

1 Differentiable Manifolds

1.1 Definition

1.1.1 Coordinate Charts

Definition 1.1 (Coordinate Charts).

An m -dimensional, $m \neq \infty$ coordinate chart on a topological space \mathcal{M} is a pair

$$(U, \phi) \begin{cases} U \subseteq \mathcal{M}, U \text{ open} \\ \phi : U \rightarrow \mathbb{R}^m, \phi \text{ homeomorphism} \end{cases}$$

↗ **Remark.**

If $U = \mathcal{M}$, then we say the coordinate chart ϕ is globally defined; if not, then it is locally defined. Few manifolds have globally defined property.

ℳ

Definition 1.2 (Overlap Function).

Let $(U_1, \phi_1), (U_2, \phi_2)$ be a pair of m -dimensional coordinate charts with $U_1 \cap U_2 \neq \emptyset$. Then the overlap function is defined as

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \subseteq \mathbb{R}^m \rightarrow \phi_2(U_1 \cap U_2) \subseteq \mathbb{R}^m.$$

Definition 1.3 (Atlas).

An m -dimensional atlas on \mathcal{M} is a family of m -dimensional coordinate charts $(U_i, \phi_i), i \in I$ s.t.

1. $\mathcal{M} = \bigcup_{i \in I} U_i$.
2. Each overlap function $\phi_j \circ \phi_i^{-1}, i, j \in I$ is C^∞ .

Definition 1.4 (Differentiable Manifolds).

An m -dimensional differentiable manifold is a topological space \mathcal{M} equipped with an atlas.

↗ Remark.

We didn't define a differentiable manifold by regulating the differentiability of the coordinate charts themselves. That's because differentiation is not defined on a manifold, so we need to rely on Euclidean spaces.

ℳ

Definition 1.5 (Coordinate Functions).

The coordinate functions are the (Euclidean) components of coordinate.

$$\begin{aligned}\phi : U \rightarrow \mathbb{R}^m & \qquad p \mapsto \phi(p), \\ \phi^\mu : U \rightarrow \mathbb{R} & \qquad \text{s.t. } \phi(p) = \begin{pmatrix} \phi^1(p) \\ \vdots \\ \phi^m(p) \end{pmatrix}.\end{aligned}$$

An alternative notation is

$$x^\mu := \phi^\mu.$$

↗ Remark.

There are (Euclidean) projection functions,

$$u^\mu : \mathbb{R}^m \rightarrow \mathbb{R}.$$

But I think mention it will cause a lot of confusion. Just remember in the future when we say $\frac{\partial}{\partial u^\mu}$, we are referring to the Euclidean partial derivative wrt the μ -th component.

ℳ

2 Tangent Spaces

2.1 The Curve Formulation of Tangent Spaces

↗ **Remark.**

The definition of manifold do not require the entity to be embeded in a higher dimensional space. Therefore, the traditional view of tangency is not valid here.

ℳ

↗ **Remark.**

The curve formulation remains valid in the infinite-dimensional case, while the algebraic formulation is not. However, in the finite-dimensional case, they are isomorphic.

ℳ

2.1.1 Curves and Vectors

Definition 2.1 (Curve).

A curve on \mathcal{M} is a C^∞ map,

$$\sigma : (-\epsilon, \epsilon) \rightarrow \mathcal{M}.$$

Definition 2.2 (Curve Tangency).

Two curves σ_1, σ_2 are tangent at $p \in \mathcal{M}$ if

1. $\sigma_1(0) = \sigma_2(0) = p$.
2. $\frac{d}{dt}(x^i \circ \sigma_1(t))|_{t=0} = \frac{d}{dt}(x^i \circ \sigma_2(t))|_{t=0}, \quad 1 \leq i \leq m$.

↗ **Remark.**

Written more compactly,

$$\frac{d}{dt}(\phi \circ \sigma_1) \Big|_{t=0} = \frac{d}{dt}(\phi \circ \sigma_2) \Big|_{t=0}$$

\mathfrak{M}

Definition 2.3 (Tangent Vectors).

A tangent vector at $p \in \mathcal{M}$ is an equivalence class of curves where the equivalence relation is that they are tangent. It will be denoted as

$$v = [\sigma].$$

Definition 2.4 (Tangent Space).

The tangent space $T_p\mathcal{M}$ at point p is the set of all tangent vectors at point p .

Definition 2.5 (Tangent Bundle).

The tangent bundle $T\mathcal{M}$ is

$$T\mathcal{M} := \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}.$$

2.1.2 Addition and Scalar Multiplication

Definition 2.6 (Addition and Scalar Multiplication).

Let $v_1 = [\sigma_1], v_2 = [\sigma_2] \in T_p\mathcal{M}$, and $r \in \mathbb{R}$. Then define

$$\begin{aligned} v_1 + v_2 &:= [\phi^{-1} \circ (\phi \circ \sigma_1 + \phi \circ \sigma_2)], \\ rv_1 &:= [\phi^{-1} \circ (r\phi \circ \sigma_1)]. \end{aligned}$$

Theorem 2.1.

The definition Definition 2.6 is well-defined. That is, they are independent of

the choice of chart (U, ϕ) and σ_1, σ_2 as long as $v_1 = [\sigma_1]$ and $v_2 = [\sigma_2]$.

Therefore, $T_p\mathcal{M}$ is a real vector space.

Proof. Let $v_1 = [\sigma_1] = v'_1 := [\tau_1], v_2 = [\sigma_2] = v'_2 := [\tau_2]$. First check (1) of [Definition 2.2](#),

$$\begin{aligned} (rv_1 + v_2)(0) &= (\phi^{-1} \circ (r\phi \circ \sigma_1(0) + \phi \circ \sigma_2(0))) \\ &= (\phi^{-1} \circ (r\phi \circ \tau_1(0) + \phi \circ \tau_2(0))) \\ &= (rv'_1 + v'_2)(0), \end{aligned}$$

since $\phi \circ \sigma_1(0) = \phi \circ \tau_1(0) = \phi(p)$ by equivalence, and the same for σ_2 .

Now consider

$$\begin{aligned} \frac{d}{dt}(\phi \circ (rv_1 + v_2)) \Big|_{t=0} &= \frac{d}{dt}(r\phi \circ \sigma_1 + \phi \circ \sigma_2) \Big|_{t=0} \\ &= r \frac{d}{dt}(\phi \circ \sigma_1) \Big|_{t=0} + \frac{d}{dt}(\phi \circ \sigma_2) \Big|_{t=0} \\ &= r \frac{d}{dt}(\phi \circ \tau_1) \Big|_{t=0} + \frac{d}{dt}(\phi \circ \tau_2) \Big|_{t=0} \\ &= \frac{d}{dt}(\phi \circ (rv'_1 + v'_2)) \Big|_{t=0}, \end{aligned}$$

since $\frac{d}{dt}(\phi \circ \sigma_1) \Big|_{t=0} = \frac{d}{dt}(\phi \circ \tau_1) \Big|_{t=0}$ by equivalence, and the same for σ_2 . ◻

2.1.3 Curves and Derivation

Definition 2.7 (Directional Derivative).

For any $f : \mathcal{M} \rightarrow \mathbb{R}$ s.t. $f \in C^\infty$, we define

$$v(f) := \frac{d}{dt}(f \circ \sigma(t)) \Big|_{t=0},$$

where $v = [\sigma]$.

Theorem 2.2.

The definition [Definition 2.7](#) is well-defined. That is, $v(f)$ is independent of the curve σ chosen as well as $v = [\sigma]$.

Proof. Let $v_1 = [\sigma_1] = [\sigma_2] = v_2$. Then

$$\begin{aligned} v_1(f) &= \frac{d}{dt}(f \circ \sigma_1)\Big|_{t=0}, \\ v_2(f) &= \frac{d}{dt}(f \circ \sigma_2)\Big|_{t=0}, \\ \frac{d}{dt}(\phi \circ \sigma_1)\Big|_{t=0} &= \frac{d}{dt}(\phi \circ \sigma_2)\Big|_{t=0}. \end{aligned}$$

Then

$$\begin{aligned} v_1(f) &= \frac{d}{dt}\left(\underbrace{(f \circ \phi^{-1})}_{\mathbb{R} \leftarrow \mathbb{R}^m} \circ \underbrace{(\phi \circ \sigma_1)}_{\mathbb{R}^m \leftarrow \mathbb{R}}\right)\Big|_{t=0} \\ &= (f \circ \phi^{-1})' \circ (\phi \circ \sigma_1) \cdot (\phi \circ \sigma_1)'|_{t=0} \\ &= (f \circ \phi^{-1})' \circ (\phi \circ \sigma_2) \cdot (\phi \circ \sigma_2)'|_{t=0} \\ &= v_2(f), \end{aligned}$$

since $\phi \circ \sigma_1(0) = \phi \circ \sigma_2(0) = \phi(p)$, and $(\phi \circ \sigma_1)' = (\phi \circ \sigma_2)'$ by equivalence. ■

2.2 The Algebraic Formulation of Tangent Spaces

2.2.1 The Space of Derivations

Definition 2.8 (Derivation).

A derivation at $p \in \mathcal{M}$ is a map $v : C^\infty(\mathcal{M}) \rightarrow \mathbb{R}$ s.t.

1. $v(rf + g) = rv(f) + v(g)$, (Linear)
2. $v(fg) = f(p)v(g) + g(p)v(f)$, (Leibniz)

where $f, g \in C^\infty$.

Definition 2.9 (Tangent Space (Algebraic)).

The space of all derivations at $p \in \mathcal{M}$ is denoted $D_p\mathcal{M}$.

Definition 2.10 (Addition and Scalar Multiplication).

Given $v_1, v_2 \in D_p\mathcal{M}$, define

$$(v_1 + v_2)(f) := v_1(f) + v_2(f)$$

$$(rv)(f) := rv(f).$$

Theorem 2.3.

$D_p\mathcal{M}$ is a real vector space.

2.2.2 The Basis Tangent Vectors

Definition 2.11 (Basis Tangent Vectors).

We define the basis tangent vectors via derivations by

$$(\partial_\mu)_p f := \frac{\partial}{\partial u^\mu} (f \circ \phi^{-1}(\vec{u})) \Big|_{\vec{u}=\phi(p)}, \quad 1 \leq \mu \leq \dim \mathcal{M}.$$

where $u \in \mathbb{R}^m$, $f : \mathcal{M} \rightarrow \mathbb{R}$, $f \in C^\infty$. For the use of u^μ , see [Definition 1.5](#).

Theorem 2.4.

$$(\partial_\mu)_p x^\nu = \delta^\nu_\mu.$$

Proof. Although a simple exercise, it was a good chance to explain the sophisticated notation.

$$(\partial_\mu)_p x^\mu = \frac{\partial}{\partial u^\mu} \left(x^\mu \circ \phi^{-1} \begin{pmatrix} u^1 \\ \vdots \\ u^m \end{pmatrix} \right) \Big|_{\phi(p)}.$$

The coordinate $u \in \mathbb{R}^m$ was brought to \mathcal{M} and projected to \mathbb{R}^m again and taken out the μ -th component. So

$$= \frac{\partial}{\partial u^\mu} (u^\mu) \Big|_{\phi(p)} = 1.$$



Theorem 2.5 (Linear Independence of Basis Tangent Vectors).

The basis tangent vectors $(\partial_\mu)_p$, $1 \leq \mu \leq \dim \mathcal{M}$ are linear independent.

Proof. Suppose $a^\mu (\partial_\mu)_p = 0$. Then

$$a^\mu (\partial_\mu)_p (x^\nu) = a^\mu \delta^\nu_\mu = 0(x^\nu) = 0.$$

So $a^\mu = 0$. □

Theorem 2.6 (Coordinate Expansion of Tangent Vectors).

For all $v \in D_p \mathcal{M}$, we have

$$v = v^\mu (\partial_\mu)_p,$$

where Einstein notation was used, and $v^\mu = v(x^\mu)$.

↗ **Remark.**

The proof was sophisticated and did not teach me much.

ℳ

2.3 Isomorphism of Curves and Derivations

Theorem 2.7 (Isomorphism of Curves and Derivations).

Similar to [Definition 2.7](#), we define the linear map $\iota : T_p \mathcal{M} \rightarrow D_p \mathcal{M}$ acting on $v = [\sigma] \in T_p \mathcal{M}$ by

$$\iota(v)(f) := \left. \frac{d}{dt} (f \circ \sigma(t)) \right|_{t=0}.$$

Then ι is a linear isomorphism. Note that RHS $\in D_p \mathcal{M}$.

Proof. (linearity) Choose ϕ s.t. $\phi(p) = 0$.

$$\begin{aligned}
\iota(rv_1 + v_2)(f) &= \frac{d}{dt}(f \circ \phi^{-1} \circ (r\phi \circ \sigma_1 + \phi \circ \sigma_2)) \Big|_{t=0} \\
&= ((f \circ \phi^{-1})' \circ (r\phi \circ \sigma_1 + \phi \circ \sigma_2) \cdot (r\phi \circ \sigma_1 + \phi \circ \sigma_2)') \Big|_{t=0} \\
&= ((f \circ \phi^{-1})'(0) \cdot ((r\phi \circ \sigma_1)' + (\phi \circ \sigma_2)')) \Big|_{t=0} \\
&= (r(f \circ \phi^{-1})' \circ (\phi \circ \sigma_1) \cdot (\phi \circ \sigma_1)' + (f \circ \phi^{-1})' \circ (\phi \circ \sigma_2) \cdot (\phi \circ \sigma_2)') \Big|_{t=0} \\
&= r\iota(v_1) + \iota(v_2)(f).
\end{aligned}$$

(surjectivity) To show surjectivity, we need to construct a curve for all $v' \in D_p\mathcal{M}$ s.t. $\iota(v) = v'$.

Let $v' \in D_p\mathcal{M}$ and construct $\sigma : (-\epsilon, \epsilon) \rightarrow \mathcal{M}$ s.t.

$$\begin{aligned}
\sigma(0) &= p, \\
v^\mu = v(x^\mu) &= \frac{d}{dt}(x^\mu \circ \sigma(t)) \Big|_{t=0}.
\end{aligned}$$

Then

$$v(f) = v^\mu (\partial_\mu)_p f = \frac{d}{dt}(x^\mu \circ \sigma(t)) \Big|_{t=0} (\partial_\mu)_p f.$$

Also,

$$\begin{aligned}
\frac{d}{dt}(f \circ \sigma(t)) \Big|_{t=0} &= \frac{d}{dt}(f \circ \phi^{-1} \circ \phi \circ \sigma(t)) \Big|_{t=0} \\
&= \sum_{\mu=1}^m \frac{\partial}{\partial u^\mu} (f \circ \phi^{-1}) \Big|_{\phi(p)} \frac{d}{dt}(u^\mu \circ \phi \circ \sigma) \Big|_{t=0} \quad (\text{component-wise}) \\
&= \sum_{\mu=1}^m (\partial_\mu)_p f \frac{d}{dt}(x^\mu \circ \sigma) \Big|_{t=0} \\
&= v(f).
\end{aligned}$$

Thus completing the proof. ◻

↗ Remark.

It's tempting to use the powerful theorem that surjectivity of linear transformations is equivalent to injectivity. However, the requirement of that theorem is that **the dimensions of the two spaces are equal**. At this point, we don't know the dimension of $T_p\mathcal{M}$, so we cannot do this. ℳ

2.4 Pushforward

2.4.1 Definition and Linearity

↗ **Remark.**

The pushforward $h_* : T_p \mathcal{M} \rightarrow T_{h(p)} \mathcal{N}$ of a specific function $h : \mathcal{M} \rightarrow \mathcal{N}$ can be thought of as local linearization of the function.

ℳ

Definition 2.12 (Pushforward).

Given a function $h : \mathcal{M} \rightarrow \mathcal{N}$ and $v \in T_p \mathcal{M}$, then we define the pushforward $h_* : T_p \mathcal{M} \rightarrow T_{h(p)} \mathcal{N}$ by

$$h_*(v) := [h \circ \sigma], \quad v = [\sigma].$$

Theorem 2.8.

The pushforward operation [Definition 2.12](#) is well-defined. That is, $h_*(v_1) = h_*(v_2)$ if $v_1 = [\sigma_1] = [\sigma_2] = v_2$.

Theorem 2.9 (Algebraic Definition of Pushforward).

The definition of pushforward [Definition 2.12](#) is equivalent to the following: let $h : \mathcal{M} \rightarrow \mathcal{N}$, $h_* : D_p \mathcal{M} \rightarrow D_{h(p)} \mathcal{M}$ is defined by,

$$(h_* v)(f) := v(f \circ h).$$

Proof. (\rightarrow)

$$\begin{aligned} h_*(v)(f) &= [h \circ \sigma](f) = \frac{d}{dt}(f \circ h \circ \sigma(t)) \Big|_{t=0} \\ &= \frac{d}{dt}((f \circ h) \circ \sigma(t)) \Big|_{t=0} \\ &:= v(f \circ h). \end{aligned}$$

(\leftarrow) This direction is similar. ◻

Theorem 2.10 (Linearity of Pushforward).

The pushforward map $h_* : T_p \mathcal{M} \rightarrow T_{h(p)} \mathcal{N}$ is linear.

$$h_*(rv_1 + v_2) = rh_*(v_1) + h_*(v_2).$$

Proof. (Using Definition 2.12) Let $p \in (U, \phi) \subseteq \mathcal{M}$, and $h(p) \in (V, \psi) \subseteq \mathcal{N}$. Choose ϕ s.t. $\phi(p) = 0$. It is obvious that $h_*(rv_1 + v_2)(0) = (rh_*(v_1) + h_*(v_2))(0) = h(p)$.

Consider

$$\begin{aligned} \frac{d}{dt} \underbrace{(\psi \circ h_*(rv_1 + v_2))}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathbb{R}} \Big|_{t=0} &= \frac{d}{dt} \left(\underbrace{\psi \circ h \circ (\phi^{-1})}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathcal{M} \leftarrow \mathbb{R}^m} \circ \underbrace{(r\phi \circ \sigma_1 + \phi \circ \sigma_2)}_{\mathbb{R}^m \leftarrow \mathcal{M} \leftarrow \mathbb{R}} \right) \Big|_{t=0} \\ &= (\psi \circ h \circ \phi^{-1})' \circ (r\phi \circ \sigma_1 + \phi \circ \sigma_2) \cdot (r\phi \circ \sigma_1 + \phi \circ \sigma_2)' \Big|_{t=0} \\ &= (\psi \circ h \circ \phi^{-1})'(0) \cdot ((r\phi \circ \sigma_1)' + (\phi \circ \sigma_2)') \Big|_{t=0}. \end{aligned}$$

And

$$\begin{aligned} \frac{d}{dt} \left(\underbrace{\psi}_{\mathbb{R}^n \leftarrow} \circ \underbrace{(rh_*(v_1) + h_*(v_2))}_{\mathcal{N} \leftarrow \mathbb{R}} \right) \Big|_{t=0} &= \frac{d}{dt} \underbrace{(r\psi \circ h \circ \sigma_1 + \psi \circ h \circ \sigma_2)}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathcal{M} \leftarrow \mathbb{R}} \Big|_{t=0} \\ &= (\underbrace{r\psi \circ h \circ \phi^{-1}}_{\mathbb{R}^n \leftarrow \mathcal{N} \leftarrow \mathcal{M} \leftarrow \mathbb{R}^m} \circ \underbrace{\phi \circ \sigma_1}_{\mathbb{R}^m \leftarrow \mathcal{M} \leftarrow \mathbb{R}} + \psi \circ h \circ \phi^{-1} \circ \phi \circ \sigma_2)' \Big|_{t=0} \\ &= (r(\psi \circ h \circ \phi^{-1})' \circ (\phi \circ \sigma_1) \cdot (\phi \circ \sigma_1)') \Big|_{t=0} \\ &\quad + ((\psi \circ h \circ \phi^{-1})' \circ (\phi \circ \sigma_2) \cdot (\phi \circ \sigma_2)') \Big|_{t=0} \\ &= (\psi \circ h \circ \phi^{-1})'(0) \cdot (r(\phi \circ \sigma_1)' + (\phi \circ \sigma_2)') \Big|_{t=0}. \end{aligned}$$

So we see the two are equal.

(Using Theorem 2.9)

$$\begin{aligned} (h_*(rv_1 + v_2))(f) &= (rv_1 + v_2)(f \circ h) \\ &= rv_1(f \circ h) + v_2(f \circ h) \\ &= r(h_*v_1)f + (h_*v_2)f. \end{aligned}$$



Theorem 2.11 (Associativity of Pushforwards).

Given manifolds $\mathcal{M}, \mathcal{N}, \mathcal{P}$ and $h : \mathcal{M} \rightarrow \mathcal{N}$, $k : \mathcal{N} \rightarrow \mathcal{P}$, then

$$(k \circ h)_* = k_* \circ h_*,$$

2.4.2 Jacobian

Theorem 2.12 (Local Representative of Pushforward).

Let $\dim \mathcal{M} = m$, $\dim \mathcal{N} = n$, $h : \mathcal{M} \rightarrow \mathcal{N}$, $\{x^1, \dots, x^m\}$ be the local coordinates of \mathcal{M} around p , and $\{y^1, \dots, y^n\}$ be the local coordinates of \mathcal{N} around $h(p)$.

Then

$$h_* v = \sum_{\mu=1}^m \sum_{\nu=1}^n (\partial_\nu)_{h(p)} \left. \frac{\partial h^\nu}{\partial x^\mu} \right|_p v^\mu,$$

where $J^\nu_\mu := \left. \frac{\partial h^\nu}{\partial x^\mu} \right|_p := (\partial_\mu)_p (y^\nu \circ h)$ is the Jacobian matrix.

Proof. First expand v in terms of local coordinates and use linearity,

$$h_* v = h_* (v^\mu (\partial_\mu)_p) = v^\mu h_* ((\partial_\mu)_p).$$

Expand the result in local coordinates of \mathcal{N} ,

$$h_* ((\partial_\mu)_p) = \left(h_* (\partial_\mu)_p \right)^\nu (\partial_\nu)_{h(p)}.$$

Using [Theorem 2.9](#),

$$\begin{aligned} \left(h_* (\partial_\mu)_p \right)^\nu &= \left(h_* (\partial_\mu)_p \right) \circ y^\nu \\ &= (\partial_\mu)_p (y^\nu \circ h) \\ &:= (\partial_\mu)_p h^\nu. \end{aligned}$$

So,

$$h_* ((\partial_\mu)_p) = (\partial_\mu)_p h^\nu (\partial_\nu)_{h(p)}.$$

And,

$$h_* v = v^\mu (\partial_\mu)_p h^\nu (\partial_\nu)_{h(p)}.$$



Theorem 2.13 (Using Curve to Pushforward).

Given $c : (-\epsilon, \epsilon) \rightarrow \mathcal{M}$ a curve, and choose the coordinate chart of \mathbb{R} to be the identity, then

$$c_* \left(\frac{d}{dt} \right)_0 = [c] \in T_p \mathcal{M}.$$

Proof. First we clarify what is $(\frac{d}{dt})_0$. Since on the trivial manifold \mathbb{R} there is only one coordinate, namely t , we need not specify the number. Also, considering our functions are scalar valued $f : \mathcal{M} \rightarrow \mathbb{R}$, this motivates us to write "total differential".

For all $f \in C^\infty$,

$$c_* \left(\frac{d}{dt} \right)_0 f = \left(\frac{d}{dt} \right)_0 (f \circ c).$$

Since the coordinate chart is the identity,

$$\begin{aligned} \left(\frac{d}{dt} \right)_0 (f \circ c) &= \frac{d}{dt} (f \circ c \circ I) \Big|_{I(t)=0} \\ &= \frac{d}{dt} (f \circ c) \Big|_{t=0} \\ &= [c] f. \end{aligned}$$

■

Theorem 2.14 (Contravariancy of Tangent Vectors).

The components of tangent vectors are contravariant, i.e., given two coordinate charts (U, ϕ) and (U', ϕ') s.t. $U \cap U' = S \neq \emptyset$, then on S ,

$$v'^\nu = \sum_{\mu=1}^m v^\mu \frac{\partial x'^\nu}{\partial x^\mu}.$$

Proof. Given that we have the local representative of pushforward at hand, consider the identity pushforward $\text{id}_* : T_p \mathcal{M} \rightarrow T_p \mathcal{M}$,

$$\text{id}_* v = \sum_{\mu=1}^m \sum_{\nu=1}^m v^\mu \frac{\partial x'^\nu}{\partial x^\mu} \Big|_p (\partial_{\nu'})_p.$$

We see immediately that the result holds. ■

3 Vector Fields

3.1 Definition

Definition 3.1 (Vector Fields).

A vector field X on \mathcal{M} is a smooth assignment of a tangent vector $X_p \in T_p\mathcal{M} \forall p \in \mathcal{M}$.

"Smooth" assignment is defined to be that the Lie derivative **Definition 3.2** is smooth.

Definition 3.2 (Lie Derivative).

The Lie-derivative of function f with respect to vector field X is defined as

$$\mathcal{L}_X f := Xf,$$

and at a specific point $p \in \mathcal{M}$,

$$\mathcal{L}_X f(p) := Xf(p) := X_p f.$$

Theorem 3.1 (Properties of Lie Derivative).

The Lie derivative has the following properties,

1. $X(rf + g) = rXf + Xg$
2. $X(fg) = fXg + gXf.$

Theorem 3.2 (Component of Vector Field).

Given a chart (U, ϕ) on \mathcal{M} , we can write

$$X_U = X_U x^\mu \partial_\mu.$$

When the context is clear or for convenience, we write

$$X = X x^\mu \partial_\mu := X^\mu \partial_\mu.$$

Proof. We know

$$(Xf)(p) = X_p f = X_p x^\mu (\partial_\mu)_p f = (Xx^\mu)(p) (\partial_\mu)_p f.$$



↗ **Remark.**

∂_μ is a vector field that assigns each point $p \in \mathcal{M}$ with the vector $(\partial_\mu)_p \in T_p \mathcal{M}$.



Theorem 3.3 (Contravariancy of Vector Fields).

Given two coordinate charts (U, ϕ) and (U', ϕ') s.t. $U \cap U' = S \neq \emptyset$. On S ,

$$X^{\nu'} = \sum_{\mu=1}^m X^\mu \frac{\partial x'^\nu}{\partial x^\mu}.$$

Analogous to [Theorem 2.14](#).

3.2 Lie Bracket

Definition 3.3 (Composition of Vector Fields).

We can view $X : C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$, and so does Y . Therefore, we define

$$(X \circ Y)(f) := X(Yf).$$

Definition 3.4 (Lie Bracket (Commutator)).

We define the Lie Bracket of two vector fields X, Y to be

$$[X, Y] := X \circ Y - Y \circ X.$$

↗ **Remark.**

Lie Bracket [Definition 3.4](#) is a vector field, while the expression $X \circ Y$ is not, because it contains second differential terms. See the following proof.

Theorem 3.4 (Lie Bracket Components).

$$[X, Y]^\mu = (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu).$$

Proof. Given $X = X^\mu \partial_\mu, Y = Y^\nu \partial_\nu$, we try to write the component of $X \circ Y$.

$$X \circ Y(f) = X^\mu \partial_\mu (Y^\nu \partial_\nu f).$$

However, notice that

$$\begin{aligned} Y^\nu &:= Yx^\nu \in C^\infty(\mathcal{M}); \\ \partial_\nu : C^\infty(\mathcal{M}) &\rightarrow C^\infty(\mathcal{M}), \\ \implies \partial_\nu f &\in C^\infty(\mathcal{M}). \end{aligned}$$

So we need to use the Leibniz property of ∂_μ [Definition 2.8](#) in order to evaluate the second term. Doing this for $X \circ Y(f)$ and $Y \circ X(f)$, we have

$$\begin{aligned} X \circ Y(f) &= X^\mu ((\partial_\mu Y^\nu)(\partial_\nu f) + Y^\nu \partial_\mu \partial_\nu f). \\ Y \circ X(f) &= Y^\nu ((\partial_\nu X^\mu)(\partial_\mu f) + X^\mu \partial_\nu \partial_\mu f). \end{aligned}$$

So if $\partial_\mu \partial_\nu f = \partial_\nu \partial_\mu f$, then by subtracting, we can cancel the second order terms, and we are done. We prove so now.

$$\begin{aligned} (\partial_\mu \partial_\nu f)(p) &= \frac{\partial}{\partial u^\mu} ((\partial_\nu f) \circ \phi^{-1}) \Big|_{\phi(p)} \\ &= \frac{\partial}{\partial u^\mu} \left((\partial_\nu)_{\phi^{-1}(u)} f \right) \Big|_{\phi(p)} \\ &= \frac{\partial}{\partial u^\mu} \left(\frac{\partial}{\partial u^\nu} (f \circ \phi^{-1}) \Big|_u \right) \Big|_{\phi(p)} \\ &= \frac{\partial}{\partial u^\nu} \left(\frac{\partial}{\partial u^\mu} (f \circ \phi^{-1}) \Big|_u \right) \Big|_{\phi(p)} \\ &= (\partial_\nu \partial_\mu f)(p). \end{aligned}$$



Theorem 3.5 (Properties of Lie Brackets).

1. $[X, Y] = -[Y, X]$ (antisymmetry)
2. $\sum_{\text{cyc}} [X, [Y, Z]] = 0.$ (Jacobi Identity)

3.3 Integral Curves and Flows

Definition 3.5 (Integral Curve).

Let X be a vector field on \mathcal{M} , $p \in \mathcal{M}$. Then an integral curve of X through p is a curve $\sigma : (-\epsilon, \epsilon) \rightarrow \mathcal{M}$ s.t.

$$\begin{aligned}\sigma(0) &= p, \\ \sigma_* \left(\frac{d}{dt} \right)_t &= X_{\sigma(t)}.\end{aligned}$$

↗ **Remark.**

Qualitatively, using [Theorem 2.13](#), this pushforward is just $[\sigma] \in T_{\sigma(t)}\mathcal{M}$. Therefore, the second condition is saying in some sense that the curve is tangent to the vector field on the manifold. For quantitative description, see below.

ℳ

Definition 3.6 (Differential Equations of Integral Curve).

The components X^μ of X determine the integral curve σ by the following ODE with boundary conditions,

$$\begin{aligned}X^\mu(\sigma(t)) &= \frac{d}{dt}x^\mu(\sigma(t)) \\ x^\mu(\sigma(0)) &= x^\mu(p), \mu = 1, 2, \dots, m.\end{aligned}$$

3.3.1 One-parameter Family of Diffeomorphisms

Definition 3.7 (Local 1D Family of Local Diffeomorphisms).

A local, 1D family of local diffeomorphisms at $p \in \mathcal{M}$ is made up of (1) an open neighborhood U of p , (2) $\epsilon > 0$ (3) a family of diffeomorphisms $\{\phi_t \mid |t| < \epsilon\}$, $\phi_t : U \rightarrow \mathcal{M}$ s.t.

1. Every ϕ_t is a smooth function in t and q .
2. $\forall t, s \in \mathbb{R}$ and $|t|, |s|, |t+s| < \epsilon$, and $\forall q \in U$ s.t. $\phi_t(q), \phi_s(q), \phi_{t+s}(q) \in U$, we have
$$\phi_s(\phi_t(q)) = \phi_{s+t}(q).$$
3. $\phi_0(q) = q$.

Remark.

The first "local" refers to the parameter t , which is limited to $(-\epsilon, \epsilon)$. The second "local" refers to the spatial limitation to U . You can view $\phi_t(q)$ as a curve that brings $t \in (-\epsilon, \epsilon)$ to $\phi_t(q) \in \mathcal{M}$.

ℳ

Definition 3.8 (Induced Vector Field).

By taking tangents to the curve family Definition 3.7, we have the induced vector field X^ϕ given by

$$X_q^\phi(f) := \left. \frac{d}{dt} (f(\phi_t(q))) \right|_{t=0}$$

Theorem 3.6.

The curve family $t \mapsto \phi_t(q)$ is the integral curve of the induced vector field Definition 3.8 X_q^ϕ .

Proof.

$$\begin{aligned} X_{\phi_s(q)}^\phi &= \frac{d}{dt}(f \circ \phi_t \circ \phi_s(q)) \Big|_{t=0} \\ &= \frac{d}{dt}(f \circ \phi_{t+s}(q)) \Big|_{t=0}. \end{aligned}$$

Let $u = t + s$. Then

$$\begin{aligned} X_{\phi_s(q)}^\phi &= \frac{d}{du}(f \circ \phi_u(q)) \Big|_{u=s} \\ &= \phi_{q*} \left(\frac{d}{dt} \right)_s f. \end{aligned}$$



3.3.2 Local Flows

Definition 3.9 (Local Flow).

Let X be a vector field on open $U \subseteq \mathcal{M}$, and $p \in U$. A local flow at p is a local one-parameter family of local diffeomorphisms [Definition 3.7](#) defined on some open $V \subseteq U$ s.t. $p \in V$ and the induced vector field [Definition 3.8](#) is X .

↗ **Remark.**

Local flows always exist and are unique. In contrast, global flows (which means $t \in \mathbb{R}$ instead of a restricted interval) may not exist.



3.3.3 Lie Derivative

Theorem 3.7 (Interpretation of Lie Bracket).

If X, Y are two vector fields on \mathcal{M} , and define the following quantity, which can be interpreted as the change of Y when following the integral curves of X , as

$$\frac{d}{dt}(\phi_{-t*}^X(Y)) \Big|_{t=0} := \lim_{\epsilon \rightarrow 0} \frac{\phi_{-\epsilon*}^X(Y_{\phi_\epsilon^X(p)}) - Y_p}{\epsilon}.$$

Then,

$$\frac{d}{dt}(\phi_{-t*}^X(Y)) \Big|_{t=0} = [X, Y].$$

4 Cotangent Spaces

4.1 Cotangent Vectors

Definition 4.1 (Cotangent Spaces).

The cotangent space $T_p^*\mathcal{M}$ at $p \in \mathcal{M}$ is the set of all linear functions $f : T_p\mathcal{M} \rightarrow \mathbb{R}$.

Its member is called a cotangent vector.

$$\dim T_p^*\mathcal{M} = \dim T_p\mathcal{M}.$$

Definition 4.2 (One-Form).

A one-form on \mathcal{M} is a smooth assignment of cotangent vectors $\omega : p \mapsto \omega_p$.

It may be understood as a covector field.

Definition 4.3 (Basis Cotangent Vectors).

The basis cotangent vectors is chosen to be the dual basis of the basis tangent vectors [Definition 2.11](#),

$$(dx^\mu)_p((\partial_\nu)_p) = \delta^\mu_\nu.$$

Theorem 4.1 (Coordinate Expression of Cotangent Vectors).

Any $f \in T_p^*\mathcal{M}$ can be expanded as

$$f = f_\mu (dx^\mu)_p.$$

Any one-form ω can be expressed as

$$\omega = \omega_\mu dx^\mu.$$

4.2 Pullback

4.2.1 Definition

Definition 4.4 (Pullback).

Given a function and its pushforward, we define pullback to be the dual of pushforward, i.e.,

$$\begin{array}{ccc} h : & \mathcal{M} & \rightarrow & \mathcal{N}, \\ h_* : & T_p \mathcal{M} & \rightarrow & T_{h(p)} \mathcal{N}, \\ h^* : & T_{h(p)}^* \mathcal{N} & \rightarrow & T_p^* \mathcal{M}, \end{array}$$

s.t. given $f \in T_{h(p)}^* \mathcal{N}$ and $v \in T_p \mathcal{M}$,

$$(h^* f)(v) := f(h_* v).$$

↗ **Remark.**

Note especially on the direction of original function and its induced pullback. This is crucial to the covariancy of one-forms. ℳ

Theorem 4.2.

Given ω a one-form on \mathcal{N} , and a function $h : \mathcal{M} \rightarrow \mathcal{N}$, the pullback $h^* \omega$ is defined as

$$(h^* \omega)(v)_p = \omega(h_* v)_{h(p)}.$$

Theorem 4.3 (Associativity of Pullbacks).

Analogous to [Theorem 2.11](#), given manifolds $\mathcal{M}, \mathcal{N}, \mathcal{P}$ and $h : \mathcal{M} \rightarrow \mathcal{N}$, $k : \mathcal{N} \rightarrow \mathcal{P}$, then

$$(k \circ h)^* = k^* \circ h^*.$$

4.2.2 Jacobian

Theorem 4.4 (Local Representative of Pullback).

Let $\dim \mathcal{M} = m$, $\dim \mathcal{N} = n$, $h : \mathcal{M} \rightarrow \mathcal{N}$, $\{x^1, \dots, x^m\}$ be the local coordinates of \mathcal{M} around p , and $\{y^1, \dots, y^n\}$ be the local coordinates of \mathcal{N} around $h(p)$. Then

$$h^*\omega = \sum_{\mu=1}^m \sum_{\nu=1}^n \omega_\nu \frac{\partial h^\nu}{\partial x^\mu} \Big|_p (dx^\mu)_p,$$

where $J^\nu_\mu := \frac{\partial h^\nu}{\partial x^\mu} \Big|_p := (\partial_\mu)_p (y^\nu \circ h)$ is the Jacobian matrix.

Proof. We know by [Definition 4.4](#),

$$(h^*\omega)_\mu(p) = h^*\omega(\partial_\mu) = \omega(h_*\partial_\mu).$$

Expand it in local coordinates of \mathcal{N} ,

$$(h^*\omega)_\mu(p) = \omega_\nu dy^\nu(h_*\partial_\mu).$$

Via similar procedure in [Theorem 2.12](#), we arrive at

$$(h^*\omega)_\mu(p) = \omega_\nu \frac{\partial h^\nu}{\partial x^\mu}.$$

□

4.3 Transformation Properties

Theorem 4.5 (Covariancy and Contravariancy).

Given two coordinate charts (U, ϕ) and (U', ϕ') s.t. $U \cap U' = S \neq \emptyset$, then on S ,

$$\begin{aligned} X^{\nu'} &= \sum_{\mu=1}^m \frac{\partial x'^\nu}{\partial x^\mu} X^\mu, \\ \omega_{\nu'} &= \sum_{\mu=1}^m \omega_\mu \frac{\partial x^\mu}{\partial x'^\nu}. \end{aligned}$$

If Jacobian matrix is given,

$$J^{\nu'}_{\mu} := \left. \frac{\partial x'^{\nu}}{\partial x^{\mu}} \right|_p := (\partial_{\mu})_p x'^{\nu},$$

$$(J^{-1})^{\mu}_{\nu'} := \left. \frac{\partial x^{\mu}}{\partial x'^{\nu}} \right|_p := (\partial_{\nu'})_p x^{\mu},$$

then,

$$X^{\nu'} = J^{\nu'}_{\mu} X^{\mu}, \quad (\text{contravariant})$$

$$\omega_{\nu'} = \omega_{\mu} (J^{-1})^{\mu}_{\nu'}. \quad (\text{covariant})$$

Proof. The contravariant part is proved in [Theorem 2.14](#). Now we turn to the covariant part.

Let $h = \text{id} : (U, \phi) \subseteq \mathcal{M} \rightarrow (U', \phi') \subseteq \mathcal{M}$, consider its pullback.

$$(\text{id}^* \omega)_p = \omega_{\nu'} \frac{\partial x'^{\nu}}{\partial x^{\mu}} dx^{\mu}.$$

Then,

$$\omega_{\mu} = \omega_{\nu'} \frac{\partial x'^{\nu}}{\partial x^{\mu}}.$$

Inverting the matrix equation above, we get the desired result. ■

5 Tensors

Definition 5.1 (Tensors).

If $\dim \mathcal{M} \neq \infty$, the tensors of type (r, s) $T_p^{r,s}\mathcal{M}$ are all the linear functions

$$f : \bigtimes^r T_p^*\mathcal{M} \times \bigtimes^s T_p\mathcal{M} \rightarrow \mathbb{R}.$$

I.e., it eats r covectors and s vectors.

Theorem 5.1 (Dimensions of General Tensor Space).

The dimension of $T_p^{r,s}\mathcal{M}$ is $m^r m^s$. In particular, a basis for the space is,

$$\bigotimes_{1 \leq \mu_1 \cdots \mu_r \leq m} (\partial_{\mu_i})_p \otimes \bigotimes_{1 \leq \nu_1 \cdots \nu_s \leq m} (dx^{\nu_i})_p$$

6 n-Forms