Intra-Party Politics and Dynamic Policy Polarization*

Xiaoye Liao† and Andrew McClellan‡

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Abstract

This paper presents a model of dynamic policy choices by two political parties with alternating political control and the presence of within-party heterogeneity. Moderate party leaders must appease radical party members to prevent them from attempting to seize control of the party. We find that the interaction between moderates and radicals may lead to dynamic polarization, in which parties continually choose more extreme policies over time. Dynamic polarization relies on radicals being present in both parties and may become more pronounced when players are more patient or the persistence of political control decreases. Extensions of the baseline model illustrate the robustness of our results.

Keywords: dynamic political economy, dynamic policy polarization, intra-party politics.

JEL Classifications: C73, D74, P16,

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†NYU Shanghai, Email: `xl775@nyu.edu`.

‡University of Chicago, Booth School of Business, Email: `Andrew.McClellan@chicagobooth.edu`

1 Introduction

The shifting nature of political power introduces a complex series of interactions between parties, which allows them to sustain different paths of policy choices. Looking back through history, we find rich patterns of policy dynamics with party politics, some of which exhibit long run stability through multilateral compromises, but others feature escalating antagonism and end up with sharp partisan divides. One may naturally wonder what factors determine which type of policy dynamics arise. This paper offers an answer to this question by investigating the effect of repeated inter-party and intra-party interactions.

Our main novelty is to incorporate the intra-party dimension. Parties are not single agents but may represent groups with heterogeneous preferences. Party leaders then face the problem of appeasing the various wings of their party, which we term *intra-party politics*. We demonstrate that this is a crucial mechanism for generating polarization and increasing antagonism between parties. This mechanism can help explain why polarization has been shown to be increasing both across parties and within parties. Groenendyk et al. (2020) found increasing differences in voters' feelings towards their own party and the opposing party, as well as evidence of increasing dissatisfaction of moderates with their own party. This dissatisfaction is widespread—Pew Research Center (2019) found that "47% of Democrats and 45% of Republicans say their own party is described very or somewhat well by the phrase 'too extreme in its positions.'"

To better understand this pattern of polarization, we consider an infinite horizon dynamic model in which two parties alternate power according to an exogenous Markov process. In each period, the leaders of the party in power decide the government policy, which determines the stage payoffs of both parties. To capture intra-party politics, we assume that each party consists of two types of agents, *moderates* and *radicals*. Radicals have more extreme preferences than the moderates in their party on which policy to implement. We assume that the current party leaders share the same preferences as the moderates but face a threat of being replaced by radicals. In addition to considering how the other party will respond to a particular policy choice, which we term inter-party politics, party leaders must also consider whether a policy choice will induce or dissuade radicals in their own party from attempting to replace them. The need for party leaders to mollify their radical wing plays a significant role in determining the dynamics of policies. Assuming that a takeover by radicals is sufficiently costly to the party leaders, we characterize the Pareto frontier of the set of sequential equilibria (which are called *party-efficient equilibria*) payoffs to party leaders subject to the sequential participation constraints that ensure radicals do not find it profitable to take over their parties.

Say that a sequence of policies *polarizes* if, after losing and then regaining power, the policy each party chooses moves more towards their radical wing's most preferred one. A sequence of policies is said to *settle* if eventually, each party, when ruling, implements the same policy (which may vary across parties). We begin by showing that all party-efficient equilibria are composed of *term-stationary* policies; that is, each party chooses the same policy until it loses power, after which it may change its policy the next time it regains power. Using this property, we first solve the model, assuming away radicals, to set the benchmark against which we can evaluate the effect of introducing intra-party politics. When intra-party politics are not present, we find that all equilibria settle by the first power transition and the corresponding policy sequences feature *compromise*, in which the first party in power moves its policy towards the other party's preferred policy after losing and regaining power.

We then investigate the implications of intra-party politics by reincorporating radicals. Our main result shows that every party-efficient equilibrium either settles by the first power switch or never settles and features continual polarization. The equilibrium dynamics exhibit a vicious cycle: In order to deter a takeover by radicals in a party, the party will move its policies towards their radicals' bliss point in the future. But this makes radicals' in the other party worse off and the other party must then make their own future policies more extreme in order to placate their radicals and prevent a takeover. Over time, radicals' stage payoff increases when their party is in power and decrease when their party is not. We interpret this as increasing antagonism between the two types of radicals, matching recent polling patterns on the changes in attitudes towards political opponents.[1]

There are two major implications of our results. First, intra-party politics are a mechanism for generating dynamic polarization while inter-party politics alone leads to compromise. These results illustrate the importance of understanding the heterogeneity of preferences within a party; treating parties as single agents can lead to starkly different results. Second, this mechanism generates polarization in an environment that, except for the transition of power between parties, is completely stationary; all players' preferences depend only on the policy chosen and party in power and do not change over time. This stands in contrast to much of the previous literature which looks at changes in preferences or in beliefs as the driving force for polarization. This is not to say that these are not important drivers of real-world polarization, only that intra-party politics may also generate polarization even without these other factors. What we emphasize here is that we may have polarizing policies even under stationary preferences and beliefs, and changing policies do not necessarily imply changes in parties' preferences.

We then study how the policy dynamics in party-efficient equilibria rely on some parameters of the model. In particular, our analysis is focused on the role played by patience, in order to understand whether less myopic players are less inclined to generate dynamic polarization; and the role played by the persistence of political power, in order to speak to the debate over whether stable distributions of political control lead to more socially efficient policies. We find that, unlike what the intuition based on folk theorems would suggest, increasing the players' discount factor may exacerbate the degree of dynamic polarization, leading to more extreme policies. In contrast, increasing the expected length of time a party remains in power may help mitigate dynamic polarization, leading to less extreme policies. These results show that some of our standard intuitions for the comparative statics of dynamic interactions no longer hold when we incorporate intra-party politics.

[1] For example, according to Pew Research Center (2019), "the level of division and animosity—including negative sentiments among partisans toward the members of the opposing party—has only deepened" in recent years.

We then examine some extensions of our baseline model. Our first extension shows that intra-party politics in both parties is necessary for generating continual polarization. We find that when only one party has radical members, the equilibrium settles by the first power transition and so strict polarization does not occur. The vicious cycle that generates continual polarization relies on the fact that appeasing radicals of one party makes radicals in the other party worse off; with only one set of radicals, this cycle does not arise. We also illustrate the robustness of our result by considering more general policy spaces. Under some appropriate extensions that ensure that radicals are more extreme than moderates on every dimension of the policy, the main insights still go through—that is, we show that radicals' stage payoffs are decreasing over time when the other party is ruling but are increasing when their party is.

The remainder of this paper is structured as follows. We first review some related literature, and then introduce our model in Section 2. In Section 3 we present our main results. Some extensions of the baseline model are discussed in Section 4. Section 5 concludes. All proofs are relegated to the Appendix.

Related Literature

Our paper is related to the literature on dynamic interactions between parties. The closest paper to our own is Dixit et al. (2000), in which the authors study how repeated interactions between parties shape the long-run dynamics of policies. The main difference between our paper and theirs is that we treat a party as consisting of heterogeneous agents rather than as a single player. This generates a stark difference in policy dynamics between our paper and theirs. Our model without radicals is essentially a special case of theirs, and so our results show that intra-party politics are necessary for polarization. Other papers such as Acemoglu et al. (2011), Aguiar & Amador (2011), Bowen et al. (2014) and Dziuda & Loeper (2016) also study the implications of dynamic party interactions. Again, our main difference with these papers is our focus on polarization and the inclusion of intra-party politics.

Our paper also contributes to the literature looking at the effects of intra-party interactions (e.g., Caillaud & Tirole (2002), Levy (2004), Persico et al. (2011)) and the effect of pressure groups on politicians (e.g., Dal Bó & Di Tella (2003)). Most of these papers look at static models of parties and focus on the effect of parties vis-à-vis election outcomes. Our model, in contrast, abstracts away from electoral concerns and focuses on the dynamics of policies that are implemented.

The increase in political polarization has been well empirically documented (e.g., Barber et al. (2015)) and been widely studied. The role of parties in generating political polarization has been studied by Canen et al. (2021), who find that parties have exerted increasing effort in corralling moderate Congressional members to vote for more extreme positions. Various explanations for political polarization have been proposed–e.g., belief polarization with Bayesian players (e.g, Dixit & Weibull (2007)), differences in voters' media consumption (e.g., Levendusky (2013), Perego & Yuksel (2021)), and changes in voters' preferences (e.g., Callander et al. (2020)).

Our paper contributes to this literature by identifying intra-party politics as a mechanism for polarization even when preferences and beliefs are stable.

Our work is also related to the literature on deriving efficient dynamic allocations subject to limited commitment–e.g, risk-sharing models (e.g., Thomas & Worrall (1988), Kocherlakota (1996)), principal-agent models (e.g., Ray (2002)) and political economy models (e.g., Dixit et al. (2000), Acemoglu et al. (2011)). Like these papers, we find a similar type of stationarity in the equilibrium allocation until some player is tempted to deviate, after which the equilibrium allocation moves in that player's preferred direction. Our model differs from those in the literature by adding multiple players with heterogeneous preferences over the same action; making one radical better off makes the other radical worse off, which drives our polarization dynamics that are new relative to the previous literature.

2 Model

Consider an infinite horizon discrete time game with dates $t = 0, 1, 2, \ldots$. There are two parties indexed by L and R. In each period t, Nature randomly chooses a state θ_t, called the *ruling state*, which determines the *ruling party* for that period. Abusing notation slightly, we assume that $\theta_t \in \{L, R\}$ so that party i is selected as the ruling party in period t if and only if $\theta_t = i$. The ruling party of period t decides how to allocate a given dollar: it chooses a *policy* $g_t \in [0, 1]$, which is interpreted as the period t budget for a public good, and the remainder $1 - g_t$ is the ruling party's private good for period t.

Preferences

Each party is composed of two types of agents, *moderates* (tagged by m) and *radicals* (tagged by r), who have different preferences. For each $i \in \{L, R\}$, moderates in party i (i-moderates hereafter) rank a policy sequence $\{g_t\}_{t=0}^{\infty}$ according to

$$\mathbf{E}\left[\sum_{t=0}^{\infty} \delta^t u_i(\theta_t, g_t)\right],$$

while radicals in party i (i-radicals hereafter) rank the policy sequence according to

$$\mathbf{E}\left[\sum_{t=0}^{\infty} \delta^t v_i(\theta_t, g_t)\right].$$

Here $\delta \in (0, 1)$ is the common discount factor and $u_i(\theta, g)$ ($v_i(\theta, g)$, resp.) is i-moderates' (i-radicals', resp.) period utility function, which is strictly concave and continuous in g for either θ. Each party i does not benefit from the private good of party $-i$, and so both $u_i(\theta, g)$ and $v_i(\theta, g)$ are strictly increasing in g when $\theta \neq i$ (i.e., when party i is not in power). Furthermore, we assume that the party in power has a lower demand for the public good than party out of power and moderates hold less extreme stances than their party comrades. Precisely, letting $g^*_{i,m}(\theta) = \arg\max_{g\in[0,1]} u_i(\theta, g)$ and $g^*_{i,r}(\theta) = \arg\max_{g\in[0,1]} v_i(\theta, g)$ ($i, \theta \in \{L, R\}$) be, respectively, the most preferred level of public good provision for i-moderates and i-radicals in state θ, we have

$$\begin{aligned} 1 = g^*_{L,r}(R) = g^*_{L,m}(R) > g^*_{R,m}(R) > g^*_{R,r}(R) \geqslant 0, \\ 1 = g^*_{R,r}(L) = g^*_{R,m}(L) > g^*_{L,m}(L) > g^*_{L,r}(L) \geqslant 0. \end{aligned}$$

On the range of Pareto-efficient policies $[g^*_{i,r}(\theta), 1]$ for each $i, \theta \in \{L, R\}$ with $i = \theta$, our assumptions so far imply *strict antagonism* between the radicals: that is, making one type of radicals strictly better off necessarily renders the other type of radicals strictly worse off. The assumption that radicals' and moderates' bliss points when out of power are both 1 is not necessary; our results continue to hold as long as the radicals' bliss point is higher than the moderates' when not in power.

Timing of Policy Choice

It is always the moderates of the ruling party who choose the policy unless the radicals of the opposite party choose to a *takeover* action, in which case the radicals will control their party; and i-radicals can only succeed in their takeover in state $-i$. We interpret this as the ruling moderates of a party being sufficiently strong when in power to thwart any takeover. If a takeover in period t occurs, i-radicals become the policymaker of their party thereafter and choose g_s in each period $s > t$ with $\theta_s = i$.

The order of play is as follows. At the beginning of each period t, the state θ_t is first realized, after which the opposite radicals decide whether to take over. Then the ruling party chooses g_t and the stage payoffs are realized.

State Evolution

The state variable θ_t evolves from period to period according to a time-homogeneous and transient Markov chain. In particular, for each $\theta \in \{L, R\}$ and $t \in \mathbf{N}$,[2]

$$\mathbf{P}(\theta_{t+1} = \theta \,|\, \theta_t = \theta) = \pi_\theta \in (0, 1),$$

and we presume that $\theta_0 = L$, that is, the initially selected ruling party is L. We recursively define the *nth inauguration* of party L, denoted by τ^L_n, as the nth time θ_t switches from R to L,

[2]In this paper, we use $\mathbf{N}$ to denote the set of all nonnegative integers, and $\mathbf{N}^*$ to denote the set of positive integers, that is, $\mathbf{N} = \{0, 1, 2, \dots\}$ and $\mathbf{N}^* = \{1, 2, 3, \dots\}$.

that is, for each $n \geqslant 2$,

$$\tau_n^L = \inf\{t > \tau_{n-1}^L \,|\, \theta_{t-1} = R,\, \theta_t = L\}, \;\; \tau_1^L = 0.$$

The nth inauguration of party R, denoted by τ_R^n, is similarly defined as the nth time θ_t switches from L to R, namely, for each $n \in \mathbf{N}^*$,

$$\tau_n^R = \inf\{t > \tau_n^L \,|\, \theta_{t-1} = L,\, \theta_t = R\}.$$

Accordingly, the *nth term* of party L (R, resp.), denoted by T_n^L, is defined as the set of periods between τ_n^L and $\tau_n^R - 1$, that is, $T_n^L = \{\tau_n^L, \tau_n^L + 1, \ldots, \tau_n^R - 1\}$. Symmetrically, the *nth term* of party R is $T_n^R = \{\tau_n^R, \tau_n^R + 1, \ldots, \tau_{n+1}^L - 1\}$. Since the underlying Markov chain for the state evolution is transient, the probability for any τ_n^L or τ_n^R to be $+\infty$ is zero. For the sake of simplicity, all terms are assumed to be finite, and so $\tau_n^L < \tau_n^R < \tau_{n+1}^L$ for all $n \geqslant 1$. Figure 1 below illustrates these definitions with a typical realization of $\{\theta_t\}_{t=0}^{\infty}$.

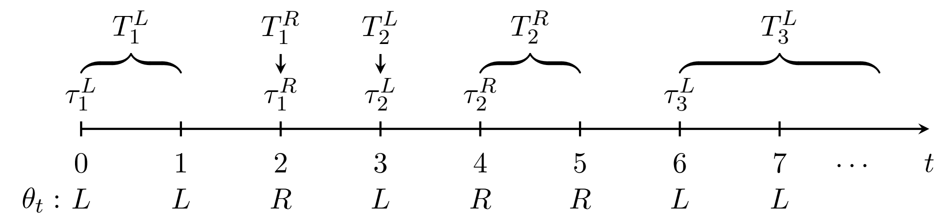

FIGURE 1: AN ILLUSTRATION OF POWER SWITCHES, INAUGURATIONS, AND TERMS

Let χ_t be an indicator for a radical takeover in period t. A typical t-period history is a list $h_t = (\theta_0, \chi_0, g_0; \theta_1, \chi_1, g_1; \ldots; \theta_{t-1}, \chi_{t-1}, g_{t-1}; \theta_t, \chi_t)$. The set of all possible t-period histories is denoted by $\mathbf{H}_t$, and the set of all histories with each term being finite is denoted by $\mathbf{H}$, where $\mathbf{H} = \bigcup_{t=0}^{\infty} \mathbf{H}_t$. For each h_t and $n \in \mathbf{N}^*$, the corresponding nth inauguration date of party i is denoted by $\tau_n^i(h_t)$ (given that h_t is long enough to have reached such a date), and the nth term of party i in h_t is denoted by $T_n^i(h_t)$.

Equilibrium Concept

Our main objective is to characterize the set of sequential equilibria that are constrained efficient for the two parties[3], which will be called the *party-efficient equilibrium* (or *equilibrium* for short). Note that the solution concept is constrained efficiency.[4] Given our setup, by Abreu (1988), for each player a worst sequential equilibrium exists, which is employed in the *optimal penal code* to support all sequential equilibria. We denote by U_i the value of the optimal penal code for

[3]That is, the set of sequential equilibria that are not Pareto dominated (with respect to parties' utilities) by any other sequential equilibria.

[4]Given that moderates are initially in control, they will agree on the set of norms to maximize their joint utility and will not consider the radical's utility except through the takeover threat that the radicals present.

i-moderates if they deviate in state i. This value includes both the maximal value of deviating today, namely $\max_{g\in[0,1]}\ u_i(i,g)$, and the lowest equilibrium continuation value from tomorrow onwards. We also denote by V_i the continuation value of i-radicals in the optimal penal code after they takeover.

We assume that the parties' continuation payoffs after a radical takeover are sufficiently low so that no takeover will be triggered on a party-efficient equilibrium path, and so we only consider on-path histories without any radical takeovers. For simplicity, we take party leaders' continuation value after a radical takeover to be $-\infty$.[5] As is well known, each party-efficient equilibrium corresponds to maximizing a utilitarian social welfare function for the moderates subject to ensuring that moderates and radicals do not deviate from the prescribed equilibrium actions. Specifically, defining

$$u_\eta(\theta,g) = \eta u_L(\theta,g) + (1-\eta)u_R(\theta,g)$$

as the per-period social welfare function, with $\eta \in [0,1]$ being the Pareto weight for L-moderates, a party-efficient equilibrium solves

$$\max_{\{g_t\}_{t=0}^{\infty}} \mathbf{E}\left[\sum_{t=0}^{\infty} \delta^t u_\eta(\theta_t, g_t)\right] \tag{1}$$

subject to i-moderates' incentive constraints: for $i \in \{L,R\}$ and every history $h_s \in \mathbf{H}$ ($s \in \mathbf{N}$) such that $\theta_s = i$,

$$IC^i(h_s): \quad \mathbf{E}\left[\sum_{t=s}^{\infty} \delta^{t-s} u_i(\theta_t, g_t) \,\middle|\, h_s\right] \geqslant U_i; \tag{2}$$

and to $-i$-radicals participation constraints: for $i \in \{L,R\}$ and every history $h_s \in \mathbf{H}$ ($s \in \mathbf{N}$) such that $\theta_s = i$,

$$PC^{-i}(h_s): \quad \mathbf{E}\left[\sum_{t=s}^{\infty} \delta^{t-s} v_{-i}(\theta_t, g_t) \,\middle|\, h_s\right] \geqslant V_{-i}. \tag{3}$$

Restricting attention to pure strategies, each profile of on-path dynamic policies can be summarized by a function $g : \mathbf{H} \to [0,1]$ such that the policy implemented after each history $h_t \in \mathbf{H}$ is $g(h_t)$.[6] To ensure that this problem is well-posed and for technical convenience, we assume that there exists a policy g for which the all constraints are slack.[7]

[5] The exact value of $-\infty$ as their continuation value is not important, only that it is sufficiently low.

[6] Since all utility functions are strictly concave, it is without loss of generality to restrict attention to pure strategies only.

[7] The assumption that participation constraints are slack is used to verify a Slater condition in our constrained optimization problem.

Discussion of the Model

Before going to derive the equilibrium structure, we stop briefly and discuss some aspects of our model.

Interpretation of Policy Choice

Although we have interpreted $1-g$ as a party specific good, it bears other interpretations. For example, we could think of g as the amount of time spent legislating on public goods and $1-g$ as that on the ruling party's ideological platform. With this interpretation, a decrease in g corresponds to more time devoted towards partisan ideological interests.

Our model allows for a rich set of specifications of each players' stage payoff, which can depend on both the policy g and the state θ. It is worth mentioning that our model can nest standard spatial models for policy choices, in which players have preferences that are invariant to the state θ. For example, we can let $[0,1]$ be the space of a policy choice a and all utilities be independent of θ, where i-radicals receiving utility $\hat{v}_i(a)$ and i-moderates $\hat{u}_i(a)$. The bliss points are ranked according to $a^*_{L,r} \leqslant a^*_{L,m} \leqslant a^*_{R,m} \leqslant a^*_{R,r}$ where $a^*_{i,r}, a^*_{i,m}$ are the respective bliss points for i-radicals and i-moderates. To see that this can be covered by our setup, take the choice of g in state R in our model to be the distance of the policy a from 1 and the choice of g in state L to be the distance of the policy a from 0: that is, $v_i(R,g) = \hat{v}_i(1-g)$ and $v_i(L,g) = \hat{v}_i(g)$ (with similar translations for u_i). With this interpretation, however, some additional restrictions may need to be made on the policy choices available to parties in order to avoid some unnatural equilibria dynamics. This point will be discussed in Subsection 3.2.

Interpretations of Radical Takeovers

In our current interpretation, a takeover by i-radicals leaves them in charge of their own party forever. However, we can immediately fit other stories into our model without any difficulty. For example, we can consider the case in which a takeover corresponds to starting a civil war or a coup to take control of the government. We can also allow for a radical takeover to have a fixed cost or only probabilistically succeed and for moderates to seize back control of the party from radicals after some time. All that matters for the radicals when making their decision is the value they receive after a takeover, which is summarized completely by V_i. We emphasize that in order to characterize party-efficient equilibria on-path, we only need to know the value of V_i rather than the exact details of the off-path equilibrium used to deliver V_i. This observation follows from our assumption that takeover will never occur on an efficient equilibrium path.

The threat of a radical takeover may provide a useful punishment for sustaining moderates' equilibrium actions. However, while we assume that a takeover by radicals is sufficiently bad for moderates, our setup makes no assumptions on the credibility of such a takeover. If radicals' utility following a takeover is not high enough, there may not exist a sequential equilibrium in which radicals takeover. In such a case, the extreme punishment for moderates from a radical takeover cannot be used to enforce equilibrium actions for moderates.

The assumption that i-radicals can only take control in state $-i$ can be thought of as a constraint on the ability of i-radicals to attract support in their party for such a takeover. For example, it may be difficult to motivate members of a party to start a civil war when they are in control of the government.

3 Equilibrium Characterization

Our setup allows parties to choose among a wide array of history-dependent policies. For example, a party in power may moderate their policies more or less the longer their term continues or a party regaining power may adjust their policies depending on the length of time they were out of power. Nevertheless, we find that equilibrium policies always possess a simple structure, using the same policy at every period in a term regardless of the history prior to the term. In each party-efficient equilibrium, the policies must therefore remain constant until a switch of power. Moreover, such policies must be term-specific, that is, the policy of party i in its nth term will be the same across all on path histories that lead to τ_n^i. This property, which is called *term-stationarity*, will significantly simply our further analysis.

Proposition 1. *Let $g : \mathbf{H} \to [0,1]$ be a party-efficient equilibrium policy sequence. Then $g(h_t) = g(h_s)$ if $t \in T_n^i(h_t)$ and $s \in T_n^i(h_s)$ for some $i \in \{L, R\}$ and $n \in \mathbf{N}^*$.*

The driving force of this result is the strict concavity of the utility functions. Strict concavity implies that, for any non-constant policy sequence, all players will prefer a constant "average" policy sequence. This force naturally pushes towards stationary policies. We show that, for any sequence of equilibrium policies in a term, we can replace each of them with a *term-average* policy, which is equal to the average discounted policy across all histories for that term, and achieve a welfare improvement for all agents. This policy depends only on the index of the term and not on the length of the term or history prior to the term.

As mentioned above, with Proposition 1 our problem (1)-(3) can be reformulated in a much simpler form. In particular, we can restrict attention to sequences of polices of the form $\mathbf{g} = \{(g_n^L, g_n^R)\}_{n=1}^{\infty}$, where for each $i \in \{L, R\}$ and $n \in \mathbf{N}^*$, g_n^i represents the policy for the nth term of party i. As a result, for a fixed Pareto weight η, the corresponding party-efficient equilibrium is characterized by the following sequence problem:

$$\max_{\{(g_n^L, g_n^R)\}_{n=1}^{\infty}} \sum_{n=1}^{\infty} \sum_{\theta \in \{L,R\}} \mathbf{E}\Bigg[\sum_{t \in T_n^\theta} \delta^t\Bigg] u_\eta(\theta, g_n^\theta) \tag{4}$$
$$\text{s.t. } IC_n^i,\ PC_n^i \text{ for all } n \in \mathbf{N}^* \text{ and } \theta \in \{L, R\},$$

where IC_n^i is i-moderates' incentive constraint at τ_n^i, and PC_n^i is i-radicals' participation constraint at τ_n^{-i}, that is,

$$IC_n^R: \quad \sum_{m\geqslant n}\sum_{\theta\in\{L,R\}} \mathbf{E}\Bigg[\sum_{t\in T_m^\theta}\delta^t\Bigg] u_R(\theta, g_m^\theta)\mathbf{1}_{\{\theta=R \text{ or } m>n\}} \geqslant \mathbf{E}\left[\delta^{\tau_n^R}\right] U_R, \tag{5}$$

$$IC_n^L: \quad \sum_{m\geqslant n}\sum_{\theta\in\{L,R\}} \mathbf{E}\Bigg[\sum_{t\in T_m^\theta}\delta^t\Bigg] u_L(\theta, g_m^\theta) \geqslant \mathbf{E}\left[\delta^{\tau_n^L}\right] U_L, \tag{6}$$

$$PC_n^R: \quad \sum_{m\geqslant n}\sum_{\theta\in\{L,R\}} \mathbf{E}\Bigg[\sum_{t\in T_m^\theta}\delta^t\Bigg] v_R(\theta, g_m^\theta) \geqslant \mathbf{E}\left[\delta^{\tau_n^L}\right] V_R, \tag{7}$$

$$PC_n^L: \quad \sum_{m\geqslant n}\sum_{\theta\in\{L,R\}} \mathbf{E}\Bigg[\sum_{t\in T_m^\theta}\delta^t\Bigg] v_L(\theta, g_m^\theta)\, \mathbf{1}_{\{\theta=R \text{ or } m>n\}} \geqslant \mathbf{E}\left[\delta^{\tau_n^R}\right] V_L. \tag{8}$$

A sequence of policies $\mathbf{g} = \{(g_n^L, g_n^R)\}_{n=1}^\infty$ that satisfies (5)-(8) for all $n \in \mathbf{N}^*$ is called *admissible*. Note that hereafter each subscript n refers to the index of a term rather than the calendar time. Since each party's utility is strictly concave in g for each state, it is easy to see that the solution to (4) is unique for each Pareto weight η. To understand the dynamics of the solution to (4), we first define some properties the equilibrium policies may possess.

Definition 1 (Settlement)**.** *A sequence of policies has settled by τ_m^i if each party chooses the same policy in each of its term after τ_m^i.*

More concretely, the definition says that if $\mathbf{g} = \{(g_n^L, g_n^R)\}_{n=1}^\infty$ has settled by τ_m^L, then for each $n \geqslant m$ and $i \in \{L, R\}$ one has $g_n^i = g_m^i$, while if it has settled by τ_m^R, then one has $g_n^R = g_m^R$ for all $n \geqslant m$ and $g_n^L = g_{m+1}^L$ for all $n \geqslant m+1$, but settlement does not require parties to choose the same policies. The asymmetry between settlement at τ_m^L and that at τ_m^R is simply an artifact of our convention for numbering terms. Once a sequence of policies settles, the continuation play then features a Markovian structure, depending only on the identity of the party in power.

Definition 2 (Compromise and Polarization)**.** *A sequence of policies* $\mathbf{g} = \{(g_n^L, g_n^R)\}_{n=1}^\infty$

(i) *compromises if $g_n^i \leqslant g_{n+1}^i$ for each $i \in \{L, R\}$ and all $n \in \mathbf{N}^*$;*

(ii) *polarizes if $g_n^i \geqslant g_{n+1}^i$ for each $i \in \{L, R\}$ and all $n \in \mathbf{N}^*$, and at least one of these inequalities is strict;*

(iii) *strictly polarizes if $g_n^i > g_{n+1}^i$ for each $i \in \{L, R\}$ and every $n \in \mathbf{N}^*$.*

The idea behind the definition of compromise is that once party i loses power and regains power, it compromises on its policy choice by moving closer to party $-i$'s preferred position. For polarization, the idea is the opposite: a party, after losing power, chooses to implement more extreme policies and offer less of the public good in the future. The other party responds in kind, leading to a decrease in public good expenditure over terms.

Notice that our definitions for compromise and polarization can be restated in terms of the radicals' stage payoffs over time, if, as will occur in any efficient equilibrium, the policy for each term lies between the radicals' bliss points.[8] More concretely, a sequence of policies $\mathbf{g} = \{(g_n^L, g_n^R)\}_{n=1}^{\infty}$ compromises if for each $i, \theta \in \{L, R\}$ and $n \in \mathbf{N}^*$, $v_i(\theta, g_n^{-i}) \leqslant v_i(\theta, g_{n+1}^{-i})$ for $i \neq \theta$ and $v_i(\theta, g_n^i) \geqslant v_i(\theta, g_{n+1}^i)$ for $i = \theta$; and that it polarizes if for each $i, \theta \in \{L, R\}$ and $n \in \mathbf{N}^*$, $v_i(\theta, g_n^{-i}) \geqslant v_i(\theta, g_{n+1}^{-i})$ for $i \neq \theta$ and $v_i(\theta, g_n^i) \leqslant v_i(\theta, g_{n+1}^i)$ for $i = \theta$, and at least one of these inequalities is strict.

With these definitions in hand, we study below in Subsection 3.1 a benchmark version of our model in which radicals are absent and show that no polarization occurs. Instead, each equilibrium features settlement and a compromise is achieved.

3.1 Party Homogeneity

A party i is said to be *homogeneous* if i-radicals are not present or, if they are present, their continuation value following a takeover is low enough that there exists no sequential equilibrium in which they takeover. Mathematically, these two are equivalent. When both parties are homogeneous, radicals place no constraints on the allocation parties can implement in equilibrium. This version of our model therefore isolates the effect of *inter-party* politics on the dynamics of the optimal policy. The result below shows that policies are either Markovian or "almost" Markovian in the state θ_t: the policy for T_1^L may be different from that for T_n^L when $n \geqslant 2$ but policies will have settled by the first power switch at τ_1^R. We also show that the equilibrium features compromise, so that the L party (weakly) increases public good spending after it loses power for the first time.

Proposition 2. *If both parties are homogeneous, then in every party-efficient equilibrium, we must have $g_1^L \leqslant g_2^L = g_n^L$ for all $n \geqslant 2$ and $g_1^R = g_n^R$ for all $n \in \mathbf{N}^*$.*

To understand this result, consider a social planner who is deciding the policies and ignores the IC constraints, in which case the first-best allocation is achievable. The concavity of u_i then implies that such an allocation would involve each party using the same policy in each of their terms—namely, immediate settlement. If such an allocation turns out to satisfy the IC constraints, then we have found a party-efficient equilibrium in which settlement occurs immediately.

If, however, such an allocation violates some of the IC constraints, then when it first binds we must increase the continuation value of the party whose IC constraint is binding. That party still has an interest in smoothing consumption. Suppose that IC_n^L is the first binding constraint. Taking a policy which uses a discounted average of g_m^i for $n \geqslant m$, we can construct an improvement for both parties. However, this smoothed allocation may violate the IC of party R in the future, in which case R's continuation value must be increased. We show in the

[8] Notice that we are interested only in compromise or polarization *on the equilibrium path*. It is easy to see that no g outside this range will ever be implemented on the equilibrium path.

proof that if a party-efficient equilibrium does not settle immediately, then some IC_n^i must be binding for each $n \geqslant 1$ and that if IC_n^i binds, then IC_m^i binds for all $m > n$. Building upon this result, we further observe that at τ_n^i, given that party i will receive a continuation value of U_i, the efficient equilibrium chooses a sequence of continuation policies to maximize party $-i$'s continuation value subject to delivering U_i to party i. But once we have reached τ_{n+1}^i, the problem of the social planner is the same as that at τ_n^i, meaning that the policy for T_{n+1}^i will be the same as that for T_n^i. Repeating this argument, we establish the stationarity of the sequence of policies in a party-efficient equilibrium with party homogeneity.

One may naturally wonder what is different about the first term of party L that allows for $g_1^L < g_n^L$ for $n \geqslant 2$. This difference is driven by the fact that party R has no opportunity to deviate prior to τ_1^R. When IC_1^R is binding, the equilibrium policy for T_n^L with $n > 1$ must take into account its impact on party R's continuation value at τ_1^R and therefore must deliver the corresponding continuation value to party R. However, no such concern is present during T_1^L. This means that the social planner can choose g_1^L without considering party R's utility other than its inclusion in the objective function. For $t \in T_n^L$ with $n > 1$, the need to provide additional continuation value to party R leads to an increase in the provision of public goods.

The equilibrium dynamics identified in Proposition 2, in particular the compromise and settlement at τ_1^R, are essentially the same as those in Dixit et al. (2000) in the special case of their model where the state space is binary. Our setup without radicals differs from their setup in that we consider a slightly more general utility function and restrict attention to a binary state space for θ. The similarity between our results in Proposition 2 and those in Dixit et al. (2000) emphasizes that it is the new element, namely intra-party politics, that drives the polarization results that are new in our model.

3.2 Polarization

Having characterized the dynamics caused by inter-party politics, we now incorporate intra-party politics (i.e., radicals whose interests are misaligned with those of the moderates') and investigate its implications on policy dynamics. Assume that both sets of radicals may credibly threaten to take over—that is, for each set of radicals, there is a sequential equilibrium in which that set of radicals takes over their party on-path. By punishing any deviation by i-moderates with a reversion to such a takeover equilibrium, we know that the optimal penal code yields i-moderates a continuation value of $U_i = -\infty$, ensuring that the IC constraints will be slack. The interpretation of this is that i-moderates' continuation value after a deviation is much lower than i-radicals', which implies that moderates are more inclined to compromise than radicals are. This fits naturally with the idea that radicals are more dogmatic with their preferences. For a concrete example in which a takeover will leave moderates with a continuation value of $-\infty$, for any $\sigma, \alpha \geqslant 0$ and $\beta \geqslant 1$, take[9]

$$u_i(\theta, g) = \ln(g) + \alpha \mathbf{1}_{\{\theta=i\}}(1-g),$$
$$v_i(\theta, g) = \frac{(g+1)^{1-\sigma} - 1}{1-\sigma} + \beta \mathbf{1}_{\{\theta=i\}}(1-g).$$

Any deviation by i-moderates is punished with a takeover by $-i$-radicals who then use an "autarky" equilibrium in which they implement their bliss point whenever in power. Because radicals' bliss point when in power is $g = 0$, such an equilibrium leads to $U_i = -\infty$.

Our main result below provides a complete characterization of the dynamics of that may arise on the efficient equilibrium path when the IC constraints are slack. In contrast to Proposition 2, the presence of radicals may lead to polarization on the equilibrium path rather than compromise and a much richer set of dynamics than the essentially Markovian dynamics generated under one-party homogeneity. Interestingly, unlike many other explanations for political polarization where the major driving force is changes in agents' preferences and/or beliefs, our model generates the polarization with fixed preferences and complete information.

Theorem 1. *Every party-efficient equilibrium features either immediate settlement or polarization, that is, $g_n^i \geqslant g_{n+1}^i$ for all $i \in \{L, R\}$ and $n \in \mathbf{N}^*$. Moreover, if an equilibrium has not settled by τ_1^R, then it strictly polarizes, that is, $g_n^i > g_{n+1}^i$ for all $i \in \{L, R\}$ and $n \in \mathbf{N}^*$.*

The direct implications of Theorem 1 are three-fold. To start with, although the space of admissible sequences of policies is large and admissible policies can feature many patterns in addition to the compromise and polarization dynamics we have defined (for example, cycles in which party i raises g for some terms and then lowers it for others), the result effectively leaves us with only three possibilities. First, it could be that the sequence of policies settles immediately at τ_1^L (i.e., at $t = 0$), in which case each party chooses the same policy whenever it is in power (although the policies for different parties may differ). Second, it could be that the sequence of policies first settles at τ_1^R, in which case the only change in a policy across terms involves party L choosing a higher g in its first term and a lower g thereafter. Third, an equilibrium may feature strict polarization, in which case each party reduces g whenever it regains power, as compared with its previous term.

If the equilibrium ever settles, it will do so no later than τ_1^R. Therefore, parties either reach a stationary outcome almost immediately or continue to change their policies forever in an efficient equilibrium, and so polarization that continues for some intermediate length of time will necessarily continue forever. The inability of radicals to commit to not pursue a takeover is crucial for generating strict polarization; because of the concavity of each player's utility, for any strict polarization equilibrium, there exists another "smoothed" path of policies that settles at $t = 0$ and yields strictly higher continuation values for both moderates and radicals at $t = 0$.

[9] This choice of u_i is similar to those in Bowen et al. (2014). Any choice of v_i with $g^*_{i,r}(\theta) = 0$ when $\theta = i$ will suffice.

However, such a smoothed policy path will violate the radicals' participation constraint, forcing the moderates to adjust the policy to prevent a takeover.

Finally, Theorem 1 identifies the dynamics for radicals' stage payoffs. Specifically, in each party-efficient equilibrium and for each $i \in \{L, R\}$, the i-radicals' stage payoff is increasing over the terms of party i but is decreasing over that of party $-i$, the monotonicity being strict with strict polarization. The difference in radicals' stage payoffs between when their party is in power and out of power is increasing over terms, matching the real-world growth in the difference between partisan voters' feelings for their own party and the opposing party. Looking at the i-moderates' stage utilities, our result predicts that they are decreasing over terms when party $-i$ is ruling and are single-peaked when party i is ruling—that is, moderates' utility when their own party is in power may be initially increasing, but if they begin to decrease they will decrease over all future terms. The dynamics of moderates' utilities also match empirical evidence which finds decreasing own party satisfaction among moderates (see Groenendyk et al. (2020)).

The structure of the set of binding participation constraints is the key determinant of the different patterns of dynamics found in Theorem 1. Strict polarization occurs when both sets of radicals' participation constraints are binding in every term after τ_1^R. Once a new term for party i is reached, the $-i$-radicals must be incentivized not to take over. This is then done by choosing a more extreme policy (i.e., less public good provision) when their party regains control. But, once party $-i$ does indeed regain control, the i-radicals participation constraint must then be satisfied. The more extreme policy by party $-i$ makes it harder to satisfy the i-radicals' participation constraint, necessitating the promise of a more extreme policy once party i returns to power. These dynamics create a vicious cycle in which appeasing the radicals of one party makes it harder to appease the radicals of the opposing party.

The proof of Theorem 1 roughly consists of three steps. First, we look at the monotonicity of radicals' stage payoffs based on the set of binding participation constraints. In particular, we show that policies across terms in state θ are constant as long as no participation constraints bind and that i-radicals' stage payoffs weakly decrease when their participation constraints are slack, with the decrease being strict if $-i$-radicals' participation constraints are binding in the meantime. This property is intuitive: when participation constraints are not binding, concavity of utilities implies that policies in the same state should be constant, and when $-i$-radicals' participation constraints do bind, we must tilt the policy towards $-i$-radicals' preferred level so as to satisfy these constraints. The second step consists of pinning down the patterns of binding participation constraints that can arise in a party-efficient equilibrium. We show that if i-radicals' participation constraint is binding in a term in state $\theta = -i$, then it will be binding in all future terms of that state. Similar to the intuition for the case of party homogeneity, we further find that for each $i \in \{L, R\}$, i-radicals' participation constraints are either always binding or always slack after τ_1^R. We finally use this characterization of participation constraints to generate a dynamic programming equation that then determines the dynamics of g_n^i and allows us to establish its monotonicity.

Figure 2 provides an illustration of the results presented in this section. As we can see in the

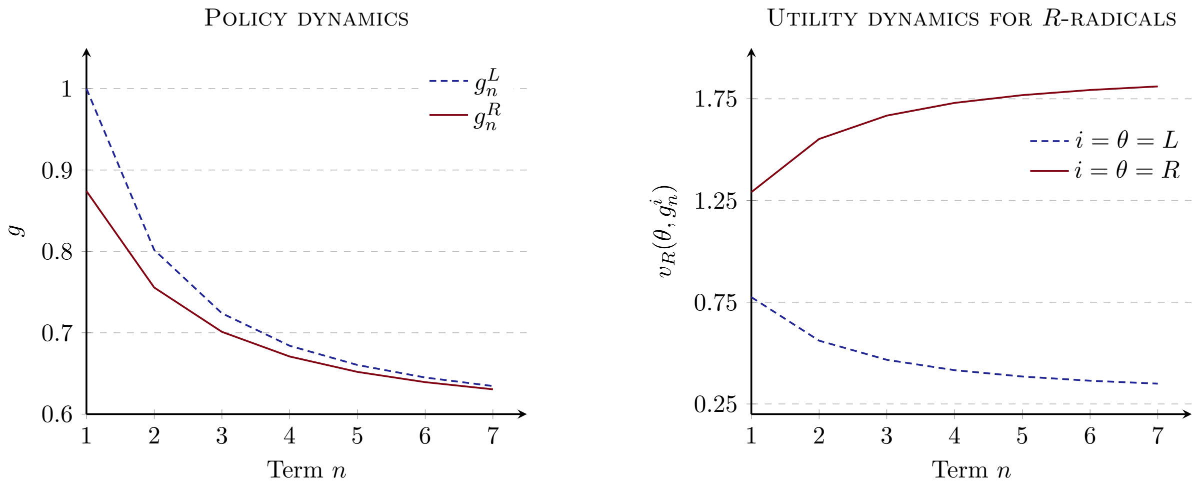

Figure 2: The above figures take $u_i(\theta, g) = \ln(g) + \alpha \mathbf{1}_{\{\theta=i\}}(1-g)$ and $v_i(\theta, g) = \frac{(g+1)^{1-\sigma}-1}{1-\sigma} + \beta \mathbf{1}_{\{\theta=i\}}(1-g)$ with $\sigma = 0.7$, $\beta = 3$, $\alpha = 2$, $\pi_L = \pi_R = 0.5$, $\delta = 0.9$, and $\eta = 0.5$.

left panel, each party (strictly) reduces the public good provision over its terms. Because g_n^i are monotonic in n, public good provision for each party will converge in the long run. In Figure 2 they converge to the same level due to the symmetry of the utility functions but in general different parties may have different asymptotic levels of public good provision. By allowing for different utilities across the two types of radicals, our model can capture asymmetric movements in parties policies. In the right panel we plot the dynamics of R-radicals' stage utilities over terms. In party R's terms, the utilities are increasing while they decrease in party L's terms, leading to larger swings in their utilities across terms, escalating antagonism and generating dynamic polarization. Although not plotted, the dynamics of L-radicals' stage payoffs follow a similar pattern.

As briefly mentioned at the end of Section 2, our setup nests a spatial model with invariant preferences by reinterpreting g in different states. When party L is ruling, the policy choice a is equal to g while it is equal to $1-g$ when party R is ruling. As natural as it looks, odd results may be obtained without further restrictions. In particular, we can end up with counter-cyclical policy transitions in which party R gaining power leads to a leftward movement of the policy and vice versa for party L. To rule out such unnatural dynamics, we place the restriction that for some constant $c \in (0,1)$, party R can only choose actions $a \in [c, 1]$ and party L can only choose $a \in [0, c]$. Interpreting the power transitions as reflecting underlying changes in voters' preferences, such a restriction reflects the idea that voters will only elect party R if they choose a sufficiently rightward policy and only elect party L if they choose a sufficiently leftward policy. This restriction can also be motivated as a constraint on the policies party leaders will be able to implement (e.g., conservative party leaders may be unable to convince their party members to support extremely liberal policies once in power). Imposing this constraint, our model predicts that party R will choose higher a over terms and party L will choose lower a over terms, leading to more extreme policies and wider swings of the policy choice over time.

3.3 Does patience or political persistence facilitate compromise?

We now look at the effect of changing the patience of players and the persistence of party control (i.e., π_θ) on the degree of equilibrium polarization. We will interpret a decrease in g as an increase in the degree of polarization.

A general intuition from the repeated game literature is that it is easier to sustain cooperation when people care more about future payoffs. Applying this to our model, one might naturally conjecture that polarization is less likely to occur when agents are more patient. It turns out that, however, this is not necessarily true. We will show this point, provided some additional assumptions, by looking at the comparative statics of equilibrium policies for a fixed Pareto weight η. Our assumptions are formally stated below.

Assumption 1. *We assume that*

(i) *for each $i \in \{L, R\}$, V_i is generated by an equilibrium in which i-radicals and $-i$-moderates choose static Nash equilibrium policies in each future period.*

(ii) *$u_\eta(\theta, g)$ is maximized with $g = 1$ for both $\theta \in \{L, R\}$.*

(iii) *All utility functions are differentiable in g.*

Assumption 1 (i) states that reversion to a static Nash equilibrium is used to generate the optimal penal code value V_i and is a commonly made assumption in the repeated games literature. Assumption 1 (ii) states that the efficient solution chooses full public good spending, which we can interpret as saying that the moderates are sufficiently moderate in their preferences that full public good spending is jointly optimal for them. It simplifies our derivation of the optimal policy dynamics by ensuring that moderates always prefer higher g. Assumption 1 (iii) is made purely for convenience in the analysis. For expositional ease, we denote by $\mathbf{g}(\eta; \delta) = \{(g_n^L(\eta; \delta), g_n^R(\eta; \delta))\}_{n=1}^{\infty}$ the party-efficient equilibrium under Pareto weight η and discount factor δ.

Proposition 3. *If Assumption 1 holds and $\delta < \delta'$, then $g_n^i(\eta; \delta') \leqslant g_n^i(\eta; \delta)$ for all $i \in \{L, R\}$ and $n \in \mathbf{N}^*$, with strict inequality for $n > 1$ if strict polarization occurs at δ.*

To understand this result, consider the case when the equilibrium features strict polarization. The key step for this result lies in showing that increasing the discount factor has a more significant impact on the radicals' outside option values V_i than on their on-path continuation values. This makes it more difficult to meet their participation constraints, and so yields a lower level of public good provision for each term. The key driving force for this result is that deviations by radicals occur when their party is out of power, and so the benefit of seizing control through the ability to choose the policy is not realized immediately. As δ increases, the delay costs from waiting to return to power (i.e., when the benefit of seizing control is realized) decrease, making a deviation more attractive. In fact, strict polarization necessarily relies on a high enough discount factor. For low discount factors, settlement will occur immediately at $t = 0$.

We then move on to investigate how the persistence of political control affects policy dynamics. In addition to Assumption 1, we also assume that the Markov process for the ruling state is symmetric, that is, $\pi_L = \pi_R$. It turns out that a higher level of persistence helps mitigate polarization in our model. We similarly denote, for ease of exposition, by $\mathbf{g}(\eta;\pi) = \{(g_n^L(\eta;\pi), g_n^R(\eta;\pi))\}_{n=1}^{\infty}$ the party-efficient equilibrium under Pareto weight η and persistence parameter $\pi = \pi_L = \pi_R$.

Proposition 4. *If Assumption 1 holds and $\pi < \pi'$, then $g_n^i(\eta;\pi') \geqslant g_n^i(\eta;\pi)$ for all $i \in \{L, R\}$ and $n \in \mathbf{N}^*$, with strict inequality for $n > 1$ if strict polarization occurs at π.*

Intuitively, increasing political persistence increases the expected time spent in each term and so is similar to decreasing the discount factor in its impact on players' utilities. These comparative statics speak to the debate on the effects of political persistence on the efficiency of policies. They differ from the comparative statics found in Acemoglu et al. (2011), who show that in a model with only inter-party interactions, the set of first-best achievable policies is decreasing in political persistence and increasing in the discount factor. An important difference in our setups is that radicals can only seize control of their party when their party is not in control, while when radicals are absent, a deviation by a party only happens when that party is in power. In our model, an increase in political persistence increases the expected length of time before radicals who have seized control can benefit from setting the policy, while when radicals are absent an increase in persistence lengthens the amount of time a party can enjoy setting the policy after deviating. Comparing the differences in our results highlights the role played by radicals and intra-party politics.

4 Extensions

In this section we extend the baseline model on two fronts.

4.1 One-party homogeneity

Having studied the problem with no radicals or radicals on both sides, it is natural to wonder what will happen when only one party has radical members—that is, one party is homogeneous. If party i is homogeneous, we assume a takeover by $-i$-radicals is credible (otherwise we are essentially back in the case where both parties are homogeneous). This means that the incentive constraints for the homogeneous party may now bind because they no longer face the threat of a radical takeover (i.e., $U_i > -\infty$). Including both types of radicals turns out to be substantial for generating dynamic polarization. We find that strict polarization *never* occurs when at least one party is homogeneous. In fact, if party L is homogeneous, then policies will settle at $t = 0$, while if party R is homogeneous, then policies will settle by τ_1^R.

Proposition 5. *If party L is homogeneous, then $g_n^i = g_{n+1}^i$ for all $i \in \{L, R\}$ and $n \in \mathbf{N}^*$. If party R is homogeneous, then $g_n^L = g_{n+1}^L$ for all $n \geqslant 2$ and $g_n^R = g_{n+1}^R$ for all $n \in \mathbf{N}^*$.*

This result emphasizes that both parties must face a threat from their radical wings for strict polarization to occur. The vicious cycle that leads to strict polarization does not arise when radicals on only one side need to be mollified. Our model then predicts that there will be relatively little dynamics to any polarization that occurs in countries where at least one side of the political divide has little disagreement on policy outcomes or is strong enough to resist any radicals.

4.2 Higher-dimensional policy space

In all real-world policy setting by parties, policies are multi-dimensional. A party must decide what position to take on many issues from taxation, military spending, immigration, cultural issues, etc. Our model so far has focused on a one-dimensional policy space for expositional simplicity, but under an appropriate extension of the definition of polarization we can show that our results generalize to richer policies spaces. To this end, we denote by $\mathcal{G}$ the policy space, which is assumed to be a hyperrectangle, namely,

$$\mathcal{G} = \prod_{j=1}^{k} [0, a_j]$$

for some fixed $(a_1, a_2, \cdots, a_k) \in \mathbf{R}_+^k$ and $k \geqslant 2$.

For a pair of arbitrary policies g, g' in $\mathcal{G}$, outside of the one-dimensional case it is not obvious how to judge whether one policy is more extreme than the other (e.g., how to compare $g = (1, 0)$ and $g' = (0, 1)$). We therefore use the definition of polarization in terms of the dynamics of radicals' utilities mentioned after Definition 2, which we will now call *polarization of radicals*. Moreover, we place some additional restrictions on the utility functions, precisely stated in Assumption 2 below, to capture the idea that the radicals have more extreme bliss points than the moderates.

Assumption 2. (Regularity) *The utility functions $\{u_i(\theta, g), v_i(\theta, g)\}_{i,\theta \in \{L,R\}}$ are regular if the following two conditions are satisfied for each $i, \theta \in \{L, R\}$:*

(i) *$u_i(\theta, g)$ and $v_i(\theta, g)$ are additively separable across all dimensions of g;*

(ii) *For each dimension, the moderates' bliss points always lie between that of the radicals'.*

Assumption 2 imposes strict antagonism dimension by dimension. However, strict antagonism may not hold overall when considering all policies together: it may be possible to make both radicals better off while making moderates worse off. Nevertheless, we find that Assumption 2 is sufficient for ensuring that polarization still occurs for the case with higher dimensional policy spaces.

Proposition 6. *With a policy space as specified above and regular utility functions, every party-efficient equilibrium features either immediate settlement or polarization of radicals, that is, for all*

$n \in \mathbf{N}^*$, $v_i(\theta, g^i_{n+1}) \geqslant v_i(\theta, g^i_n)$ when $\theta = i$ and $v_i(\theta, g^i_{n+1}) \leqslant v_i(\theta, g^i_n)$ when $\theta \neq i$. If $g^i_n = g^i_{n+1}$ for both $i \in \{L, H\}$ at some n, then the equilibrium has settled by τ^L_n.

It should be emphasized that Proposition 6 does not establish a strict polarization result parallel with that in Theorem 1. However, we can show such a result under an appropriate extension of strict antagonism to the multi-dimensional policy space. We say utilities feature strict antagonism if for any policies g, g', if g makes both radicals better off than g', then moderates are better off as well; that is, in the region of Pareto optimal policies, it is not possible to appease both sets of radicals simultaneously. This condition is satisfied for example when moderates' utilities are convex combinations of the two radicals' utilities. With this stronger assumption, we can extend Proposition 6 to show that, if the equilibrium does not settle by τ^R_1, then all inequalities are strict—that is, $v_i(\theta, g^i_{n+1}) > v_i(\theta, g^i_n)$ for $\theta = i$ and $v_i(\theta, g^i_{n+1}) < v_i(\theta, g^i_n)$ for $\theta \neq i$. The proof is an immediate corollary of Theorem 1 once we note that showing this polarization of radicals in the proof Theorem 1 relied only on strict antagonism and not the one-dimensional policy space.

The extension to a higher-dimensional policy space opens up the way for richer dynamics of policies. Illustrated in Figure 3 is an example with a two-dimensional policy space (the policy path in state R, while omitted, is similar). Although we eventually see polarization on both dimensions, polarization on the second dimension is delayed until the fourth term of party R. Polarization occurs first on the dimension that radicals care more about relative to moderates, as this dimension provides the cheaper way to increase radicals' continuation value. Only after sufficient polarization occurs on the cheaper dimension, increasing the marginal cost of decreases in this dimension, does polarization begin on the other dimension. This example shows how parties may be able to reach stable agreements on some dimensions of their policy choices for multiple terms, only to have polarization eventually spill over onto these dimension even without any changes in underlying preferences.

5 Concluding Remarks

This paper proposes a tractable framework to analyze how intra-party politics, in the form of heterogeneous preferences within parties, affect policy dynamics. When at least one party is homogeneous, policy settlement is always achieved even with the presence of inter-party politics. In contrast, with heterogeneity in both parties, intra-party politics can cause dynamic polarization, in which both parties continually choose more extreme policies over time, leading to ever increasing swings in radicals' stage payoffs across time. We have demonstrated that intra-party politics may serve as a reason for dynamic polarization and that polarization can be generated in a world with alternating political control but stationary beliefs and preferences. Our results thus speak to the growing debate on the causes and consequences of political polarization.

There are a number of potentially interesting directions to extend our framework. Incorporating electoral concerns, where policy choices impact the power transition process, and richer

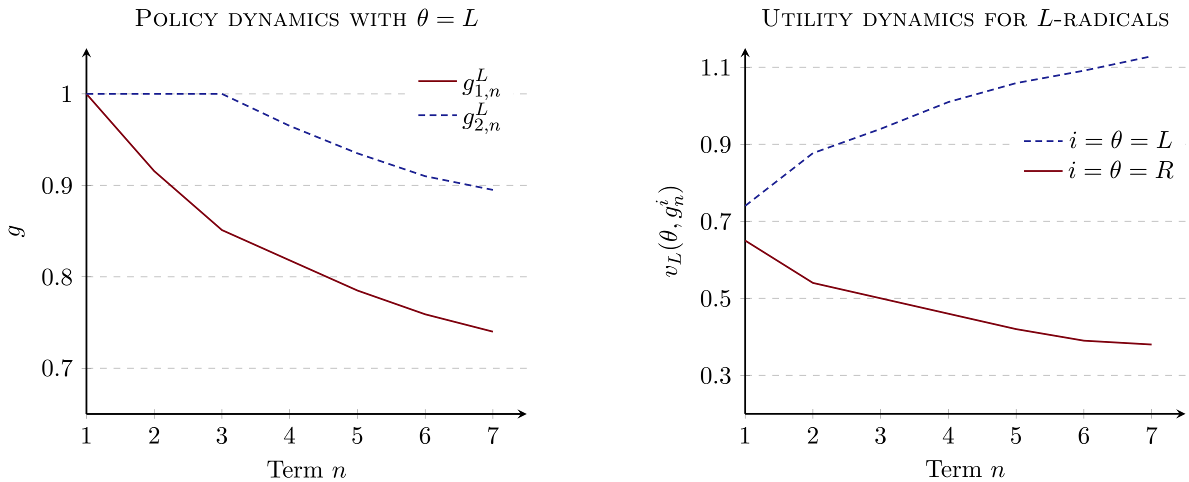

Figure 3: The above figure takes $\mathcal{G} = [0,1]^2$, $u_i(\theta, g) = \sum_{j=1}^{2}[\ln(g_j) + 2\mathbf{1}_{\{\theta=i\}}(1-g_j)]$ and $v_i(\theta, g) = \frac{1}{2}\sum_{j=1}^{2}[\frac{(g_j+1)^{1-\sigma}-1}{1-\sigma} + \beta_j \mathbf{1}_{\{\theta=i\}}(1-g_j)]$ with $\sigma = 0.7$, $\beta_1 = 3$, $\beta_2 = 2$, $\pi_L = \pi_R = 0.6$, $\delta = 0.8$, and $\eta = 0.5$. We write $g^i_n = (g^i_{1,n}, g^i_{2,n})$.

heterogeneity within parties (namely, more than one type of radical) are two natural directions. Our model has also assumed that whenever radicals' may credibly threaten to take over their party, such a takeover is sufficiently harmful for moderates so as to ensure they do not deviate. While it is technically challenging to allow for all radicals' participation constraints and moderates' incentive constraints simultaneously to bind, relaxing this assumption may yield new insights on the interactions between moderates and radicals within a party. We leave this for future research.

Appendices

For notational simplicity, in the Appendix we will use i (the index for a party) and θ (the index for a state) interchangeably. For example, we may write g_n^θ for the policy of the nth term of party i, where $\theta = i$.

The proofs will be presented in the order found in the text except for Proposition 2, which will be delayed until after the proof of Theorem 1 as it relies on many of the same results as in the proof of Theorem 1.

A Proof of Proposition 1

Suppose there is a party-efficient equilibrium $\widehat{g} : \mathbf{H} \to [0,1]$ that fails the property in Proposition 1. For simplicity, we write $\widehat{g}(h_t)$ as $\widehat{g}_t$ for each $t \in \mathbf{N}$. For each $i \in \{L, R\}$ and $n \in \mathbf{N}^*$, we define the *term-average* policy as

$$\bar{g}_n^i \equiv \frac{\mathbf{E}\left[\sum_{t \in T_n^i} \delta^t \widehat{g}_t\right]}{\mathbf{E}\left[\sum_{t \in T_n^i} \delta^t\right]},$$

where the expectation is taken over all possible histories. Let $\bar{g} : \mathbf{H} \to [0,1]$ be the corresponding strategy profile, that is, $\bar{g}(h_t) = \bar{g}_n^i$ if $t \in T_n^i(h_t)$. To reach a contradiction, we show that $\bar{g}$ is admissible and Pareto-dominates $\widehat{g}$.

To this end, note that i-moderates' expected utility under $\widehat{g}$ is

$$\mathbf{E}\left[\sum_{t=0}^{\infty} \delta^t u_i(\theta_t, \widehat{g}_t)\right] = \sum_{j \in \{L,R\}} \sum_{n=1}^{\infty} \mathbf{E}\left[\sum_{t \in T_n^j} \delta^t u_i(j, \widehat{g}_t)\right].$$

Every finite ruling history, due to the transiency of the underlying Markov chain, is reached with a strictly positive probability, and so we obtain from the strict concavity of u_i that

$$\begin{aligned}
\mathbf{E}\left[\sum_{t \in T_n^j} \delta^t u_i(j, \widehat{g}_t)\right] &= \mathbf{E}\left\{\left(\sum_{t \in T_n^j} \delta^t\right) \sum_{t \in T_n^j} \left[\frac{\delta^t}{\sum_{t \in T_n^j} \delta^t}\right] u_i(j, \widehat{g}_t)\right\} \\
&< \mathbf{E}\left[\sum_{t \in T_n^j} \delta^t\right] u_i\left(j, \frac{\mathbf{E}\left[\sum_{t \in T_n^i} \delta^t \widehat{g}_t\right]}{\mathbf{E}\left[\sum_{t \in T_n^i} \delta^t\right]}\right) \\
&= \mathbf{E}\left[\sum_{t \in T_n^j} \delta^t u_i(j, \bar{g}_n^j)\right]. \qquad \text{(A.1)}
\end{aligned}$$

As a result, $\bar{g}$ yields a strictly higher expected payoff to i-moderates. A similar argument shows that i-radicals will also strictly benefit from moving to $\bar{g}$. Therefore, $\bar{g}$ Pareto-dominates $\widehat{g}$.

It remains to be verified that, for each $i \in \{L, R\}$, $IC^i(h_s)$ and $PC^{-i}(h_s)$ are satisfied by $\bar{g}$ for every history $h_s \in \mathbf{H}$ with $\theta_s = i$. We let $i = L$ (the argument when $i = R$ is analogous). For this, note that for each $n \in \mathbf{N}^*$, using the fact that $\widehat{g}$ is incentive compatible, we have

$$\mathbf{E}\left[\sum_{t=\tau_L^n}^{\infty} \delta^{t-\tau_n^L} u_L(\theta_t, \widehat{g}_t) \,\middle|\, h_{\tau_n^L}\right] = \sum_{j\in\{L,R\}} \sum_{m=n}^{\infty} \mathbf{E}\left[\sum_{t\in T_m^j} \delta^{t-\tau_n^L} u_L(j, \widehat{g}_t) \,\middle|\, h_{\tau_n^L}\right] \geqslant U_L.$$

Taking expectation over all possible $h_{\tau_n^L}$ and employing the law of iterated expectations, one concludes that

$$\mathbf{E}\left[\sum_{t=\tau_L^n}^{\infty} \delta^{t-\tau_n^L} u_L(\theta_t, \widehat{g}_t)\right] = \sum_{j\in\{L,R\}} \sum_{m=n}^{\infty} \mathbf{E}\left[\sum_{t\in T_m^j} \delta^{t-\tau_n^L} u_L(j, \widehat{g}_t)\right] \geqslant U_L.$$

Using (A.1), we arrive at

$$\sum_{j\in\{L,R\}} \sum_{m=n}^{\infty} \mathbf{E}\left[\sum_{t\in T_m^j} \delta^{t-\tau_n^L} u_L(j, \bar{g}_m^j)\right] > U_L. \tag{A.2}$$

By definition, $\bar{g}$ is constant on T_m^j for every history that leads to T_m^j. Moreover, since the underlying state is Markovian, for each $m \geqslant n$, $\mathbf{E}\big[\sum_{t\in T_m^j} \delta^{t-\tau_n^L} \mid h_{\tau_n^L}\big]$ is constant over all possible $h_{\tau_n^L}$. Therefore, we can convert (A.2) into its conditional version without violating the inequality, that is,

$$\sum_{j\in\{L,R\}} \sum_{m=n}^{\infty} \mathbf{E}\left[\sum_{t\in T_m^j} \delta^{t-\tau_n^L} u_L(j, \bar{g}_m^j) \,\middle|\, h_{\tau_n^L}\right] > U_L. \tag{A.3}$$

Since the LHS of (A.3) is the continuation value from policy $\bar{g}$ to L-moderates after history $h_{\tau_n^L}$, $IC^L(h_{\tau_n^L})$ holds. Replacing u_L by v_R in our argument above justifies $PC^R(h_{\tau_n^L})$. For an arbitrary history h_t with $t \in T_k^j$ for some $j \in \{L, R\}$, and $k \in \mathbf{N}^*$, notice that the continuation value from $\bar{g}$ to either L-moderates or R-radicals is the same as that after $h_{\tau_k^j}$, the truncation of h_t up to period τ_k^j. Therefore, $IC^L(h_t)$ is satisfied by $\bar{g}$ if $IC^L(h_{\tau_k^j})$ is satisfied, and similarly for $PC^R(h_t)$. Now the proof is complete.

B Proof of Theorem 1

The proof will be built upon Lemma 1 - Lemma 7. Before moving further, it is useful to define and derive some expressions for radicals' continuation values. In particular, we denote by $V_t^i(\mathbf{g})$ the continuation value to i-radicals at (the beginning of) period t under a sequence of policies $\mathbf{g}$. With a term-stationary sequence of policies $\mathbf{g} = \{(g_n^L, g_n^R)\}_{n=1}^{\infty}$, we have, recursively, that for each $i \in \{L, R\}$,

$$V^i_{\tau^L_n}(\mathbf{g}) = v_i(L, g^L_n) + \delta \left[\pi_L V^i_{\tau^L_n}(\mathbf{g}) + (1-\pi_L)V^i_{\tau^R_n}(\mathbf{g})\right],$$
$$V^i_{\tau^R_n}(\mathbf{g}) = v_i(L, g^R_n) + \delta\left[\pi_R V^i_{\tau^R_n}(\mathbf{g}) + (1-\pi_R)V^i_{\tau^L_{n+1}}(\mathbf{g})\right].$$

Substituting in $V^i_{\tau^R_n}(\mathbf{g})$ and $V^i_{\tau^L_{n+1}}(\mathbf{g})$, we get

$$V^i_{\tau^L_n}(\mathbf{g}) = \frac{1}{1-\delta\pi_L}v_i(L, g^L_n) + \frac{\delta(1-\pi_L)}{1-\delta\pi_L}\left[\frac{1}{1-\delta\pi_R}v_i(R, g^R_n) + \frac{\delta(1-\pi_R)}{1-\delta\pi_R}V^i_{\tau^L_{n+1}}(\mathbf{g})\right],$$
$$V^i_{\tau^R_n}(\mathbf{g}) = \frac{1}{1-\delta\pi_R}v_i(R, g^R_n) + \frac{\delta(1-\pi_R)}{1-\delta\pi_R}\left[\frac{1}{1-\delta\pi_L}v_i(L, g^L_{n+1}) + \frac{\delta(1-\pi_L)}{1-\delta\pi_L}V^i_{\tau^R_{n+1}}(\mathbf{g})\right].$$

For each $\theta \in \{L, R\}$, let

$$A_\theta \equiv \frac{1}{1-\delta\pi_\theta}, \quad B_\theta \equiv \frac{\delta(1-\pi_\theta)}{(1-\delta\pi_L)(1-\delta\pi_R)}, \quad C \equiv \frac{\delta^2(1-\pi_L)(1-\pi_R)}{(1-\delta\pi_L)(1-\delta\pi_R)}. \tag{B.1}$$

Then we can write for each $i \in \{L, R\}$

$$\begin{aligned} V^i_{\tau^L_n}(\mathbf{g}) &= A_L v_i(L, g^L_n) + B_L v_i(R, g^R_n) + C V^i_{\tau^L_{n+1}}(\mathbf{g}) \\ &= \sum_{k=0}^{m-1} C^k[A_L v_i(L, g^L_{n+k}) + B_L v_i(R, g^R_{n+k})] + C^m V^i_{\tau^L_{n+m}}(\mathbf{g}) \end{aligned} \tag{B.2}$$

for an arbitrary $m \geqslant 1$. Since $C \in (0, 1)$, we can let $m \to \infty$ in (B.2) to obtain

$$V^i_{\tau^L_n}(\mathbf{g}) = \sum_{k=0}^{\infty} C^k[A_L v_i(L, g^L_{n+k}) + B_L v_i(R, g^R_{n+k})]. \tag{B.3}$$

Similarly, for every $m \geqslant 1$ we also have

$$\begin{aligned} V^i_{\tau^R_n}(\mathbf{g}) &= \sum_{k=0}^{m-1} C^k[A_R v_i(R, g^R_{n+k}) + B_R v_i(L, g^L_{n+k+1})] + C^m V^i_{\tau^R_{n+m}}(\mathbf{g}) \\ &= \sum_{k=0}^{\infty} C^k[A_R v_i(R, g^R_{n+k}) + B_R v_i(L, g^L_{n+k+1})]. \end{aligned} \tag{B.4}$$

Our first lemma studies, in a party-efficient equilibrium, the monotonicity of radicals' stage payoffs when certain participation constraints are slack. In words, it roughly says that a slack participation constraint for a type of radicals in a term will lead to a (weak) decrease in their stage payoff as we move to the next term of the same ruling party. The statement of the result differs slightly when the participation constraint for R-radicals and L-radicals is slack due to the notation we use when defining terms.

Lemma 1. *Let* $\mathbf{g} = \{(g_n^L, g_n^R)\}_{n=1}^{\infty}$ *be a party-efficient equilibrium.*

(i) *For each* $n \in \mathbf{N}^*$, *if* PC_{n+1}^R *is slack, then* $v_R(\theta, g_n^\theta) \geqslant v_R(\theta, g_{n+1}^\theta)$ *and* $v_L(\theta, g_n^\theta) \leqslant v_L(\theta, g_{n+1}^\theta)$ *for each* $\theta \in \{L, R\}$.

(ii) *For each* $n \in \mathbf{N}^*$, *if* PC_n^L *is slack, then* $v_L(L, g_n^L) \geqslant v_L(L, g_{n+1}^L)$ *and* $v_R(L, g_n^L) \leqslant v_R(L, g_{n+1}^L)$; *if* PC_{n+1}^L *is slack, then* $v_L(R, g_n^R) \geqslant v_L(R, g_{n+1}^R)$ *and* $v_R(R, g_n^R) \leqslant v_R(R, g_{n+1}^R)$.

Proof. Assume $\mathbf{g}$ solves the sequence problem (4) for some Pareto weight η, where, by our assumption that $U_i = -\infty$, we can safely drop the incentive constraints for moderates. The corresponding Lagrangian is

$$\mathcal{L} = \sum_{n=1}^{\infty} \sum_{\theta \in \{L,R\}} \mathbf{E}\Big[\sum_{t \in T_n^\theta} \delta^t\Big] u_\eta(\theta, \hat{g}_n^\theta) + \sum_{n=1}^{\infty} \lambda_n^R \Bigg\{ \Bigg[\sum_{m \geqslant n}^{\infty} \sum_{\theta \in \{L,R\}} \mathbf{E}\Bigg(\sum_{t \in T_m^\theta} \delta^t \Bigg) v_R(\theta, \hat{g}_m^\theta) \Bigg] - \mathbf{E}\Big[\delta^{\tau_n^L}\Big] V_R \Bigg\} + \sum_{n=1}^{\infty} \lambda_n^L \Bigg\{ \Bigg[\sum_{m \geqslant n}^{\infty} \sum_{\theta \in \{L,R\}} \mathbf{E}\Bigg(\sum_{t \in T_m^\theta} \delta^t \Bigg) v_L(\theta, \hat{g}_m^\theta) \mathbf{1}_{\{m > n \text{ or } \theta = R\}} \Bigg] - \mathbf{E}\Big[\delta^{\tau_n^R}\Big] V_L \Bigg\},$$

where $\lambda_n^i \geqslant 0$ is the multiplier associated with PC_n^i for each $i \in \{L, R\}$ and $n \in \mathbf{N}^*$. Rearranging according to the index n, we can rewrite the Lagrangian as

$$\mathcal{L} = \sum_{n=1}^{\infty} \sum_{\theta \in \{L,R\}} \Bigg\{ \mathbf{E}\Big[\sum_{t \in T_n^\theta} \delta^t\Big] \Bigg[u_\eta(\theta, \hat{g}_n^\theta) + \Bigg(\sum_{\ell \leqslant n} \lambda_\ell^R \Bigg) v_R(\theta, \hat{g}_n^\theta) + \Bigg(\sum_{\ell \leqslant n} \lambda_\ell^L \mathbf{1}_{\{\ell < n \text{ or } \theta = R\}} \Bigg) v_L(\theta, \hat{g}_n^\theta) \Bigg] - \lambda_n^R \mathbf{E}\Big[\delta^{\tau_n^L}\Big] V_R - \lambda_n^L \mathbf{E}\Big[\delta^{\tau_n^R}\Big] V_L \Bigg\}. \tag{B.5}$$

For notational simplicity, we hereafter write $\sum_{\ell \leqslant n} \lambda_\ell^\theta$ as Λ_n^θ for each $\theta \in \{L, R\}$ and $n \in \mathbf{N}$ (with the interpretation that $\lambda_0^\theta = 0$). Since all of the multipliers are nonnegative, Λ_n^θ is weakly increasing in n.

Since we have assumed there is an interior admissible sequence of policies, using (B.5), we have that[10]

$$g_n^L = \arg\max_{g \in [0,1]} \Big\{ u_\eta(L, g) + \Lambda_n^R v_R(L, g) + \Lambda_{n-1}^L v_L(L, g) \Big\}, \tag{B.6}$$

$$g_n^R = \arg\max_{g \in [0,1]} \Big\{ u_\eta(R, g) + \Lambda_n^R v_R(R, g) + \Lambda_n^L v_L(R, g) \Big\}. \tag{B.7}$$

Consider the solutions to (B.6) at n and $n + 1$, respectively. Since $\mathbf{g}$ solves our problem, one

[10]See, among others, Luenberger (1997), Chapter 8, Theorem 1, which says that, given a Slater condition, an optimal solution must maximize the Lagrangian, admit nonnegative multipliers, and satisfy complementary slackness.

concludes that

$$u_\eta(L, g_n^L) + \Lambda_n^R v_R(L, g_n^L) + \Lambda_{n-1}^L v_L(L, g_n^L) \geqslant u_\eta(L, g_{n+1}^L) + \Lambda_n^R v_R(L, g_{n+1}^L) + \Lambda_{n-1}^L v_L(L, g_{n+1}^L),$$
$$u_\eta(L, g_{n+1}^L) + \Lambda_{n+1}^R v_R(L, g_{n+1}^L) + \Lambda_n^L v_L(L, g_{n+1}^L) \geqslant u_\eta(L, g_n^L) + \Lambda_{n+1}^R v_R(L, g_n^L) + \Lambda_n^L v_L(L, g_n^L).$$

If PC_{n+1}^R is slack, then $\lambda_{n+1}^R = 0$ by complementary slackness, and so $\Lambda_n^R = \Lambda_{n+1}^R$. Adding the two inequalities and simplifying, we obtain

$$\left(\Lambda_{n-1}^L - \Lambda_n^L\right)\left[v_L(L, g_n^L) - v_L(L, g_{n+1}^L)\right] \geqslant 0.$$

We now argue that $v_L(L, g_n^L) \leqslant v_L(L, g_{n+1}^L)$. If $\Lambda_{n-1}^L \neq \Lambda_n^L$, then we must have, by monotonicity, $\Lambda_{n-1}^L < \Lambda_n^L$, and one can conclude that $v_L(L, g_n^L) \leqslant v_L(L, g_{n+1}^L)$. If, however, $\Lambda_{n-1}^L = \Lambda_n^L$, then it is clear that the optimization problem at τ_n^R is that same as that at τ_{n+1}^R, which, due to the uniqueness of the solutions to (B.6) for n and $n+1$, respectively, implies that $g_n^L = g_{n+1}^L$, and so $v_L(L, g_n^L) = v_L(L, g_{n+1}^L)$. The fact that $v_R(L, g_n^L) \geqslant v_R(L, g_{n+1}^L)$ then follows from strict antagonism. With a similar argument with respect to (B.7), one can show that $v_L(R, g_n^R) \leqslant v_L(R, g_{n+1}^R)$ and $v_R(R, g_n^R) \geqslant v_R(R, g_{n+1}^R)$. This completes the proof for Lemma 1 (i). Lemma 1 (ii) can be argued in an analogous way. ■

Lemma 1 will play an important role in identifying possible patterns of policy dynamics in party-efficient equilibria. Building upon it, we now argue that if a participation constraint for some type of radicals binds, then all future participation constraints for that radical also bind.

Lemma 2. *For each party-efficient equilibrium sequence of policies $\mathbf{g} = \{(g_n^L, g_n^R)\}_{n=1}^\infty$, if PC_n^i is binding, then PC_m^i must be binding for all $m \geqslant n$.*

Proof. We first let $i = R$. Suppose PC_n^R is binding but PC_{n+1}^R is slack for some $n \in \mathbf{N}^*$. Then it follows that

$$V_R = V_{\tau_n^L}^R(\mathbf{g}) = A_L v_R(L, g_n^L) + B_L v_R(R, g_n^R) + C V_{\tau_{n+1}^L}^R(\mathbf{g}). \tag{B.8}$$

Let $m = \inf\{k > n+1 \mid PC_k^R \text{ is binding}\}$. There are two cases to consider.

- **Case 1:** $m < \infty$. By definition, we know that PC_k^R is slack for all $n+1 \leqslant k < m$. Lemma 1 (i) then implies that $v_R(\theta, g_n^\theta) \geqslant v_R(\theta, g_{n+1}^\theta) \geqslant \cdots \geqslant v_R(\theta, g_{m-1}^\theta)$ for each $\theta \in \{L, R\}$. Since PC_{m-1}^R and PC_{n+1}^R are slack while PC_m^R binds, we have $V_{\tau_{m-1}^L}^R > V_R$ and $V_{\tau_{n+1}^L}^R > V_R = V_{\tau_m^L}^R$. One thus concludes

$$\begin{aligned} V_R < V_{\tau_{m-1}^L}^R(\mathbf{g}) &= A_L v_R(L, g_{m-1}^L) + B_L v_R(R, g_{m-1}^R) + C V_{\tau_m^L}^R(\mathbf{g}) \\ &\leqslant A_L v_R(L, g_n^L) + B_L v_R(R, g_n^R) + C V_{\tau_{n+1}^L}^R(\mathbf{g}) \\ &= V_{\tau_n^1}^R(\mathbf{g}) = V_R, \end{aligned}$$

  a contradiction.

- **Case 2:** $m = \infty$. Because PC_k^R is slack for $k \geqslant n+1$, we have $v_R(\theta, g_k^\theta) \geqslant v_R(\theta, g_{k+1}^\theta)$ for each $k \geqslant n$, and so, using (B.3) we have

$$\begin{aligned}
V_R = V_{\tau_n^L}^R(\mathbf{g}) &= \sum_{k=0}^{\infty} C^k [A_L v_L(L, g_{n+k}^L) + B_L v_R(R, g_{n+k}^R)] \\
&\geqslant \sum_{k=0}^{\infty} C^k [A_L v_L(L, g_{n+k+1}^L) + B_L v_R(R, g_{n+k+1}^R)] \\
&= V_{\tau_{n+1}^L}^R(\mathbf{g}) > V_R,
\end{aligned}$$

  a contradiction.

The case with $i = L$ can be shown analogously. The proof is now complete. ■

The next lemma shows that for a sequence of policies to solve (4), we cannot have both participation constraints slack during the initial term (i.e., T_1^L and T_1^R) only to start binding later. Formally, we show that either one participation constraint is binding at some τ_1^θ or the parties' first-best outcome is achieved, namely, g_n^θ will be equal to g_η^θ for each term, where for each $\theta \in \{L, R\}$

$$g_\eta^\theta \equiv \arg\max_{g \in [0,1]} u_\eta(\theta, g).$$

The sequence of policies that achieves the parties' first-best outcome is denoted by $\mathbf{g}_\eta^*$, that is, for each $\theta \in \{L, R\}$ and $n \in \mathbf{N}^*$ it prescribes policy g_η^θ at all $t \in T_n^\theta$.

Lemma 3. *If a party-efficient equilibrium $\mathbf{g} = \{(g_n^L, g_n^R)\}_{n=1}^\infty$ does not achieve the parties' first-best outcome, then PC_1^i is binding for some $i \in \{L, R\}$.*

Proof. It is clear that the parties' first-best outcome will be achieved if none of the participation constraints are binding. We therefore assume that PC_n^i is binding for some $i \in \{L, R\}$ and $n \in \mathbf{N}^*$, and let $n^* = \inf\{k \geqslant 1 \,|\, PC_k^i \text{ is binding for some } i \in \{L, R\}\}$. We argue that $n^* = 1$. To this end, first suppose that $n^* \geqslant 2$. We discuss the two possible cases.

- **Case 1:** $PC_{n^*}^R$ is binding. Because PC_k^L and PC_k^R are slack for $k = 1, 2, \ldots, n^* - 1$, it is easy to see that $g_k^\theta = g_\eta^\theta$ for such k. To reach a contradiction, we argue that the parties' first-best outcome $\mathbf{g}_\eta^*$ can be achieved in this case.

  First, we prove that $\mathbf{g}_\eta^*$ fulfills PC_n^R for all $n \in \mathbf{N}^*$. Since $\mathbf{g}_\eta^*$ is Markovian in θ_t and the state process $\{\theta_t\}_{t=1}^\infty$ is Markovian, R-radicals' continuation value under $\mathbf{g}_\eta^*$ is also Markovian in θ_t. So it suffices to show that $\mathbf{g}_\eta^*$ satisfies PC_1^R. If $\mathbf{g}_\eta^*$ is implemented, R-radicals' continuation payoff as evaluated at τ_1^L, $V_{\tau_1^L}^R(\mathbf{g}_\eta^*)$, satisfies (by (B.3))

$$V_{\tau_1^L}^R(\mathbf{g}_\eta^*) = A_L v_R(L, g_\eta^L) + B_L v_R(R, g_\eta^R) + C V_{\tau_2^L}^R(\mathbf{g}_\eta^*),$$

which, because $V^R_{\tau^L_1}(\mathbf{g}^*_\eta) = V^R_{\tau^L_2}(\mathbf{g}^*_\eta)$, yields

$$V^R_{\tau^L_1}(\mathbf{g}^*_\eta) = \frac{1}{1-C}\left[A_L v_R(L, g^L_\eta) + B_L v_R(R, g^R_\eta)\right]. \tag{B.9}$$

However, under $\mathbf{g}$, R-radicals' continuation value at $\tau^L_{n^*-1}$, because $PC^R_{n^*-1}$ is slack and $PC^R_{n^*}$ is binding, satisfies

$$V^R_{\tau^L_{n^*-1}}(\mathbf{g}) = A_L v_R(L, g^L_\eta) + B_L v_R(R, g^R_\eta) + CV_R > V_R.$$

Simplifying, we obtain

$$V_R < \frac{1}{1-C}\left[A_L v_R(L, g^L_\eta) + B_L v_R(R, g^R_\eta)\right]. \tag{B.10}$$

Comparing (B.9) and (B.10), it follows immediately that $V^R_{\tau^L_1}(\mathbf{g}^*_\eta) > V_R$, as was to be shown.

We next argue that $\mathbf{g}^*_\eta$ satisfies PC^L_n for all $n \in \mathbf{N}^*$. Again, it suffices to show that $V^L_{\tau^R_1}(\mathbf{g}^*_\eta) \geqslant V_L$. Let $m^* = \inf\{k \geqslant n^* \mid PC^L_k \text{ is binding}\}$.

If $m^* = \infty$, then PC^L_k is slack for every $k \in \mathbf{N}^*$, and we can use Lemma 1 (ii) to conclude that $v_L(\theta, g^\theta_\eta) = v_L(\theta, g^\theta_1) \geqslant v_L(\theta, g^\theta_k) \geqslant v_L(\theta, g^\theta_{k+1})$ for all $k \in \mathbf{N}^*$ and $\theta \in \{L, R\}$. Therefore, by (B.4), we have

$$\begin{aligned} V_L < V^L_{\tau^R_1}(\mathbf{g}) &= \sum_{k=0}^{\infty} C^k \left[A_R v_L(R, g^R_{1+k}) + B_R v_L(L, g^L_{2+k})\right] \\ &\leqslant \sum_{k=0}^{\infty} C^k \left[A_R v_L(R, g^R_\eta) + B_R v_L(L, g^L_\eta)\right] \\ &= V^L_{\tau^R_1}(\mathbf{g}^*_\eta), \end{aligned}$$

which shows that $\mathbf{g}^*_\eta$ satisfies all participation constraints.

If, instead, $m^* < \infty$, then using the fact that $PC^L_1, PC^L_2, \ldots, PC^L_{m^*-1}$ are slack and Lemma 1 (i), we have $v_L(L, g^L_1) \geqslant v_L(L, g^L_2) \geqslant \cdots \geqslant v_L(L, g^L_{m^*})$ and $v_L(R, g^R_1) \geqslant v_L(R, g^R_2) \geqslant \cdots \geqslant v_L(R, g^R_{m^*-1})$. Also, since $g^\theta_1 = g^\theta_\eta$ for $\theta \in \{L, R\}$, we have

$$\begin{aligned} V_L < V^L_{\tau^R_{m^*-1}}(\mathbf{g}) &= A_R v_L(R, g^R_{m^*-1}) + B_R v_L(L, g^L_{m^*}) + CV_L \\ &\leqslant A_R v_L(R, g^R_\eta) + B_R v_L(L, g^L_\eta) + CV_L, \end{aligned}$$

which implies

$$V_L < \frac{1}{1-C}\left[A_R v_L(R, g^R_\eta) + B_R v_L(L, g^L_\eta)\right]. \tag{B.11}$$

It is easy to see that the RHS of (B.11) is exactly $V^L_{\tau^R_1}(\mathbf{g}^*_\eta)$, and so $\mathbf{g}^*_\eta$ fulfills all participation constraints for L-radicals as well. Hence, $\mathbf{g}^*_\eta$ is admissible and yields higher party utility than $\mathbf{g}$, a contradiction.

- **Case 2:** $PC^L_{n^*}$ is binding and $PC^R_{n^*}$ is slack. The argument is analogous to that for case 1.

Now our proof is complete. ■

According to Lemma 3, if the parties' first-best outcome is not achievable, then either PC^L_1 or PC^R_1 is binding. The following two lemmas derive some further implications of this fact.

Lemma 4. *Let* $\mathbf{g} = \{(g^L_n, g^R_n)\}_{n=1}^{\infty}$ *be a party-efficient equilibrium. If* PC^R_1 *is binding, then either* PC^L_1 *is binding or* PC^L_n *is slack for all* $n \in \mathbf{N}^*$.

Proof. Suppose PC^R_1 is binding while PC^L_1 is slack. We will show that PC^L_n must be slack for all $n \in \mathbf{N}^*$. To this end, we let $n^* = \inf\{k \in \mathbf{N}^* \,|\, PC^L_k \text{ is binding}\}$ and suppose $n^* < \infty$, that is, the L-radicals' participation constraint is not binding initially but becomes binding in some later term.

We first argue that $g^\theta_n = g^\theta_1$ for all $n \leqslant n^* - 1$ and $\theta \in \{L, R\}$. This follows trivially if $n^* = 2$ and so we assume $n^* \geqslant 3$ in below. To reach a contradiction, suppose there is some $m \leqslant n^* - 1$ and $\theta' \in \{L, R\}$ such that $g^{\theta'}_m \neq g^{\theta'}_1$. We argue that this implies $v_R(\theta', g^{\theta'}_1) < v_R(\theta', g^{\theta'}_m)$. Since PC^L_k is slack for all $k \leqslant n^* - 1$, Lemma 1 (ii) implies that $v_R(\theta', g^{\theta'}_1) \leqslant v_R(\theta', g^{\theta'}_m)$. To justify the strict inequality, we first suppose that $v_R(\theta', g^{\theta'}_1) = v_R(\theta', g^{\theta'}_m)$. For an arbitrary $\alpha \in (0, 1)$, define $g^{\theta'}_\alpha = \alpha g^{\theta'}_1 + (1 - \alpha) g^{\theta'}_m$. By strict concavity, one must have

$$u_\eta(\theta', g^{\theta'}_\alpha) > \alpha u_\eta(\theta', g^{\theta'}_1) + (1 - \alpha) u_\eta(\theta', g^{\theta'}_m),$$
$$v_R(\theta', g^{\theta'}_\alpha) > \alpha v_R(\theta', g^{\theta'}_1) + (1 - \alpha) v_R(\theta', g^{\theta'}_m).$$

Let $u_\eta(\theta', g^{\theta'}_s) = \min\{u_\eta(\theta', g^{\theta'}_1), u_\eta(\theta', g^{\theta'}_m)\}$, where $s \in \{1, m\}$. Then it is clear that

$$u_\eta(\theta', g^{\theta'}_\alpha) + \Lambda^R_s v_R(\theta', g^{\theta'}_\alpha) > u_\eta(\theta', g^{\theta'}_s) + \Lambda^R_s v_R(\theta', g^{\theta'}_s),$$

which contradicts the fact that

$$g^{\theta'}_s = \arg\max_{g \in [0,1]} \left\{ u_\eta(\theta', g) + \Lambda^R_s v_R(\theta', g) \right\}$$

(note that this is essentially (B.6) or (B.7): since PC^L_k is slack for all $k \leqslant m$, we have $\Lambda^L_k = 0$ for all $k = 0, 1, \ldots, m$). Thus, we must have $v_R(\theta', g^{\theta'}_1) < v_R(\theta', g^{\theta'}_m)$.

By Lemma 2, PC^R_n is binding for all $n \in \mathbf{N}^*$, and so we can employ the fact that $v_R(\theta, g^\theta_1) \leqslant v_R(\theta, g^\theta_m)$ (by Lemma 1 (ii)) and $v_R(\theta', g^{\theta'}_1) < v_R(\theta', g^{\theta'}_m)$ to reach

$$\begin{aligned} V_R = V^R_{\tau^L_m}(\mathbf{g}) &= A_L v_R(L, g^L_m) + B_L v_R(R, g^R_m) + C V_R \\ &> A_L v_R(L, g^L_1) + B_L v_R(R, g^R_1) + C V_R = V^R_{\tau^L_1}(\mathbf{g}) = V_R, \end{aligned}$$

a contradiction. Therefore, we must have $g_n^\theta = g_1^\theta$ for each $n \leqslant n^* - 1$.

Next, we argue that the policy sequence $\widehat{\mathbf{g}} = \{(\widehat{g}_n^L, \widehat{g}_n^R)\}_{n=1}^\infty$ with $\widehat{g}_n^\theta = g_1^\theta$ for each $n \in \mathbf{N}^*$ and $\theta \in \{L, R\}$ is admissible and yields a higher party utility than $\mathbf{g}$, a contradiction. To this end, we first show that $\widehat{\mathbf{g}}$ is admissible. Indeed, since $g_n^\theta = g_1^\theta$ for each $n \leqslant n^* - 1$ and all PC_n^R's are binding, the continuation value of R-radicals at τ_1^L, $V_{\tau_1^L}^R(\mathbf{g})$, satisfies

$$V_R = V_{\tau_1^L}^R(\mathbf{g}) = A_L v_R(L, g_1^L) + B_L v_R(R, g_1^R) + CV_R,$$

which implies that

$$V_R = \frac{1}{1-C}\left[A_L v_R(L, g_1^L) + B_L v_R(R, g_1^R)\right] = V_{\tau_n^L}^R(\widehat{\mathbf{g}}) \text{ for each } n \in \mathbf{N}^*. \tag{B.12}$$

Hence, $\widehat{\mathbf{g}}$ fulfills PC_n^R for all $n \in \mathbf{N}^*$.

Now we consider the participation constraints of L-radicals. Since $PC_{n^*-1}^L$ is slack, our argument above and Lemma 1 (ii) imply that $v_L(R, g_{n^*-1}^R) = v_L(R, g_1^R)$ and $v_L(L, g_{n^*}^L) \leqslant v_L(L, g_{n^*-1}^L) = v_L(L, g_1^L)$. Thus, the continuation value of L-radicals at $\tau_{n^*-1}^R$, $V_{\tau_{n^*-1}^R}^L(\mathbf{g})$, satisfies

$$V_L < V_{\tau_{n^*-1}^R}^L(\mathbf{g}) = A_R v_L(R, g_{n^*-1}^R) + B_R v_L(L, g_{n^*}^L) + CV_L \leqslant A_R v_L(R, g_1^R) + B_R v_L(L, g_1^L) + CV_L.$$

Rearranging, we arrive at

$$V_L < \frac{1}{1-C}\left[A_R v_L(R, g_1^R) + B_R v_L(L, g_1^L)\right] = V_{\tau_n^R}^L(\widehat{\mathbf{g}}) \text{ for all } n \in \mathbf{N}^*,$$

which shows that $\widehat{\mathbf{g}}$ satisfies PC_n^L for all $n \in \mathbf{N}^*$.

We now prove that $\widehat{\mathbf{g}}$ yields a (weakly) higher value to the parties than $\mathbf{g}$. To see this, we denote by $U_{\tau_n^\theta}^\eta(\cdot)$ the parties' continuation value at τ_n^θ under a sequence of policies and Pareto weight η. Then, using our argument above, we have

$$U_{\tau_1^L}^\eta(\mathbf{g}) = \sum_{n=1}^{n^*-1} \sum_{\theta \in \{L,R\}} \mathbf{E}\Bigg[\sum_{t \in T_n^\theta} \delta^t\Bigg] u_\eta(\theta, g_1^\theta) + \mathbf{E}\left[\delta^{\tau_{n^*}^L}\right] U_{\tau_{n^*}^L}^\eta(\mathbf{g}).$$

Since the parties could have chosen to start from using the continuation policies starting from $\tau_{n^*}^L$,[11] it must be that $U_{\tau_1^L}^\eta(\mathbf{g}) \geqslant U_{\tau_{n^*}^L}^\eta(\mathbf{g})$, which implies that

$$U_{\tau_1^L}^\eta(\mathbf{g}) \leqslant \frac{\sum_{n=1}^{n^*-1} \sum_{\theta \in \{L,R\}} \mathbf{E}[\sum_{t \in T_n^\theta} \delta^t] u_\eta(\theta, g_1^\theta)}{1 - \mathbf{E}\left[\delta^{\tau_{n^*}^L}\right]}.$$

[11]That is, the parties can employ the sequence of policies $\widetilde{\mathbf{g}} = \{(\widetilde{g}_n^L, \widetilde{g}_n^R)\}_{n=1}^\infty$ such that $\widetilde{g}_n^\theta = g_{n^*+n-1}^\theta$ for each $\theta \in \{L, R\}$ and $n \in \mathbf{N}^*$.

Using the stationarity of $\widehat{\mathbf{g}}$, we have

$$\begin{aligned} U^{\eta}_{\tau_1^L}(\widehat{\mathbf{g}}) &= \sum_{n=1}^{n^*-1}\sum_{\theta\in\{L,R\}}\mathbf{E}\Big[\sum_{t\in T_n^\theta}\delta^t\Big]u_\eta(\theta,g_1^\theta)+\mathbf{E}\left[\delta^{\tau^L_{n^*}}\right]U^{\eta}_{\tau^L_{n^*}}(\widehat{\mathbf{g}}) \\ &= \sum_{n=1}^{n^*-1}\sum_{\theta\in\{L,R\}}\mathbf{E}\Big[\sum_{t\in T_n^\theta}\delta^t\Big]u_\eta(\theta,g_1^\theta)+\mathbf{E}\left[\delta^{\tau^L_{n^*}}\right]U^{\eta}_{\tau_1^L}(\widehat{\mathbf{g}}), \end{aligned}$$

and so

$$U^{\eta}_{\tau_1^L}(\widehat{\mathbf{g}}) = \frac{\sum_{n=1}^{n^*-1}\sum_{\theta\in\{L,R\}}\mathbf{E}[\sum_{t\in T_n^\theta}\delta^t]u_\eta(\theta,g_1^\theta)}{1-\mathbf{E}\left[\delta^{\tau^L_{n^*}}\right]}.$$

We therefore conclude that $U_{\tau_1^L}(\widehat{\mathbf{g}}) \geqslant U_{\tau_1^L}(\mathbf{g})$, the afore mentioned contradiction being reached. The desired result thus follows. ■

Lemma 5. *Let $\mathbf{g}=\{(g_n^L,g_n^R)\}_{n=1}^\infty$ be a party-efficient equilibrium. If PC_1^R is slack while PC_1^L is binding, then either PC_2^R is binding or PC_n^R is slack for all $n\geqslant 2$.*

Proof. Assume that PC_1^L is binding while PC_2^R is slack. We will show that PC_n^R must be slack for all $n\geqslant 2$. To that end, let $n^*=\inf\{k\geqslant 3\,|\,PC_k^R \text{ is binding}\}$. For the sake of contradiction, suppose that n^* is finite. Notice that we must have $g_1^L=g_\eta^L$ since the participation constraint PC_1^R is slack. Because PC_2^R is slack, it follows from Lemma 1 (i) that $v_R(L,g_2^L)\leqslant v_R(L,g_\eta^L)$. By analogous arguments in the proof for Lemma 4, we know that

$$\begin{cases} g_n^L=g_2^L, & n=2,3,\dots,n^*-1 \\ g_n^R=g_1^R, & n=1,2,\dots,n^*-1 \end{cases}$$

and a sequence of policies that implements g_2^L in T_n^L for all $n\geqslant 2$ and g_1^R in T_n^R for all $n\in\mathbf{N}^*$ will fulfill PC_n^L for each $n\in\mathbf{N}^*$ and PC_n^R for each $n\geqslant 2$. Let $\widehat{\mathbf{g}}=\{(\widehat{g}_n^L,\widehat{g}_n^R)\}_{n=1}^\infty$ be such a sequence of policies with $\widehat{g}_1^L=g_\eta^L$. To establish its admissibility, it therefore suffices to show that $\widehat{\mathbf{g}}$ satisfies PC_1^R. To see this, notice that we can use (B.2) to write (employing the supposition that $PC_{n^*}^R$ is binding)

$$V^R_{\tau_1^L}(\mathbf{g}) = \left[A_Lv_R(L,g_\eta^L)+B_Lv_R(R,g_1^R)\right]+\sum_{n=1}^{n^*-2}C^n\left[A_Lv_R(L,g_{n+1}^L)+B_Lv_R(R,g_{n+1}^R)\right]+C^{n^*-1}V_R,$$

$$V^R_{\tau_1^L}(\widehat{\mathbf{g}}) = [A_Lv_R(L,g_\eta^L)+B_Lv_R(R,g_1^R)]+\sum_{n=1}^{n^*-2}C^n\left[A_Lv_R(L,g_2^L)+B_Lv_R(R,g_1^R)\right]+C^{n^*-1}V^R_{\tau^L_{n^*}}(\widehat{\mathbf{g}}).$$

We know $V^R_{\tau^L_{n^*}}(\widehat{\mathbf{g}})\geqslant V_R$; the fact that PC_n^R is slack for $n=1,2,\dots,n^*-1$ implies, via Lemma 1 (i), that $v_R(L,g_2^\theta)\geqslant v_R(L,g_n^\theta)$ and $v_R(R,g_1^R)\geqslant v_R(R,g_n^R)$ for each $n=2,\cdots,n^*-1$. We hence have $V^R_{\tau_1^L}(\widehat{\mathbf{g}})\geqslant V^R_{\tau_1^L}(\mathbf{g})$, which implies that $\widehat{\mathbf{g}}$ is admissible.

We next argue that $\widehat{\mathbf{g}}$ yields a (weakly) higher payoff to the parties than $\mathbf{g}$. We first show that $U^{\eta}_{\tau^R_{n^*-1}}(\mathbf{g}) \leqslant U^{\eta}_{\tau^R_1}(\mathbf{g})$. To do so, we establish the admissibility of a sequence of policies $\widetilde{\mathbf{g}} = \{(\widetilde{g}^L_n, \widetilde{g}^R_n)\}_{n=1}^{\infty}$, defined as

$$\widetilde{g}^{\theta}_n = \begin{cases} g^L_{\eta}, & n = 1,\, \theta = L \\ g^L_{n^*+n-2}, & n \geqslant 2,\, \theta = L \\ g^R_{n^*+n-2}, & n \in \mathbf{N}^*,\, \theta = R \end{cases}.$$

Notice that $\widetilde{\mathbf{g}}$ is obtained from $\mathbf{g}$ by "appending" the policies of $\mathbf{g}$ starting from $\tau^R_{n^*-1}$ to τ^R_1 while maintaining the policy for T^L_1. If $U^{\eta}_{\tau^R_{n^*-1}}(\mathbf{g}) > U^{\eta}_{\tau^R_1}(\mathbf{g})$ and $\widetilde{\mathbf{g}}$ was admissible, then because $U^{\eta}_{\tau^R_{n^*-1}}(\mathbf{g}) = U^{\eta}_{\tau^R_1}(\widetilde{\mathbf{g}})$, the parties would be better off using $\widetilde{\mathbf{g}}$. Clearly, since $\mathbf{g}$ is admissible, we only need to show that PC^R_1 is satisfied by $\widetilde{\mathbf{g}}$. To see this, note that

$$V^R_{\tau^L_1}(\widetilde{\mathbf{g}}) = A_L v_R(L, g^L_{\eta}) + B_L v_R(R, g^R_{n^*-1}) + CV_R,$$

where we have used the fact that $PC^R_{n^*}$ is binding (now this continuation value is pushed forward to τ^L_2). In the meantime, we also conclude that

$$\begin{aligned} V_R < V^R_{\tau^L_{n^*-1}}(\mathbf{g}) &= A_L v_R(L, g^L_{n^*-1}) + B_L v_R(R, g^R_{n^*-1}) + CV_R \\ &\leqslant A_L v_R(L, g^L_{\eta}) + B_L v_R(R, g^R_{n^*-1}) + CV_R, \end{aligned}$$

where the first line employs the supposition that $PC^R_{n^*}$ is binding, and the second line uses that PC^R_n is slack for each $n = 2, 3, \ldots, n^* - 1$ and Lemma 1 (i). It is then easy to see that $V^R_{\tau^L_1}(\widetilde{\mathbf{g}}) \geqslant V_R$ and so $\widetilde{\mathbf{g}}$ is admissible. Because $\widetilde{\mathbf{g}}$ was not the party-efficient equilibrium, we conclude that $U^{\eta}_{\tau^R_{n^*-1}}(\mathbf{g}) \leqslant U^{\eta}_{\tau^R_1}(\mathbf{g})$.

Finally, notice that

$$\begin{aligned} U^{\eta}_{\tau^R_1}(\mathbf{g}) &= \sum_{n=1}^{n^*-2} \mathbf{E}\Bigg[\sum_{t \in T^R_n} \delta^t\Bigg] u_{\eta}(R, g^R_n) + \sum_{n=2}^{n^*-1} \mathbf{E}\Bigg[\sum_{t \in T^L_n} \delta^t\Bigg] u_{\eta}(L, g^L_n) + \mathbf{E}\left[\delta^{\tau^R_{n^*-1}}\right] U^{\eta}_{\tau^R_{n^*-1}}(\mathbf{g}) \\ &= \sum_{n=1}^{n^*-2} \mathbf{E}\Bigg[\sum_{t \in T^R_n} \delta^t\Bigg] u_{\eta}(R, g^R_1) + \sum_{n=2}^{n^*-1} \mathbf{E}\Bigg[\sum_{t \in T^L_n} \delta^t\Bigg] u_{\eta}(L, g^L_2) + \mathbf{E}\left[\delta^{\tau^R_{n^*-1}}\right] U^{\eta}_{\tau^R_{n^*-1}}(\mathbf{g}). \end{aligned}$$

Using $U^{\eta}_{\tau^R_{n^*-1}}(\mathbf{g}) \leqslant U^{\eta}_{\tau^R_1}(\mathbf{g})$, it follows that

$$U^{\eta}_{\tau^R_1}(\mathbf{g}) \leqslant \frac{\sum_{n=1}^{n^*-2} \mathbf{E}\left[\sum_{t \in T^R_n} \delta^t\right] u_{\eta}(R, g^R_1) + \sum_{n=2}^{n^*-1} \mathbf{E}\left[\sum_{t \in T^L_n} \delta^t\right] u_{\eta}(L, g^L_2)}{1 - \mathbf{E}\left[\delta^{\tau^R_{n^*-1}}\right]}. \tag{B.13}$$

Also, since $\widehat{\mathbf{g}}$ is stationary after T_1^L, we have $U^\eta_{\tau^R_{n^*-1}}(\widehat{\mathbf{g}}) = U^\eta_{\tau^R_1}(\widehat{\mathbf{g}})$ and so

$$\begin{aligned} U^\eta_{\tau_1^R}(\widehat{\mathbf{g}}) &= \sum_{n=1}^{n^*-2} \mathbf{E}\Big[\sum_{t\in T_n^R} \delta^t\Big] u_\eta(R, g_1^R) + \sum_{n=2}^{n^*-1} \mathbf{E}\Big[\sum_{t\in T_n^L} \delta^t\Big] u_\eta(L, g_2^L) + \mathbf{E}\left[\delta^{\tau^R_{n^*-1}}\right] U^\eta_{\tau^R_{n^*-1}}(\widehat{\mathbf{g}}) \\ &= \sum_{n=1}^{n^*-2} \mathbf{E}\Big[\sum_{t\in T_n^R} \delta^t\Big] u_\eta(R, g_1^R) + \sum_{n=2}^{n^*-1} \mathbf{E}\Big[\sum_{t\in T_n^L} \delta^t\Big] u_\eta(L, g_2^L) + \mathbf{E}\left[\delta^{\tau^R_{n^*-1}}\right] U^\eta_{\tau^R_1}(\widehat{\mathbf{g}}), \end{aligned}$$

which yields that

$$U^\eta_{\tau_1^R}(\widehat{\mathbf{g}}) = \frac{\sum_{n=1}^{n^*-2} \mathbf{E}\Big[\sum_{t\in T_n^R} \delta^t\Big] u_\eta(R, g_1^R) + \sum_{n=2}^{n^*-1} \mathbf{E}\Big[\sum_{t\in T_n^L} \delta^t\Big] u_\eta(L, g_2^L)}{1 - \mathbf{E}\left[\delta^{\tau^R_{n^*-1}}\right]}. \tag{B.14}$$

Comparing (B.13) and (B.14), we conclude that $U_{\tau_1^R}(\widehat{\mathbf{g}}) \geqslant U_{\tau_1^R}(\mathbf{g})$, and so

$$\begin{aligned} U^\eta_{\tau_1^L}(\mathbf{g}) &= \mathbf{E}\Big[\sum_{t\in T_1^L} \delta^t\Big] u_\eta(L, g_\eta^L) + \mathbf{E}\left[\delta^{\tau_1^R}\right] U^\eta_{\tau_1^R}(\mathbf{g}) \\ &\leqslant \mathbf{E}\Big[\sum_{t\in T_1^L} \delta^t\Big] u_\eta(L, g_\eta^L) + \mathbf{E}\left[\delta^{\tau_1^R}\right] U^\eta_{\tau_1^R}(\widehat{\mathbf{g}}) = U^\eta_{\tau_1^L}(\widehat{\mathbf{g}}), \end{aligned}$$

the desired result following. ■

The following lemma looks at how the equilibrium policies change if they have not settled by some date τ_m^R.

Lemma 6. *Let $\mathbf{g} = \{(g_\ell^L, g_\ell^R)\}_{\ell=1}^\infty$ be a party-efficient equilibrium. If both PC^R_{m+1} and PC^L_m are binding for some $m \in \mathbf{N}^*$ and the policy has not settled by τ_m^R, then $g^\theta_{\ell+1} < g^\theta_\ell$ for all $\theta \in \{L, R\}$ and $\ell \geqslant m$.*

Proof. Because $v_i(\theta, g)$ is increasing in g for $i \neq \theta$, the claimed result is equivalent to $v_L(R, g_\ell^R)$ and $v_R(L, g_\ell^L)$ being strictly decreasing in ℓ for all $\ell \geqslant m$. This can be further equivalently stated in terms of the monotonicity of the radicals' continuation values in terms. Indeed, by Lemma 2, PC_k^R is binding for all $k \geqslant m+1$ and PC_k^L is binding for all $k \geqslant m$. Thus, for each $k \geqslant m$ and $i = \theta \in \{L, R\}$,

$$V^i_{\tau_k^\theta}(\mathbf{g}) = v_i(\theta, g_k^\theta) + \delta \pi_\theta V^i_{\tau_k^\theta}(\mathbf{g}) + \delta(1-\pi_\theta)V_i,$$

from which we obtain

$$V^i_{\tau_k^\theta}(\mathbf{g}) = A_\theta v_i(\theta, g_k^\theta) + D_\theta V_i \text{ for all } i = \theta \in \{L, R\} \text{ and } k \geqslant m, \tag{B.15}$$

where for each $\theta \in \{L, R\}$

$$D_\theta \equiv \frac{\delta(1-\pi_\theta)}{1-\delta\pi_\theta} > 0.$$

Similarly, we have

$$V^L_{\tau^R_k}(\mathbf{g}) = A_R v_L(R, g^R_k) + D_R V^L_{\tau^L_{k+1}}(\mathbf{g}) = V_L \text{ for all } k \geqslant m, \tag{B.16}$$

$$V^R_{\tau^L_k}(\mathbf{g}) = A_L v_R(L, g^L_k) + D_L V^R_{\tau^R_k}(\mathbf{g}) = V_R \text{ for all } k \geqslant m+1, \tag{B.17}$$

$$V^R_{\tau^L_m}(\mathbf{g}) = A_L v_R(L, g^L_m) + D_L V^R_{\tau^R_m}(\mathbf{g}) \geqslant V_R. \tag{B.18}$$

Therefore, by (B.16), $v_L(R, g^R_\ell)$ is strictly decreasing in ℓ for all $\ell \geqslant m$ if and only if $V^L_{\tau^L_\ell}(\mathbf{g})$ is strictly increasing in ℓ for all $\ell \geqslant m+1$; and, by (B.17), $v_R(L, g^L_\ell)$ is strictly decreasing in ℓ for all $\ell \geqslant m$ if and only if $V^R_{\tau^R_\ell}(\mathbf{g})$ is strictly increasing in ℓ for all $\ell \geqslant m+1$ and $v_R(L, g^L_m) > v_R(L, g^L_{m+1})$. This can be further simplified, according to the following observation.

Observation 1. *If $V^L_{\tau^L_\ell}(\mathbf{g})$ is strictly increasing in ℓ for all $\ell \geqslant m+1$, then $v_i(\theta, g^\theta_\ell)$ is strictly decreasing in ℓ for all $\ell \geqslant m$ and $i \neq \theta$, where $i, \theta \in \{L, R\}$.*

Proof of Observation 1. $V^L_{\tau^L_\ell}(\mathbf{g}) < V^L_{\tau^L_{\ell+1}}(\mathbf{g})$ for all $\ell \geqslant m+1$ implies through (B.16) that $v_L(R, g^R_\ell) > v_L(R, g^R_{\ell+1})$ for all $\ell \geqslant m$, which, due to strict antagonism, implies that $v_R(R, g^R_\ell) < v_R(R, g^R_{\ell+1})$ for all $\ell \geqslant m$. Using this in (B.15), one concludes that $V^R_{\tau^R_\ell}(\mathbf{g}) < V^R_{\tau^R_{\ell+1}}(\mathbf{g})$ holds for all $\ell \geqslant m$. Using this in (B.17), we conclude that $v_R(L, g^L_\ell) > v_R(L, g^L_{\ell+1})$ for all $\ell \geqslant m+1$. All that is left to show is that $v_R(L, g^L_m) > v_R(L, g^L_{m+1})$. To this end, suppose, to the contrary, that $v_R(L, g^L_m) \leqslant v_R(L, g^L_{m+1})$. By (B.17) and (B.18),

$$A_L v_R(L, g^L_m) + D_L V^R_{\tau^R_m}(\mathbf{g}) \geqslant V_R = A_L v_R(L, g^L_{m+1}) + D_L V^R_{\tau^R_{m+1}}(\mathbf{g}),$$

via which our supposition implies that $V^R_{\tau^R_m}(\mathbf{g}) \geqslant V^R_{\tau^R_{m+1}}(\mathbf{g})$. Using this in (B.15), one concludes that $v_R(R, g^R_m) \geqslant v_R(R, g^R_{m+1})$, which further implies via strict antagonism that $v_L(R, g^R_m) \leqslant v_L(R, g^R_{m+1})$. Employing this in (B.16), we get $V^L_{\tau^L_{m+1}}(\mathbf{g}) \geqslant V^L_{\tau^L_{m+2}}(\mathbf{g})$, which contradicts our assumption that $V^L_{\tau^L_\ell}(\mathbf{g})$ is strictly increasing in ℓ for all $\ell \geqslant m+1$, as was to be shown. ||

We now argue that if both PC^R_{m+1} and PC^L_m are binding for some $m \in \mathbf{N}^*$ and the policy has not settled by τ^R_m, then $V^L_{\tau^L_\ell}(\mathbf{g})$ is strictly increasing in ℓ for all $\ell \geqslant m+1$. To this end, we suppose, to the contrary, that there exists some $n \geqslant m+1$ such that $V^L_{\tau^L_n}(\mathbf{g}) \geqslant V^L_{\tau^L_{n+1}}(\mathbf{g})$. The consequence of this supposition is stated in the following observation.

Observation 2. *If $V^L_{\tau^L_n}(\mathbf{g}) \geqslant V^L_{\tau^L_{n+1}}(\mathbf{g})$ for some $n \geqslant m+1$, then $V^L_{\tau^L_\ell}(\mathbf{g})$ is weakly decreasing in ℓ for all $\ell \geqslant m+1$.*

Proof of Observation 2. One concludes from (B.15) that $v_L(L, g^L_n) \geqslant v_L(L, g^L_{n+1})$, which, by strict antagonism, implies that $v_R(L, g_n^L) \leqslant v_R(L, g_{n+1}^L)$. By (B.17), this yields $V^R_{\tau^R_n}(\mathbf{g}) \geqslant V^R_{\tau^R_{n+1}}(\mathbf{g})$, and so we have $v_R(R, g_n^R) \geqslant v_R(R, g_{n+1}^R)$ by (B.15). Using strict antagonism again, we conclude that $v_L(R, g_n^R) \leqslant v_L(R, g_{n+1}^R)$, which in turn implies $V^L_{\tau^L_{n+1}}(\mathbf{g}) \geqslant V^L_{\tau^L_{n+2}}(\mathbf{g})$ by (B.16). Proceeding by induction, we end up with $V^L_{\tau^L_\ell}(\mathbf{g}) \geqslant V^L_{\tau^L_{\ell+1}}(\mathbf{g})$ for all $\ell \geqslant n$. If $n \geqslant m+2$, then by (B.16), we get $v_L(R, g_{n-1}^R) \leqslant v_L(R, g_n^R)$, and so $v_R(R, g_{n-1}^R) \geqslant v_R(R, g_n^R)$ according to strict antagonism, which then implies via (B.15) that $V^R_{\tau^R_{n-1}}(\mathbf{g}) \geqslant V^R_{\tau^R_n}(\mathbf{g})$. Using (B.17), we get $v_R(L, g_{n-1}^L) \leqslant v_R(L, g_n^L)$, and by strict antagonism we further conclude that $v_L(L, g_{n-1}^L) \geqslant v_L(L, g_n^L)$, which, by (B.15), implies that $V^L_{\tau^L_{n-1}}(\mathbf{g}) \geqslant V^L_{\tau^L_n}(\mathbf{g})$. Proceeding by backward induction, we end up with $V^L_{\tau^L_{m+1}}(\mathbf{g}) \geqslant V^L_{\tau^L_{m+2}}(\mathbf{g}) \geqslant \cdots \geqslant V^L_{\tau^L_n}(\mathbf{g})$. Piecing together the results obtained from forward and backward induction justifies Observation 2. ||

The following observation is based on Observation 2.

Observation 3. *Given the condition in Observation 2, $v_R(R, g_\ell^R)$ is weakly decreasing in ℓ for all $\ell \geqslant m$.*

Proof of Observation 3. By Observation 2, $V^L_{\tau^L_\ell}(\mathbf{g})$ is weakly decreasing in ℓ for all $\ell \geqslant m+1$. By (B.16), we have

$$A_R v_L(R, g_m^R) + D_R V^L_{\tau^L_{m+1}}(\mathbf{g}) = V_L = A_R v_L(R, g_{m+1}^R) + D_R V^L_{\tau^L_{m+2}}(\mathbf{g}),$$

which implies that $v_L(R, g_m^R) \leqslant v_L(R, g_{m+1}^R)$. Using strict antagonism, we conclude that $v_R(R, g_m^R) \geqslant v_R(R, g_{m+1}^R)$. The monotonicity can be established in a similar way for each $\ell \geqslant m$. ||

To reach a contradiction, we now argue that any $\mathbf{g}$ with $V^L_{\tau^L_n}(\mathbf{g})$ decreasing in n for $n \geqslant m+1$ is not efficient. For each $\theta \in \{L, R\}$, define

$$\overline{g}^\theta = \frac{\sum_{\ell=m}^\infty \mathbf{E}\left[\sum_{t \in T_\ell^\theta} \delta^t\right] \mathbf{1}_{\{\ell > m \text{ or } \theta = R\}} g_\ell^\theta}{\sum_{\ell=m}^\infty \mathbf{E}\left[\sum_{t \in T_\ell^\theta} \delta^t\right] \mathbf{1}_{\{\ell > m \text{ or } \theta = R\}}}.$$

In words, $\overline{g}^\theta$ is the average level of all policies used in state θ starting from τ_m^R. Consider an alternative sequence of policies $\overline{\mathbf{g}} = \{(\overline{g}_\ell^L, \overline{g}_\ell^R)\}_{\ell=1}^\infty$ such that

$$\overline{g}_\ell^\theta = \begin{cases} g_\ell^L, & \ell \leqslant m,\, \theta = L \\ g_\ell^R, & \ell < m,\, \theta = R \\ \overline{g}^L, & \ell > m,\, \theta = L \\ \overline{g}^R, & \ell \geqslant m,\, \theta = R \end{cases}.$$

Since the policy has not settled by τ_m^R, $\overline{\mathbf{g}}$ is not equal to $\mathbf{g}$, and the difference, by our definition, lies in some term(s) after τ_m^R. Since the parties' utility function u_η is strictly concave, we have $U^\eta_{\tau_m^R}(\overline{\mathbf{g}}) > U^\eta_{\tau_m^R}(\mathbf{g})$. Since $\overline{\mathbf{g}}$ and $\mathbf{g}$ are identical before τ_m^R, this implies that $U^\eta_{\tau_1^L}(\overline{\mathbf{g}}) > U^\eta_{\tau_1^L}(\mathbf{g})$. To show the inefficiency of $\mathbf{g}$, it only remains to show that $\overline{\mathbf{g}}$ is admissible.

To this end, we first note that $\overline{\mathbf{g}}$ fulfills PC_m^L. Indeed, since v_L is concave, one concludes that $V^L_{\tau_m^R}(\overline{\mathbf{g}}) \geqslant V^L_{\tau_m^R}(\mathbf{g})$. The admissibility of $\mathbf{g}$ gives $V^L_{\tau_m^R}(\mathbf{g}) \geqslant V_L$, by which we know that $V^L_{\tau_m^R}(\overline{\mathbf{g}}) \geqslant V_L$. Since $\overline{\mathbf{g}}$ is stationary after τ_m^R, it satisfies PC_k^L for all $k \geqslant m$. Moreover, $\overline{\mathbf{g}}$ also satisfies other participation constraints for L-radicals prior to τ_m^R, since $\overline{\mathbf{g}}$ is identical to $\mathbf{g}$ before τ_m^R and generates an improvement for L-radicals at τ_m^R.

For the participation constraints of R-radicals, by the strict concavity of v_R, we can easily see that $\overline{\mathbf{g}}$ satisfies PC_k^R for all $k \leqslant m$. Making use of the stationarity of $\overline{\mathbf{g}}$ after τ_m^R, to establish that PC_k^R holds for $k \geqslant m+1$, it suffices to show that $\overline{\mathbf{g}}$ improves, as compared to $\mathbf{g}$, the life-time expected payoff to R-radicals at τ_{m+1}^L.

By definition, we have

$$V^R_{\tau_{m+1}^L}(\overline{\mathbf{g}}) = \sum_{k=m+1}^{\infty} \mathbf{E}\Big[\sum_{t\in T_k^L} \delta^{t-\tau_{m+1}^L}\Big] v_R(L, \overline{g}^L) + \sum_{\ell=m+1}^{\infty} \mathbf{E}\Big[\sum_{t\in T_\ell^R} \delta^{t-\tau_{m+1}^L}\Big] v_R(R, \overline{g}^R).$$

Since PC_{m+1}^R is binding under $\mathbf{g}$, we have, using the concavity of v_R, that

$$\begin{aligned}
V^R_{\tau_{m+1}^L}(\mathbf{g}) &= \sum_{k=m+1}^{\infty} \mathbf{E}\Big[\sum_{t\in T_k^L} \delta^{t-\tau_{m+1}^L}\Big] v_R(L, g_n^L) + \sum_{\ell=m+1}^{\infty} \mathbf{E}\Big[\sum_{t\in T_\ell^R} \delta^{t-\tau_{m+1}^L}\Big] v_R(R, g_\ell^R) \\
&\leqslant \sum_{k=m+1}^{\infty} \mathbf{E}\Big[\sum_{t\in T_k^L} \delta^{t-\tau_{m+1}^L}\Big] v_R(L, \overline{g}^L) + \sum_{\ell=m+1}^{\infty} \mathbf{E}\Big[\sum_{t\in T_\ell^R} \delta^{t-\tau_{m+1}^L}\Big] v_R(R, g_\ell^R).
\end{aligned}$$

Then $\overline{\mathbf{g}}$ yields a higher continuation value at τ_{m+1}^L to R-radicals than $\mathbf{g}$ if

$$\sum_{\ell=m+1}^{\infty} \mathbf{E}\Big[\sum_{t\in T_\ell^R} \delta^{t-\tau_{m+1}^L}\Big] v_R(R, g_\ell^R) \leqslant \sum_{\ell=m+1}^{\infty} \mathbf{E}\Big[\sum_{t\in T_\ell^R} \delta^{t-\tau_{m+1}^L}\Big] v_R(R, \overline{g}^R),$$

which is equivalent to

$$\sum_{\ell=m+1}^{\infty} \mathbf{E}\Big[\sum_{t\in T_\ell^R} \delta^{t}\Big] v_R(R, g_\ell^R) \leqslant \sum_{\ell=m+1}^{\infty} \mathbf{E}\Big[\sum_{t\in T_\ell^R} \delta^{t}\Big] v_R(R, \overline{g}^R). \tag{B.19}$$

If $v_R(R, \overline{g}^R) \geqslant v_R(R, g_\ell^R)$ for all $\ell \geqslant m$, then (B.19) holds trivially. So we assume below that $v_R(R, \overline{g}^R) < \sup_{\ell \geqslant m} v_R(R, g_\ell^R)$. Note that by Observation 3, $v_R(R, g_\ell^R)$ is weakly decreasing in ℓ for all $\ell \geqslant m$, which implies that $\sup_{\ell\geqslant m} v_R(R, g_\ell^R) = v_R(R, g_m^R)$, and so $v_R(R, \overline{g}^R) < v_R(R, g_m^R)$.

By the concavity of v_R, we have

$$
\begin{aligned}
&\mathbf{E}\Big[\sum_{t\in T_m^R}\delta^t\Big]v_R(R,\overline{g}^R)+\sum_{\ell=m+1}^{\infty}\mathbf{E}\Big[\sum_{t\in T_\ell^R}\delta^t\Big]v_R(R,\overline{g}^R)\\
&=\sum_{\ell=m}^{\infty}\mathbf{E}\Big[\sum_{t\in T_\ell^R}\delta^t\Big]v_R(R,\overline{g}^R)\\
&>\sum_{\ell=m}^{\infty}\mathbf{E}\Big[\sum_{t\in T_\ell^R}\delta^t\Big]v_R(R,g_\ell^R)\\
&=\mathbf{E}\Big[\sum_{t\in T_m^R}\delta^t\Big]v_R(R,g_m^R)+\sum_{\ell=m+1}^{\infty}\mathbf{E}\Big[\sum_{t\in T_\ell^R}\delta^t\Big]v_R(R,g_\ell^R),
\end{aligned}
$$

from which we can immediately obtain (B.19), a contradiction thus following. As a result, $V^L_{\tau^L_\ell}(\mathbf{g})$ is strictly increasing in ℓ for all $\ell \geqslant m+1$, which completes our proof. ■

The following lemma describes all possible patterns for party-efficient equilibria when the participation constraints for one type of radicals are all binding but those for the other type are all slack.

Lemma 7. *Given a party-efficient equilibrium $\mathbf{g} = \{(g_n^L, g_n^R)\}_{n=1}^{\infty}$,*

(i) *if all participation constraints for R-radicals are binding while all those for L-radicals are slack, then $g_n^\theta = g_{n+1}^\theta$ for all $\theta \in \{L, R\}$ and $n \in \mathbf{N}^*$;*

(ii) *if all participation constraints for L-radicals are binding while all those for R-radicals are slack, then $g_n^R = g_{n+1}^R$ for all $n \in \mathbf{N}^*$ and $g_1^L \geqslant g_2^L = g_n^L$ for all $n \geqslant 2$.*

Proof. We first consider (i). To reach a contradiction, we suppose that there exist some $\theta' \in \{L, R\}$ and $m \in \mathbf{N}^*$ such that $g_m^{\theta'} \neq g_{m+1}^{\theta'}$. Since PC_k^L is slack for all $k \in \mathbf{N}^*$, Lemma 1 (ii) implies that $v_R(\theta, g_k^\theta) \leqslant v_R(\theta, g_{k+1}^\theta)$ for all $\theta \in \{L, R\}$ and $k \in \mathbf{N}^*$. Using arguments analogous to that in the second paragraph of the proof of Lemma 4, one concludes that $v_R(\theta', g_m^{\theta'}) < v_R(\theta', g_{m+1}^{\theta'})$, which, together with the fact that PC_m^R and PC_{m+1}^R are binding, imply that

$$
\begin{aligned}
V_R = V^R_{\tau^L_m}(\mathbf{g}) &= A_L v_R(L, g_m^L) + B_L v_R(R, g_m^R) + CV_R\\
&< A_L v_R(L, g_{m+1}^L) + B_L v_R(R, g_{m+1}^R) + CV_R = V_R,
\end{aligned}
$$

a contradiction being reached.

For (ii), the proof for the property that $g_n^R = g_{n+1}^R$ for all $n \in \mathbf{N}^*$ and that $g_2^L = g_n^L$ for all $n \geqslant 2$ is analogous to the proof of (i). It thus only remains to show that $g_1^L \geqslant g_2^L$. Indeed, because PC_2^R is slack, one can conclude from Lemma 1 (i) that $v_R(L, g_1^L) \geqslant v_R(L, g_2^L)$. Since $v_R(L, g)$ is increasing in g, the desired result follows. ■

We now conclude the proof of Theorem 1 based on lemmas presented above. As we have seen, there are four possible patterns of binding/slack participation constraints:

(i) All participation constraints are slack (and so the parties' first-best outcome is achievable);

(ii) One type of participation constraints (i.e., either L or R) is always binding while the other type is always slack;

(iii) All participation constraints but PC_1^R are binding;

(iv) All participation constraints are binding.

In the first two cases, settlement occurs by τ_1^R, according to Lemma 7. The last two cases correspond the case of $m = 1$ in Lemma 6, where strict polarization occurs. Theorem 1 is hence proven.

C Proof of Proposition 2

Notice that since only the ruling party is active in a period, our problem with party homogeneity is essentially the same as one with radicals where we set $u_i = v_{-i}$ and $U_i = V_{-i}$ for each $i \in \{L, R\}$. Then, IC_n^i in the current case is simply PC_n^{-i} in the case with radicals. Notice that now there is strict antagonism between the moderates on the range of Pareto-efficient policies $[g_{i,m}^*(\theta), 1]$ for each $i = \theta \in \{L, R\}$. It is straightforward to see that Lemma 2 - Lemma 5 in Appendix B (with $u_i = v_{-i}$ and $U_i = V_{-i}$) still hold for the case with party homogeneity.

The claimed result is trivially true if no incentive constraint ever binds, and so we consider below the other cases. Assume IC_n^i is binding for some $i \in \{L, R\}$ and $n \in \mathbf{N}^*$. Then by Lemma 2, IC_m^i will be binding for each $m \geqslant n$. Therefore, at each τ_m^θ with $\theta = i$ and $m \geqslant n$, the solution (i.e., the continuation sequence of policies) to our problem is to maximize $-i$-moderates' expected utility among admissible policies subject to delivering exactly U_i to i-moderates since IC_m^i is binding.[12] One observes that these problems are all identical, and thus concludes that $g_m^\theta = g_n^\theta$ for each $\theta \in \{L, R\}$ and all $m \geqslant n$. As a result, we can safely conclude that an equilibrium will have settled the first time a binding incentive constraint is reached. Lemma 3 then implies that settlement must have occurred by τ_1^R.

Finally, we argue that $g_1^L \leqslant g_n^L$. It only remains to show this when an equilibrium has not settled by τ_1^L. Let $\mathbf{g} = \{(g_n^L, g_n^R)\}_{n=1}^\infty$ be such an equilibrium. Clearly, now IC_1^L must be slack. Given that IC_1^R binds, Lemma 5 then implies that we either have IC_m^L binding for all $m \geqslant 2$ or being slack for all $m \in \mathbf{N}^*$. The claimed property obviously holds in the latter case, and so we focus on the former case below.

[12]If the optimal policy delivered a lower continuation value to $-i$ then it must be delivering a higher continuation value to i than U_i, in which case IC_m^i would not be binding.

Since IC_1^L is slack, we have (using $U^i_{\tau_k^\theta}(\cdot)$ to denote the value for i-moderates at τ_k^θ under a sequence of policies, where $\theta \in \{L, R\}$ and $k \in \mathbf{N}^*$)

$$U^L_{\tau_1^L}(\mathbf{g}) = u_L(L, g_1^L) + \delta \pi_L U^L_{\tau_1^L}(\mathbf{g}) + \delta(1-\pi_L) U^L_{\tau_1^R}(\mathbf{g}) > U_L,$$

and so

$$U^L_{\tau_1^L}(\mathbf{g}) = A_L u_L(L, g_1^L) + B_L U^L_{\tau_1^R}(\mathbf{g}) > U_L. \tag{C.1}$$

Since IC_2^L is binding, we have

$$U^L_{\tau_2^L}(\mathbf{g}) = u_L(L, g_2^L) + \delta \pi_L U^L_{\tau_2^L}(\mathbf{g}) + \delta(1-\pi_L) U^L_{\tau_2^R}(\mathbf{g}) = U_L,$$

which implies that

$$U^L_{\tau_2^L}(\mathbf{g}) = A_L u_L(L, g_2^L) + B_L U^L_{\tau_2^R}(\mathbf{g}) = U_L. \tag{C.2}$$

Since the equilibrium has settled by τ_1^R, $U^L_{\tau_1^R}(\mathbf{g}) = U^L_{\tau_2^R}(\mathbf{g})$, so (C.1) and (C.2) yield that $u_L(L, g_2^L) < u_L(L, g_1^L)$. Since $g_1^L, g_2^L \in [g^*_{L,m}(L), 1]$ while $u_L(L, \cdot)$ is strictly concave, one must have $g_1^L < g_2^L$, as was to be shown.

D Proof of Propositions 3 and 4

We start with Proposition 3. For simplicity, we normalize the stage payoff for i-radicals out of power to 0 in the autarky equilibrium, which implies that

$$V_L = \frac{B_R}{1-C} v_L(L, g^*_{L,r}(L)), \quad V_R = \frac{B_L}{1-C} v_R(R, g^*_{R,r}(R)), \tag{D.1}$$

where B_θ ($\theta \in \{L, R\}$) and C are defined in (B.1). According to the proof of Theorem 1, we only need to consider three possible cases.

Case 1: *All participation constraints are slack.* In this case, the policy choice is $g_n^\theta = 1$ for both $\theta \in \{L, R\}$ and so is constant in δ. The claimed monotonicity trivially holds.

Case 2: *All participation constraints are binding from τ_1^R onwards.* In this case strict polarization will occur. We first consider the situation where all participation constraints bind. Then, for all $n \in \mathbf{N}^*$

$$\begin{aligned} V_R &= A_L v_R(L, g_n^L) + B_L v_R(R, g_n^R) + C V_R, \\ V_L &= A_R v_L(R, g_n^R) + B_R v_L(L, g_{n+1}^L) + C V_L. \end{aligned}$$

Replacing V_R and V_L with the expressions in (D.1), we obtain, after simplifying, that

$$\begin{aligned} v_R(L, g_n^L) + \frac{\delta(1-\pi_L)}{1-\delta\pi_R}\left[v_R(R, g_n^R) - v_R(R, g_{R,r}^*(R))\right] = 0, \\ v_L(R, g_n^R) + \frac{\delta(1-\pi_R)}{1-\delta\pi_L}[v_L(L, g_{n+1}^L) - v_L(L, g_{L,r}^*(L))] = 0. \end{aligned} \tag{D.2}$$

Differentiating each of the two equations in (D.2) with respect to δ and rearranging, one arrives at

$$\frac{\mathrm{d}g_n^R}{\mathrm{d}\delta} = \frac{v_R(R, g_{R,r}^*(R)) - v_R(R, g_n^R)}{\delta(1-\delta\pi_R)\frac{\partial v_R(R,g)}{\partial g}\big|_{g=g_n^R}} - \frac{(1-\delta\pi_R)\frac{\partial v_R(L,g)}{\partial g}\big|_{g=g_n^L}}{\delta(1-\pi_L)\frac{\partial v_R(R,g)}{\partial g}\big|_{g=g_n^R}} \cdot \frac{\mathrm{d}g_n^L}{\mathrm{d}\delta}, \tag{D.3}$$

$$\frac{\mathrm{d}g_{n+1}^L}{\mathrm{d}\delta} = \frac{v_L(L, g_{L,r}^*(L)) - v_L(L, g_{n+1}^L)}{\delta(1-\delta\pi_L)\frac{\partial v_L(L,g)}{\partial g}\big|_{g=g_{n+1}^L}} - \frac{(1-\delta\pi_L)\frac{\partial v_L(R,g)}{\partial g}\big|_{g=g_n^R}}{\delta(1-\pi_R)\frac{\partial v_L(L,g)}{\partial g}\big|_{g=g_{n+1}^L}} \cdot \frac{\mathrm{d}g_n^R}{\mathrm{d}\delta}. \tag{D.4}$$

Notice that by (B.6) we must have $g_1^L = 1$ regardless of the value of δ as both $u_\eta(L, g)$ and $v_R(L, g)$ achieve their maximums at $g = 1$, which implies that $\mathrm{d}g_1^L/\mathrm{d}\delta = 0$. Since, by strict polarization we have $v_R(R, g_{R,r}^*(R)) > v_R(R, g_1^R)$, while $\partial v_R(R, g)/\partial g\big|_{g=g_1^R} < 0$, we conclude in (D.3) that $\mathrm{d}g_1^R/\mathrm{d}\delta < 0$. Using this in (D.4) (i.e., letting $n = 1$) and noticing that $v_L(L, g_{L,r}^*(L)) > v_L(L, g_{n+1}^L)$, $\partial v_L(L, g)/\partial g\big|_{g=g_{n+1}^L} < 0$, and $\partial v_L(R, g)/\partial g > 0$, we further conclude that $\mathrm{d}g_2^L/\mathrm{d}\delta < 0$. Proceeding by induction, we conclude that $\mathrm{d}g_n^L/\mathrm{d}\delta < 0$ for all $n \geqslant 2$ and $\mathrm{d}g_n^R/\mathrm{d}\delta < 0$ for all $n \in \mathbf{N}^*$.

Then consider the case where all participation constraints but PC_1^R bind. Then (D.3) holds for all $n \geqslant 2$ while (D.4) still holds for all $n \in \mathbf{N}^*$. Thus, the desired result will hold if $\mathrm{d}g_1^R/\mathrm{d}\delta \leqslant 0$. Indeed, given that PC_1^R is slack, $\Lambda_1^R = 0$, and so (B.7) yields that

$$g_1^R = \underset{g\in[0,1]}{\arg\max}\left\{u_\eta(R, g) + \Lambda_1^L v_L(R, g)\right\} = 1$$

since both $u_\eta(R, g)$ and $v_L(R, g)$ are maximized at $g = 1$. Thus, $\mathrm{d}g_1^R/\mathrm{d}\delta = 0$, as was to be shown.

Case 3: *One type of participation constraints is always binding and the other is always slack.* Suppose that the binding participation constraints are for i-radicals ($i \in \{L, R\}$). Then by Assumption 1, we must have $g_n^{-i} = 1$ for all $n \in \mathbf{N}^*$, since we have $\Lambda_k^\theta = 0$ for all $k \in \mathbf{N}$ in (B.6) and (B.7) while both $u_\eta(\theta, g)$ and $v_i(\theta, g)$ achieves their maximums at $g = 1$. Also note that we still have $g_1^L = 1$ for all δ. Thus, we conclude that $\mathrm{d}g_n^i/\mathrm{d}\delta \leqslant 0$, employing (D.2) or (D.3).

Now we prove Proposition 4. We still discuss in the three cases above. The conclusion follows immediately in Case 1. In Case 2, we can differentiate each equation in (D.2) with respect to π

(note that now $\pi_L = \pi_R = \pi$) to obtain

$$\frac{\mathrm{d}g_n^R}{\mathrm{d}\pi} = \frac{(1-\delta)[v_R(R,g_n^R) - v_R(R,g_{R,r}^*(R))]}{(1-\delta\pi)(1-\pi)\frac{\partial v_R(R,g)}{\partial g}\big|_{g=g_n^R}} - \frac{(1-\delta\pi)\frac{\partial v_R(L,g)}{\partial g}\big|_{g=g_n^L}}{\delta(1-\pi)\frac{\partial v_R(L,g)}{\partial g}\big|_{g=g_n^R}} \cdot \frac{\mathrm{d}g_n^L}{\mathrm{d}\pi}, \tag{D.5}$$

$$\frac{\mathrm{d}g_{n+1}^L}{\mathrm{d}\pi} = \frac{(1-\delta)[v_L(L,g_{n+1}^L) - v_L(L,g_{L,r}^*(L))]}{(1-\pi)(1-\delta\pi)\frac{\partial v_L(L,g)}{\partial g}\big|_{g=g_{n+1}^L}} - \frac{(1-\delta\pi)\frac{\partial v_L(R,g)}{\partial g}\big|_{g=g_{n+1}^L}}{\delta(1-\pi)\frac{\partial v_R(R,g)}{\partial g}\big|_{g=g_n^R}} \cdot \frac{\mathrm{d}g_n^R}{\mathrm{d}\pi}. \tag{D.6}$$

Using the facts presented in the proof for Proposition 3 (Case 2), we see from (D.5) and (D.6) that $\mathrm{d}g_1^L/\mathrm{d}\pi = 0$, $\mathrm{d}g_n^R/\mathrm{d}\pi > 0$ for all $n \in \mathbf{N}^*$, and $\mathrm{d}g_n^L/\mathrm{d}\pi > 0$ for all $n \geqslant 2$, which yields the desired result. The case with all participation constraints but PC_1^R binding can be argued in an analogous way.

Finally, for Case 3, if all participation constraints for i-radicals bind while all those for $-i$-radicals are slack, then we still have $g_n^{-i} = 1$ for all $n \in \mathbf{N}^*$ and $g_1^L = 1$ for all δ. The rest of the argument is a straightforward application of (D.5) or (D.6), noting that $g_1^L = 1$ for all δ and so $\mathrm{d}g_n^{-i}/\mathrm{d}\pi = 0$ for all $n \in \mathbf{N}^*$ and $\theta = -i$.

E Proof of Proposition 5

If party L is homogeneous, the constraints we need to take care of are the incentive constraints for L-moderates and the participation constraints for R-radicals when party L is ruling. Let $\mathbf{g} = \{(g_n^L, g_n^R)\}_{n=1}^\infty$ be a party-efficient equilibrium in this case; our goal is to show that $g_n^i = g_{n+1}^i$ for all $i \in \{L, R\}$ and $n \in \mathbf{N}^*$. To this end, suppose that there exist some $j \in \{L, R\}$ and m such that $g_m^j \neq g_{m+1}^j$. For the sake of a contradiction, consider an alternative sequence of policies $\overline{\mathbf{g}} = \{(\overline{g}_n^L, \overline{g}_n^R)\}_{n=1}^\infty$ such that for each $n \in \mathbf{N}^*$ and $i \in \{L, R\}$

$$\overline{g}_n^i = \overline{g}^i \equiv \frac{\sum_{\ell=1}^\infty \mathbf{E}\left[\sum_{t \in T_\ell^\theta} \delta^t\right] g_\ell^i}{\sum_{\ell=1}^\infty \mathbf{E}\left[\sum_{t \in T_\ell^\theta} \delta^t\right]}.$$

Under our supposition, it is clear that $\overline{g}^j \neq g_m^j$. Employing the strict concavity of the utility functions, we immediately obtain that

$$U_{\tau_1^L}^L(\overline{\mathbf{g}}) > U_{\tau_1^L}^L(\mathbf{g}), \quad U_{\tau_1^L}^\eta(\overline{\mathbf{g}}) > U_{\tau_1^L}^\eta(\mathbf{g}), \quad V_{\tau_1^L}^R(\overline{\mathbf{g}}) > V_{\tau_1^L}^R(\mathbf{g}). \tag{E.1}$$

Since $\mathbf{g}$ is admissible, we have $V_{\tau_1^L}^R(\mathbf{g}) \geqslant V_R$ and $U_{\tau_1^L}^L(\mathbf{g}) \geqslant U_L$, and so by (E.1), $\overline{\mathbf{g}}$ satisfies PC_1^R and IC_1^L. By the stationarity of $\overline{\mathbf{g}}$, we further conclude that $\overline{\mathbf{g}}$ also fulfills PC_k^R and IC_k^L for all $k \in \mathbf{N}^*$. Therefore, $\overline{\mathbf{g}}$ generates an improvement for the parties, contradicting the efficiency of $\mathbf{g}$. Thus, settlement must have been achieved by τ_1^L.

Now we consider the case in which party R is homogeneous. The constraints we need to take care of are the incentive constraints for R-moderates and the participation constraints for L-radicals. For any party-efficient equilibrium $\mathbf{g}$ that does not achieve settlement by τ_1^R, we can define an alternative sequence of policies $\widetilde{\mathbf{g}} = \{(\widetilde{g}_n^L, \widetilde{g}_n^R)\}_{n=1}^{\infty}$ such that $\widetilde{g}_1^L = g_1^L$,

$$\widetilde{g}_n^L = \frac{\sum_{\ell=2}^{\infty} \mathbf{E}\left[\sum_{t\in T_\ell^L} \delta^t\right] g_\ell^L}{\sum_{\ell=2}^{\infty} \mathbf{E}\left[\sum_{t\in T_\ell^L} \delta^t\right]} \text{ for all } n \geqslant 2, \quad \text{and} \quad \widetilde{g}_n^R = \frac{\sum_{\ell=1}^{\infty} \mathbf{E}\left[\sum_{t\in T_\ell^R} \delta^t\right] g_\ell^R}{\sum_{\ell=1}^{\infty} \mathbf{E}\left[\sum_{t\in T_\ell^R} \delta^t\right]} \text{ for all } n \in \mathbf{N}^*.$$

Then it is easy to see that $\widetilde{\mathbf{g}}$ generates an improvement for the parties at τ_1^R, and hence also at τ_1^L, while it satisfies all incentive and participation constraints due to the admissibility of $\mathbf{g}$ and the stationarity of $\widetilde{\mathbf{g}}$ after τ_1^R. This contradicts the efficiency of $\mathbf{g}$, as was to be shown.

F Proof of Proposition 6

For Proposition 6, notice that the proof of Lemma 1, given Assumption 2, applies to each dimension of the policy, by which we obtain the same result as in Lemma 1 on the monotonicity for the radicals' stage payoffs over terms. Lemmas 2-5 and 7 are therefore the same as in the one-dimensional case. The proof is then immediate in the case in which at most one radicals' PC constraints bind. Let us therefore focus on the case in which PC_1^L and PC_2^F bind.

To proceed, consider the recursive version of the problem at the beginning of the mth term of party R, given that PC_m^L is binding for some $m \geqslant 1$ and that L's continuation value at PC_{m+1}^L will be V_L. Denote by $\overline{W}_R$ the highest possible continuation value that can be delivered to R-radicals by an admissible sequence of policies. The solution of the problem will determine the policy for the mth term of party R, denoted by g^R, that for the $(m+1)$th term of party L, denoted by g^L, and the continuation value for R for the $(m+1)$th term of party R, provided that a continuation value of at least W_R will have to be delivered to R-radicals. Denote the value of the optimal solution to the parties by $F : (-\infty, \overline{W}_R] \to \mathbf{R}$, we have the problem recursively formulated as

$$\begin{aligned} F(W_R) = \max_{g^R, g^L, W_R'} &\left\{A_R u_\eta(R, g^R) + B_R u_\eta(L, g^L) + CF(W_R')\right\} \\ \text{subject to} \quad PK^R: &\quad A_R v_R(R, g^R) + B_R v_R(L, g^L) + CW_R' \geqslant W_R \\ PC^R: &\quad A_L v_R(L, g^L) + D_L W_R' \geqslant V_R \\ PC^L: &\quad A_R v_L(R, g^R) + B_R v_L(L, g^L) + CV_L \geqslant V_L \\ Feas: &\quad \overline{W}_R \geqslant W_R' \end{aligned} \tag{F.1}$$

Standard arguments establish the uniqueness, concavity, continuity, and monotonicity (weakly decreasing) of F, for which it is easy to see that there exists $W^c \in [-\infty, \overline{W}_R)$ such that F is constant on $(-\infty, W^c)$ and strictly decreasing on $[W^c, \overline{W}_R]$. For each $W \in (-\infty, \overline{W}_R]$, denote by $\mathbf{s}(W) = (g^R(W), g^L(W), W_R'(W))$ the solution to (F.1). When evaluated at $W_R = V_{\tau_m^R}^R$, we will have $U^{\eta}_{\tau^R_m} = F(V^R_{\tau^R_m})$ and $V^R_{\tau^R_{m+1}} = W'_R(W)$.

We next state and prove the following lemma.

Lemma 8. *F is strictly concave on $[W^c, \overline{W}_R]$.*

Proof. For the sake of contradiction, we assume, without loss of generality, that F is linear on some $[W_1, W_2] \subset [W^c, \overline{W}_R]$, which is *maximal* in the sense that F is not linear on any interval $\mathcal{I}$ such that $[W_1, W_2] \subsetneq \mathcal{I}$.

We first argue that $g^\theta(W) = g^\theta(W_1)$ for each $W \in [W_1, W_2]$ and $\theta \in \{L, R\}$. Suppose this were not the case for some $W_3 \in (W_1, W_2)$ with $W_3 = \alpha W_1 + (1-\alpha)W_2$. Take some $W_4 \in (W_1, W_3)$ and $\beta \in (0,1)$ such that $W_4 = \beta W_1 + (1-\beta)W_3$. Because, as is easily seen, $(g^R_\beta, g^L_\beta, W'_{R,\beta}) \equiv \beta s(W_1) + (1-\beta)s(W_3)$ satisfies all the constraints in (F.1) at W_4, we know that

$$\begin{aligned}
F(W_4) &\geqslant A_R u_\eta(R, g^R_\beta) + B_R u_\eta(L, g^L_\beta) + CF(W'_{R,\beta}) \\
&> \beta\big[A_R u_\eta(R, g^R(W_1)) + B_R u_\eta(L, g^L(W_1)) + CF(W'_R(W_1))\big] \\
&\quad + (1-\beta)\big[A_R u_\eta(R, g^R(W_3)) + B_R u_\eta(L, g^L(W_3)) + CF(W'_R(W_3))\big] \\
&= \beta F(W_1) + (1-\beta)F(W_3),
\end{aligned}$$

where the strict inequality follows from the strict concavity of u_η and weak concavity of F. However, this contradicts the linearity of F on $[W_1, W_2]$.

Based on the above property justified, one can further conclude that $F(W'_R(W))$ is strictly decreasing in W on $[W_1, W_2]$, because the choices of g^R and g^L are the same for all $W \in [W_1, W_2]$ while F is strictly decreasing on $[W_1, W_2]$.

With the observations above, we next show that PK^R must be binding at $\mathbf{s}(W)$ for each $W \in [W_1, W_2]$. Suppose PK^R failed to bind for some $\widehat{W} \in (W_1, W_2]$, then one can achieve an improvement by choosing instead $\gamma\mathbf{s}(W_1) + (1-\gamma)\mathbf{s}(\widehat{W})$ for some $\gamma \in (0,1)$ sufficiently close to 0 so that PK^R is satisfied. Given this, the case with $\widehat{W} = W_1$ is a consequence of continuity.

Since PK^R is binding for each $W \in [W_1, W_2]$, there exists an open interval $\mathcal{I}' \subseteq [W_1, W_2]$ such that $W'_R(W) \notin [W_1, W_2]$ for each $W \in \mathcal{I}'$. To see this, notice that we have

$$\begin{aligned}
A_R v_R(R, g^R(W_1)) + B_R v_R(L, g^L(W_1)) + CW'_R(W_1) = W_1, \\
A_R v_R(R, g^R(W_2)) + B_R v_R(L, g^L(W_2)) + CW'_R(W_2) = W_2,
\end{aligned}$$

which, with the fact that $g^\theta(W_1) = g^\theta(W_2)$ for all $\theta \in \{L, R\}$, implies that

$$\frac{1}{C}(W_2 - W_1) = W'_R(W_2) - W'_R(W_1).$$

Since $C \in (0,1)$, it is impossible to have both $W'_R(W_1)$ and $W'_R(W_2)$ in $[W_1, W_2]$, and so $W'_R(W) \notin [W_1, W_2]$ for some $W \in (W_1, W_2)$. Due to continuity, some neighborhood of W will meet our demand. We thus let $W_0 \in \mathcal{I}'$. Since PK^R is binding at $\mathbf{s}(W_0)$, we have

$$W_R'(W_0) = \frac{1}{C}\left[W_0 - A_R v_R(R, g^R(W_0)) - B_R v_R(L, g^L(W_0)\right],$$

which then implies that

$$\begin{aligned} F(W_0) =& A_R u_\eta(R, g^R(W_0)) + B_R u_\eta(L, g^L(W_0)) \\ &+ CF\left(\frac{1}{C}\left[W_0 - A_R v_R(R, g^R(W_0)) - B_R v_R(L, g^L(W_0))\right]\right). \end{aligned} \tag{F.2}$$

Therefore, taking the derivative with respect to W_0 in (F.2) we have $F'(W_0) = F'(W_R'(W_0))$, which contradicts the condition that $[W_1, W_2]$ is maximal and the fact that F is concave. Our proof is now complete. ■

We then show that the continuation value for R-radicals at τ_ℓ^R is weakly increasing in ℓ for all $\ell \geqslant m$. The result is formally stated in the lemma below.

Lemma 9. *For $\ell \geq m$, $V^R_{\tau^R_{\ell+1}} = W_R'(W_R) \geq W_R = V^R_{\tau^R_\ell}$.*

Proof. Without loss of generality, we can assume that $W_R, W_R'(W_R) \geqslant W^c$ and discuss two cases.

Case 1: $\max\{W_R, W_R'(W_R)\} = \overline{W}_R$. The desired result will follow trivially if $W_R'(W_R) = \overline{W}_R$, and so we assume instead that $W_R = \overline{W}_R$. In this case, if $W_R'(W_R) < \overline{W}_R$, then there is some $\varepsilon > 0$ such that $W_R'(W_R) + \varepsilon \leqslant \overline{W}_R$ and so $\mathbf{s}_\varepsilon \equiv (g^R(W_R), g^L(W_R), W_R'(W_R) + \varepsilon)$ is feasible to (F.1). Notice that $\mathbf{s}_\varepsilon$ would deliver to R-radicals a continuation value higher than $\overline{W}_R$, which contradicts the definition of $\overline{W}_R$. As a result, we must have $W_R \leqslant W_R'(W_R) = \overline{W}_R$.

Case 2: $\max\{W_R, W_R'(W_R)\} < \overline{W}_R$. We first argue that there is some $\mathbf{s}_0 = (g_0^R, g_0^L, W_{R,0}')$ that makes all constraints in (F.1) slack (i.e., Slater's condition is satisfied for (F.1)). Indeed, consider the solution to (F.1) at W_R. By our assumption, the feasibility constraint ($Feas$) must be slack at $\mathbf{s}(W_R)$. Let $\Delta \equiv \overline{W}_R - W_R'(W_R) > 0$. Then by continuity, one can find a neighborhood of $(g^R(W_R), g^L(W_R))$, denoted by $\mathscr{N}$, such that for each $(g^R, g^L) \in \mathscr{N}$,

$$\max\left\{\begin{array}{l} A_R|v_R(R, g^R) - v_R(R, g^R(W_R))| + B_R|v_R(L, g^L) - v_R(L, g^L(W_R))|, \\ A_R|v_L(R, g^R) - v_L(R, g^R(W_R))| + B_R|v_L(L, g^L) - v_L(L, g^L(W_R))|, \\ A_L|v_R(L, g^L) - v_R(L, g^L(W_R))| \end{array}\right\} < \frac{\Delta}{2}\min\{C, D_L\}.$$

Since there is a policy for which all constraints are slack for the sequence problem, for each neighborhood of $(g^R(W_R), g^L(W_R))$ there must be some pair (g_0^L, g_0^R) such that PC_L is slack. If PC_L is slack at $(g^R(W_R), g^L(W_R))$, then we can take $(g_0^L, g_0^R) = (g^R(W_R), g^L(W_R))$. Suppose PC_L binds at $(g^R(W_R), g^L(W_R))$. We want to find (g_0^R, g_0^L) that yields a higher expected payoff to L-radicals in PC_L than $\mathbf{s}(W_R)$. If no such pair existed, then $(g^R(W_R), g^\cdot(W_R)) = (g^*_{L,r}(R), g^*_{L,r}(L))$ and, by PC_L binding, we have $V_L = \frac{1}{C}[A_R v_L(R, g^*_{L,r}(R)) + B_R v_L(L, g^*_{L,r}(L))]$, which is equal to L-radicals first-best payoff. This would violate our assumption that there exists a policy for which all constraints are slack as any other policy would yield L-radicals a payoff lower than V_L. Thus, we can pick (g_0^L, g_0^R) from $\mathscr{N}$ with such a property and let $\mathbf{s}_0 = (g_0^R, g_0^L, W_R'(W_R) + 0.5\Delta)$ to fulfill our requirement.

Provided the Slater's condition, we consider the Lagrangian for (F.1). Write constraint $x \in \{PK^R, PC^R, PC^L, Feas\}$ as $h_x(g^R, g^L, W_R'; W_R) \geqslant 0$ for some function h_x and let ζ_x be its Lagrange multiplier. We then have

$$F(W_R) = \min_{\zeta \in \mathbf{R}_+^4} \max_{g^L, g^R, W_R'} \Big\{ A_R u_\eta(R, g^R) + B_R u_\eta(L, g^L) + CF(W_R') + \sum_x \zeta_x h_x(g^R, g^L, W_R'; W_R) \Big\}.$$

Let $Z(W_R) \subseteq \mathbf{R}_+^4 \backslash \{\mathbf{0}\}$ be the set of Lagrange multiplier(s) for the problem above. By Corollary 5 of Milgrom & Shannon (1994), we know that the directional derivatives $F_+'(W_R) = \min_{\zeta' \in Z(W_R)} -\zeta'_{PK_R}$ and $F_-'(W_R) = \max_{\zeta' \in Z(W_R)} -\zeta'_{PK_R}$. The optimality condition prescribes that for each $\zeta \in Z$, there is some $f_\zeta \in [F_+'(W_R'(W_R)), F_-'(W_R'(W_R))]$ such that

$$f_\zeta = -\zeta_{PK_R} - \frac{D_L}{C}\zeta_{PC_R} + \frac{1}{C}\zeta_{Feas}.$$

Since $W_R'(W_R) < \overline{W}_R$, it follows from complementary slackness that $\zeta_{Feas} = 0$, and so one concludes that $f_\zeta \leqslant -\zeta_{PK_R} \leqslant F_-(W_R)$. Since F is strictly concave on $[W^c, \overline{W}_R]$, we must have $W_R \leqslant W_R'(W_R)$, as was to be shown. ■

In terms of the sequence problem, Lemma 9 shows that $V^R_{\tau^R_\ell}(\mathbf{g})$ is weakly increasing in ℓ for each party-efficient equilibrium $\mathbf{g}$ and $\ell \geqslant m$, given that PC^L_m is binding. Symmetrically, given that PC^R_{m+1} is binding, we can show with analogous arguments that $V^L_{\tau^L_\ell}(\mathbf{g})$ is weakly increasing in ℓ for each $\ell \geqslant m$ and party-efficient equilibrium $\mathbf{g}$.

Finally, we argue that if $g^i_n = g^i_{n+1}$ for both $i \in \{L, R\}$, some $n \in \mathbf{N}^*$, and a party-efficient equilibrium $\mathbf{g} = \{(g^L_\ell, g^R_\ell)\}_{\ell=1}^\infty$, then settlement has been achieved by τ^i_n. According to our arguments as in Theorem 1, settlement occurs by τ^R_1 if at most one type of radicals' PC binds. All that remains to consider is the case in which both PC^L_1 and PC^R_2 are binding. To this end, we consider the recursive problem (F.1) at τ^R_n. Since PC^R_{n+1} is binding, one has

$$A_L v_R(L, g^L_{n+1}) + D_L V^R_{\tau^R_{n+1}}(\mathbf{g}) = V_R. \tag{F.3}$$

Also, due to PC^R_n, one has

$$A_L v_R(L, g^L_n) + D_L V^R_{\tau^R_n}(\mathbf{g}) \geqslant V_R. \tag{F.4}$$

Since $g^L_n = g^L_{n+1}$, (F.3) and (F.4) imply that $V^R_{\tau^R_{n+1}}(\mathbf{g}) \leqslant V^R_{\tau^R_n}(\mathbf{g})$. By our argument before, since PC^L_n is binding, we must have $V^R_{\tau^R_{n+1}}(\mathbf{g}) \geqslant V^R_{\tau^R_n}(\mathbf{g})$. We thus conclude that $V^R_{\tau^R_{n+1}}(\mathbf{g}) = V^R_{\tau^R_n}(\mathbf{g})$.

Therefore, one concludes that

$$U^{\eta}_{\tau_n^R}(\mathbf{g}) = F\left(V^R_{\tau_n^R}(\mathbf{g})\right) = \frac{1}{1-C}\left[A_R u_\eta(R, g_n^R) + B_R u_\eta(L, g_{n+1}^L)\right],$$

which says that $F\left(V^R_{\tau_n^R}(\mathbf{g})\right)$ can be achieved by repeating (g_n^R, g_{n+1}^L) from τ_n^R onwards. Thus, polices have settled by τ_n^R. An analogous argument shows settlement by τ_n^L, completing the proof.

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