Section 3.3   ALEKS

natural log + exp fcns.
using logs to solve problems
properties of log fcns.
word problems.

exponential functions: natural logs:

$y = a^x$ $y = e^x$

$e \approx 2.718\ldots$

Inv. of a function..

$y = f(x)$ then $x = f^{-1}(y)$

and $x = f^{-1}(f(x))$

and $y = f(f^{-1}(y))$

Def: The inv. of $\exp(x) = e^x$ is the natural log, $x = \ln(y)$.

so, if $y = e^x$ then $x = \ln(y)$ (and vice versa)

and $\ln(e^x) = x$

and $e^{\ln(x)} = x$.

ex/   if $y = e^3$ then $\ln(y) = 3$

The # you have to raise e to, to get y.

ex/ $y = 5^x$ recall $\ln(a^r) = r\ln(a)$

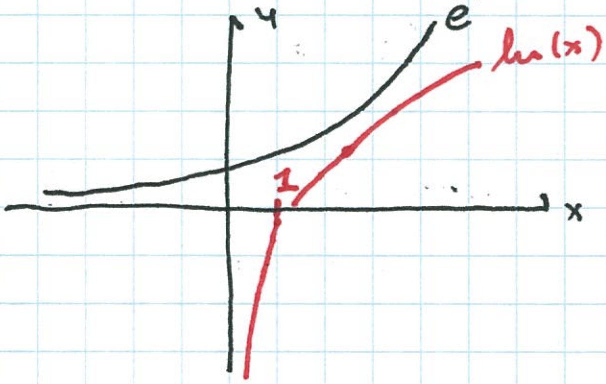
so $\ln(y) = \ln(5^x)$

$\ln(y) = x\ln(5)$

or $x = \ln(y)/\ln(5)$

do onto the left as you do onto the right (and ln is 1-1).

The graph of $\ln(x)$



dom: $(0,\infty)$
range: $(-\infty,\infty)$
and is 1-1.

consistent w/ e^x. (!

why is $\ln(1)=0$? coz $e^0=1$.

ex/ $\ln(3x+1) = 2$ exponentiate both sides.

$e^{\ln(3x+1)} = e^2$

$3x+1 = e^2 \Rightarrow x = \frac{e^2-1}{3}$

by the way, there are many ways to do exponentials

ex: $4 = 5^x$ ← exponent. ← base

e is a good base, but we can use any base.

For example many scientists use base 10.

we could work in base 5,

$4 = 5^x$ * you raise 5 to, to get 4.

$\Rightarrow \log_5(4) = x$ ← base

It just happens that base e has some nice properties w.r.t. its ~~change~~ average rate of change

If $y = 2^x$ then $x = \log_2(y)$

For ex/ $8 = 2^3$ or $3 = \log_2(8)$

$16 = 2^4$ or $4 = \log_2(16)$

ex/ $\log_8(3x+1) = 2 + \log_8(x)$

$$\Rightarrow \quad 8^{\log_8(3x+1)} = 8^{2+\log_8(x)}$$

$$3x+1 = 8^2 \cdot 8^{\log_8(x)}$$

$$3x+1 = 64(x)$$

$$1 = 61(x) \quad \text{or} \quad x = 1/61.$$

ex/ $\ln(x^2) = 4$

$e^{\ln(x^2)} = e^4$ so $x^2 = e^4$

and $x = \pm\sqrt{e^4} = \pm e^2$

which one?

either / both! cuz $x^2 = e^4$ and $\ln(e^4) = 4$ ✓

This needs to be checked, and we have to be careful!

why? $\log_a(x) = b \iff a^b = x$

but $\underline{\underline{a > 0}}$ so $a^b > 0$

strict.

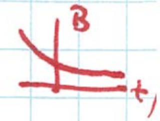
$\Rightarrow x > 0.$

Recall: dom of $\log_a(x)$ is $(0, \infty)$

range of $\log_a(x)$ is $(-\infty, \infty)$.

ex/ A radioactive isotope has a decay rate of 0.05. What is the half-life?

i.e. How long till half goes away?

Amount $= B_0 e^{-rt} = B_0 e^{-.05t}$ (decay! 

Problem: we do not know B_0

Answer: we do not care!

$$A(0) = B_0 e^0 = B_0$$

when is $A(T) = \frac{1}{2} A(0) = \frac{1}{2} B_0$?

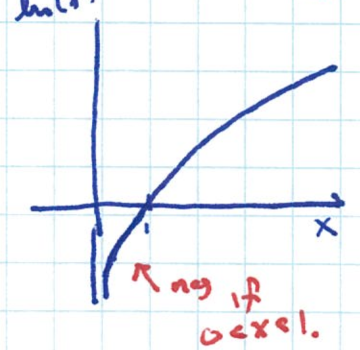
$$B_0 e^{-.05T} = \frac{1}{2} B_0$$ divide by B_0

$$e^{-.05T} = 1/2$$

$$\ln(e^{-.05T}) = \ln(1/2)$$ note $\ln(1/2) < 0$

$$-.05T = \ln(1/2)$$

$$T = -\frac{1}{.05} \ln(1/2)$$ This is a pos #



Recall Avg. Rate of change/Diff. quotient.

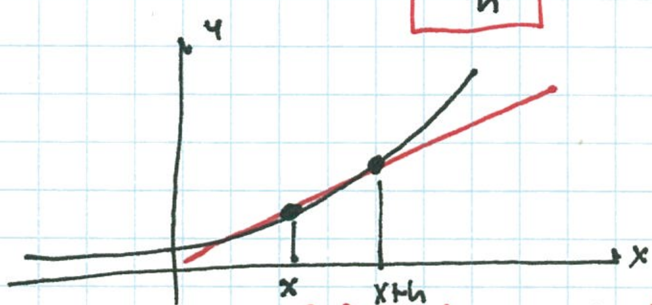
let $\exp(x) = e^x$

so ~~Avg~~ the Diff. Quotient

$$= \frac{\exp(x+h) - \exp(x)}{h}$$ (This is the def.)

$$= \frac{e^{x+h} - e^x}{h} = \frac{e^x \cdot e^h - e^x}{h}$$

$$= e^x \boxed{\frac{e^h - 1}{h}}$$ what's up w/ this?



let's shrink this...

h	$\frac{e^h-1}{h}$
1	1.7183
[illegible]	1.2974
1/4	1.1361
1/8	1.0652
1/16	1.0315
$\frac{1}{128}$	1.0039 (!)

for kicks...

h	$\frac{5^h-1}{h}$
1	4
1/2	2.472
1/4	1.981
1/8	1.7828
1/16	1.693
$\frac{1}{128}$	1.6196 (?)

$e^{1.6196} \approx 5.05$ (!!!)

There is something about e!..

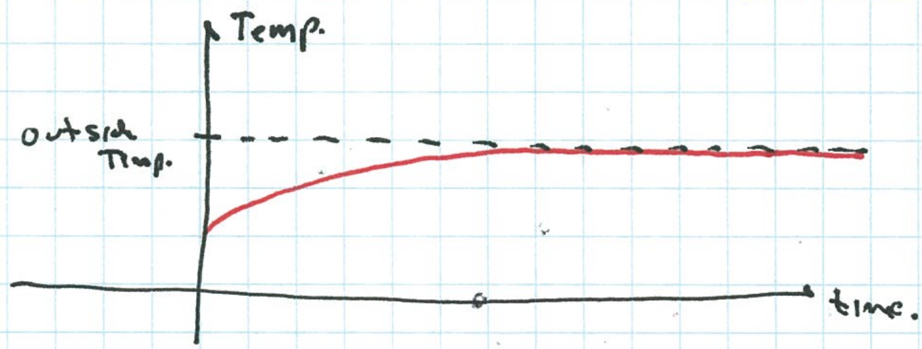
ex/ Newton's Law of cooling.

The rate that temp. changes is prop. to the temperature difference of the object and its surroundings.

we get

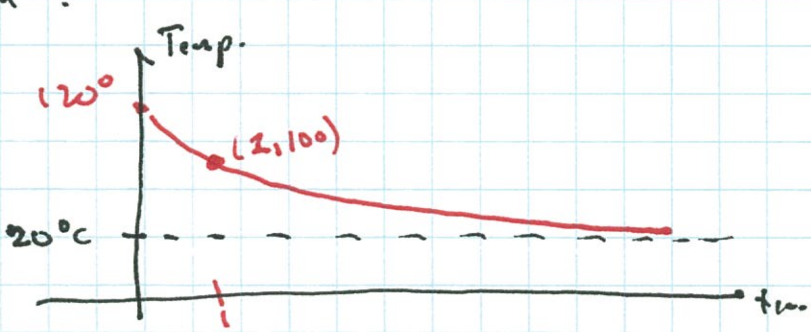
$$\text{Temp}(t) = \text{outside Temp} + B_0 e^{-rt}$$

(This is a calc. problem)



cuz B_0e^{-rt} gets close to zero.

An object has an initial temp of 120°C. It is placed outside when the temp. is 20°C. After 1 hour the temp. is 100°C. What is its temp. at any time?



(5)

$$\text{Temp}(t) = 20 + Be^{-rt}$$

(A) $120 = \text{Temp}(0) = 20 + Be^{0} = 20 + B$

(B) $100 = \text{Temp}(1) = 20 + Be^{-r\cdot 1}$

1st write down unknowns decide how to proceed.

use A to solve for B

$$120 = 20 + B \quad \text{so} \quad B = 100.$$

$$100 = 20 + 100e^{-r}$$

$$80 = 100e^{-r}$$

$$\frac{80}{100} = \frac{4}{5} = e^{-r}$$

$$\ln(4/5) = -r \quad \text{so} \quad r = -\ln(4/5)$$

$$\text{Temp}(t) = 20 + 100\, e^{t \ln(4/5)} \qquad (\ln(4/5) < 0).$$

Time permitting.

ex/ The population of a country is estimated to be

$$\text{Pop}(t) = Ae^{t(.0002)}$$

where $t=0$ is 1970 and t is in years. How long will it take to double the pop?

$$\text{Pop}(T + \text{double time}) = \text{Pop}(T) * 2$$

or $\text{Pop}(T + d) = \text{Pop}(T) * 2.$

$$Ae^{(T+d)(.0002)} = 2Ae^{T(.0002)}$$

$$\cancel{A}e^{T.0002 + d.0002} = 2\cancel{A}e^{T(.0002)}$$

$$\cancel{e^{T.0002}} \cdot e^{d\cdot.0002} = 2\cancel{e^{T.0002}}$$

$$e^{d\cdot.0002} = 2$$

$$\ln(e^{d\cdot.0002}) = \ln(2)$$

$$d \cdot .0002 = \ln(2)$$

$$d = \underbrace{\frac{1}{.0002}\ln(2)}_{\text{pop. doubles every 3,470 years.}} \approx 3,465.7 \text{ years.}$$

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Time Permitting

ex/ The amount of ~~drug~~ drug in a patient's blood stream is estimated to be

$$\text{Amount}(t) = .2e^{-rt} \text{ mg.} \qquad (r>0)$$

It takes 3 hours to go from .2mg to .15 mg. what is r? (assume @ t=0 there is .2mg in the patient's system).

$$\text{Amount}(0) = .2e^{-r(0)} = .2 \checkmark \qquad \text{assumption ok.}$$

$$\text{Amount}(3) = .15 = .2e^{-r\cdot 3} \qquad 1 \text{ eqn} / 1 \text{ unknowns}$$

$$\frac{.15}{.2} = e^{-3r}$$

$$\ln(.15/.2) = \ln(e^{-3r}) = -3r$$

$$r = \underbrace{-\tfrac{1}{3}}_{\text{neg}} \underbrace{\ln(.15/.2)}_{\text{neg}} \Rightarrow r>0$$