3.1 Inverse functions.

HW ALEKS

1-1 functions.
Definition of inverse
Determine inverse functions
domain and range of inverse functions.

We have — A function is a method or rule used to determine a <u>single</u> output in its range given a <u>single</u> input from its domain.

Q1 can we go backwards?

A1 maybe? — it depends, but <u>not</u> <u>always</u>

Restate the question:

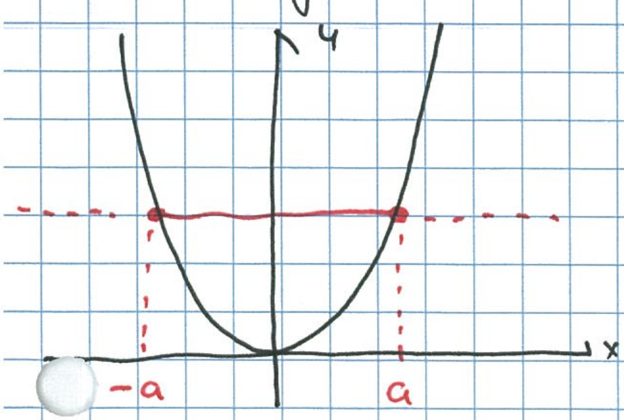
If we have $y = f(x)$

we can determine y given x.

If we have y can we get x?

— not always — :(

ex/ $y = x^2$



if $x = a$ then $y = a^2$. Fine

if $x = -a$ then $y = (-a)^2 = a^2$
uh oh...

if $a \neq 0$ then there are always two inputs corresponding to this one output.

:(

When can we do this? We need some help.

<u>Def</u>: A function is one-to-one (1-1) if

(A) if $a \neq b$ then $f(a) \neq f(b)$
for any a, b in the domain of f.

<u>or</u>

(B) if $f(a) = f(b)$ then $a = b$ for any a, b in the domain of f.

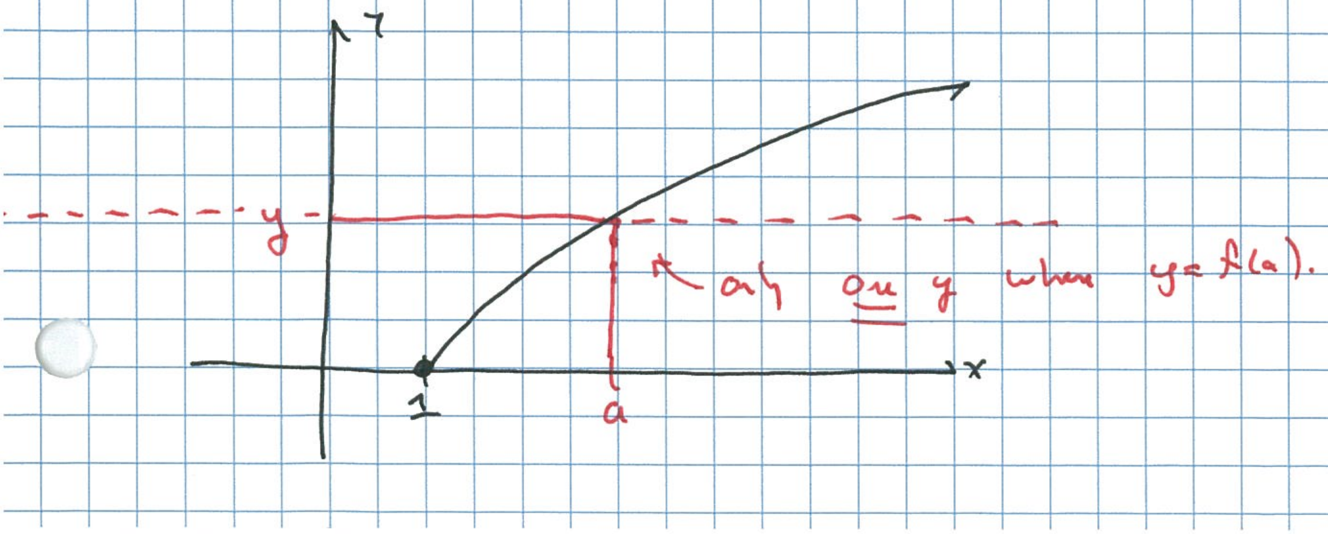
[these are equivalent definitions]
- which to use - depends on the context

So what?
if $y = f(x)$, ~~then~~ and if we know ~~that~~ y, then
there is <u>only</u> <u>one</u> possible x when $y = f(x)$.

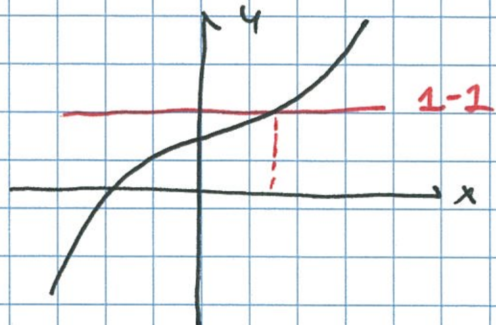
ex/ $y = \sqrt{x-1}$
suppose there are two values, a and b, where
$f(a) = f(b)$. (use part B def)

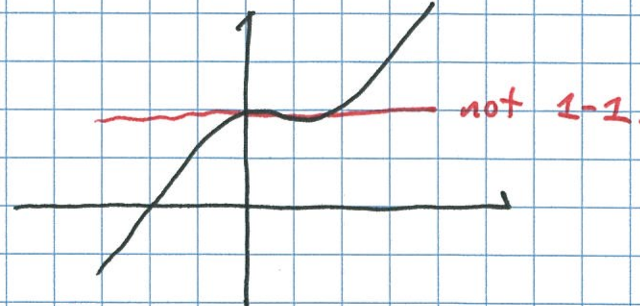
$$\Rightarrow \quad \sqrt{a-1} = \sqrt{b-1} \quad \text{(subst.)}$$
$$a-1 = b-1 \quad \text{(square both side)}$$
$$a = b \quad \text{(add 1 to both)}$$

$\Rightarrow$ The fn. is 1-1.



The idea: if $f(a) = f(b)$ then this can only happen if $a = b$. So given $x = a$ ~~then~~ then it is the only x to give the resulting y.





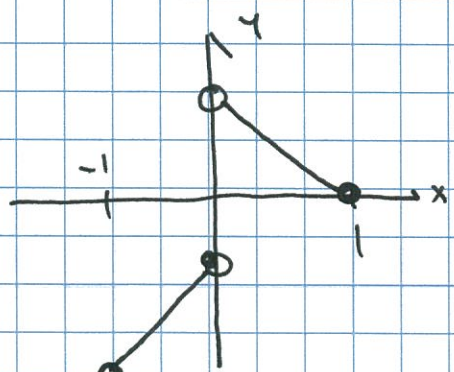
horizontal line test.

Note: If a function is strictly inc. then it is 1-1. (or strictly decreasing.)

Is this necessary?

ex/ Nope!

$$\text{Lenny}(x) = \begin{cases} +x-1 & -1 \le x < 0 \\ -x+1 & 0 < x \le 1 \end{cases}$$



This is 1-1 but not strictly inc. nor strictly dec.

So what? This is fun and all but why do it?

Recall: given $y = f(x)$ if you know y what is x? (Good Question)

First, we need a name for this process, cuz it is awkward to talk about.

- we have y and know that $y = f(x)$.
- given y we want to determine x.

This describes a function!

$$\Rightarrow \quad x = g(y)$$ a rule to get x given y.

The function g is called the inverse of f.

ex/ we had $y = \sqrt{x-1}$ + showed it was 1-1.

call this $y = \text{Dan}(x) = \sqrt{x-1}$

given y what is x?

$y = \sqrt{x-1}$ (square both sides)

$y^2 = \cancel{x+1}\ x-1$ (add 1 to both sides)

$x = y^2+1$

call this Wanda(y).

Wanda is the inverse of Dan.

note $\text{Wanda} \circ \text{Dan}(x) = \text{Wanda}(\text{Dan}(x)) = \text{Wanda}(\sqrt{x-1})$

$= (\sqrt{x-1})^2+1 = x-1+1 = x.$

and $\text{Dan}(\text{Wanda}(y)) = \text{Dan}(y^2+1) = \sqrt{y^2+1-1} = \sqrt{y^2} = |y|$

Uh oh!

We need to be careful!

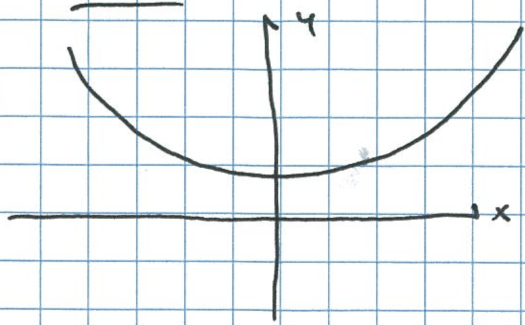
domain of Dan: $[1, \infty)$

range of Dan: $[0, \infty)$

so if y is in $[0,\infty)$ then $\text{Dan} \circ \text{Wanda}(y) = |y| = y$

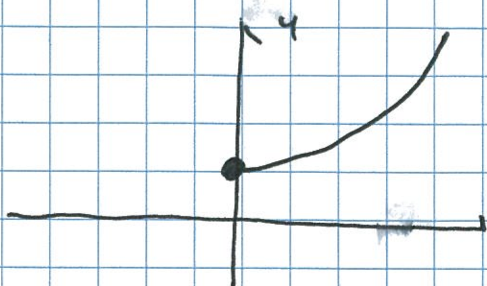
cuz y is not negative.

note:



This fcn is ~~not~~ 1-1.

So... let's cheat!



We restrict the domain so that the new function is 1-1.

For this new fn. Life is good!

This is what we do w/ $\sqrt{\cdot}$ we ~~cheat~~ cheat!

~~Notato~~ Notation

Disclaimer: This is the worstest notation in the history of mankind. It is not my fault. I am sorry for this, but...

If $g(x)$ is the inverse of $f(x)$ we denote it as $f^{-1}(x)$. (sorry)

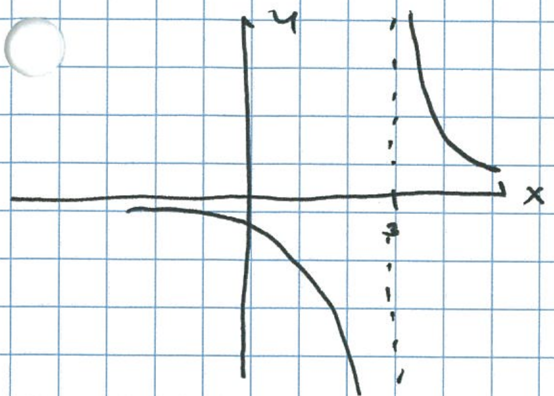
USUALLY (But not always)

$f(x)$ has domain D and range R.

Then $f^{-1}(x)$ has domain R & ~~domain~~ range D.

Always true: $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$.

ex/ $\text{Gabble}(x) = \frac{5}{x-3}$. det $\text{Gabble}^{-1}(y)$.



- passes horizontal line test (1-1)
- is a fcn.
- domain: $(-\infty,3)\cup(3,\infty)$
- range: $(-\infty,0)\cup(0,\infty)$

$y = 5/(x-3) \Rightarrow$ solve for x

$$(x-3)y = 5$$
$$xy - 3y = 5$$
$$xy = 5+3y \Rightarrow x = \frac{5+3y}{y} \quad \text{if } (y \neq 0)$$

So $\text{Gabble}^{-1}(y) = \frac{5+3y}{y}$

$$\text{Gabble}(\text{Gabble}^{-1}(y)) = \text{Gabble}\left(\frac{5+3y}{y}\right) = \frac{5}{\left(\frac{5+3y}{y}\right)-3}$$
$$= \frac{5y}{5+3y-3y} = \frac{5y}{5} = y \checkmark$$

$$\text{Gabble}^{-1}(\text{Gabble}(x)) = \text{Gabble}^{-1}\left(\frac{5}{x-3}\right) = \frac{5+3\cdot\frac{5}{x-3}}{5/(x-3)}$$
$$= \frac{5(x-3)+15}{5} = \frac{5x-15+15}{5} = x \checkmark$$

ex/

x	f
1	5
2	6
3	-2
4	5

not 1-1
no inverse!

try...

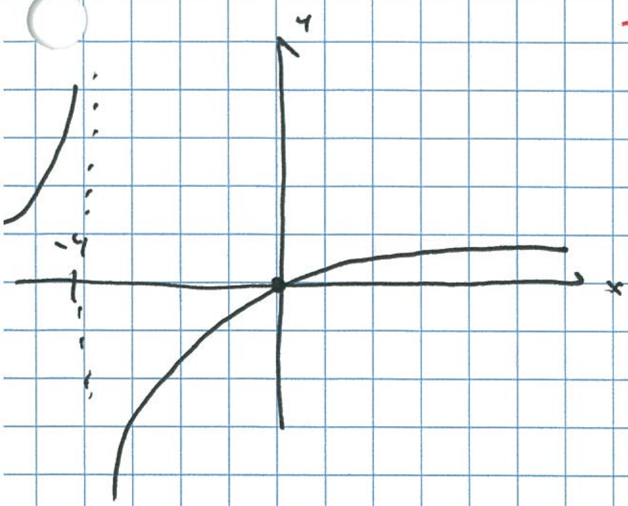
x	f
1	5
2	6
3	-2
4	3

$\Rightarrow$

y	f^{-1}
-2	3
3	4
5	1
6	2

Time permitting

eg/ $y = \frac{x}{4+x} = \text{Leon}(x)$



- 1-1 ✓ is a fcn.

Domain: $(-\infty, -4) \cup (-4, \infty)$

range $(-\infty, 1) \cup (1, \infty)$

$y = \frac{x}{4+x} \Rightarrow y(4+x) = x$

$4y + xy = x$

$4y = x - xy$

$4y = x(1-y)$

$\Rightarrow x = \frac{4y}{1-y} \quad y \neq 1.$

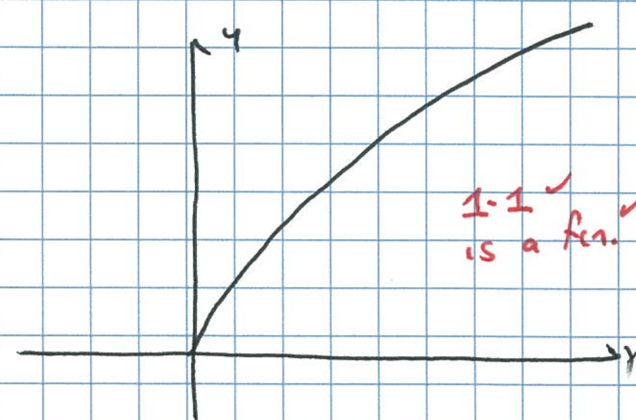
$\Rightarrow \text{Leon}^{-1}(y) = \frac{4y}{1-y}.$

Time permitting

eg/ $y = \sqrt{x} + x$

D: $[0, \infty)$

R: $[0, \infty)$



1-1 ✓

is a fcn. ✓

$y = \sqrt{x} + x$ solve for x:

$x + \sqrt{x} - y = 0$ let $u = \sqrt{x}$

$u^2 + u - y = 0$

$u = \frac{-1 \pm \sqrt{1^2 + 4y}}{2}$ umm... which one? + or − ??

well... ~~Dom.~~ Range of ~~Fn.~~ $\sqrt{x} = u$ is $[0, \infty)$ so range of this must be $[0, \infty)$ so take the + ver.

$u = \frac{-1 + \sqrt{1+4y}}{2}$ so $\sqrt{x} = \frac{-1 + \sqrt{1+4y}}{2}$

$\Rightarrow x = \left[\frac{-1 + \sqrt{1+4y}}{2}\right]^2$ inv. fcn.