

And,

$$h_* v = v^\mu \left(\partial_\mu\right)_p h^\nu (\partial_\nu)_{h(p)}.$$

▨

Theorem 2.13 (Using Curve to Pushforward)**.**
Given $c : (-\epsilon, \epsilon) \to \mathcal{M}$ a curve, and choose the coordinate chart of $\mathbb{R}$ to be the identity, then

$$c_* \left(\frac{d}{dt}\right)_0 = [c] \in T_p\mathcal{M}.$$

Proof. First we clarify what is $\left(\frac{d}{dt}\right)_0$. Since on the trivial manifold $\mathbb{R}$ there is only one coordinate, namely t, we need not specify the number. Also, considering our functions are scalar valued $f : \mathcal{M} \to \mathbb{R}$, this motivates us to write ”total differential”.

For all $f \in C^\infty$,

$$c_* \left(\frac{d}{dt}\right)_0 f = \left(\frac{d}{dt}\right)_0 (f \circ c).$$

Since the coordinate chart is the identity,

$$\begin{aligned}\left(\frac{d}{dt}\right)_0 (f \circ c) &= \frac{d}{dt}(f \circ c \circ I)\Big|_{I(t)=0} \\ &= \frac{d}{dt}(f \circ c)\Big|_{t=0} \\ &= [c]f.\end{aligned}$$

▨

Theorem 2.14 (Contravariancy of Tangent Vectors)**.**
The components of tangent vectors are contravariant, i.e., given two coordinate charts (U, ϕ) and (U', ϕ') s.t. $U \cap U' = S \neq \varnothing$, then on S,

$$v'^\nu = \sum_{\mu=1}^{m} v^\mu \frac{\partial x'^\nu}{\partial x^\mu}.$$

Proof. Given that we have the local representative of pushforward at hand, consider the identity pushforward $\mathrm{id}_* : T_p\mathcal{M} \to T_p\mathcal{M}$,

$$\mathrm{id}_*\, v = \sum_{\mu=1}^{m} \sum_{\nu=1}^{m} v^\mu \left.\frac{\partial x'^\nu}{\partial x^\mu}\right|_p \left(\partial_{\nu'}\right)_p .$$

We see immediately that the result holds. ▨

3 Vector Fields

3.1 Definition

Definition 3.1 (Vector Fields)**.**
A vector field X on $\mathcal{M}$ is a smooth assignment of a tangent vector $X_p \in T_p\mathcal{M}\ \forall p \in \mathcal{M}$.

”Smooth” assignment is defined to be that the Lie derivative 3.2 is smooth.

Definition 3.2 (Lie Derivative)**.**
The Lie-derivative of function f with respect to vector field X is defined as

$$\mathcal{L}_X f := Xf,$$

and at a specific point $p \in \mathcal{M}$,

$$\mathcal{L}_X f(p) := Xf(p) := X_p f.$$

Theorem 3.1 (Properties of Lie Derivative)**.**
The Lie derivative has the following properties,

1. $X(rf + g) = rXf + Xg$
2. $X(fg) = fXg + gXf.$

Theorem 3.2 (Component of Vector Field)**.**
Given a chart (U, ϕ) on $\mathcal{M}$, we can write

$$X_U = X_U x^\mu \partial_\mu.$$

When the context is clear or **for convenience**, we write

$$X = Xx^\mu \partial_\mu := X^\mu \partial_\mu.$$

Proof. We know

$$(Xf)(p) = X_p f = X_p x^\mu \left(\partial_\mu\right)_p f = (Xx^\mu)(p) \left(\partial_\mu\right)_p f.$$

▨

Remark.
∂_μ is a vector field that assigns each point $p \in \mathcal{M}$ with the vector $\left(\partial_\mu\right)_p \in T_p\mathcal{M}$.

♍

Theorem 3.3 (Contravariancy of Vector Fields)**.**
Given two coordinate charts (U, ϕ) and (U', ϕ') s.t. $U \cap U' = S \neq \varnothing$. On S,

$$X^{\nu'} = \sum_{\mu=1}^{m} X^\mu \frac{\partial x'^\nu}{\partial x^\mu}.$$

Analogous to 2.14.

3.2 Lie Bracket

Definition 3.3 (Composition of Vector Fields)**.**
We can view $X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, and so does Y. Therefore, we define

$$(X \circ Y)(f) := X(Yf).$$

Definition 3.4 (Lie Bracket (Commutator))**.**
We define the Lie Bracket of two vector fields X, Y to be

$$[X, Y] := X \circ Y - Y \circ X.$$

Remark.
Lie Bracket 3.4 is a vector field, while the expression $X \circ Y$ is not, because it contains second differential terms. See the following proof.

Theorem 3.4 (Lie Bracket Components)**.**

$$[X,Y]^\mu = (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu).$$

Proof. Given $X = X^\mu \partial_\mu, Y = Y^\nu \partial_\nu$, we try to write the component of $X \circ Y$.

$$X \circ Y(f) = X^\mu \partial_\mu \left(Y^\nu \partial_\nu f\right).$$

However, notice that

$$\begin{aligned}
&Y^\nu := Y x^\nu \in C^\infty(\mathcal{M});\\
&\partial_\nu : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}),\\
&\implies \partial_\nu f \in C^\infty(\mathcal{M}).
\end{aligned}$$

So we need to use the Leibniz property of ∂_μ 2.8 in order to evaluate the second term. Doing this for $X \circ Y(f)$ and $Y \circ X(f)$, we have

$$\begin{aligned}
X \circ Y(f) &= X^\mu \big((\partial_\mu Y^\nu)(\partial_\nu f) + Y^\nu \partial_\mu \partial_\nu f\big).\\
Y \circ X(f) &= Y^\nu \big((\partial_\nu X^\mu)(\partial_\mu f) + X^\mu \partial_\nu \partial_\mu f\big).
\end{aligned}$$

So if $\partial_\mu \partial_\nu f = \partial_\nu \partial_\mu f$, then by subtracting, we can cancel the second order terms, and we are done. We prove so now.

$$\begin{aligned}
(\partial_\mu \partial_\nu f)(p) &= \frac{\partial}{\partial u^\mu} \left((\partial_\nu f) \circ \phi^{-1}\right)\Big|_{\phi(p)}\\
&= \frac{\partial}{\partial u^\mu} \Big((\partial_\nu)_{\phi^{-1}(u)} f\Big)\Big|_{\phi(p)}\\
&= \frac{\partial}{\partial u^\mu} \left(\frac{\partial}{\partial u^\nu}(f \circ \phi^{-1})\Big|_u\right)\Bigg|_{\phi(p)}\\
&= \frac{\partial}{\partial u^\nu} \left(\frac{\partial}{\partial u^\mu}(f \circ \phi^{-1})\Big|_u\right)\Bigg|_{\phi(p)}\\
&= (\partial_\nu \partial_\mu f)(p).
\end{aligned}$$

▨

Theorem 3.5 (Properties of Lie Brackets).
1. $[X, Y] = -[Y, X]$ (antisymmetry)
2. $\sum_{\text{cyc}}[X, [Y, Z]] = 0$. (Jacobi Identity)

3.3 Integral Curves and Flows

Definition 3.5 (Intergral Curve).
Let X be a vector field on $\mathcal{M}$, $p \in \mathcal{M}$. Then an integral curve of X through p is a curve $\sigma : (-\epsilon, \epsilon) \to \mathcal{M}$ s.t.

$$\sigma(0) = p,$$
$$\sigma_* \left(\frac{d}{dt}\right)_t = X_{\sigma(t)}.$$

Remark.
Qualitatively, using 2.13, this pushforward is just $[\sigma] \in T_{\sigma(t)}\mathcal{M}$. Therefore, the second condition is saying in some sense that the curve is tangent to the vector field on the manifold. For quantitative description, see below.

♍

Definition 3.6 (Differential Equations of Integral Curve).
The components X^μ of X determine the integral curve σ by the following ODE with boundary conditions,

$$X^\mu(\sigma(t)) = \frac{d}{dt} x^\mu(\sigma(t))$$
$$x^\mu(\sigma(0)) = x^\mu(p), \mu = 1, 2, \dots, m.$$

3.3.1 One-parameter Family of Diffeomorphisms

Definition 3.7 (Local 1D Family of Local Diffeomorphisms).
A local, 1D family of local diffeomorphisms at $p \in \mathcal{M}$ is made up of (1) an open neighborhood U of p, (2) $\epsilon > 0$ (3) a family of diffeomorphisms $\{ \phi_t \mid |t| < \epsilon \}$, $\phi_t : U \to \mathcal{M}$ s.t.

1. Every ϕ_t is a smooth function in t and q.
2. $\forall t, s \in \mathbb{R}$ and $|t|, |s|, |t+s| < \epsilon$, and $\forall q \in U$ s.t. $\phi_t(q), \phi_s(q), \phi_{t+s}(q) \in U$, we have
$$\phi_s(\phi_t(q)) = \phi_{s+t}(q).$$
3. $\phi_0(q) = q$.

Remark.
The first "local" refers to the parameter t, which is limited to $(-\epsilon, \epsilon)$. The second "local" refers to the spatial limitation to U.

You can view $\phi_t(q)$ as a curve that brings $t \in (-\epsilon, \epsilon)$ to $\phi_t(q) \in \mathcal{M}$.

♍

Definition 3.8 (Induced Vector Field).
By taking tangents to the curve family 3.7, we have the induced vector field X^ϕ given by
$$X_q^\phi(f) := \frac{d}{dt}(f(\phi_t(q)))\Big|_{t=0}$$

Theorem 3.6.
The curve family $t \mapsto \phi_t(q)$ is the integral curve of the induced vector field 3.8 X_q^ϕ.

Proof.

$$\begin{aligned} X^{\phi}_{\phi_s(q)} &= \frac{d}{dt}(f \circ \phi_t \circ \phi_s(q))\Big|_{t=0} \\ &= \frac{d}{dt}(f \circ \phi_{t+s}(q))\Big|_{t=0}. \end{aligned}$$

Let $u = t + s$. Then

$$\begin{aligned} X^{\phi}_{\phi_s(q)} &= \frac{d}{du}(f \circ \phi_u(q))\Big|_{u=s}. \\ &= \phi_{q*}\left(\frac{d}{dt}\right)_s f. \end{aligned}$$

▨

3.3.2 Local Flows

Definition 3.9 (Local Flow)**.**
Let X be a vector field on open $U \subseteq \mathcal{M}$, and $p \in U$. A local flow at p is a local one-parameter family of local diffeomorphisms 3.7 defined on some open $V \subseteq U$ s.t. $p \in V$ and the induced vector field 3.8 is X.

Remark.
Local flows always exist and are unique. In contrast, global flows (which means $t \in \mathbb{R}$ instead of a restricted interval) may not exist.

♍

3.3.3 Lie Derivative

Theorem 3.7 (Interpretation of Lie Bracket)**.**
If X, Y are two vector fields on $\mathcal{M}$, and define the following quantity, which can be interpreted as the change of Y when following the integral curves of X, as

$$\frac{d}{dt}(\phi^X_{-t*}(Y))\Big|_{t=0} := \lim_{\epsilon\to 0} \frac{\phi^X_{-\epsilon*}(Y_{\phi^X_\epsilon(p)}) - Y_p}{\epsilon}.$$

Then,

$$\frac{d}{dt}(\phi^X_{-t*}(Y))\Big|_{t=0} = [X, Y].$$

4 Cotangent Spaces

Definition 4.1 (Cotangent Spaces)**.**
The cotangent space $T_p^*\mathcal{M}$ at $p \in \mathcal{M}$ is the set of all linear functions $f : T_p\mathcal{M} \to \mathbb{R}$.

Its member is called a cotangent vector.

$\dim T_p^*\mathcal{M} = \dim T_p\mathcal{M}$.

Definition 4.2 (One-Form)**.**
A one-form on $\mathcal{M}$ is a smooth assignment of cotangent vectors $\omega : p \mapsto \omega_p$.

It may be understood as a covector field.

Definition 4.3 (Basis Cotangent Vectors)**.**
The basis cotangent vectors is chosen to be the dual basis of the basis tangent vectors 2.11,

$$(dx^\mu)_p((\partial_\nu)_p) = \delta^\mu{}_\nu.$$

Theorem 4.1 (Coordinate Expression of Cotangent Vectors)**.**
Any $f \in T_p^*\mathcal{M}$ can be expanded as

$$f = f_\mu (dx^\mu)_p.$$

Any one-form ω can be expressed as

$$\omega = \omega_\mu dx^\mu.$$