3.2 exponential functions — ALEKS HW

Def of exponential fcns
graphs of exp. fcns.
base e
compound interest
word problems.

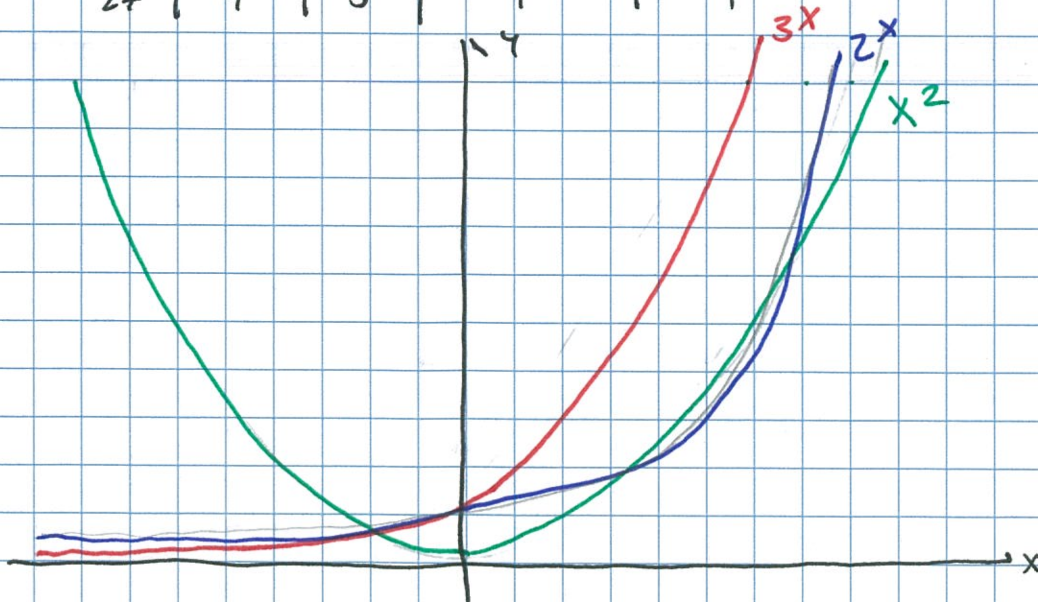
So far we have done this
$$1,\ x,\ x\cdot x = x^2,\ x\cdot x\cdot x = x^3, \dots$$

What about the other way?

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

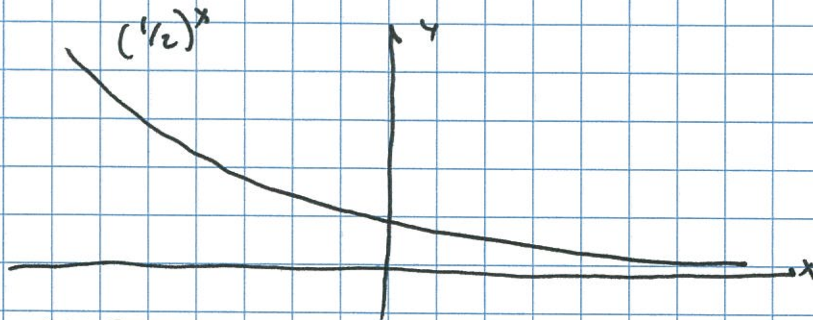
what about 3^x?

x	-3	-2	-1	0	1	2	3
3^x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27



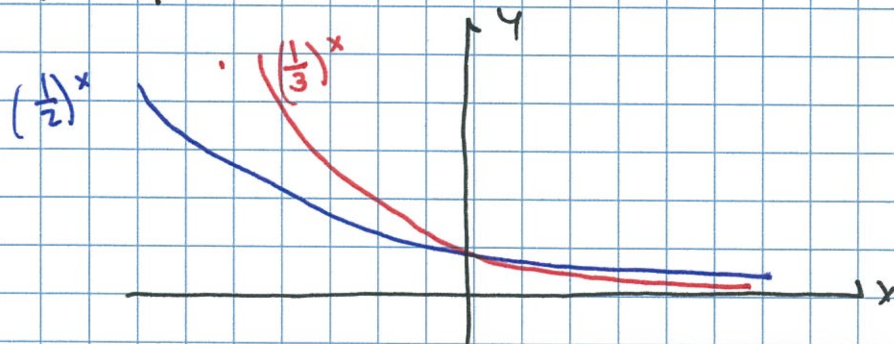
What about $(1/2)^x$?

x	-3	-2	-1	0	1	2	3
$\left(\frac{1}{2}\right)^x$	8	4	2	1	$1/2$	$1/4$	$1/8$



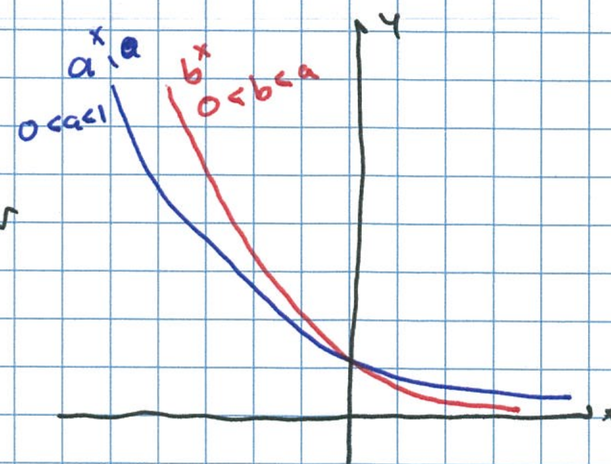
and $(1/3)^x$?

x	-3	-2	-1	0	1	2	3
$\left(\frac{1}{3}\right)^x$	27	9	3	1	$1/3$	$1/9$	$1/27$



In general, a "simple exponential" can be written as $y = a^x$ ($a \neq 1$ is a constant)

or



<u>note</u> $f(x) = a^x$ is a 1-1 function.

$\Rightarrow$ it has an inverse.

so if $a^x = a^c$ then $x = c$.

ex/ $f(x) = 4^x$

$g(x) = 4^{2x+1}$

det when $f(x) = g(x)$

$$4^x = 4^{2x+1}$$

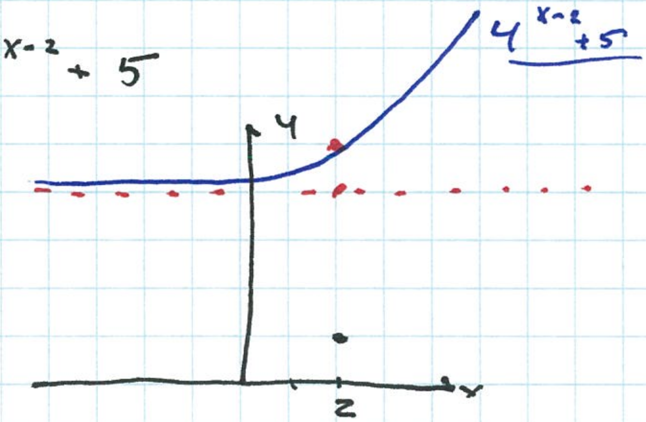
both are 4° (so base = 4)

$$\Rightarrow \quad x = 2x+1$$

or $\underline{x = -1}$

ex/ graph $Amy(x) = 4^{x-2} + 5$

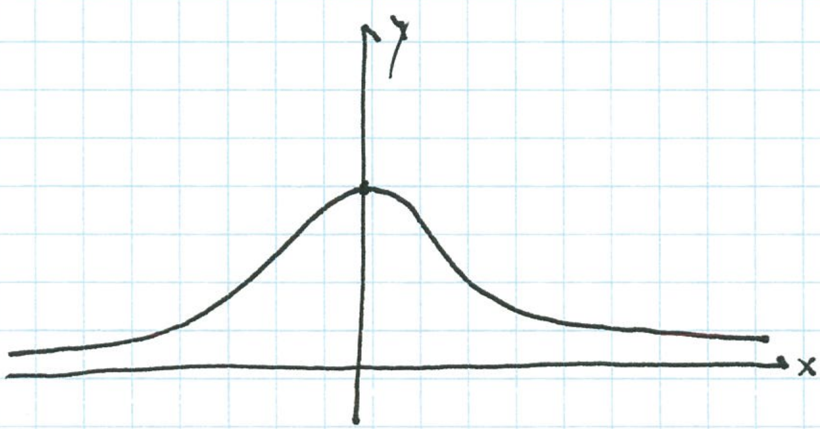
right 2
up. 5



ex/ graph $Cindy(x) = 2^{-x^2}$

x	-3	-2	-1	0	1	2	3
2^{-x^2}	$\frac{1}{2^9}$	$\frac{1}{4}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2^9}$

It is an even fcn.
as x gets big fcn. gets close to zero.



used in probability.

<u>Note</u> Banks pay compounded interest.
If you put 100\$ in a bank account
then in 1 month you earn ~~interest~~ interest.
Next month you do the same but use
the <u>new</u> balance!

<u>But</u> they calculate the interest in a way to confuse you!

ex) If int. rate is advertised as being "5% annual compounded monthly" then...

12 months = 1 annual so

rate = .05/12.

(this was not my idea.)

So what happens?

B_0 = initial balance suppose

$B_0 = 100$

1 month balance $= B_1 = 100 + 100 \cdot \frac{.05}{12} = 100\left(1 + \frac{.05}{12}\right)$

2 month balance $= B_2 = B_1 + B_1 \cdot \frac{.05}{12} = B_1\left(1+\frac{.05}{12}\right) = 100\left(1+\frac{.05}{12}\right)^2$

3 month Balance $= B_3 = B_2 + B_2 \cdot \frac{.05}{12} = B_2\left(1+\frac{.05}{12}\right) = 100\left(1+\frac{.05}{12}\right)^3$

$\vdots$

m^{th} month Balance $= B_m = 100 \cdot \big(\underbrace{1 + \tfrac{.05}{12}}_{a>1 \text{ growth.}}\big)^m$

we assume you can do algebra w/ exponentials!

In ~~general~~ general (Sorry but this banker stuff)

r = <u>annual</u> interest rate

n = # of periods per year (ex $n=12$ = monthly)

t = # of years

P_0 = original principal

current amount = "Balance" = ?

(assume <u>no</u> withdrawals - yet)

current balance $= P_0 (1 + r/n)^{nt}$

In the prev. example $r = .05$, $n = 12$; $P_0 = 100$, $t = \frac{1}{4}$ (3 months)

ex/ $500^{\$}$ is put in an account that ~~earns~~ earns 1.3% annual int. compounded weekly. How much will be in the account after 18 months

$r = .013$
$n = 52$
$t = 3/2$
$P_0 = 500$

$$\left.\begin{array}{l} r=.013 \\ n=52 \\ t=3/2 \\ P_0=500 \end{array}\right\} \text{balance} = \left(1 + \frac{.013}{52}\right)^{52 \cdot 3/2} \cdot 500 \approx 509.84 \,\$$$

Radioactive Decay

The half-life of an isotope is the time it takes for 1/2 of the amount to decay.

For example: Start w/ 3 kgs of an isotope. The half-life is 200 days. How much is left after 600 days?

Time progresses linearly

$@\ t = 0 \Rightarrow$ Amount $= A_0 = 3$ kgs

$@\ t = 200$ Amount $= A_1 = \frac{1}{2} A_0 = \frac{1}{2} \cdot 3$

$@\ t = 400$ Amount $= A_2 = \frac{1}{2} A_1 = (\frac{1}{2})^2 \cdot 3$

$@\ t = 600$ Amount $= A_3 = \frac{1}{2} A_2 = (1/2)^3 \cdot 3$

Amount cut in 1/2 each time.

In general, an isotope has a half-life of M days

P_0 = initial amount

Amount @ time $t = A(t)$

$A(0) = P_0$

$A(m) = \frac{1}{2} P_0$

$A(2m) = (\frac{1}{2}) \cdot \frac{1}{2} P_0 = \frac{1}{2^2} P_0$

$A(3m) = \frac{1}{2} (\frac{1}{2^2} P_0) = \frac{1}{2^3} P_0$

$\vdots$

$A(n \cdot m) = \frac{1}{2^n} P_0$ $\qquad$ n = # of time period whose length is m days

$\Rightarrow t = n \cdot m$ or $n = t/m$

so $A(t) = (\frac{1}{2})^{t/m} \cdot P_0$.

exponential decay!
$a = 1/2 < 1$

ex/ if the half-life of an isotope is 800 days and we start w/ 10 kg then

$$A(t) = (1/2)^{t/800} \cdot 10$$

How long till $\frac{1}{16}$ the original amount?

$$\frac{1}{16} A(0) = \frac{1}{16} \cdot 10 = (\frac{1}{2})^{t/800} \cdot 10$$

$$\Rightarrow \frac{1}{16} = (\frac{1}{2})^{t/800}$$

$$(\frac{1}{2})^4 = (1/2)^{t/800} \Rightarrow 4 = t/800$$

common bases!

so $t = 3200$ days.

Time Permitting

ex/ An isotope has a half-life of 1200 days. How long does it take to reduce the amount to 1/8 its original amount?

$$A(t) = (1/2)^{t/1200} \cdot P_0$$

when is $A(t) = \frac{1}{8} P_0 = (1/2)^{t/1200} P_0$

$$\frac{1}{8} = \left(\frac{1}{2}\right)^{t/1200}$$

$$\left(\frac{1}{2}\right)^3 = (1/2)^{t/1200} \Rightarrow 3 = t/1200$$

or $t = 3600$ days.

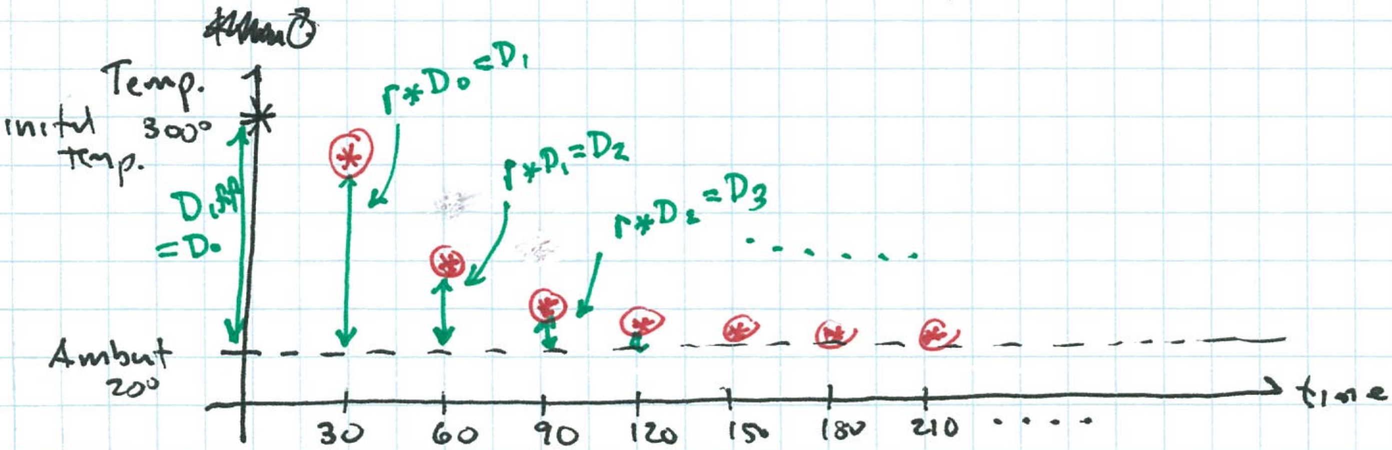
Time Permitting.

Newton's Law of Cooling.

An object has some temp. and is placed in a location w/ some constant ambient temperature. The object will cool/warm to match the surrounding temp. Over each constant time period the difference between the object's Temp. and the ambient temp. will be reduced by the same fraction.

Newton's Law of cooling.

ex/ A clay brick is fired to 300°C, and it is placed outside where the temp. is a constant 20°C. After 30 min the temp. is 275°C. What will the temp. be at any Time?



From $t=0$ to $t=30$ min,

Temp = 300 °C to 275 °C but Ambient = 20

so temp diff

goes from 280 °C to 255 °C

or $255 = r * 280$

$\Rightarrow r = \frac{255}{280}$

so $Diff_0 = 280°C$

$Diff_1 = r * Diff_0 = \frac{255}{280} \cdot (280)$

$Diff_2 = r * Diff_1 = \left(\frac{255}{280}\right)\left(\frac{255}{280}\right) 280 = \left(\frac{255}{280}\right)^2 \cdot 280$

$Diff_3 = r * Diff_2 = \left(\frac{255}{280}\right)\left(\frac{255}{280}\right)^2 280 = \left(\frac{255}{280}\right)^3 \cdot 280$

$\vdots$

$Diff_n = r * Diff_{n-1} = \left(\frac{255}{280}\right)^n \cdot 280$

so Temp @ time t = Ambient temp + $\left(\frac{255}{280}\right)^{n} \cdot \underline{\underline{280}}$

original temp diff.

n = # time periods

so $t = 30 \cdot n$

or $n = t/30$

$$\text{Temp @ time } t = \underbrace{20°C}_{\text{ambient temp.}} + \left(\frac{255}{280}\right)^{t/30} \cdot 280$$

r time/time period   original temp diff