Sect 3.2 (continued) ALEKS

The number e.
natural exponential
modeling w/ exp. functions.

So we did this - the "exponential function"

$$f(x) = \underline{c} \cdot a^x$$

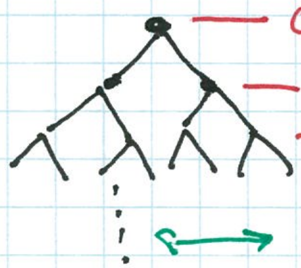
c ← constant
a ← constant.

ex/ $f(x) = 3 \cdot 2^x$

ex/ A bacterial colony doubles in size every 4 hours. It initially contains 1,000 microbes.

doubles ← geometric inc.

Time inc. linearly. → 4 hours



— 0 hours
— 4 hours.
— 8 hours
— 16 hours.

const. rate of inc. in time.

↔ grows faster each time step.

$$B(0) = 1000$$
$$B(4) = 2000 = 2 \cdot B(0) = 2 \times 1000$$
$$B(8) = 2 \cdot B(4) = 4{,}000 = 2^2 \times 1000$$
$$B(12) = 2 \cdot B(8) = 8{,}000 = 2^3 \times 1000$$
$$\vdots$$
$$B(n \cdot 4) = 2^n \cdot 1000 \qquad \text{but } t = n \cdot 4 \text{ so } n = t/4$$
$$B(t) = 2^{t/4} \cdot 1{,}000 .$$

what is so special about doubling?
nothing - it is just easier to introduce the idea!

ex/ Suppose we have P bacteria initially.

After 1 scaled unit time they increase by a factor of r.

so $B(0) = P$

$B(1) = P + rP = (1+r)P$

$B(2) = B(1) + rB(1) = (1+r)B(1) = (1+r)^2 P$

$B(3) = B(2) + rB(2) = (1+r)B(2) = (1+r)^3 P$

$\vdots$

$B(n \text{ units}) = (1+r)^n P$

time $= t = n \cdot r$

coz $\boxed{r} = \text{inc.}/\text{time}$

so $r = t/n$

$\Rightarrow \quad B(t) = (1 + t/n)^n \cdot P$

what happens for diff't n? if $t = 1$ ...

n	$(1 + 1/n)^n$
1	2
10	2.59
100	2.705
1000	2.717
10,000	2.7181
100,000	2.7183
1,000,000	2.71828
$\vdots$	$\vdots$

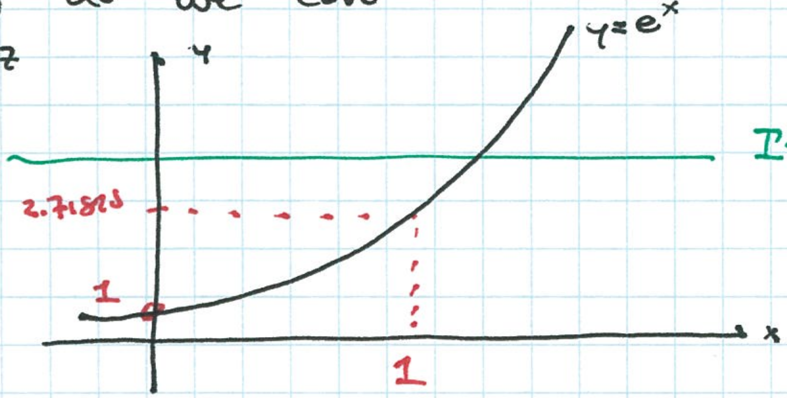
as n gets biggerer and biggerer more this number gets close to 2.71828...

we call this # e.
coz Euler got it first.

Note: we call it "e" coz Euler was the first person to torture his students with it. In math we name things after the first person to inflict pain with an idea.

why do we care?

cuz

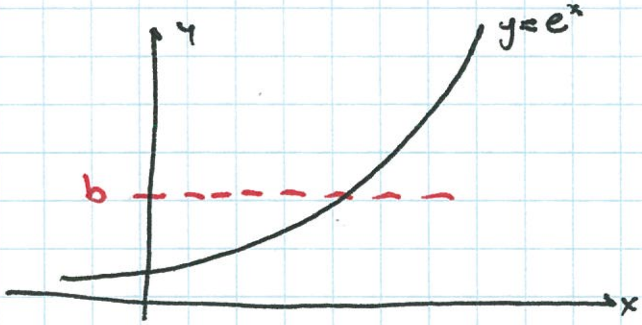


It is 1-1.
so if $3 = e^r$
There is only one r that satisfies the relationship.

note $3^x = (e^r)^x = e^{r \cdot x}$

no need for fancy bases!

more generally



if $b = e^r$
There is only one r that satisfies this given b.

so... $b^x = (e^r)^x = e^{rx}$

no need for base b. Just use e.

We can write any exponential in terms of e.

Definition: Exponential growth is

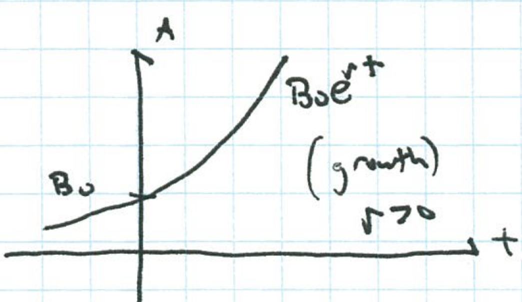
$$A(t) = B_0 e^{rt}$$

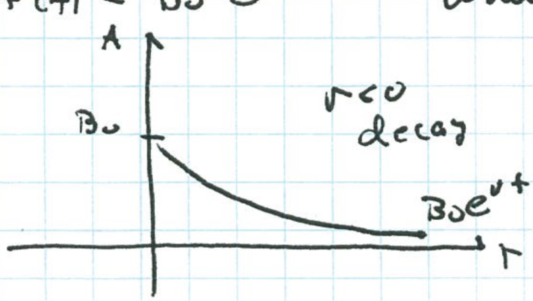
where $r > 0$.

Definition: Exponential decay is

$$A(t) = B_0 e^{rt}$$

where $r < 0$.





r is called the growth (decay) rate.

ex/ The decay rate of carbon 14 is approx. $.000121$ per year. What percentage of carbon 14 will decay from a sample after 10 years?

Assume units = years?

$$\text{Amount} = A_0 e^{-.000121t}$$ decay!

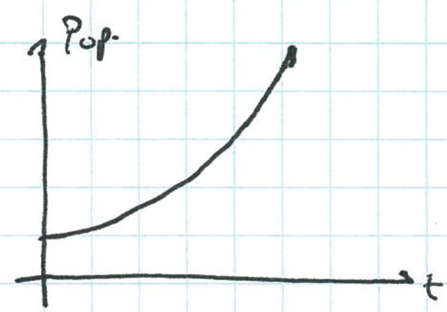
$$\text{Initial Amount} = A_0 e^0 = A_0$$

$\Rightarrow$ % after 10 years $= \dfrac{\cancel{A_0} e^{-.000121 \cdot 10}}{\cancel{A_0}} \approx .9988$ or 99.88%

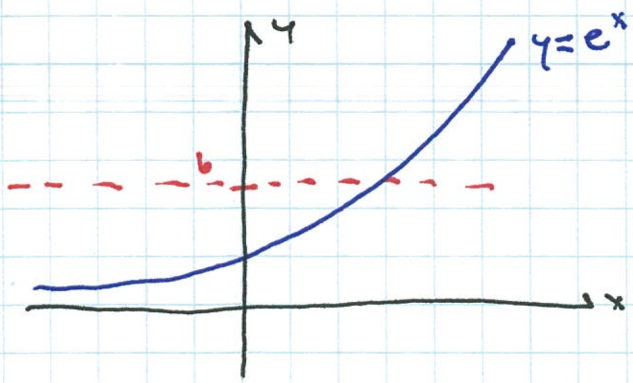
ex/ The growth rate of a pop. of animals is 2.1% per year. If the population is 2,500 individuals in the 1st observation how many will be present in 4 years?

$$P(t) = 2{,}500\, e^{+.021t}$$ growth!

$$P(4) = 2{,}500\, e^{.021(4)}$$
$$\approx 2{,}719$$



Recall - we said that e^x is 1-1.



given $b = e^x$ what is x?

A: it's complicated.

we need some notation and time to explore

let $y = e^x = \underbrace{\exp(x)}_{\text{its full name}}$ $\quad$ $\exp(x) = e^x$ is the simplest exponential.

if $y = \exp(x)$ then $x = \underbrace{\text{inv. of the exp}}_{\text{This name sucks.}}(y)$.

Historical note: we are doing this backwords. The inverse predates the exp. function. We started w/ the exponential cuz it is easier to motivate.

we will call the inverse the logarithm™ or $\ln(y)$ for short

Logos – reason and logic
+ "arithmes" – number

so, yeah, the notation is awkward, but here we ~~are~~ are.

so, we have special properties of exponentials.

$$e^a \cdot e^b = e^{a+b} \qquad \text{ex: } e^3 \cdot e^{2.5} = e^{5.5}$$

$$(e^a)^b = e^{ab} \qquad \text{ex: } (e^{2.5})^3 = e^{7.5}$$

$$e^a / e^b = e^{a-b} \qquad \text{ex: } e^3/e^{2.5} = e^{.5}$$

what about logarithms?

suppose that $y = \exp(x) = e^x$ $\quad$ equivalent statements.
$\iff \ln(y) = x$

and $\exp(\ln(x)) = x$ $\quad$ def. of inv. functions.
and $\ln(\exp(x)) = x$

so... $y = e^x \iff x = \ln(y)$

or $a = e^b \iff b = \ln(a)$

or $c = e^d \iff d = \ln(c)$

assume a, b, c, d are all constants.

well...

$$a \cdot c = e^b \cdot e^d = e^{b+d} \Rightarrow \ln(ac) = b+d$$
$$= \ln(a) + \ln(c)$$

so... $\ln(15) = \ln(3 \cdot 5) = \ln(3) + \ln(5)$!

and $e^3 \cdot e^5 = e^8$

also $a = e^b$ so $b = \ln(a)$

$$a^r = \cancel{(e^b)^r} \Rightarrow \ln(a^r) = \ln(e^{rb}) = rb$$
$$= (e^b)^r = e^{rb}$$

$$\ln(a^r) = r\ln(a)$$

wot? $\ln(4^2) = 2\ln(4)$

$\ln(3^8) = 8\ln(3)$

and $\ln(5 \cdot 6^7) = \ln(5) + \ln(6^7)$

$= \ln(5) + 7\ln(6)$

note $2 \cdot 4^x = 3 \cdot 6^{3x}$

$$\Rightarrow \ln(2 \cdot 4^x) = \ln(3 \cdot 6^{3x})$$
$$\ln(2) + \ln(4^x) = \ln(3) + \ln(6^{3x})$$
$$\ln(2) + x\ln(4) = \ln(3) + 3x\ln(6)$$
$$x\ln(4) - 3x\ln(6) = \ln(3) - \ln(2)$$
$$x = \frac{\ln(3) - \ln(2)}{\ln(4) - 3\ln(6)}$$

!

Time Permitting

ex/ "simplify" the expression.

$$(e^{2x}-e^{-2x})^2 - (e^{2x}+e^{-2x})^2$$

$$= (e^{2x})^2 - 2e^{2x}e^{-2x} + (e^{-2x})^2$$

$$- \left[(e^{2x})^2 + 2e^{2x}e^{-2x} + (e^{-2x})^2\right]$$

$$= e^{4x} - 2e^0 + e^{-4x} - (e^{4x} + 2e^0 + e^{-4x})$$

$$= -4.$$

Time Permitting

ex/ $$\frac{1}{e^x+e^{-x}} + \frac{1}{e^x-e^{-x}}$$ bring together into one fraction.

$$= \frac{e^x-e^{-x}}{(e^x+e^{-x})(e^x-e^{-x})} + \frac{e^x+e^{-x}}{(e^x+e^{-x})(e^x-e^{-x})}$$

$$= \frac{e^x-e^{-x}}{(e^x)^2 + e^{-x}e^x - e^{-x}e^x - (e^{-x})^2} + \frac{e^x+e^{-x}}{(e^x)^2 - e^{-x}e^x - e^xe^{-x} - (e^{-x})^2}$$

$$= \frac{e^x-e^{-x}}{e^{2x}-e^{-2x}} + \frac{e^x+e^{-x}}{e^{2x}-e^{-2x}} = \frac{2e^x}{e^{2x}-e^{-2x}}.$$

Time Permitting

ex/ $e^x = (e^x-1)^2$ solve for x.

$$e^x = (e^x)^2 - 2e^x + 1$$

$$0 = e^{2x} - 3e^x + 1 \qquad \text{let } u = e^x \Rightarrow u^2 = e^{2x}$$

$$u^2 - 3u + 1 = 0 \qquad u = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

so $e^x = \frac{3+\sqrt{5}}{2} > 0$ ✓ or $e^x = \frac{3-\sqrt{5}}{2} > 0$ ✓ ok!

$$x = \ln\left(\frac{3+\sqrt{5}}{2}\right) \quad \text{or} \quad x = \ln\left(\frac{3-\sqrt{5}}{2}\right)$$

Time Permitting

ex/

$$(e^x + 1/2)^2 = 1$$

either $e^x + 1/2 = 1$ or $e^x + 1/2 = -1$

$e^x = 1/2$ $\qquad$ $\underbrace{e^x = -3/2}$ not in range of $\exp(\cdot)$

$\Rightarrow x = \ln(1/2)$ or Nope!

so $x = \ln(1/2)$

Time Permitting

ex/

$\sqrt{e^x + 1} = e^x$ solve for x

$e^x + 1 = (e^x)^2 = e^{2x}$

$e^{2x} - e^x - 1 = 0$ let $u = e^x$

$u^2 - u - 1 = 0$

$$u = \frac{+1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$u = e^x = \frac{1+\sqrt{5}}{2}$ or $u = e^x = \frac{1-\sqrt{5}}{2} < 0$ not possible.

$$x = \ln\left(\frac{1+\sqrt{5}}{2}\right)$$

Time Permitting

ex/ suppose $y = 3e^{4x}$ what is x when $y = 5$?

$5 = 3e^{4x}$

$5/3 = e^{4x}$

$\ln(5/3) = 4x$ so $x = \frac{1}{4}\ln(5/3)$

check work: $3e^{4 \cdot \frac{1}{4}\ln(5/3)} = 3e^{\ln(5/3)} = 3 \cdot \frac{5}{3} = 5$ ✓

note: $3e^{t\ln(1/2)} = 3(e^{\ln(1/2)})^t = 3(\frac{1}{2})^t$

so $y = A_0 e^{rt}$

if $r = \ln(b)$

$= A_0 e^{t\ln(b)}$

$= A_0 (e^{\ln(b)})^t = A_0 b^t$

any base can be written in terms of e!