Section 3.6 ALEKS

Math Models + exponentials
Growth vs. decay
Logistic growth
solving system of equations w/ exp + log terms.

- Algebra still matters!
  - give equation identify variables to solve for.
  - get them all on one side.

addition ⟷ subtract
mult. ⟷ divide
exp. ⟷ log.

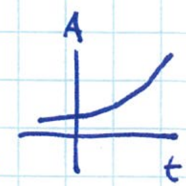
- what/when is an exp. model?
- Logistic growth?

Recall Problem solving (It takes a village to solve a problem! do not just think of magic inspiration!)

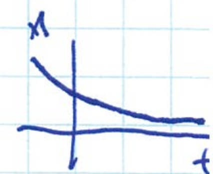
0) Read! (parse)
1) draw picture / graph
2) identify variables + knowns
3) explore relationships
4) plan
5) execute
6) check work.

Exponential Models

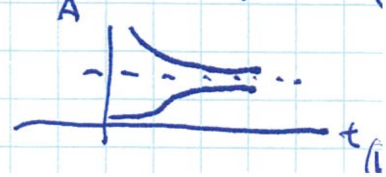
Growth — Interest / bank acct / simple bio model



Decay — radioactivity / degradation of things



Logistic — balance of both!
ex: bio systems.



ex) The pop. of a ~~city~~ country in 2000 is 10.5 million. In 2010 the pop. is 10.9 mill. What will the pop. be in 2012?

0) Read.

① 

expect growth!

② let $t=0$ @ 2000. t in years.

$A(0) = 10.5$ mill.

$A(10) = 10.9$ mill

$A(12) = ?$

③ $P(t) = Ae^{rt}$ simple (growth) $r>0$

$P(0) = 10.5 = Ae^{r(0)}$

$P(10) = 10.9 = Ae^{r(10)}$

$P(12) = Ae^{12r} = ?$

④ 2 eqn's 2 unknowns – soln for A + r then plug in to get $P(12)$.

⑤ $10.5 = Ae^0 = A \Rightarrow A = 10.5$

$10.9 = Ae^{10r} \Rightarrow 10.9 = 10.5\,e^{10r}$

$\frac{10.9}{10.5} = e^{10r}$

$\ln(10.9/10.5) = \ln(e^{10r}) = 10r$

$r = \frac{1}{10}\ln(10.9/10.5)$

$P(12) = 10.5\,e^{12 \cdot \frac{1}{10}\ln(10.9/10.5)}$

≈ 10.98 mill.

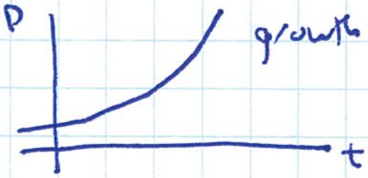
⑥ check work!

Q) is this reasonable beyond 2050?

growth w/o bound!

ex/ A company will put money into an account that will earn 1.7% annual interest compounded monthly. How long will it take for the principle to inc. by half?

0) Read.

① 

growth

② $r = .017$, $n = 12$, $t = ?$, $P_0 = ?$

③ $\text{Principle}(t) = P_0\left(1 + .017/12\right)^{12t}$

$\text{Principle}(0) = P_0\left(1 + .017/12\right)^{0} = P_0$

$\text{Principle}(T) = 1.5 \ \text{principle}(0) = 1.5 P_0$

④ Set $\text{Principle}(T) = 1.5P_0$ & solve for T.

⑤ $$1.5 \cancel{P_0} = \cancel{P_0}\left(1 + .017/12\right)^{12t}$$

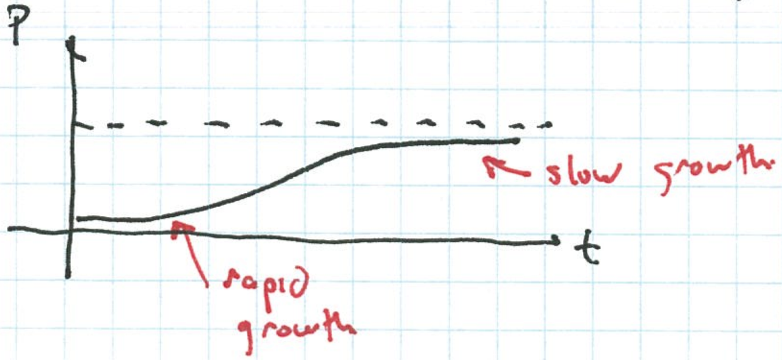
$$1.5 = \left(1 + .017/12\right)^{12t}$$

$$\ln(1.5) = \ln\left(\left(1 + .017/12\right)^{12T}\right) = 12T \ln\left(1 + .017/12\right)$$

$$T = \frac{\ln(1.5)}{12 \ln\left(1 + .017/12\right)} \approx 23.9 \text{ years.}$$

⑥ ~~Check~~. check.

By the way... In some systems things try to reach an "equilibrium". Ex. Bacteria colony eventually level out due to lack of resources.



we have a model for this!

Logistic Eqn: $$y = \frac{C}{1 + \underbrace{ae^{-bt}}_{\text{gets close to zero.}}}$$

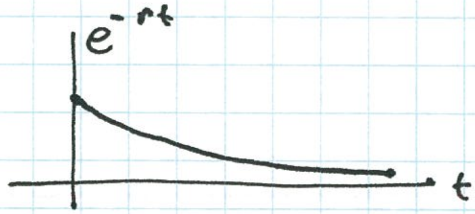
a, C is const. $b > 0$.

ex/ suppose The # of bacteria in a colony follows a logistic growth given by

$$N(t) = \frac{2{,}000{,}000}{1 - 2/3\, e^{-rt}}$$ (# of bacteria

after 10 hours there are 5,000,000 bacteria. How many will there be in 20 hours?

(0) • The idea

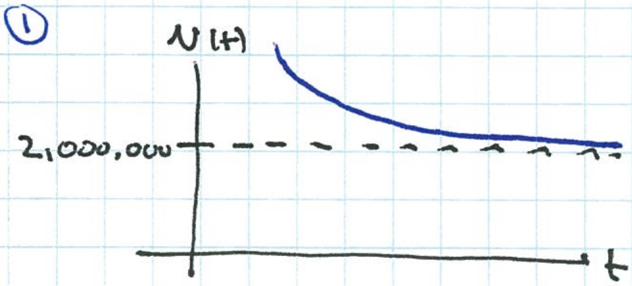


• so $-2/3\, e^{-rt}$ will get close to zero

$\Rightarrow$ • $1 - 2/3\, e^{-rt}$ will get close to 1

• so $N(t)$ gets close to 2,000,000.

(1)



(2)/(3) $N(0) \approx \dfrac{2{,}000{,}000}{1 - 2/3\, e^{0}} = 6{,}000{,}000$ not helpful :(

$N(10) = \dfrac{2{,}000{,}000}{1 - 2/3\, e^{-10r}} = 5{,}000{,}000$ Soln for r

$2{,}000{,}000 = 5{,}000{,}000\,(1 - 2/3\, e^{-10r})$

$\frac{2}{5} = 1 - 2/3\, e^{-10r}$

$-3/5 = -2/3\, e^{-10r}$

$\frac{9}{10} = e^{-10r}$

$\ln(9/10) = \ln(e^{-10r}) = -10r$ $\quad r = -\frac{1}{10}\ln(9/10)$

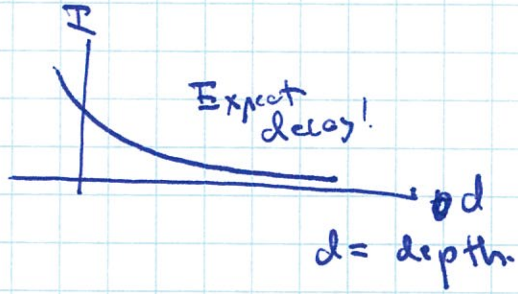
↑ neg!

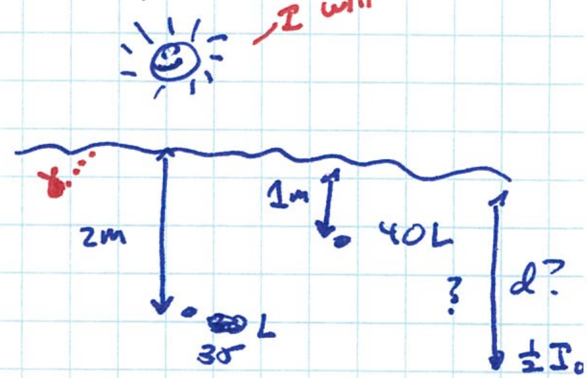
$N(20) = \dfrac{2{,}000{,}000}{1 - 2/3\, e^{-20\cdot(-1/10)\ln(9/10)}} \approx 4{,}347{,}826$

↑ That 6 bacterium matter!

ex/ It is estimated that the intensity of light decays as an exp. fcn of the depth of water. The intensity of light is measured @ 1m and 2m, and the intensity is 40 Lumens and 35 lumens respectively. How deep should you go so that the light ~~is~~ intensity is half the surface ~~int~~ intensity?

⓪ Read. Yikes!

① 



② I = intensity @ depth d.

③ $I = Ae^{-rd}$

$40 = Ae^{-r}$ Ⓘ

$35 = Ae^{-2r}$ Ⓘⓘ

$I(0) = Ae^{0} = A$

$I(d) = \frac{1}{2} I(0) = 1/2\, A.$ $\quad d = ?$

3 eqns 3 unknowns

④ well... I need $Ae^{-rd} = 1/2\, Ae^{0}$ or $e^{-rd} = 1/2$ Ⓘⓘⓘ

2 eqn's / 2 unknowns. Figure out A, r & then solve

⑤ $40 = Ae^{-r}$ so $A = 40e^{r}$

$35 = Ae^{-2r} = 40\, e^{r}e^{-2r} = 40e^{-r}$

$\frac{35}{40} = e^{-r}$

Ha Ha! I do not care about A!

$\ln(35/40) = \ln(e^{-r}) = -r$ $\qquad r = -\ln(35/40)$

$e^{-rd} = 1/2$

$e^{+\ln(35/40)d} = 1/2$

$\ln(e^{\ln(35/40)d}) = \ln(1/2)$

$\ln(35/40)\, d = \ln(1/2)$

$d = \frac{\ln(1/2)}{\ln(35/40)} \approx \underline{\underline{5.2m.}}$

⑥ check.

ex/ The hyperbolic tangent is defined to be

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Det. the inverse function.

ugh...

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$ (try to get e^x alone?)

$$y(e^x + e^{-x}) = e^x - e^{-x}$$

$$y e^x + y e^{-x} = e^x - e^{-x}$$

$$y e^x - e^x + \;= -e^{-x} - y e^{-x}$$

$$e^x [y-1] = e^{-x} [-1-y)$$

$$e^{2x} [y-1] = [-1-y) \Rightarrow e^{2x} = \frac{-1-y}{y-1}$$

$$\ln(e^{2x}) = 2x = \ln\left(\frac{-y-1}{y-1}\right)$$

$$x = \tfrac{1}{2} \ln\left((-y-1)/y-1\right)$$

need $\cdot \frac{-y-1}{y-1} > 0$

$\Rightarrow -\frac{y+1}{y-1} > 0$

$\frac{y+1}{y-1} < 0$

$\Rightarrow \underline{\underline{-1<y<1}}$

ex/ $\text{Ricky}(x) = \sqrt[3]{e^x + 1}$ det the inv. fcn.

$$y = \sqrt[3]{e^x + 1}$$

$$y^3 = e^x + 1$$

$$y^3 - 1 = e^x \Rightarrow \underline{\ln(y^3-1)} = \ln(e^x) = x.$$

dom. of Ricky : $(-\infty, \infty)$

range of Ricky : $(1, \infty)$

---

dom. of Ricky^{-1} is $(1, \infty)$

Rng of Ricky^{-1} is $(-\infty, \infty)$.

Time Permits

ex/ The energy released by an earth quake is approximated by

$$\log(\text{Energy}) = 11.4 + 1.5R$$

R = "magnitude" of quake (Richter Scale)

Q/ How much energy is released if R goes up by 1?

$$\log(\text{Energy}) = 11.4 + 1.5R$$

$$\log(\text{more energy}) = 11.4 + 1.5(R+1)$$
$$= 11.4 + 1.5R + 1.5$$

$$\Rightarrow \log(\text{More Energy}) - \log(\text{Energy}) = 11.4 + 1.5R + 1.5 - 11.4 - 1.5R$$
$$= 1.5$$

$$\Rightarrow \log\left(\frac{\text{more energy}}{\text{energy}}\right) = 1.5$$

$$\frac{\text{more Energy}}{\text{energy}} = 10^{1.5} \Rightarrow \text{More energy} = \text{energy} * 10^{1.5}$$

Time Permitting

ex/ A sample of radioactive material is taken. Some time after the sample was taken there was 3g of an isotope. 24 hours later there was 2.5g of the isotope. It is est. that there was originally 5g of the isotope. When was the sample taken?

0) Read. ① 

② $A(0) = 5$ $\qquad T = ?$
$A(T) = 3$
$A(T+24) = 2.5$

③ $A(T) = ce^{rt}$

$$\left.\begin{aligned} 5 &= ce^{0} \\ 3 &= ce^{r\cdot T} \\ 2.5 &= ce^{r(T+2.5)} \end{aligned}\right\}$$

3 unknowns r, c, T.
3 equations.

Solve the system, by ~~subst~~ solve + subst.

I do not care what r, c are!
Just T.

(4) $C=5$

$3 = 5e^{rT}$

$$2.5 = 5e^{r(T+24)} = 5e^{rT+24r} = 5e^{rT}\cdot e^{24r}$$

with $5e^{rT} = 3$:

$$2.5 = 3e^{24r}$$

$$\frac{2.5}{3} = e^{24r}$$

$$\ln(2.5/3) = \ln(e^{24r}) = 24r$$

$$r = \frac{1}{24}\ln(2.5/3)$$

$A(t) = 5e^{t/24 \ln(2.5/3)}$ $\quad A(T)=3 \quad 80$

$$3 = 5e^{T/24 \ln(2.5/3)}$$

$$3/5 = e^{T/24 \ln(2.5/3)}$$

$$\ln(3/5) = \ln(e^{T/24 \ln(2.5/3)}) = \frac{T}{24}\ln(2.5/3)$$

$$T = 24\ln(3/5) / \ln(2.5/3)$$

Time permitting.

ex/ A body is found, and its temp. is 85°F. The outside temp. is 70°F, and 1 hour later the temp. of the body is 80°F. When was the person killed? By the way

$$T(\text{time}) = \text{Ambient temp} + Ce^{-rt}$$

$\text{Temp}(0) = 99°F.$
$\text{Temp}(T) = 85°F$
$\text{Temp}(T+1) = 80$

$\text{Temp}(0) = 99 = 70 + Ce^0 = 70 + C = 99$
$\text{Temp}(T) = 85 = 70 + Ce^{rT}$
$\text{Temp}(T+1) = 80 = 70 + Ce^{r(T+1)}$

3 Eqn's 3-unkn.

$C = 99 - 70 = 29$

$85 = 70 + 29e^{rT}$

$15/29 = e^{rT}$

and

$80 = 70 + \overset{29}{C}e^{rT}\cdot e^{r}$

$10/29 = e^{rT}\cdot e^{r} = \cancel{10/29}\cdot\frac{15}{29}\cdot e^{r}$

$e^{r} = \frac{10}{15} \qquad r = \ln(10/15)$

$$15/29 = e^{T\ln(10/15)}$$

$$\ln(15/29) = \ln(e^{T\ln(10/15)}) = T\ln(10/15)$$

$$T = \frac{\ln(15/29)}{\ln(10/15)}$$

I did this in my head! probably wrong.