Functions and Relations

Chapter Outline

Each year the IRS (Internal Revenue Service) publishes tax rates that tell us how much federal income tax we need to pay based on our taxable income. For example, for a recent year, a single person with taxable income of more than $36,250 but not more than $87,850 pays $4991.25 plus 25% of the amount over $36,250 in federal income tax. However, finding taxable income is not always trivial. There are numerous variables that come into play. The IRS takes into account exemptions, deductions, and tax credits among other things.

In Chapter 1, we will look at mathematical relationships involving two or more variables, including the relationship between taxable income and federal income tax. To fully appreciate the connection among several variables, we will investigate their relationships algebraically, numerically, and graphically.

Schedule X—If your filing status is Single

If your taxable income is			
over—	but not over—	The tax is	of the amount over—
$0	$8925	$0 + 10%	$0
$8925	$36,250	$892.50 + 15%	$8925
$36,250	$87,850	$4991.25 + 25%	$36,250
$87,850	$183,250	$17,891.25 + 28%	$87,850
$183,250	$398,350	$44,603.25 + 33%	$183,250
$398,350	$400,000	$115,586.25 + 35%	$398,350
$400,000	———	$116,163.75 + 39.6%	$400,000

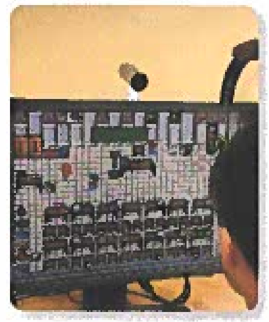
Source: Internal Revenue Service, www.irs.gov

SECTION 1.1 The Rectangular Coordinate System and Graphing Utilities

OBJECTIVES

1. Plot Points on a Rectangular Coordinate System
2. Use the Distance and Midpoint Formulas
3. Graph Equations by Plotting Points
4. Identify x- and y-Intercepts
5. Graph Equations Using a Graphing Utility

Websites, newspapers, sporting events, and the workplace all utilize graphs and tables to present data. Therefore, it is important to learn how to create and interpret meaningful graphs. Understanding how points are located relative to a fixed origin is important for many graphing applications. For example, computer game developers use a rectangular coordinate system to define the locations of objects moving around the screen.



1. Plot Points on a Rectangular Coordinate System

Mathematician René Descartes (pronounced "day cart") (1597–1650) was the first to identify points in a plane by a pair of coordinates. He did this by intersecting two perpendicular number lines with the point of intersection called the **origin.** These lines form a **rectangular coordinate system** (also known in his honor as the **Cartesian coordinate system**) or simply a **coordinate plane.** The horizontal line is called the **x-axis** and the vertical line is called the **y-axis.** The x- and y-axes divide the plane into four **quadrants.** The quadrants are labeled counterclockwise as I, II, III, and IV (Figure 1-1).

Every point in the plane can be uniquely identified by using an ordered pair (x, y) to specify its coordinates with respect to the origin. In an ordered pair, the first coordinate is called the **x-coordinate,** and the second is called the **y-coordinate.** The origin is identified as $(0, 0)$. In Figure 1-2, six points have been graphed. The point $(-3, 5)$, for example, is placed 3 units in the negative x direction (to the left) and 5 units in the positive y direction (upward).

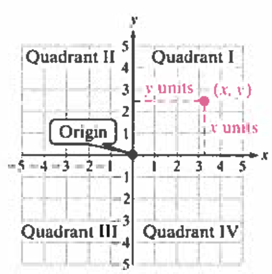


Figure 1-1

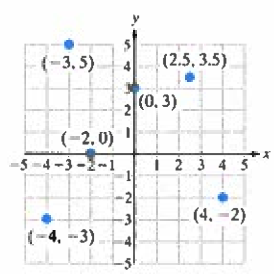


Figure 1-2

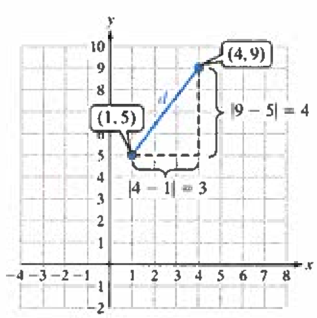


Figure 1-3

2. Use the Distance and Midpoint Formulas

Recall that the distance between two points A and B on a number line can be represented by $|A - B|$ or $|B - A|$. Now we want to find the distance between two points in a coordinate plane. For example, consider the points $(1, 5)$ and $(4, 9)$. The distance d between the points is labeled in Figure 1-3. The dashed horizontal and vertical line segments form a right triangle with hypotenuse d.

The horizontal distance between the points is $|4 - 1| = 3$.

The vertical distance between the points is $|9 - 5| = 4$.

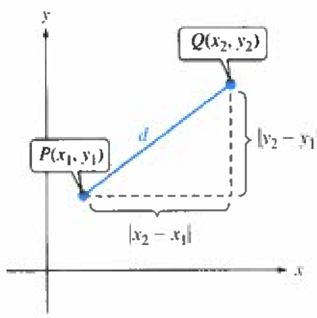


Figure 1-4

Applying the Pythagorean theorem, we have

$$d^2 = (3)^2 + (4)^2$$
$$d = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$$

Since d is a distance, reject the negative square root.

The distance between the points is 5 units.

We can make this process generic by labeling the points $P(x_1, y_1)$ and $Q(x_2, y_2)$. See Figure 1-4.

- The horizontal leg of the right triangle is $|x_2 - x_1|$ or equivalently $|x_1 - x_2|$.
- The vertical leg of the right triangle is $|y_2 - y_1|$ or equivalently $|y_1 - y_2|$.

Applying the Pythagorean theorem, we have

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We can drop the absolute value bars because $|a|^2 = (a)^2$ for all real numbers a. Likewise $|x_2 - x_1|^2 = (x_2 - x_1)^2$ and $|y_2 - y_1|^2 = (y_2 - y_1)^2$.

TIP Since

$(x_2 - x_1)^2 = (x_1 - x_2)^2$ and $(y_2 - y_1)^2 = (y_1 - y_2)^2$,

the distance formula can also be expressed as

$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Distance Formula

The distance between points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 1 Finding the Distance Between Two Points

Find the distance between the points $(-5, 1)$ and $(7, -3)$. Give the exact distance and an approximation to 2 decimal places.

Solution:

$(-5, 1)$ and $(7, -3)$
(x_1, y_1) and (x_2, y_2)

Label the points. Note that the choice for (x_1, y_1) and (x_2, y_2) will not affect the outcome.

$$d = \sqrt{[7 - (-5)]^2 + (-3 - 1)^2}$$

Apply the distance formula. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(12)^2 + (-4)^2}$$
$$= \sqrt{160}$$
$$= 4\sqrt{10} \approx 12.65$$

Simplify the radical.

The exact distance is $4\sqrt{10}$ units. This is approximately 12.65 units.

Skill Practice 1 Find the distance between the points $(-1, 4)$ and $(3, -6)$. Give the exact distance and an approximation to 2 decimal places.

Avoiding Mistakes

A statement of the form "if p, then q" is called a **conditional statement.** Its **converse** is the statement "if q, then p." The converse of a statement is not necessarily true. However, in the case of the Pythagorean theorem, the converse is a true statement.

The Pythagorean theorem tells us that if a right triangle has legs of lengths a and b and hypotenuse of length c, then $a^2 + b^2 = c^2$. The following related statement is also true: If $a^2 + b^2 = c^2$, then a triangle with sides of lengths a, b, and c is a right triangle. We use this important concept in Example 2.

Answer

1. $2\sqrt{29}$ units ≈ 10.77 units

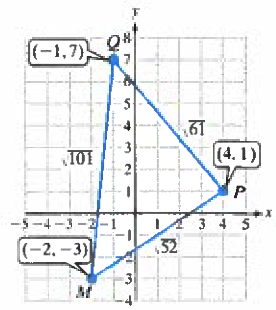
EXAMPLE 2 Determining if Three Points Form the Vertices of a Right Triangle

Determine if the points $M(-2, -3)$, $P(4, 1)$, and $Q(-1, 7)$ form the vertices of a right triangle.

Solution:

Determine the distance between each pair of points.

$$d(M, P) = \sqrt{[4 - (-2)]^2 + [1 - (-3)]^2} = \sqrt{52}$$
$$d(P, Q) = \sqrt{(-1 - 4)^2 + (7 - 1)^2} = \sqrt{61}$$
$$d(M, Q) = \sqrt{[-1 - (-2)]^2 + [7 - (-3)]^2} = \sqrt{101}$$



The line segment $\overline{MQ}$ is the longest and would potentially be the hypotenuse, c. Label the shorter sides as a and b.

Check the condition that $a^2 + b^2 = c^2$.

$$(\sqrt{52})^2 + (\sqrt{61})^2 \stackrel{?}{=} (\sqrt{101})^2$$
$$52 + 61 \neq 101$$

The points M, P, and Q do not form the vertices of a right triangle.

Skill Practice 2 Determine if the points $X(-6, -4)$, $Y(2, -2)$, and $Z(0, 5)$ form the vertices of a right triangle.

TIP We denote the distance between points P and Q as

$d(P, Q)$ or PQ.

The second notation is the length of the line segment with endpoints P and Q.

Now suppose that we want to find the midpoint of the line segment between the distinct points (x_1, y_1) and (x_2, y_2). The **midpoint** of a line segment is the point equidistant (the same distance) from the endpoints (Figure 1-5).

The x-coordinate of the midpoint is the average of the x-coordinates from the endpoints. Likewise, the y-coordinate of the midpoint is the average of the y-coordinates from the endpoints.

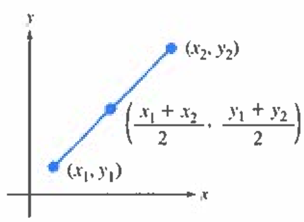


Figure 1-5

Midpoint Formula

The midpoint of the line segment with endpoints (x_1, y_1) and (x_2, y_2) is

$$M = \left(\underbrace{\frac{x_1 + x_2}{2}}_{\text{average of } x\text{-coordinates}}, \underbrace{\frac{y_1 + y_2}{2}}_{\text{average of } y\text{-coordinates}}\right)$$

Avoiding Mistakes

The midpoint of a line segment is an ordered pair (with two coordinates), not a single number.

EXAMPLE 3 Finding the Midpoint of a Line Segment

Find the midpoint of the line segment with endpoints $(4.2, -4)$ and $(-2.8, 3)$.

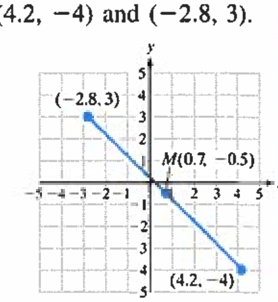
Solution:

$(4.2, -4)$ and $(-2.8, 3)$

(x_1, y_1) and (x_2, y_2) Label the points.

$$M = \left(\frac{4.2 + (-2.8)}{2}, \frac{-4 + 3}{2}\right)$$ Apply the midpoint formula.

$$= \left(0.7, -\frac{1}{2}\right) \text{ or } (0.7, -0.5)$$ Simplify.



Answer

2. No

Skill Practice 3 Find the midpoint of the line segment with endpoints $(-1.5, -9)$ and $(-8.7, 4)$.

3. Graph Equations by Plotting Points

The relationship between two variables can often be expressed as a graph or expressed algebraically as an equation. For example, suppose that two variables, x and y, are related such that y is 2 more than x. An equation to represent this relationship is $y = x + 2$. A **solution to an equation** in the variables x and y is an ordered pair (x, y) that when substituted into the equation makes the equation a true statement.

For example, the following ordered pairs are solutions to the equation $y = x + 2$.

Solution	$y = x + 2$
$(0, 2)$	$2 = 0 + 2$ ✓
$(-4, -2)$	$-2 = -4 + 2$ ✓
$(2, 4)$	$4 = 2 + 2$ ✓

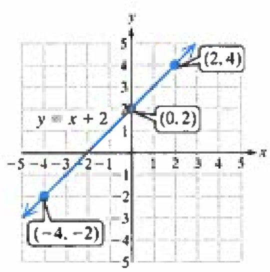


Figure 1-6

The set of all solutions to an equation is called the **solution set of the equation.** The graph of all solutions to an equation is called the **graph of the equation.** The graph of $y = x + 2$ is shown in Figure 1-6.

One of the goals of this text is to identify families of equations and the characteristics of their graphs. As we proceed through the text, we will develop tools to graph equations efficiently. For now, we present the point-plotting method to graph the solution set of an equation. In Example 4, we start by selecting several values of x and using the equation to calculate the corresponding values of y. Then we plot the points to form a general outline of the curve.

EXAMPLE 4 Graphing an Equation by Plotting Points

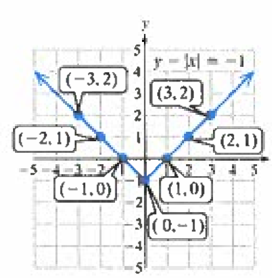
Graph the equation by plotting points. $y - |x| = -1$

Solution:

$y - |x| = -1$ Solve for y in terms of x.

$y = |x| - 1$ Arbitrarily select negative and positive values for x such as -3, -2, -1, 0, 1, 2, and 3. Then use the equation to calculate the corresponding y values.

x	y	$y = \|x\| - 1$	Ordered pair
-3	2	$y = \|-3\| - 1 = 2$	$(-3, 2)$
-2	1	$y = \|-2\| - 1 = 1$	$(-2, 1)$
-1	0	$y = \|-1\| - 1 = 0$	$(-1, 0)$
0	-1	$y = \|0\| - 1 = -1$	$(0, -1)$
1	0	$y = \|1\| - 1 = 0$	$(1, 0)$
2	1	$y = \|2\| - 1 = 1$	$(2, 1)$
3	2	$y = \|3\| - 1 = 2$	$(3, 2)$

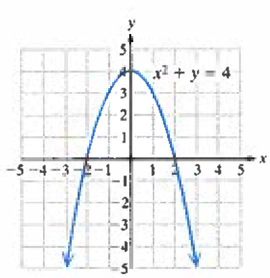


Skill Practice 4 Graph the equation by plotting points. $x^2 + y = 4$

Answers

3. $\left(-5.1, -\frac{5}{2}\right)$ or $(-5.1, -2.5)$

4.



The graph of an equation in the variables x and y represents a relationship between a real number x and a corresponding real number y. Therefore, the values of x must be chosen so that when substituted into the equation, they produce a real number for y. Sometimes the values of x must be restricted to produce real numbers for y. This is demonstrated in Example 5.

EXAMPLE 5 **Graphing an Equation by Plotting Points**

Graph the equation by plotting points. $y^2 - 1 = x$

Solution:

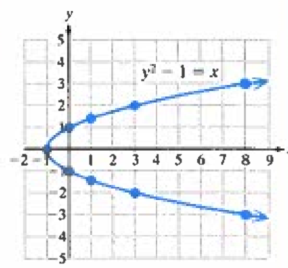
$y^2 - 1 = x$ Solve for y in terms of x.

$y^2 = x + 1$

$y = \pm\sqrt{x+1}$ Apply the square root property.

Choose $x \geq -1$ so that the radicand is nonnegative.

x	y	$y = \pm\sqrt{x+1}$	Ordered pairs
-1	0	$y = \pm\sqrt{(-1)+1} = 0$	$(-1, 0)$
0	± 1	$y = \pm\sqrt{(0)+1} = \pm 1$	$(0, 1), (0, -1)$
1	$\pm\sqrt{2}$	$y = \pm\sqrt{(1)+1} = \pm\sqrt{2} \approx 1.4$	$(1, \sqrt{2}), (1, -\sqrt{2})$
3	± 2	$y = \pm\sqrt{(3)+1} = \pm 2$	$(3, 2), (3, -2)$
8	± 3	$y = \pm\sqrt{(8)+1} = \pm 3$	$(8, 3), (8, -3)$



TIP In Example 5, we choose several convenient values of x such as -1, 0, 3, and 8 so that the radicand will be a perfect square.

Skill Practice 5 Graph the equation by plotting points. $x + y^2 = 2$

4. Identify x- and y-Intercepts

When analyzing graphs, we want to examine their most important features. Two key features are the x- and y-intercepts of a graph. These are the points where a graph intersects the x- and y-axes.

Any point on the x-axis has a y-coordinate of zero. Therefore, an **x-intercept** is a point $(a, 0)$ where a graph intersects the x-axis (Figure 1-7). Any point on the y-axis has an x-coordinate of zero. Therefore, a **y-intercept** is a point $(0, b)$ where a graph intersects the y-axis (Figure 1-7).

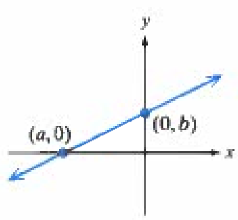


Figure 1-7

TIP In some applications, we may refer to an x-intercept as the *x-coordinate* of a point of intersection that a graph makes with the x-axis. For example, if an x-intercept is $(-4, 0)$, then the x-intercept may be stated simply as -4 (the y-coordinate is understood to be zero). Similarly, we may refer to a y-intercept as the *y-coordinate* of a point of intersection that a graph makes with the y-axis. For example, if a y-intercept is $(0, 2)$, then it may be stated simply as 2.

To find the x- and y-intercepts from an equation in x and y, follow these steps.

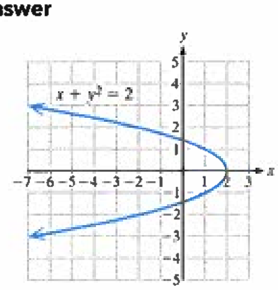
Determining x- and y-Intercepts from an Equation

Given an equation in x and y,

- Find the x-intercept(s) by substituting 0 for y in the equation and solving for x.
- Find the y-intercept(s) by substituting 0 for x in the equation and solving for y.

Answer

5.



EXAMPLE 6 Finding x- and y-Intercepts

Given the equation $y = |x| - 1$,

a. Find the x-intercept(s). **b.** Find the y-intercept(s).

Solution:

a. $y = |x| - 1$

$0 = |x| - 1$ To find the x-intercept(s), substitute 0 for y and solve for x.

$|x| = 1$ Isolate the absolute value.

$x = 1$ or $x = -1$ Recall that for $k > 0$, $|x| = k$ is equivalent to $x = k$ or $x = -k$.

The x-intercepts are $(1, 0)$ and $(-1, 0)$.

b. $y = |x| - 1$

$= |0| - 1$ To find the y-intercept(s), substitute 0 for x and solve for y.

$= -1$

The y-intercept is $(0, -1)$.

The intercepts $(1, 0)$, $(-1, 0)$, and $(0, -1)$ are consistent with the graph of the equation $y = |x| - 1$ found in Example 4 (Figure 1-8).

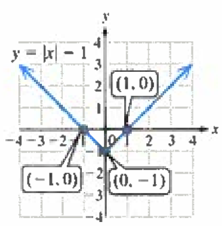
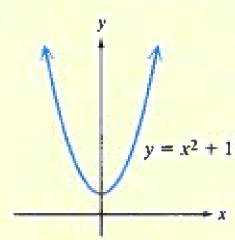


Figure 1-8

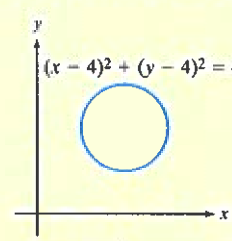
Skill Practice 6 Given the equation $y = x^2 - 4$,

a. Find the x-intercept(s). **b.** Find the y-intercept(s).

TIP Sometimes when solving for an x- or y-intercept, we encounter an equation with an imaginary solution. In such a case, the graph has no x- or y-intercept.



No x-intercept



No x- or y-intercept

5. Graph Equations Using a Graphing Utility

Graphing by the point-plotting method should only be considered a beginning strategy for creating the graphs of equations in two variables. We will quickly enhance this method with other techniques that are less cumbersome and use more analysis and strategy.

One weakness of the point-plotting method is that it may be slow to execute by pencil and paper. Also, the selected points must fairly represent the shape of the graph. Otherwise the sketch will be inaccurate. Graphing utilities can help with both of these weaknesses. They can graph many points quickly, and the more points that are plotted, the greater the likelihood that we see the key features of the graph. Graphing utilities include graphing calculators, spreadsheets, specialty graphing programs, and apps on phones.

Figures 1-9 and 1-10 show a table and a graph for $y = x^2 - 3$.

Answers

6. a. $(2, 0)$ and $(-2, 0)$
b. $(0, -4)$

TECHNOLOGY CONNECTIONS

Using the Table Feature and Graphing an Equation

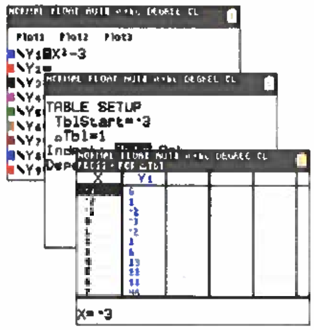


Figure 1-9

In Figure 1-9, we first enter the equation into the graphing editor. Notice that the calculator expects the equation represented with the y variable isolated.

To set up a table, enter the starting value for x, in this case, -3. Then set the increment by which to increase x, in this case 1. The x-increment is entered as ΔTbl (read "delta table"). Using the "Auto" setting means that the table of values for X and Y_1 will be automatically generated.

The table shows eleven x-y pairs but more can be accessed by using the up and down arrow keys on the keypad.

The graph in Figure 1-10 is shown between x and y values from -10 to 10. The tick marks on the axes are 1 unit apart. The viewing window with these parameters is denoted $[-10, 10, 1]$ by $[-10, 10, 1]$.

minimum x value | maximum x value | minimum y value | maximum y value

$[-10, 10, 1]$ by $[-10, 10, 1]$.

distance between tick marks x-axis | distance between tick marks y-axis

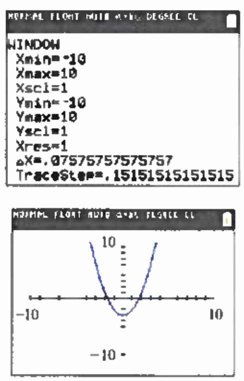


Figure 1-10

TIP The Greek letter Δ ("delta") written before a variable represents an increment of change in that variable. In this context, it represents the change from one value of x to the next.

TIP The calculator plots a large number of points and then connects the points. So instead of graphing a single smooth curve, it graphs a series of short line segments. This may give the graph a jagged look (Figure 1-10).

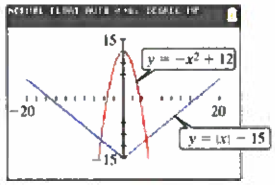
EXAMPLE 7 Graphing Equations Using a Graphing Utility

Use a graphing utility to graph $y = |x| - 15$ and $y = -x^2 + 12$ on the viewing window defined by $[-20, 20, 2]$ by $[-15, 15, 3]$.

Solution:

Enter the equations using the Y= editor.

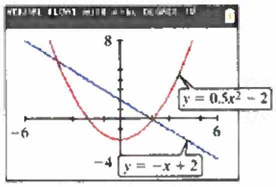
Use the WINDOW editor to change the viewing window parameters. The variables Xmin, Xmax, and Xscl relate to $[-20, 20, 2]$. The variables Ymin, Ymax, and Yscl relate to $[-15, 15, 3]$.



Select the GRAPH feature. Notice that the graphs of both equations appear. This provides us with a tool for visually examining two different models at the same time.

Skill Practice 7 Use a graphing utility to graph $y = -x + 2$ and $y = 0.5x^2 - 2$ on the viewing window $[-6, 6, 1]$ by $[-4, 8, 1]$.

Answer

7.


SECTION 1.1 Practice Exercises

Prerequisite Review

R.1. Simplify the radical. $\sqrt{48}$

R.2. Given a right triangle with a leg of length 7 km and hypotenuse of length 25 km, find the length of the unknown leg.

R.3. Solve for y. $ax + by = c$

R.4. Evaluate $x^2 + 4x + 5$ for $x = -5$

Concept Connections

1. In a rectangular coordinate system, the point where the x- and y-axes meet is called the __________.
2. The x- and y-axes divide the coordinate plane into four regions called __________.
3. The distance between two distinct points (x_1, y_1) and (x_2, y_2) is given by the formula __________.
4. The midpoint of the line segment with endpoints (x_1, y_1) and (x_2, y_2) is given by the formula __________.
5. A __________ to an equation in the variables x and y is an ordered pair (x, y) that makes the equation a true statement.
6. An x-intercept of a graph has a y-coordinate of __________.
7. A y-intercept of a graph has an x-coordinate of __________.
8. Given an equation in the variables x and y, find the y-intercept by substituting __________ for x and solving for __________.

Objective 1: Plot Points on a Rectangular Coordinate System

For Exercises 9–10, plot the points on a rectangular coordinate system.

9. $A(-3, -4)$ $B\left(\frac{5}{3}, \frac{7}{4}\right)$ $C(-1.2, 3.8)$ $D(\pi, -5)$ $E(0, 4.5)$ $F(\sqrt{5}, 0)$

10. $A(-2, -5)$ $B\left(\frac{9}{2}, \frac{7}{3}\right)$ $C(-3.6, 2.1)$ $D(5, -\pi)$ $E(3.4, 0)$ $F(0, \sqrt{3})$

Objective 2: Use the Distance and Midpoint Formulas

For Exercises 11–18,

a. Find the exact distance between the points. **(See Example 1)**

b. Find the midpoint of the line segment whose endpoints are the given points. **(See Example 3)**

11. $(-2, 7)$ and $(-4, 11)$

12. $(-1, -3)$ and $(3, -7)$

13. $(-7, -4)$ and $(2, 5)$

14. $(3, 6)$ and $(-4, -1)$

15. $(2.2, -2.4)$ and $(5.2, -6.4)$

16. $(37.1, -24.7)$ and $(31.1, -32.7)$

17. $(\sqrt{5}, -\sqrt{2})$ and $(4\sqrt{5}, -7\sqrt{2})$

18. $(\sqrt{7}, -3\sqrt{5})$ and $(2\sqrt{7}, \sqrt{5})$

For Exercises 19–22, determine if the given points form the vertices of a right triangle. (See Example 2)

19. $(1, 3)$, $(3, 1)$, and $(0, -2)$

20. $(1, 2)$, $(3, 0)$, and $(-3, -2)$

21. $(-2, 4)$, $(5, 0)$, and $(-5, 1)$

22. $(-6, 2)$, $(3, 1)$, and $(1, -2)$

Objective 3: Graph Equations by Plotting Points

For Exercises 23–24, determine if the given points are solutions to the equation.

23. $x^2 + y = 1$

a. $(-2, -3)$ **b.** $(4, -17)$ **c.** $\left(\frac{1}{2}, \frac{3}{4}\right)$

24. $|x - 3| - y = 4$

a. $(1, -2)$ **b.** $(-2, -3)$ **c.** $\left(\frac{1}{10}, -\frac{11}{10}\right)$

For Exercises 25–30, identify the set of values *x* for which *y* will be a real number.

25. $y = \dfrac{2}{x - 3}$

26. $y = \dfrac{2}{x + 7}$

27. $y = \sqrt{x - 10}$

28. $y = \sqrt{x + 11}$

29. $y = \sqrt{1.5 - x}$

30. $y = \sqrt{2.2 - x}$

For Exercises 31–44, graph the equations by plotting points. (See Examples 4–5)

31. $y = x$

32. $y = x^2$

33. $y = \sqrt{x}$

34. $y = |x|$

35. $y = x^3$

36. $y = \dfrac{1}{x}$

37. $y - |x| = 2$

38. $|x| + y = 3$

39. $y^2 - x - 2 = 0$

40. $y^2 - x + 1 = 0$

41. $x = |y| + 1$

42. $x = |y| - 3$

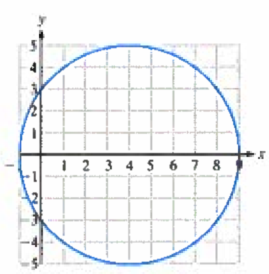
43. $y = |x + 1|$

44. $y = |x - 2|$

Objective 4: Identify *x*- and *y*-Intercepts

For Exercises 45–50, estimate the *x*- and *y*-intercepts from the graph.

45.



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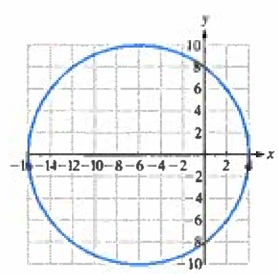

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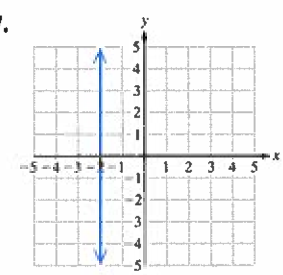

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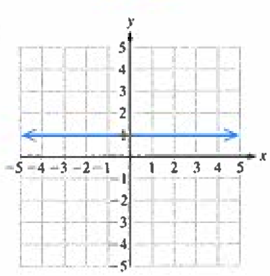

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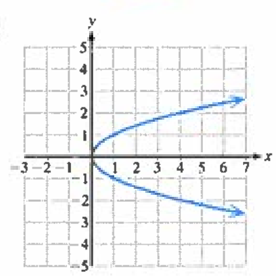

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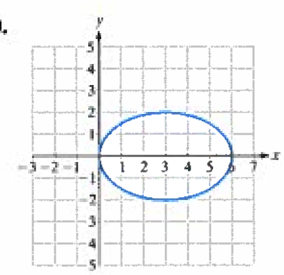


For Exercises 51–62, find the x- and y-intercepts. (See Example 6)

51. $-2x + 4y = 12$

52. $-3x - 5y = 60$

53. $x^2 + y = 9$

54. $x^2 = -y + 16$

55. $y = |x - 5| - 2$

56. $y = |x + 4| - 3$

57. $x = y^2 - 1$

58. $x = y^2 - 4$

59. $|x| = |y|$

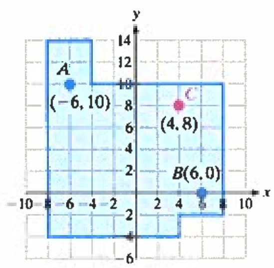
60. $x = |5y|$

61. $\dfrac{(x-3)^2}{4} + \dfrac{(y-4)^2}{9} = 1$

62. $\dfrac{(x+6)^2}{16} + \dfrac{(y+3)^2}{4} = 1$

Mixed Exercises

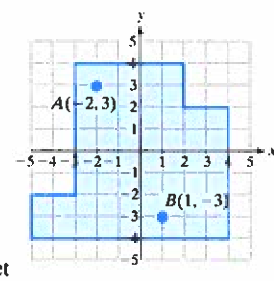
63. A map of a wilderness area is drawn with the origin placed at the parking area. Two fire observation platforms are located at points A and B. If a fire is located at point C, determine the distance to the fire from each observation platform.



64. A map of a state park is drawn so that the origin is placed at the visitor center. The distance between grid lines is 1 mi. Suppose that two hikers are located at points A and B.

a. Determine the distance between the hikers.

b. If the hikers want to meet for lunch, determine the location of the midpoint between the hikers.



The position of an object in a video game is represented by an ordered pair. The coordinates of the ordered pair give the number of pixels horizontally and vertically from the origin. Use this scenario for Exercises 65–66.

65. a. Suppose that player A is located at (36, 315) and player B is located at (410, 53). How far apart are the players? Round to the nearest pixel.

b. If the two players move directly toward each other at the same speed, where will they meet?

c. If player A moves three times faster than player B, where will they meet? Round to the nearest pixel.

66. Suppose that a player is located at point $A(460, 420)$ and must move in a direct line to point $B(80, 210)$ and then in a direct line to point $C(120, 60)$ to pick up prizes before a 5-sec timer runs out. If the player moves at 120 pixels per second, will the player have enough time to pick up both prizes? Explain.

67. Verify that the points $A(0, 0)$, $B(x, 0)$, and $C\left(\frac{1}{2}x, \frac{\sqrt{3}}{2}x\right)$ make up the vertices of an equilateral triangle.

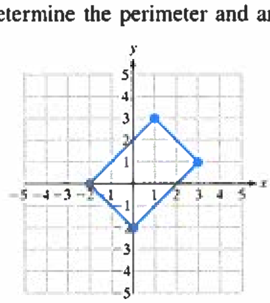
68. Verify that the points $A(0, 0)$, $B(x, 0)$, and $C(0, x)$ make up the vertices of an isosceles right triangle (an isosceles triangle has two sides of equal length).

For Exercises 69–70, assume that the units shown in the grid are in feet.

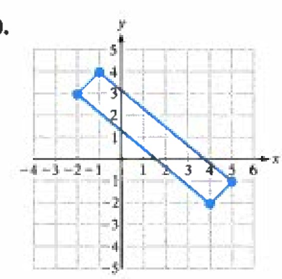
a. Determine the exact length and width of the rectangle shown.

b. Determine the perimeter and area.

69.

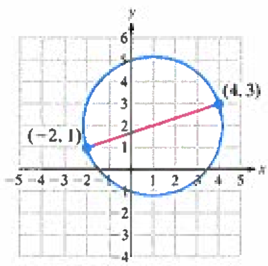


70.

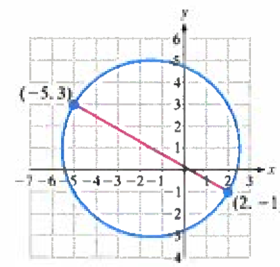


For Exercises 71–72, the endpoints of a diameter of a circle are shown. Find the center and radius of the circle.

71.

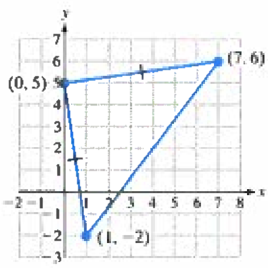


72.

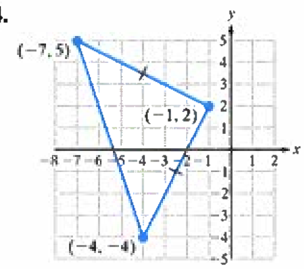


For Exercises 73–74, an isosceles triangle is shown. Find the area of the triangle. Assume that the units shown in the grid are in meters.

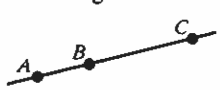
73.



74.



For Exercises 75–78, determine if points A, B, and C are collinear. Three points are collinear if they all fall on the same line. There are several ways that we can determine if three points, A, B, and C are collinear. One method is to determine if the sum of the lengths of the line segments $\overline{AB}$ and $\overline{BC}$ equals the length of $\overline{AC}$.



75. (2, 2), (4, 3), and (8, 5)

76. (2, 1.5), (4, 2), and (8, 3)

77. (−2, 8), (1, 2), and (4, −3)

78. (−1, 5), (0, 3), and (5, −13)

Write About It

79. Suppose that d represents the distance between two points (x_1, y_1) and (x_2, y_2). Explain how the distance formula is developed from the Pythagorean theorem.

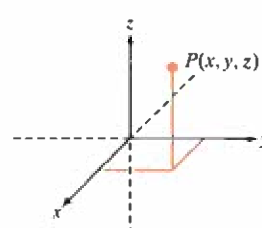
80. Explain how you might remember the midpoint formula to find the midpoint of the line segment between (x_1, y_1) and (x_2, y_2).

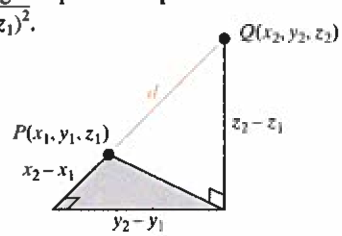
81. Explain how to find the x- and y-intercepts from an equation in the variables x and y.

82. Given an equation in the variables x and y, what does the graph of the equation represent?

Expanding Your Skills

A point in three-dimensional space can be represented in a three-dimensional coordinate system. In such a case, a z-axis is taken perpendicular to both the x- and y-axes. A point P is assigned an ordered triple $P(x, y, z)$ relative to a fixed origin where the three axes meet. For Exercises 83–86, determine the distance between the two given points in space. Use the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.





83. (5, −3, 2) and (4, 6, −1)

84. (6, −4, −1) and (2, 3, 1)

85. (3, 7, −2) and (0, −5, 1)

86. (9, −5, −3) and (2, 0, 1)

Objective 5: Graph Equations Using a Graphing Utility (Technology Connections)

87. What is meant by a viewing window on a graphing device?

88. Which of the viewing windows would show both the x- and y-intercepts of the graph of $780x - 42y = 5460$?

a. $[-20, 20, 2]$ by $[-40, 40, 10]$

b. $[-10, 10, 1]$ by $[-10, 10, 1]$

c. $[-10, 10, 1]$ by $[-10, 150, 10]$

d. $[-10, 10, 1]$ by $[-150, 10, 10]$

For Exercises 89–92, graph the equation with a graphing utility on the given viewing window. (See Example 7)

89. $y = 2x - 5$ on $[-10, 10, 1]$ by $[-10, 10, 1]$

90. $y = -4x + 1$ on $[-10, 10, 1]$ by $[-10, 10, 1]$

91. $y = 1400x^2 - 1200x$ on $[-5, 5, 1]$ by $[-1000, 2000, 500]$

92. $y = -800x^2 + 600x$ on $[-5, 5, 1]$ by $[-1000, 500, 200]$

For Exercises 93–94, graph the equations on the standard viewing window. (See Example 7)

93. a. $y = x^3$

b. $y = |x| - 9$

94. a. $y = \sqrt{x + 4}$

b. $y = |x - 2|$

SECTION 1.3 Functions and Relations

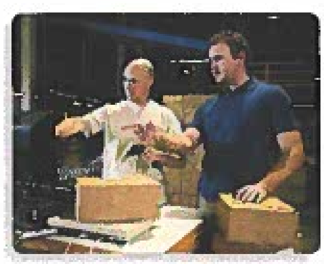
OBJECTIVES

1. Determine Whether a Relation Is a Function
2. Apply Function Notation
3. Determine x- and y-Intercepts of a Function Defined by $y = f(x)$
4. Determine Domain and Range of a Function
5. Interpret a Function Graphically

1. Determine Whether a Relation Is a Function

In the physical world, many quantities that are subject to change are related to other variables. For example:

- The cost of mailing a package is related to the weight of a package.
- The minimum braking distance of a car depends on the speed of the car.
- The perimeter of a rectangle is a function of its length and width.
- The test score that a student earns is related to the number of hours of study.



In mathematics we can express the relationship between two values as a set of ordered pairs.

Definition of a Relation

A set of ordered pairs (x, y) is called a **relation** in x and y.

- The set of x values in the ordered pairs is called the **domain** of the relation.
- The set of y values in the ordered pairs is called the **range** of the relation.

EXAMPLE 1 Writing a Relation from Observed Data Points

Table 1-1 shows the score y that a student earned on an algebra test based on the number of hours x spent studying one week prior to the test.

a. Write the set of ordered pairs that defines the relation given in Table 1-1.

b. Write the domain.

c. Write the range.

Hours of Study, x	Test Score, y
8	92
3	58
11	98
5	72
8	86

Table 1-1

Solution:

a. Relation: $\{(8, 92), (3, 58), (11, 98), (5, 72), (8, 86)\}$

b. Domain: $\{8, 3, 11, 5\}$

c. Range: $\{92, 58, 98, 72, 86\}$

Avoiding Mistakes

Do not list the elements in a set more than once. The value 8 is listed in the domain one time only.

Skill Practice 1 For the table shown,

a. Write the set of ordered pairs that defines the relation.

b. Write the domain.

c. Write the range.

x	3	-2	5	1
y	-4	0	3	0

The data in Table 1-1 show two different test scores for 8 hr of study. That is, for $x = 8$, there are two different y values. In many applications, we prefer to work with relations that assign one and only one y value for a given value of x. Such a relation is called a function.

Definition of a Function

Given a relation in x and y, we say that **y is a function of x** if for each value of x in the domain, there is exactly one value of y in the range.

EXAMPLE 2 Determining if a Relation Is a Function

Determine if the relation defines y as a function of x.

a. $\{(3, 1), (2, 5), (-4, 2), (-1, 0), (3, -4)\}$

b. $\{(-1, 4), (2, 3), (3, 4), (-4, 5)\}$

Solution:

a.

same x values

$\{(3, 1), (2, 5), (-4, 2), (-1, 0), (3, -4)\}$

different y values

When $x = 3$, there are two different y values: $y = 1$ and $y = -4$.

This relation is *not* a function.

b. $\{(-1, 4), (2, 3), (3, 4), (-4, 5)\}$

No two ordered pairs have the same x value but different y values.

This relation *is* a function.

TIP A function may not have the same x value paired with different y values. However, it is acceptable for a function to have two or more x values paired with the same y value, as shown in Example 2(b).

Skill Practice 2 Determine if the relation defines y as a function of x.

a. $\{(8, 4), (3, -1), (5, 4)\}$ **b.** $\{(-3, 2), (9, 5), (1, 0), (-3, 1)\}$

Answers

1. a. $\{(3, -4), (-2, 0), (5, 3), (1, 0)\}$
   b. Domain: $\{3, -2, 5, 1\}$
   c. Range: $\{-4, 0, 3\}$
2. a. Yes b. No

A relation that is not a function has at least one domain element x paired with more than one range element y. For example, the ordered pairs (3, 1) and (3, −4) do not make up a function. On a graph, these two points are aligned vertically. A vertical line drawn through one point also intersects the other point (Figure 1-14). This observation leads to the vertical line test.

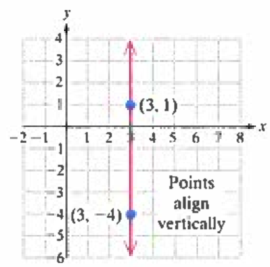


Figure 1-14

Using the Vertical Line Test

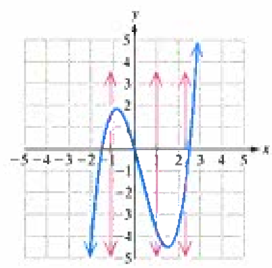
Consider a relation defined by a set of points (x, y) graphed on a rectangular coordinate system. The graph defines y as a function of x if no vertical line intersects the graph in more than one point.

EXAMPLE 3 Applying the Vertical Line Test

The graphs of three relations are shown in blue. In each case, determine if the relation defines y as a function of x.

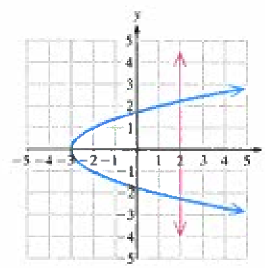
Solution:

a.



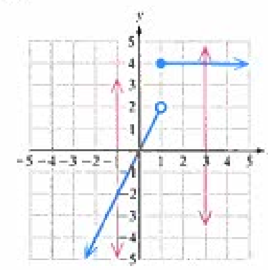
This is a function.
No vertical line intersects the graph in more than one point.

b.



This is not a function.
There is at least one vertical line that intersects the graph in more than one point.

c.

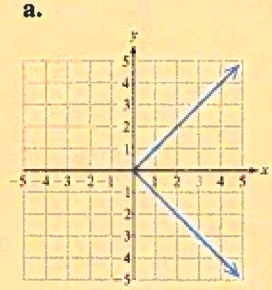


This is a function.
No vertical line intersects the graph in more than one point.

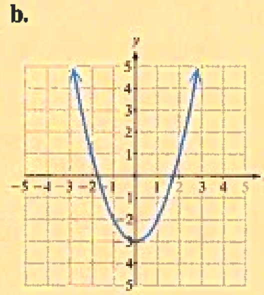
TIP In Example 3(c) there is only one y value assigned to $x = 1$. This is because the point (1, 2) is *not* included in the graph of the function as denoted by the open dot.

Skill Practice 3 Determine if the given relation defines y as a function of x.

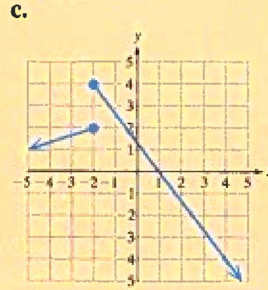
a.



b.



c.



A relation can also be defined by a figure showing a "mapping" between x and y, or by an equation in x and y.

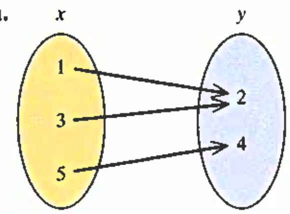
Answers

3. a. No **b.** Yes **c.** No

EXAMPLE 4 **Determining if a Relation Is a Function**

Determine if the relation defines y as a function of x.

a.



b. $y^2 = x$ **c.** $(x - 2)^2 + (y + 1)^2 = 9$

Solution:

a. This mapping defines the set of ordered pairs: $\{(1, 2), (3, 2), (5, 4)\}$. This relation *is* a function.

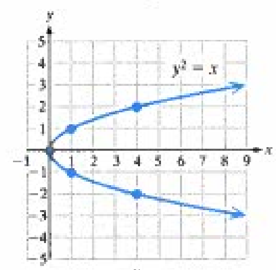
No two ordered pairs have the same x value but different y values.

b. $y^2 = x$

$y = \pm\sqrt{x}$

Solve the equation for y. For any $x > 0$, there are two corresponding y values.

x	y	Ordered pairs
0	0	(0, 0)
1	1, −1	(1, 1), (1, −1)
4	2, −2	(4, 2), (4, −2)
9	3, −3	(9, 3), (9, −3)

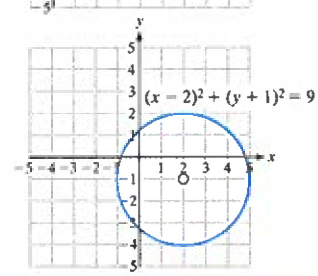


This relation is *not* a function.

c. $(x - 2)^2 + (y + 1)^2 = 9$

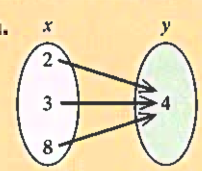
This equation represents the graph of a circle with center $(2, -1)$ and radius 3.

This relation is *not* a function because it fails the vertical line test.



Skill Practice 4 Determine if the relation defines y as a function of x.

a.



b. $|y + 1| = x$ **c.** $x^2 + y^2 = 25$

2. Apply Function Notation

A function may be defined by an equation with two variables. For example, the equation $y = x - 2$ defines y as a function of x. This is because for any real number x, the value of y is the unique number that is 2 less than x.

When a function is defined by an equation, we often use function notation. For example, the equation $y = x - 2$ may be written in function notation as

$f(x) = x - 2$ read as "f of x equals $x - 2$."

Answers

4. a. Yes b. No c. No

With function notation,

- f is the name of the function,
- x is an input variable from the domain,
- $f(x)$ is the function value (or y value) corresponding to x.

Avoiding Mistakes
The notation $f(x)$ does *not* imply multiplication of f and x.

A function may be evaluated at different values of x by using substitution.

$f(x) = x - 2$
$f(4) = (4) - 2 = 2$ — $f(4) = 2$ can be interpreted as $(4, 2)$.
$f(1) = (1) - 2 = -1$ — $f(1) = -1$ can be interpreted as $(1, -1)$.

EXAMPLE 5 Evaluating a Function

Evaluate the function defined by $g(x) = 2x + 1$ for the given values of x.

a. $g(-2)$ **b.** $g(-1)$ **c.** $g(0)$ **d.** $g(1)$ **e.** $g(2)$

Solution:

a. $g(-2) = 2(-2) + 1 = -3$ — Substitute -2 for x. $g(-2) = -3$

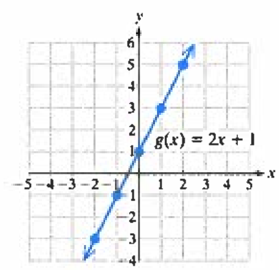
b. $g(-1) = 2(-1) + 1 = -1$ — Substitute -1 for x. $g(-1) = -1$

c. $g(0) = 2(0) + 1 = 1$ — Substitute 0 for x. $g(0) = 1$

d. $g(1) = 2(1) + 1 = 3$ — Substitute 1 for x. $g(1) = 3$

e. $g(2) = 2(2) + 1 = 5$ — Substitute 2 for x. $g(2) = 5$

The function values represent the ordered pairs $(-2, -3)$, $(-1, -1)$, $(0, 1)$, $(1, 3)$, and $(2, 5)$. The line through the points represents all ordered pairs defined by this function. This is the graph of the function.



TIP The name of a function can be represented by any letter or symbol. However, lowercase letters such as f, g, h, and so on are often used.

Skill Practice 5 Evaluate the function defined by $h(x) = 4x - 3$ for the given values of x.

a. $h(-3)$ **b.** $h(-1)$ **c.** $h(0)$ **d.** $h(1)$ **e.** $h(3)$

EXAMPLE 6 Evaluating a Function

Evaluate the function defined by $f(x) = 3x^2 + 2x$ for the given values of x.

a. $f(a)$ **b.** $f(x + h)$

Solution:

a. $f(a) = 3a^2 + 2a$ — Substitute a for x.

b. $f(x + h) = 3(x + h)^2 + 2(x + h)$ — Substitute $x + h$ for x.

$= 3(x^2 + 2xh + h^2) + 2x + 2h$ — Simplify. Recall: $(a + b)^2 = a^2 + 2ab + b^2$

$= 3x^2 + 6xh + 3h^2 + 2x + 2h$

Skill Practice 6 Evaluate the function defined by $f(x) = -x^2 + 4x$ for the given values of x.

a. $f(t)$ **b.** $f(a + h)$

Answers

5. a. $h(-3) = -15$ b. $h(-1) = -7$ c. $h(0) = -3$ d. $h(1) = 1$ e. $h(3) = 9$

6. a. $f(t) = -t^2 + 4t$ b. $f(a + h) = -a^2 - 2ah - h^2 + 4a + 4h$

3. Determine x- and y-Intercepts of a Function Defined by $y = f(x)$

Recall that to find an x-intercept(s) of the graph of an equation, we substitute 0 for y in the equation and solve for x. Using function notation, $y = f(x)$, this is equivalent to finding the real solutions of the equation $f(x) = 0$. To find the y-intercept, substitute 0 for x and solve the equation for y. Using function notation, this is equivalent to finding $f(0)$.

Finding Intercepts Using Function Notation

Given a function defined by $y = f(x)$,

- The x-intercepts are the real solutions to the equation $f(x) = 0$.
- The y-intercept is given by $f(0)$.

EXAMPLE 7 **Finding the x- and y-Intercepts of a Function**

Find the x- and y-intercepts of the function defined by $f(x) = x^2 - 4$.

Solution:

To find the x-intercept(s), solve the equation $f(x) = 0$.

$$f(x) = x^2 - 4$$
$$0 = x^2 - 4$$
$$x^2 = 4$$
$$x = \pm 2$$

The x-intercepts are $(2, 0)$ and $(-2, 0)$.

To find the y-intercept, evaluate $f(0)$.

$$f(0) = (0)^2 - 4$$
$$= -4$$

The y-intercept is $(0, -4)$.

Skill Practice 7 Find the x- and y-intercepts of the function defined by $f(x) = |x| - 5$.

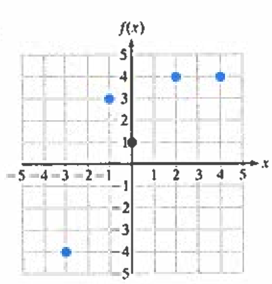
4. Determine Domain and Range of a Function

Given a relation defining y as a function of x, the **domain** is the set of x values in the function, and the **range** is the set of y values in the function. In Example 8, we find the domain and range from the graph of a function.

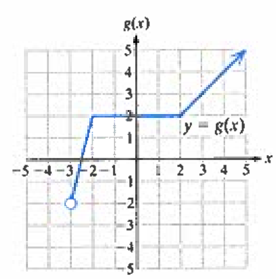
EXAMPLE 8 **Determining Domain and Range**

Determine the domain and range for the functions shown.

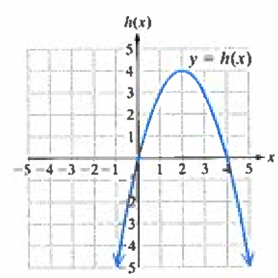
a.



b.



c.



Answer

7. x-intercepts: $(5, 0)$ and $(-5, 0)$; y-intercept: $(0, -5)$

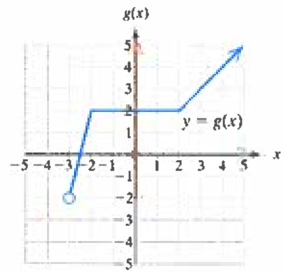
Solution:

a. The graph defines the set of ordered pairs:
$\{(-3, -4), (-1, 3), (0, 1), (2, 4), (4, 4)\}$

Domain: $\{-3, -1, 0, 2, 4\}$ The domain is the set of x values.

Range: $\{-4, 1, 3, 4\}$ The range is the set of y values.

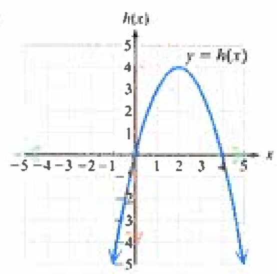
b.



The domain is shown on the x-axis in green tint.
Domain: $\{x \mid x > -3\}$ or in interval notation: $(-3, \infty)$.

The range is shown on the y-axis in red tint.
Range: $\{y \mid y > -2\}$ or in interval notation: $(-2, \infty)$.

c.



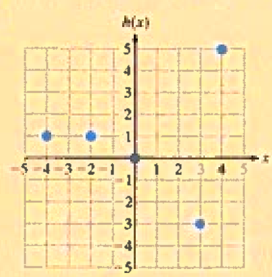
The graph extends infinitely far downward and infinitely far to the left and right. Therefore, the domain is the set of all real numbers, x.

The domain is shown on the x-axis in green tint.
Domain: $\mathbb{R}$ or in interval notation: $(-\infty, \infty)$.

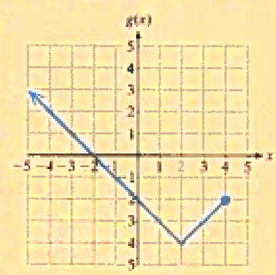
The range is shown on the y-axis in red tint.
Range: $\{y \mid y \le 4\}$ or in interval notation: $(-\infty, 4]$.

Skill Practice 8 Determine the domain and range for the functions shown.

a.



b.



In some cases, a function may have restrictions on the domain. For example, consider the function defined by

$$f(x) = x^2 + 2 \quad \text{for} \quad x \ge 0$$

The restriction on x (that is, $x \ge 0$) is explicitly stated along with the definition of the function. If no such restriction is stated, then by default, the domain is all real numbers that when substituted into the function produce real numbers in the range.

Guidelines to Find Domain of a Function

To determine the implied domain of a function defined by $y = f(x)$,

- Exclude values of x that make the denominator of a fraction zero.
- Exclude values of x that make the radicand negative within an even-indexed root.

Answers

8. a. Domain: $\{-4, -2, 0, 3, 4\}$; Range: $\{-3, 0, 1, 5\}$

b. Domain: $\{x \mid x \le 4\}$ or $(-\infty, 4]$; Range: $\{y \mid y \ge -4\}$ or $[-4, \infty)$

EXAMPLE 9 Determining the Domain of a Function

Write the domain of each function in interval notation.

a. $f(x) = \dfrac{x + 3}{2x - 5}$ **b.** $g(x) = \dfrac{x}{x^2 + 4}$

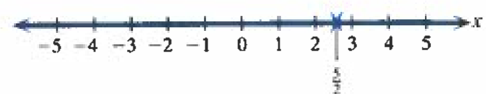
c. $h(t) = \sqrt{2 - t}$ **d.** $m(a) = |4 + a|$

Solution:

a. $f(x) = \dfrac{x + 3}{2x - 5}$

The domain is all real numbers except those that make the denominator zero.

The variable x has the restriction that $2x - 5 \neq 0$. Therefore, $x \neq \frac{5}{2}$.



Domain: $\left(-\infty, \dfrac{5}{2}\right) \cup \left(\dfrac{5}{2}, \infty\right)$

b. $g(x) = \dfrac{x}{x^2 + 4}$ Denominator always positive (never zero)

Domain: $(-\infty, \infty)$

The expression $x^2 \geq 0$ for all real numbers x. Therefore, $x^2 + 4 > 0$ for all real numbers x.

c. $h(t) = \sqrt{2 - t}$

The domain is restricted to the real numbers that make the radicand greater than or equal to zero.

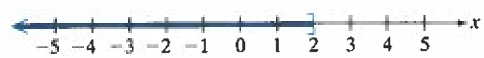
$2 - t \geq 0$

$-t \geq -2$

Divide by -1 and reverse the inequality sign.

$t \leq 2$

Domain: $(-\infty, 2]$



d. $m(a) = |4 + a|$

Domain: $(-\infty, \infty)$

There are no fractions or radicals that would restrict the domain. The expression $|4 + a|$ is a real number for all real numbers a.

Skill Practice 9 Write the domain of each function in interval notation.

a. $f(x) = \dfrac{x - 2}{3x + 1}$ **b.** $g(x) = \dfrac{x^2}{5}$

c. $k(x) = \sqrt{x + 3}$ **d.** $p(x) = 2x^2 + 3x$

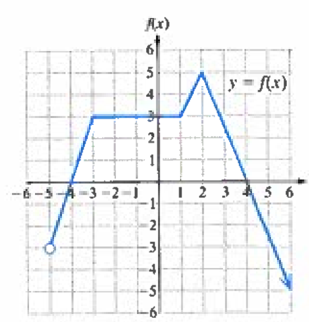
5. Interpret a Function Graphically

In Example 10, we will review the key concepts studied in this section by identifying characteristics of a function based on its graph.

EXAMPLE 10 Identifying Characteristics of a Function

Use the function f pictured to answer the questions.

a. Determine $f(2)$.
b. Determine $f(-5)$.
c. Find all x for which $f(x) = 0$.
d. Find all x for which $f(x) = 3$.
e. Determine the x-intercept(s).
f. Determine the y-intercept.
g. Determine the domain of f.
h. Determine the range of f.



Answers

9. a. $\left(-\infty, -\frac{1}{3}\right) \cup \left(-\frac{1}{3}, \infty\right)$
b. $(-\infty, \infty)$
c. $[-3, \infty)$
d. $(-\infty, \infty)$

Solution:

a. $f(2) = 5$

$f(2) = 5$ because the function contains the point $(2, 5)$.

b. $f(-5)$ is not defined.

The point $(-5, -3)$ is not included in the function as indicated by the open dot.

c. $f(x) = 0$ for $x = -4$ and $x = 4$.

The points $(-4, 0)$ and $(4, 0)$ represent the points where $f(x) = 0$.

d. $f(x) = 3$ for all x on the interval $[-3, 1]$ and for $x = \frac{14}{5}$.

e. The x-intercepts are $(-4, 0)$ and $(4, 0)$.

f. The y-intercept is $(0, 3)$.

g. The domain is $(-5, \infty)$.

h. The range is $(-\infty, 5]$.

Skill Practice 10 Use the function f pictured to find:

a. $f(-2)$.

b. $f(4)$.

c. All x for which $f(x) = 3$.

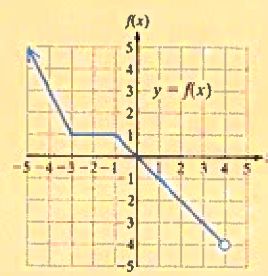
d. All x for which $f(x) = 1$.

e. The x-intercept(s).

f. The y-intercept.

g. The domain of f.

h. The range of f.



Answers

10. **a.** $f(-2) = 1$
 b. $f(4)$ is not defined.
 c. $x = -4$
 d. All x on the interval $[-3, -1]$
 e. $(0, 0)$
 f. $(0, 0)$
 g. $(-\infty, 4)$
 h. $(-4, \infty)$

SECTION 1.3 Practice Exercises

Prerequisite Review

R.1. Solve the equation using the square root property. $8x^2 - 40 = 0$

R.2. Solve. $4x^2 - 7x - 15 = 0$

R.3. Solve. Write the solution set in interval notation. $-3y - 9 \leq 15$

R.4. Solve. $|2n + 5| = 2$

R.5. Given $2x - 5y = 20$,

a. Find the x-intercept. **b.** Find the y-intercept.

Concept Connections

1. A set of ordered pairs (x, y) is called a __________ in x and y. The set of x values in the relation is called the __________ of the relation. The set of __________ values is called the range of the relation.
2. Given a function defined by $y = f(x)$, the statement $f(2) = 4$ is equivalent to what ordered pair?
3. Given a function defined by $y = f(x)$, to find the __________-intercept, evaluate $f(0)$.
4. Given a function defined by $y = f(x)$, to find the x-intercept(s), substitute 0 for __________ and solve for x.
5. Given $f(x) = \dfrac{x+1}{x+5}$, the domain is restricted so that $x \neq$ __________.
6. Given $g(x) = \sqrt{x - 5}$, the domain is restricted so that $x \geq$ __________.
7. Consider a relation that defines the height y of a tree for a given time t after it is planted. Does this relation define y as a function of t? Explain.
8. Consider a relation that defines a time y during the course of a year when the temperature T in Fort Collins, Colorado, is 70°. Does this relation define y as a function of T? Explain.

Objective 1: Determine Whether a Relation Is a Function

For Exercises 9–12,

a. Write a set of ordered pairs (x, y) that defines the relation.

b. Write the domain of the relation.

c. Write the range of the relation.

d. Determine if the relation defines y as a function of x. **(See Examples 1–2)**

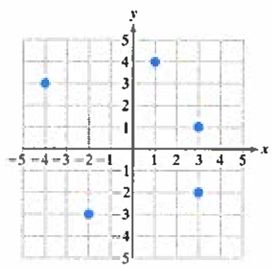
9.

Actor x	Number of Oscar Nominations y
Tom Hanks	5
Jack Nicholson	12
Sean Penn	5
Dustin Hoffman	7

10.

City x	Elevation at Airport (ft) y
Albany	285
Denver	5883
Miami	11
San Francisco	11

11.



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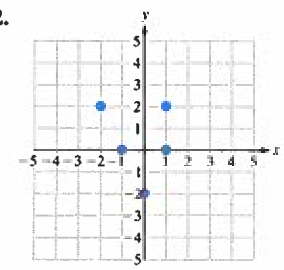
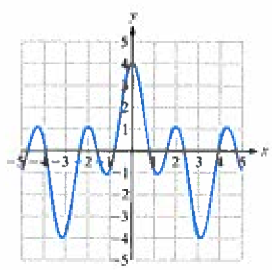


13. Answer true or false. All relations are functions.

14. Answer true or false. All functions are relations.

For Exercises 15–32, determine if the relation defines y as a function of x. (See Examples 3–4)

15.



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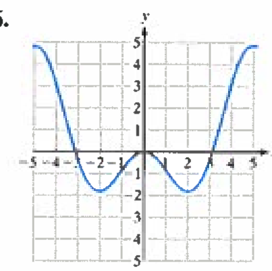

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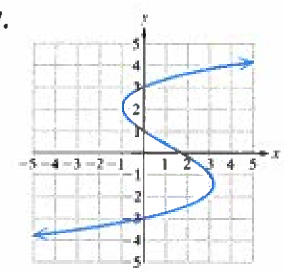

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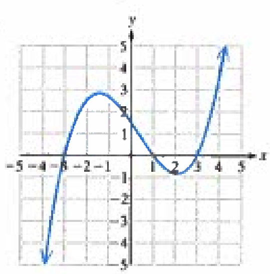

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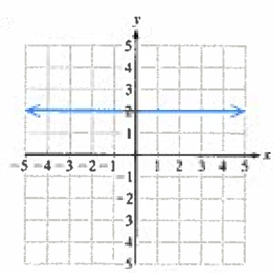

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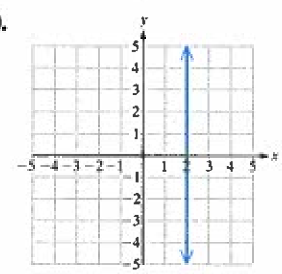

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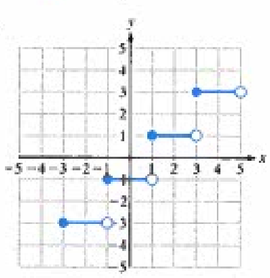

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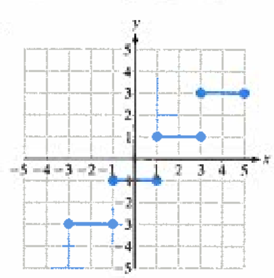

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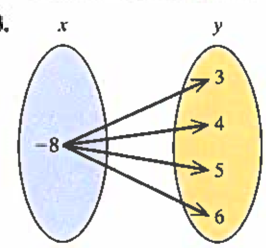

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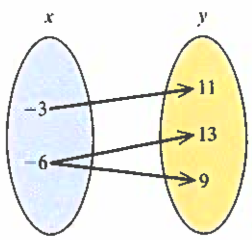

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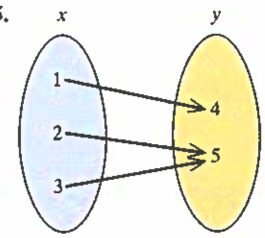

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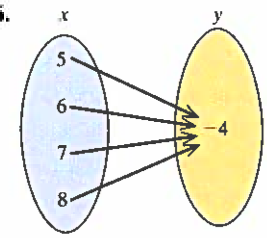


27. $(x + 1)^2 + (y + 5)^2 = 25$

28. $(x + 3)^2 + (y + 4)^2 = 1$

29. $y = x + 3$

30. $y = x - 4$

31. **a.** $y = x^2$
b. $x = y^2$

32. **a.** $y = |x|$
b. $x = |y|$

Objective 2: Apply Function Notation

33. The statement $f(4) = 1$ corresponds to what ordered pair?

34. The statement $g(7) = -5$ corresponds to what ordered pair?

For Exercises 35–56, evaluate the function for the given value of x. (See Examples 5–6)

$f(x) = x^2 + 3x$ $\quad g(x) = \dfrac{1}{x}$ $\quad h(x) = 5$ $\quad k(x) = \sqrt{x + 1}$

35. **a.** $f(-2)$ **b.** $f(-1)$ **c.** $f(0)$ **d.** $f(1)$ **e.** $f(2)$

36. **a.** $g(-2)$ **b.** $g(-1)$ **c.** $g\left(-\frac{1}{2}\right)$ **d.** $g\left(\frac{1}{2}\right)$ **e.** $g(2)$

37. **a.** $h(-2)$ **b.** $h(-1)$ **c.** $h(0)$ **d.** $h(1)$ **e.** $h(2)$

38. **a.** $k(-2)$ **b.** $k(-1)$ **c.** $k(0)$ **d.** $k(1)$ **e.** $k(3)$

39. $g(3)$

40. $h(-7)$

41. $g\left(\frac{1}{3}\right)$

42. $h(7)$

43. $k(-5)$

44. $f(5)$

45. $k(8)$

46. $f(-5)$

47. $g(t)$

48. $f(a)$

49. $k(x + h)$

50. $h(x + h)$

51. $f(a + 4)$

52. $f(t - 3)$

53. $g(0)$

54. $k(-10)$

55. $f(x + h)$

56. $g(x + h)$

For Exercises 57–62, find and simplify $f(x + h)$. (See Example 6)

57. $f(x) = -4x^2 - 5x + 2$

58. $f(x) = -2x^2 + 6x - 3$

59. $f(x) = 7 - 3x^2$

60. $f(x) = 11 - 5x^2$

61. $f(x) = x^3 + 2x - 5$

62. $f(x) = x^3 - 4x + 2$

For Exercises 63–70, refer to the function $f = \{(2, 3), (9, 7), (3, 4), (-1, 6)\}$.

63. Determine $f(9)$.

64. Determine $f(-1)$.

65. Determine $f(3)$.

66. Determine $f(2)$.

67. For what value of x is $f(x) = 6$?

68. For what value of x is $f(x) = 7$?

69. For what value of x is $f(x) = 3$?

70. For what value of x is $f(x) = 4$?

71. Joe rides his bicycle an average of 18 mph. The distance Joe rides $d(t)$ (in mi) is given by $d(t) = 18t$, where t is the time in hours that he rides.

a. Evaluate $d(2)$ and interpret the meaning.

b. Determine the distance Joe travels in 40 min.

72. Frank needs to drive 250 mi from Daytona Beach to Miami. After having driven x miles, the distance remaining $r(x)$ (in mi) is given by $r(x) = 250 - x$.

a. Evaluate $r(50)$ and interpret the meaning.

b. Determine the distance remaining after 122 mi.

73. At a restaurant, if a party has eight or more people, the gratuity is automatically added to the bill. If x is the cost of the meal, then the total bill $C(x)$ with an 18% gratuity and a 6% sales tax is given by: $C(x) = x + 0.06x + 0.18x$. Evaluate $C(225)$ and interpret the meaning in the context of this problem.

74. A bookstore marks up the price of a book by 40% of the cost from the publisher. Therefore, the bookstore's price to the student, $P(x)$ (in \$) after a 7.5% sales tax, is given by $P(x) = 1.075(x + 0.40x)$, where x is the cost of the book from the publisher. Evaluate $P(60)$ and interpret the meaning in the context of this problem.

Objective 3: Determine x- and y-Intercepts of a Function Defined by $y = f(x)$

For Exercises 75–84, determine the x- and y-intercepts for the given function. (See Example 7)

75. $f(x) = 2x - 4$

76. $g(x) = 3x - 12$

77. $h(x) = |x| - 8$

78. $k(x) = -|x| + 2$

79. $p(x) = -x^2 + 12$

80. $q(x) = x^2 - 8$

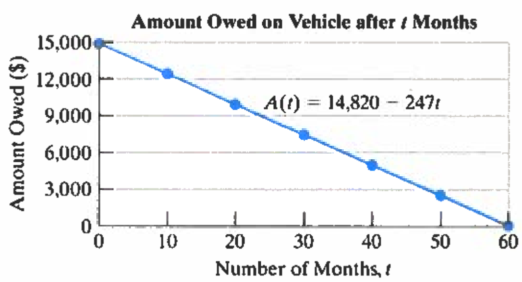
81. $r(x) = |x - 8|$

82. $s(x) = |x + 3|$

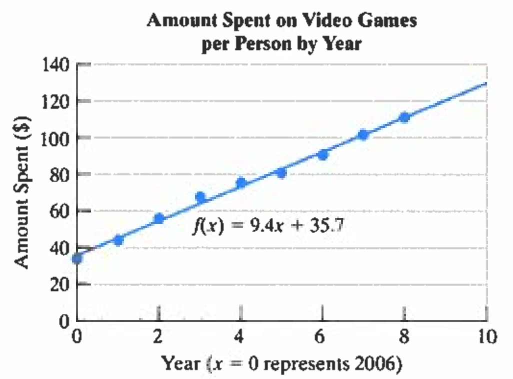
83. $f(x) = \sqrt{x} - 2$

84. $g(x) = -\sqrt{x} + 3$

85. A student decides to finance a used car over a 5-yr (60-month) period. After making a down payment of \$2000, the remaining cost of the car including tax and interest is \$14,820. The amount owed $y = A(t)$ (in \$) is given by $A(t) = 14{,}820 - 247t$, where t is the number of months after purchase and $0 \le t \le 60$. Determine the t-intercept and y-intercept and interpret their meanings in context.



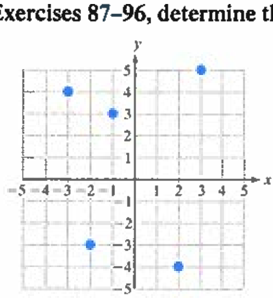
86. The amount spent on video games per person in the United States has been increasing since 2006. (*Source*: www.census.gov) The function defined by $f(x) = 9.4x + 35.7$ represents the amount spent $f(x)$ (in \$) x years since 2006. Determine the y-intercept and interpret its meaning in context.



Objective 4: Determine Domain and Range of a Function

For Exercises 87–96, determine the domain and range of the function. (See Example 8)

87.



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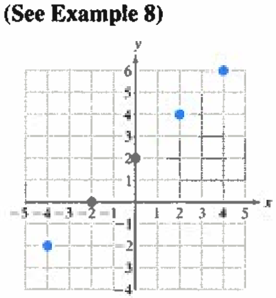

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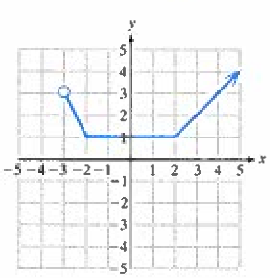

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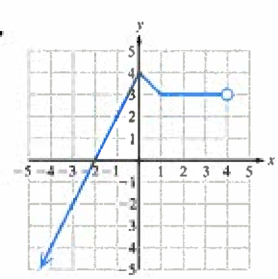

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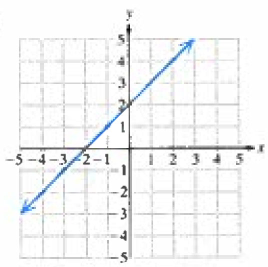

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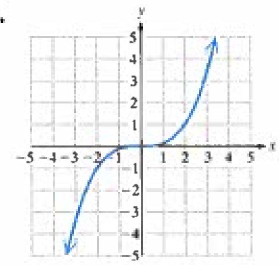

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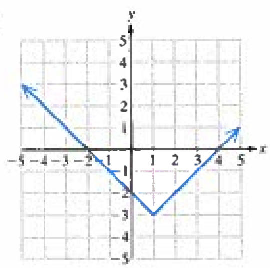

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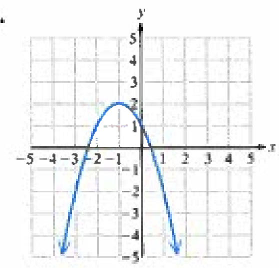

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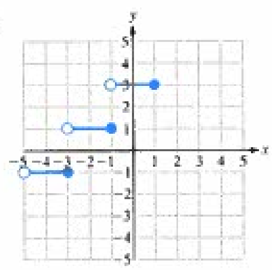

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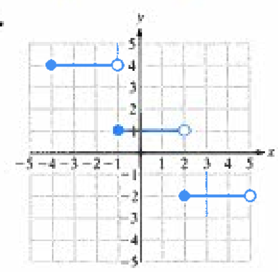


For Exercises 97–110, write the domain in interval notation. (See Example 9)

97. a. $f(x) = \dfrac{x-3}{x-4}$ **b.** $g(x) = \dfrac{x-3}{x^2-4}$ **c.** $h(x) = \dfrac{x-3}{x^2+4}$

98. a. $k(x) = \dfrac{x+6}{x-2}$ **b.** $j(x) = \dfrac{x+6}{x^2+2}$ **c.** $p(x) = \dfrac{x+6}{x^2-2}$

99. a. $a(x) = \sqrt{x+9}$ **b.** $b(x) = \sqrt{9-x}$ **c.** $c(x) = \dfrac{1}{\sqrt{x+9}}$

100. a. $y(t) = \sqrt{16-t}$ **b.** $w(t) = \sqrt{t-16}$ **c.** $z(t) = \dfrac{1}{\sqrt{16-t}}$

101. a. $f(t) = \sqrt[3]{t-5}$ **b.** $g(t) = \sqrt[3]{5-t}$ **c.** $h(t) = \dfrac{1}{\sqrt[3]{t-5}}$

102. a. $k(x) = \sqrt[5]{3+x}$ **b.** $m(x) = \sqrt[5]{x-3}$ **c.** $n(x) = \dfrac{1}{\sqrt[5]{x-3}}$

103. a. $f(x) = x^2 - 3x - 28$ **b.** $g(x) = \dfrac{x+2}{x^2-3x-28}$ **c.** $h(x) = \dfrac{x^2-3x-28}{x+2}$

104. a. $r(x) = x^2 - 4x - 12$ **b.** $s(x) = \dfrac{x^2-4x-12}{x+1}$ **c.** $t(x) = \dfrac{x+1}{x^2-4x-12}$

105. a. $w(x) = |x+1| + 4$ **b.** $y(x) = \dfrac{x}{|x+1|+4}$ **c.** $z(x) = \dfrac{x}{|x+1|-4}$

106. a. $f(a) = 8 - |a-2|$ **b.** $g(a) = \dfrac{5}{8-|a-2|}$ **c.** $h(a) = \dfrac{5}{8+|a-2|}$

107. **a.** $f(x) = \sqrt{x + 15}$ **b.** $g(x) = \sqrt{x + 15} - 2$ **c.** $k(x) = \dfrac{5}{\sqrt{x + 15} - 2}$

108. **a.** $f(c) = \sqrt{c + 20}$ **b.** $g(c) = \sqrt{c + 20} - 1$ **c.** $h(c) = \dfrac{-4}{\sqrt{c + 20} - 1}$

109. **a.** $p(x) = 2x + 1$ **b.** $q(x) = 2x + 1;\ x \geq 0$ **c.** $r(x) = 2x + 1;\ 0 \leq x < 7$

110. **a.** $m(x) = 3x - 7$ **b.** $n(x) = 3x - 7;\ x < 0$ **c.** $n(x) = 3x - 7;\ -2 < x < 2$

Objective 5: Interpret a Function Graphically

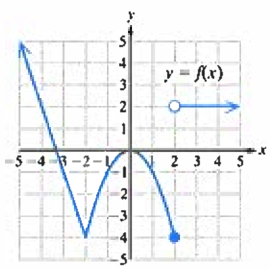
For Exercises 111–114, use the graph of $y = f(x)$ to answer the following. (See Example 10)

a. Determine $f(-2)$. **b.** Determine $f(3)$. **c.** Find all x for which $f(x) = -1$.

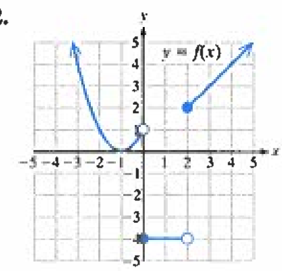
d. Find all x for which $f(x) = -4$. **e.** Determine the x-intercept(s). **f.** Determine the y-intercept.

g. Determine the domain of f. **h.** Determine the range of f.

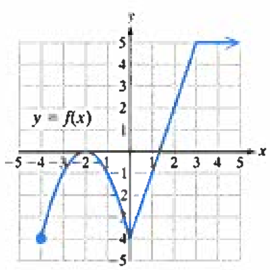
111.



112.



113.



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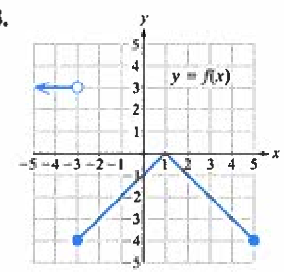


Mixed Exercises

For Exercises 115–122, write a function that represents the given statement.

115. Suppose that a phone card has 400 min. Write a relationship that represents the number of minutes remaining $r(x)$ as a function of the number of minutes already used x.

116. Suppose that a roll of wire has 200 ft. Write a relationship that represents the amount of wire remaining $w(x)$ as a function of the number of feet of wire x already used.

117. Given an equilateral triangle with sides of length x, write a relationship that represents the perimeter $P(x)$ as a function of x.

118. In an isosceles triangle, two angles are equal in measure. If the third angle is x degrees, write a relationship that represents the measure of one of the equal angles $A(x)$ as a function of x.

119. Two adjacent angles form a right angle. If the measure of one angle is x degrees, write a relationship representing the measure of the other angle $C(x)$ as a function of x.

120. Two adjacent angles form a straight angle ($180°$). If the measure of one angle is x degrees, write a relationship representing the measure of the other angle $S(x)$ as a function of x.

121. Write a relationship for a function whose $f(x)$ values are 2 less than three times the square of x.

122. Write a relationship for a function whose $f(x)$ values are 3 more than the principal square root of x.

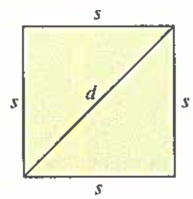
Write About It

123. If two points align vertically then the points do not define y as a function of x. Explain why.

124. Given a function defined by $y = f(x)$, explain how to determine the x- and y-intercepts.

Expanding Your Skills

125. Given a square with sides of length s, diagonal of length d, perimeter P, and area A,



a. Write P as a function of s.

b. Write A as a function of s.

c. Write A as a function of P.

d. Write P as a function of A.

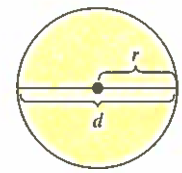
e. Write d as a function of s.

f. Write s as a function of d.

g. Write P as a function of d.

h. Write A as a function of d.

126. Given a circle with radius r, diameter d, circumference C, and area A,



a. Write C as a function of r.

b. Write A as a function of r.

c. Write r as a function of d.

d. Write d as a function of r.

e. Write C as a function of d.

f. Write A as a function of d.

g. Write A as a function of C.

h. Write C as a function of A.

SECTION 1.4 Linear Equations in Two Variables and Linear Functions

OBJECTIVES

1. Graph Linear Equations in Two Variables
2. Determine the Slope of a Line
3. Apply the Slope-Intercept Form of a Line
4. Compute Average Rate of Change
5. Solve Equations and Inequalities Graphically

1. Graph Linear Equations in Two Variables

The median incomes for individuals for all levels of education have shown an increasing trend since 1990. However, the median income for individuals with a bachelor's degree is consistently greater than for individuals whose highest level of education is a high school degree or equivalent (Figure 1-15). (*Source*: U.S. Census Bureau, www.census.gov)

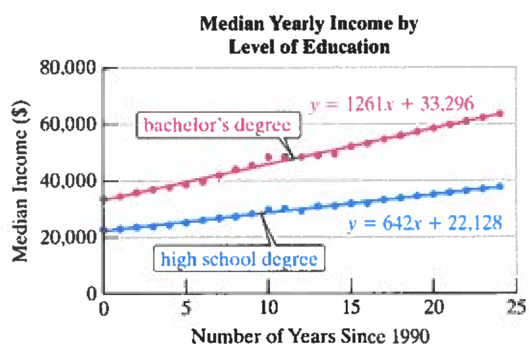


Figure 1-15

The graph in Figure 1-15 is called a scatter plot. A **scatter plot** is a visual representation of a set of points. In this case, the x values represent the number of years since 1990, and the y values represent the median income in dollars. The line that models each set of data is called a **regression line** and is found by using techniques taught in a first course in statistics. The equations that represent the two lines are called linear equations in two variables.

TIP For an equation in standard form, the value of A, B, and C are usually taken to be integers where A, B, and C share no common factors.

Linear Equation in Two Variables

Let A, B, and C represent real numbers such that A and B are not both zero. A **linear equation** in the variables x and y is an equation that can be written in the form:

$Ax + By = C$ This is called the **standard form** of an equation of a line.

Note: A linear equation $Ax + By = C$ has variables x and y each of first degree.

In Example 1, we demonstrate that the graph of a linear equation $Ax + By = C$ is a line. The line may be slanted, horizontal, or vertical depending on the coefficients A, B, and C.

EXAMPLE 1 Graphing Linear Equations

Graph the line represented by each equation.

a. $2x + 3y = 6$ **b.** $x = -3$ **c.** $2y = 4$

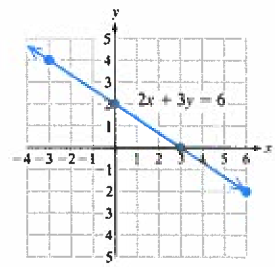
Solution:

a. Solve the equation for y. Then substitute arbitrary values of x into the equation and solve for the corresponding values of y.

$$2x + 3y = 6 \qquad \text{Solve the equation for } y.$$
$$3y = -2x + 6$$
$$y = -\frac{2}{3}x + 2$$

In the table we have selected convenient values of x that are multiples of 3.

x	y
−3	4
0	2
3	0
6	−2

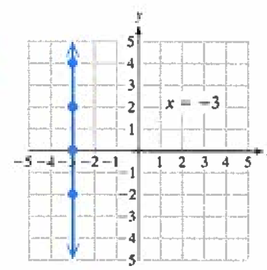


b. $x = -3$

The solutions to this equation must have an x-coordinate of −3. The y variable can be *any* real number.

x	y
−3	−2
−3	0
−3	2
−3	4

x must be −3. y can be any real number.



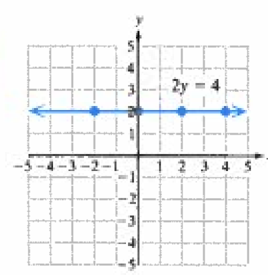
c. $2y = 4$ Solve for y.

$y = 2$

The solutions to this equation must have a y-coordinate of 2. The x variable can be *any* real number.

x	y
−2	2
0	2
2	2
4	2

x can be any real number. y must be 2.



Skill Practice 1 Graph the line represented by each equation.

a. $4x + 2y = 2$ **b.** $y = 1$ **c.** $-3x = 12$

Avoiding Mistake

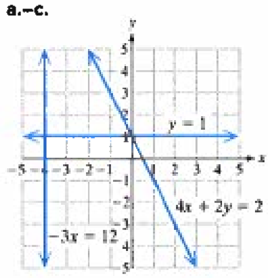
The graph of a linear equation is a line. Therefore, a minimum of two points is needed to graph the line. A third point can be used to verify that the line is graphed correctly. The points must all line up.

TIP The graph of a vertical line will have no y-intercept unless the line is the y-axis itself.

TIP The graph of a horizontal line will have no x-intercept unless the line is the x-axis itself.

Answer

1. a.–c.



2. Determine the Slope of a Line

One of the important characteristics of a nonvertical line is that for every 1 unit of change in the horizontal variable, the vertical change is a constant m called the **slope** of the line. For example, consider the line representing the median income for individuals with a bachelor's degree, x years since the year 1990. The line in Figure 1-16 has a slope of \$1261. This means that median income for individuals with a bachelor's degree increased on average by \$1261 per year during this time period.

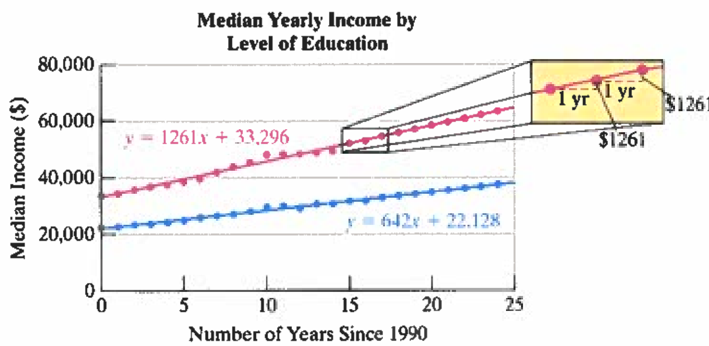


Figure 1-16

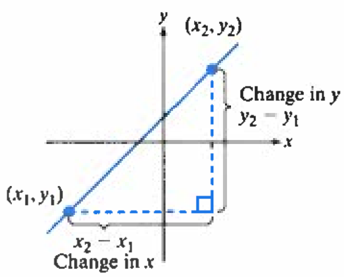


Figure 1-17

Consider any two distinct points (x_1, y_1) and (x_2, y_2) on a line (Figure 1-17). The slope m of the line through the points is the ratio between the change in the y values $(y_2 - y_1)$ and the change in the x values $(x_2 - x_1)$. In many applications in the sciences, the change in a variable is denoted by the Greek letter Δ (delta). Therefore, $(y_2 - y_1)$ can be represented by Δy and $(x_2 - x_1)$ can be represented by Δx.

Slope Formula

The **slope** of a line passing through the distinct points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{\overbrace{y_2 - y_1}^{\text{change in } y \text{ (rise)}}}{\underbrace{x_2 - x_1}_{\text{change in } x \text{ (run)}}} \quad \text{provided that } x_2 - x_1 \neq 0$$

EXAMPLE 2 Finding the Slope of a Line Through Two Points

Find the slope of the line passing through the given points.

a. $(-3, -2)$ and $(2, 5)$ **b.** $\left(-\frac{5}{2}, 0\right)$ and $(1, -7)$

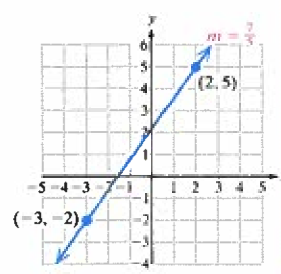
Solution:

a. $(-3, -2)$ and $(2, 5)$

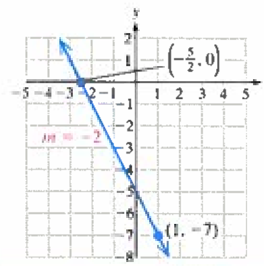
(x_1, y_1) and (x_2, y_2) Label the points.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-2)}{2 - (-3)} = \frac{7}{5}$$

A line with a positive slope "*rises*" upward from left to right.



b. $\left(-\frac{5}{2}, 0\right)$ and $(1, -7)$



(x_1, y_1) and (x_2, y_2) Label the points.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7 - 0}{1 - (-\frac{5}{2})} = \frac{-7}{\frac{7}{2}} = -7 \cdot \frac{2}{7} = -2$$

A line with a negative slope "*falls*" downward from left to right.

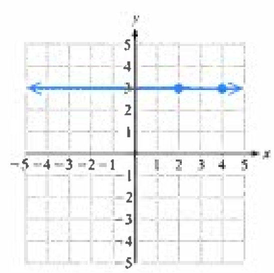
Skill Practice 2 Find the slope of the line passing through the given points.

a. $(-4, 1)$ and $(2, -2)$ **b.** $\left(\frac{3}{4}, 2\right)$ and $(-3, 17)$

EXAMPLE 3 Finding the Slope of Horizontal and Vertical Lines

Find the slope of each line.

a.



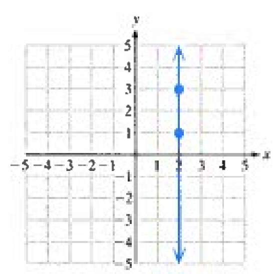
Solution:

By inspection, we see that between any two points on the graph, the vertical change is zero, so the slope is zero.

To compute this numerically, select any two points on the line such as (2, 3) and (4, 3).

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 3}{4 - 2} = \frac{0}{2} = 0$$

b.



To find the slope, select any two points on the line such as (2, 1) and (2, 3).

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 1}{2 - 2} = \frac{2}{0} \quad \text{(undefined)}$$

By inspection, we see that between any two points on the line, the change in x is zero. This makes the slope undefined because the ratio representing the slope has a divisor of zero.

Skill Practice 3 Fill in the blank.

a. The slope of a vertical line is ________.

b. The slope of a horizontal line is ________.

From Example 1, we see that a linear equation may represent the graph of a slanted line, a horizontal line, or a vertical line. From Examples 2 and 3, we see that a line may have a positive slope, a negative slope, a zero slope, or an undefined slope.

Answers

2. a. $-\frac{1}{2}$ **b.** -4

3. a. Undefined **b.** 0

Linear Equations and Slopes of Lines

$Ax + By = C$ $(A \neq 0, B \neq 0)$	$y = k$ (k is a constant)	$x = k$ (k is a constant)
Slanted line	Horizontal line	Vertical line
		
Positive slope Negative slope	Zero slope	Undefined slope

3. Apply the Slope-Intercept Form of a Line

The slope formula can be used to develop the slope-intercept form of a line. Suppose that a line has a slope m and y-intercept $(0, b)$. Let (x, y) be any other point on the line. From the slope formula, we have:

$$\frac{y - b}{x - 0} = m \qquad \text{Slope formula}$$

$$y - b = mx \qquad \text{Multiply by } x.$$

$$y = mx + b \qquad \text{This is slope-intercept form. Slope intercept-form has the } y \text{ variable isolated.}$$

Avoiding Mistakes

An equation of a vertical line takes the form $x = k$, where k is a constant. Because there is no y variable and because the slope is undefined, an equation of a vertical line cannot be written in slope-intercept form.

Slope-Intercept Form of a Line

Given a line with slope m and y-intercept $(0, b)$, the **slope-intercept form** of the line is given by $y = mx + b$.

The slope-intercept form of a line is particularly useful because we can identify the slope and y-intercept by inspection. For example:

$y = \frac{2}{3}x - 5$	$m = \frac{2}{3}$	y-intercept: $(0, -5)$
$y = x + 4$	$m = 1$	y-intercept: $(0, 4)$
$y = 2x$ (or $y = 2x + 0$)	$m = 2$	y-intercept: $(0, 0)$
$y = 6$ (or $y = 0x + 6$)	$m = 0$	y-intercept: $(0, 6)$

If the slope and y-intercept of a line are known, we can graph the line. This is demonstrated in Example 4.

EXAMPLE 4 **Using the Slope and y-Intercept to Graph a Line**

Given $3x + 4y = 4$,

a. Write the equation in slope-intercept form.
b. Determine the slope and y-intercept.
c. Graph the line by using the slope and y-intercept.

Solution:

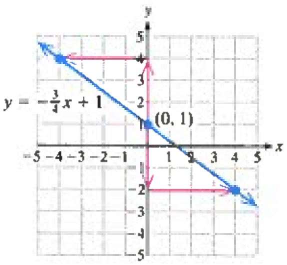
a. $3x + 4y = 4$

$$4y = -3x + 4 \qquad \text{To write an equation in slope-intercept form, isolate the } y \text{ variable.}$$

$$y = -\frac{3}{4}x + 1 \qquad \text{Slope-intercept form}$$

b. $m = -\frac{3}{4}$ and the y-intercept is $(0, 1)$. The slope is the coefficient on x. The constant term gives the y-intercept.

c.



To graph the line, first plot the y-intercept (0, 1).

Then begin at the y-intercept, and use the slope to find a second point on the line. In this case, the slope can be interpreted as the following two ratios:

$$m = \frac{-3}{4} \begin{array}{l} \leftarrow \text{Move down 3 units.} \\ \leftarrow \text{Move to the right 4 units.} \end{array}$$

$$m = \frac{3}{-4} \begin{array}{l} \leftarrow \text{Move up 3 units.} \\ \leftarrow \text{Move to the left 4 units.} \end{array}$$

Skill Practice 4 Given $2x + 4y = 8$,

a. Write the equation in slope-intercept form.
b. Determine the slope and y-intercept.
c. Graph the line by using the slope and y-intercept.

Notice that the slope-intercept form of a line $y = mx + b$ has the y variable isolated and defines y in terms of x. Therefore, an equation written in slope-intercept form defines y as a function of x. In Example 4, $y = -\frac{3}{4}x + 1$ can be written using function notation as $f(x) = -\frac{3}{4}x + 1$.

Definition of Linear and Constant Functions

Let m and b represent real numbers where $m \neq 0$. Then,

- A function defined by $f(x) = mx + b$ is a **linear function.** The graph of a linear function is a slanted line.
- A function defined by $f(x) = b$ is a **constant function.** The graph of a constant function is a horizontal line.

The slope-intercept form of a line can be used as a tool to define a linear function given a point on the line and the slope.

EXAMPLE 5 Writing an Equation of a Line Given a Point and the Slope

Write an equation of the line that passes through the point $(2, -3)$ and has slope -4. Then write the linear equation using function notation, where $y = f(x)$.

Solution:

Given $m = -4$ and $(2, -3)$. We need to find an equation of the form $y = mx + b$.

$y = mx + b$

$y = -4x + b$ The value of m is given as -4.

$-3 = -4(2) + b$ Substitute $x = 2$ and $y = -3$ from the given point $(2, -3)$.

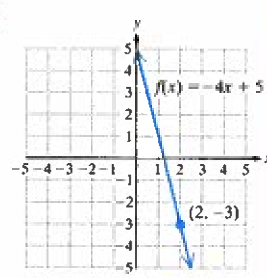
$-3 = -8 + b$ Solve for b.

$b = 5$

$y = mx + b$

$y = -4x + 5$ Substitute $m = -4$ and $b = 5$ into the equation $y = mx + b$.

$f(x) = -4x + 5$ Write the relation using function notation.



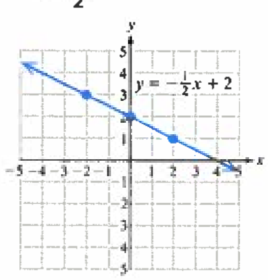
From the graph, we see that the graph of $f(x) = -4x + 5$ does indeed pass through the point $(2, -3)$ and has slope -4.

Answers

4. a. $y = -\frac{1}{2}x + 2$

b. $m = -\frac{1}{2}$; y-intercept: (0, 2)

c.



Skill Practice 5 Write an equation of the line that passes through the point $(-1, -4)$ and has slope 3. Then write the equation using function notation.

4. Compute Average Rate of Change

The graphs of many functions are not linear. However, we often use linear approximations to analyze nonlinear functions on small intervals. For example, the graph in Figure 1-18 shows the blood alcohol concentration (BAC) for an individual over a period of 9 hr.

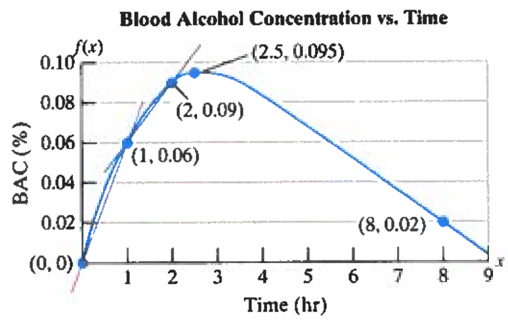


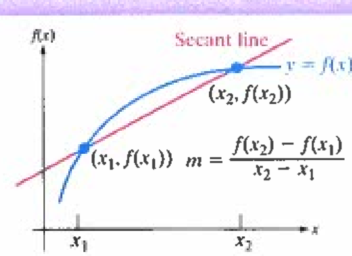
Figure 1-18

A line drawn through two points on a curve is called a **secant line.** In Figure 1-18, the average rate of change in BAC between two points on the graph is the slope of the secant line through the points. Notice that the slope of the secant line between $x = 0$ and $x = 1$ (shown in red) is greater than the slope of the secant line between $x = 1$ and $x = 2$ (shown in green). This means that the average increase in BAC is greater over the first hour than over the second hour.

Average Rate of Change of a Function

Suppose that the points (x_1, y_1) and (x_2, y_2) are points on the graph of a function f. Using function notation, these are the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

If f is defined on the interval $[x_1, x_2]$, then the **average rate of change** of f on the interval $[x_1, x_2]$ is the slope of the secant line containing $(x_1, f(x_1))$ and $(x_2, f(x_2))$.



Average rate of change: $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$ or $m = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

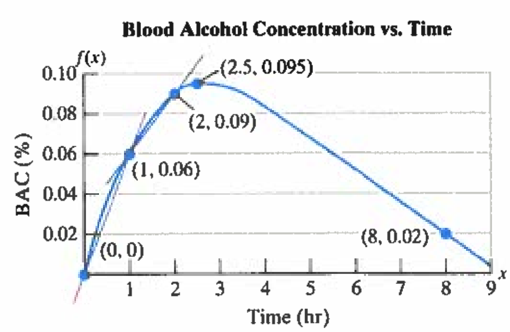
EXAMPLE 6 Computing Average Rate of Change

Determine the average rate of change of blood alcohol level

a. from $x_1 = 0$ to $x_2 = 1$.

b. from $x_1 = 1$ to $x_2 = 2$.

c. Interpret the results from parts (a) and (b).



Answer

5. $y = 3x - 1$; $f(x) = 3x - 1$

Solution:

a. Average rate of change $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{f(1) - f(0)}{1 - 0} = \dfrac{0.06 - 0}{1}$

$= 0.06$

b. Average rate of change $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{f(2) - f(1)}{2 - 1} = \dfrac{0.09 - 0.06}{1}$

$= 0.03$

c. The blood alcohol concentration rose by an average of 0.06% per hour during the first hour.

The blood alcohol concentration rose by an average of 0.03% per hour during the second hour.

Skill Practice 6 Refer to the graph in Example 6.

a. Determine the average rate of change of blood alcohol level from $x_1 = 2.5$ to $x_2 = 8$. Round to 3 decimal places.

b. Interpret the results from part (a).

EXAMPLE 7 Computing Average Rate of Change

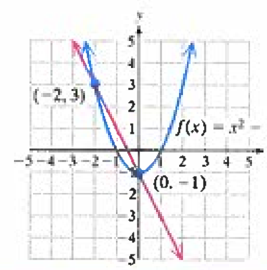
Given the function defined by $f(x) = x^2 - 1$, determine the average rate of change from $x_1 = -2$ to $x_2 = 0$.

Solution:

Average rate of change $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$= \dfrac{f(0) - f(-2)}{0 - (-2)} = \dfrac{-1 - 3}{2} = -2$

The average rate of change is -2.



Skill Practice 7 Given the function defined by $f(x) = x^3 + 2$, determine the average rate of change from $x_1 = -3$ to $x_2 = 0$.

5. Solve Equations and Inequalities Graphically

In many settings, the use of technology can provide a numerical and visual interpretation of an algebraic problem. For example, consider the equation $-x - 1 = x + 5$.

$$-x - 1 = x + 5$$
$$-6 = 2x$$
$$-3 = x \qquad \text{The solution set is } \{-3\}.$$

Now suppose that we create two functions from the left and right sides of the equation. We have $Y_1 = -x - 1$ and $Y_2 = x + 5$. Figure 1-19 shows that the graphs

Answers

6. a. -0.014

b. The blood alcohol concentration decreased by an average of 0.014% per hour during this time interval.

7. 9

of the two lines intersect at $(-3, 2)$. The x-coordinate of the point of intersection is the solution to the equation $-x - 1 = x + 5$. That is, $Y_1 = Y_2$ when $x = -3$.

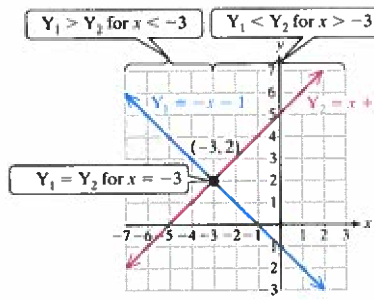


Figure 1-19

The graphs $Y_1 = -x - 1$ and $Y_2 = x + 5$ can also be used to find the solution sets to the related inequalities.

$-x - 1 < x + 5$ The solution set is the set of x values for which $Y_1 < Y_2$. This is the interval where the blue line is below the red line. The solution set is $(-3, \infty)$.

$-x - 1 > x + 5$ The solution set is the set of x values for which $Y_1 > Y_2$. This is the interval where the blue line is *above* the red line. The solution set is $(-\infty, -3)$.

EXAMPLE 8 Solving Equations and Inequalities Graphically

Solve the equations and inequalities graphically.

a. $2x - 3 = x - 1$ **b.** $2x - 3 < x - 1$ **c.** $2x - 3 > x - 1$

Solution:

a. The left side of the equation is graphed as $Y_1 = 2x - 3$. The right side of the equation is graphed as $Y_2 = x - 1$. The point of intersection is $(2, 1)$. Therefore, $Y_1 = Y_2$ for $x = 2$.

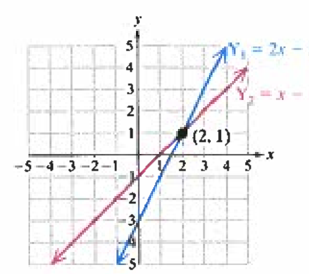
The solution set is $\{2\}$.

b. $Y_1 < Y_2$ to the *left* of $x = 2$. (That is, the blue line is below the red line for $x < 2$.)

In interval notation the solution set is $(-\infty, 2)$.

c. $Y_1 > Y_2$ to the *right* of $x = 2$. (That is, the blue line is above the red line for $x > 2$.)

In interval notation the solution set is $(2, \infty)$.



TIP The solution set to the inequality $2x - 3 \le x - 1$ includes equality, so the right endpoint would be included: $(-\infty, 2]$.

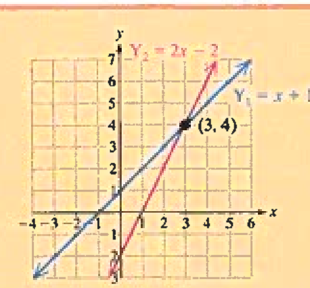
The solution set to the inequality $2x - 3 \ge x - 1$ includes equality, so the left endpoint would be included: $[2, \infty)$.

Skill Practice 8 Use the graph to solve the equations and inequalities.

a. $x + 1 = 2x - 2$

b. $x + 1 \le 2x - 2$

c. $x + 1 \ge 2x - 2$



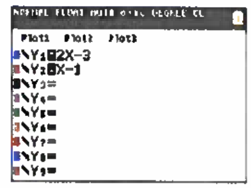
Answers

8. a. $\{3\}$ **b.** $[3, \infty)$ **c.** $(-\infty, 3]$

TECHNOLOGY CONNECTIONS

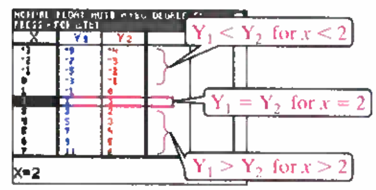
Verifying Solutions to an Equation

We can verify the solutions to the equations and inequalities from Example 8 on a graphing calculator.

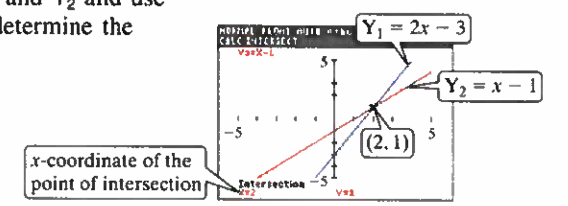


The solutions can be verified numerically by using the **Table** feature on the calculator. First enter $Y_1 = 2x - 3$ and $Y_2 = x - 1$.

Then display the table values for Y_1 and Y_2 for $x = 2$ and for x values less than and greater than 2.



Display the graphs of Y_1 and Y_2 and use the **Intersect** feature to determine the point of intersection.



In Example 9 we solve the equation $6x - 2(x + 2) - 5 = 0$. Notice that one side is zero. We can check the solution graphically by determining where the related function $Y_1 = 6x - 2(x + 2) - 5$ intersects the x-axis.

EXAMPLE 9 Solving Equations and Inequalities Graphically

a. Solve the equation $6x - 2(x + 2) - 5 = 0$ and verify the solution graphically on a graphing utility.

b. Use the graph to find the solution set to the inequality $6x - 2(x + 2) - 5 \leq 0$.

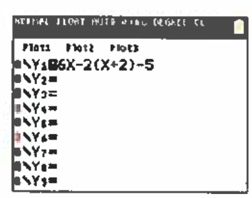
c. Use the graph to find the solution set to the inequality $6x - 2(x + 2) - 5 \geq 0$.

Solution:

a.
$$\begin{aligned} 6x - 2(x + 2) - 5 &= 0 \\ 6x - 2x - 4 - 5 &= 0 \\ 4x - 9 &= 0 \\ x &= \frac{9}{4} \end{aligned}$$

The solution set is $\left\{\frac{9}{4}\right\}$.

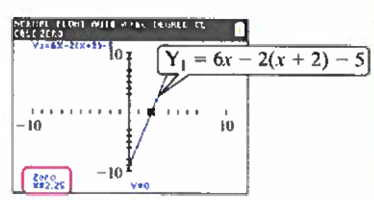
To verify the solution graphically enter the left side of the equation as $Y_1 = 6x - 2(x + 2) - 5$.



Using the **Zero** feature, we have $Y_1 = 0$ for $x = 2.25$. This is consistent with the solution $x = \frac{9}{4}$.

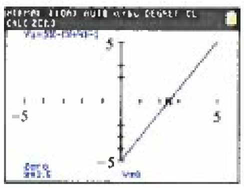
b. To solve $6x - 2(x + 2) - 5 \leq 0$ determine the values of x for which $Y_1 \leq 0$ (where the function is on or below the x-axis).

The solution set is $\left(-\infty, \frac{9}{4}\right]$.



Answers

9. a. $\left\{\frac{5}{2}\right\}$



b. $\left(-\infty, \frac{5}{2}\right]$

c. $\left[\frac{5}{2}, \infty\right)$

c. To solve $6x - 2(x + 2) - 5 \geq 0$ determine the values of x for which $Y_1 \geq 0$ (where the function is on or above the x-axis).

The solution set is $\left[\frac{9}{4}, \infty\right)$.

Skill Practice 9

a. Solve the equation $3x - (x + 4) - 1 = 0$ and verify the solution graphically on a graphing utility.

b. Use the graph to find the solution set to the inequality $3x - (x + 4) - 1 \leq 0$.

c. Use the graph to find the solution set to the inequality $3x - (x + 4) - 1 \geq 0$.

SECTION 1.4 Practice Exercises

Prerequisite Review

R.1. Determine the x- and y-intercepts for $h(x) = 6x - 42$.

R.2. Solve $-7x - 8y = 1$ for y.

For Exercises R.3–R.4, solve the inequality. Write the solution set in interval notation.

R.3. $-4t + 5 < 13$

R.4. $6p - 2 \geq 5p + 8$

R.5. Given the function defined by $g(x) = -x^2 + 3x + 2$, find $g(-1)$.

Concept Connections

1. A ______________ equation in the variables x and y can be written in the form $Ax + By = C$, where A and B are not both zero.
2. An equation of the form $x = k$ where k is a constant represents the graph of a ______________ line.
3. An equation of the form $y = k$ where k is a constant represents the graph of a ______________ line.
4. Write the formula for the slope of a line between the two distinct points (x_1, y_1) and (x_2, y_2).
5. The slope of a horizontal line is ______________ and the slope of a vertical line is ______________.
6. A function f is a linear function if $f(x) =$ ______________, where m represents the slope and $(0, b)$ represents the y-intercept.
7. If f is defined on the interval $[x_1, x_2]$, then the average rate of change of f on the interval $[x_1, x_2]$ is given by the formula ______________.
8. The graph of a constant function defined by $f(x) = b$ is a (horizontal/vertical) line.

Objective 1: Graph Linear Equations in Two Variables

For Exercises 9–20, graph the equation and identify the x- and y-intercepts. (See Example 1)

9. $-3x + 4y = 12$ **10.** $-2x + y = 4$ **11.** $2y = -5x + 2$

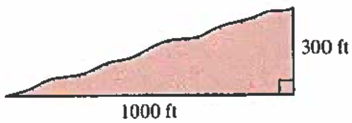
12. $3y = -4x + 6$ **13.** $x = -6$ **14.** $y = 4$

15. $5y + 1 = 11$ **16.** $3x - 2 = 4$ **17.** $0.02x + 0.05y = 0.1$

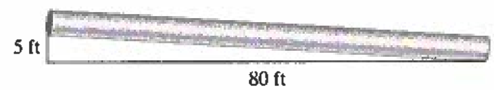
18. $0.03x + 0.07y = 0.21$ **19.** $2x = 3y$ **20.** $2x = -5y$

Objective 2: Determine the Slope of a Line

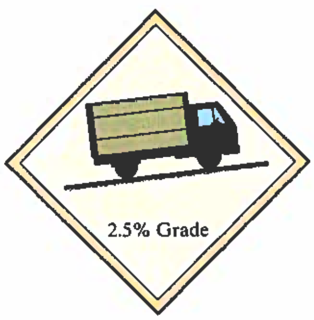
21. Find the average slope of the hill.



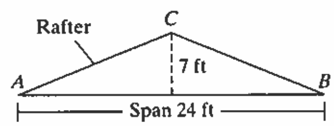
22. Find the absolute value of the slope of the storm drainage pipe.



23. The road sign shown in the figure indicates the percent grade of a hill. This gives the slope of the road as the change in elevation per 100 horizontal feet. Given a 2.5% grade, write this as a slope in fractional form.



24. The pitch of a roof is defined as $\dfrac{\text{rafter rise}}{\text{rafter run}}$ and the fraction is typically written with a denominator of 12. Determine the pitch of the roof from point A to point C.



For Exercises 25–36, determine the slope of the line passing through the given points. (See Example 2)

25. $(4, -7)$ and $(2, -1)$

26. $(-3, -8)$ and $(4, 6)$

27. $(17, 9)$ and $(42, -6)$

28. $(-9, 4)$ and $(-1, -6)$

29. $(30, -52)$ and $(-22, -39)$

30. $(-100, -16)$ and $(84, 30)$

31. $(2.6, 4.1)$ and $(9.5, -3.7)$

32. $(8.5, 6.2)$ and $(-5.1, 7.9)$

33. $\left(\frac{3}{4}, 6\right)$ and $\left(\frac{5}{2}, 1\right)$

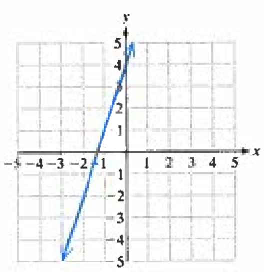
34. $\left(-3, \frac{2}{5}\right)$ and $\left(4, \frac{3}{10}\right)$

35. $(3\sqrt{6}, 2\sqrt{5})$ and $(\sqrt{6}, \sqrt{5})$

36. $(2\sqrt{11}, -3\sqrt{3})$ and $(\sqrt{11}, -5\sqrt{3})$

For Exercises 37–42, determine the slope of the line. (See Examples 2–3)

37.



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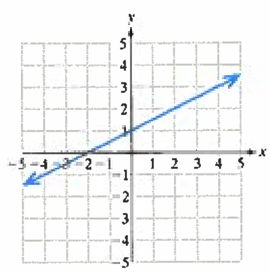

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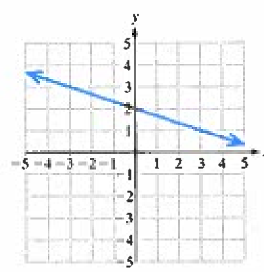

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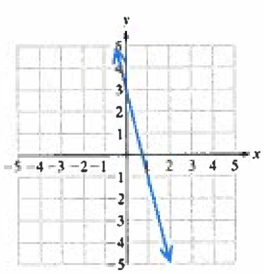

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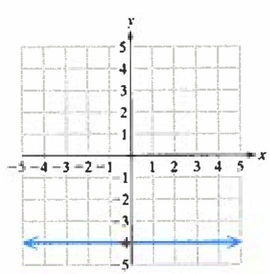

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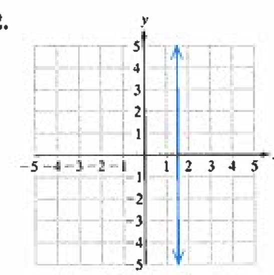


43. What is the slope of a line perpendicular to the x-axis?

44. What is the slope of a line parallel to the x-axis?

45. What is the slope of a line defined by $y = -7$?

46. What is the slope of a line defined by $x = 2$?

47. If the slope of a line is $\frac{4}{5}$, how much vertical change will be present for a horizontal change of 52 ft?

48. If the slope of a line is $\frac{5}{8}$, how much horizontal change will be present for a vertical change of 216 m?

49. Suppose that $y = P(t)$ represents the population of a city at time t. What does $\frac{\Delta P}{\Delta t}$ represent?

50. Suppose that $y = d(t)$ represents the distance that an object travels in time t. What does $\frac{\Delta d}{\Delta t}$ represent?

Objective 3: Apply the Slope-Intercept Form of a Line

For Exercises 51–62,

a. Write the equation in slope-intercept form if possible, and determine the slope and y-intercept.

b. Graph the equation using the slope and y-intercept. **(See Example 4)**

51. $2x - 4y = 8$ **52.** $3x - y = 6$ **53.** $3x = 2y - 4$ **54.** $5x = 3y - 6$

55. $3x = 4y$ **56.** $-2x = 3y$ **57.** $2y - 6 = 8$ **58.** $3y + 9 = 6$

59. $0.02x + 0.06y = 0.06$ **60.** $0.03x + 0.04y = 0.12$ **61.** $\frac{x}{4} + \frac{y}{7} = 1$ **62.** $\frac{x}{3} + \frac{y}{4} = 1$

For Exercises 63–64, determine if the function is linear, constant, or neither.

63. a. $f(x) = -\frac{3}{4}x$ **b.** $g(x) = -\frac{3}{4}x - 3$ **c.** $h(x) = -\frac{3}{4x}$ **d.** $k(x) = -\frac{3}{4}$

64. a. $m(x) = 5x + 1$ **b.** $n(x) = \frac{5}{x} + 1$ **c.** $p(x) = 5$ **d.** $q(x) = 5x$

For Exercises 65–74,

a. Use slope-intercept form to write an equation of the line that passes through the given point and has the given slope.

b. Write the equation using function notation where $y = f(x)$. **(See Example 5)**

65. $(0, 9)$; $m = \frac{1}{2}$ **66.** $(0, -4)$; $m = \frac{1}{3}$ **67.** $(1, -6)$; $m = -3$ **68.** $(2, -8)$; $m = -5$

69. $(-5, -3)$; $m = \frac{2}{3}$ **70.** $(-4, -2)$; $m = \frac{3}{2}$ **71.** $(2, 5)$; $m = 0$ **72.** $(-1, -3)$; $m = 0$

73. $(3.6, 5.1)$; $m = 1.2$ **74.** $(1.2, 2.8)$; $m = 2.4$

For Exercises 75–78,

a. Use slope-intercept form to write an equation of the line that passes through the two given points.

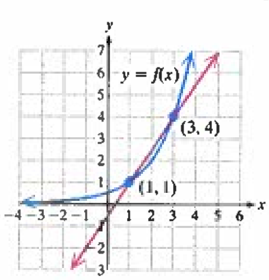
b. Then write the equation using function notation where $y = f(x)$.

75. $(4, 2)$ and $(0, -6)$ **76.** $(-8, 1)$ and $(0, -3)$ **77.** $(7, -3)$ and $(4, 1)$ **78.** $(2, -4)$ and $(-1, 3)$

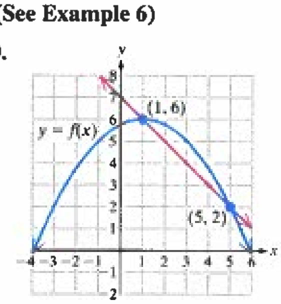
Objective 4: Compute Average Rate of Change

For Exercises 79–80, find the slope of the secant line pictured in red. (See Example 6)

79.

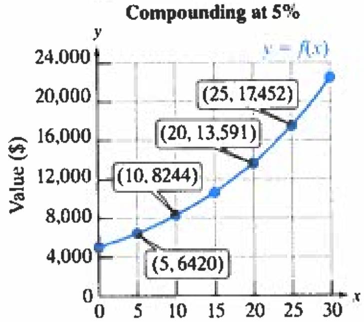


80.



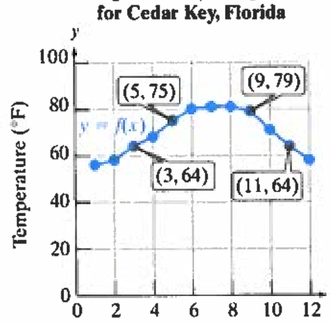
81. The function given by $y = f(x)$ shows the value of \$5000 invested at 5% interest compounded continuously, x years after the money was originally invested.
   a. Find the average amount earned per year between the 5th year and 10th year.
   b. Find the average amount earned per year between the 20th year and the 25th year.
   c. Based on the answers from parts (a) and (b), does it appear that the rate at which annual income increases is increasing or decreasing with time?

Value of \$5000 with Continuous Compounding at 5%



82. The function given by $y = f(x)$ shows the average monthly temperature (in °F) for Cedar Key. The value of x is the month number and $x = 1$ represents January.
   a. Find the average rate of change in temperature between months 3 and 5 (March and May).
   b. Find the average rate of change in temperature between months 9 and 11 (September and November).
   c. Comparing the results in parts (a) and (b), what does a positive rate of change mean in the context of this problem? What does a negative rate of change mean?

Average Monthly Temperature for Cedar Key, Florida



83. The number $N(t)$ of new cases of a flu outbreak for a given city is given by $N(t) = 5000 \cdot 2^{-0.04t^2}$, where t is the number of months since the outbreak began.
   a. Find the average rate of change in the number of new flu cases between months 0 and 2, and interpret the result. Round to the nearest whole unit.
   b. Find the average rate of change in the number of new flu cases between months 4 and 6, and between months 10 and 12.
   c. Use a graphing utility to graph the function. Use the graph and the average rates of change found in parts (a) and (b) to discuss the pattern of the number of new flu cases.

84. The speed $v(L)$ (in m/sec) of an ocean wave in deep water is approximated by $v(L) = 1.2\sqrt{L}$, where L (in meters) is the wavelength of the wave. (The wavelength is the distance between two consecutive wave crests.)
   a. Find the average rate of change in speed between waves that are between 1 m and 4 m in length.
   b. Find the average rate of change in speed between waves that are between 4 m and 9 m in length.
   c. Use a graphing utility to graph the function. Using the graph and the results from parts (a) and (b), what does the difference in the rates of change mean?

For Exercises 85–90, determine the average rate of change of the function on the given interval. (See Example 7)

85. $f(x) = x^2 - 3$
   a. on $[0, 1]$
   b. on $[1, 3]$
   c. on $[-2, 0]$

86. $g(x) = 2x^2 + 2$
   a. on $[0, 1]$
   b. on $[1, 3]$
   c. on $[-2, 0]$

87. $h(x) = x^3$
   a. on $[-1, 0]$
   b. on $[0, 1]$
   c. on $[1, 2]$

88. $k(x) = x^3 - 2$
   a. on $[-1, 0]$
   b. on $[0, 1]$
   c. on $[1, 2]$

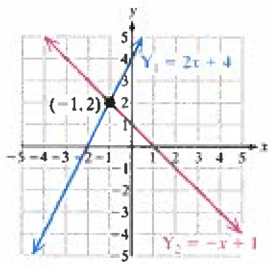
89. $m(x) = \sqrt{x}$
   a. $[0, 1]$
   b. $[1, 4]$
   c. $[4, 9]$

90. $n(x) = \sqrt{x - 1}$
   a. $[1, 2]$
   b. $[2, 5]$
   c. $[5, 10]$

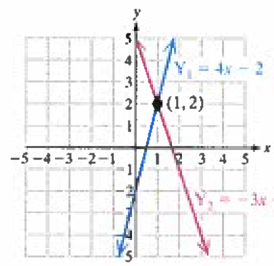
Objective 5: Solve Equations and Inequalities Graphically

For Exercises 91–98, use the graph to solve the equation and inequalities. Write the solutions to the inequalities in interval notation. (See Examples 8–9)

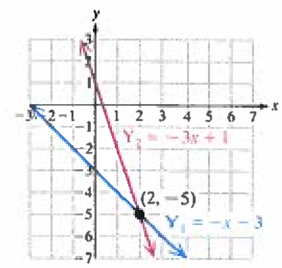
91. a. $2x + 4 = -x + 1$
b. $2x + 4 < -x + 1$
c. $2x + 4 \geq -x + 1$



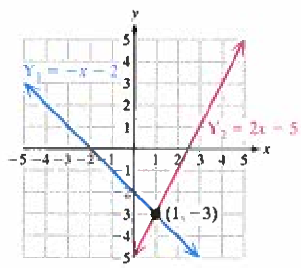
92. a. $4x - 2 = -3x + 5$
b. $4x - 2 < -3x + 5$
c. $4x - 2 \geq -3x + 5$



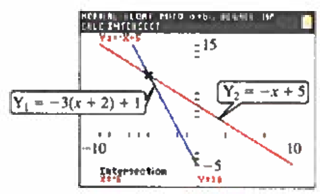
93. a. $-3x + 1 = -x - 3$
b. $-3x + 1 > -x - 3$
c. $-3x + 1 \leq -x - 3$



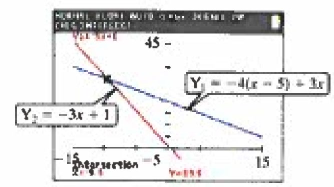
94. a. $-x - 2 = 2x - 5$
b. $-x - 2 \leq 2x - 5$
c. $-x - 2 > 2x - 5$



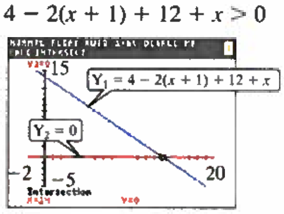
95. a. $-3(x + 2) + 1 = -x + 5$
b. $-3(x + 2) + 1 \leq -x + 5$
c. $-3(x + 2) + 1 \geq -x + 5$



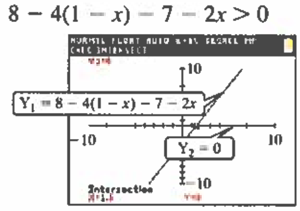
96. a. $-4(x - 5) + 3x = -3x + 1$
b. $-4(x - 5) + 3x \leq -3x + 1$
c. $-4(x - 5) + 3x \geq -3x + 1$



97. a. $4 - 2(x + 1) + 12 + x = 0$
b. $4 - 2(x + 1) + 12 + x < 0$
c. $4 - 2(x + 1) + 12 + x > 0$



98. a. $8 - 4(1 - x) - 7 - 2x = 0$
b. $8 - 4(1 - x) - 7 - 2x < 0$
c. $8 - 4(1 - x) - 7 - 2x > 0$



Write About It

99. Explain how you can determine from a linear equation $Ax + By = C$ (A and B not both zero) whether the line is slanted, horizontal, or vertical.

100. Explain how you can determine from a linear equation $Ax + By = C$ (A and B not both zero) whether the line passes through the origin.

101. What is the benefit of writing an equation of a line in slope-intercept form?

102. Explain how the average rate of change of a function f on the interval $[x_1, x_2]$ is related to slope.

Expanding Your Skills

103. Determine the area in the second quadrant enclosed by the equation $y = 2x + 4$ and the x- and y-axes.

104. Determine the area enclosed by the equations.

$$y = x + 6$$
$$y = -2x + 6$$
$$y = 0$$

105. Determine the area enclosed by the equations.

$$y = -\frac{1}{2}x - 2$$
$$y = \frac{1}{3}x - 2$$
$$y = 0$$

106. Determine the area enclosed by the equations.

$$y = \sqrt{4 - (x - 2)^2}$$
$$y = 0$$

107. Consider the standard form of a linear equation $Ax + By = C$ in the case where $B \neq 0$.

a. Write the equation in slope-intercept form.

b. Identify the slope in terms of the coefficients A and B.

c. Identify the y-intercept in terms of the coefficients B and C.

108. Use the results from Exercise 107 to determine the slope and y-intercept for the graphs of the lines.

a. $5x - 9y = 6$

b. $0.052x - 0.013y = 0.39$

Technology Connections

For Exercises 109–112, solve the equation in part (a) and verify the solution on a graphing calculator. Then use the graph to find the solution set to the inequalities in parts (b) and (c). Write the solution sets to the inequalities in interval notation. (See Example 9)

109. a. $3.1 - 2.2(t + 1) = 6.3 + 1.4t$

b. $3.1 - 2.2(t + 1) > 6.3 + 1.4t$

c. $3.1 - 2.2(t + 1) < 6.3 + 1.4t$

110. a. $-11.2 - 4.6(c - 3) + 1.8c = 0.4(c + 2)$

b. $-11.2 - 4.6(c - 3) + 1.8c > 0.4(c + 2)$

c. $-11.2 - 4.6(c - 3) + 1.8c < 0.4(c + 2)$

111. a. $|2x - 3.8| - 4.6 = 7.2$

b. $|2x - 3.8| - 4.6 \geq 7.2$

c. $|2x - 3.8| - 4.6 \leq 7.2$

112. a. $|x - 1.7| + 4.95 = 11.15$

b. $|x - 1.7| + 4.95 \geq 11.15$

c. $|x - 1.7| + 4.95 \leq 11.15$

For Exercises 113–114, graph the lines in (a)–(c) on the standard viewing window. Compare the graphs. Are they exactly the same? If not, how are they different?

113. a. $y = 3x + 1$

b. $y = 2.99x + 1$

c. $y = 3.01x + 1$

114. a. $y = x + 3$

b. $y = x + 2.99$

c. $y = x + 3.01$

SECTION 1.5 Applications of Linear Equations and Modeling

OBJECTIVES

1. Apply the Point-Slope Formula
2. Determine the Slopes of Parallel and Perpendicular Lines
3. Create Linear Functions to Model Data
4. Create Models Using Linear Regression

1. Apply the Point-Slope Formula

The slope formula can be used to develop the point-slope form of an equation of a line. Suppose that a line has a slope m and passes through a known point (x_1, y_1). Let (x, y) be any other point on the line. From the slope formula, we have

$$\frac{y - y_1}{x - x_1} = m \qquad \text{Slope formula}$$

$$\left(\frac{y - y_1}{x - x_1}\right)(x - x_1) = m(x - x_1) \qquad \text{Clear fractions.}$$

$$y - y_1 = m(x - x_1) \qquad \text{This is called the point slope formula for a line.}$$

The point-slope formula is useful to build an equation of a line given a point on the line and the slope of the line.

Point-Slope Formula

The **point-slope formula** for a line is given by $y - y_1 = m(x - x_1)$, where m is the slope of the line and (x_1, y_1) is a point on the line.

EXAMPLE 1 Writing an Equation of a Line Given a Point on the Line and the Slope

Use the point-slope formula to find an equation of the line passing through the point $(2, -3)$ and having slope -4. Write the answer in slope-intercept form.

Solution:

Label $(2, -3)$ as (x_1, y_1) and $m = -4$.

$$y - y_1 = m(x - x_1) \qquad \text{Apply the point-slope formula.}$$

$$y - (-3) = -4(x - 2) \qquad \text{Substitute } x_1 = 2,\ y_1 = -3, \text{ and } m = -4.$$

$$y + 3 = -4x + 8 \qquad \text{Simplify.}$$

$$y = -4x + 5 \quad \text{(slope-intercept form)}$$

TIP The slope-intercept form of a line can also be used to write an equation of a line if a point on the line and the slope are known. See Example 5 in Section 1.4.

Skill Practice 1 Use the point-slope formula to find an equation of the line passing through the point $(-5, 2)$ and having slope -3. Write the answer in slope-intercept form.

EXAMPLE 2 Writing an Equation of a Line Given Two Points

Use the point-slope formula to write an equation of the line passing through the points $(4, -6)$ and $(-1, 2)$. Write the answer in slope-intercept form.

Answer

1. $y = -3x - 13$

Solution:

To apply the point-slope formula, we first need to know the slope of the line.

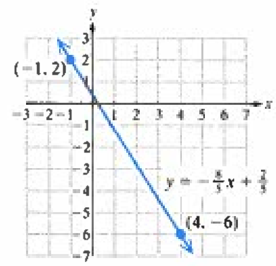
$(4, -6)$ and $(-1, 2)$ (x_1, y_1) and (x_2, y_2)	Label the points. Either point can be labeled (x_1, y_1).
$m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2 - (-6)}{-1 - 4} = \dfrac{8}{-5} = -\dfrac{8}{5}$	Apply the slope formula.
$y - y_1 = m(x - x_1)$	Apply the point-slope formula.
$y - (-6) = -\dfrac{8}{5}(x - 4)$	Substitute $y_1 = -6$, $x_1 = 4$, and $m = -\frac{8}{5}$.

$$y + 6 = -\frac{8}{5}x + \frac{32}{5}$$

$$y = -\frac{8}{5}x + \frac{32}{5} - 6$$

$$y = -\frac{8}{5}x + \frac{32}{5} - \frac{30}{5}$$

$$y = -\frac{8}{5}x + \frac{2}{5} \quad \text{(slope-intercept form)}$$



To check, we see that the graph of the line passes through $(4, -6)$ and $(-1, 2)$ as expected.

TIP In Example 2, the slope-intercept form of a line can also be used to find an equation of the line. Substitute $-\frac{8}{5}$ for m and $(4, -6)$ for (x, y).

$$y = mx + b$$
$$-6 = -\tfrac{8}{5}(4) + b$$
$$-6 = -\tfrac{32}{5} + b$$
$$-6 + \tfrac{32}{5} = b$$
$$\tfrac{2}{5} = b$$

Therefore, $y = mx + b$ is $y = -\frac{8}{5}x + \frac{2}{5}$.

Skill Practice 2 Write an equation of the line passing through the points $(2, -5)$ and $(7, -3)$.

2. Determine the Slopes of Parallel and Perpendicular Lines

Lines in the same plane that do not intersect are **parallel lines.** Nonvertical parallel lines have the same slope and different y-intercepts (Figure 1-20).

Lines that intersect at a right angle are **perpendicular lines.** If two nonvertical lines are perpendicular, then the slope of one line is the opposite of the reciprocal of the slope of the other line (Figure 1-21).

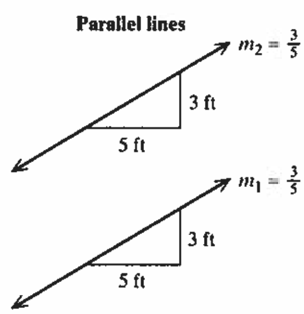


Figure 1-20

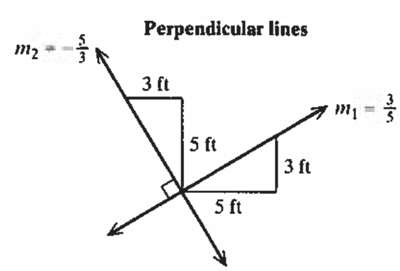


Figure 1-21

Slopes of Parallel and Perpendicular Lines

- If m_1 and m_2 represent the slopes of two nonvertical parallel lines, then $m_1 = m_2$.
- If m_1 and m_2 represent the slopes of two nonvertical perpendicular lines, then $m_1 = -\dfrac{1}{m_2}$ or equivalently $m_1 m_2 = -1$.

Answer

2. $y = \dfrac{2}{5}x - \dfrac{29}{5}$

In Examples 3 and 4, we use the point-slope formula to find an equation of a line through a specified point and parallel or perpendicular to another line.

EXAMPLE 3 Writing an Equation of a Line Parallel to Another Line

Write an equation of the line passing through the point $(-4, 1)$ and parallel to the line defined by $x + 4y = 3$. Write the answer in slope-intercept form and in standard form.

Solution:

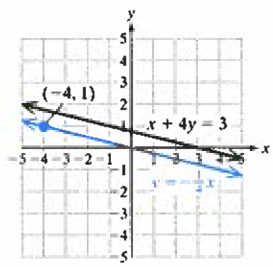
$$x + 4y = 3$$
$$4y = -x + 3$$
$$y = -\frac{1}{4}x + \frac{3}{4}$$

The slope of the given line can be found from its slope-intercept form. Solve for y.

The slope of both lines is $-\frac{1}{4}$.

Apply the point-slope formula with $x_1 = -4$, $y_1 = 1$, and $m = -\frac{1}{4}$.

$$y - y_1 = m(x - x_1)$$
$$y - 1 = -\frac{1}{4}[x - (-4)]$$
$$y - 1 = -\frac{1}{4}(x + 4)$$
$$y - 1 = -\frac{1}{4}x - 1$$
$$y = -\frac{1}{4}x \text{ (slope-intercept form)}$$



From the graph, we see that the line $y = -\frac{1}{4}x$ passes through the point $(-4, 1)$ and is parallel to the graph of $x + 4y = 3$.

$$4(y) = 4\left(-\frac{1}{4}x\right)$$
$$4y = -x$$
$$x + 4y = 0 \text{ (standard form)}$$

Clearing fractions, and collecting the x and y terms on one side of the equation gives us standard form.

Skill Practice 3 Write an equation of the line passing through the point $(-3, 2)$ and parallel to the line defined by $x + 3y = 6$. Write the answer in slope-intercept form and in standard form.

EXAMPLE 4 Writing an Equation of a Line Perpendicular to Another Line

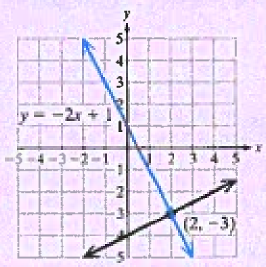
Write an equation of the line passing through the point $(2, -3)$ and perpendicular to the line defined by $y = \frac{1}{2}x - 4$. Write the answer in slope-intercept form and in standard form.

Answer

3. $y = -\frac{1}{3}x + 1$; $x + 3y = 3$

Avoiding Mistakes

The solution to Example 4 can be checked by graphing both lines and verifying that they are perpendicular and that the line $y = -2x + 1$ passes through the point $(2, -3)$.



Solution:

From the slope-intercept form, $y = \frac{1}{2}x - 4$, the slope of given line is $\frac{1}{2}$.

$y - y_1 = m(x - x_1)$ — The slope of a line perpendicular to the given line is -2.

$y - (-3) = -2(x - 2)$ — Apply the point-slope formula with $x_1 = 2$, $y_1 = -3$, and $m = -2$.

$y + 3 = -2x + 4$ — Simplify.

$y = -2x + 1$ (slope-intercept form) — Write the equation in slope-intercept form by solving for y.

$2x + y = 1$ (standard form) — Write the equation in standard form by collecting the x and y terms on one side of the equation.

Skill Practice 4 Write an equation of the line passing through the point $(-8, -4)$ and perpendicular to the line defined by $y = \frac{1}{6}x + 3$.

3. Create Linear Functions to Model Data

In many day-to-day applications, two variables are related linearly. By finding an equation of the line, we produce a model that relates the two variables. This is demonstrated in Example 5.

EXAMPLE 5 **Using a Linear Function in an Application**

A family plan for a cell phone has a monthly base price of \$99 plus \$12.99 for each additional family member added beyond the primary account holder.

a. Write a linear function to model the monthly cost $C(x)$ (in \$) of a family plan for x additional family members added.

b. Evaluate $C(4)$ and interpret the meaning in the context of this problem.

Solution:

a. $C(x) = mx + b$ — The base price \$99 is the fixed cost with zero additional family members added. So the constant b is 99.

$C(x) = 12.99x + 99$ — The rate of increase, \$12.99 per additional family member, is the slope.

b. $C(4) = 12.99(4) + 99$ — Substitute 4 for x.

$= 150.96$

The total monthly cost of the plan with 4 additional family members beyond the primary account holder is \$150.96.

Skill Practice 5 A speeding ticket is \$100 plus \$5 for every 1 mph over the speed limit.

a. Write a linear function to model the cost $S(x)$ (in \$) of a speeding ticket for a person caught driving x mph over the speed limit.

b. Evaluate $S(15)$ and interpret the meaning in the context of this problem.

Answers

4. $y = -6x - 52$; $6x + y = -52$

5. a. $S(x) = 5x + 100$

b. $S(15) = 175$ means that a ticket costs \$175 for a person caught speeding 15 mph over the speed limit.

Linear functions can sometimes be used to model the cost, revenue, and profit of producing and selling x items.

Linear Cost, Revenue, and Profit Functions

A **linear cost function** models the cost $C(x)$ to produce x items.

$C(x) = mx + b$

m is the variable cost per item.
b is the fixed cost.

The fixed cost does not change relative to the number of items produced. For example, the cost to rent an office is a fixed cost. The variable cost per item is the rate at which cost increases for each additional unit produced. Variable costs include labor, material, and shipping.

A **linear revenue function** models revenue $R(x)$ for selling x items.

$R(x) = px$

The product px represents the price per item p times the number of items sold x.

A **linear profit function** models the profit for producing and selling x items.

$P(x) = R(x) - C(x)$

Subtract the cost to produce x items from the revenue brought in from selling x items.

EXAMPLE 6 Writing Linear Cost, Revenue, and Profit Functions

At a summer art show a vendor sells lemonade for \$2.00 per cup. The cost to rent the booth is \$120. Furthermore, the vendor knows that the lemons, sugar, and cups collectively cost \$0.50 for each cup of lemonade produced.

a. Write a linear cost function to produce x cups of lemonade.
b. Write a linear revenue function for selling x cups of lemonade.
c. Write a linear profit function for producing and selling x cups of lemonade.
d. How much profit will the vendor make if 50 cups of lemonade are produced and sold?
e. How much profit will be made for producing and selling 128 cups?
f. Determine the break-even point.

Solution:

a. $C(x) = 0.50x + 120$

The fixed cost is \$120 because it does not change relative to the number of cups of lemonade produced. The variable cost is \$0.50 per lemonade.

b. $R(x) = 2.00x$

The price per cup of lemonade is \$2.00. Therefore, the product $2.00x$ gives the amount of revenue for x cups of lemonade sold.

c. $P(x) = R(x) - C(x)$

Profit is defined as the difference of revenue and cost.

$P(x) = 2.00x - (0.50x + 120)$
$P(x) = 1.50x - 120$

d. $P(50) = 1.50(50) - 120$

Substitute 50 for x.

$= -45$

The vendor will lose \$45.

e. $P(128) = 1.50(128) - 120$ Substitute 128 for x.

$= 72$ The vendor will make \$72.

f. For what value of x will $R(x) = C(x)$ or $P(x) = 0$?

The break-even point is defined as the point where revenue equals cost. Alternatively, this can be stated as the point where profit equals zero: $P(x) = 0$.

$$P(x) = 0$$

$$1.50x - 120 = 0 \quad \text{Solve for } x.$$

$$1.50x = 120$$

$$x = 80$$

If the vendor produces and sells 80 cups of lemonade, the cost and revenue will be equal, resulting in a profit of \$0. This is the break-even point.

Skill Practice 6 Repeat Example 6 in the case where the vendor can cut the cost to \$0.40 per cup of lemonade, and sell lemonades for \$1.50 per cup.

Figure 1-22 shows the graphs of the revenue and cost functions from Example 6. Notice that R and C intersect at (80, 160). This means that if 80 cups of lemonade are produced and sold, the revenue and cost are both \$160. That is, $R(x) = C(x)$ and the company breaks even. The graph of the profit function P is consistent with this result. The value of $P(x)$ is 0 for 80 lemonades produced and sold (Figure 1-23).

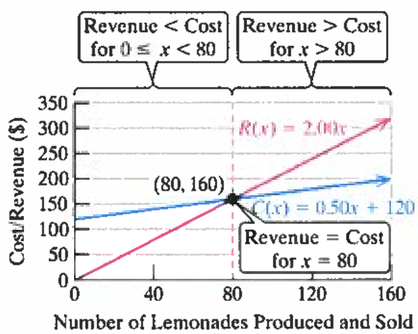


Figure 1-22

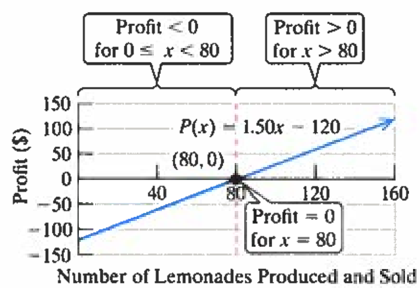


Figure 1-23

From Figures 1-22 and 1-23, we can draw the following conclusions.

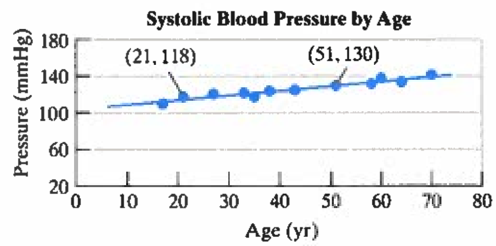
- The company experiences a loss if fewer than 80 cups of lemonade are produced and sold. That is, $R(x) < C(x)$, or equivalently $P(x) < 0$.
- The company experiences a profit if more than 80 cups of lemonade are produced and sold. That is, $R(x) > C(x)$, or equivalently $P(x) > 0$.
- The company breaks even if exactly 80 cups of lemonade are produced and sold. That is, $R(x) = C(x)$, or equivalently $P(x) = 0$.

Answers

6. a. $C(x) = 0.40x + 120$
b. $R(x) = 1.50x$
c. $P(x) = 1.10x - 120$
d. −\$65
e. \$20.80
f. Approximately 109 cups

EXAMPLE 7 Writing a Linear Model to Relate Two Variables

The data shown in the graph represent the age and systolic blood pressure for a sample of 12 randomly selected healthy adults.



a. Suppose that x represents the age of an adult (in yr), and y represents the systolic blood pressure (in mmHg). Use the points (21, 118) and (51, 130) to write a linear model relating y as a function of x.

b. Interpret the meaning of the slope in the context of this problem.

c. Use the model to estimate the systolic blood pressure for a 55-year-old. Round to the nearest whole unit.

TIP The equation $y = -0.4x + 109.6$ can also be expressed in function notation. For example, we can rename y as $S(x)$.

$S(x) = -0.4x + 109.6$

The value $S(x)$ represents the estimated systolic blood pressure for an adult of age x years.

Solution:

a. (21, 118) and (51, 130)

(x_1, y_1) and (x_2, y_2) — Label the points.

$m = \dfrac{130 - 118}{51 - 21} = 0.4$ — Determine the slope of the line.

$y - y_1 = m(x - x_1)$ — Apply the point-slope formula.

$y - 118 = 0.4(x - 21)$

$y = 0.4x + 109.6$ — The equation $y = 0.4x + 109.6$ relates an individual's age to an estimated systolic blood pressure for that age.

b. The slope is 0.4. This means that the average increase in systolic blood pressure for adults is 0.4 mmHg per year of age.

c. $y = 0.4x + 109.6$

$y = 0.4(55) + 109.6$ — Substitute 55 for x.

$y = 131.6$

Based on the sample of data, the estimated systolic blood pressure for a 55-year-old is 132 mmHg.

Skill Practice 7 Suppose that y represents the average consumer spending on television services per year (in dollars), and that x represents the number of years since 2004.

a. Use the data points (2, 308) and (6, 408) to write a linear equation relating y to x.

b. Interpret the meaning of the slope in the context of this problem.

c. Interpret the meaning of the y-intercept in the context of this problem.

d. Use the model from part (a) to estimate the average consumer spending on television services for the year 2007.

Answers

7. **a.** $y = 25x + 258$
   b. The slope is 25 and means that consumer spending on television services rose \$25 per year during this time period.
   c. (0, 258); The average consumer spending on television services for the year 2004 was \$258.
   d. \$333

4. Create Models Using Linear Regression

In Example 7, we used two given data points to determine a linear model for systolic blood pressure versus age. There are two drawbacks to this method. First, the equation is not necessarily unique. If we use two different data points, we may get a different equation. Second, it is generally preferable to write a model that is based on *all* the data points, rather than just two points. One such model is called the least-squares regression line.

The procedure to find the least-squares regression line is discussed in detail in a statistics course. Here we will give the basic premise and use a graphing utility to perform the calculations. Consider a set of data points (x_1, y_1), (x_2, y_2), (x_3, y_3), ... , (x_n, y_n). The **least-squares regression line,** $\hat{y} = mx + b$, is the unique line that minimizes the sum of the squared vertical deviations from the observed data points to the line (Figure 1-24).

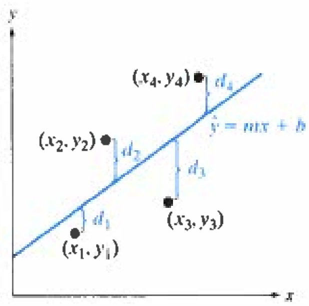


Figure 1-24

On a calculator or spreadsheet, the equation $\hat{y} = mx + b$ may be denoted as $y = ax + b$ or as $y = b_0 + b_1x$. In any event, the coefficient of x is the slope of the line, and the constant gives us the y-intercept. Although the exact keystrokes on different calculators and graphing utilities may vary, we will use the following guidelines to find the least-squares regression line.

Creating a Linear Regression Model

1. Graph the data in a scatter plot.
2. Inspect the data visually to determine if the data suggest a linear trend.
3. Invoke the linear regression feature on a calculator, graphing utility, or spreadsheet.
4. Check the result by graphing the line with the data points to verify that the line passes through or near the data points.

EXAMPLE 8 Finding a Least-Squares Regression Line

The data given in the table represent the age and systolic blood pressure for a sample of 12 randomly selected healthy adults.

Age (yr)	17	21	27	33	35	38	43	51	58	60	64	70
Systolic blood pressure (mmHg)	110	118	121	122	118	124	125	130	132	138	134	142

a. Make a scatter plot of the data using age as the independent variable x and systolic pressure as the dependent variable y.

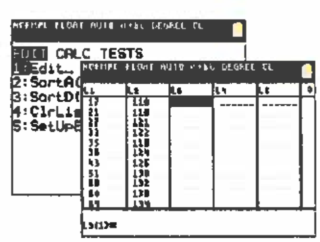
b. Based on the graph, does a linear model seem appropriate?

c. Determine the equation of the least-squares regression line.

d. Use the least-squares regression line to approximate the systolic blood pressure for a healthy 55-year-old. Round to the nearest whole unit.

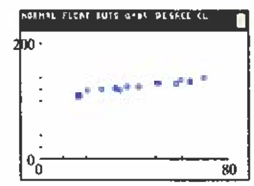
Solution:

a. On a graphing calculator hit the STAT button and select EDIT to enter the x and y data into two lists (shown here as L1 and L2).



Select the STAT PLOT option and turn Plot1 to On. For the type of graph, select the scatter plot image.

Be sure that the window is set to accommodate x values between 17 and 70, and y values between 110 and 142, inclusive. Then hit the GRAPH key. The window settings shown here are [0, 80, 10] by [0, 200, 20].



b. From the graph, the data appear to follow a linear trend.

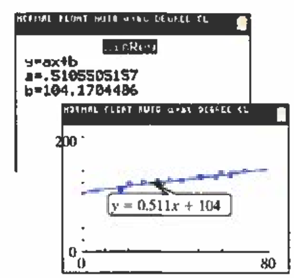
c. Under the STAT menu, select CALC and then the LinReg(ax + b) option.

The command LinReg(ax + b) prompts the user to enter the list names (L_1 and L_2) containing the x and y data values. Then highlight Calculate and hit ENTER.

In the regression model $y = ax + b$, the values for the coefficients a and b are placed on the home screen.

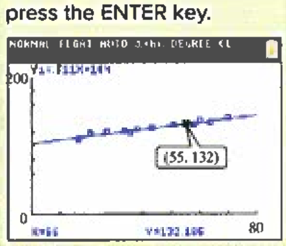
Rounding the values of a and b gives us $y = 0.511x + 104$.

Enter the equation $Y_1 = 0.511x + 104$ into the equation editor and hit the GRAPH key. The graph of the regression line passes near or through the observed data points.



TIP The linear equation found in Example 7 was based on two data points. The least-squares regression line is based on all available data points. The estimate from each model for systolic blood pressure for a 55-year-old rounds to 132 mmHg.

TIP In Example 8(d), the value of the function at $x = 55$ can also be found by selecting the CALC menu and selecting the VALUE function. Enter 55 for x and press the ENTER key.



d. $y = 0.511x + 104$

$y = 0.511(55) + 104$ — To approximate the systolic blood pressure for a 55-year-old, substitute 55 for x.

$= 132.105$

The systolic blood pressure for a healthy 55-year-old would be approximately 132 mmHg.

Skill Practice 8 The data given represent the class averages for individual students based on the number of absences from class.

Number of Absences (x)	3	7	1	11	2	14	2	5
Average in Class (y)	88	67	96	62	90	56	97	82

a. Find the equation of the least-squares regression line.

b. Use the model from part (a) to approximate the average for a student who misses 6 classes.

Answers

8. a. $y = -3.27x + 98.1$

b. The student's average would be approximately 78.5.

SECTION 1.5 Practice Exercises

Prerequisite Review

R.1. Use slope-intercept form to write an equation of the line that passes through $(3, -7)$ with slope -5.

R.2. Write the equation in slope-intercept form and determine the slope and y-intercept $3x - 5y = -15$.

R.3. Determine the slope of the line containing the points $(-4, -2)$ and $(-4, -7)$.

R.4. Determine the slope of the line containing the points $(3, -2)$ and $(5, -2)$.

R.5. The cost C (in dollars) to rent an apartment is \$850 per month, plus a \$450 nonrefundable security deposit, plus a \$250 deposit for each dog or cat. Write a formula for the total cost to rent an apartment for m months with n cats/dogs.

Concept Connections

1. Given a point (x_1, y_1) on a line with slope m, the point-slope formula is given by ______________.

2. If two nonvertical lines have the same slope but different y-intercepts, then the lines are (parallel/perpendicular).

3. If m_1 and m_2 represent the slopes of two nonvertical perpendicular lines, then $m_1m_2 =$ ______________.

4. Suppose that $y = C(x)$ represents the cost to produce x items, and that $y = R(x)$ represents the revenue for selling x items. The profit $P(x)$ of producing and selling x items is defined by $P(x) =$ ______________.

Objective 1: Apply the Point-Slope Formula

For Exercises 5–20, use the point-slope formula to write an equation of the line having the given conditions. Write the answer in slope-intercept form (if possible). (See Examples 1–2)

5. Passes through $(-3, 5)$ and $m = -2$.

6. Passes through $(4, -6)$ and $m = 3$.

7. Passes through $(-1, 0)$ and $m = \frac{2}{3}$.

8. Passes through $(-4, 0)$ and $m = \frac{3}{5}$.

9. Passes through $(3.4, 2.6)$ and $m = 1.2$.

10. Passes through $(2.2, 4.1)$ and $m = 2.4$.

11. Passes through $(6, 2)$ and $(-3, 1)$.

12. Passes through $(-4, 8)$ and $(-7, -3)$.

13. Passes through $(0, 8)$ and $(5, 0)$.

14. Passes through $(0, -6)$ and $(11, 0)$.

15. Passes through $(2.3, 5.1)$ and $(1.9, 3.7)$.

16. Passes through $(1.6, 4.8)$ and $(0.8, 6)$.

17. Passes through $(3, -4)$ and $m = 0$.

18. Passes through $(-5, 1)$ and $m = 0$.

19. Passes through $\left(\frac{2}{3}, \frac{1}{5}\right)$ and the slope is undefined.

20. Passes through $\left(-\frac{4}{7}, \frac{3}{10}\right)$ and the slope is undefined.

21. Given a line defined by $x = 4$, what is the slope of the line?

22. Given a line defined by $y = -2$, what is the slope of the line?

Objective 2: Determine the Slopes of Parallel and Perpendicular Lines

For Exercises 23–28, the slope of a line is given.

a. Determine the slope of a line parallel to the given line, if possible.

b. Determine the slope of a line perpendicular to the given line, if possible.

23. $m = \frac{3}{11}$

24. $m = \frac{6}{7}$

25. $m = -6$

26. $m = -10$

27. $m = 1$

28. m is undefined

For Exercises 29–36, determine if the lines defined by the given equations are parallel, perpendicular, or neither.

29. $y = 2x - 3$
$y = -\frac{1}{2}x + 7$

30. $y = \frac{4}{3}x - 1$
$y = -\frac{3}{4}x + 5$

31. $8x - 5y = 3$
$2x = \frac{5}{4}y + 1$

32. $2x + 3y = 7$
$4x = -6y + 2$

33. $2x = 6$
$5 = y$

34. $3y = 5$
$x = 1$

35. $6x = 7y$
$\frac{7}{2}x - 3y = 0$

36. $5y = 2x$
$\frac{5}{2}x - y = 0$

For Exercises 37–44, write an equation of the line satisfying the given conditions. Write the answer in slope-intercept form (if possible) and in standard form ($Ax + By = C$) with no fractional coefficients. (See Examples 3–4)

37. Passes through (2, 5) and is parallel to the line defined by $2x + y = 6$.

38. Passes through $(3, -1)$ and is parallel to the line defined by $-3x + y = 4$.

39. Passes through $(6, -4)$ and is perpendicular to the line defined by $x - 5y = 1$.

40. Passes through (5, 4) and is perpendicular to the line defined by $x - 2y = 7$.

41. Passes through (6, 8) and is parallel to the line defined by $3x = 7y + 5$.

42. Passes through $(7, -6)$ and is parallel to the line defined by $2x = 5y - 4$.

43. Passes through (2.2, 6.4) and is perpendicular to the line defined by $2x = 4 - y$.

44. Passes through (3.6, 1.2) and is perpendicular to the line defined by $4x = 9 - y$.

For Exercises 45–50, write an equation of the line that satisfies the given conditions.

45. Passes through (8, 6) and is parallel to the x-axis.

46. Passes through $(-11, 13)$ and is parallel to the y-axis.

47. Passes through $\left(\frac{5}{11}, -\frac{3}{4}\right)$ and is perpendicular to the y-axis.

48. Passes through $\left(-\frac{7}{9}, \frac{7}{3}\right)$ and is perpendicular to the x-axis.

49. Passes through $(-61.5, 47.6)$ and is parallel to the line defined by $x = -12$.

50. Passes through $(-0.004, 0.009)$ and is parallel to the line defined by $y = 6$.

Objective 3: Create Linear Functions to Model Data

51. A sales person makes a base salary of \$400 per week plus 12% commission on sales. **(See Example 5)**

a. Write a linear function to model the sales person's weekly salary $S(x)$ for x dollars in sales.

b. Evaluate $S(8000)$ and interpret the meaning in the context of this problem.

52. At a parking garage in a large city, the charge for parking consists of a flat fee of \$2.00 plus \$1.50/hr.

a. Write a linear function to model the cost for parking $P(t)$ for t hours.

b. Evaluate $P(1.6)$ and interpret the meaning in the context of this problem.

53. Millage rate is the amount per \$1000 that is often used to calculate property tax. For example, a home with a \$60,000 taxable value in a municipality with a 19 mil tax rate would require (0.019)(\$60,000) = \$1140 in property taxes. In one county, homeowners pay a flat tax of \$172 plus a rate of 19 mil on the taxable value of a home.

a. Write a linear function that represents the total property tax $T(x)$ for a home with a taxable value of x dollars.

b. Evaluate $T(80{,}000)$ and interpret the meaning in the context of this problem.

54. The average water level in a retention pond is 6.8 ft. During a time of drought, the water level decreases at a rate of 3 in./day.

a. Write a linear function W that represents the water level $W(t)$ (in ft) t days after a drought begins.

b. Evaluate $W(20)$ and interpret the meaning in the context of this problem.

For Exercises 55–56, the fixed and variable costs to produce an item are given along with the price at which an item is sold. (See Example 6)

a. Write a linear cost function that represents the cost $C(x)$ to produce x items.

b. Write a linear revenue function that represents the revenue $R(x)$ for selling x items.

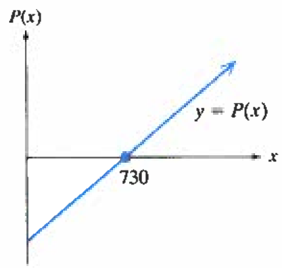
c. Write a linear profit function that represents the profit $P(x)$ for producing and selling x items.

d. Determine the break-even point.

55. Fixed cost: \$2275
Variable cost per item: \$34.50
Price at which the item is sold: \$80.00

56. Fixed cost: \$5625
Variable cost per item: \$0.40
Price at which the item is sold: \$1.30

57. The profit function P is shown for producing and selling x items. Determine the values of x for which

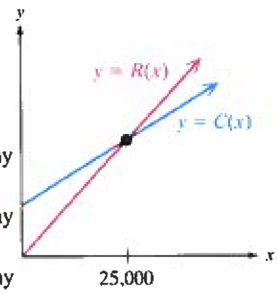


a. $P(x) = 0$ (the company breaks even)

b. $P(x) < 0$ (the company experiences a loss)

c. $P(x) > 0$ (the company makes a profit)

58. The cost and revenue functions C and R are shown for producing and selling x items. Determine the values of x for which



a. $R(x) = C(x)$ (the company breaks even)

b. $R(x) < C(x)$ (the company experiences a loss)

c. $R(x) > C(x)$ (the company makes a profit)

59. A small business makes cookies and sells them at the farmer's market. The fixed monthly cost for use of a Health Department–approved kitchen and rental space at the farmer's market is \$790. The cost of labor, taxes, and ingredients for the cookies amounts to \$0.24 per cookie, and the cookies sell for \$6.00 per dozen. **(See Example 6)**

a. Write a linear cost function representing the cost $C(x)$ to produce x dozen cookies per month.

b. Write a linear revenue function representing the revenue $R(x)$ for selling x dozen cookies.

c. Write a linear profit function representing the profit for producing and selling x dozen cookies in a month.

d. Determine the number of cookies (in dozens) that must be produced and sold for a monthly profit.

e. If 150 dozen cookies are sold in a given month, how much money will the business make or lose?

60. A lawn service company charges \$60 for each lawn maintenance call. The fixed monthly cost of \$680 includes telephone service and depreciation of equipment. The variable costs include labor, gasoline, and taxes and amount to \$36 per lawn.

a. Write a linear cost function representing the monthly cost $C(x)$ for x maintenance calls.

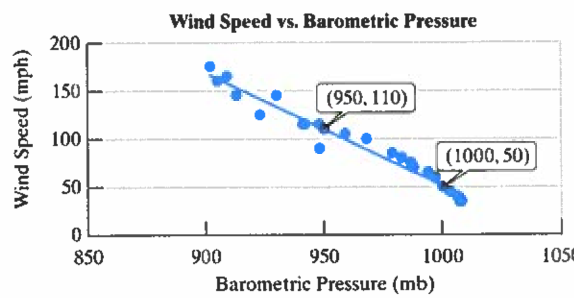
b. Write a linear revenue function representing the monthly revenue $R(x)$ for x maintenance calls.

c. Write a linear profit function representing the monthly profit $P(x)$ for x maintenance calls.

d. Determine the number of lawn maintenance calls needed per month for the company to make money.

e. If 42 maintenance calls are made for a given month, how much money will the lawn service make or lose?

61. The data in the graph show the wind speed y (in mph) for Hurricane Katrina versus the barometric pressure x (in millibars, mb). (*Source*: NOAA: www.noaa.gov) **(See Example 7)**



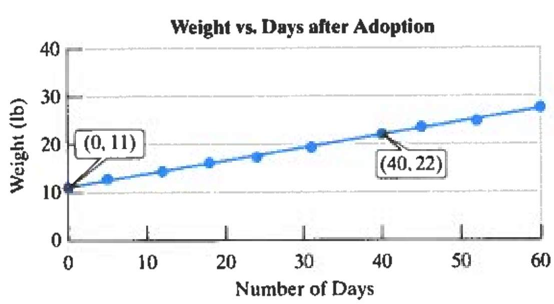
a. Use the points (950, 110) and (1000, 50) to write a linear model for these data.

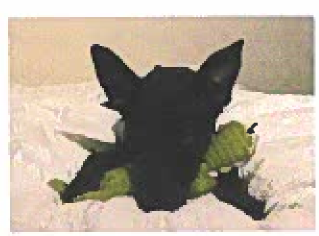
b. Interpret the meaning of the slope in the context of this problem.

c. Use the model from part (a) to estimate the wind speed for a hurricane with a pressure of 900 mb.

d. The lowest barometric pressure ever recorded for an Atlantic hurricane was 882 mb for Hurricane Wilma in 2005. Would it be reasonable to use the model from part (a) to estimate the wind speed for a hurricane with a pressure of 800 mb?

62. Caroline adopted a puppy named Dodger from an animal shelter in Chicago. She recorded Dodger's weight during the first two months. The data in the graph show Dodger's weight y (in lb), x days after adoption.





a. Use the points (0, 11) and (40, 22) to write a linear model for these data.

b. Interpret the meaning of the slope in context.

c. Interpret the meaning of the y-intercept in context.

d. If this linear trend continues during Dodger's growth period, how long will it take Dodger to reach 90% of his expected full-grown weight of 70 lb? Round to the nearest day.

e. Is the model from part (a) reasonable long term?

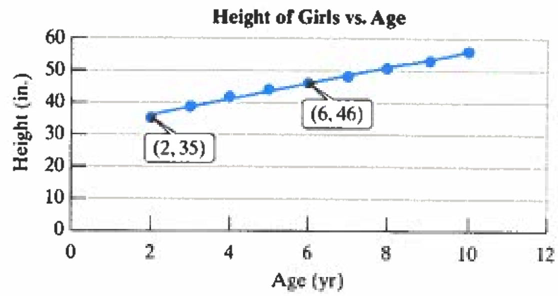
63. A pediatrician records the age x (in yr) and average height y (in inches) for girls between the ages of 2 and 10.

a. Use the points (2, 35) and (6, 46) to write a linear model for these data.

b. Interpret the meaning of the slope in context.

c. Use the model to forecast the average height of 11-yr-old girls.

d. If the height of a girl at age 11 is 90% of her full-grown adult height, use the result of part (c) to estimate the average height of adult women. Round to the nearest tenth of an inch.



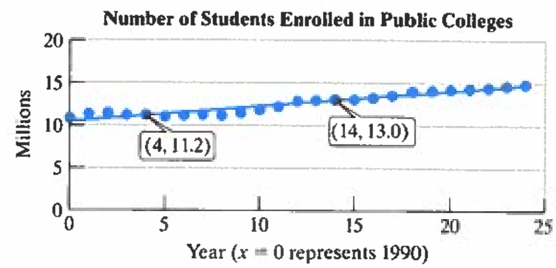
64. The graph shows the number of students enrolled in public colleges for selected years (*Source*: U.S. National Center for Education Statistics, www.nces.ed.gov). The x variable represents the number of years since 1990 and the y variable represents the number of students (in millions).

a. Use the points (4, 11.2) and (14, 13.0) to write a linear model for these data.

b. Interpret the meaning of the slope in the context of this problem.

c. Interpret the meaning of the y-intercept in the context of this problem.

d. In the event that the linear trend continues beyond the last observed data point, use the model in part (a) to predict the number of students enrolled in public colleges for the year 2020.



65. The table gives the number of calories and the amount of cholesterol for selected fast food hamburgers.

a. Graph the data in a scatter plot using the number of calories as the independent variable x and the amount of cholesterol as the dependent variable y.

b. Use the data points (480, 60) and (720, 90) to write a linear function that defines the amount of cholesterol $c(x)$ as a linear function of the number of calories x.

c. Interpret the meaning of the slope in the context of this problem.

d. Use the model from part (b) to predict the amount of cholesterol for a hamburger with 650 calories.

Hamburger Calories	Cholesterol (mg)
220	35
420	50
460	50
480	60
560	70
590	105
610	65
680	80
720	90

66. The table gives the average gestation period for selected animals and their corresponding average longevity.

a. Graph the data in a scatter plot using the number of days for gestation as the independent variable x and the longevity as the dependent variable y.

b. Use the data points (44, 8.5) and (620, 35) to write a linear function that defines longevity $L(x)$ as a linear function of the length of the gestation period x. Round the slope to 3 decimal places and the y-intercept to 2 decimal places.

c. Interpret the meaning of the slope in the context of this problem.

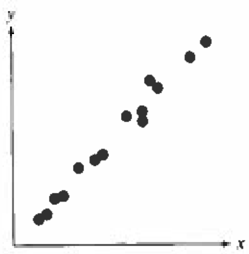
d. Use the model from part (b) to predict the longevity for an animal with an 80-day gestation period. Round to the nearest year.

Animal	Gestation Period (days)	Longevity (yr)
Rabbit	33	7.0
Squirrel	44	8.5
Fox	57	9.0
Cat	60	11.0
Dog	62	11.0
Lion	109	10.0
Pig	115	10.0
Goat	148	12.0
Horse	337	23.0
Elephant	620	35.0

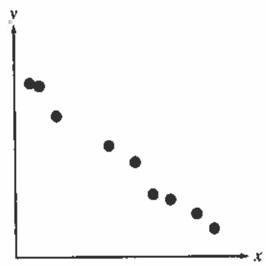
Objective 4: Create Models Using Linear Regression

For Exercises 67–70, use the scatter plot to determine if a linear regression model appears to be appropriate.

67.



68.



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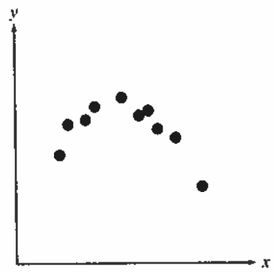
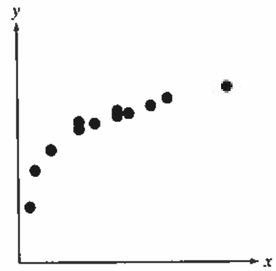


70.



71. The graph in Exercise 61 shows the wind speed y (in mph) of a hurricane versus the barometric pressure x (in mb). The table gives a partial list of data from the graph. (**See Example 8**)

a. Use the data in the table to find the least-squares regression line. Round the slope to 2 decimal places and the y-intercept to the nearest whole unit.

b. Use a graphing utility to graph the regression line and the observed data.

c. Use the model in part (a) to approximate the wind speed of a hurricane with a barometric pressure of 900 mb.

d. By how much do the results of part (c) differ from the result of Exercise 61(c)?

Barometric Pressure (mb) (x)	Wind Speed (mph) (y)
1007	35
1003	45
1000	50
994	65
983	80
968	100
950	110
930	145
905	160

72. The graph in Exercise 62 shows the weight of Dodger, a puppy recently adopted from an animal shelter. The data in the table give Dodger's weight y (in lb), x days after adoption.

a. Use the data in the table to find the least-squares regression line. Round the slope to 2 decimal places and the y-intercept to 1 decimal place.

b. Use a graphing utility to graph the regression line and the observed data.

c. Use the model in part (a) to approximate the time required for Dodger to reach 90% of his full-grown weight of 70 lb. Round to the nearest day.

d. By how much do the results of part (c) differ from the result of Exercise 62(d)?

Number of Days (x)	Weight (lb) (y)
0	11.0
5	12.8
12	14.3
18	16.1
24	17.2
31	19.2
40	22.0
45	23.4
52	24.7
60	27.5

73. The graph in Exercise 63 shows the average height of girls based on their age. The data in the table give the average height y (in inches) for girls of age x (in yr).

a. Use the data in the table to find the least-squares regression line. Round the slope to 2 decimal places and the y-intercept to 1 decimal place.

b. Use a graphing utility to graph the regression line and the observed data.

c. Use the model in part (a) to approximate the average height of 11-yr-old girls.

d. If the height of a girl at age 11 is 90% of her full-grown adult height, use the result of part (c) to estimate the average height of adult women. Round to the nearest tenth of an inch.

e. By how much do the results of part (d) differ from the result of Exercise 63(d)?

Age (yr) (x)	Height (in.) (y)
2	35.00
3	38.50
4	41.75
5	44.00
6	46.00
7	48.00
8	50.50
9	53.00
10	56.00

74. The graph in Exercise 64 shows the number of students y enrolled in public colleges for selected years x, where x is the number of years since 1990. The table gives a partial list of data from the graph.

a. Use the data in the table to find the least-squares regression line. Round the slope to 2 decimal places and the y-intercept to 1 decimal place.

b. Use a graphing utility to graph the regression line and the observed data.

c. Assuming that the linear trend continues, use the model from part (a) to predict the number of students enrolled in public colleges for the year 2020.

d. By how much do the results of part (c) differ from the result of Exercise 64(d)?

Number of Years Since 1990 (x)	Enrollment (millions) (y)
0	10.8
4	11.2
8	11.1
12	12.8
16	13.2
20	14.2
24	14.8

75. The data in Exercise 65 give the amount of cholesterol y for a hamburger with x calories.

a. Use these data to find the least-squares regression line. Round the slope to 3 decimal places and the y-intercept to 2 decimal places.

b. Use a graphing utility to graph the regression line and the observed data.

c. Use the regression line to predict the amount of cholesterol in a hamburger with 650 calories. Round to the nearest milligram.

76. The data in Exercise 66 give the average gestation period x (in days) for selected animals and their corresponding average longevity y (in yr).

a. Use these data to find the least-squares regression line. Round the slope to 3 decimal places and the y-intercept to 2 decimal places.

b. Use a graphing utility to graph the regression line and the observed data.

c. Use the regression line to predict the longevity for an animal with an 80-day gestation period. Round to the nearest year.

Mixed Exercises

77. Suppose that a line passes through the points $(4, -6)$ and $(2, -1)$. Where will it pass through the x-axis?

78. Suppose that a line passes through the point $(2, -5)$ and $(-4, 7)$. Where will it pass through the x-axis?

79. Write a rule for a linear function $y = f(x)$, given that $f(0) = 4$ and $f(3) = 11$.

80. Write a rule for a linear function $y = g(x)$, given that $g(0) = 7$ and $g(-2) = 4$.

81. Write a rule for a linear function $y = h(x)$, given that $h(1) = 6$ and $h(-3) = 2$.

82. Write a rule for a linear function $y = k(x)$, given that $k(-2) = 10$ and $k(5) = -18$.

Write About It

83. Explain how you can use slope to determine if two nonvertical lines are parallel or perpendicular.

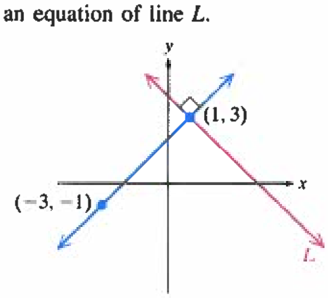
84. State one application of using the point-slope formula.

85. Explain how cost and revenue are related to profit.

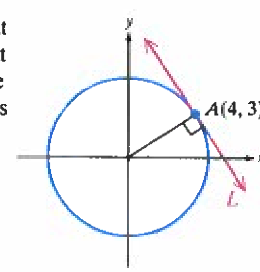
86. Explain how to determine the break-even point.

Expanding Your Skills

87. Find an equation of line L.



88. In geometry, it is known that the tangent line to a circle at a given point A on the circle is perpendicular to the radius drawn to point A. Suppose that line L is tangent to the given circle at the point $(4, 3)$. Write an equation representing line L.



89. In calculus, we can show that the slope of the line drawn tangent to the curve $y = x^3 + 1$ at the point $(c, c^3 + 1)$ is given by $3c^2$. Find an equation of the line tangent to $y = x^3 + 1$ at the point $(-2, -7)$.

90. In calculus, we can show that the slope of the line drawn tangent to the curve $y = \frac{1}{x}$ at the point $\left(c, \frac{1}{c}\right)$ is given by $-\frac{1}{c^2}$. Find an equation of the line tangent to $y = \frac{1}{x}$ at the point $\left(2, \frac{1}{2}\right)$.

For Exercises 91–92, use the fact that a median of a triangle is a line segment drawn from a vertex of the triangle to the midpoint of the opposite side of the triangle.

91. Find an equation of the median of a triangle drawn from vertex $A(5, -2)$ to the side formed by $B(-2, 9)$ and $C(4, 7)$.

92. Find an equation of the median of a triangle drawn from vertex $A(6, -5)$ to the side formed by $B(-4, 1)$ and $C(12, 3)$.

PROBLEM RECOGNITION EXERCISES

Comparing Graphs of Equations

In Section 1.6, we will learn additional techniques to graph functions by recognizing characteristics of the functions. In many cases, we can also graph families of functions by relating them to one of several basic graphs. To prepare for the discussion in Section 1.6, use a graphing utility or plot points to graph the basic functions in Exercises 1–8.

1. $y = 1$

2. $y = x$

3. $y = x^2$

4. $y = x^3$

5. $y = \sqrt{x}$

6. $y = \sqrt[3]{x}$

7. $y = |x|$

8. $y = \dfrac{1}{x}$

For Exercises 9–18, graph the functions by plotting points or by using a graphing utility. Explain how the graphs are related.

9. **a.** $f(x) = x^2$
b. $g(x) = x^2 + 2$
c. $h(x) = x^2 - 4$

10. **a.** $f(x) = |x|$
b. $g(x) = |x| + 2$
c. $h(x) = |x| - 4$

11. **a.** $f(x) = \sqrt{x}$
b. $g(x) = \sqrt{x - 2}$
c. $h(x) = \sqrt{x + 4}$

12. **a.** $f(x) = x^2$
b. $g(x) = (x - 2)^2$
c. $h(x) = (x + 3)^2$

13. **a.** $f(x) = |x|$
b. $g(x) = -|x|$

14. **a.** $f(x) = \sqrt{x}$
b. $g(x) = -\sqrt{x}$

15. **a.** $f(x) = x^2$
b. $g(x) = \dfrac{1}{2}x^2$
c. $h(x) = 2x^2$

16. **a.** $f(x) = |x|$
b. $g(x) = \dfrac{1}{3}|x|$
c. $h(x) = 3|x|$

17. **a.** $f(x) = \sqrt{x}$
b. $g(x) = \sqrt{-x}$

18. **a.** $f(x) = \sqrt[3]{x}$
b. $g(x) = \sqrt[3]{-x}$

SECTION 1.6 Transformations of Graphs

OBJECTIVES

1. Recognize Basic Functions
2. Apply Vertical and Horizontal Translations (Shifts)
3. Apply Vertical and Horizontal Shrinking and Stretching
4. Apply Reflections Across the x- and y-Axes
5. Summarize Transformations of Graphs

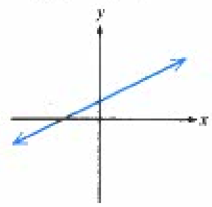
1. Recognize Basic Functions

A function defined by $f(x) = mx + b$ is a linear function and its graph is a line in a rectangular coordinate system. In addition to linear functions, we will learn to identify other categories of functions and the shapes of their graphs (Table 1-2).

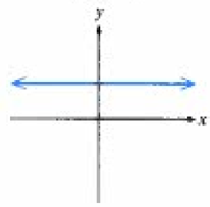
TIP The functions given in Table 1-2 were introduced in Section 1.1, Exercises 31–36, and in the Problem Recognition Exercises on page 182.

Table 1-2 Basic Functions and Their Graphs

1. Linear functions $f(x) = mx + b$

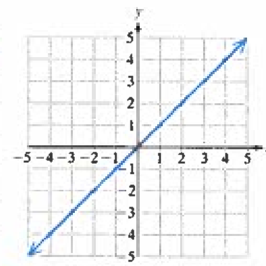


Constant functions $f(x) = b$



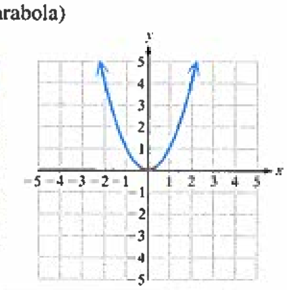
2. Identity function: $f(x) = x$

x	$f(x)$
-2	-2
-1	-1
0	0
1	1
2	2



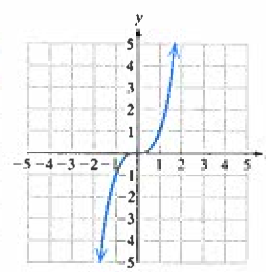
3. Quadratic function: $f(x) = x^2$ (graph is a parabola)

x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4



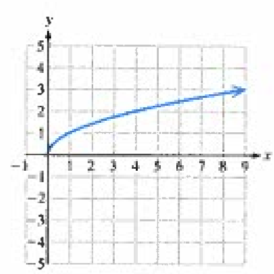
4. Cube function: $f(x) = x^3$

x	$f(x)$
-2	-8
-1	-1
0	0
1	1
2	8



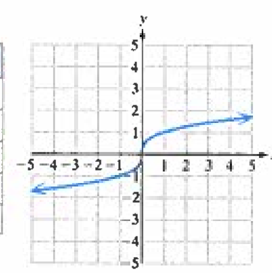
5. Square root function: $f(x) = \sqrt{x}$

x	$f(x)$
0	0
1	1
4	2
9	3
16	4



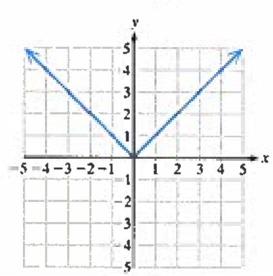
6. Cube root function: $f(x) = \sqrt[3]{x}$

x	$f(x)$
-8	-2
-1	-1
0	0
1	1
8	2



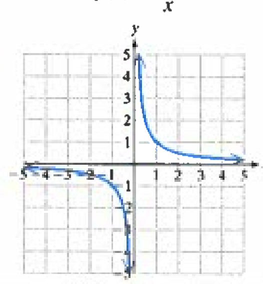
7. Absolute value function: $f(x) = |x|$

x	$f(x)$
-2	2
-1	1
0	0
1	1
2	2



8. Reciprocal function: $f(x) = \dfrac{1}{x}$

x	$f(x)$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$



Notice that the graph of $f(x) = \frac{1}{x}$ gets close to (but never touches) the y-axis as x gets close to zero. Likewise, as x approaches ∞ and $-\infty$, the graph approaches the x-axis without touching the x-axis. The x- and y-axes are called **asymptotes** of f and will be studied in detail in Section 2.5.

2. Apply Vertical and Horizontal Translations (Shifts)

We will call the eight basic functions pictured in Table 1-2 "parent" functions. Other functions that share the characteristics of a parent function are grouped as a "family" of functions. For example, consider the functions defined by $g(x) = x^2 + 2$ and $h(x) = x^2 - 4$, pictured in Figure 1-25.

x	$f(x) = x^2$	$g(x) = x^2 + 2$	$h(x) = x^2 - 4$
-3	9	11	5
-2	4	6	0
-1	1	3	-3
0	0	2	-4
1	1	3	-3
2	4	6	0
3	9	11	5

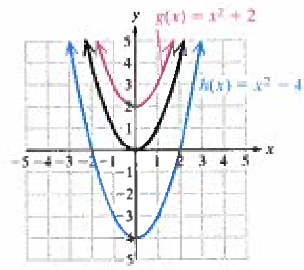


Figure 1-25

The graphs of g and h both resemble the graph of $f(x) = x^2$, but are shifted vertically upward or downward. The table of points reveals that for corresponding x values, the values of $g(x)$ are 2 more than the values of $f(x)$. Thus, the graph is shifted *upward* 2 units. Likewise, the values of $h(x)$ are 4 less than the values of $f(x)$ and the graph is shifted *downward* 4 units. Such shifts are called **translations.** These observations are consistent with the following rules.

TIP For each ordered pair (x, y) on the graph of $y = f(x)$, the corresponding point
- $(x, y + k)$ is on the graph of $y = f(x) + k$.
- $(x, y - k)$ is on the graph of $y = f(x) - k$.

Vertical Translations of Graphs

Consider a function defined by $y = f(x)$. Let k represent a positive real number.

- The graph of $y = f(x) + k$ is the graph of $y = f(x)$ shifted k units *upward.*
- The graph of $y = f(x) - k$ is the graph of $y = f(x)$ shifted k units *downward.*

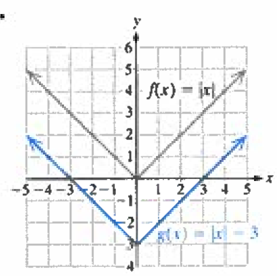
EXAMPLE 1 Translating a Graph Vertically

Use translations to graph the given functions.

a. $g(x) = |x| - 3$ **b.** $h(x) = x^3 + 2$

Solution:

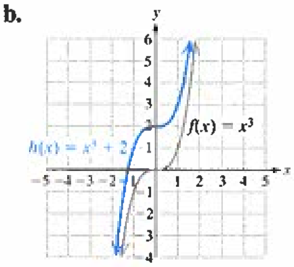
a.



The parent function for $g(x) = |x| - 3$ is $f(x) = |x|$.

The graph of g (shown in blue) is the graph of f shifted *downward* 3 units. For example the point (0, 0) on the graph of $f(x) = |x|$ corresponds to (0, -3) on the graph of $g(x) = |x| - 3$.

b.



The parent function for $h(x) = x^3 + 2$ is $f(x) = x^3$.

The graph of h (shown in blue) is the graph of f shifted *upward* 2 units. For example:

The point (0, 0) on the graph of $f(x) = x^3$ corresponds to (0, 2) on the graph of $h(x) = x^3 + 2$.

The point (1, 1) on the graph of $f(x) = x^3$ corresponds to (1, 3) on the graph of $h(x) = x^3 + 2$.

Skill Practice 1 Use translations to graph the given functions.

a. $g(x) = \sqrt{x} - 2$ **b.** $h(x) = \sqrt{x} + 3$

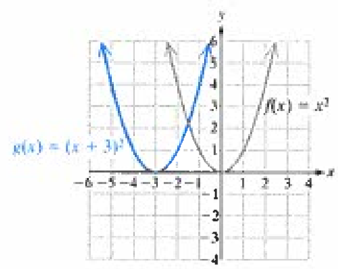
The graph of a function will be shifted to the right or left if a constant is added to or subtracted from the input variable x. In Example 2, we consider $g(x) = (x + 3)^2$.

EXAMPLE 2 Translating a Graph Horizontally

Graph the function defined by $g(x) = (x + 3)^2$.

Solution:

Because 3 is added to the x variable, we might expect the graph of $g(x) = (x + 3)^2$ to be the same as the graph of $f(x) = x^2$, but shifted in the x direction (horizontally). To determine whether the shift is to the left or right, we can locate the x-intercept of the graph of $g(x) = (x + 3)^2$. Substituting 0 for $g(x)$, we have:



$$0 = (x + 3)^2$$

$$x = -3 \qquad \text{The } x\text{-intercept is } (-3, 0).$$

Therefore, the new x-intercept (and also the vertex of the parabola) is $(-3, 0)$. This means that the graph is shifted to the left.

Skill Practice 2 Graph the function defined by $g(x) = |x + 2|$.

Using similar logic as in Example 2, we can show that the graph of $h(x) = (x - 3)^2$ is the graph of $f(x) = x^2$ translated to the *right* 3 units. These observations are consistent with the following rules.

Horizontal Translations of Graphs

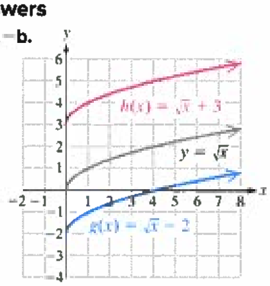
Consider a function defined by $y = f(x)$. Let h represent a positive real number.

- The graph of $y = f(x - h)$ is the graph of $y = f(x)$ shifted h units to the *right*.
- The graph of $y = f(x + h)$ is the graph of $y = f(x)$ shifted h units to the *left*.

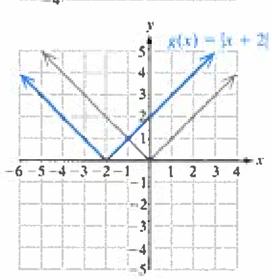
TIP Consider a positive real number h. To graph $y = f(x - h)$ or $y = f(x + h)$, shift the graph of $y = f(x)$ horizontally in the opposite direction of the sign within parentheses. The graph of $y = f(x - h)$ is a shift in the positive x direction. The graph of $y = f(x + h)$ is a shift in the negative x direction.

Answers

1. a.–b.



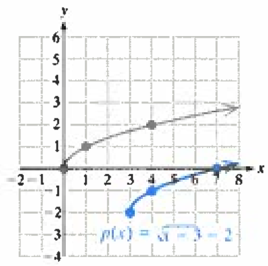
2.



EXAMPLE 3 Translating a Function Horizontally and Vertically

Use translations to graph the function defined by $p(x) = \sqrt{x-3} - 2$.

Solution:



The parent function for $p(x) = \sqrt{x-3} - 2$ is $f(x) = \sqrt{x}$.

The graph of p (shown in blue) is the graph of f shifted to the right 3 units and downward 2 units. We can plot several strategic points as an outline for the new curve.

- The point (0, 0) on the graph of f corresponds to $(0 + 3, 0 - 2) = (3, -2)$ on the graph of p.
- The point (1, 1) on the graph of f corresponds to $(1 + 3, 1 - 2) = (4, -1)$ on the graph of p.
- The point (4, 2) on the graph of f corresponds to $(4 + 3, 2 - 2) = (7, 0)$ on the graph of p.

Skill Practice 3 Use translations to graph the function defined by $q(x) = \sqrt{x+2} - 5$.

3. Apply Vertical and Horizontal Shrinking and Stretching

Horizontal and vertical translations of functions are called **rigid transformations** because the shape of the graph is not affected. We now look at **nonrigid transformations.** These operations cause a distortion of the graph (either an elongation or contraction in the horizontal or vertical direction). We begin by investigating the functions defined by $y = f(x)$ and $y = a \cdot f(x)$, where a is a positive real number.

EXAMPLE 4 Graphing a Function with a Vertical Stretch or Shrink

Graph the functions.

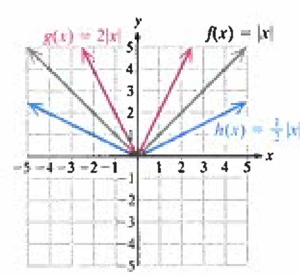
a. $f(x) = |x|$ **b.** $g(x) = 2|x|$ **c.** $h(x) = \frac{1}{2}|x|$

Solution:

x	$f(x) = \|x\|$	$g(x) = 2\|x\|$	$h(x) = \frac{1}{2}\|x\|$
−3	3	6	$\frac{3}{2}$
−2	2	4	1
−1	1	2	$\frac{1}{2}$
0	0	0	0
1	1	2	$\frac{1}{2}$
2	2	4	1
3	3	6	$\frac{3}{2}$

double

multiply by $\frac{1}{2}$



For a given value of x, the value of $g(x)$ is twice the value of $f(x)$. Therefore, the graph of g is elongated or stretched vertically by a factor of 2.

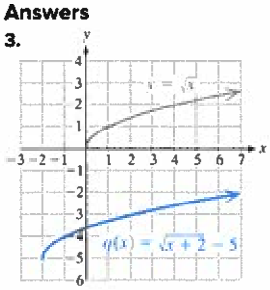
For a given value of x, the value of $h(x)$ is one-half that of $f(x)$. Therefore, the graph of h is shrunk vertically.

Skill Practice 4 Graph the functions.

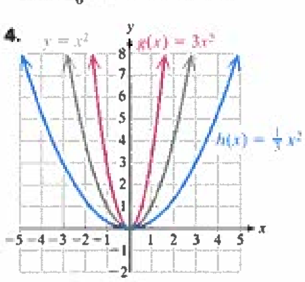
a. $f(x) = x^2$ **b.** $g(x) = 3x^2$ **c.** $h(x) = \frac{1}{3}x^2$

Answers

3.



4.



Vertical Shrinking and Stretching of Graphs

Consider a function defined by $y = f(x)$. Let a represent a positive real number.

- If $a > 1$, then the graph of $y = af(x)$ is the graph of $y = f(x)$ stretched vertically by a factor of a.
- If $0 < a < 1$, then the graph of $y = af(x)$ is the graph of $y = f(x)$ shrunk vertically by a factor of a.

Note: For any point (x, y) on the graph of $y = f(x)$, the point (x, ay) is on the graph of $y = af(x)$.

A function may also be stretched or shrunk horizontally.

Horizontal Shrinking and Stretching of Graphs

Consider a function defined by $y = f(x)$. Let a represent a positive real number.

- If $a > 1$, then the graph of $y = f(ax)$ is the graph of $y = f(x)$ shrunk horizontally by a factor of $\frac{1}{a}$.
- If $0 < a < 1$, then the graph of $y = f(ax)$ is the graph of $y = f(x)$ stretched horizontally by a factor of $\frac{1}{a}$.

Note: For any point (x, y) on the graph of $y = f(x)$, the point $\left(\frac{x}{a}, y\right)$ is on the graph of $y = f(ax)$.

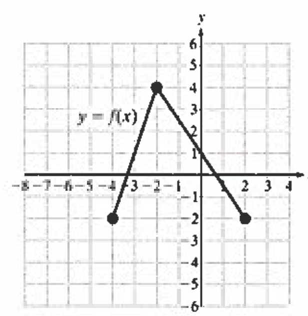
A point (x, y) on the graph of $y = f(x)$ corresponds to the point $\left(\frac{x}{a}, y\right)$ on the graph of $y = f(ax)$. Since the x-coordinate is multiplied by the *reciprocal* of a, values of a greater than 1 actually compress (shrink) the graph horizontally toward the y-axis. Values of a between 0 and 1 *stretch* the graph horizontally away from the y-axis. This is demonstrated in Example 5.

EXAMPLE 5 Graphing a Function with a Horizontal Shrink or Stretch

The graph of $y = f(x)$ is shown. Graph

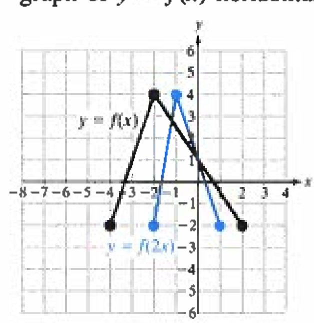
a. $y = f(2x)$

b. $y = f\left(\frac{1}{2}x\right)$



Solution:

a. $f(2x)$ is in the form $f(ax)$ with $a = 2 > 1$. The graph of $y = f(2x)$ is the graph of $y = f(x)$ horizontally compressed.



The graph of f has the following "strategic" points that define the shape of the function: $(-4, -2)$, $(-2, 4)$, and $(2, -2)$.

To graph $y = f(2x)$, divide each x value by 2.

$(-4, -2)$ becomes $\left(\frac{-4}{2}, -2\right) = (-2, -2)$.

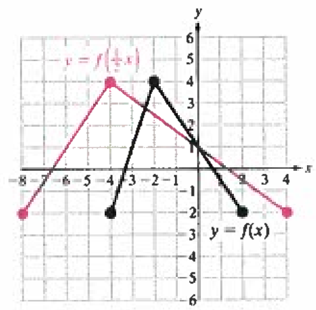
$(-2, 4)$ becomes $\left(\frac{-2}{2}, 4\right) = (-1, 4)$.

$(2, -2)$ becomes $\left(\frac{2}{2}, -2\right) = (1, -2)$.

The graph of $y = f(2x)$ is shown in blue.

TIP Dividing the x values by $\frac{1}{2}$ is the same as multiplying the x values by 2.

b. $f(\frac{1}{2}x)$ is in the form $f(ax)$ with $a = \frac{1}{2}$. The graph of $y = f(\frac{1}{2}x)$ is the graph of $y = f(x)$ stretched horizontally.



To graph $y = f(\frac{1}{2}x)$, divide each x value on the graph of $y = f(x)$ by $\frac{1}{2}$. For example:

$(-4, -2)$ becomes $\left(\frac{-4}{\frac{1}{2}}, -2\right) = (-8, -2)$.

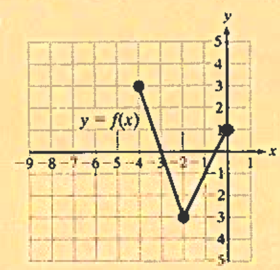
$(-2, 4)$ becomes $\left(\frac{-2}{\frac{1}{2}}, 4\right) = (-4, 4)$.

$(2, -2)$ becomes $\left(\frac{2}{\frac{1}{2}}, -2\right) = (4, -2)$.

The graph of $y = f(\frac{1}{2}x)$ is shown in red.

Skill Practice 5 The graph of $y = f(x)$ is shown.

Graph. **a.** $y = f(2x)$ **b.** $y = f\left(\frac{1}{2}x\right)$



4. Apply Reflections Across the x- and y-Axes

The graphs of $f(x) = x^2$ (in black) and $g(x) = -x^2$ (in blue) are shown in Figure 1-26. Notice that a point (x, y) on the graph of f corresponds to the point $(x, -y)$ on the graph of g. Therefore, the graph of g is the graph of f reflected across the x-axis.

The graphs of $f(x) = \sqrt{x}$ (in black) and $g(x) = \sqrt{-x}$ (in blue) are shown in Figure 1-27. Notice that a point (x, y) on the graph of f corresponds to the point $(-x, y)$ on g. Therefore, the graph of g is the graph of f reflected across the y-axis.

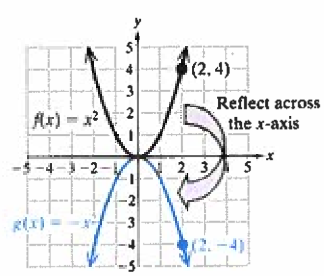


Figure 1-26

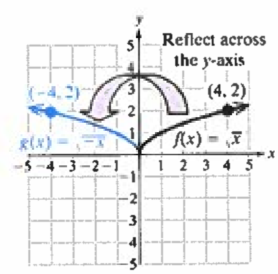
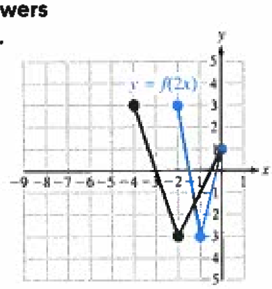


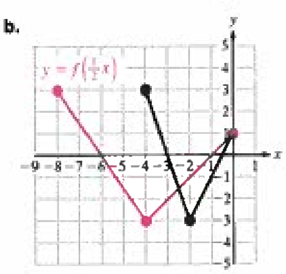
Figure 1-27

Answers

5. a.



b.



Reflections Across the x- and y-Axes

Consider a function defined by $y = f(x)$.

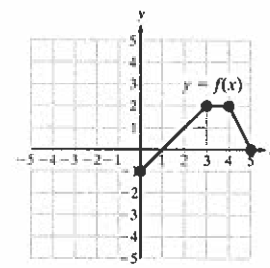
- The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected across the x-axis.
- The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected across the y-axis.

EXAMPLE 6 Reflecting the Graph of a Function Across the x- and y-Axes

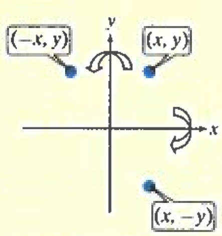
The graph of $y = f(x)$ is given.

a. Graph $y = -f(x)$.

b. Graph $y = f(-x)$.

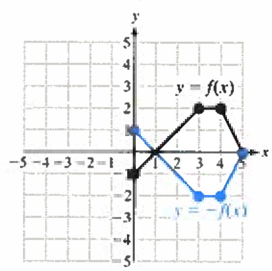


TIP For a given point (x, y), notice that $(-x, y)$ is on the opposite side of and equidistant to the y-axis. Likewise, $(x, -y)$ is on the opposite side of and equidistant from the x-axis as (x, y).

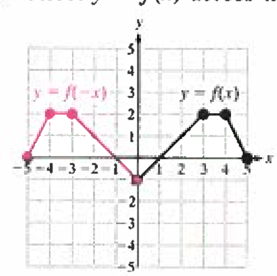


Solution:

a. Reflect $y = f(x)$ across the x-axis.



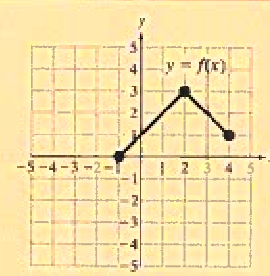
b. Reflect $y = f(x)$ across the y-axis.



Skill Practice 6 The graph of $y = f(x)$ is given.

a. Graph $y = -f(x)$.

b. Graph $y = f(-x)$.

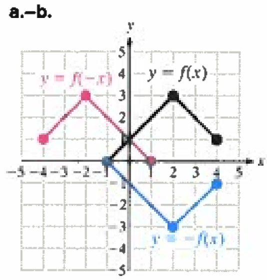


5. Summarize Transformations of Graphs

The operations of reflecting a graph of a function about an axis and shifting, stretching, and shrinking a graph are called **transformations.** Transformations give us tools to graph families of functions that are built from basic "parent" functions.

Answers

6. a.–b.



Transformations of Functions

Consider a function defined by $y = f(x)$. If h, k, and a represent positive real numbers, then the graphs of the following functions are related to $y = f(x)$ as follows.

Transformation	Effect on the Graph of f	Changes to Points on f
Vertical translation (shift) $y = f(x) + k$ $y = f(x) - k$	Shift upward k units Shift downward k units	Replace (x, y) by $(x, y + k)$. Replace (x, y) by $(x, y - k)$.
Horizontal translation (shift) $y = f(x - h)$ $y = f(x + h)$	Shift to the right h units Shift to the left h units	Replace (x, y) by $(x + h, y)$. Replace (x, y) by $(x - h, y)$.
Vertical stretch/shrink $y = a[f(x)]$	Vertical stretch (if $a > 1$) Vertical shrink (if $0 < a < 1$) Graph is stretched/shrunk vertically by a factor of a.	Replace (x, y) by (x, ay).
Horizontal stretch/shrink $y = f(a \cdot x)$	Horizontal shrink (if $a > 1$) Horizontal stretch (if $0 < a < 1$) Graph is shrunk/stretched horizontally by a factor of $\frac{1}{a}$.	Replace (x, y) by $\left(\frac{x}{a}, y\right)$.
Reflection $y = -f(x)$ $y = f(-x)$	Reflection across the x-axis Reflection across the y-axis	Replace (x, y) by $(x, -y)$. Replace (x, y) by $(-x, y)$.

When graphing a function requiring multiple transformations on the parent function, it is important to follow the correct sequence of steps.

Steps for Graphing Multiple Transformations of Functions

To graph a function requiring multiple transformations, use the following order.

1. Horizontal translation (shift)
2. Horizontal and vertical stretch and shrink
3. Reflections across the x- and y-axes
4. Vertical translation (shift)

EXAMPLE 7 Using Transformations to Graph a Function

Use transformations to graph the function defined by $n(x) = -\frac{1}{2}(x - 2)^2 + 3$.

Solution:

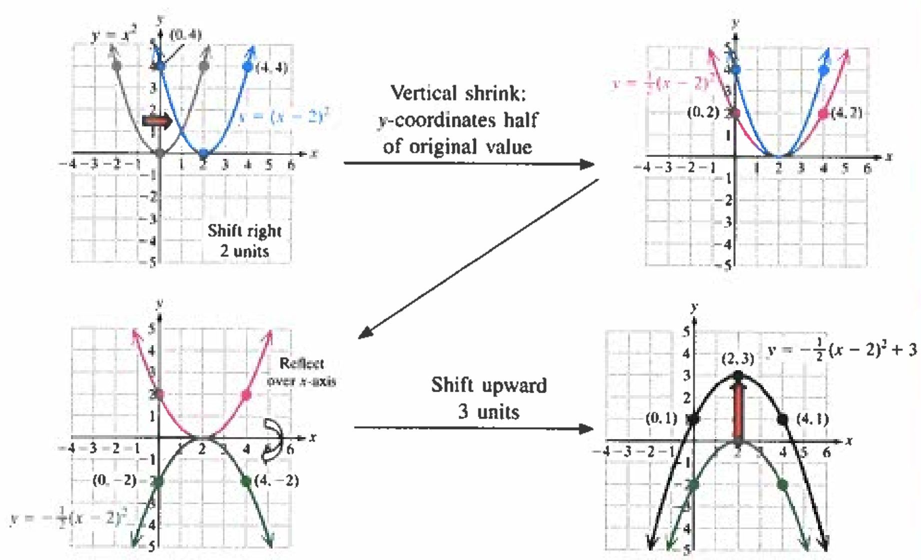
The graph of $n(x) = -\frac{1}{2}(x - 2)^2 + 3$ is the same as the graph of $f(x) = x^2$, with four transformations in the following order.

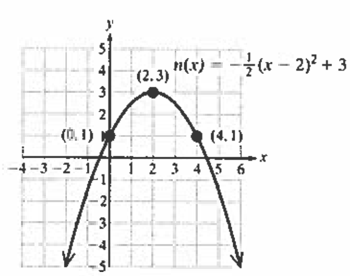
1. Shift the graph to the right 2 units.
2. Apply a vertical shrink (multiply the y values by $\frac{1}{2}$).
3. Reflect the graph over the x-axis.
4. Shift the graph upward 3 units.

shift right   shift up

$$n(x) = -\frac{1}{2}(x - 2)^2 + 3$$

reflect over x-axis   vertical shrink





Avoiding Mistakes

As a means to check the graph of $y = n(x)$, substitute the x-coordinates of the strategic points $(0, 1)$, $(2, 3)$, and $(4, 1)$ into the function.

$n(0) = -\frac{1}{2}(0 - 2)^2 + 3 = 1$ ✓
$n(2) = -\frac{1}{2}(2 - 2)^2 + 3 = 3$ ✓
$n(4) = -\frac{1}{2}(4 - 2)^2 + 3 = 1$ ✓

Skill Practice 7 Use transformations to graph the function defined by $m(x) = -3|x - 2| - 4$.

EXAMPLE 8 Using Transformations to Graph a Function

Use transformations to graph the function defined by $v(x) = -\sqrt{-x + 2}$.

Solution:

The graph of $v(x) = -\sqrt{-x + 2}$ is the same as the graph of $f(x) = \sqrt{x}$, with three transformations in the following order.

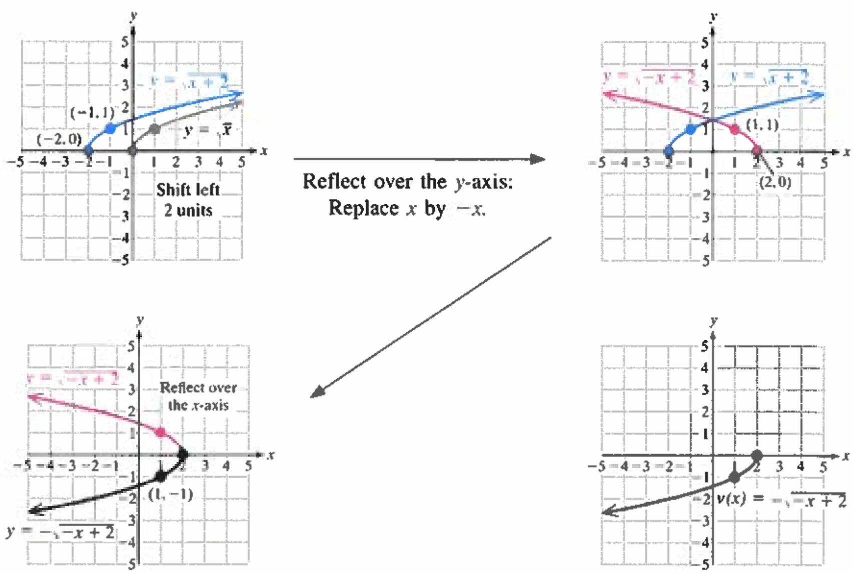
1. Shift the graph to the left 2 units.
2. Reflect the graph across the y-axis.
3. Reflect the graph across the x-axis.

   (Note that the reflections in steps 2 and 3 can be applied in either order.)

reflect across y-axis shift left

$v(x) = -\sqrt{-x + 2}$

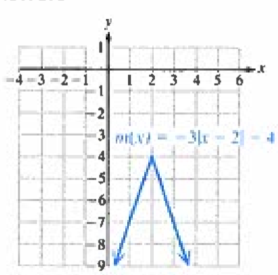
reflect across x-axis



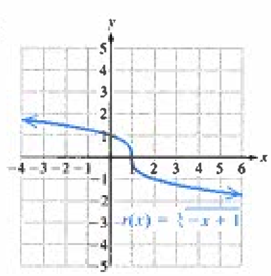
Skill Practice 8 Use transformations to graph the function defined by $r(x) = \sqrt[3]{-x + 1}$.

Answers

7.



8.



Avoiding Mistakes

Transformations involving a horizontal shrink, stretch, or reflection often introduce confusion when coupled with a horizontal shift. To further illustrate the rationale for the order of steps taken in Example 8, begin with the parent function $y = \sqrt{x}$. Performing a horizontal shift first means that we replace x by $x + 2$. This gives us $y = \sqrt{x + 2}$. Then to perform the reflection across the y-axis, we replace x by $-x$ to get $y = \sqrt{-x + 2}$. Performing these two transformations in the reverse order, would *not* result in the function we want. We would first have $y = \sqrt{-x}$, and then replacing x by $x + 2$ would give $y = \sqrt{-(x + 2)} = \sqrt{-x - 2}$ rather than $y = \sqrt{-x + 2}$.

SECTION 1.6 Practice Exercises

Prerequisite Review

For Exercises R.1–R.3, graph each equation.

R.1. $y = -3x - 1$ **R.2.** $y = \frac{3}{5}x + 2$ **R.3.** $y = 1$

Concept Connections

1. Let c represent a positive real number. The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted (up/down/left/right) c units.

2. Let c represent a positive real number. The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted (up/down/left/right) c units.

3. Let c represent a positive real number. The graph of $y = f(x) - c$ is the graph of $y = f(x)$ shifted (up/down/left/right) c units.

4. The graph of $y = 3f(x)$ is the graph of $y = f(x)$ with a (choose one: vertical stretch, vertical shrink, horizontal stretch, horizontal shrink).

5. The graph of $y = f(3x)$ is the graph of $y = f(x)$ with a (choose one: vertical stretch, vertical shrink, horizontal stretch, horizontal shrink).

6. The graph of $y = f\left(\frac{1}{3}x\right)$ is the graph of $y = f(x)$ with a (choose one: vertical stretch, vertical shrink, horizontal stretch, horizontal shrink).

7. The graph of $y = \frac{1}{3}f(x)$ is the graph of $y = f(x)$ with a (choose one: vertical stretch, vertical shrink, horizontal stretch, horizontal shrink).

8. The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected across the ______________ -axis.

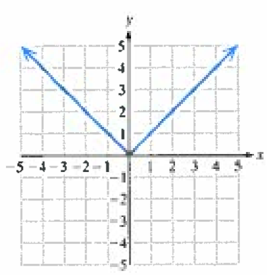
Objective 1: Recognize Basic Functions

For Exercises 9–14, from memory match the equation with its graph.

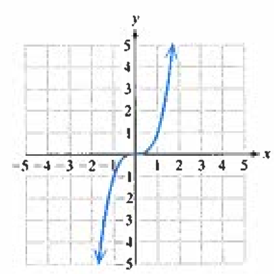
9. $f(x) = \sqrt{x}$ **10.** $f(x) = \sqrt[3]{x}$ **11.** $f(x) = x^3$

12. $f(x) = x^2$ **13.** $f(x) = |x|$ **14.** $f(x) = \frac{1}{x}$

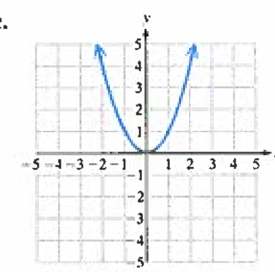
a.



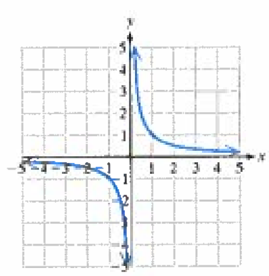
b.



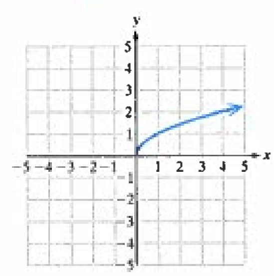
c.



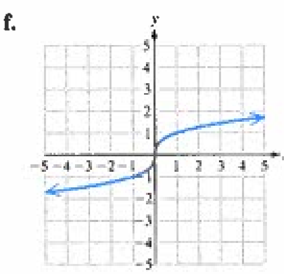
d.



e.



f.



Objective 2: Apply Vertical and Horizontal Translations (Shifts)

For Exercises 15–26, use translations to graph the given functions. (See Examples 1–3)

15. $f(x) = |x| + 1$ **16.** $g(x) = \sqrt{x} + 2$ **17.** $k(x) = x^3 - 2$ **18.** $h(x) = \frac{1}{x} - 2$

19. $g(x) = \sqrt{x + 5}$ **20.** $m(x) = |x + 1|$ **21.** $r(x) = (x - 4)^2$ **22.** $t(x) = \sqrt[3]{x - 2}$

23. $a(x) = \sqrt{x + 1} - 3$ **24.** $b(x) = |x - 2| + 4$ **25.** $c(x) = \frac{1}{x - 3} + 1$ **26.** $d(x) = \frac{1}{x + 4} - 1$

Objective 3: Apply Vertical and Horizontal Shrinking and Stretching

For Exercises 27–32, use transformations to graph the functions. (See Example 4)

27. $m(x) = 4\sqrt[3]{x}$ **28.** $n(x) = 3|x|$ **29.** $r(x) = \frac{1}{2}x^2$

30. $t(x) = \frac{1}{3}|x|$ **31.** $p(x) = |2x|$ **32.** $q(x) = \sqrt{2x}$

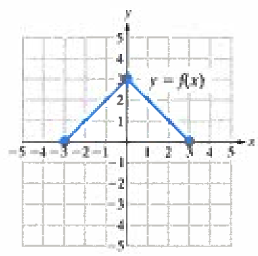
For Exercises 33–40, use the graphs of $y = f(x)$ and $y = g(x)$ to graph the given function. (See Example 5)

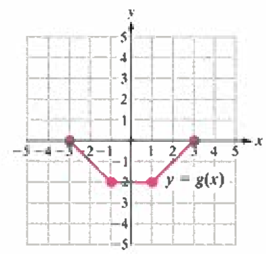
33. $y = \frac{1}{3}f(x)$ **34.** $y = \frac{1}{2}g(x)$

35. $y = 3f(x)$ **36.** $y = 2g(x)$

37. $y = f(3x)$ **38.** $y = g(2x)$

39. $y = f\left(\frac{1}{3}x\right)$ **40.** $y = g\left(\frac{1}{2}x\right)$





Objective 4: Apply Reflections Across the x- and y-Axes

For Exercises 41–46, graph the function by applying an appropriate reflection.

41. $f(x) = -\frac{1}{x}$ **42.** $g(x) = -\sqrt{x}$ **43.** $h(x) = -x^3$

44. $k(x) = -|x|$ **45.** $p(x) = (-x)^3$ **46.** $q(x) = \sqrt[3]{-x}$

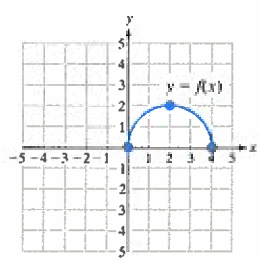
For Exercises 47–50, use the graphs of $y = f(x)$ and $y = g(x)$ to graph the given function. (See Example 6)

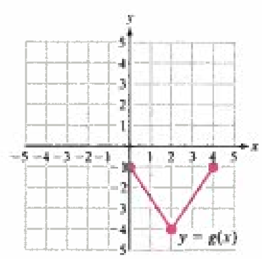
47. $y = f(-x)$

48. $y = g(-x)$

49. $y = -f(x)$

50. $y = -g(x)$





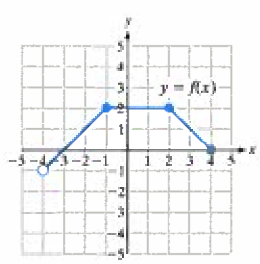
For Exercises 51–54, use the graphs of $y = f(x)$ and $y = g(x)$ to graph the given function. (See Example 6)

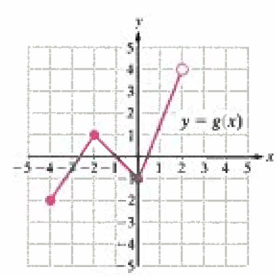
51. $y = f(-x)$

52. $y = g(-x)$

53. $y = -f(x)$

54. $y = -g(x)$





Objective 5: Summarize Transformations of Graphs

For Exercises 55–62, a function g is given. Identify the parent function from Table 1-2 on page 183. Then use the steps for graphing multiple transformations of functions on page 190 to list, in order, the transformations applied to the parent function to obtain the graph of g.

55. $g(x) = \frac{3}{1 + x} - 2$ **56.** $g(x) = \frac{5}{x - 4} + 1$ **57.** $g(x) = \frac{1}{3}(x - 2.1)^2 + 7.9$

58. $g(x) = \frac{1}{2}\sqrt{x + 4.3} - 8.4$ **59.** $g(x) = 2\sqrt{-2x + 5}$ **60.** $g(x) = 3\left|-\frac{1}{2}x - 4\right|$

61. $g(x) = -\sqrt{\frac{1}{3}x} - 6$ **62.** $g(x) = -|2x| + 8$

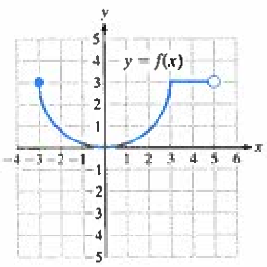
For Exercises 63–78, use transformations to graph the functions. (See Examples 7–8)

63. $v(x) = -(x + 2)^2 + 1$
64. $u(x) = -(x - 1)^2 - 2$
65. $f(x) = 2\sqrt{x + 3} - 1$
66. $g(x) = 2\sqrt{x - 1} + 3$
67. $p(x) = \frac{1}{2}|x - 1| - 2$
68. $q(x) = \frac{1}{3}|x + 2| - 1$
69. $r(x) = -\sqrt{-x} + 1$
70. $s(x) = -\sqrt{-x} - 2$
71. $f(x) = \sqrt{-x + 3}$
72. $g(x) = \sqrt{-x - 4}$
73. $n(x) = -\left|\frac{1}{2}x - 3\right|$
74. $m(x) = -\left|\frac{1}{3}x + 1\right|$
75. $f(x) = -\frac{1}{2}(x - 3)^2 + 8$
76. $g(x) = -\frac{1}{3}(x + 2)^2 + 3$
77. $p(x) = -4|x + 2| - 1$
78. $q(x) = -2|x - 1| + 4$

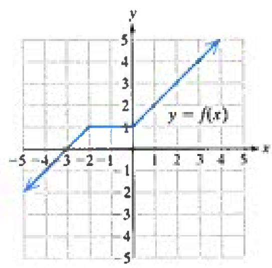
Mixed Exercises

For Exercises 79–86, the graph of $y = f(x)$ is given. Graph the indicated function.

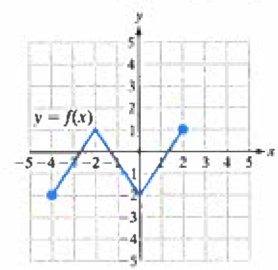
79. Graph $y = -f(x - 1) + 2$.



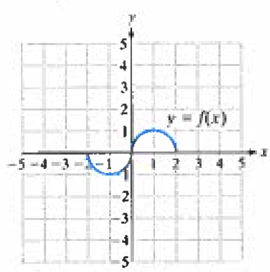
80. Graph $y = -f(x + 1) - 2$.



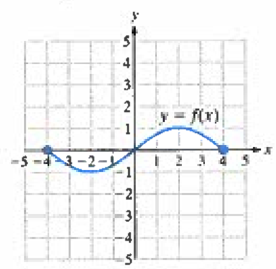
81. Graph $y = 2f(x - 2) - 3$.



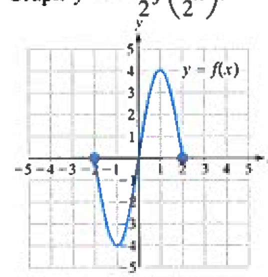
82. Graph $y = 2f(x + 2) - 4$.



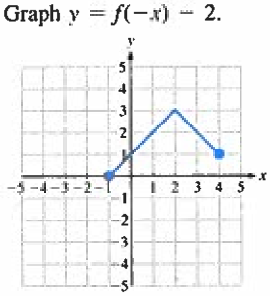
83. Graph $y = -3f(2x)$.



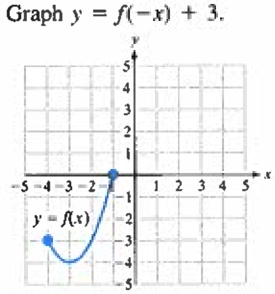
84. Graph $y = -\frac{1}{2}f\left(\frac{1}{2}x\right)$.



85. Graph $y = f(-x) - 2$.



86. Graph $y = f(-x) + 3$.



For Exercises 87–90, write a function based on the given parent function and transformations in the given order.

87. Parent function: $y = x^3$
1. Shift 4.5 units to the left.
2. Reflect across the y-axis.
3. Shift upward 2.1 units.

88. Parent function $y = \sqrt[3]{x}$
1. Shift 1 unit to the left.
2. Stretch horizontally by a factor of 4.
3. Reflect across the x-axis.

89. Parent function: $y = \frac{1}{x}$
1. Stretch vertically by a factor of 2.
2. Reflect across the x-axis.
3. Shift downward 3 units.

90. Parent function: $y = |x|$
1. Shift 3.7 units to the right.
2. Shrink horizontally by a factor of $\frac{1}{3}$.
3. Reflect across the y-axis.

Write About It

91. Explain why the graph of $g(x) = |2x|$ can be interpreted as a horizontal shrink of the graph of $f(x) = |x|$ or as a vertical stretch of the graph of $f(x) = |x|$.

92. Explain why the graph of $h(x) = \sqrt{\frac{1}{2}x}$ can be interpreted as a horizontal stretch of the graph of $f(x) = \sqrt{x}$ or as a vertical shrink of the graph of $f(x) = \sqrt{x}$.

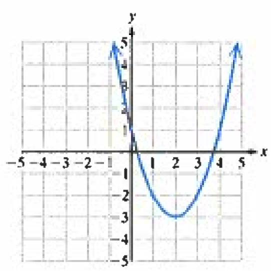
93. Explain the difference between the graphs of $f(x) = |x - 2| - 3$ and $g(x) = |x - 3| - 2$.

94. Explain why $g(x) = \dfrac{1}{-x + 1}$ can be graphed by shifting the graph of $f(x) = \dfrac{1}{x}$ one unit to the left and reflecting across the y-axis, or by shifting the graph of f one unit to the right and reflecting across the x-axis.

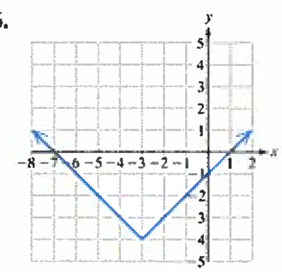
Expanding Your Skills

For Exercises 95–100, use transformations on the basic functions presented in Table 1-2 to write a rule $y = f(x)$ that would produce the given graph.

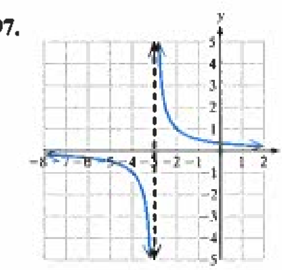
95.



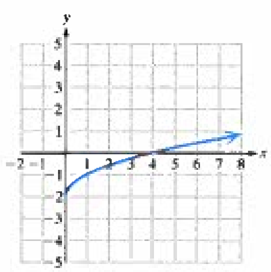
96.



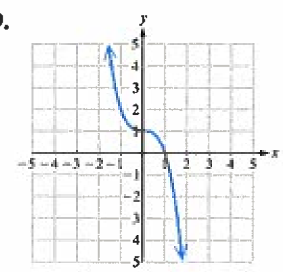
97.



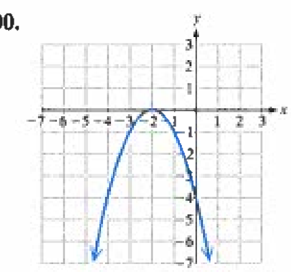
98.



99.

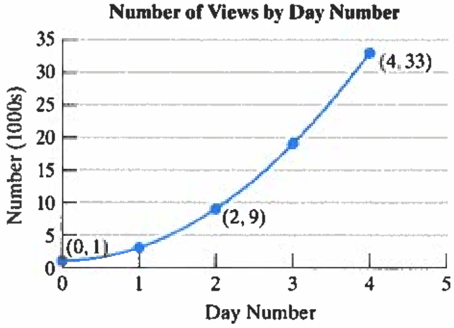


100.



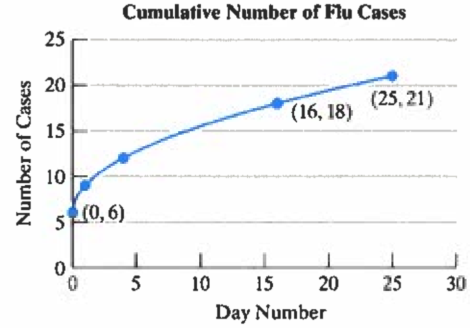
101. The graph shows the number of views y (in thousands) for a new online video, t days after it was posted. Use transformations on one of the parent functions from Table 1-2 on page 183 to model these data.

Number of Views by Day Number



102. The graph shows the cumulative number y of flu cases among passengers on a 25-day cruise, t days after the cruise began. Use transformations on one of the parent functions from Table 1-2 on page 183 to model these data.

Cumulative Number of Flu Cases



Technology Connections

103. a. Graph the functions on the viewing window $[-5, 5, 1]$ by $[-2, 8, 1]$.

$y = x^2$
$y = x^4$
$y = x^6$

b. Graph the functions on the viewing window $[-4, 4, 1]$ by $[-10, 10, 1]$.

$y = x^3$
$y = x^5$
$y = x^7$

c. Describe the general shape of the graph of $y = x^n$ where n is an even integer greater than 1.

d. Describe the general shape of the graph of $y = x^n$ where n is an odd integer greater than 1.

SECTION 1.7 Analyzing Graphs of Functions and Piecewise-Defined Functions

OBJECTIVES

1. Test for Symmetry
2. Identify Even and Odd Functions
3. Graph Piecewise-Defined Functions
4. Investigate Increasing, Decreasing, and Constant Behavior of a Function
5. Determine Relative Minima and Maxima of a Function

1. Test for Symmetry

The photos in Figures 1-28 through 1-30 each show a type of symmetry.

Figure 1-28

Figure 1-29

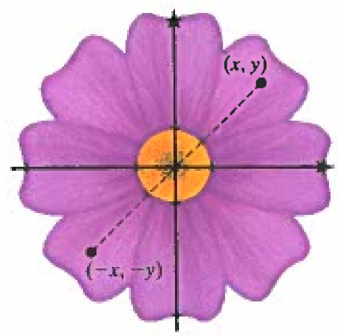


Figure 1-30

The photo of the kingfisher (Figure 1-28) shows an image of the bird reflected in the water. Suppose that we superimpose the x-axis at the waterline. Every point (x, y) on the bird has a mirror image $(x, -y)$ below the x-axis. Therefore, this image is symmetric with respect to the x-axis.

A human face is symmetric with respect to a vertical line through the center (Figure 1-29). If we place the y-axis along this line, a point (x, y) on one side has a mirror image at $(-x, y)$. This image is symmetric with respect to the y-axis.

The flower shown in Figure 1-30 is symmetric with respect to the point at its center. Suppose that we place the origin at the center of the flower. Notice that a point (x, y) on the image has a corresponding point $(-x, -y)$ on the image. This image is symmetric with respect to the origin.

Given an equation in the variables x and y, use the following rules to determine if the graph is symmetric with respect to the x-axis, the y-axis, or the origin.

Tests for Symmetry

Consider an equation in the variables x and y.

- The graph of the equation is symmetric with respect to the y-axis if substituting $-x$ for x in the equation results in an equivalent equation.
- The graph of the equation is symmetric with respect to the x-axis if substituting $-y$ for y in the equation results in an equivalent equation.
- The graph of the equation is symmetric with respect to the origin if substituting $-x$ for x and $-y$ for y in the equation results in an equivalent equation.

EXAMPLE 1 Testing for Symmetry

Determine whether the graph is symmetric with respect to the y-axis, x-axis, or origin.

a. $y = |x|$ **b.** $x = y^2 - 4$

Solution:

a.

$y = |x|$
$y = |-x|$
$y = |x|$

Same equation: Graph is symmetric with respect to the y-axis.

Test for symmetry with respect to the y-axis. Replace x by $-x$. Note that $|-x| = |x|$. The resulting equation *is* equivalent to the original equation.

$y = |x|$
$-y = |x|$
$y = -|x|$

not the same

Test for symmetry with respect to the x-axis. Replace y by $-y$. The resulting equation is *not* equivalent to the original equation.

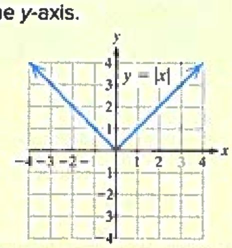
$y = |x|$
$-y = |-x|$
$-y = |x|$
$y = -|x|$

not the same

Test for symmetry with respect to the origin. Replace x by $-x$ and y by $-y$. The resulting equation is *not* equivalent to the original equation.

The graph is symmetric with respect to the y-axis only.

TIP The graph of $y = |x|$ is one of the basic graphs presented in Section 1.6. From our familiarity with the graph we can visualize the symmetry with respect to the y-axis.



b.

$x = y^2 - 4$
$-x = y^2 - 4$
$x = -y^2 + 4$

not the same

Test for symmetry with respect to the y-axis. Replace x by $-x$. The resulting equation is *not* equivalent to the original equation.

$x = y^2 - 4$
$x = (-y)^2 - 4$
$x = y^2 - 4$

Same equation: Graph is symmetric with respect to the x-axis.

Test for symmetry with respect to the x-axis. Replace y by $-y$. The resulting equation *is* equivalent to the original equation.

$x = y^2 - 4$
$-x = (-y)^2 - 4$
$-x = y^2 - 4$
$x = -y^2 + 4$

not the same

Test for symmetry with respect to the origin. Replace x by $-x$ and y by $-y$. The resulting equation is *not* equivalent to the original equation.

The graph is symmetric with respect to the x-axis only (Figure 1-31).

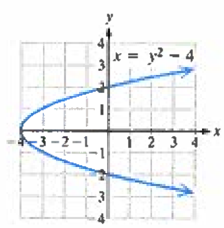


Figure 1-31

Skill Practice 1 Determine whether the graph is symmetric with respect to the y-axis, x-axis, or origin.

a. $y = x^2$ **b.** $|y| = x + 1$

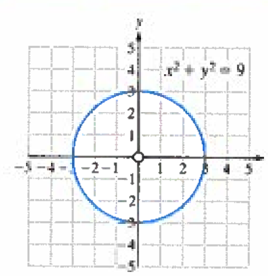
EXAMPLE 2 Testing for Symmetry

Determine whether the graph is symmetric with respect to the y-axis, x-axis, or origin.

$$x^2 + y^2 = 9$$

Solution:

The graph of $x^2 + y^2 = 9$ is a circle with center at the origin and radius 3. By inspection, we can see that the graph is symmetric with respect to both axes and the origin.



Answers

1. **a.** y-axis
 b. x-axis

Test for y-axis symmetry.
Replace x by $-x$.

$$x^2 + y^2 = 9$$
$$(-x)^2 + y^2 = 9$$
$$x^2 + y^2 = 9$$

same

Test for x-axis symmetry.
Replace y by $-y$.

$$x^2 + y^2 = 9$$
$$x^2 + (-y)^2 = 9$$
$$x^2 + y^2 = 9$$

same

Test for origin symmetry.
Replace x by $-x$ and y by $-y$.

$$x^2 + y^2 = 9$$
$$(-x)^2 + (-y)^2 = 9$$
$$x^2 + y^2 = 9$$

same

The graph is symmetric with respect to the y-axis, the x-axis, and the origin.

Skill Practice 2 Determine whether the graph is symmetric with respect to the y-axis, x-axis, or origin.

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

2. Identify Even and Odd Functions

A function may be symmetric with respect to the y-axis or to the origin. A function that is symmetric with respect to the y-axis is called an *even* function. A function that is symmetric with respect to the origin is called an *odd* function.

Avoiding Mistakes

The only functions that are symmetric with respect to the x-axis are functions whose points lie solely on the x-axis.

Even and Odd Functions

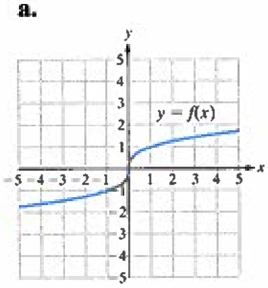
- A function f is an **even function** if $f(-x) = f(x)$ for all x in the domain of f. The graph of an even function is symmetric with respect to the y-axis.
- A function f is an **odd function** if $f(-x) = -f(x)$ for all x in the domain of f. The graph of an odd function is symmetric with respect to the origin.

EXAMPLE 3 **Identifying Even and Odd Functions**

By inspection determine if the function is even, odd, or neither.

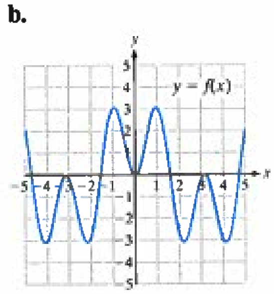
Solution:

a.



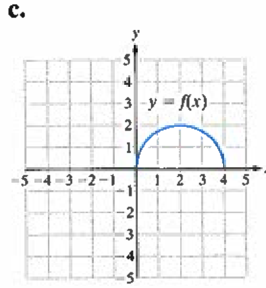
The function is symmetric with respect to the origin. Therefore, the function is an *odd* function.

b.



The function is symmetric with respect to the y-axis. Therefore, the function is an *even* function.

c.



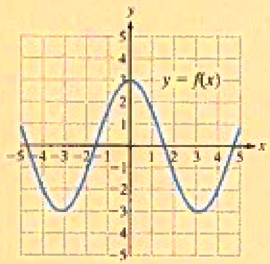
The function is not symmetric with respect to either the y-axis or the origin. Therefore, the function is *neither* even nor odd.

Answer

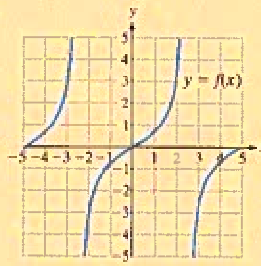
2. y-axis, x-axis, and origin

Skill Practice 3 Determine if the function is even, odd, or neither.

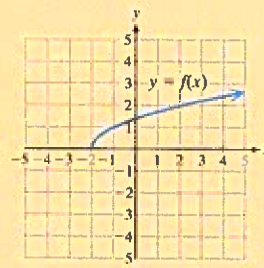
a.



b.



c.



EXAMPLE 4 Identifying Even and Odd Functions

Determine if the function is even, odd, or neither.

a. $f(x) = -2x^4 + 5|x|$ **b.** $g(x) = 4x^3 - x$ **c.** $h(x) = 2x^2 + x$

Solution:

a.
$$f(x) = -2x^4 + 5|x|$$
$$f(-x) = -2(-x)^4 + 5|-x|$$
$$= -2x^4 + 5|x|$$

same

Determine whether the function is even. Replace x by $-x$ to determine if $f(-x) = f(x)$.

Since $f(-x) = f(x)$, the function f is an even function.

There is no need to test whether f is an odd function because a function cannot be both even and odd unless all points are on the x-axis.

b. $g(x) = 4x^3 - x$

Each term has x raised to an odd power. Therefore, replacing x by $-x$ will result in the *opposite* of the original term. Therefore, test whether g is an odd function. That is, test whether $g(-x) = -g(x)$.

Evaluate: $g(-x)$
$$g(-x) = 4(-x)^3 - (-x)$$
$$= -4x^3 + x$$

same

Evaluate: $-g(x)$
$$-g(x) = -(4x^3 - x)$$
$$= -4x^3 + x$$

Since $g(-x) = -g(x)$, the function g is an odd function.

c.
$$h(x) = 2x^2 + x$$
$$h(-x) = 2(-x)^2 + (-x)$$
$$= 2x^2 - x$$

not the same

Determine whether the function is even. Replace x by $-x$ to determine if $h(-x) = h(x)$.

Since $h(-x) \neq h(x)$, the function is not even.

Next, test whether h is an odd function. Test whether $h(-x) = -h(x)$.

Evaluate: $h(-x)$
$$h(-x) = 2(-x)^2 + (-x)$$
$$= 2x^2 - x$$

not the same

Evaluate: $-h(x)$
$$-h(x) = -(2x^2 + x)$$
$$= -2x^2 - x$$

Since $h(-x) \neq -h(x)$, the function is not an odd function. Therefore, h is neither even nor odd.

Skill Practice 4 Determine if the function is even, odd, or neither.

a. $m(x) = -x^5 + x^3$ **b.** $n(x) = x^2 - |x| + 1$ **c.** $p(x) = 2|x| + x$

TIP In Example 4(a), we suspect that f is an even function because each term is of the form x^{even} or $|x|$. In each case, replacing x by $-x$ results in an equivalent term.

TIP In Example 4(b), we suspect that g is an odd function because each term is of the form x^{odd}. In each case, replacing x by $-x$ results in the *opposite* of the original term.

TIP In Example 4(c), $h(x)$ has a mixture of terms of the form x^{odd} and x^{even}. Therefore, we might suspect that the function is neither even nor odd.

Answers

3. a. Even function
   b. Odd function
   c. Neither even nor odd
4. a. Odd function
   b. Even function
   c. Neither even nor odd

3. Graph Piecewise-Defined Functions

Suppose that a car is stopped for a red light. When the light turns green, the car undergoes a constant acceleration for 20 sec until it reaches a speed of 45 mph. It travels 45 mph for 1 min (60 sec), and then decelerates for 30 sec to stop at another red light. The graph of the car's speed y (in mph) versus the time x (in sec) after leaving the first red light is shown in Figure 1-32.

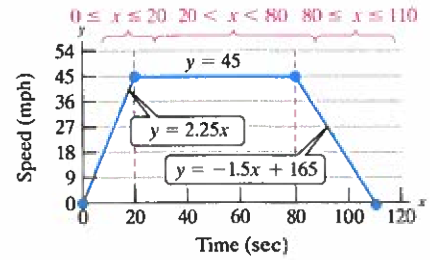


Figure 1-32

Notice that the graph can be segmented into three pieces. The first 20 sec is represented by a linear function with a positive slope, $y = 2.25x$. The next 60 sec is represented by the constant function $y = 45$. And the last 30 sec is represented by a linear function with a negative slope, $y = -1.5x + 165$.

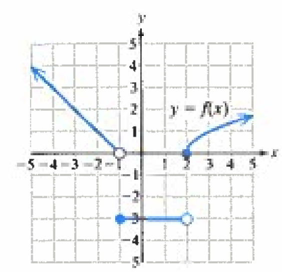
To write a rule defining this function we use a **piecewise-defined function** in which we define each "piece" on a restricted domain.

$$f(x) = \begin{cases} 2.25x & \text{for } 0 \le x \le 20 \\ 45 & \text{for } 20 < x < 80 \\ -1.5x + 165 & \text{for } 80 \le x \le 110 \end{cases}$$

EXAMPLE 5 Interpreting a Piecewise-Defined Function

Evaluate the function for the given values of x.

$$f(x) = \begin{cases} -x - 1 & \text{for } x < -1 \\ -3 & \text{for } -1 \le x < 2 \\ \sqrt{x-2} & \text{for } x \ge 2 \end{cases}$$



a. $f(-3)$ **b.** $f(-1)$
c. $f(2)$ **d.** $f(6)$

Solution:

a. $f(x) = -x - 1$
$f(-3) = -(-3) - 1$
$= 2$

$x = -3$ is on the interval $x < -1$. Use the first rule in the function: $f(x) = -x - 1$.

b. $f(x) = -3$
$f(-1) = -3$

$x = -1$ is on the interval $-1 \le x < 2$. Use the second rule in the function: $f(x) = -3$.

c. $f(x) = \sqrt{x-2}$
$f(2) = \sqrt{2-2}$
$= 0$

$x = 2$ is on the interval $x \ge 2$. Use the third rule in the function: $f(x) = \sqrt{x-2}$.

d. $f(x) = \sqrt{x-2}$
$f(6) = \sqrt{6-2}$
$= 2$

$x = 6$ is on the interval $x \ge 2$. Use the third rule in the function: $f(x) = \sqrt{x-2}$.

Skill Practice 5 Evaluate the function for the given values of x.

$$f(x) = \begin{cases} x + 7 & \text{for } x < -2 \\ x^2 & \text{for } -2 \le x < 1 \\ 3 & \text{for } x \ge 1 \end{cases}$$

a. $f(-3)$ **b.** $f(-2)$ **c.** $f(1)$ **d.** $f(4)$

Answers
5. a. 4 **b.** 4 **c.** 3 **d.** 3

TECHNOLOGY CONNECTIONS

Graphing a Piecewise-Defined Function

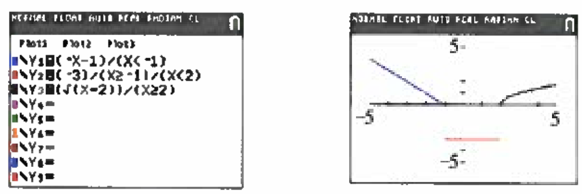
A graphing calculator can be used to graph a piecewise-defined function. The format to enter the function is as follows.

Y_1 = (first piece)/(first condition)
Y_2 = (second piece)/(second condition)
⋮

Each condition in parentheses is an inequality and the calculator assigns it a value of 1 or 0 depending on whether the inequality is true or false. If an inequality is true, the function is divided by 1 on that interval and is "turned on." If an inequality is false, then the function is divided by 0. Since division by zero is undefined, the calculator does not graph the function on that interval, and the function is effectively "turned off."

Enter the function from Example 5 as shown. Note that the inequality symbols can be found in the TEST menu.

$$f(x) = \begin{cases} -x - 1 & \text{for } x < -1 \\ -3 & \text{for } -1 \le x < 2 \\ \sqrt{x - 2} & \text{for } x \ge 2 \end{cases}$$



Notice that the individual "pieces" of the graph do not "hook-up." For this reason, it is also a good practice to put the calculator in DOT mode in the MODE menu.

In Examples 6 and 7, we graph piecewise-defined functions.

EXAMPLE 6 Graphing a Piecewise-Defined Function

Graph the function defined by $f(x) = \begin{cases} -3x & \text{for } x < 1 \\ -3 & \text{for } x \ge 1 \end{cases}$.

Solution:

- The first rule $f(x) = -3x$ defines a line with slope -3 and y-intercept $(0, 0)$. This line should be graphed only to the left of $x = 1$. The point $(1, -3)$ is graphed as an open dot, because the point is not part of the rule $f(x) = -3x$. See the blue portion of the graph in Figure 1-33.
- The second rule $f(x) = -3$ is a horizontal line for $x \ge 1$. The point $(1, -3)$ is a closed dot to show that it is part of the rule $f(x) = -3$. The closed dot from the red segment of the graph "overrides" the open dot from the blue segment. Taken together, the closed dot "plugs" the hole in the graph.

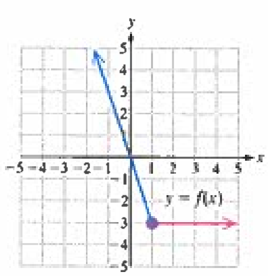


Figure 1-33

Skill Practice 6 Graph the function.

$$f(x) = \begin{cases} 2 & \text{for } x \leq -1 \\ -2x & \text{for } x > -1 \end{cases}$$

TIP The function in Example 6 has no "gaps," and therefore we say that the function is **continuous.** Informally, this means that we can draw the function without lifting our pencil from the page. The formal definition of a continuous function will be studied in calculus.

EXAMPLE 7 Graphing a Piecewise-Defined Function

Graph the function. $f(x) = \begin{cases} x + 3 & \text{for } x < -1 \\ x^2 & \text{for } -1 \leq x < 2 \end{cases}$

Solution:

The first rule $f(x) = x + 3$ defines a line with slope 1 and y-intercept $(0, 3)$. This line should be graphed only for $x < -1$ (that is to the left of $x = -1$). The point $(-1, 2)$ is graphed as an open dot, because the point is not part of the function. See the red portion of the graph in Figure 1-34.

The second rule $f(x) = x^2$ is one of the basic functions learned in Section 1.6. It is a parabola with vertex at the origin. We sketch this function only for x values on the interval $-1 \leq x < 2$. The point $(-1, 1)$ is a closed dot to show that it is part of the function. The point $(2, 4)$ is displayed as an open dot to indicate that it is not part of the function.

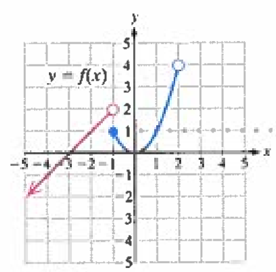


Figure 1-34

Avoiding Mistakes

Note that the function cannot have a closed dot at both $(-1, 1)$ and $(-1, 2)$ because it would not pass the vertical line test.

Skill Practice 7 Graph the function.

$$f(x) = \begin{cases} |x| & \text{for } -4 \leq x < 2 \\ -x + 2 & \text{for } x \geq 2 \end{cases}$$

TIP The function in Example 7 has a gap at $x = -1$, and therefore, we say that f is **discontinuous** at -1.

We now look at a special category of piecewise-defined functions called **step functions.** The graph of a step function is a series of discontinuous "steps." One important step function is called the **greatest integer function** or **floor function.** It is defined by

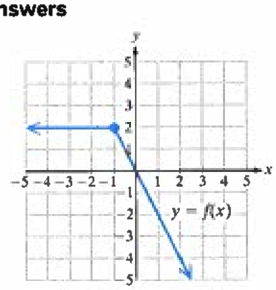
$$f(x) = [\![x]\!] \text{ where } [\![x]\!] \text{ is the greatest integer less than or equal to } x.$$

The operation $[\![x]\!]$ may also be denoted as **int(x)** or by **floor(x).** These alternative notations are often used in computer programming.

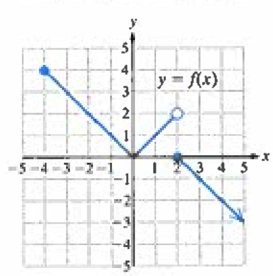
In Example 8, we graph the greatest integer function.

Answers

6.



7.



EXAMPLE 8 Graphing the Greatest Integer Function

Graph the function defined by $f(x) = [\![x]\!]$.

Solution:

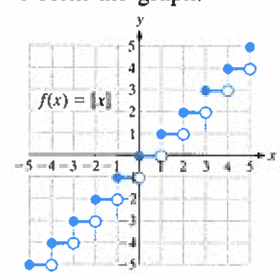
x	$f(x) = [\![x]\!]$	
−1.7	−2	Greatest integer less than or equal to −1.7 is −2.
−1	−1	Greatest integer less than or equal to −1 is −1.
−0.6	−1	Greatest integer less than or equal to −0.6 is −1.
0	0	Greatest integer less than or equal to 0 is 0.
0.4	0	Greatest integer less than or equal to 0.4 is 0.
1	1	Greatest integer less than or equal to 1 is 1.
1.8	1	Greatest integer less than or equal to 1.8 is 1.
2	2	Greatest integer less than or equal to 2 is 2.
2.5	2	Greatest integer less than or equal to 2.5 is 2.

Evaluate f for several values of x.

TIP On many graphing calculators, the greatest integer function is denoted by int( ) and is found under the MATH menu followed by NUM.

From the table, we see a pattern and from the pattern, we form the graph.

If $-3 \le x < -2$, then $[\![x]\!] = -3$
If $-2 \le x < -1$, then $[\![x]\!] = -2$
If $-1 \le x < 0$, then $[\![x]\!] = -1$
If $0 \le x < 1$, then $[\![x]\!] = 0$
If $1 \le x < 2$, then $[\![x]\!] = 1$
If $2 \le x < 3$, then $[\![x]\!] = 2$
...



Skill Practice 8 Evaluate $f(x) = [\![x]\!]$ for the given values of x.

a. $f(1.7)$ **b.** $f(5.5)$ **c.** $f(-4)$ **d.** $f(-4.2)$

In Example 9, we use a piecewise-defined function to model an application.

EXAMPLE 9 Using a Piecewise-Defined Function in an Application

A salesperson makes a monthly salary of \$3000 along with a 5% commission on sales over \$20,000 for the month. Write a piecewise-defined function to represent the salesperson's monthly income $I(x)$ (in \$) for x dollars in sales.

Solution:

Let x represent the amount in sales.
Then $x - 20{,}000$ represents the amount in sales over \$20,000.

There are two scenarios for the salesperson's income.

<u>Scenario 1</u>: The salesperson sells \$20,000 or less. In this case, the monthly income is a constant \$3000. This is represented by

$$y = 3000 \quad \text{for } 0 \le x \le 20{,}000$$

<u>Scenario 2</u>: The salesperson sells over \$20,000. In this case, the monthly income is \$3000 plus 5% of sales over \$20,000. This is represented by

$$y = 3000 + 0.05(x - 20{,}000) \quad \text{for } x > 20{,}000$$

Answers
8. a. 1 b. 5
c. −4 d. −5

Therefore, a piecewise-defined function for monthly income is

$$I(x) = \begin{cases} 3000 & \text{for } 0 \le x \le 20{,}000 \\ 3000 + 0.05(x - 20{,}000) & \text{for } x > 20{,}000 \end{cases}$$

Alternatively, we can simplify to get

$$I(x) = \begin{cases} 3000 & \text{for } 0 \le x \le 20{,}000 \\ 0.05x + 2000 & \text{for } x > 20{,}000 \end{cases}$$

A graph of $y = I(x)$ is shown in Figure 1-35. Notice that for $x = \$20{,}000$, both equations within the piecewise-defined function yield a monthly salary of \$3000. Therefore, the two line segments in the graph meet at (20,000, 3000).

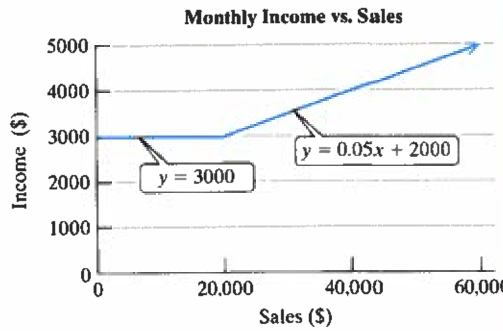


Figure 1-35

Skill Practice 9 A retail store buys T-shirts from the manufacturer. The cost is \$7.99 per shirt for 1 to 100 shirts, inclusive. Then the price is decreased to \$6.99 per shirt thereafter. Write a piecewise-defined function that expresses the cost $C(x)$ (in \$) to buy x shirts.

4. Investigate Increasing, Decreasing, and Constant Behavior of a Function

The graph in Figure 1-36 approximates the altitude of an airplane, $f(t)$, at a time t minutes after takeoff.

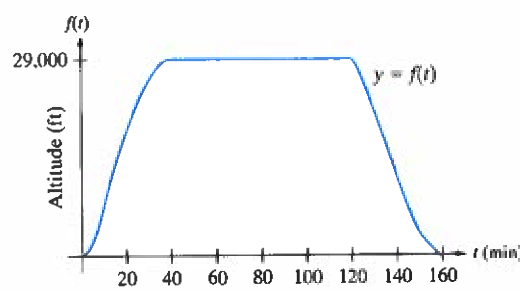


Figure 1-36

Notice that the plane's altitude increases up to the first 40 min of the flight. So we say that the function f is increasing on the interval (0, 40). The plane flies at a constant altitude for the next 1 hr 20 min, so we say that f is constant on the interval (40, 120). Finally, the plane's altitude decreases for the last 40 min, so we say that f is decreasing on the interval (120, 160).

Informally, a function is increasing on an interval in its domain if its graph rises from left to right. A function is decreasing on an interval in its domain if the graph "falls" from left to right. A function is constant on an interval in its domain if its graph is horizontal over the interval. These ideas are stated formally using mathematical notation.

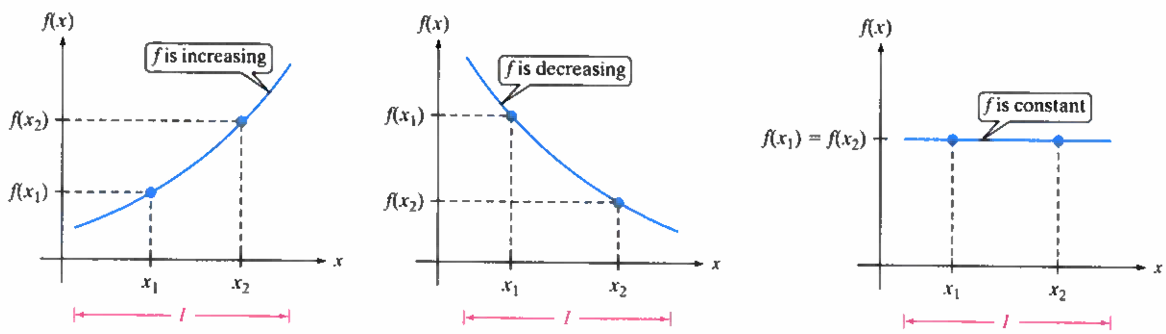
Answer

9. $C(x) = \begin{cases} 7.99x & \text{for } 1 \le x \le 100 \\ 799 + 6.99(x - 100) & \text{for } x > 100 \end{cases}$

Intervals of Increasing, Decreasing, and Constant Behavior

Suppose that I is an interval contained within the domain of a function f.

- f is increasing on I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ on I.
- f is decreasing on I if $f(x_1) > f(x_2)$ for all $x_1 < x_2$ on I.
- f is constant on I if $f(x_1) = f(x_2)$ for all x_1 and x_2 on I.

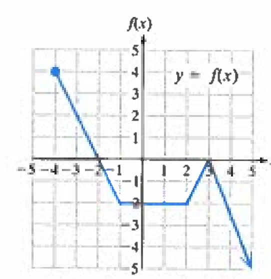


For all $x_1 < x_2$ on I, $f(x_1) < f(x_2)$ | For all $x_1 < x_2$ on I, $f(x_1) > f(x_2)$ | For all x_1 and x_2 on I, $f(x_1) = f(x_2)$

EXAMPLE 10 Determining the Intervals Over Which a Function Is Increasing, Decreasing, and Constant

Use interval notation to write the interval(s) over which f is

a. Increasing **b.** Decreasing
c. Constant

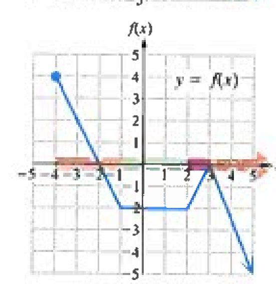


Solution:

a. f is increasing on the interval $(2, 3)$. (Highlighted in red tint.)

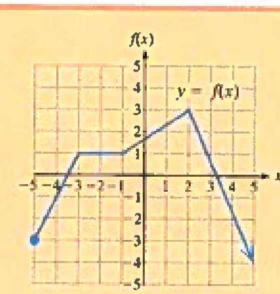
b. f is decreasing on the interval $(-4, -1) \cup (3, \infty)$. (Highlighted in orange tint.)

c. f is constant on the interval $(-1, 2)$. (Highlighted in green tint.)



Skill Practice 10 Use interval notation to write the interval(s) over which f is

a. Increasing
b. Decreasing
c. Constant



Answers

10. a. $(-5, -3) \cup (-1, 2)$
b. $(2, \infty)$ **c.** $(-3, -1)$

5. Determine Relative Minima and Maxima of a Function

The intervals over which a function changes from increasing to decreasing behavior or vice versa tell us where to look for relative maximum values and relative minimum values of a function. Consider the function pictured in Figure 1-37.

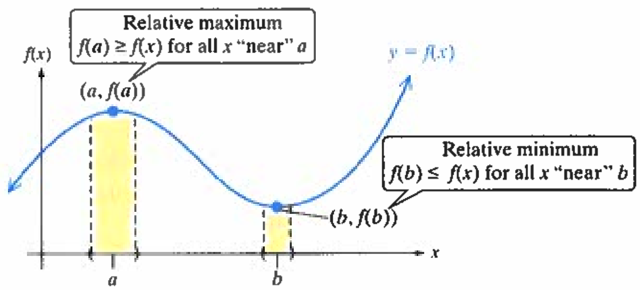


Figure 1-37

- The function has a relative maximum of $f(a)$. Informally, this means that $f(a)$ is the greatest function value relative to other points on the function nearby.
- The function has a relative minimum of $f(b)$. Informally, this means that $f(b)$ is the smallest function value relative to other points on the function nearby.

This is stated formally in the following definition.

TIP The plural of maximum and minimum are **maxima** and **minima**.

Note that relative maxima and minima are also called *local* maxima and minima.

Relative Minimum and Relative Maximum Values

- $f(a)$ is a **relative maximum** of f if there exists an open interval containing a such that $f(a) \geq f(x)$ for all x in the interval.
- $f(b)$ is a **relative minimum** of f if there exists an open interval containing b such that $f(b) \leq f(x)$ for all x in the interval.

Note: An open interval is an interval in which the endpoints are not included.

If an ordered pair $(a, f(a))$ corresponds to a relative minimum or relative maximum, we interpret the coordinates of the ordered pair as follows.

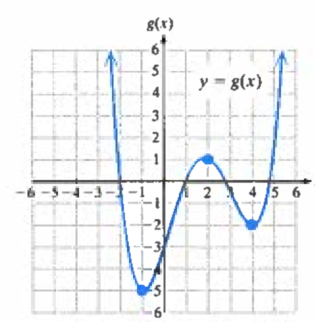
- The x-coordinate is the *location* of the relative maximum or minimum within the domain of the function.
- The y-coordinate is the *value* of the relative maximum or minimum. This tells us how "high" or "low" the graph is at that point.

EXAMPLE 11 Finding Relative Maxima and Minima

For the graph of $y = g(x)$ shown,

a. Determine the location and value of any relative maxima.

b. Determine the location and value of any relative minima.

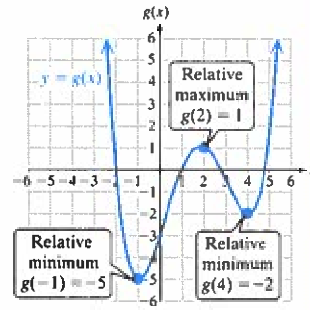


Solution:

a. The point (2, 1) is the highest point in a small interval surrounding $x = 2$. Therefore, at $x = 2$, the function has a relative maximum of 1.

b. The point $(-1, -5)$ is the lowest point in a small interval surrounding $x = -1$. Therefore, at $x = -1$, the function has a relative minimum of -5.

The point $(4, -2)$ is the lowest point in a small interval surrounding $x = 4$. Therefore, at $x = 4$, the function has a relative minimum of -2.



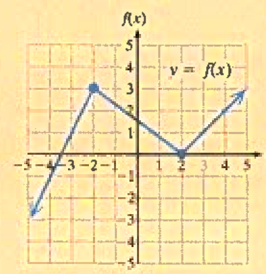
Avoiding Mistakes

Be sure to note that the value of a relative minimum or relative maximum is the *y* value of a function, not the *x* value.

Skill Practice 11 For the graph shown,

a. Determine the location and value of any relative maxima.

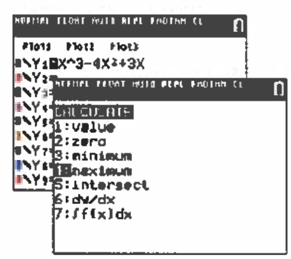
b. Determine the location and value of any relative minima.



TECHNOLOGY CONNECTIONS

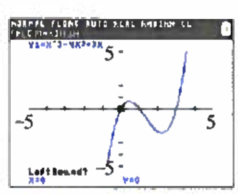
Determining Relative Maxima and Minima

Relative maxima and relative minima are often difficult to find analytically and require techniques from calculus. However, a graphing utility can be used to approximate the location and value of relative maxima and minima. To do so, we use the Minimum and Maximum features.

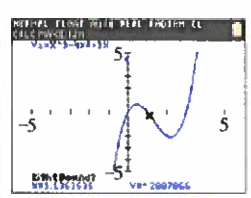


For example, enter the function defined by $Y_1 = x^3 - 4x^2 + 3x$. Then access the Maximum feature from the CALC menu.

The calculator asks for a left bound. This is a point slightly to the left of the relative maximum. Then hit ENTER.



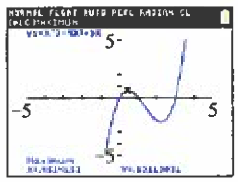
The calculator asks for a right bound. This is a point slightly to the right of the relative maximum. Hit ENTER.



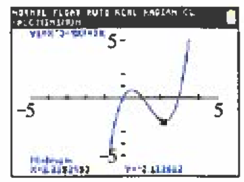
Answers

11. a. At $x = -2$, the function has a relative maximum of 3.
    b. At $x = 2$, the function has a relative minimum of 0.

The calculator asks for a guess. This is a point close to the relative maximum. Hit ENTER and the approximate coordinates of the relative maximum point are shown (0.45, 0.63).



To find the relative minimum, repeat these steps using the Minimum feature. The coordinates of the relative minimum point are approximately (2.22, −2.11).



SECTION 1.7 Practice Exercises

Prerequisite Review

R.1. Given the function defined by $f(x) = 7x - 2$, find $f(-a)$.

For Exercises R.2–R.4, graph the set and express the set in interval notation.

R.2. $\{x \mid x < 8\}$

R.3. $\{x \mid -2.4 \leq x < 5.8\}$

R.4. $\left\{x \,\middle|\, x \geq -\dfrac{9}{2}\right\}$

Concept Connections

1. A graph of an equation is symmetric with respect to the ______________-axis if replacing x by $-x$ results in an equivalent equation.

2. A graph of an equation is symmetric with respect to the ______________-axis if replacing y by $-y$ results in an equivalent equation.

3. A graph of an equation is symmetric with respect to the ______________ if replacing x by $-x$ and y by $-y$ results in an equivalent equation.

4. An even function is symmetric with respect to the ______________.

5. An odd function is symmetric with respect to the ______________.

6. The expression ______________ represents the greatest integer, less than or equal to x.

Objective 1: Test for Symmetry

For Exercises 7–18, determine whether the graph of the equation is symmetric with respect to the x-axis, y-axis, origin, or none of these. (See Examples 1–2)

7. $y = x^2 + 3$

8. $y = -|x| - 4$

9. $x = -|y| - 4$

10. $x = y^2 + 3$

11. $x^2 + y^2 = 3$

12. $|x| + |y| = 4$

13. $y = |x| + 2x + 7$

14. $y = x^2 + 6x + 1$

15. $x^2 = 5 + y^2$

16. $y^4 = 2 + x^2$

17. $y = \frac{1}{2}x - 3$

18. $y = \frac{2}{5}x + 1$

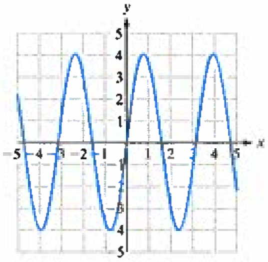
Objective 2: Identify Even and Odd Functions

19. What type of symmetry does an even function have?

20. What type of symmetry does an odd function have?

For Exercises 21–26, use the graph to determine if the function is even, odd, or neither. (See Example 3)

21.



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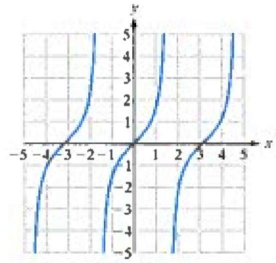

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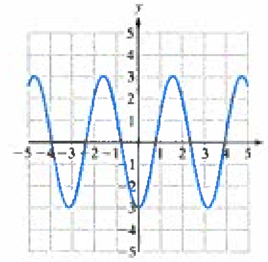

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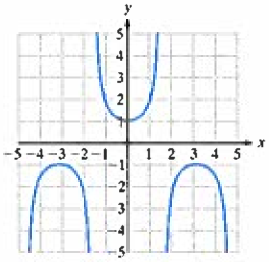

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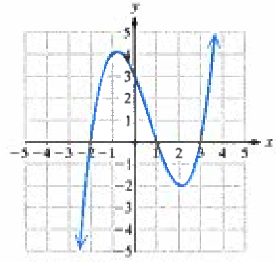

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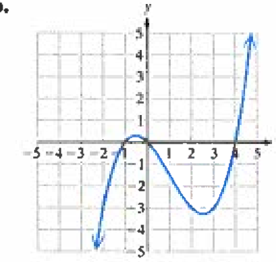


27. a. Given $f(x) = 4x^2 - 3|x|$, find $f(-x)$.
b. Is $f(-x) = f(x)$?
c. Is this function even, odd, or neither?

28. a. Given $g(x) = -x^8 + |3x|$, find $g(-x)$.
b. Is $g(-x) = g(x)$?
c. Is this function even, odd, or neither?

29. a. Given $h(x) = 4x^3 - 2x$, find $h(-x)$.
b. Find $-h(x)$.
c. Is $h(-x) = -h(x)$?
d. Is this function even, odd, or neither?

30. a. Given $k(x) = -8x^5 - 6x^3$, find $k(-x)$.
b. Find $-k(x)$.
c. Is $k(-x) = -k(x)$?
d. Is this function even, odd, or neither?

31. a. Given $m(x) = 4x^2 + 2x - 3$, find $m(-x)$.
b. Find $-m(x)$.
c. Is $m(-x) = m(x)$?
d. Is $m(-x) = -m(x)$?
e. Is this function even, odd, or neither?

32. a. Given $n(x) = 7|x| + 3x - 1$, find $n(-x)$.
b. Find $-n(x)$.
c. Is $n(-x) = n(x)$?
d. Is $n(-x) = -n(x)$?
e. Is this function even, odd, or neither?

For Exercises 33–46, determine if the function is even, odd, or neither. (See Example 4)

33. $f(x) = 3x^6 + 2x^2 + |x|$
34. $p(x) = -|x| + 12x^{10} + 5$
35. $k(x) = 13x^3 + 12x$
36. $m(x) = -4x^5 + 2x^3 + x$
37. $n(x) = \sqrt{16 - (x - 3)^2}$
38. $r(x) = \sqrt{81 - (x + 2)^2}$
39. $q(x) = \sqrt{16 + x^2}$
40. $z(x) = \sqrt{49 + x^2}$
41. $h(x) = 5x$
42. $g(x) = -x$
43. $f(x) = \dfrac{x^2}{3(x - 4)^2}$
44. $g(x) = \dfrac{x^3}{2(x - 1)^3}$
45. $v(x) = \dfrac{-x^5}{|x| + 2}$
46. $w(x) = \dfrac{-\sqrt[3]{x}}{x^2 + 1}$

Objective 3: Graph Piecewise-Defined Functions

For Exercises 47–50, evaluate the function for the given values of x. (See Example 5)

47. $f(x) = \begin{cases} -3x + 7 & \text{for } x < -1 \\ x^2 + 3 & \text{for } -1 \le x < 4 \\ 5 & \text{for } x \ge 4 \end{cases}$

a. $f(3)$ **b.** $f(-2)$ **c.** $f(-1)$
d. $f(4)$ **e.** $f(5)$

48. $g(x) = \begin{cases} -2|x| - 3 & \text{for } x \le -2 \\ 5x + 6 & \text{for } -2 < x < 3 \\ 4 & \text{for } x \ge 3 \end{cases}$

a. $g(-3)$ **b.** $g(3)$ **c.** $g(-2)$
d. $g(0)$ **e.** $g(4)$

49. $h(x) = \begin{cases} 2 & \text{for } -3 \le x < -2 \\ 1 & \text{for } -2 \le x < -1 \\ 0 & \text{for } -1 \le x < 0 \\ -1 & \text{for } 0 \le x < 1 \end{cases}$

a. $h(-1.7)$ **b.** $h(-2.5)$ **c.** $h(0.05)$
d. $h(-2)$ **e.** $h(0)$

50. $t(x) = \begin{cases} x & \text{for } 0 < x \le 1 \\ 2x & \text{for } 1 < x \le 2 \\ 3x & \text{for } 2 < x \le 3 \\ 4x & \text{for } 3 < x \le 4 \end{cases}$

a. $t(1.99)$ **b.** $t(0.4)$ **c.** $t(3)$
d. $t(1)$ **e.** $t(3.001)$

51. A sled accelerates down a hill and then slows down after it reaches a flat portion of ground. The speed of the sled $s(t)$ (in ft/sec) at a time t (in sec) after movement begins can be approximated by:

$$s(t) = \begin{cases} 1.5t & \text{for } 0 \le t \le 20 \\ \dfrac{30}{t - 19} & \text{for } 20 < t \le 40 \end{cases}$$

Determine the speed of the sled after 10 sec, 20 sec, 30 sec, and 40 sec. Round to 1 decimal place if necessary.

52. A car starts from rest and accelerates to a speed of 60 mph in 12 sec. It travels 60 mph for 1 min and then decelerates for 20 sec until it comes to rest. The speed of the car $s(t)$ (in mph) at a time t (in sec) after the car begins motion can be modeled by:

$$s(t) = \begin{cases} \dfrac{5}{12}t^2 & \text{for } 0 \le t \le 12 \\ 60 & \text{for } 12 < t \le 72 \\ \dfrac{3}{20}(92 - t)^2 & \text{for } 72 < t \le 92 \end{cases}$$

Determine the speed of the car 6 sec, 12 sec, 45 sec, and 80 sec after the car begins motion.

For Exercises 53–56, match the function with its graph.

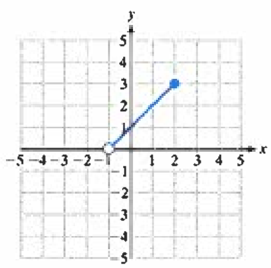
53. $f(x) = x + 1$ for $x < 2$

54. $f(x) = x + 1$ for $-1 < x \le 2$

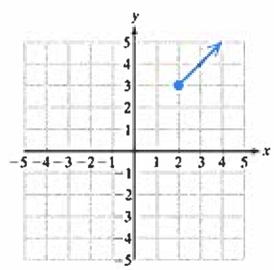
55. $f(x) = x + 1$ for $-1 \le x < 2$

56. $f(x) = x + 1$ for $x \ge 2$

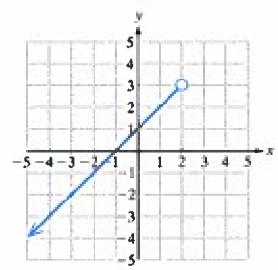
a.



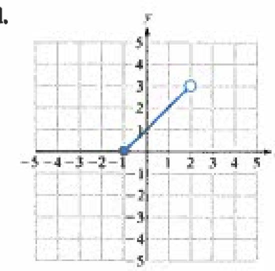
b.



c.



d.



57. **a.** Graph $p(x) = x + 2$ for $x \le 0$. **(See Examples 6–7)**

b. Graph $q(x) = -x^2$ for $x > 0$.

c. Graph $r(x) = \begin{cases} x + 2 & \text{for } x \le 0 \\ -x^2 & \text{for } x > 0 \end{cases}$

58. **a.** Graph $f(x) = |x|$ for $x < 0$.

b. Graph $g(x) = \sqrt{x}$ for $x \ge 0$.

c. Graph $h(x) = \begin{cases} |x| & \text{for } x < 0 \\ \sqrt{x} & \text{for } x \ge 0 \end{cases}$

59. **a.** Graph $m(x) = \frac{1}{2}x - 2$ for $x \le -2$.

b. Graph $n(x) = -x + 1$ for $x > -2$.

c. Graph $t(x) = \begin{cases} \frac{1}{2}x - 2 & \text{for } x \le -2 \\ -x + 1 & \text{for } x > -2 \end{cases}$

60. **a.** Graph $a(x) = x$ for $x < 1$.

b. Graph $b(x) = \sqrt{x - 1}$ for $x \ge 1$.

c. Graph $c(x) = \begin{cases} x & \text{for } x < 1 \\ \sqrt{x - 1} & \text{for } x \ge 1 \end{cases}$

For Exercises 61–70, graph the function. (See Examples 6–7)

61. $f(x) = \begin{cases} |x| & \text{for } x < 2 \\ -x + 4 & \text{for } x \ge 2 \end{cases}$

62. $h(x) = \begin{cases} -2x & \text{for } x < 0 \\ \sqrt{x} & \text{for } x \ge 0 \end{cases}$

63. $g(x) = \begin{cases} x + 2 & \text{for } x < -1 \\ -x + 2 & \text{for } x \ge -1 \end{cases}$

64. $k(x) = \begin{cases} 3x & \text{for } x < 1 \\ -3x & \text{for } x \ge 1 \end{cases}$

65. $r(x) = \begin{cases} x^2 - 4 & \text{for } x \le 2 \\ 2x - 4 & \text{for } x > 2 \end{cases}$

66. $s(x) = \begin{cases} -x - 1 & \text{for } x \le -1 \\ \sqrt{x + 1} & \text{for } x > -1 \end{cases}$

67. $t(x) = \begin{cases} -3 & \text{for } -4 \le x < -2 \\ -1 & \text{for } -2 \le x < 0 \\ 1 & \text{for } 0 \le x < 2 \end{cases}$

68. $z(x) = \begin{cases} -1 & \text{for } -3 < x \le -1 \\ 1 & \text{for } -1 < x \le 1 \\ 3 & \text{for } 1 < x \le 3 \end{cases}$

69. $m(x) = \begin{cases} 3 & \text{for } -4 < x < -1 \\ -x & \text{for } -1 \le x < 3 \\ \sqrt{x-3} & \text{for } x \ge 3 \end{cases}$

70. $n(x) = \begin{cases} -4 & \text{for } -3 < x < -1 \\ x & \text{for } -1 \le x < 2 \\ -x^2 + 4 & \text{for } x \ge 2 \end{cases}$

71. a. Graph $f(x) = \begin{cases} -x & \text{for } x < 0 \\ x & \text{for } x \ge 0 \end{cases}$

b. To what basic function from Section 1.6 is the graph of f equivalent?

For Exercises 72–80, evaluate the step function defined by $f(x) = [\![x]\!]$ for the given values of x. (See Example 8)

72. $f(-3.7)$ **73.** $f(-4.2)$ **74.** $f(-0.5)$ **75.** $f(-0.09)$ **76.** $f(0.5)$

77. $f(0.09)$ **78.** $f(6)$ **79.** $f(-9)$ **80.** $f(-5)$

For Exercises 81–84, graph the function. (See Example 8)

81. $f(x) = [\![x + 3]\!]$ **82.** $g(x) = [\![x - 3]\!]$ **83.** $k(x) = \text{int}\left(\frac{1}{2}x\right)$ **84.** $h(x) = \text{int}(2x)$

85. For a recent year, the rate for first class postage was as follows. (See Example 9)

Weight not Over	Price
1 oz	\$0.44
2 oz	\$0.61
3 oz	\$0.78
3.5 oz	\$0.95

Write a piecewise-defined function to model the cost $C(x)$ to mail a letter first class if the letter is x ounces.

86. The water level in a retention pond started at 5 ft (60 in.) and decreased at a rate of 2 in./day during a 14-day drought. A tropical depression moved through at the beginning of the 15th day and produced rain at an average rate of 2.5 in./day for 5 days. Write a piecewise-defined function to model the water level $L(x)$ (in inches) as a function of the number of days x since the beginning of the drought.

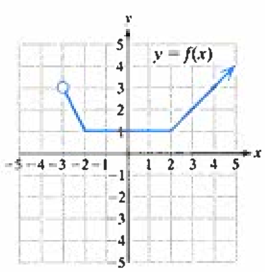
87. A salesperson makes a base salary of \$2000 per month. Once he reaches \$40,000 in total sales, he earns an additional 5% commission on the amount in sales over \$40,000. Write a piecewise-defined function to model the salesperson's total monthly salary $S(x)$ (in \$) as a function of the amount in sales x.

88. A cell phone plan charges \$49.95 per month plus \$14.02 in taxes, plus \$0.40 per minute for calls beyond the 600-min monthly limit. Write a piecewise-defined function to model the monthly cost $C(x)$ (in \$) as a function of the number of minutes used x for the month.

Objective 4: Investigate Increasing, Decreasing, and Constant Behavior of a Function

For Exercises 89–96, use interval notation to write the intervals over which f is (a) increasing, (b) decreasing, and (c) constant. (See Example 10)

89.



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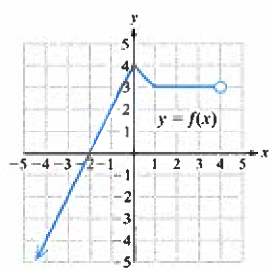

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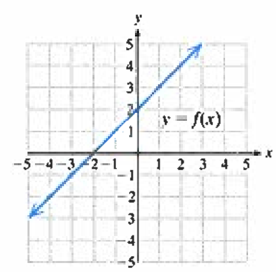

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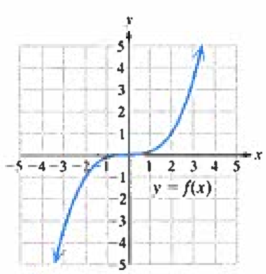

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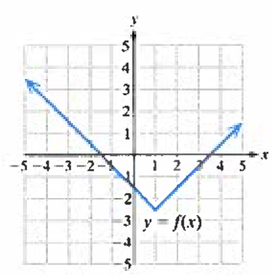

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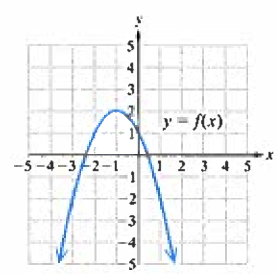

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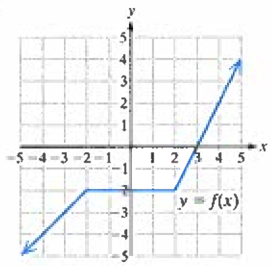
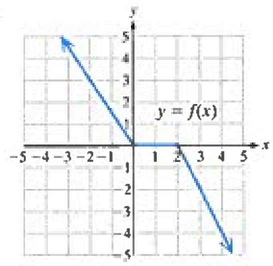


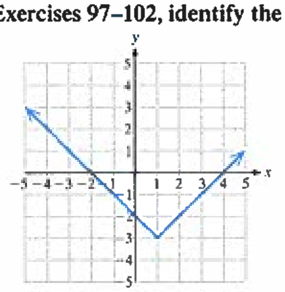
96.



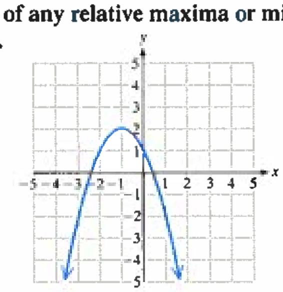
Objective 5: Determine Relative Minima and Maxima of a Function

For Exercises 97–102, identify the location and value of any relative maxima or minima of the function. (See Example 11)

97.



98.



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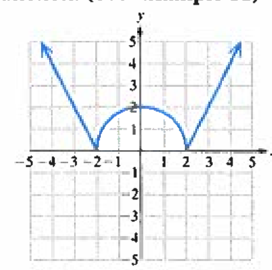
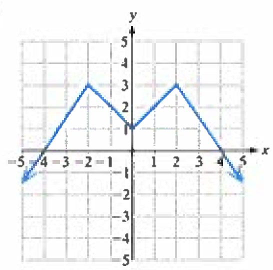
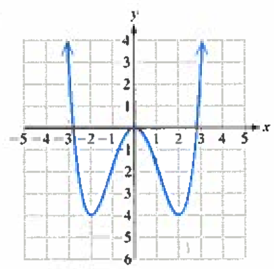


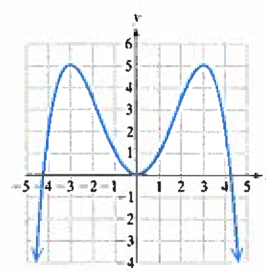
100.



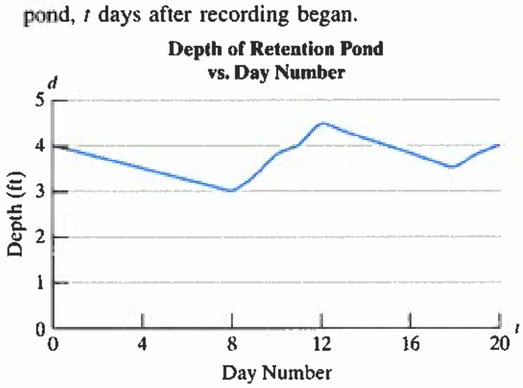
101.



102.



103. The graph shows the depth d (in ft) of a retention pond, t days after recording began.



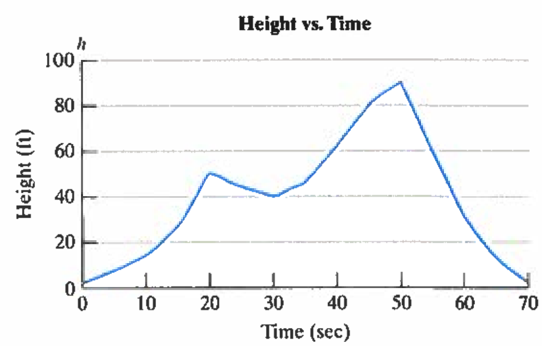
a. Over what interval(s) does the depth increase?

b. Over what interval(s) does the depth decrease?

c. Estimate the times and values of any relative maxima or minima on the interval (0, 20).

d. If rain is the only water that enters the pond, explain what the intervals of increasing and decreasing behavior mean in the context of this problem.

104. The graph shows the height h (in meters) of a roller coaster t seconds after the ride starts.



a. Over what interval(s) does the height increase?

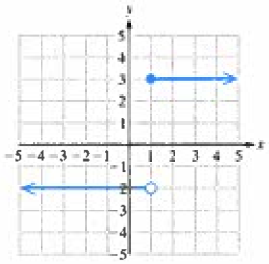
b. Over what interval(s) does the height decrease?

c. Estimate the times and values of any relative maxima or minima on the interval (0, 70).

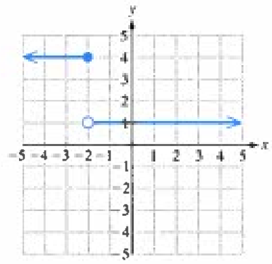
Mixed Exercises

For Exercises 105–110, produce a rule for the function whose graph is shown. (*Hint*: Consider using the basic functions learned in Section 1.6 and transformations of their graphs.)

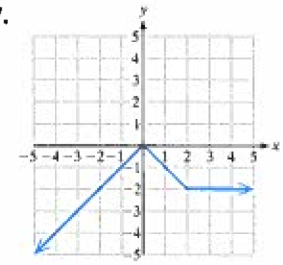
105.



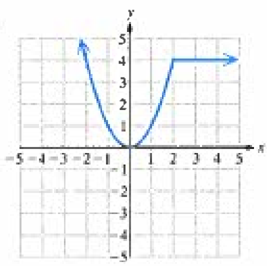
106.



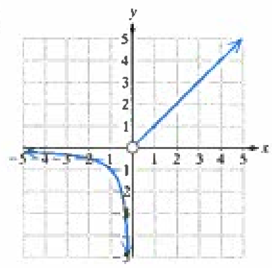
107.



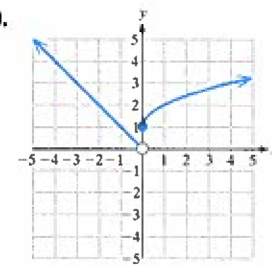
108.



109.



110.



For Exercises 111–112,

a. Graph the function.

b. Write the domain in interval notation.

c. Write the range in interval notation.

d. Evaluate $f(-1)$, $f(1)$, and $f(2)$.

e. Find the value(s) of x for which $f(x) = 6$.

f. Find the value(s) of x for which $f(x) = -3$.

g. Use interval notation to write the intervals over which f is increasing, decreasing, or constant.

111. $f(x) = \begin{cases} -x^2 + 1 & \text{for } x \le 1 \\ 2x & \text{for } x > 1 \end{cases}$

112. $f(x) = \begin{cases} |x| & \text{for } x < 2 \\ -x & \text{for } x \ge 2 \end{cases}$

In computer programming, the greatest integer function is sometimes called the "floor" function. Programmers also make use of the "ceiling" function, which returns the smallest integer not less than x. For example: ceil(3.1) = 4. For Exercises 113–114, evaluate the floor and ceiling functions for the given value of x.

floor(x) is the greatest integer less than or equal to x.
ceil(x) is the smallest integer not less than x.

113. a. floor(2.8) **b.** floor(-3.1) **c.** floor(4)
d. ceil(2.8) **e.** ceil(-3.1) **f.** ceil(4)

114. a. floor(5.5) **b.** floor(-0.1) **c.** floor(-2)
d. ceil(5.5) **e.** ceil(-0.1) **f.** ceil(-2)

Write About It

115. From an equation in x and y, explain how to determine whether the graph of the equation is symmetric with respect to the x-axis, y-axis, or origin.

116. From the graph of a function, how can you determine if the function is even or odd?

117. Explain why the relation defined by

$$y = \begin{cases} 2x & \text{for } x \le 1 \\ 3 & \text{for } x \ge 1 \end{cases}$$

is not a function.

118. Explain why the function is discontinuous at $x = 1$.

$$f(x) = \begin{cases} 3x & \text{for } x < 1 \\ 3 & \text{for } x > 1 \end{cases}$$

119. Provide an informal explanation of a relative maximum.

120. Explain what it means for a function to be increasing on an interval.

Expanding Your Skills

121. Suppose that the average rate of change of a continuous function between any two points to the left of $x = a$ is negative, and the average rate of change of the function between any two points to the right of $x = a$ is positive. Does the function have a relative minimum or maximum at a?

122. Suppose that the average rate of change of a continuous function between any two points to the left of $x = a$ is positive, and the average rate of change of the function between any two points to the right of $x = a$ is negative. Does the function have a relative minimum or maximum at a?

A graph is *concave up* on a given interval if it "bends" upward. A graph is *concave down* on a given interval if it "bends" downward. For Exercises 123–126, determine whether the curve is (a) concave up or concave down and (b) increasing or decreasing.

123. 

124. 

125. 

126. 

127. For a recent year, the federal income tax owed by a taxpayer (single—no dependents) was based on the individual's taxable income. (*Source*: Internal Revenue Service, www.irs.gov)

If your taxable income is			
over—	**but not over—**	**The tax is**	**of the amount over—**
\$0	\$8925	\$0 + 10%	\$0
\$8925	\$36,250	\$892.50 + 15%	\$8925
\$36,250	\$87,850	\$4991.25 + 25%	\$36,250

Write a piecewise-defined function that expresses an individual's federal income tax $f(x)$ (in \$) as a function of the individual's taxable income x (in \$).

Technology Connections

For Exercises 128–131, use a graphing utility to graph the piecewise-defined function.

128. $f(x) = \begin{cases} 2.5x + 2 & \text{for } x \le 1 \\ x^2 - x - 1 & \text{for } x > 1 \end{cases}$

129. $g(x) = \begin{cases} -3.1x - 4 & \text{for } x < -2 \\ -x^3 + 4x - 1 & \text{for } x \ge -2 \end{cases}$

130. $k(x) = \begin{cases} -2.7x - 4.1 & \text{for } x \le -1 \\ -x^3 + 2x + 5 & \text{for } -1 < x < 2 \\ 1 & \text{for } x \ge 2 \end{cases}$

131. $z(x) = \begin{cases} 2.5x + 8 & \text{for } x < -2 \\ -2x^2 + x + 4 & \text{for } -2 \le x < 2 \\ -2 & \text{for } x \ge 2 \end{cases}$

For Exercises 132–135, use a graphing utility to

a. Find the locations and values of the relative maxima and relative minima of the function on the standard viewing window. Round to 3 decimal places.

b. Use interval notation to write the intervals over which f is increasing or decreasing.

132. $f(x) = -0.6x^2 + 2x + 3$

133. $f(x) = 0.4x^2 - 3x - 2.2$

134. $f(x) = 0.5x^3 + 2.1x^2 - 3x - 7$

135. $f(x) = -0.4x^3 - 1.1x^2 + 2x + 3$

SECTION 1.8 Algebra of Functions and Function Composition

OBJECTIVES

1. Perform Operations on Functions
2. Evaluate a Difference Quotient
3. Compose and Decompose Functions

1. Perform Operations on Functions

In Section 1.5, we learned that a profit function can be constructed from the difference of a revenue function and a cost function according to the following rule.

$$P(x) = R(x) - C(x)$$

As this example illustrates, the difference of two functions makes up a new function. New functions can also be formed from the sum, product, and quotient of two functions.

Sum, Difference, Product, and Quotient of Functions

Given functions f and g, the functions $f + g$, $f - g$, $f \cdot g$, and $\frac{f}{g}$ are defined by

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{provided that } g(x) \neq 0$$

The domains of the functions $f + g$, $f - g$, $f \cdot g$, and $\frac{f}{g}$ are all real numbers in the intersection of the domains of the individual functions f and g. For $\frac{f}{g}$, we further restrict the domain to exclude values of x for which $g(x) = 0$.

EXAMPLE 1 Adding Two Functions

Given $f(x) = \sqrt{25 - x^2}$ and $g(x) = 5$, find $(f + g)(x)$.

Solution:

By definition $(f + g)(x) = f(x) + g(x)$.

$$= \sqrt{25 - x^2} + 5$$

Skill Practice 1 Given $m(x) = -|x|$ and $n(x) = 4$, find $(m + n)(x)$.

In Example 1, the graph of function f is a semicircle and the graph of function g is a horizontal line (Figure 1-38). Therefore, the graph of $y = (f + g)(x)$ is the graph of f with a vertical shift (shown in blue). Notice that each y value on $f + g$ is the sum of the y values from the individual functions f and g.

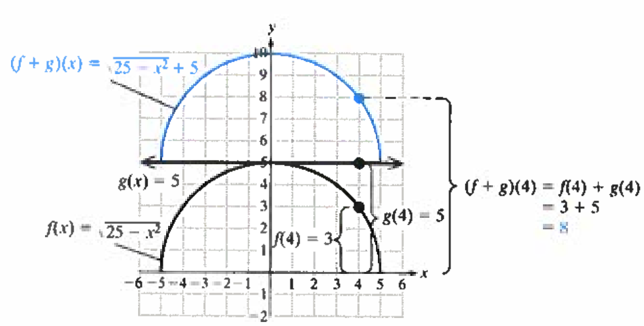


Figure 1-38

Answer

1. $(m + n)(x) = -|x| + 4$

In Example 2, we evaluate the difference, product, and quotient of functions for given values of x.

EXAMPLE 2 Evaluating Functions for Given Values of x

Given $m(x) = 4x$, $n(x) = |x - 3|$, and $p(x) = \dfrac{1}{x+1}$, determine the function values if possible.

a. $(m - n)(-2)$ **b.** $(m \cdot p)(1)$ **c.** $\left(\dfrac{p}{n}\right)(3)$

Solution:

a. $$\begin{aligned}(m - n)(-2) &= m(-2) - n(-2)\\ &= 4(-2) - |-2 - 3|\\ &= -8 - 5\\ &= -13\end{aligned}$$

b. $$\begin{aligned}(m \cdot p)(1) &= m(1) \cdot p(1)\\ &= 4(1) \cdot \frac{1}{1+1}\\ &= 2\end{aligned}$$

c. $$\left(\frac{p}{n}\right)(3) = \frac{p(3)}{n(3)} = \frac{\frac{1}{3+1}}{|3-3|} = \frac{\frac{1}{4}}{0} \text{ (undefined)}$$

The domain of $\frac{p}{n}$ excludes any values of x that make $n(x) = 0$. In this case, $x = 3$ is excluded from the domain.

Skill Practice 2 Use the functions defined in Example 2 to find

a. $(n - m)(-6)$ **b.** $(n \cdot p)(0)$ **c.** $\left(\dfrac{p}{m}\right)(0)$

When combining two or more functions to create a new function, always be sure to determine the domain of the new function. Notice that in Example 2(c), the function $\frac{p}{n}$ is not defined for $x = -1$ or for $x = 3$.

$$\left(\frac{p}{n}\right)(x) = \frac{p(x)}{n(x)} = \frac{\frac{1}{x+1}}{|x-3|}$$

Denominator is zero for $x = -1$.

Denominator of the complex fraction is zero for $x = 3$.

EXAMPLE 3 Combining Functions and Determining Domain

Given $g(x) = 2x$, $h(x) = x^2 - 4x$, and $k(x) = \sqrt{x-1}$,

a. Find $(g - h)(x)$ and write the domain of $g - h$ in interval notation.

b. Find $(g \cdot k)(x)$ and write the domain of $g \cdot k$ in interval notation.

c. Find $\left(\dfrac{k}{h}\right)(x)$ and write the domain of $\dfrac{k}{h}$ in interval notation.

Solution:

a. $$\begin{aligned}(g - h)(x) &= g(x) - h(x)\\ &= 2x - (x^2 - 4x)\\ &= -x^2 + 6x\end{aligned}$$

The domain is $(-\infty, \infty)$.

The domain of g is $(-\infty, \infty)$. The domain of h is $(-\infty, \infty)$. Therefore, the intersection of their domains is $(-\infty, \infty)$.

Answers

2. a. 33 **b.** 3 **c.** Undefined

b. $(g \cdot k)(x) = g(x) \cdot k(x)$

$= 2x\sqrt{x - 1}$

The domain is $[1, \infty)$.

The domain of g is $(-\infty, \infty)$.
The domain of k is $[1, \infty)$.
Therefore, the intersection of their domains is $[1, \infty)$.

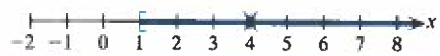
c. $\left(\dfrac{k}{h}\right)(x) = \dfrac{k(x)}{h(x)} = \dfrac{\sqrt{x-1}}{x^2 - 4x}$

$= \dfrac{\sqrt{x-1}}{x(x-4)}$

The domain of k is $[1, \infty)$.
The domain of h is $(-\infty, \infty)$.
The intersection of their domains is $[1, \infty)$.

However, we must also exclude values of x that make the denominator zero. In this case, exclude $x = 0$ and $x = 4$. The value $x = 0$ is already excluded because it is not on the interval $[1, \infty)$. Excluding $x = 4$, the domain of $\frac{k}{h}$ is $[1, 4) \cup (4, \infty)$.

The domain is $[1, 4) \cup (4, \infty)$.



Skill Practice 3 Given $m(x) = x + 3$, $n(x) = x^2 - 9$, and $p(x) = \sqrt{x + 1}$

a. Find $(n - m)(x)$ and write the domain of $n - m$ in interval notation.

b. Find $(m \cdot p)(x)$ and write the domain of $m \cdot p$ in interval notation.

c. Find $\left(\dfrac{p}{n}\right)(x)$ and write the domain of $\dfrac{p}{n}$ in interval notation.

2. Evaluate a Difference Quotient

In Section 1.4, we learned that if f is defined on an interval $[x_1, x_2]$, then the average rate of change of f between $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \quad \text{(Figure 1-39)}$$

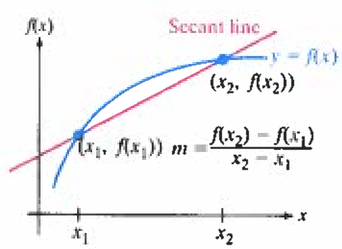


Figure 1-39

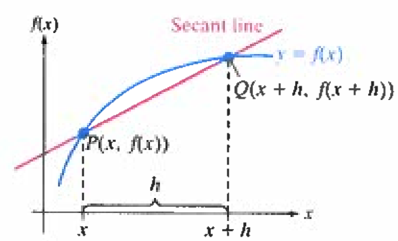


Figure 1-40

TIP h is taken to be a positive real number, implying that $h \neq 0$.

Now we look at a related idea. Let P be an arbitrary point $(x, f(x))$ on the function f. Let h be a positive real number and let Q be the point $(x + h, f(x + h))$. See Figure 1-40. The average rate of change between P and Q is the slope of the secant line and is given by:

$$m = \frac{f(x + h) - f(x)}{(x + h) - x}$$

$$= \frac{f(x + h) - f(x)}{h} \quad \text{(Difference quotient)}$$

The expression on the right is called the **difference quotient** and is very important for the foundation of calculus. In Examples 4 and 5, we practice evaluating the difference quotient for two functions.

Answers

3. a. $(n - m)(x) = x^2 - x - 12$;
Domain: $(-\infty, \infty)$

b. $(m \cdot p)(x) = (x + 3)\sqrt{x + 1}$;
Domain: $[-1, \infty)$

c. $\left(\dfrac{p}{n}\right)(x) = \dfrac{\sqrt{x + 1}}{x^2 - 9}$;
Domain: $[-1, 3) \cup (3, \infty)$

EXAMPLE 4 Finding a Difference Quotient

Given $f(x) = 3x - 5$,

a. Find $f(x + h)$.

b. Find the difference quotient, $\dfrac{f(x+h) - f(x)}{h}$.

Solution:

a. $f(x + h) = 3(x + h) - 5$ Substitute $(x + h)$ for x.

$= 3x + 3h - 5$

b. $$\frac{f(x+h) - f(x)}{h} = \frac{\overbrace{(3x + 3h - 5)}^{f(x+h)} - \overbrace{(3x - 5)}^{f(x)}}{h}$$

$= \dfrac{3x + 3h - 5 - 3x + 5}{h}$ Clear parentheses.

$= \dfrac{3h}{h}$ Combine like terms.

$= 3$ Simplify the fraction.

Skill Practice 4 Given $f(x) = 4x - 2$,

a. Find $f(x + h)$.

b. Find the difference quotient, $\dfrac{f(x+h) - f(x)}{h}$.

EXAMPLE 5 Finding a Difference Quotient

Given $f(x) = -2x^2 + 4x - 1$,

a. Find $f(x + h)$.

b. Find the difference quotient, $\dfrac{f(x+h) - f(x)}{h}$.

Solution:

a. $f(x + h) = -2(x + h)^2 + 4(x + h) - 1$ Substitute $(x + h)$ for x.

$= -2(x^2 + 2xh + h^2) + 4x + 4h - 1$

$= -2x^2 - 4xh - 2h^2 + 4x + 4h - 1$

b. $$\frac{f(x+h) - f(x)}{h} = \frac{\overbrace{(-2x^2 - 4xh - 2h^2 + 4x + 4h - 1)}^{f(x+h)} - \overbrace{(-2x^2 + 4x - 1)}^{f(x)}}{h}$$

$= \dfrac{-2x^2 - 4xh - 2h^2 + 4x + 4h - 1 + 2x^2 - 4x + 1}{h}$ Clear parentheses.

$= \dfrac{-4xh - 2h^2 + 4h}{h}$ Combine like terms.

$= \dfrac{\cancel{h}(-4x - 2h + 4)}{\cancel{h}}$ Factor numerator and denominator, and simplify the fraction.

$= -4x - 2h + 4$

Answers

4. a. $4x + 4h - 2$ b. 4

Skill Practice 5 Given $f(x) = -x^2 - 5x + 2$,

a. Find $f(x + h)$.

b. Find the difference quotient, $\dfrac{f(x + h) - f(x)}{h}$.

3. Compose and Decompose Functions

The next operation on functions we present is called the composition of functions. Informally, this involves a substitution process in which the output from one function becomes the input to another function.

TIP It is important to note that the notation $(f \circ g)(x)$ represents the composition of functions, *not* multiplication of f, g, and x.

Composition of Functions

The **composition of f and g,** denoted $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is the set of real numbers x in the domain of g such that $g(x)$ is in the domain of f.

To visualize the composition of functions $(f \circ g)(x) = f(g(x))$, consider Figure 1-41.

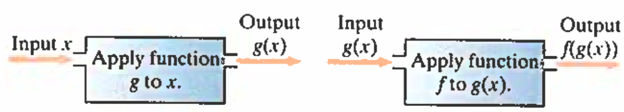


Figure 1-41

EXAMPLE 6 Composing Functions

Given $f(x) = x^2 + 2x$ and $g(x) = x - 4$, find

a. $f(g(6))$ **b.** $g(f(-3))$ **c.** $(f \circ g)(0)$ **d.** $(g \circ f)(5)$

Solution:

a. $f(g(6)) = f(2)$ Evaluate $g(6)$ first. $g(6) = 6 - 4 = 2$.
$= 8$ $f(2) = (2)^2 + 2(2) = 8$

b. $g(f(-3)) = g(3)$ Evaluate $f(-3)$ first. $f(-3) = (-3)^2 + 2(-3) = 3$.
$= -1$ $g(3) = 3 - 4 = -1$

c. $(f \circ g)(0) = f(g(0))$ Evaluate $g(0)$ first. $g(0) = 0 - 4 = -4$.
$= f(-4)$ $f(-4) = (-4)^2 + 2(-4) = 8$
$= 8$

d. $(g \circ f)(5) = g(f(5))$ Evaluate $f(5)$ first. $f(5) = (5)^2 + 2(5) = 35$.
$= g(35)$ $g(35) = 35 - 4 = 31$
$= 31$

TIP When composing functions, apply the order of operations. In Example 6(a), the value of $g(6)$ is found first.

Skill Practice 6 Refer to functions f and g given in Example 6. Find

a. $f(g(-4))$ **b.** $g(f(-5))$ **c.** $(f \circ g)(9)$ **d.** $(g \circ f)(10)$

In Example 7, we practice composing functions and identifying the domain of the composite function. This example also illustrates that function composition is not commutative. That is, $(f \circ g)(x) \neq (g \circ f)(x)$ for all functions f and g.

Answers

5. a. $-x^2 - 2xh - h^2 - 5x - 5h + 2$
b. $-2x - h - 5$

6. a. 48 **b.** 11 **c.** 35
d. 116

EXAMPLE 7 Composing Functions and Determining Domain

Given $f(x) = 2x - 6$ and $g(x) = \frac{1}{x+4}$, write a rule for each function and write the domain in interval notation.

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$

Solution:

a. $(f \circ g)(x) = f(g(x)) = 2(g(x)) - 6$

$= 2\left(\dfrac{1}{x+4}\right) - 6$

Function g has the restriction that $x \neq -4$.

$= \dfrac{2}{x+4} - 6$ provided $x \neq -4$

The domain of f is all real numbers. Therefore, no further restrictions need to be imposed.

The domain is $(-\infty, -4) \cup (-4, \infty)$.

b. $(g \circ f)(x) = g(f(x)) = \dfrac{1}{f(x) + 4}$

$= \dfrac{1}{(2x - 6) + 4}$ $f(x) \neq -4$

$= \dfrac{1}{2x - 2}$ provided $x \neq 1$

The domain is $(-\infty, 1) \cup (1, \infty)$.

The domain of f has no restrictions.

However, function g must not have an input value of -4. Therefore, we have the restriction $f(x) \neq -4$. Thus,

$2x - 6 = -4$

$2x = 2$

$x = 1$ must be excluded.

Skill Practice 7 Given $f(x) = 3x + 4$ and $g(x) = \frac{1}{x-1}$, write a rule for each function and write the domain in interval notation.

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$

EXAMPLE 8 Composing Functions and Determining Domain

Given $m(x) = \frac{1}{x-5}$ and $p(x) = \sqrt{x-2}$, find $(m \circ p)(x)$ and write the domain in interval notation.

Solution:

$(m \circ p)(x) = m(p(x)) = \dfrac{1}{p(x) - 5}$

$p(x) \neq 5$

$= \dfrac{1}{\sqrt{x-2} - 5}$

$(m \circ p)(x) = \dfrac{1}{\sqrt{x-2} - 5}$

First note that function p has the restriction that $x \geq 2$.

The input value for function m must not be 5. Therefore, $p(x) \neq 5$. Thus,

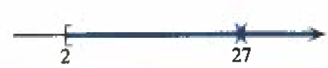
$\sqrt{x-2} = 5$

$(\sqrt{x-2})^2 = (5)^2$

$x - 2 = 25$

$x = 27$ must be excluded

The domain is $[2, 27) \cup (27, \infty)$.



Skill Practice 8 Given $f(x) = \sqrt{x-1}$ and $g(x) = \frac{1}{x-3}$, find $(g \circ f)(x)$ and write the domain of $g \circ f$ in interval notation.

Answers

7. a. $(f \circ g)(x) = \dfrac{3}{x-1} + 4$; Domain: $(-\infty, 1) \cup (1, \infty)$

b. $(g \circ f)(x) = \dfrac{1}{3x+3}$; Domain: $(-\infty, -1) \cup (-1, \infty)$

8. $(g \circ f)(x) = \dfrac{1}{\sqrt{x-1} - 3}$; Domain: $[1, 10) \cup (10, \infty)$

EXAMPLE 9 Composing Functions and Determining Domain

Given $k(x) = \dfrac{x}{x-2}$ and $m(x) = \dfrac{6}{x^2-1}$, find $(k \circ m)(x)$ and write the domain in interval notation.

Solution:

$(k \circ m)(x) = k(m(x))$ — Evaluate k at $m(x)$.

$= \dfrac{\left(\dfrac{6}{x^2-1}\right)}{\left(\dfrac{6}{x^2-1}\right) - 2}$ — Substitute $\dfrac{6}{x^2-1}$ for x in $k(x)$.

- m has the restriction that $x^2 - 1 \neq 0$. Therefore, $x \neq \pm 1$.

$= \dfrac{(x^2-1)}{(x^2-1)} \cdot \dfrac{\left(\dfrac{6}{x^2-1}\right)}{\left(\dfrac{6}{x^2-1} - 2\right)}$ — Simplify the complex fraction by multiplying numerator and denominator by the LCD $x^2 - 1$.

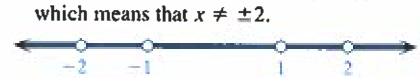
$= \dfrac{6}{6 - 2(x^2-1)}$ — Apply the distributive property.

$= \dfrac{6}{-2x^2 + 8}$ — Simplify the denominator.

$= -\dfrac{6}{2x^2 - 8}$ — Factor out -1 from the denominator.

$= -\dfrac{3}{x^2 - 4}$ — Simplify the rational expression.

- Note the added restriction that $x^2 - 4 \neq 0$ which means that $x \neq \pm 2$.



The domain is $(-\infty, -2) \cup (-2, -1) \cup (-1, 1) \cup (1, 2) \cup (2, \infty)$.

Skill Practice 9 Given $r(x) = \dfrac{x}{x+1}$ and $t(x) = \dfrac{5}{x^2-9}$, find $(r \circ t)(x)$ and write the domain in interval notation.

EXAMPLE 10 Applying Function Composition

At a popular website, the cost to download individual songs is \$1.49 per song. In addition, a first-time visitor to the website has a one-time coupon for \$1.00 off.

a. Write a function to represent the cost $C(x)$ (in \$) for a first-time visitor to purchase x songs.

b. The sales tax for online purchases depends on the location of the business and customer. If the sales tax rate on a purchase is 6%, write a function to represent the total cost $T(a)$ for a first-time visitor who buys a dollars in songs.

c. Find $(T \circ C)(x)$ and interpret the meaning in context.

d. Evaluate $(T \circ C)(10)$ and interpret the meaning in context.

Solution:

a. $C(x) = 1.49x - 1.00;\ x \geq 1$ — The cost function is a linear function with \$1.49 as the variable rate per song.

b. $T(a) = a + 0.06a$
$= 1.06a$ — The total cost is the sum of the cost of the songs plus the sales tax.

Answer

9. $(r \circ t)(x) = \dfrac{5}{x^2 - 4}$;
Domain: $(-\infty, -3) \cup (-3, -2) \cup (-2, 2) \cup (2, 3) \cup (3, \infty)$

c. $(T \circ C)(x) = T(C(x))$
$= 1.06(C(x))$
$= 1.06(1.49x - 1.00)$ Substitute $1.49x - 1.00$ for $C(x)$.
$= 1.5794x - 1.06$

$(T \circ C)(x) = 1.5794x - 1.06$ represents the total cost to buy x songs for a first-time visitor to the website.

d. $(T \circ C)(x) = 1.5794x - 1.06$
$(T \circ C)(10) = 1.5794(10) - 1.06$
$= 14.734$

The total cost for a first-time visitor to buy 10 songs is \$14.73.

Skill Practice 10 An artist shops online for tubes of watercolor paint. The cost is \$16 for each 14-mL tube.

a. Write a function representing the cost $C(x)$ (in \$) for x tubes of paint.
b. There is a 5.5% sales tax on the cost of merchandise and a fixed cost of \$4.99 for shipping. Write a function representing the total cost $T(a)$ for a dollars spent in merchandise.
c. Find $(T \circ C)(x)$ and interpret the meaning in context.
d. Evaluate $(T \circ C)(18)$ and interpret the meaning in context.

TIP The decomposition of functions is not unique. For example, $h(x) = (x - 3)^2$ can also be written as $h(x) = f(g(x))$, where $g(x) = x^2 - 6x$ and $f(x) = x + 9$.

The composition of two functions creates a new function in which the output from one function becomes the input to the other. We can also reverse this process. That is, we can decompose a composite function into two or more simpler functions.

For example, consider the function h defined by $h(x) = (x - 3)^2$. To write h as a composition of two functions, we have $h(x) = (f \circ g)(x) = f(g(x))$. The function g is the "inside" function and f is the "outside" function. So one natural choice for g and f would be:

$g(x) = x - 3$ Function g subtracts 3 from the input value.
$f(x) = x^2$ Function f squares the result.
$h(x) = f(g(x)) = (g(x))^2 = (x - 3)^2$

EXAMPLE 11 Decomposing Two Functions

Given $h(x) = |2x^2 - 5|$, find two functions f and g such that $h(x) = (f \circ g)(x)$.

Solution:

We need to find two functions f and g such that $h(x) = (f \circ g)(x) = f(g(x))$. The function h first evaluates the expression $2x^2 - 5$, and then takes the absolute value. Therefore, it would be natural to take the absolute value of $g(x) = 2x^2 - 5$.

We have: $g(x) = 2x^2 - 5$ and $f(x) = |x|$

Check: $h(x) = (f \circ g)(x) = f(g(x)) = |g(x)|$
$= |2x^2 - 5|$

Answers

10. a. $C(x) = 16x$
b. $T(a) = 1.055a + 4.99$
c. $(T \circ C)(x) = 16.88x + 4.99$ represents the total cost to buy x tubes of paint.
d. $(T \circ C)(18) = \$308.83$; The total cost to buy 18 tubes of paint is \$308.83.

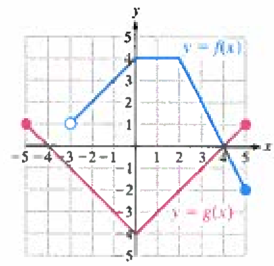
Skill Practice 11 Given $m(x) = \sqrt[3]{5x + 1}$, find two functions f and g such that $m(x) = (f \circ g)(x)$.

In Example 12, we have the graphs of two functions, and we apply function addition, subtraction, multiplication, and composition for selected values of x.

EXAMPLE 12 Estimating Function Values from a Graph

The graphs of f and g are shown. Evaluate the functions at the given values of x if possible.

a. $(f + g)(1)$
b. $(fg)(0)$
c. $(g - f)(-3)$
d. $(f \circ g)(3)$
e. $(g \circ f)(4)$
f. $f(g(1))$



Solution:

a. $(f + g)(1) = f(1) + g(1)$
$= 4 + (-3)$
$= 1$

b. $(fg)(0) = f(0) \cdot g(0)$
$= (4)(-4)$
$= -16$

c. $(g - f)(-3) = g(-3) - \underbrace{f(-3)}$ $f(-3)$ is undefined.

$(g - f)(-3)$ is undefined.

d. $(f \circ g)(3) = f(g(3))$
$= f(-1)$
$= 3$

e. $(g \circ f)(4) = g(f(4))$
$= g(0)$
$= -4$

f. $f(g(1)) = f(-3)$ is undefined.

The open dot at $(-3, 1)$ indicates that -3 is not in the domain of f. The value $g(1) = -3$, but $f(-3)$ is undefined. Therefore, $f(g(1))$ is undefined.

Skill Practice 12 Refer to the functions f and g pictured in Example 12. Evaluate the functions at the given values of x if possible.

a. $(f - g)(-2)$
b. $\left(\frac{f}{g}\right)(3)$
c. $(gf)(5)$
d. $(g \circ f)(5)$
e. $(f \circ g)(5)$
f. $f(g(0))$

Answers
11. $g(x) = 5x + 1$ and $f(x) = \sqrt[3]{x}$
12. a. 4 b. -2 c. -2 d. -2 e. 4 f. Undefined

SECTION 1.8 Practice Exercises

Prerequisite Review

For Exercises R.1–R.4, write the domain in interval notation.

R.1. $f(x) = \dfrac{x - 1}{x + 2}$

R.2. $r(x) = \sqrt{x + 3}$

R.3. $h(x) = \dfrac{4}{\sqrt{3 - x}}$

R.4. $p(x) = 2x^2 - 3x + 1$

R.5. Given $k(x) = x^2 - 2x + 3$, find $k(x + 3)$.

Concept Connections

1. The function $f + g$ is defined by $(f + g)(x) =$ ________ + ________.

2. The function $\frac{f}{g}$ is defined by $\left(\frac{f}{g}\right)(x) =$ ________ provided that ________ $\neq 0$.

3. Let h represent a positive real number. Given a function defined by $y = f(x)$, the difference quotient is given by ________________.

4. The composition of f and g, denoted by $f \circ g$, is defined by $(f \circ g)(x) =$ ________.

Objective 1: Perform Operations on Functions

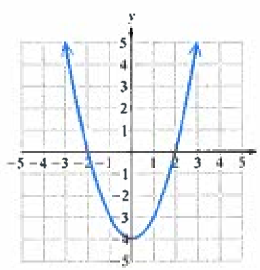
For Exercises 5–8, find $(f + g)(x)$ and identify the graph of $f + g$. (See Example 1)

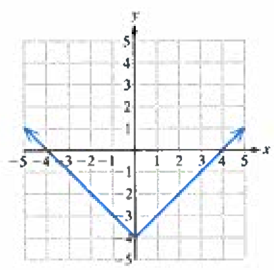
5. $f(x) = |x|$ and $g(x) = 3$

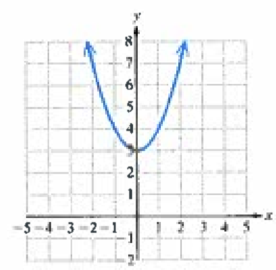
6. $f(x) = |x|$ and $g(x) = -4$

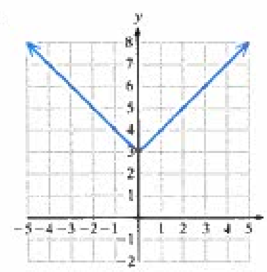
7. $f(x) = x^2$ and $g(x) = -4$

8. $f(x) = x^2$ and $g(x) = 3$

a.


b.


c.


d.


For Exercises 9–18, evaluate the functions for the given values of x. (See Example 2)

$$f(x) = -2x \qquad g(x) = |x + 4| \qquad h(x) = \frac{1}{x - 3}$$

9. $(f - g)(3)$

10. $(g - h)(2)$

11. $(f \cdot g)(-1)$

12. $(h \cdot g)(4)$

13. $(g + h)(0)$

14. $(f + h)(5)$

15. $\left(\frac{f}{g}\right)(8)$

16. $\left(\frac{h}{f}\right)(7)$

17. $\left(\frac{g}{f}\right)(0)$

18. $\left(\frac{h}{g}\right)(-4)$

For Exercises 19–26, refer to the functions r, p, and q. Find the indicated function and write the domain in interval notation. (See Example 3)

$$r(x) = -3x \qquad p(x) = x^2 + 3x \qquad q(x) = \sqrt{1 - x}$$

19. $(r - p)(x)$

20. $(p - r)(x)$

21. $(p \cdot q)(x)$

22. $(r \cdot q)(x)$

23. $\left(\frac{q}{p}\right)(x)$

24. $\left(\frac{q}{r}\right)(x)$

25. $\left(\frac{p}{q}\right)(x)$

26. $\left(\frac{r}{q}\right)(x)$

For Exercises 27–32, refer to functions s, t, and v. Find the indicated function and write the domain in interval notation. (See Example 3)

$$s(x) = \frac{x - 2}{x^2 - 9} \qquad t(x) = \frac{x - 3}{2 - x} \qquad v(x) = \sqrt{x + 3}$$

27. $(s \cdot t)(x)$

28. $\left(\frac{s}{t}\right)(x)$

29. $(s + t)(x)$

30. $(s - t)(x)$

31. $(s \cdot v)(x)$

32. $\left(\frac{v}{s}\right)(x)$

Objective 2: Evaluate a Difference Quotient

For Exercises 33–36, a function is given. (See Examples 4–5)

a. Find $f(x + h)$.

b. Find $\frac{f(x + h) - f(x)}{h}$.

33. $f(x) = 5x + 9$

34. $f(x) = 8x + 4$

35. $f(x) = x^2 + 4x$

36. $f(x) = x^2 - 3x$

For Exercises 37–44, find the difference quotient and simplify. (See Examples 4–5)

37. $f(x) = -2x + 5$ **38.** $f(x) = -3x + 8$ **39.** $f(x) = -5x^2 - 4x + 2$ **40.** $f(x) = -4x^2 - 2x + 6$

41. $f(x) = x^3 + 5$ **42.** $f(x) = x^3 - 2$ **43.** $f(x) = \frac{1}{x}$ **44.** $f(x) = \frac{1}{x + 2}$

45. Given $f(x) = 4\sqrt{x}$,

a. Find the difference quotient (do not simplify).

b. Evaluate the difference quotient for $x = 1$, and the following values of h: $h = 1, h = 0.1, h = 0.01$, and $h = 0.001$. Round to 4 decimal places.

c. What value does the difference quotient seem to be approaching as h gets close to 0?

46. Given $f(x) = \frac{12}{x}$,

a. Find the difference quotient (do not simplify).

b. Evaluate the difference quotient for $x = 2$, and the following values of h: $h = 0.1, h = 0.01, h = 0.001$, and $h = 0.0001$. Round to 4 decimal places.

c. What value does the difference quotient seem to be approaching as h gets close to 0?

Objective 3: Compose and Decompose Functions

For Exercises 47–62, refer to functions f, g, and h. Evaluate the functions for the given values of x. (See Example 6)

$f(x) = x^3 - 4x \qquad g(x) = \sqrt{2x} \qquad h(x) = 2x + 3$

47. $f(g(8))$ **48.** $h(g(2))$ **49.** $h(f(1))$ **50.** $g(f(3))$

51. $(f \circ g)(18)$ **52.** $(f \circ h)(-1)$ **53.** $(g \circ f)(5)$ **54.** $(h \circ f)(-2)$

55. $(h \circ f)(-3)$ **56.** $(h \circ g)(72)$ **57.** $(g \circ f)(1)$ **58.** $(g \circ f)(-4)$

59. $(f \circ f)(3)$ **60.** $(h \circ h)(-4)$ **61.** $(f \circ h \circ g)(2)$ **62.** $(f \circ h \circ g)(8)$

63. Given $f(x) = 2x + 4$ and $g(x) = x^2$,

a. Find $(f \circ g)(x)$. **b.** Find $(g \circ f)(x)$.

c. Is the operation of function composition commutative?

64. Given $k(x) = -3x + 1$ and $m(x) = \frac{1}{x}$,

a. Find $(k \circ m)(x)$. **b.** Find $(m \circ k)(x)$.

c. Is $(k \circ m)(x) = (m \circ k)(x)$?

For Exercises 65–76, refer to functions m, n, p, q, and r. Find the indicated function and write the domain in interval notation. (See Examples 7–9)

$m(x) = \sqrt{x + 8} \qquad n(x) = x - 5 \qquad p(x) = x^2 - 9x \qquad q(x) = \frac{1}{x - 10} \qquad r(x) = |2x + 3|$

65. $(n \circ p)(x)$ **66.** $(p \circ n)(x)$ **67.** $(m \circ n)(x)$ **68.** $(n \circ m)(x)$

69. $(q \circ n)(x)$ **70.** $(q \circ p)(x)$ **71.** $(q \circ r)(x)$ **72.** $(q \circ m)(x)$

73. $(n \circ r)(x)$ **74.** $(r \circ n)(x)$ **75.** $(q \circ q)(x)$ **76.** $(p \circ p)(x)$

For Exercises 77–80, find $(f \circ g)(x)$ and write the domain in interval notation. (See Example 9)

77. $f(x) = \frac{3}{x^2 - 16}$, $g(x) = \sqrt{2 - x}$

78. $f(x) = \frac{4}{x^2 - 9}$, $g(x) = \sqrt{3 - x}$

79. $f(x) = \frac{x}{x - 1}$, $g(x) = \frac{9}{x^2 - 16}$

80. $f(x) = \frac{x}{x + 4}$, $g(x) = \frac{3}{x^2 - 1}$

81. Given $f(x) = \frac{1}{x - 2}$, find $(f \circ f)(x)$ and write the domain in interval notation.

82. Given $g(x) = \sqrt{x - 3}$, find $(g \circ g)(x)$ and write the domain in interval notation.

For Exercises 83–86, find the indicated functions.

$f(x) = 2x + 1 \qquad g(x) = x^2 \qquad h(x) = \sqrt[3]{x}$

83. $(f \circ g \circ h)(x)$ **84.** $(g \circ f \circ h)(x)$

85. $(h \circ g \circ f)(x)$ **86.** $(g \circ h \circ f)(x)$

87. A law office orders business stationery. The cost is \$21.95 per box. **(See Example 10)**

- **a.** Write a function that represents the cost $C(x)$ (in \$) for x boxes of stationery.
- **b.** There is a 6% sales tax on the cost of merchandise and \$10.99 for shipping. Write a function that represents the total cost $T(a)$ for a dollars spent in merchandise and shipping.
- **c.** Find $(T \circ C)(x)$.
- **d.** Find $(T \circ C)(4)$ and interpret its meaning in the context of this problem.

88. The cost to buy tickets online for a dance show is \$60 per ticket.

- **a.** Write a function that represents the cost $C(x)$ (in \$) for x tickets to the show.
- **b.** There is a sales tax of 5.5% and a processing fee of \$8.00 for a group of tickets. Write a function that represents the total cost $T(a)$ for a dollars spent on tickets.
- **c.** Find $(T \circ C)(x)$.
- **d.** Find $(T \circ C)(6)$ and interpret its meaning in the context of this problem.

89. A bicycle wheel turns at a rate of 80 revolutions per minute (rpm).

- **a.** Write a function that represents the number of revolutions $r(t)$ in t minutes.
- **b.** For each revolution of the wheels, the bicycle travels 7.2 ft. Write a function that represents the distance traveled $d(r)$ (in ft) for r revolutions of the wheel.
- **c.** Find $(d \circ r)(t)$ and interpret the meaning in the context of this problem.
- **d.** Evaluate $(d \circ r)(30)$ and interpret the meaning in the context of this problem.

90. While on vacation in France, Sadie bought a box of almond croissants. Each croissant cost €2.4 (euros).

- **a.** Write a function that represents the cost $C(x)$ (in euros) for x croissants.
- **b.** At the time of the purchase, the exchange rate was \$1 = €0.80. Write a function that represents the amount $D(C)$ (in \$) for C euros spent.
- **c.** Find $(D \circ C)(x)$ and interpret the meaning in the context of this problem.
- **d.** Evaluate $(D \circ C)(12)$ and interpret the meaning in the context of this problem.

For Exercises 91–98, find two functions f and g such that $h(x) = (f \circ g)(x)$. (See Example 11)

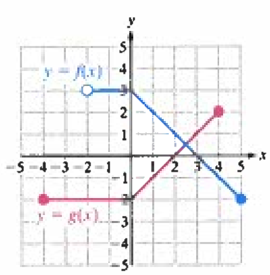
91. $h(x) = (x + 7)^2$ **92.** $h(x) = (x - 8)^2$ **93.** $h(x) = \sqrt[3]{2x + 1}$ **94.** $h(x) = \sqrt[4]{9x - 5}$

95. $h(x) = |2x^2 - 3|$ **96.** $h(x) = |4 - x^2|$ **97.** $h(x) = \dfrac{5}{x + 4}$ **98.** $h(x) = \dfrac{11}{x - 3}$

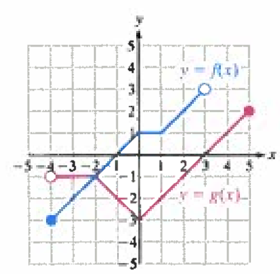
Mixed Exercises

For Exercises 99–102, the graphs of two functions are shown. Evaluate the function at the given values of x, if possible. (See Example 12)

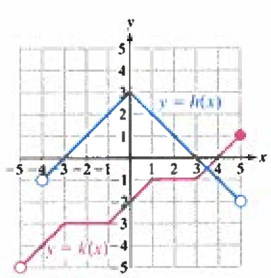
99. **a.** $(f + g)(0)$
b. $(g - f)(2)$
c. $(g \cdot f)(-1)$
d. $\left(\dfrac{g}{f}\right)(1)$
e. $(f \circ g)(4)$
f. $(g \circ f)(0)$
g. $g(f(4))$



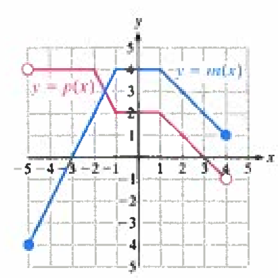
100. **a.** $(f + g)(0)$
b. $(g - f)(1)$
c. $(g \cdot f)(2)$
d. $\left(\dfrac{f}{g}\right)(-3)$
e. $(f \circ g)(3)$
f. $(g \circ f)(0)$
g. $g(f(-4))$



101. **a.** $(h + k)(-1)$
b. $(h \cdot k)(4)$
c. $\left(\dfrac{k}{h}\right)(-3)$
d. $(k - h)(1)$
e. $(k \circ h)(4)$
f. $(h \circ k)(-2)$
g. $h(k(3))$



102. **a.** $(m + p)(1)$
b. $(p - m)(-4)$
c. $\left(\dfrac{m}{p}\right)(3)$
d. $(m \cdot p)(3)$
e. $(m \circ p)(0)$
f. $(p \circ m)(0)$
g. $p(m(-4))$



For Exercises 103–110, refer to the functions f and g and evaluate the functions for the given values of x.

$$f = \{(2, 4), (6, -1), (4, -2), (0, 3), (-1, 6)\} \quad \text{and} \quad g = \{(4, 3), (0, 6), (5, 7), (6, 0)\}$$

103. $(f + g)(4)$ **104.** $(g \cdot f)(0)$ **105.** $(g \circ f)(2)$ **106.** $(f \circ g)(0)$

107. $(g \circ g)(6)$ **108.** $(f \circ f)(-1)$ **109.** $(f \circ g)(5)$ **110.** $(g \circ f)(0)$

111. The diameter d of a sphere is twice the radius r. The volume of the sphere as a function of its radius is given by $V(r) = \frac{4}{3}\pi r^3$.

a. Write the diameter d of the sphere as a function of the radius r.

b. Write the radius r as a function of the diameter d.

c. Find $(V \circ r)(d)$ and interpret its meaning.

112. Consider a right circular cone with given height h. The volume of the cone as a function of its radius r is given by $V(r) = \frac{1}{3}\pi r^2 h$. Consider a right circular cone with fixed height $h = 6$ in.

a. Write the diameter d of the cone as a function of the radius r.

b. Write the radius r as a function of the diameter d.

c. Find $(V \circ r)(d)$ and interpret its meaning. Assume that $h = 6$ in.

113. An investment earns 4.5% interest paid at the end of 1 yr. If x is the amount of money initially invested, then $A(x) = 1.045x$ represents the amount of money in the account 1 yr later. Find $(A \circ A)(x)$ and interpret the result.

114. The population in a certain town has been decreasing at a rate of 2% per year. If x is the population at a certain fixed time, then $P(x) = 0.98x$ represents the population 1 yr later. Find $(P \circ P)(x)$ and interpret the result.

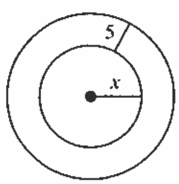
115. Suppose that a function H gives the high temperature $H(x)$ (in °F) for day x. Suppose that a function L gives the low temperature $L(x)$ (in °F) for day x. What does $\left(\frac{H + L}{2}\right)(x)$ represent?

116. For the given figure,

a. What does $A_1(x) = \pi(x + 5)^2$ represent?

b. What does $A_2(x) = \pi x^2$ represent?

c. Find $(A_1 - A_2)(x)$ and interpret its meaning.

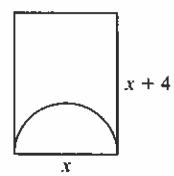


117. For the given figure,

a. Write an expression $S_1(x)$ that represents the area of the rectangle.

b. Write an expression $S_2(x)$ that represents the area of the semicircle.

c. Find $(S_1 - S_2)(x)$ and interpret its meaning.



Write About It

118. Given functions f and g, explain how to determine the domain of $\left(\frac{f}{g}\right)(x)$.

119. Given functions f and g, explain how to determine the domain of $(f \circ g)(x)$.

120. Explain what the difference quotient represents.

Expanding Your Skills

121. Given $f(x) = \sqrt{x + 3}$,

a. Find the difference quotient.

b. Rationalize the numerator of the expression in part (a) and simplify.

c. Evaluate the expression in part (b) for $h = 0$.

122. Given $f(x) = \sqrt{x - 4}$,

a. Find the difference quotient.

b. Rationalize the numerator of the expression in part (a) and simplify.

c. Evaluate the expression in part (b) for $h = 0$.

123. A car traveling 60 mph (88 ft/sec) undergoes a constant deceleration until it comes to rest approximately 9.09 sec later. The distance $d(t)$ (in ft) that the car travels t seconds after the brakes are applied is given by $d(t) = -4.84t^2 + 88t$, where $0 \le t \le 9.09$. **(See Example 5)**

a. Find the difference quotient $\frac{d(t + h) - d(t)}{h}$.

Use the difference quotient to determine the average rate of speed on the following intervals for t.

b. $[0, 2]$ (*Hint*: $t = 0$ and $h = 2$)

c. $[2, 4]$ (*Hint*: $t = 2$ and $h = 2$)

d. $[4, 6]$ (*Hint*: $t = 4$ and $h = 2$)

e. $[6, 8]$ (*Hint*: $t = 6$ and $h = 2$)

124. A car accelerates from 0 to 60 mph (88 ft/sec) in 8.8 sec. The distance $d(t)$ (in ft) that the car travels t seconds after motion begins is given by $d(t) = 5t^2$, where $0 \le t \le 8.8$.

a. Find the difference quotient $\frac{d(t + h) - d(t)}{h}$.

Use the difference quotient to determine the average rate of speed on the following intervals for t.

b. $[0, 2]$

c. $[2, 4]$

d. $[4, 6]$

e. $[6, 8]$

125. If a is b plus eight, and c is the square of a, write c as a function of b.

126. If q is r minus seven, and s is the square root of q, write s as a function of r.

127. If x is twice y, and z is four less than x, write z as a function of y.

128. If m is one-third of n, and p is two less than m, write p as a function of n.

129. Given $f(x) = \sqrt[3]{4x^2 + 1}$, define functions m, n, h, and k such that $f(x) = (m \circ n \circ h \circ k)(x)$.

130. Given $f(x) = |-2x^3 - 4|$, define functions m, n, h, and k such that $f(x) = (m \circ n \circ h \circ k)(x)$.

CHAPTER 1 KEY CONCEPTS

SECTION 1.1 The Rectangular Coordinate System and Graphing Utilities	Reference
The **distance** between two points (x_1, y_1) and (x_2, y_2) in a rectangular coordinate system is given by $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$	p. 121
The **midpoint** between the points is given by $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.	p. 122
• To find an x-intercept $(a, 0)$ of the graph of an equation, substitute 0 for y and solve for x. • To find a y-intercept $(0, b)$ of the graph of an equation, substitute 0 for x and solve for y.	p. 124

SECTION 1.2 Circles	Reference
The **standard form** of an equation of a circle with radius r and center (h, k) is $(x - h)^2 + (y - k)^2 = r^2$.	p. 132
An equation of a circle written in the form $x^2 + y^2 + Ax + By + C = 0$ is called the **general form** of an equation of a circle.	p. 133

SECTION 1.3 Functions and Relations	Reference
A set of ordered pairs (x, y) is called a **relation** in x and y. The set of x values is the **domain** of the relation, and the set of y values is the **range** of the relation.	p. 137
Given a relation in x and y, we say that **y is a function of x** if for each value of x in the domain, there is exactly one value of y in the range.	p. 138
The vertical line test tells us that the graph of a relation defines y as a function of x if no vertical line intersects the graph in more than one point.	p. 139
Given a function defined by $y = f(x)$, • The x-intercepts are the real solutions to $f(x) = 0$. • The y-intercept is given by $f(0)$.	p. 142
Given $y = f(x)$, the domain of f is the set of real numbers x that when substituted into the function produce a real number. This excludes • Values of x that make the denominator zero. • Values of x that make a radicand negative within an even-indexed root.	p. 143

SECTION 1.4 Linear Equations in Two Variables and Linear Functions	Reference
Let A, B, and C represent real numbers where A and B are not both zero. A **linear equation** in the variables x and y is an equation that can be written as $Ax + By = C$.	p. 151
The slope of a line passing through the distinct points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.	p. 153
Given a line with slope m and y-intercept $(0, b)$, the **slope-intercept form** of the line is given by $y = mx + b$.	p. 155

If f is defined on the interval $[x_1, x_2]$, then the **average rate of change** of f on the interval $[x_1, x_2]$ is the slope of the secant line containing $(x_1, f(x_1))$ and $(x_2, f(x_2))$ and is given by $$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$	p. 157
The x-coordinates of the points of intersection between the graphs of $y = f(x)$ and $y = g(x)$ are the solutions to the equation $f(x) = g(x)$.	p. 159
SECTION 1.5 Applications of Linear Equations and Modeling	**Reference**
The **point-slope formula** for a line is given by $y - y_1 = m(x - x_1)$ where m is the slope of the line and (x_1, y_1) is a point on the line.	p. 167
• If m_1 and m_2 represent the slopes of two nonvertical parallel lines, then $m_1 = m_2$. • If m_1 and m_2 represent the slopes of two nonvertical perpendicular lines, then $m_1 = -\frac{1}{m_2}$ or equivalently $m_1 m_2 = -1$.	p. 168
In many-day-to-day applications, two variables are related linearly. • A linear model can be made from two data points that represent the general trend of the data. • Alternatively, the least-squares regression line is a model that utilizes *all* observed data points.	p. 173 p. 174
SECTION 1.6 Transformations of Graphs	**Reference**
Consider a function defined by $y = f(x)$. Let h, k, and a represent positive real numbers. The graphs of the following functions are related to $y = f(x)$ as follows. • Vertical translation (shift) $y = f(x) + k$ Shift upward $y = f(x) - k$ Shift downward • Horizontal translation (shift) $y = f(x - h)$ Shift to the right $y = f(x + h)$ Shift to the left	pp. 184–185
• Vertical stretch/shrink $y = af(x)$ Vertical stretch (if $a > 1$) Vertical shrink (if $0 < a < 1$) • Horizontal stretch/shrink $y = f(ax)$ Horizontal shrink (if $a > 1$) Horizontal stretch (if $0 < a < 1$)	p. 187
• Reflection $y = -f(x)$ Reflection across the x-axis $y = f(-x)$ Reflection across the y-axis	p. 189
To graph a function requiring multiple transformations, use the following order. 1. Horizontal translation (shift) 2. Horizontal and vertical stretch and shrink 3. Reflections across the x- and y-axes 4. Vertical translation (shift)	p. 190
SECTION 1.7 Analyzing Graphs of Functions and Piecewise-Defined Functions	**Reference**
Consider the graph of an equation in x and y. • The graph of the equation is symmetric to the y-axis if substituting $-x$ for x results in an equivalent equation. • The graph of the equation is symmetric to the x-axis if substituting $-y$ for y results in an equivalent equation. • The graph of the equation is symmetric to the origin if substituting $-x$ for x and $-y$ for y results in an equivalent equation.	p. 197
• f is an even function if $f(-x) = f(x)$ for all x in the domain of f. • f is an odd function if $f(-x) = -f(x)$ for all x in the domain of f.	p. 199
To graph a piecewise-defined function, graph each individual function on its domain.	p. 201

The **greatest integer function,** denoted by $f(x) = [\![x]\!]$ or $f(x) = \text{int}(x)$ or $f(x) = \text{floor}(x)$ defines $f(x)$ as the greatest integer less than or equal to x.	p. 204
Suppose that I is an interval contained within the domain of a function f. • f is increasing on I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ on I. • f is decreasing on I if $f(x_1) > f(x_2)$ for all $x_1 < x_2$ on I. • f is constant on I if $f(x_1) = f(x_2)$ for all x_1 and x_2 on I.	p. 206
• $f(a)$ is a **relative maximum** of f if there exists an open interval containing a such that $f(a) \geq f(x)$ for all x in the interval. • $f(b)$ is a **relative minimum** of f if there exists an open interval containing b such that $f(b) \leq f(x)$ for all x in the interval.	p. 207
SECTION 1.8 Algebra of Functions and Function Composition	**Reference**
Given functions f and g, the functions $f + g$, $f - g$, $f \cdot g$, and $\frac{f}{g}$ are defined by $(f + g)(x) = f(x) + g(x)$ $(f - g)(x) = f(x) - g(x)$ $(f \cdot g)(x) = f(x) \cdot g(x)$ $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ provided that $g(x) \neq 0$	p. 216
The **difference quotient** represents the average rate of change of a function f between two points $(x, f(x))$ and $(x + h, f(x + h))$. $\frac{f(x + h) - f(x)}{h}$ Difference quotient	p. 218
The **composition of f and g,** denoted $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is the set of real numbers x in the domain of g such that $g(x)$ is in the domain of f.	p. 220

Expanded Chapter Summary available at www.mhhe.com/millerprecalculus.

CHAPTER 1 Review Exercises

SECTION 1.1

For Exercises 1–2,

a. Find the exact distance between the points.

b. Find the midpoint of the line segment whose endpoints are the given points.

1. $(-1, 8)$ and $(4, -2)$

2. $(\sqrt{3}, -\sqrt{6})$ and $(3\sqrt{3}, 4\sqrt{6})$

3. Determine if the given ordered pair is a solution to the equation $4|x - 1| + y = 18$.

a. $(-3, 2)$ **b.** $(5, -2)$

For Exercises 4–6, determine the *x*- and *y*-intercepts of the graph of the equation.

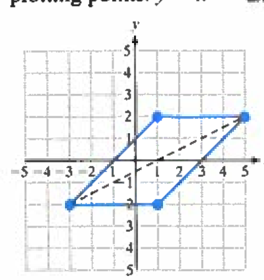
4. $-3y + 4x = 6$

5. $x = |y + 7| - 3$

6. $\dfrac{(x + 4)^2}{9} + \dfrac{y^2}{4} = 1$

7. Graph the equation by plotting points. $y = x^2 - 2x$

8. Find the length of the diagonal shown.



SECTION 1.2

For Exercises 9–10, determine the center and radius of the circle.

9. $(x - 4)^2 + (y + 3)^2 = 4$

10. $x^2 + \left(y - \dfrac{3}{2}\right)^2 = 17$

For Exercises 11–14, information about a circle is given.

a. Write an equation of the circle in standard form.

b. Graph the circle.

11. Center: $(-3, 1)$; Radius: $\sqrt{11}$

12. Center: $(0, 0)$; Radius: 3.2

13. Endpoints of a diameter $(7, 5)$ and $(1, -3)$

14. The center is in quadrant IV, the radius is 4, and the circle is tangent to both the x- and y-axes.

For Exercises 15–16, (a) write the equation of the circle in standard form and (b) identify the center and radius.

15. $x^2 + y^2 + 10x - 2y + 17 = 0$

16. $x^2 + y^2 - 8y + 3 = 0$

For Exercises 17–18, determine the solution set to the equation.

17. $(x + 3)^2 + (y - 5)^2 = 0$

18. $x^2 + y^2 + 6x - 4y + 15 = 0$

SECTION 1.3

19. The table lists four Olympic athletes and the number of Olympic medals won by the athlete.

Athlete (x)	Number of Medals (y)
Dara Torres (swimming)	12
Carl Lewis (track and field)	10
Bonnie Blair (speed skating)	6
Michael Phelps (swimming)	16

a. Write a set of ordered pairs (x, y) that defines the relation.

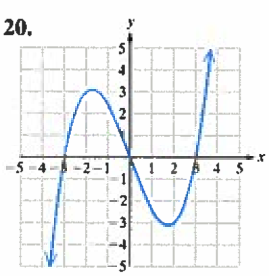
b. Write the domain of the relation.

c. Write the range of the relation.

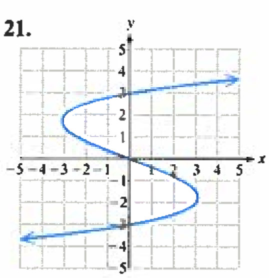
d. Determine if the relation defines y as a function of x.

For Exercises 20–23, determine if the relation defines y as a function of x.

20.



21.



22. $x^2 + (y - 3)^2 = 4$ **23.** $x^2 + y - 3 = 4$

24. Evaluate $f(x) = -2x^2 + 4x$ for the values of x given.

a. $f(0)$ **b.** $f(-1)$ **c.** $f(3)$

d. $f(t)$ **e.** $f(x + 4)$

25. Given $f = \{(3, -1), (1, 5), (-2, 4), (0, 4)\}$,

a. Determine $f(1)$.

b. Determine $f(0)$.

c. For what value(s) of x is $f(x) = -1$?

26. A department store marks up the price of a power drill by 32% of the price from the manufacturer. The price $P(x)$ (in \$) to a customer after a 6.5% sales tax is given by $P(x) = 1.065(x + 0.32x)$, where x is the cost of the drill from the manufacturer. Evaluate $P(189)$ and interpret the meaning in the context of this problem.

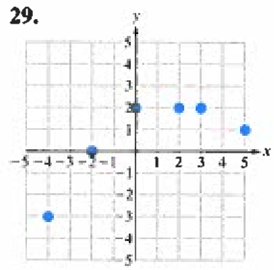
For Exercises 27–28, determine the x- and y-intercepts for the given function.

27. $p(x) = |x - 3| - 1$

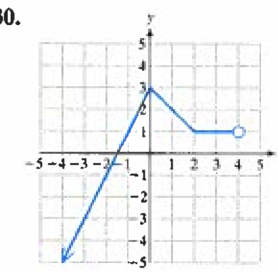
28. $q(x) = -\sqrt{x} + 2$

For Exercises 29–30, determine the domain and range of the function.

29.



30.

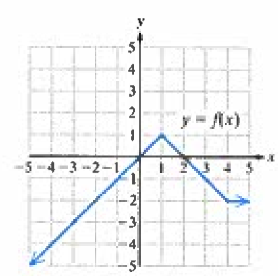


For Exercises 31–34, write the domain in interval notation.

31. $f(x) = \dfrac{x - 2}{x - 5}$ **32.** $g(x) = \dfrac{6}{|x| - 3}$

33. $m(x) = 2x^2 - 4x + 1$ **34.** $n(x) = \dfrac{10}{\sqrt{2 - x}}$

35. Use the graph of $y = f(x)$ to



a. Determine $f(-2)$.

b. Determine $f(3)$.

c. Find all x for which $f(x) = -1$.

d. Find all x for which $f(x) = -4$.

e. Determine the x-intercept(s).

f. Determine the y-intercept.

g. Determine the domain of f.

h. Determine the range of f.

36. Write a relationship for a function whose $f(x)$ value is 4 less than two times the square of x.

SECTION 1.4

For Exercises 37–40, graph the equation and determine the x- and y-intercepts.

37. $-2x + 4y = 8$
38. $-4x = 5y$
39. $y = 2$
40. $3x = 5$

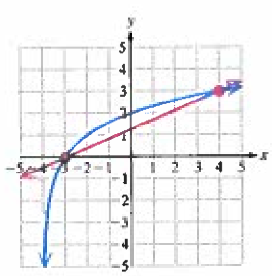
For Exercises 41–43, determine the slope of the line passing through the given points.

41. $(4, -2)$ and $(-12, -4)$
42. $\left(-3, \frac{2}{3}\right)$ and $\left(1, -\frac{4}{3}\right)$
43. $(a, f(a))$ and $(b, f(b))$
44. What is the slope of a line parallel to the x-axis?
45. What is the slope of a line with equation $x = -2$?
46. What is the slope of a line perpendicular to a line with equation $y = 1$?
47. Suppose that $y = C(t)$ represents the average cost of a gallon of milk in the United States t years since 1980. What does $\frac{\Delta C}{\Delta t}$ represent?
48. Determine if the function is linear, constant, or neither.

 a. $f(x) = -\frac{3}{2}x$ b. $g(x) = -\frac{3}{2x}$ c. $h(x) = -\frac{3}{2}$

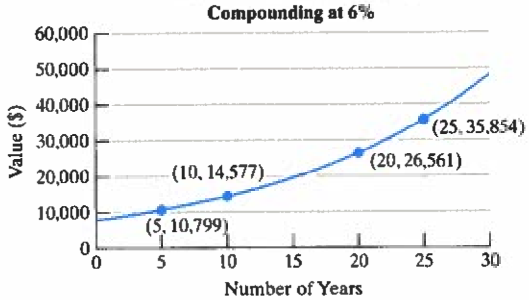
For Exercises 49–50, use slope-intercept form to write an equation of the line that passes through the given point and has the given slope. Then write the equation using function notation where $y = f(x)$.

49. $(1, -5)$ and $m = -\frac{2}{3}$
50. $\left(2, \frac{1}{4}\right)$ and $m = 0$
51. Find the slope of the secant line pictured in red.



52. The function given by $y = f(x)$ shows the value of \$8000 invested at 6% interest compounded continuously, x years after the money was originally invested.

Value of \$8000 with Continuous Compounding at 6%



 a. Find the average amount earned per year between the 5th year and the 10th year.

 b. Find the average amount earned per year between the 20th year and the 25th year.

 c. Based on the answers from parts (a) and (b), does it appear that the rate at which annual income increases is increasing or decreasing with time?

53. Given $f(x) = -x^3 + 4$, determine the average rate of change of the function on the given intervals.

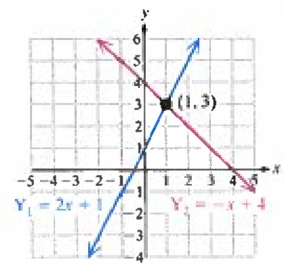
 a. $[0, 2]$ b. $[2, 4]$

54. Use the graph to solve the equation and inequalities. Write the solutions to the inequalities in interval notation.

 a. $2x + 1 = -x + 4$

 b. $2x + 1 < -x + 4$

 c. $2x + 1 \geq -x + 4$



SECTION 1.5

55. If the slope of a line is $\frac{7}{5}$,

 a. Determine the slope of a line parallel to the given line.

 b. Determine the slope of a line perpendicular to the given line.

56. Given a line L_1 defined by L_1: $2x - 4y = 3$, determine if the equations given in parts (a)–(c) represent a line parallel to L_1, perpendicular to L_1, or neither parallel nor perpendicular to L_1.

 a. $12x + 6y = 6$ b. $3y = 1.5x - 5$

 c. $4x + 8y = 8$

For Exercises 57–63, write an equation of the line having the given conditions. Write the answer in slope-intercept form if possible.

57. Passes through $(-2, -7)$ and $m = 3$.
58. Passes through $(0, 5)$ and $m = -\frac{2}{5}$.
59. Passes through $(1.1, 5.3)$ and $(-0.9, 7.1)$.
60. Passes through $(5, -7)$ and the slope is undefined.
61. Passes through $(2, -6)$ and is parallel to the line defined by $2x - y = 4$.
62. Passes through $(-2, 3)$ and is perpendicular to the line defined by $5y = 2x$.
63. The line is perpendicular to the y-axis and the y-intercept is $(0, 7)$.
64. A car has a 15-gal tank for gasoline and gets 30 mpg on a highway while driving 60 mph. Suppose that the driver starts a trip with a full tank of gas and travels 450 mi on the highway at an average speed of 60 mph.

 a. Write a linear model representing the amount of gas $G(t)$ left in the tank t hours into a trip.

b. Evaluate $G(4.5)$ and interpret the meaning in the context of this problem.

65. A dance studio has fixed monthly costs of \$1500 that include rent, utilities, insurance, and advertising. The studio charges \$60 for each private lesson, but has a variable cost for each lesson of \$35 to pay the instructor.

a. Write a linear cost function representing the cost to the studio $C(x)$ to hold x private lessons for a given month.

b. Write a linear revenue function representing the revenue $R(x)$ for holding x private lessons for the month.

c. Write a linear profit function representing the profit $P(x)$ for holding x private lessons for the month.

d. Determine the number of private lessons that must be held for the studio to make a profit.

e. If 82 private lessons are held during a given month, how much money will the studio make or lose?

66. The height y (in meters) of a volcano in the southeast Pacific Ocean is recorded in the table for selected years since 1960.

Number of Years Since 1960, x	Height (m) y
0	166
10	290
20	408
30	526
40	650
50	760
54	813

a. Graph the data in a scatter plot.

b. Use the points (0, 166) and (40, 650) to write a linear function that defines the height y of the volcano, x years since 1960.

c. Interpret the meaning of the slope in the context of this problem.

d. Use the model in part (b) to predict the height of the volcano in the year 2030 assuming that the linear trend continues.

67. Refer to the data given in Exercise 66.

a. Use a graphing utility to find the least-squares regression line. Round the slope to 1 decimal place and the y-intercept to the nearest whole unit.

b. Use a graphing utility to graph the regression line and the observed data.

c. In the event that the linear trend continues, use the model from part (a) to predict the height of the volcano in the year 2030.

SECTION 1.6

68. Write a function based on the given parent function and transformations in the given order.

Parent function: $y = x^2$

1. Shift 5 units to the left.
2. Reflect across the y-axis.
3. Shift downward 2 units.

For Exercises 69–78, use translations to graph the given functions.

69. $f(x) = |x| - 2$

70. $g(x) = \sqrt{x} + 1$

71. $h(x) = (x - 2)^2$

72. $k(x) = \sqrt[3]{x + 1}$

73. $r(x) = \sqrt{x - 3} + 1$

74. $s(x) = (x + 2)^2 - 3$

75. $t(x) = -2|x|$

76. $v(x) = -\dfrac{1}{2}|x|$

77. $m(x) = \sqrt{-x + 5}$

78. $n(x) = \sqrt{-x - 1}$

For Exercises 79–84, use the graph of $y = f(x)$ to graph the given function.

79. $y = f(2x)$

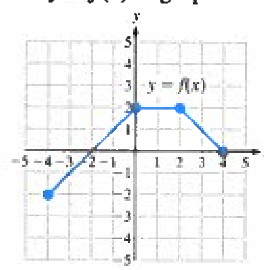
80. $y = f(\frac{1}{2}x)$

81. $y = -f(x + 1) - 3$

82. $y = -f(x - 4) - 1$

83. $y = 2f(x - 3) + 1$

84. $y = \frac{1}{2}f(x + 2) - 3$



SECTION 1.7

For Exercises 85–88, determine if the graph of the equation is symmetric to the y-axis, x-axis, origin, or none of these.

85. $y = x^4 - 3$

86. $x = |y| + y^2$

87. $y = \dfrac{1}{3}x - 1$

88. $x^2 = y^2 + 1$

For Exercises 89–94, determine if the function is even, odd, or neither.

89. $f(x) = -4x^3 + x$

90. $g(x) = \sqrt[3]{x}$

91. $p(x) = \sqrt{4 - x^2}$

92. $q(x) = -|x|$

93. $k(x) = (x - 3)^2$

94. $m(x) = |x + 2|$

95. Evaluate the function for the given values of x.

$$f(x) = \begin{cases} -4x + 2 & \text{for } x < -1 \\ x^2 & \text{for } -1 \le x \le 2 \\ 5 & \text{for } x > 2 \end{cases}$$

a. $f(-4)$ **b.** $f(-1)$ **c.** $f(3)$ **d.** $f(2)$

For Exercises 96–98, graph the function.

96. $f(x) = \begin{cases} -4x - 3 & \text{for } x < 0 \\ x^2 & \text{for } x \ge 0 \end{cases}$

97. $g(x) = \begin{cases} |x| & \text{for } x \le 2 \\ 2 & \text{for } x > 2 \end{cases}$

98. $h(x) = \begin{cases} -3 & \text{for } x < -2 \\ 1 & \text{for } -2 \le x < 0 \\ \sqrt{x} & \text{for } x \ge 0 \end{cases}$

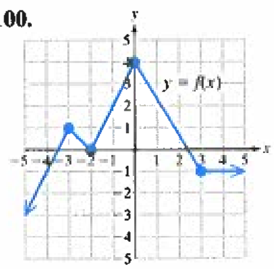
99. Evaluate $f(x) = [\![x - 1]\!]$ for the given values of x.

a. $f(-1.5)$ **b.** $f(-2)$ **c.** $f(0.1)$ **d.** $f(6.3)$

For Exercises 100–101, use interval notation to write the interval(s) over which f is

a. Increasing. **b.** Decreasing. **c.** Constant.

100.



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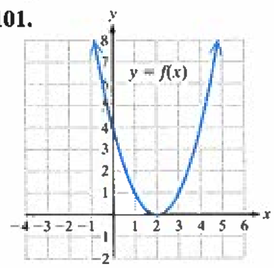
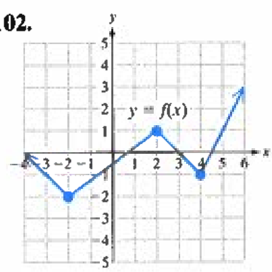
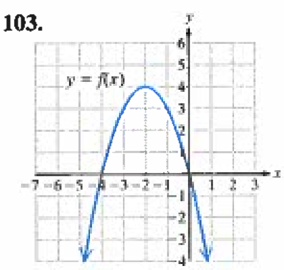


For Exercises 102–103, identify the location and value of any relative maxima or minima of the function.

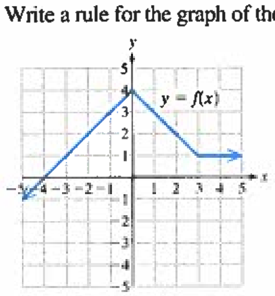
102.



103.



104. Write a rule for the graph of the function. Answers may vary.



SECTION 1.8

For Exercises 105–109, evaluate the function for the given values of x.

$f(x) = -3x$ $\quad g(x) = |x - 2|$ $\quad h(x) = \dfrac{1}{x + 1}$

105. $(f - h)(2)$ **106.** $(g \cdot h)(3)$ **107.** $\left(\dfrac{g}{h}\right)(-5)$

108. $(f \circ g)(5)$ **109.** $(g \circ f)(5)$

110. Use the graphs of f and g to find the function values for the given values of x.

a. $(f + g)(2)$

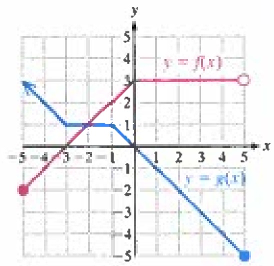
b. $(g \cdot f)(-4)$

c. $\left(\dfrac{g}{f}\right)(-3)$

d. $f[g(-4)]$

e. $(g \circ f)(-4)$

f. $(g \circ f)(5)$



For Exercises 111–116, refer to the functions m, n, p, and q. Find the function and write the domain in interval notation.

$m(x) = -4x$ $\quad n(x) = x^2 - 4x$

$p(x) = \sqrt{x - 2}$ $\quad q(x) = \dfrac{1}{x - 5}$

111. $(n - m)(x)$ **112.** $\left(\dfrac{p}{n}\right)(x)$ **113.** $\left(\dfrac{n}{p}\right)(x)$

114. $(m \cdot p)(x)$ **115.** $(q \circ n)(x)$ **116.** $(q \circ p)(x)$

For Exercises 117–118, find the difference quotient, $\dfrac{f(x + h) - f(x)}{h}$.

117. $f(x) = -6x - 5$ **118.** $f(x) = 3x^2 - 4x + 9$

For Exercises 119–120, find two functions, f and g such that $h(x) = (f \circ g)(x)$.

119. $h(x) = (x - 4)^2$ **120.** $h(x) = \dfrac{12}{x + 5}$

121. A car traveling 60 mph on the highway gets 28 mpg.

a. Write a function that represents the distance $d(t)$ (in miles) that the car travels in t hours.

b. Write a function that represents the number of gallons of gasoline $n(d)$ used for d miles traveled.

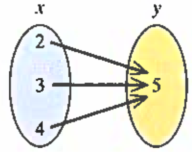
c. Find $(n \circ d)(t)$ and interpret the meaning in the context of this problem.

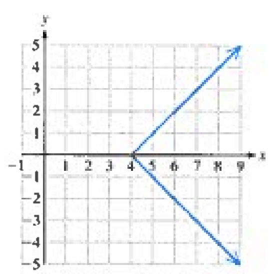
d. Evaluate $(n \circ d)(7)$ and interpret the meaning in the context of this problem.

CHAPTER 1 Test

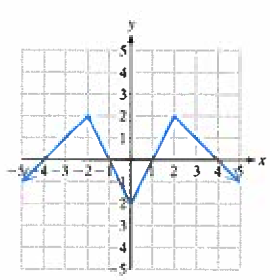
1. The endpoints of a diameter of a circle are $(-2, 3)$ and $(8, -5)$.
   a. Determine the center of the circle.
   b. Determine the radius of the circle.
   c. Write an equation of the circle in standard form.
2. Given $x = |y| - 4$,
   a. Determine the x- and y-intercepts of the graph of the equation.
   b. Does the equation define y as a function of x?
3. Given $x^2 + y^2 + 14x - 10y + 70 = 0$,
   a. Write the equation of the circle in standard form.
   b. Identify the center and radius.

For Exercises 4–5, determine if the relation defines y as a function of x.

4. 

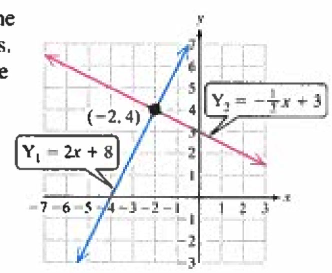
5. 

6. Given $f(x) = -2x^2 + 7x - 3$, find
   a. $f(-1)$.
   b. $f(x + h)$.
   c. The difference quotient: $\dfrac{f(x+h) - f(x)}{h}$.
   d. The x-intercepts of the graph of f.
   e. The y-intercept of the graph of f.
   f. The average rate of change of f on the interval $[1, 3]$.
7. Use the graph of $y = f(x)$ to estimate
   a. $f(0)$.
   b. $f(-4)$.
   c. The values of x for which $f(x) = 2$.
   d. The interval(s) over which f is increasing.
   e. The interval(s) over which f is decreasing.
   f. Determine the location and value of any relative minima.
   g. Determine the location and value of any relative maxima.
   h. The domain.
   i. The range.
   j. Whether f is even, odd, or neither.



For Exercises 8–9, write the domain in interval notation.

8. $f(w) = \dfrac{2w}{3w + 7}$
9. $f(c) = \sqrt{4 - c}$
10. Given $3x = -4y + 8$,
    a. Identify the slope.
    b. Identify the y-intercept.
    c. Graph the line.
    d. What is the slope of a line perpendicular to this line?
    e. What is the slope of a line parallel to this line?
11. Write an equation of the line passing through the point $(-2, 6)$ and perpendicular to the line defined by $x + 3y = 4$.
12. Use the graph to solve the equation and inequalities. Write the solutions to the inequalities in interval notation.
    a. $2x + 8 = -\frac{1}{2}x + 3$
    b. $2x + 8 < -\frac{1}{2}x + 3$
    c. $2x + 8 \geq -\frac{1}{2}x + 3$



For Exercises 13–16, graph the equation.

13. $x^2 + \left(y + \frac{5}{2}\right)^2 = 9$
14. $f(x) = 2|x + 3|$
15. $g(x) = -\sqrt{x + 4} + 3$
16. $h(x) = \begin{cases} -x + 3 & \text{for } x < 1 \\ \sqrt{x - 1} & \text{for } x \geq 1 \end{cases}$
17. Determine if the graph of the equation is symmetric to the y-axis, x-axis, origin, or none of these.
    $$x^2 + |y| = 8$$

For Exercises 18–19, determine if the function is even, odd, or neither.

18. $f(x) = x^3 - x$
19. $g(x) = x^4 + x^3 + x$
20. Evaluate the greatest integer function for the following values of x.
    a. 4.27
    b. -4.27

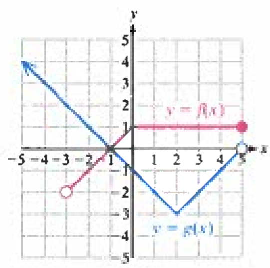
For Exercises 21–26, refer to the functions f, g, and h defined here.

$$f(x) = x - 4 \qquad g(x) = \frac{1}{x-3} \qquad h(x) = \sqrt{x-5}$$

21. Evaluate $(f - h)(6)$.
22. Evaluate $(g \cdot h)(5)$.
23. Evaluate $(h \circ f)(1)$.
24. Find $(f \cdot g)(x)$ and state the domain in interval notation.
25. Find $\left(\dfrac{g}{f}\right)(x)$ and state the domain in interval notation.
26. Find $(g \circ h)(x)$ and state the domain in interval notation.
27. Write two functions f and g such that $h(x) = (f \circ g)(x)$.

$$h(x) = \sqrt[3]{x-7}$$

28. For f and g pictured, estimate the following.



   a. $(f + g)(3)$
   b. $(f \cdot g)(0)$
   c. $g(f(3))$
   d. $(f \circ g)(2)$
   e. The interval(s) over which f is increasing.
   f. The interval(s) over which g is decreasing.

29. The number of people y that attend a weekly bingo game at an adult recreation center is given in the table for selected weeks, x.

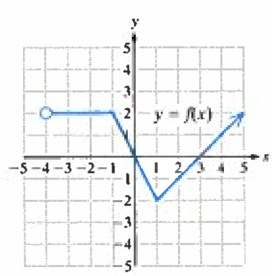
Week Number, x	Number of attendees, y
1	8
3	21
6	30
9	40
12	46
15	56
18	68

   a. Graph the data in a scatter plot.
   b. Use the points (1, 8) and (9, 40) to write a linear function that defines the number of attendees as a function of week number.
   c. Interpret the meaning of the slope in the context of this problem.
   d. Use the model in part (b) to predict the number of attendees in week 24 assuming that the linear trend continues.

30. Refer to the data given in Exercise 29.
   a. Use a graphing utility to find the least-squares regression line. Round the slope and y-intercept to 1 decimal place.
   b. Use a graphing utility to graph the regression line and the observed data.
   c. In the event that the linear trend continues, use the model from part (a) to predict the number of attendees in week 24.

CHAPTER 1 Cumulative Review Exercises

1. Use the graph of $y = f(x)$ to



   a. Evaluate $f(2)$.

   b. Find all x such that $f(x) = 0$.

   c. Determine the domain of f.

   d. Determine the range of f.

   e. Determine the interval(s) over which f is increasing.

   f. Determine the interval(s) over which f is decreasing.

   g. Determine the intervals(s) over which f is constant.

   h. Evaluate $(f \circ f)(-1)$.

2. Given the equation of the circle $x^2 + y^2 + 12x - 4y + 31 = 0$,

   a. Write the equation in standard form.

   b. Identify the center and radius.

For Exercises 3–7, refer to the functions f, g, and h defined here.

$$f(x) = -x^2 + 3x \qquad g(x) = \frac{1}{x} \qquad h(x) = \sqrt{x + 2}$$

3. Find $(g \circ f)(x)$ and write the domain in interval notation.

4. Find $(g \cdot h)(x)$ and write the domain in interval notation.

5. Find the difference quotient. $\dfrac{f(x + h) - f(x)}{h}$

6. Find the average rate of change of f over the interval $[0, 3]$.

7. Determine the x- and y-intercepts of f.

For Exercises 8–9, graph the function.

8. $f(x) = -\sqrt{x + 3}$

9. $g(x) = \begin{cases} -4 & \text{for } x < -2 \\ 1 & \text{for } -2 \leq x < 0 \\ x^2 + 1 & \text{for } x \geq 0 \end{cases}$

10. Write an equation of the line passing through the points $(8, -3)$ and $(-2, 1)$. Write the final answer in slope-intercept form.

11. Write an absolute value expression that represents the distance between the points x and 7 on the number line.

12. Factor. $2x^3 - 128$

For Exercises 13–17, solve the equation or inequality. Write the solutions to the inequalities in interval notation.

13. $-3t(t - 1) = 2t + 6$

14. $7 = |4x - 2| + 5$

15. $x^{2/5} - 3x^{1/5} + 2 = 0$

16. $|3a + 1| - 2 \leq 9$

17. $3 \leq -2x + 1 < 7$

For Exercises 18–20, perform the indicated operations and simplify.

18. $\dfrac{6}{\sqrt{15} + \sqrt{11}}$

19. $3c\sqrt{8c^2d^3} + c^2\sqrt{50d^3} - 2d\sqrt{2c^4d}$

20. $\dfrac{2u^{-1} - w^{-1}}{4u^{-2} - w^{-2}}$

Polynomial and Rational Functions

2 Polynomial and Rational Functions

Chapter Outline

Meteorology and the study of weather have a strong basis in mathematics. The factors impacting weather are not constant and change over time. For example, during the summer months, hot ocean temperatures in the Atlantic Ocean often produce breeding grounds for hurricanes off the coast of Africa or in the Caribbean. To predict the path of a hurricane, meteorologists collect data from satellites, weather stations around the world, and weather buoys in the ocean. Piecing together the data requires a variety of techniques of mathematical modeling using powerful computers. In the end, scientists combine a series of simple curves to approximate weather patterns that closely fit complicated models.

In this chapter, we study polynomial and rational functions. Both types of functions represent simple curves that can be used for modeling in a wide range of applications, including predictions for the path of a hurricane.

SECTION 2.1 Quadratic Functions and Applications

OBJECTIVES

1. Graph a Quadratic Function Written in Vertex Form
2. Write $f(x) = ax^2 + bx + c$ $(a \neq 0)$ in Vertex Form
3. Find the Vertex of a Parabola by Using the Vertex Formula
4. Solve Applications Involving Quadratic Functions
5. Create Quadratic Models Using Regression

1. Graph a Quadratic Function Written in Vertex Form

In Chapter 1, we defined a function of the form $f(x) = mx + b$ $(m \neq 0)$ as a linear function. The function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is called a *quadratic function.* Notice that a quadratic function has a leading term of second degree. We are already familiar with the graph of $f(x) = x^2$ (Figure 2-1). The graph is a parabola opening upward with vertex at the origin. Also note that the graph is symmetric with respect to the vertical line through the vertex called the **axis of symmetry.**

We can write $f(x) = ax^2 + bx + c$ $(a \neq 0)$ in the form $f(x) = a(x - h)^2 + k$ by completing the square. Furthermore, from Section 1.6 we know that the graph of $f(x) = a(x - h)^2 + k$ is related to the graph of $y = x^2$ by a vertical shrink or stretch determined by a, a horizontal shift determined by h, and a vertical shift determined by k. Therefore, the graph of a quadratic function is a parabola with vertex at (h, k).

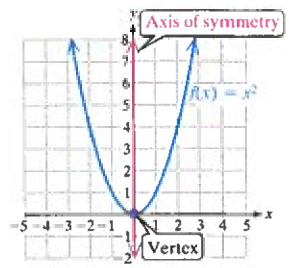
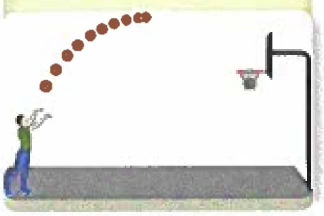


Figure 2-1

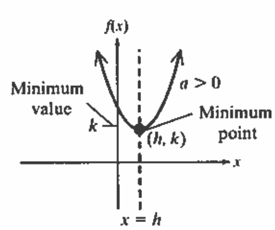
TIP A quadratic function is often used as a model for projectile motion. This is motion followed by an object influenced by an initial force and by the force of gravity.

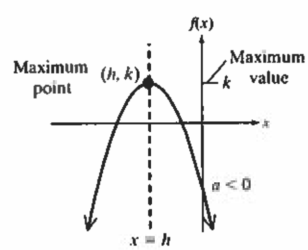


Quadratic Function

A function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is called a **quadratic function.** By completing the square, $f(x)$ can be expressed in **vertex form** as $f(x) = a(x - h)^2 + k$.

- The graph of f is a parabola with vertex (h, k).
- If $a > 0$, the parabola opens upward, and the vertex is the minimum point. The minimum *value* of f is k.
- If $a < 0$, the parabola opens downward, and the vertex is the maximum point. The maximum *value* of f is k.
- The axis of symmetry is $x = h$. This is the vertical line that passes through the vertex.





In Example 1, we analyze and graph a quadratic function by identifying the vertex, axis of symmetry, and x- and y-intercepts. From the graph, the minimum or maximum value of the function is readily apparent.

EXAMPLE 1 **Analyzing and Graphing a Quadratic Function**

Given $f(x) = -2(x - 1)^2 + 8$,

a. Determine whether the graph of the parabola opens upward or downward.
b. Identify the vertex.
c. Determine the x-intercept(s).
d. Determine the y-intercept.
e. Sketch the function.
f. Determine the axis of symmetry.
g. Determine the maximum or minimum value of f.
h. Write the domain and range in interval notation.

Solution:

a. $f(x) = -2(x - 1)^2 + 8$
The parabola opens downward.

The function is written as $f(x) = a(x - h)^2 + k$, where $a = -2$, $h = 1$, and $k = 8$. Since $a < 0$, the parabola opens downward.

b. The vertex is (1, 8).

The vertex is (h, k), which is (1, 8).

c.
$$\begin{aligned} f(x) &= -2(x - 1)^2 + 8 \\ 0 &= -2(x - 1)^2 + 8 \\ -8 &= -2(x - 1)^2 \\ 4 &= (x - 1)^2 \\ \pm\sqrt{4} &= x - 1 \\ 1 \pm 2 &= x \\ x = 3 \quad &\text{or} \quad x = -1 \end{aligned}$$

To find the x-intercept(s), find all real solutions to the equation $f(x) = 0$.

The x-intercepts are (3, 0) and (−1, 0).

d. $f(0) = -2(0 - 1)^2 + 8$
$= 6$
The y-intercept is (0, 6).

To find the y-intercept, evaluate $f(0)$.

e. The graph of f is shown in Figure 2-2.

f. The axis of symmetry is the vertical line through the vertex: $x = 1$.

g. The maximum value is 8.

h. The domain is $(-\infty, \infty)$.
The range is $(-\infty, 8]$.

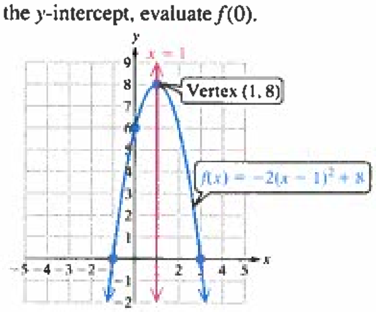


Figure 2-2

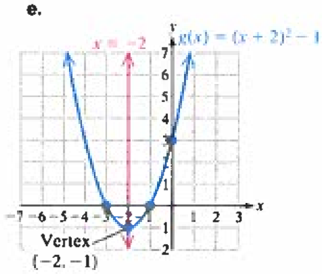
Skill Practice 1 Repeat Example 1 with $g(x) = (x + 2)^2 - 1$.

2. Write $f(x) = ax^2 + bx + c$ ($a \neq 0$) in Vertex Form

In Section 1.2, we learned how to complete the square to write an equation of a circle $x^2 + y^2 + Ax + By + C = 0$ in standard form $(x - h)^2 + (y - k)^2 = r^2$. We use the same process to write a quadratic function $f(x) = ax^2 + bx + c$ ($a \neq 0$) in vertex form $f(x) = a(x - h)^2 + k$. However, we will work on the right side of the equation only. This is demonstrated in Example 2.

Answers

1. a. Upward **b.** (−2, −1)
c. (−3, 0) and (−1, 0) **d.** (0, 3)
e.



f. $x = -2$
g. The minimum value is −1.
h. The domain is $(-\infty, \infty)$.
The range is $[-1, \infty)$.

EXAMPLE 2 Writing a Quadratic Function in Vertex Form

Given $f(x) = 3x^2 + 12x + 5$,

a. Write the function in vertex form: $f(x) = a(x - h)^2 + k$.
b. Identify the vertex.
c. Identify the x-intercept(s).
d. Identify the y-intercept.
e. Sketch the function.
f. Determine the axis of symmetry.
g. Determine the minimum or maximum value of f.
h. Write the domain and range in interval notation.

Solution:

a. $f(x) = 3x^2 + 12x + 5$

$= 3(x^2 + 4x \qquad) + 5$

Factor out the leading coefficient of the x^2 term from the two terms containing x. The leading term within parentheses now has a coefficient of 1.

$= 3(x^2 + 4x + 4 - 4) + 5$

Complete the square within parentheses. Add and subtract $[\frac{1}{2}(4)]^2 = 4$ within parentheses.

$= 3(x^2 + 4x + 4) + 3(-4) + 5$

Remove -4 from within parentheses, along with a factor of 3.

$= 3(x + 2)^2 - 7$ (vertex form)

b. The vertex is $(-2, -7)$.

c. $f(x) = 3x^2 + 12x + 5$

To find the x-intercept(s), find the real solutions to the equation $f(x) = 0$.

$0 = 3x^2 + 12x + 5$

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4(3)(5)}}{2(3)}$$

The right side is not factorable. Apply the quadratic formula.

$$= \frac{-12 \pm \sqrt{84}}{6}$$

$$= \frac{-12 \pm 2\sqrt{21}}{6}$$

$$= \frac{-6 \pm \sqrt{21}}{3} \quad \begin{matrix} \nearrow x \approx -0.47 \\ \searrow x \approx -3.53 \end{matrix}$$

The x-intercepts are $\left(\frac{-6 + \sqrt{21}}{3}, 0\right)$ and $\left(\frac{-6 - \sqrt{21}}{3}, 0\right)$ or approximately $(-0.47, 0)$ and $(-3.53, 0)$.

d. $f(0) = 3(0)^2 + 12(0) + 5$
$= 5$

To find the y-intercept, evaluate $f(0)$. The y-intercept is $(0, 5)$.

e. The graph of f is shown in Figure 2-3.

f. The axis of symmetry is $x = -2$.

g. The minimum value is -7.

h. The domain is $(-\infty, \infty)$.
The range is $[-7, \infty)$.

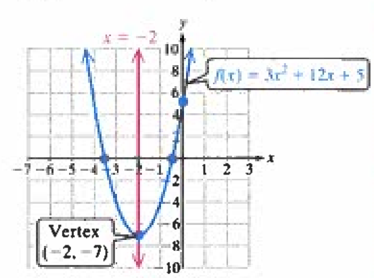


Figure 2-3

Skill Practice 2 Repeat Example 2 with $f(x) = 3x^2 - 6x + 1$.

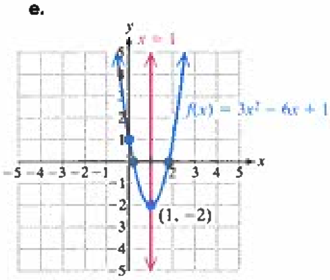
Answers

2. a. $f(x) = 3(x - 1)^2 - 2$ **b.** $(1, -2)$

c. $\left(\frac{3 \pm \sqrt{6}}{3}, 0\right)$

d. $(0, 1)$

e.



f. $x = 1$
g. The minimum value is -2.
h. The domain is $(-\infty, \infty)$.
The range is $[-2, \infty)$.

3. Find the Vertex of a Parabola by Using the Vertex Formula

Completing the square and writing a quadratic function in the form $f(x) = a(x - h)^2 + k$ is one method to find the vertex of a parabola. Another method is to use the vertex formula. The vertex formula can be derived by completing the square on $f(x) = ax^2 + bx + c$.

$f(x) = ax^2 + bx + c \ (a \neq 0)$

$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c$ — Factor out a from the x terms, and complete the square within parentheses. $\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2 = \frac{b^2}{4a^2}$

$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + a\left(-\frac{b^2}{4a^2}\right) + c$ — Remove the term $-\frac{b^2}{4a^2}$ from within parentheses along with a factor of a.

$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$ — Factor the trinomial.

$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$ — Obtain a common denominator and add the terms outside parentheses.

$= a\left[x - \left(\frac{-b}{2a}\right)\right]^2 + \frac{4ac - b^2}{4a}$ — $f(x)$ is now written in vertex form.

$f(x) = a(x - h)^2 + k$ — $h = \frac{-b}{2a}$ and $k = \frac{4ac - b^2}{4a}$

The vertex is $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$.

The y-coordinate of the vertex is given by $\frac{4ac - b^2}{4a}$ and is often hard to remember. Therefore, it is usually easier to evaluate the x-coordinate first from $\frac{-b}{2a}$, and then evaluate $f\left(\frac{-b}{2a}\right)$.

Vertex Formula to Find the Vertex of a Parabola

For $f(x) = ax^2 + bx + c \ (a \neq 0)$, the vertex is given by $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

EXAMPLE 3 Using the Vertex Formula

Given $f(x) = -x^2 + 4x - 5$,

a. State whether the graph of the parabola opens upward or downward.
b. Determine the vertex of the parabola by using the vertex formula.
c. Determine the x-intercept(s).
d. Determine the y-intercept.
e. Sketch the graph.
f. Determine the axis of symmetry.
g. Determine the minimum or maximum value of f.
h. Write the domain and range in interval notation.

Solution:

a. $f(x) = -x^2 + 4x - 5$
The parabola opens downward.

The function is written as $f(x) = ax^2 + bx + c$ where $a = -1$. Since $a < 0$, the parabola opens downward.

b. x-coordinate: $\dfrac{-b}{2a} = \dfrac{-(4)}{2(-1)} = 2$

y-coordinate: $f(2) = -(2)^2 + 4(2) - 5$
$= -1$

The vertex is $(2, -1)$.

c. Since the vertex of the parabola is below the x-axis and the parabola opens downward, the parabola cannot cross or touch the x-axis.

Therefore, there are no x-intercepts.

Solving the equation $f(x) = 0$ to find the x-intercepts results in imaginary solutions:

$0 = -x^2 + 4x - 5$

$x = \dfrac{-(4) \pm \sqrt{(4)^2 - 4(-1)(-5)}}{2(-1)}$

$x = 2 \pm i$

TIP For more accuracy in the graph, plot one or two points near the vertex. Then use the symmetry of the curve to find additional points on the graph.

For example, the points $(1, -2)$ and $(0, -5)$ are on the left branch of the parabola. The corresponding points to the right of the axis of symmetry are $(3, -2)$ and $(4, -5)$.

d. To find the y-intercept, evaluate $f(0)$.
$f(0) = -(0)^2 + 4(0) - 5 = -5$
The y-intercept is $(0, -5)$.

e. The graph of f is shown in Figure 2-4.

f. The axis of symmetry is $x = 2$.

g. The maximum value of f is -1.

h. The domain is $(-\infty, \infty)$.
The range is $(-\infty, -1]$.

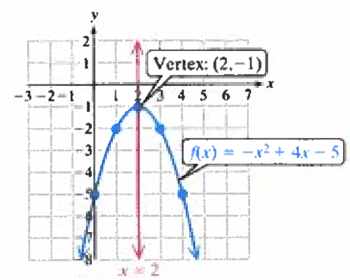


Figure 2-4

Skill Practice 3 Repeat Example 3 with $f(x) = -x^2 - 4x - 7$.

The x-intercepts of a quadratic function defined by $f(x) = ax^2 + bx + c$ are the real solutions to the equation $f(x) = 0$. The discriminant $b^2 - 4ac$ enables us to determine the number of real solutions to the equation and thus, the number of x-intercepts of the graph of the function.

Using the Discriminant to Determine the Number of x-Intercepts

Given a quadratic function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$,

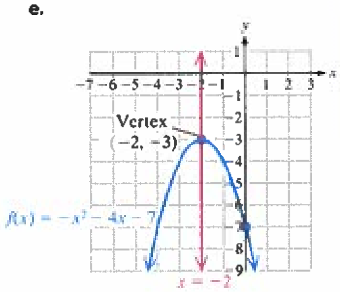
- If $b^2 - 4ac = 0$, the graph of $y = f(x)$ has one x-intercept.
- If $b^2 - 4ac > 0$, the graph of $y = f(x)$ has two x-intercepts.
- If $b^2 - 4ac < 0$, the graph of $y = f(x)$ has no x-intercept.

From Example 2, the discriminant of $3x^2 + 12x + 5 = 0$ is $(12)^2 - 4(3)(5) = 84 > 0$. Therefore, the graph of $f(x) = 3x^2 + 12x + 5$ has two x-intercepts (Figure 2-3).

From Example 3, the discriminant of $-x^2 + 4x - 5 = 0$ is $(4)^2 - 4(-1)(-5) = -4 < 0$. Therefore, the graph of $f(x) = -x^2 + 4x - 5$ has no x-intercept (Figure 2-4).

Answers

3. a. Downward **b.** $(-2, -3)$
c. No x-intercepts **d.** $(0, -7)$
e.



f. $x = -2$
g. The maximum value is -3.
h. The domain is $(-\infty, \infty)$.
The range is $(-\infty, -3]$.

4. Solve Applications Involving Quadratic Functions

Quadratic functions can be used in a variety of applications in which a variable is optimized. That is, the vertex of a parabola gives the maximum or minimum value of the dependent variable. We show three such applications in Examples 4–6.

EXAMPLE 4 Using a Quadratic Function for Projectile Motion

A stone is thrown from a 100-m cliff at an initial speed of 20 m/sec at an angle of 30° from the horizontal. The height of the stone can be modeled by $h(t) = -4.9t^2 + 10t + 100$, where $h(t)$ is the height in meters and t is the time in seconds after the stone is released.

a. Determine the time at which the stone will be at its maximum height. Round to 2 decimal places.
b. Determine the maximum height. Round to the nearest meter.
c. Determine the time at which the stone will hit the ground.

Point of Interest

The movie *Apollo 13* starring Tom Hanks was filmed in part in a "Vomit Comet," an aircraft that uses a parabolic flight trajectory to produce weightlessness. As the plane climbs toward the top of the parabolic path, occupants experience a force of nearly 2 Gs (twice their body weight). Once the plane goes over the vertex of the parabola, flyers free fall inside the plane. Such motion often produces motion sickness, thus earning the aircraft its name.

Solution:

a. The time at which the stone will be at its maximum height is the t-coordinate of the vertex.

Given $h(t) = -4.9t^2 + 10t + 100$, the coefficients are $a = -4.9$, $b = 10$, and $c = 100$.

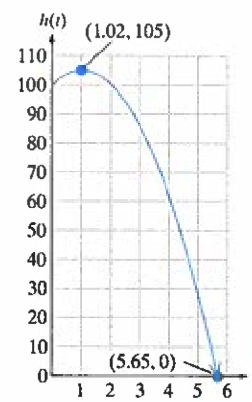
$$t = \frac{-b}{2a} = \frac{-10}{2(-4.9)} \approx 1.02$$

The vertex is given by $\left(\frac{-b}{2a}, h\left(\frac{-b}{2a}\right)\right)$.

The stone will be at its maximum height approximately 1.02 sec after release.

b. The maximum height is the value of $h(t)$ at the vertex.

$$h(1.02) = -4.9(1.02)^2 + 10(1.02) + 100$$
$$\approx 105$$ The maximum height is 105 m.



c. The stone will hit the ground when $h(t) = 0$.

$$h(t) = -4.9t^2 + 10t + 100$$
$$0 = -4.9t^2 + 10t + 100$$
$$t = \frac{-10 \pm \sqrt{(10)^2 - 4(-4.9)(100)}}{2(-4.9)}$$

$t \approx 5.65$ or $t \approx -3.61$ Reject the negative solution.

The stone will hit the ground in approximately 5.65 sec.

Skill Practice 4 A quarterback throws a football with an initial velocity of 72 ft/sec at an angle of 25°. The height of the ball can be modeled by $h(t) = -16t^2 + 30.4t + 5$, where $h(t)$ is the height (in ft) and t is the time in seconds after release.

a. Determine the time at which the ball will be at its maximum height.
b. Determine the maximum height of the ball.
c. Determine the amount of time required for the ball to reach the receiver's hands if the receiver catches the ball at a point 3 ft off the ground.

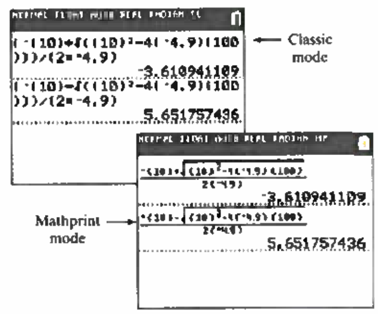
Answers

4. **a.** 0.95 sec **b.** 19.44 ft
c. Approximately 1.96 sec

TECHNOLOGY CONNECTIONS

Compute Solutions to a Quadratic Equation

The syntax to compute the expressions from Example 4(c) is shown for a calculator in Classic mode and in Mathprint mode. In Classic mode, parentheses are required around the numerator and denominator of the fraction and around the radicand within the square root. In Mathprint mode, select the ALPHA key followed by F1 to access the fraction template.



In Example 5, we present a type of application called an optimization problem. The goal is to maximize or minimize the value of the dependent variable by finding an optimal value of the independent variable.

EXAMPLE 5 Applying a Quadratic Function to Geometry

A parking area is to be constructed adjacent to a road. The developer has purchased 340 ft of fencing. Determine dimensions for the parking lot that would maximize the area. Then find the maximum area.

Solution:

Let x represent the width of the parking area.
Let y represent the length.
Let A represent the area.

Read the problem carefully, draw a representative diagram, and label the unknowns.

We need to find the values of x and y that maximize the area A of the rectangular region. The area is given by $A = (\text{length})(\text{width}) = yx = xy$.

To write the area as a function of one variable only, we need an equation that relates x and y. We know that the parking area is limited by a fixed amount of fencing. That is, the sum of the lengths of the three sides to be fenced can be at most 340 ft.

$2x + y = 340$ — The equation $2x + y = 340$ is called a **constraint equation.** This equation gives an implied restriction on x and y due to the limited amount of fencing.

Solve for y.

$y = 340 - 2x$ — Solve the constraint equation, $2x + y = 340$ for either x or y. In this case, we have solved for y.

$A = xy$

$A(x) = x(340 - 2x)$ — Substitute $340 - 2x$ for y in the equation $A = xy$.

$= -2x^2 + 340x$ — Function A is a quadratic function with a negative leading coefficient. The graph of the parabola opens downward, so the vertex is the maximum point on the function.

Avoiding Mistakes

To check, verify that the value $A(85)$ is the same as the product of length and width, xy.

$A(85) = 14{,}450 \text{ ft}^2$

$xy = (85 \text{ ft})(170 \text{ ft})$

$= 14{,}450 \text{ ft}^2$ ✓

x-coordinate of vertex:

$$x = \frac{-b}{2a} = \frac{-340}{2(-2)} = 85$$

The x-coordinate of the vertex $\frac{-b}{2a}$ is the value of x that will maximize the area.

$$y = 340 - 2(85) = 170$$

The second dimension of the parking lot can be determined from the constraint equation.

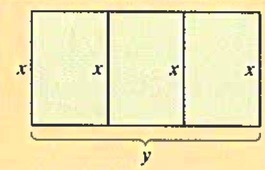
The values of x and y that would maximize the area are $x = 85$ ft and $y = 170$ ft.

$$A(85) = -2(85)^2 + 340(85) = 14{,}450$$

The value of the function at $x = 85$ gives the maximum area.

The maximum area is $14{,}450 \text{ ft}^2$.

Skill Practice 5 A farmer has 200 ft of fencing and wants to build three adjacent rectangular corrals. Determine the dimensions that should be used to maximize the area, and find the area of each individual corral.



5. Create Quadratic Models Using Regression

In Section 1.5, we introduced linear regression. A regression line is a linear model based on all observed data points. In a similar fashion, we can create a quadratic function using regression. For example, suppose that a scientist growing bacteria measures the population of bacteria as a function of time. A scatter plot reveals that the data follow a curve that is approximately parabolic (Figure 2-5). In Example 6, we use a graphing calculator to find a quadratic function that models the population of the bacteria as a function of time.

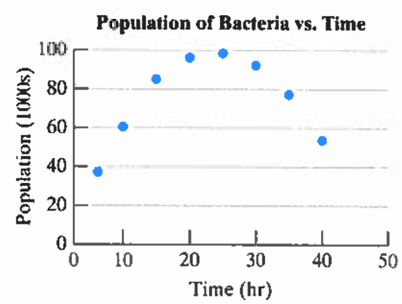


Figure 2-5

EXAMPLE 6 Creating a Quadratic Function Using Regression

The data in the table represent the population of bacteria $P(t)$ (in 1000s) versus the number of hours t since the culture was started.

a. Use regression to find a quadratic function to model the data. Round the coefficients to 3 decimal places.

b. Use the model to determine the time at which the population is the greatest. Round to the nearest hour.

c. What is the maximum population? Round to the nearest hundred.

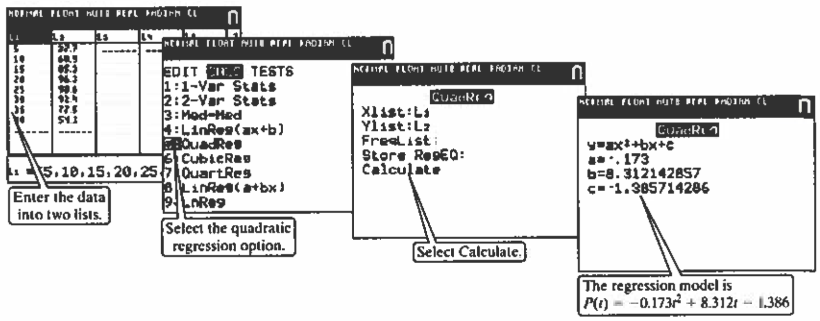
Time (hr) t	Population (1000s) $P(t)$
5	37.7
10	60.9
15	85.3
20	96.3
25	98.6
30	92.4
35	77.5
40	54.1

Answer

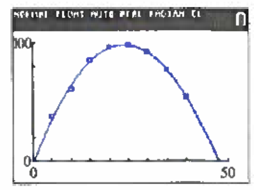
5. The dimensions should be $x = 25$ ft and $y = 50$ ft. The area of each individual corral is $\frac{1250}{3} = 416.\overline{6} \text{ ft}^2$.

Solution:

a. From the graph in Figure 2-5, it appears that the data follow a parabolic curve. Therefore, a quadratic model would be reasonable.



b. From the graph, the time when the population is greatest is the t-coordinate of the vertex.



$$t = \frac{-b}{2a} = \frac{-(8.312)}{2(-0.173)} \approx 24$$

The population is greatest 24 hr after the culture is started.

c. The maximum population of the bacteria is the $P(t)$ value at the vertex.

$$P(24) = -0.173(24)^2 + 8.312(24) - 1.386$$
$$\approx 98.5$$ The maximum number of bacteria is approximately 98,500.

Skill Practice 6 The funding $f(t)$ (in \$ millions) for a drug rehabilitation center is given in the table for selected years t.

t	0	3	6	9	12	15
$f(t)$	3.5	2.2	2.1	3	4.9	8

a. Use regression to find a quadratic function to model the data.
b. During what year is the funding the least? Round to the nearest year.
c. What is the minimum yearly amount of funding received? Round to the nearest million.

Answers
6. **a.** $f(t) = 0.060t^2 - 0.593t + 3.486$
b. Year 5 **c.** \$2 million

SECTION 2.1 Practice Exercises

Prerequisite Review

R.1. Solve the equation. $x^2 + 3x - 18 = 0$

R.2. a. Find the values of x for which $f(x) = 0$.

b. Find $f(0)$.

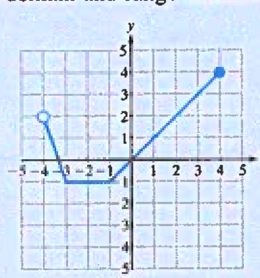
$$f(x) = 3x^2 - 7x - 20$$

R.3. Solve the equation by completing the square and applying the square root property.

$$x^2 + 8x + 12 = 0$$

R.4. Find $g\left(-\frac{1}{2}\right)$ for $g(x) = -x^2 + 2x - 4$.

R.5. Write the domain and range in interval notation.



Concept Connections

1. A function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is called a ______________ function.
2. The vertical line drawn through the vertex of a quadratic function is called the ______________ of symmetry.
3. Given $f(x) = a(x - h)^2 + k$ $(a \neq 0)$, the vertex of the parabola is the point ______________.
4. Given $f(x) = a(x - h)^2 + k$, if $a < 0$, the parabola opens (upward/downward) and the (minimum/maximum) value is ______________.
5. Given $f(x) = a(x - h)^2 + k$, if $a > 0$, the parabola opens (upward/downward) and the (minimum/maximum) value is ______________.
6. The graph of $f(x) = a(x - h)^2 + k$, $a \neq 0$, is a parabola and the axis of symmetry is the line given by ______________.

Objective 1: Graph a Quadratic Function Written in Vertex Form

For Exercises 7–14,

a. Determine whether the graph of the parabola opens upward or downward.
b. Identify the vertex.
c. Determine the x-intercept(s).
d. Determine the y-intercept.
e. Sketch the function.
f. Determine the axis of symmetry.
g. Determine the minimum or maximum value of the function.
h. Write the domain and range in interval notation. **(See Example 1)**

7. $f(x) = -(x - 4)^2 + 1$
8. $g(x) = -(x + 2)^2 + 4$
9. $h(x) = 2(x + 1)^2 - 8$
10. $k(x) = 2(x - 3)^2 - 2$
11. $m(x) = 3(x - 1)^2$
12. $n(x) = \frac{1}{2}(x + 2)^2$
13. $p(x) = -\frac{1}{5}(x + 4)^2 + 1$
14. $q(x) = -\frac{1}{3}(x - 1)^2 + 1$

Objective 2: Write $f(x) = ax^2 + bx + c$ $(a \neq 0)$ in Vertex Form

For Exercises 15–24,

a. Write the function in vertex form.
b. Identify the vertex.
c. Determine the x-intercept(s).
d. Determine the y-intercept.
e. Sketch the function.
f. Determine the axis of symmetry.
g. Determine the minimum or maximum value of the function.
h. Write the domain and range in interval notation. **(See Example 2)**

15. $f(x) = x^2 + 6x + 5$
16. $g(x) = x^2 + 8x + 7$
17. $p(x) = 3x^2 - 12x - 7$
18. $q(x) = 2x^2 - 4x - 3$
19. $c(x) = -2x^2 - 10x + 4$
20. $d(x) = -3x^2 - 9x + 8$
21. $h(x) = -2x^2 + 7x$
22. $k(x) = 3x^2 - 8x$
23. $p(x) = x^2 + 9x + 17$
24. $q(x) = x^2 + 11x + 26$

Objective 3: Find the Vertex of a Parabola by Using the Vertex Formula

For Exercises 25–32, find the vertex of the parabola by applying the vertex formula.

25. $f(x) = 3x^2 - 42x - 91$
26. $g(x) = 4x^2 - 64x + 107$
27. $k(a) = -\frac{1}{3}a^2 + 6a + 1$
28. $j(t) = -\frac{1}{4}t^2 + 10t - 5$
29. $f(c) = 4c^2 - 5$
30. $h(a) = 2a^2 + 14$
31. $P(x) = 1.2x^2 + 1.8x - 3.6$ (Write the coordinates of the vertex as decimals.)
32. $Q(x) = 7.5x^2 - 2.25x + 4.75$ (Write the coordinates of the vertex as decimals.)

For Exercises 33–42,

a. State whether the graph of the parabola opens upward or downward.

b. Identify the vertex.

c. Determine the x-intercept(s).

d. Determine the y-intercept.

e. Sketch the graph.

f. Determine the axis of symmetry.

g. Determine the minimum or maximum value of the function.

h. Write the domain and range in interval notation. **(See Example 3)**

33. $g(x) = -x^2 + 2x - 4$

34. $h(x) = -x^2 - 6x - 10$

35. $f(x) = 5x^2 - 15x + 3$

36. $k(x) = 2x^2 - 10x - 5$

37. $f(x) = 2x^2 + 3$

38. $g(x) = -x^2 - 1$

39. $f(x) = -2x^2 - 20x - 50$

40. $m(x) = 2x^2 - 8x + 8$

41. $n(x) = x^2 - x + 3$

42. $r(x) = x^2 - 5x + 7$

Objective 4: Solve Applications Involving Quadratic Functions

43. The monthly profit for a small company that makes long-sleeve T-shirts depends on the price per shirt. If the price is too high, sales will drop. If the price is too low, the revenue brought in may not cover the cost to produce the shirts. After months of data collection, the sales team determines that the monthly profit is approximated by $f(p) = -50p^2 + 1700p - 12{,}000$, where p is the price per shirt and $f(p)$ is the monthly profit based on that price. **(See Example 4)**

a. Find the price that generates the maximum profit.

b. Find the maximum profit.

c. Find the price(s) that would enable the company to break even.

44. The monthly profit for a company that makes decorative picture frames depends on the price per frame. The company determines that the profit is approximated by $f(p) = -80p^2 + 3440p - 36{,}000$, where p is the price per frame and $f(p)$ is the monthly profit based on that price.

a. Find the price that generates the maximum profit.

b. Find the maximum profit.

c. Find the price(s) that would enable the company to break even.

45. A long jumper leaves the ground at an angle of 20° above the horizontal, at a speed of 11 m/sec. The height of the jumper can be modeled by $h(x) = -0.046x^2 + 0.364x$, where h is the jumper's height in meters and x is the horizontal distance from the point of launch.

a. At what horizontal distance from the point of launch does the maximum height occur? Round to 2 decimal places.

b. What is the maximum height of the long jumper? Round to 2 decimal places.

c. What is the length of the jump? Round to 1 decimal place.

46. A firefighter holds a hose 3 m off the ground and directs a stream of water toward a burning building. The water leaves the hose at an initial speed of 16 m/sec at an angle of 30°. The height of the water can be approximated by $h(x) = -0.026x^2 + 0.577x + 3$, where $h(x)$ is the height of the water in meters at a point x meters horizontally from the firefighter to the building.

a. Determine the horizontal distance from the firefighter at which the maximum height of the water occurs. Round to 1 decimal place.

b. What is the maximum height of the water? Round to 1 decimal place.

c. The flow of water hits the house on the downward branch of the parabola at a height of 6 m. How far is the firefighter from the house? Round to the nearest meter.

47. The population $P(t)$ of a culture of the bacterium *Pseudomonas aeruginosa* is given by $P(t) = -1718t^2 + 82{,}000t + 10{,}000$, where t is the time in hours since the culture was started.

a. Determine the time at which the population is at a maximum. Round to the nearest hour.

b. Determine the maximum population. Round to the nearest thousand.

48. The gas mileage $m(x)$ (in mpg) for a certain vehicle can be approximated by $m(x) = -0.028x^2 + 2.688x - 35.012$, where x is the speed of the vehicle in mph.

a. Determine the speed at which the car gets its maximum gas mileage.

b. Determine the maximum gas mileage.

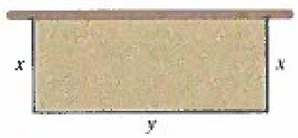
49. The sum of two positive numbers is 24. What two numbers will maximize the product? **(See Example 5)**

50. The sum of two positive numbers is 1. What two numbers will maximize the product?

51. The difference of two numbers is 10. What two numbers will minimize the product?

52. The difference of two numbers is 30. What two numbers will minimize the product?

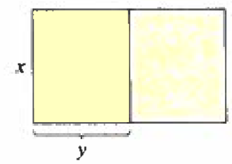
53. Suppose that a family wants to fence in an area of their yard for a vegetable garden to keep out deer. One side is already fenced from the neighbor's property. (**See Example 5**)



a. If the family has enough money to buy 160 ft of fencing, what dimensions would produce the maximum area for the garden?

b. What is the maximum area?

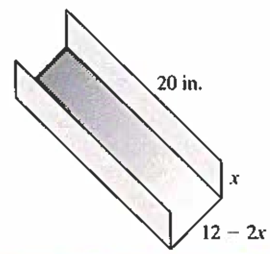
54. Two chicken coops are to be built adjacent to one another using 120 ft of fencing.



a. What dimensions should be used to maximize the area of an individual coop?

b. What is the maximum area of an individual coop?

55. A trough at the end of a gutter spout is meant to direct water away from a house. The homeowner makes the trough from a rectangular piece of aluminum that is 20 in. long and 12 in. wide. He makes a fold along the two long sides a distance of x inches from the edge.

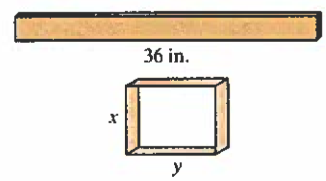


a. Write a function to represent the volume in terms of x.

b. What value of x will maximize the volume of water that can be carried by the gutter?

c. What is the maximum volume?

56. A rectangular frame of uniform depth for a shadow box is to be made from a 36-in. piece of wood.



a. Write a function to represent the display area in terms of x.

b. What dimensions should be used to maximize the display area?

c. What is the maximum area?

Objective 5: Create Quadratic Models Using Regression

57. *Tetanus bacillus* bacteria are cultured to produce tetanus toxin used in an inactive form for the tetanus vaccine. The amount of toxin produced per batch increases with time and then decreases as the culture becomes unstable. The variable t is the time in hours after the culture has started, and $y(t)$ is the yield of toxin in grams. (**See Example 6**)

t	8	16	24	32	40	48
$y(t)$	0.60	1.12	1.60	1.78	1.90	2.00

t	56	64	72	80	88	96
$y(t)$	1.94	1.80	1.48	1.30	0.66	0.10

a. Use regression to find a quadratic function to model the data.

b. At what time is the yield the greatest? Round to the nearest hour.

c. What is the maximum yield? Round to the nearest gram.

58. Gas mileage is tested for a car under different driving conditions. At lower speeds, the car is driven in stop-and-go traffic. At higher speeds, the car must overcome more wind resistance. The variable x given in the table represents the speed (in mph) for a compact car, and $m(x)$ represents the gas mileage (in mpg).

x	25	30	35	40	45
$m(x)$	22.7	25.1	27.9	30.8	31.9

x	50	55	60	65
$m(x)$	30.9	28.4	24.2	21.9

a. Use regression to find a quadratic function to model the data.

b. At what speed is the gas mileage the greatest? Round to the nearest mile per hour.

c. What is the maximum gas mileage? Round to the nearest mile per gallon.

59. Fluid runs through a drainage pipe with a 10-cm radius and a length of 30 m (3000 cm). The velocity of the fluid gradually decreases from the center of the pipe toward the edges as a result of friction with the walls of the pipe. For the data shown, $v(x)$ is the velocity of the fluid (in cm/sec) and x represents the distance (in cm) from the center of the pipe toward the edge.

x	0	1	2	3	4
$v(x)$	195.6	195.2	194.2	193.0	191.5

x	5	6	7	8	9
$v(x)$	189.8	188.0	185.5	183.0	180.0

a. The pipe is 30 m long (3000 cm). Determine how long it will take fluid to run the length of the pipe through the center of the pipe. Round to 1 decimal place.

b. Determine how long it will take fluid at a point 9 cm from the center of the pipe to run the length of the pipe. Round to 1 decimal place.

c. Use regression to find a quadratic function to model the data.

d. Use the model from part (c) to predict the velocity of the fluid at a distance 5.5 cm from the center of the pipe. Round to 1 decimal place.

60. The braking distance required for a car to stop depends on numerous variables such as the speed of the car, the weight of the car, reaction time of the driver, and the coefficient of friction between the tires and the road. For a certain vehicle on one stretch of highway, the braking distances $d(s)$ (in ft) are given for several different speeds s (in mph).

s	30	35	40	45	50
$d(s)$	109	134	162	191	223

s	55	60	65	70	75
$d(s)$	256	291	328	368	409

a. Use regression to find a quadratic function to model the data.

b. Use the model from part (a) to predict the stopping distance for the car if it is traveling 62 mph before the brakes are applied. Round to the nearest foot.

c. Suppose that the car is traveling 53 mph before the brakes are applied. If a deer is standing in the road at a distance of 245 ft from the point where the brakes are applied, will the car hit the deer?

Mixed Exercises

For Exercises 61–64, given a quadratic function defined by $f(x) = ax^2 + bx + c$ ($a \neq 0$), answer true or false. If an answer is false, explain why.

61. The graph of f can have two y-intercepts.

62. The graph of f can have two x-intercepts.

63. If $a < 0$, then the vertex of the parabola is the maximum point on the graph of f.

64. The axis of symmetry of the graph of f is the line defined by $y = c$.

For Exercises 65–70, determine the number of x-intercepts of the graph of $f(x) = ax^2 + bx + c$ ($a \neq 0$), based on the discriminant of the related equation $f(x) = 0$. (*Hint*: Recall that the discriminant is $b^2 - 4ac$.)

65. $f(x) = 4x^2 + 12x + 9$

66. $f(x) = 25x^2 - 20x + 4$

67. $f(x) = -x^2 - 5x + 8$

68. $f(x) = -3x^2 + 4x + 9$

69. $f(x) = -3x^2 + 6x - 11$

70. $f(x) = -2x^2 + 5x - 10$

For Exercises 71–78, given a quadratic function defined by $f(x) = a(x - h)^2 + k$ ($a \neq 0$), match the graph with the function based on the conditions given.

71. $a > 0, h < 0, k > 0$

72. $a > 0, h < 0, k < 0$

73. $a < 0, h < 0, k < 0$

74. $a < 0, h < 0, k > 0$

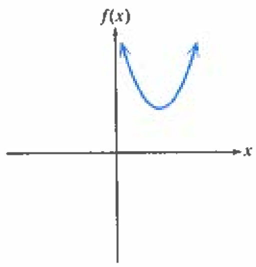
75. $a > 0$, axis of symmetry $x = 2$, $k < 0$

76. $a < 0$, axis of symmetry $x = 2$, $k > 0$

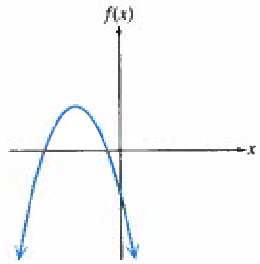
77. $a < 0$, $h = 2$, maximum value equals -2

78. $a > 0$, $h = 2$, minimum value equals 2

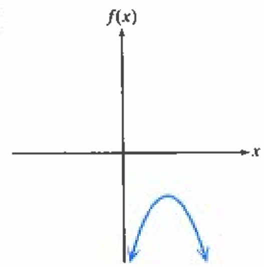
a.



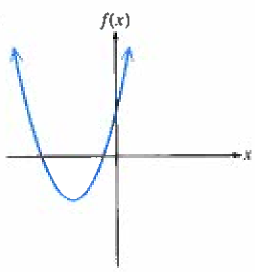
b.



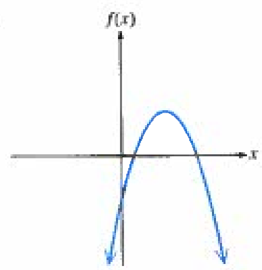
c.



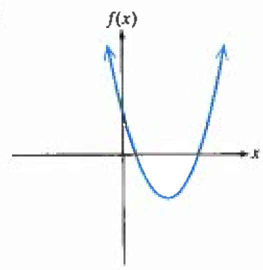
d.



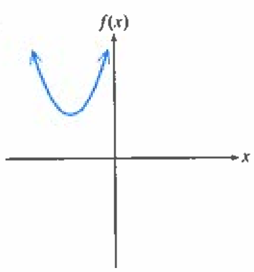
e.



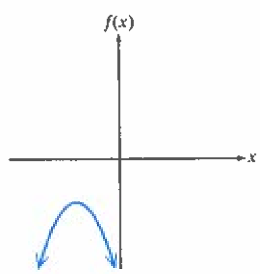
f.



g.



h.



Write About It

79. Explain why a parabola opening upward has a minimum value but no maximum value. Use the graph of $f(x) = x^2$ to explain.

80. Explain why a quadratic function whose graph opens downward with vertex $(4, -3)$ has no x-intercept.

81. Explain why a quadratic function given by $f(x) = ax^2 + bx + c$ cannot have two y-intercepts.

82. Explain how to use the discriminant to determine the number of x-intercepts for the graph of $f(x) = ax^2 + bx + c$.

83. If a quadratic function given by $y = f(x)$ has x-intercepts of $(2, 0)$ and $(6, 0)$, explain why the vertex must be $(4, f(4))$.

84. Given an equation of a parabola in the form $y = a(x - h)^2 + k$, explain how to determine by inspection if the parabola has no x-intercepts.

Expanding Your Skills

For Exercises 85–88, define a quadratic function $y = f(x)$ that satisfies the given conditions.

85. Vertex $(2, -3)$ and passes through $(0, 5)$

86. Vertex $(-3, 1)$ and passes through $(0, -17)$

87. Axis of symmetry $x = 4$, maximum value 6, passes through $(1, 3)$

88. Axis of symmetry $x = -2$, minimum value 5, passes through $(2, 13)$

For Exercises 89–92, find the value of b or c that gives the function the given minimum or maximum value.

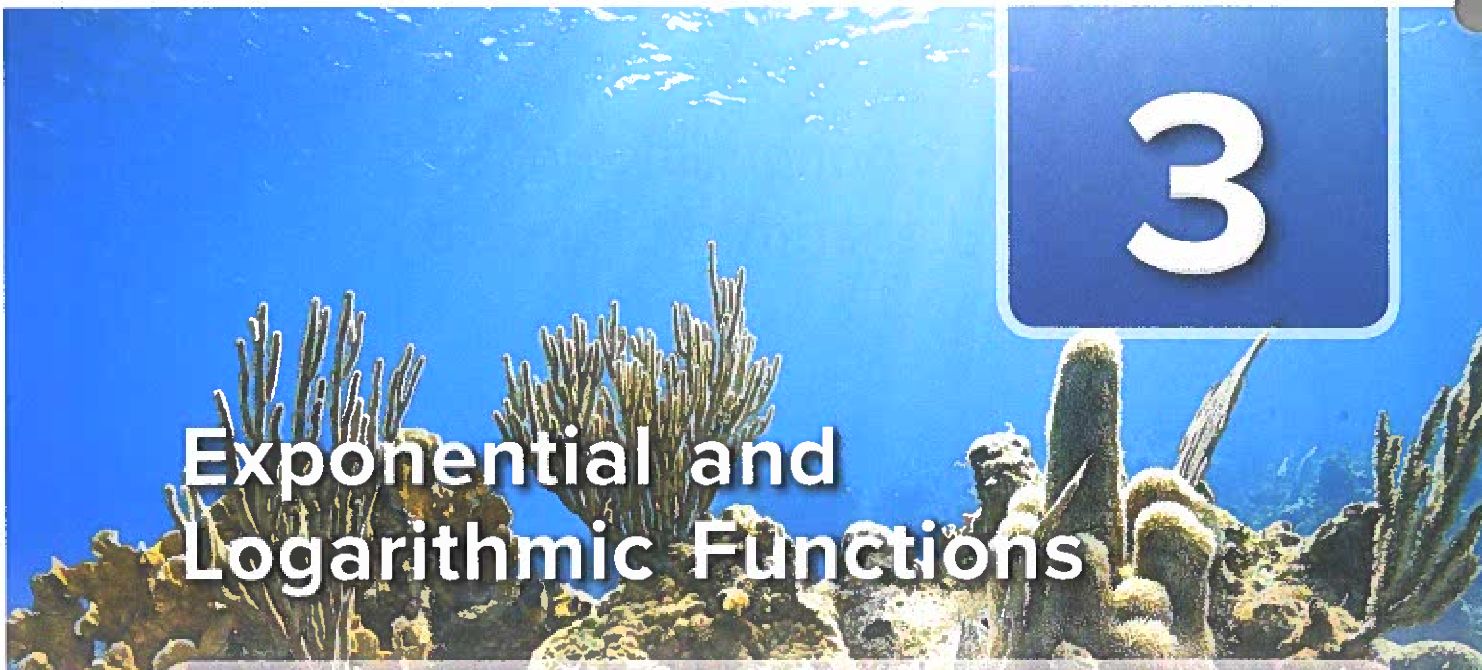
89. $f(x) = 2x^2 + 12x + c$; minimum value -9

90. $f(x) = 3x^2 + 12x + c$; minimum value -4

91. $f(x) = -x^2 + bx + 4$; maximum value 8

92. $f(x) = -x^2 + bx - 2$; maximum value 7

Exponential and Logarithmic Functions



3 Exponential and Logarithmic Functions

Chapter Outline

Visible light from the Sun is vitally important for the health of an ocean, lake, or any body of water. In particular, light penetrating through a body of water provides the energy to fuel vast amounts of microscopic plants called phytoplankton that are an essential source of food and oxygen for an aquatic ecosystem. Phytoplankton converts energy from the Sun to usable energy for plant growth. Thus, the amount of light directly affects plant productivity at the base of the food chain and ultimately animal life farther up the food chain.

With increasing depth, the percentage of visible light from the surface of a body of water drops exponentially. This means that light intensity drops quickly at first and then drops more slowly with increasing depth. (More specifically, light intensity decreases at a rate proportional to the intensity at a particular depth.) To study this phenomenon, scientists use exponential functions and their inverses, logarithmic functions. These two important categories of functions have many applications including the study of the decay of radioactive substances, short-term population growth, and the growth of investments subject to compound interest.

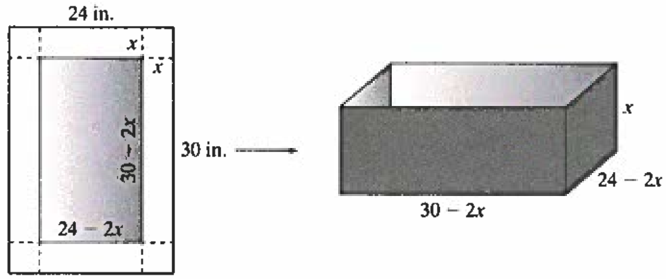
SECTION 2.2 Introduction to Polynomial Functions

OBJECTIVES

1. Determine the End Behavior of a Polynomial Function
2. Identify Zeros and Multiplicities of Zeros
3. Apply the Intermediate Value Theorem
4. Sketch a Polynomial Function

1. Determine the End Behavior of a Polynomial Function

A solar oven is to be made from an open box with reflective sides. Each box is made from a 30-in. by 24-in. rectangular sheet of aluminum with squares of length x (in inches) removed from each corner. Then the flaps are folded up to form an open box.



The volume $V(x)$ (in cubic inches) of the box is given by

$$V(x) = 4x^3 - 108x^2 + 720x, \text{ where } 0 < x < 12.$$

From the graph of $y = V(x)$ (Figure 2-6), the maximum volume appears to occur when squares of approximately 4 inches in length are cut from the corners of the sheet of aluminum. See Exercise 99.

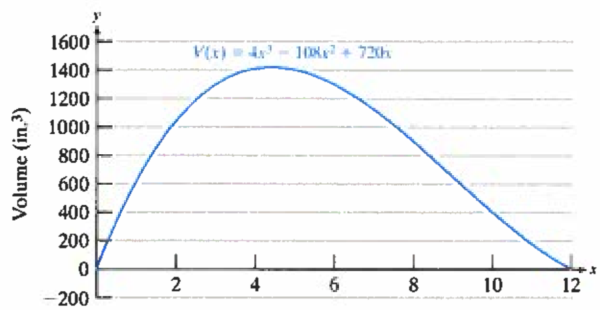


Figure 2-6

The function defined by $V(x) = 4x^3 - 108x^2 + 720x$ is an example of a polynomial function of degree 3.

Definition of a Polynomial Function

Let n be a whole number and $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ be real numbers, where $a_n \neq 0$. Then a function defined by

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$$

is called a **polynomial function of degree n.**

The coefficients of each term of a polynomial function are real numbers, and the exponents on x must be whole numbers.

Polynomial Function	**Not a Polynomial Function**
$f(x) = 4x^5 - 3x^4 + 2x^2$	$f(x) = 4\sqrt{x} - \dfrac{3}{x} + (3 + 2i)x^2$

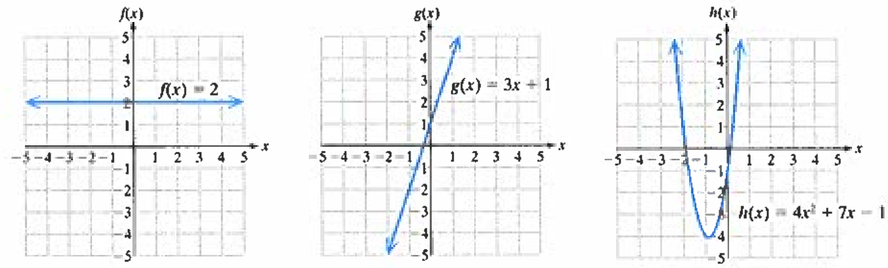
$\sqrt{x} = x^{1/2}$ Exponent not a whole number	$3/x = 3x^{-1}$ Exponent not a whole number	$(3 + 2i)$ Coefficient not a real number

TIP A third-degree polynomial function is referred to as a *cubic* polynomial function.

A fourth-degree polynomial function is referred to as a *quartic* polynomial function.

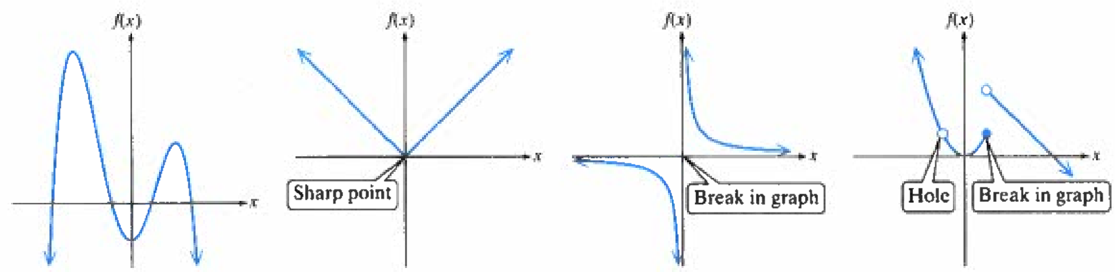
We have already studied several special cases of polynomial functions. For example:

$f(x) = 2$	constant function	(polynomial function, degree 0)
$g(x) = 3x + 1$	linear function	(polynomial function, degree 1)
$h(x) = 4x^2 + 7x - 1$	quadratic function	(polynomial function, degree 2)



The domain of a polynomial function is all real numbers. Furthermore, the graph of a polynomial function is both continuous and smooth. Informally, a continuous function can be drawn without lifting the pencil from the paper. A smooth function has no sharp corners or points. For example, the first curve shown here could be a polynomial function, but the last three are not polynomial functions.

Smooth and Continuous | Not Smooth | Not Continuous | Not Continuous



To begin our analysis of polynomial functions, we first consider the graphs of functions of the form $f(x) = ax^n$, where a is a real number and n is a positive integer. These fall into a category of functions called **power functions.** The graphs of three power functions with even degrees and positive coefficients are shown in Figure 2-7. The graphs of three power functions with odd degrees and positive coefficients are shown in Figure 2-8.

TIP For a positive integer n, the graph of the power function $y = x^n$ becomes "flatter" near the x-intercept for higher powers of n.

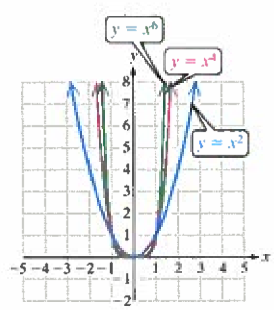


Figure 2-7

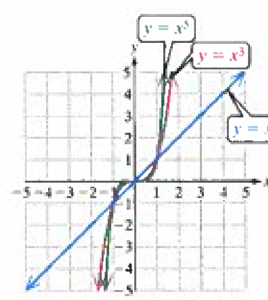


Figure 2-8

From Figure 2-7, notice that for even powers of n, the behavior of $y = x^n$ is similar to the graph of $y = x^2$ with variations on the "steepness" of the curve. Figure 2-8 shows that for odd powers, the behavior of $y = x^n$ with $n \geq 3$ is similar to the graph of $y = x^3$. For any power function $y = ax^n$, the coefficient a will impose a vertical shrink or stretch on the graph of $y = x^n$ by a factor of $|a|$. If $a < 0$, then the graph is reflected across the x-axis.

Power functions are helpful to analyze the "end behavior" of a polynomial function with multiple terms. The end behavior is the general direction that the fuction follows as x approaches ∞ or $-\infty$. To describe end behavior, we have the following notation.

Notation for Infinite Behavior of $y = f(x)$

$x \to \infty$	is read as "x approaches infinity." This means that x becomes infinitely large in the positive direction.
$x \to -\infty$	is read as "x approaches negative infinity." This means that x becomes infinitely "large" in the negative direction.
$f(x) \to \infty$	is read as "$f(x)$ approaches infinity." This means that the y value becomes infinitely large in the positive direction.
$f(x) \to -\infty$	is read as "$f(x)$ approaches negative infinity." This means that the y value becomes infinitely "large" in the negative direction.

Consider the function defined by

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$$

The leading term has the greatest exponent on x.

The leading term has the greatest exponent on x. Therefore, as $|x|$ gets large (that is, as $x \to \infty$ or as $x \to -\infty$), the leading term will be relatively larger in absolute value than all other terms. In fact, x^n will eventually be greater in absolute value than the *sum* of all other terms. Therefore, the end behavior of the function is dictated only by the leading term, and the graph of the function far to the left and far to the right will follow the general behavior of the power function $y = ax^n$.

The Leading Term Test

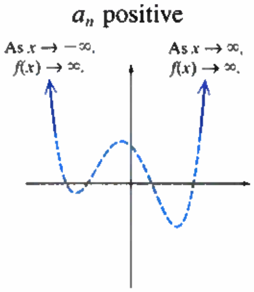
Consider a polynomial function given by

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0.$$

As $x \to \infty$ or as $x \to -\infty$, f eventually becomes forever increasing or forever decreasing and will follow the general behavior of $y = a_nx^n$.

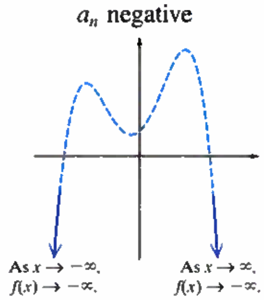
n is even

a_n positive



End behavior: up left/up right

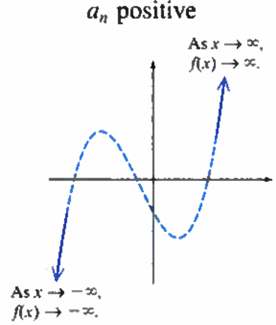
a_n negative



End behavior: down left/down right

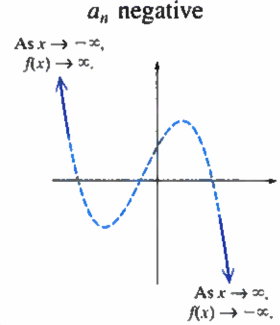
n is odd

a_n positive



End behavior: down left/up right

a_n negative



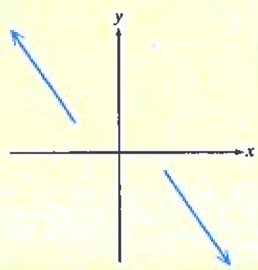
End behavior: up left/down right

EXAMPLE 1 Determining End Behavior

Use the leading term to determine the end behavior of the graph of the function.

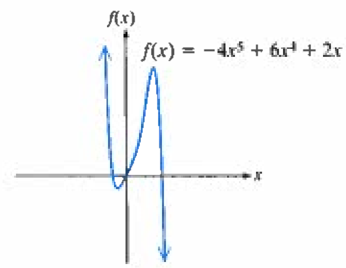
a. $f(x) = -4x^5 + 6x^4 + 2x$ **b.** $g(x) = \frac{1}{4}x(2x - 3)^3(x + 4)^2$

TIP The graph of $y = f(x)$ from Example 1(a) will exhibit the same behavior as the graph of the power function $y = -4x^5$ for values of x far to the right and far to the left. This is similar to the graph of $y = x^5$ reflected across the x-axis.



Solution:

a. $f(x) = \underbrace{-4}_{\text{negative}}x^{\overbrace{5}^{\text{odd}}} + 6x^4 + 2x$



The leading coefficient is negative and the degree is odd. By the leading term test, the end behavior is up to the left and down to the right.

As $x \to -\infty$, $f(x) \to \infty$.
As $x \to \infty$, $f(x) \to -\infty$.

b. $g(x) = \frac{1}{4}x(2x - 3)^3(x + 4)^2$

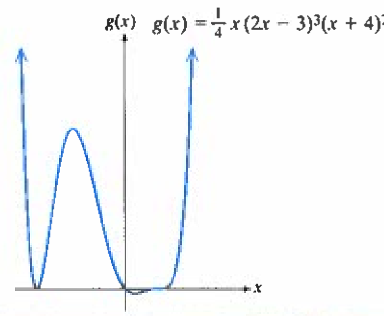
$g(x) = \frac{1}{4}x(2x - 3)^3(x + 4)^2 = \underbrace{2}_{\text{positive}}x^{\overbrace{6}^{\text{even}}} + \cdots$

To determine the leading term, multiply the leading terms from each factor. That is,

$\frac{1}{4}x(2x)^3(x)^2 = 2x^6$.

The leading coefficient is positive and the degree is even. By the leading term test, the end behavior is up to the left and up to the right.

As $x \to -\infty$, $f(x) \to \infty$.
As $x \to \infty$, $f(x) \to \infty$.



Skill Practice 1 Use the leading term to determine the end behavior of the graph of the function.

a. $f(x) = -0.3x^4 - 5x^2 - 3x + 4$ **b.** $g(x) = \frac{6}{7}(x - 9)^4(x + 4)^2(3x - 5)$

2. Identify Zeros and Multiplicities of Zeros

Consider a polynomial function defined by $y = f(x)$. The values of x in the domain of f for which $f(x) = 0$ are called the **zeros** of the function. These are the real solutions (or **roots**) of the equation $f(x) = 0$ and correspond to the x-intercepts of the graph of $y = f(x)$.

Answers

1. **a.** Down to the left, down to the right.
   As $x \to -\infty$, $f(x) \to -\infty$.
   As $x \to \infty$, $f(x) \to -\infty$.

   b. Down to the left, up to the right.
   As $x \to -\infty$, $f(x) \to -\infty$.
   As $x \to \infty$, $f(x) \to \infty$.

EXAMPLE 2 Determining the Zeros of a Polynomial Function

Find the zeros of the function defined by $f(x) = x^3 + x^2 - 9x - 9$.

Solution:

$f(x) = x^3 + x^2 - 9x - 9$

$0 = x^3 + x^2 - 9x - 9$ — To find the zeros of f, set $f(x) = 0$ and solve for x.

$0 = x^2(x + 1) - 9(x + 1)$ — Factor by grouping.

$0 = (x + 1)(x^2 - 9)$

$0 = (x + 1)(x - 3)(x + 3)$ — Factor the difference of squares.

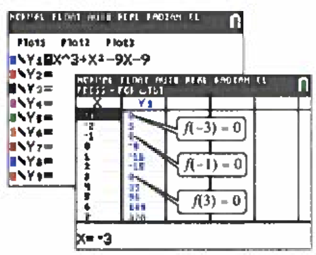
$x = -1, x = 3, x = -3$ — Set each factor equal to zero and solve for x.

The zeros of f are -1, 3, and -3.

The graph of f is shown in Figure 2-9. The zeros of the function are real numbers and correspond to the x-intercepts of the graph. By inspection, we can evaluate $f(0) = -9$, indicating that the y-intercept is $(0, -9)$.

Check:

A table of points can be used to check that $f(-1)$, $f(3)$, and $f(-3)$ all equal 0.



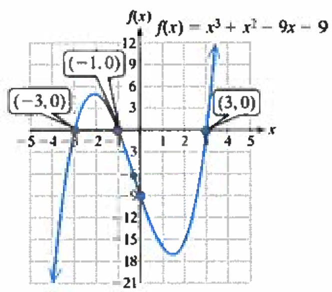


Figure 2-9

Skill Practice 2 Find the zeros of the function defined by $f(x) = 4x^3 - 4x^2 - 25x + 25$.

EXAMPLE 3 Determining the Zeros of a Polynomial Function

Find the zeros of the function defined by $f(x) = -x^3 + 8x^2 - 16x$.

Solution:

$f(x) = -x^3 + 8x^2 - 16x$ — To find the zeros of f, set $f(x) = 0$ and solve for x.

$0 = -x(x^2 - 8x + 16)$ — Factor out the GCF.

$0 = -x(x - 4)^2$ — Factor the perfect square trinomial.

$x = 0, x = 4$ — Set each factor equal to zero and solve for x.

The zeros of f are 0 and 4.

The graph of f is shown in Figure 2-10. The zeros of the function are real numbers and correspond to the x-intercepts $(0, 0)$ and $(4, 0)$.

The leading term of $f(x)$ is $-x^3$. The coefficient is negative and the exponent is odd. The graph shows the end behavior up to the left and down to the right as expected.

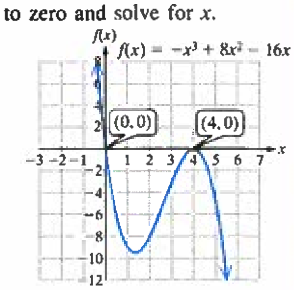


Figure 2-10

Skill Practice 3 Find the zeros of the function defined by $f(x) = x^3 + 10x^2 + 25x$.

Answers

2. $1, \frac{5}{2}, -\frac{5}{2}$

3. $0, -5$

From Example 3, $f(x) = -x^3 + 8x^2 - 16x$ can be written as a product of linear factors:

$$f(x) = -x(x - 4)^2$$

Notice that the factor $(x - 4)$ appears to the second power. Therefore, we say that the corresponding zero, 4, has a multiplicity of 2. In general, we say that if a polynomial function has a factor $(x - c)$ that appears exactly k times, then c is a **zero of multiplicity k.** For example, consider:

$g(x) = x^2(x - 2)^3(x + 4)^7$ — 0 is a zero of multiplicity 2.
2 is a zero of multiplicity 3.
-4 is a zero of multiplicity 7.

The graph of a polynomial function behaves in the following manner based on the multiplicity of the zeros.

Touch Points and Cross Points

Let f be a polynomial function and let c be a real zero of f. Then the point $(c, 0)$ is an x-intercept of the graph of f. Furthermore,

- If c is a zero of odd multiplicity, then the graph *crosses* the x-axis at c. The point $(c, 0)$ is called a **cross point.**
- If c is a zero of even multiplicity, then the graph *touches* the x-axis at c and turns back around (does not cross the x-axis). The point $(c, 0)$ is called a **touch point.**

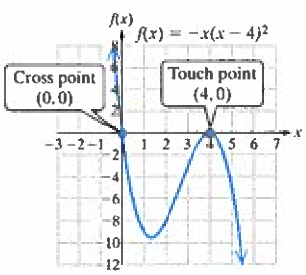


Figure 2-11

To illustrate the behavior of a polynomial function at its real zeros, consider the graph of $f(x) = -x(x - 4)^2$ from Example 3 (Figure 2-11).

- 0 has a multiplicity of 1 (odd multiplicity). The graph *crosses* the x-axis at $(0, 0)$.
- 4 has a multiplicity of 2 (even multiplicity). The graph *touches* the x-axis at $(4, 0)$ and turns back around.

EXAMPLE 4 Determining Zeros and Multiplicities

Determine the zeros and their multiplicities for the given functions.

a. $m(x) = \frac{1}{10}(x - 4)^2(2x + 5)^3$ **b.** $n(x) = x^4 - 2x^2$

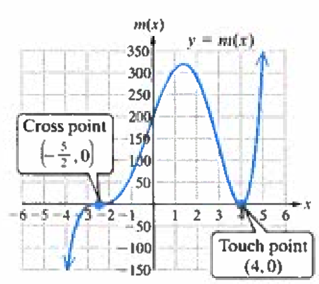
Solution:

a. $m(x) = \frac{1}{10}(x - 4)^{\overset{\text{even}}{2}}(2x + 5)^{\overset{\text{odd}}{3}}$

The function is factored into linear factors. The zeros are 4 and $-\frac{5}{2}$.

The function has a zero of 4 with multiplicity 2 (even). The graph has a touch point at $(4, 0)$.

The function has a zero of $-\frac{5}{2}$ with multiplicity 3 (odd). The graph has a cross point at $\left(-\frac{5}{2}, 0\right)$.

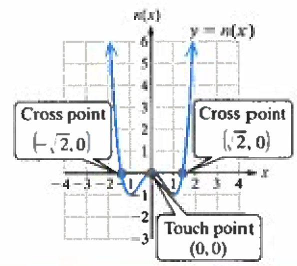


b. $n(x) = x^4 - 2x^2$
$= x^2(x^2 - 2)$
$= x^2(x - \sqrt{2})^1(x + \sqrt{2})^1$

The function has a zero of 0 with multiplicity 2 (even). The graph has a touch point at (0, 0).

The function has a zero of $\sqrt{2}$ with multiplicity 1 (odd). The graph has a cross point at $(\sqrt{2}, 0) \approx (1.41, 0)$.

The function has a zero of $-\sqrt{2}$ with multiplicity 1 (odd). The graph has a cross point at $(-\sqrt{2}, 0) \approx (-1.41, 0)$.

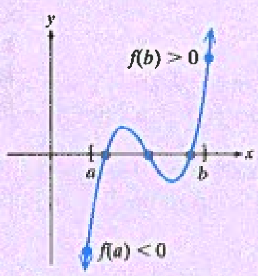


Skill Practice 4 Determine the zeros and their multiplicities for the given functions.

a. $p(x) = -\frac{3}{5}(x + 3)^4(5x - 1)^5$ **b.** $q(x) = 2x^6 - 14x^4$

Avoiding Mistakes

For a polynomial function f, if $f(a)$ and $f(b)$ have opposite signs, then f must have at least one zero on the interval $[a, b]$. This includes the possibility that f may have more than one zero on $[a, b]$.



3. Apply the Intermediate Value Theorem

In Examples 2–4, the zeros of the functions were easily identified by first factoring the polynomial. However, in most cases, the real zeros of a polynomial are difficult or impossible to determine algebraically. For example, the function given by $f(x) = x^4 + 6x^3 - 26x + 15$ has zeros of $-1 \pm \sqrt{6}$ and $-2 \pm \sqrt{7}$. At this point, we do not have the tools to find the zeros of this function analytically. However, we can use the intermediate value theorem to help us search for zeros of a polynomial function and approximate their values.

Intermediate Value Theorem

Let f be a polynomial function. For $a < b$, if $f(a)$ and $f(b)$ have opposite signs, then f has at least one zero on the interval $[a, b]$.

EXAMPLE 5 Applying the Intermediate Value Theorem

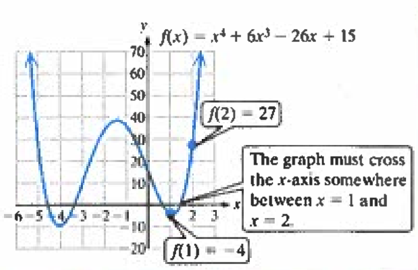
Show that $f(x) = x^4 + 6x^3 - 26x + 15$ has a zero on the interval [1, 2].

Solution:

$f(x) = x^4 + 6x^3 - 26x + 15$
$f(1) = (1)^4 + 6(1)^3 - 26(1) + 15 = -4$
$f(2) = (2)^4 + 6(2)^3 - 26(2) + 15 = 27$

Since $f(1)$ and $f(2)$ have opposite signs, then by the intermediate value theorem, we know that the function must have at least one zero on the interval [1, 2].

The actual value of the zero on the interval [1, 2] is $-1 + \sqrt{6} \approx 1.45$.



Skill Practice 5 Show that $f(x) = x^4 + 6x^3 - 26x + 15$ has a zero on the interval $[-4, -3]$.

TIP It is important to note that if the signs of $f(a)$ and $f(b)$ are the same, then the intermediate value theorem is inconclusive.

Answers

4. a. -3 (multiplicity 4) and $\frac{1}{5}$ (multiplicity 5)

b. 0 (multiplicity 4), $\sqrt{7}$ (multiplicity 1), and $-\sqrt{7}$ (multiplicity 1)

5. $f(-4) = -9$ and $f(-3) = 12$. Since $f(-4)$ and $f(-3)$ have opposite signs, then the intermediate value theorem guarantees the existence of at least one zero on the interval $[-4, -3]$.

The intermediate value theorem can be used repeatedly in a technique called the bisection method to approximate the value of a zero. See the online group activity "Investigating the Bisection Method for Finding Zeros."

Point of Interest

The modern definition of a computer is a programmable device designed to carry out a sequence of arithmetic or logical operations. However, the word "computer" originally referred to a person who did such calculations using paper and pencil. "Human computers" were notably used in the eighteenth century to predict the path of Halley's comet and to produce astronomical tables critical to surveying and navigation. Later, during World Wars I and II, human computers developed ballistic firing tables that would describe the trajectory of a shell.

Computing tables of values was very time consuming, and the "computers" would often interpolate to find intermediate values within a table. Interpolation is a method by which intermediate values between two numbers are estimated. Often the interpolated values were based on a polynomial function.

4. Sketch a Polynomial Function

TIP Even with advanced techniques from calculus or the use of a graphing utility, it is often difficult or impossible to find the exact location of the turning points of a polynomial function.

The graph of a polynomial function may also have "turning points." These correspond to relative maxima and minima. For example, consider $f(x) = x(x + 2)(x - 2)^2$. See Figure 2-12.

Multiplying the leading terms within the factors, we have a leading term of $(x)(x)(x)^2 = x^4$. Therefore, the end behavior of the graph is up to the left and up to the right.

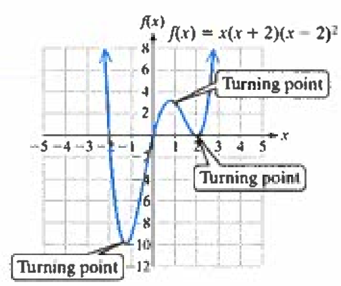
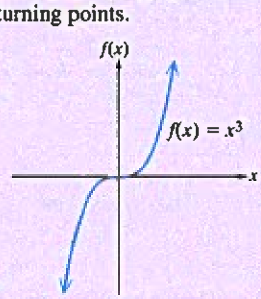


Figure 2-12

Avoiding Mistakes

A polynomial of degree n may have fewer than $n - 1$ turning points. For example, $f(x) = x^3$ is a degree 3 polynomial function (indicating that it could have a maximum of two turning points), yet the graph has no turning points.



Starting from the far left, the graph of f decreases to the x-intercept of -2. Since -2 is a zero with an odd multiplicity, the graph must cross the x-axis at -2. For the same reason, the graph must cross the x-axis again at the origin. Therefore, somewhere between $x = -2$ and $x = 0$, the graph must "turn around." This point is called a "turning point."

The turning points of a polynomial function are the points where the function changes from increasing to decreasing or vice versa.

Number of Turning Points of a Polynomial Function

Let f represent a polynomial function of degree n. Then the graph of f has at most $n - 1$ turning points.

At this point we are ready to outline a strategy for sketching a polynomial function.

Graphing a Polynomial Function

To graph a polynomial function defined by $y = f(x)$,

1. Use the leading term to determine the end behavior of the graph.
2. Determine the y-intercept by evaluating $f(0)$.
3. Determine the real zeros of f and their multiplicities (these are the x-intercepts of the graph of f).
4. Plot the x- and y-intercepts and sketch the end behavior.
5. Draw a sketch starting from the left-end behavior. Connect the x- and y-intercepts in the order that they appear from left to right using these rules:
   - The curve will cross the x-axis at an x-intercept if the corresponding zero has an odd multiplicity.
   - The curve will touch but not cross the x-axis at an x-intercept if the corresponding zero has an even multiplicity.
6. If a test for symmetry is easy to apply, use symmetry to plot additional points. Recall that
   - f is an even function (symmetric to the y-axis) if $f(-x) = f(x)$.
   - f is an odd function (symmetric to the origin) if $f(-x) = -f(x)$.
7. Plot more points if a greater level of accuracy is desired. In particular, to estimate the location of turning points, find several points between two consecutive x-intercepts.

In Examples 6 and 7, we demonstrate the process of graphing a polynomial function.

EXAMPLE 6 Graphing a Polynomial Function

Graph $f(x) = x^3 - 9x$.

Solution:

$f(x) = x^3 - 9x$

1. The leading term is x^3. The end behavior is down to the left and up to the right.

   The exponent on the leading term is odd and the leading coefficient is positive.

2. $f(0) = (0)^3 - 9(0) = 0$

   The y-intercept is $(0, 0)$.

   Determine the y-intercept by evaluating $f(0)$.

3. $0 = x^3 - 9x$

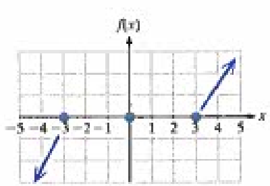
   $0 = x(x^2 - 9)$

   $0 = x(x - 3)(x + 3)$

   The zeros of the function are 0, 3, and -3, and each has a multiplicity of 1.

   Find the real zeros of f by solving for the real solutions to the equation $f(x) = 0$.

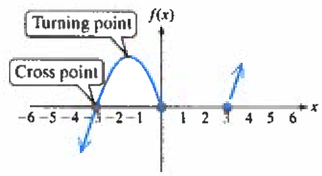
   The zeros are real numbers and correspond to x-intercepts on the graph. Since the multiplicity of each zero is an odd number, the graph will cross the x-axis at the zeros.

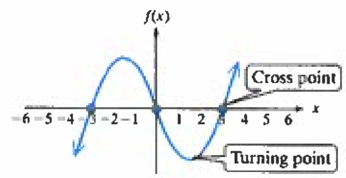
4. 

   Plot the x- and y-intercepts and sketch the end behavior.

5. Moving from left to right, the curve increases from the far left and then crosses the x-axis at -3. The graph must have a turning point between $x = -3$ and $x = 0$ so that the curve can pass through the next x-intercept of $(0, 0)$.

The graph crosses the x-axis at $x = 0$. The graph must then have another turning point between $x = 0$ and $x = 3$ so that the curve can pass through the next x-intercept of $(3, 0)$. Finally, the graph crosses the x-axis at $x = 3$ and continues to increase to the far right.





6. $f(x) = x^3 - 9x$

$$f(-x) = (-x)^3 - 9(-x) \qquad -f(x) = -(x^3 - 9x)$$
$$= -x^3 + 9x \xleftrightarrow[\text{(same)}]{f(-x) = -f(x)} = -x^3 + 9x$$

Testing for symmetry, we see that $f(-x) = -f(x)$. Therefore, f is an odd function and is symmetric with respect to the origin.

TIP Techniques of calculus can be used to find the exact coordinates of the turning points of the polynomial function in Example 6.

7. If more accuracy is desired, plot additional points. In this case, since f is symmetric to the origin, if a point (x, y) is on the graph, then so is $(-x, -y)$. The graph of f is shown in Figure 2-13.

x	$f(x)$
1	-8
2	-10
4	28

Use symmetry.

x	$f(x)$
-1	8
-2	10
-4	-28

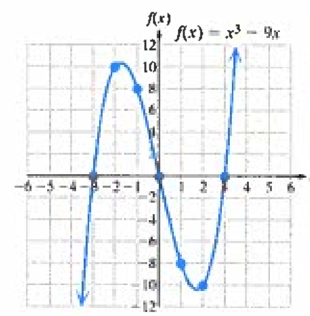


Figure 2-13

Skill Practice 6 Graph $g(x) = -x^3 + 4x$.

EXAMPLE 7 Graphing a Polynomial Function

Graph $g(x) = -0.1(x - 1)(x + 2)(x - 4)^2$.

Solution:

$g(x) = -0.1(x - 1)(x + 2)(x - 4)^2$

1. Multiplying the leading terms within the factors, we have a leading term of $-0.1(x)(x)(x)^2 = -0.1x^4$. The end behavior is down to the left and down to the right.

The exponent on the leading term is even and the leading coefficient is negative.

2. $g(0) = -0.1(0 - 1)(0 + 2)(0 - 4)^2 = 3.2$
The y-intercept is $(0, 3.2)$.

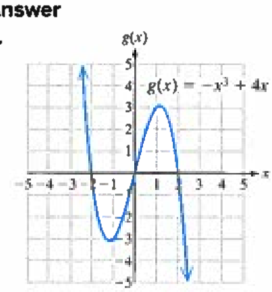
Determine the y-intercept by evaluating $g(0)$.

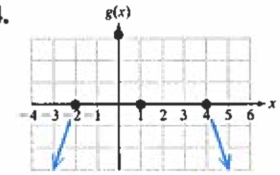
3. $0 = -0.1(x - 1)(x + 2)(x - 4)^2$
The zeros of the function are 1, -2, and 4.
The multiplicity of 1 is 1.
The multiplicity of -2 is 1.
The multiplicity of 4 is 2.

Find the real zeros of g by solving for the real solutions of the equation $g(x) = 0$.
The zeros are real numbers and correspond to x-intercepts on the graph: $(1, 0)$, $(-2, 0)$, and $(4, 0)$.

Answer

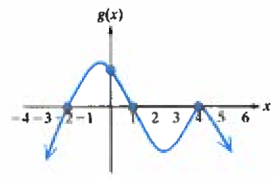
6.



4. 

Plot the x- and y-intercepts and sketch the end behavior.

5. Moving from left to right, the curve increases from the far left. It then crosses the x-axis at $x = -2$ and turns back around to pass through the next x-intercept at $x = 1$.



The curve has another turning point between $x = 1$ and $x = 4$ so that it can touch the x-axis at 4. From there it turns back downward and continues to decrease to the far right.

6. From our preliminary sketch in step 5, we see that the function is not symmetric with respect to either the y-axis or origin.

7. If more accuracy is desired, plot additional points. The graph is shown in Figure 2-14.

x	$g(x)$
-3	-19.6
-1	5
2	-1.6
3	-1
5	-2.8

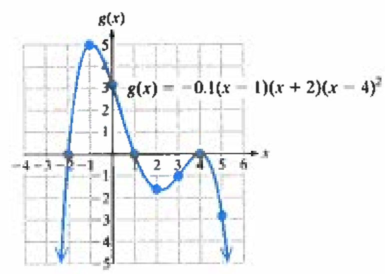


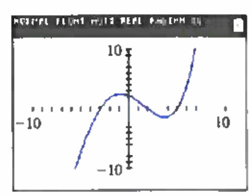
Figure 2-14

Skill Practice 7 Graph $h(x) = 0.5x(x - 1)(x + 3)^2$.

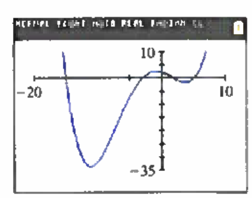
TECHNOLOGY CONNECTIONS

Using a Graphing Utility to Graph a Polynomial Function

It is important to have a strong knowledge of algebra to use a graphing utility effectively. For example, consider the graph of $f(x) = 0.005(x - 2)(x + 3)(x - 5)(x + 15)$ on the standard viewing window.



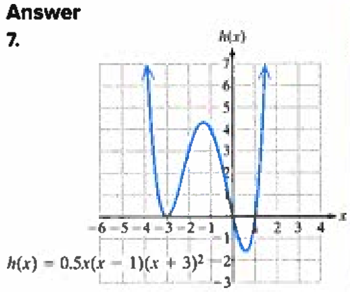
From the leading term, $0.005x^4$, we know that the end behavior should be up to the left and up to the right. Furthermore, the function has four real zeros (2, -3, 5, and -15), and should have four corresponding x-intercepts. Therefore, on the standard viewing window, the calculator does not show the key features of the graph.



By graphing f on the window $[-20, 10, 2]$ by $[-35, 10, 5]$, we see the end behavior displayed correctly, all four x-intercepts, and the turning points (there should be at most 3).

Answer

7.



SECTION 2.2 Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, solve the equation.

R.1. $3x^3 + 21x^2 - 54x = 0$

R.2. $5x^3 + 6x^2 - 20x - 24 = 0$

For Exercises R.3–R.5, use transformations to graph the given function.

R.3. $m(x) = x^3 - 5$

R.4. $f(x) = (x + 2)^2 - 4$

R.5. $g(x) = (3x - 6)^2$

Concept Connections

1. A function defined by $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ where $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ are real numbers and $a_n \neq 0$ is called a ______________ function.
2. The function given by $f(x) = -3x^5 + \sqrt{2}x + \frac{1}{2}x$ (is/is not) a polynomial function.
3. The function given by $f(x) = -3x^5 + 2\sqrt{x} + \frac{2}{x}$ (is/is not) a polynomial function.
4. A quadratic function is a polynomial function of degree ______________.
5. A linear function is a polynomial function of degree ______________.
6. The values of x in the domain of a polynomial function f for which $f(x) = 0$ are called the ______________ of the function.
7. What is the maximum number of turning points of the graph of $f(x) = -3x^6 - 4x^5 - 5x^4 + 2x^2 + 6$?
8. If the graph of a polynomial function has 3 turning points, what is the minimum degree of the function?
9. If c is a real zero of a polynomial function and the multiplicity is 3, does the graph of the function cross the x-axis or touch the x-axis (without crossing) at $(c, 0)$?
10. If c is a real zero of a polynomial function and the multiplicity is 6, does the graph of the function cross the x-axis or touch the x-axis (without crossing) at $(c, 0)$?
11. Suppose that f is a polynomial function and that $a < b$. If $f(a)$ and $f(b)$ have opposite signs, then what conclusion can be drawn from the intermediate value theorem?
12. What is the leading term of $f(x) = -\frac{1}{3}(x - 3)^4(3x + 5)^2$?

Objective 1: Determine the End Behavior of a Polynomial Function

For Exercises 13–20, determine the end behavior of the graph of the function. (See Example 1)

13. $f(x) = -3x^4 - 5x^2 + 2x - 6$

14. $g(x) = -\frac{1}{2}x^6 + 8x^4 - x^3 + 9$

15. $h(x) = 12x^5 + 8x^4 - 4x^3 - 8x + 1$

16. $k(x) = 11x^7 - 4x^2 + 9x + 3$

17. $m(x) = -4(x - 2)(2x + 1)^2(x + 6)^4$

18. $n(x) = -2(x + 4)(3x - 1)^3(x + 5)$

19. $p(x) = -2x^2(3 - x)(2x - 3)^3$

20. $q(x) = -5x^4(2 - x)^3(2x + 5)$

Objective 2: Identify Zeros and Multiplicities of Zeros

21. Given the function defined by $g(x) = -3(x - 1)^3(x + 5)^4$, the value 1 is a zero with multiplicity ______________, and the value -5 is a zero with multiplicity ______________.
22. Given the function defined by $h(x) = \frac{1}{2}x^5(x + 0.6)^3$, the value 0 is a zero with multiplicity ______________, and the value -0.6 is a zero with multiplicity ______________.

For Exercises 23–38, find the zeros of the function and state the multiplicities. (See Examples 2–4)

23. $f(x) = x^3 + 2x^2 - 25x - 50$

24. $g(x) = x^3 + 5x^2 - x - 5$

25. $h(x) = -6x^3 - 9x^2 + 60x$

26. $k(x) = -6x^3 + 26x^2 - 28x$

27. $m(x) = x^5 - 10x^4 + 25x^3$

28. $n(x) = x^6 + 4x^5 + 4x^4$

29. $p(x) = -3x(x + 2)^3(x + 4)$

30. $q(x) = -2x^4(x + 1)^3(x - 2)^2$

31. $t(x) = 5x(3x - 5)(2x + 9)(x - \sqrt{3})(x + \sqrt{3})$

32. $z(x) = 4x(5x - 1)(3x + 8)(x - \sqrt{5})(x + \sqrt{5})$

33. $c(x) = [x - (3 - \sqrt{5})][x - (3 + \sqrt{5})]$

34. $d(x) = [x - (2 - \sqrt{11})][x - (2 + \sqrt{11})]$

35. $f(x) = 4x^4 - 37x^2 + 9$

36. $k(x) = 4x^4 - 65x^2 + 16$

37. $n(x) = x^6 - 7x^4$

38. $m(x) = x^5 - 5x^3$

Objective 3: Apply the Intermediate Value Theorem

For Exercises 39–40, determine whether the intermediate value theorem guarantees that the function has a zero on the given interval. (See Example 5)

39. $f(x) = 2x^3 - 7x^2 - 14x + 30$

a. $[1, 2]$ **b.** $[2, 3]$ **c.** $[3, 4]$ **d.** $[4, 5]$

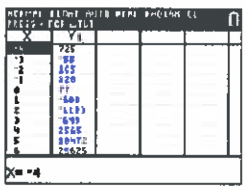
40. $g(x) = 2x^3 - 13x^2 + 18x + 5$

a. $[1, 2]$ **b.** $[2, 3]$ **c.** $[3, 4]$ **d.** $[4, 5]$

For Exercises 41–42, a table of values is given for $Y_1 = f(x)$. Determine whether the intermediate value theorem guarantees that the function has a zero on the given interval.

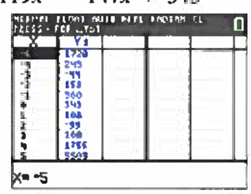
41. $Y_1 = 21x^4 + 46x^3 - 238x^2 - 506x + 77$

a. $[-4, -3]$ **b.** $[-3, -2]$ **c.** $[-2, -1]$ **d.** $[-1, 0]$



42. $Y_1 = 10x^4 + 21x^3 - 119x^2 - 147x + 343$

a. $[-4, -3]$ **b.** $[-3, -2]$ **c.** $[-2, -1]$ **d.** $[-1, 0]$



43. Given $f(x) = 4x^3 - 8x^2 - 25x + 50$,

a. Determine if f has a zero on the interval $[-3, -2]$.

b. Find a zero of f on the interval $[-3, -2]$.

44. Given $f(x) = 9x^3 - 18x^2 - 100x + 200$,

a. Determine if f has a zero on the interval $[-4, -3]$.

b. Find a zero of f on the interval $[-4, -3]$.

Objective 4: Sketch a Polynomial Function

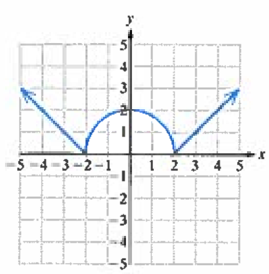
For Exercises 45–52, determine if the graph can represent a polynomial function. If so, assume that the end behavior and all turning points are represented in the graph.

a. Determine the minimum degree of the polynomial.

b. Determine whether the leading coefficient is positive or negative based on the end behavior and whether the degree of the polynomial is odd or even.

c. Approximate the real zeros of the function, and determine if their multiplicities are even or odd.

45.



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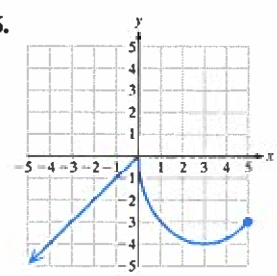

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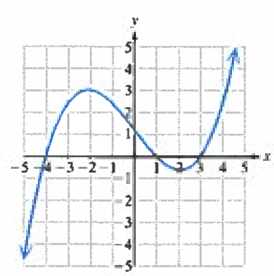

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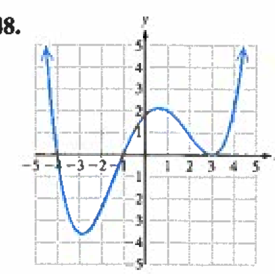

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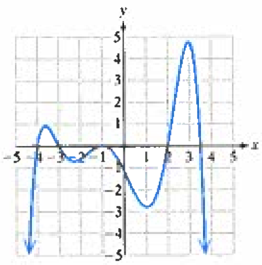

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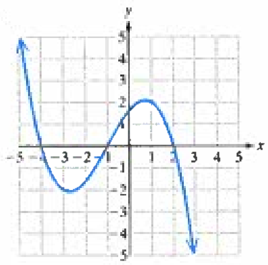

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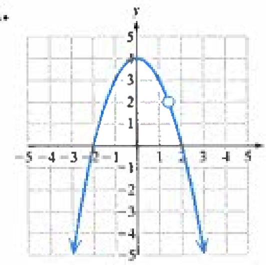

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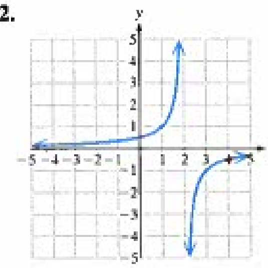


For Exercises 53–58,

a. Identify the power function of the form $y = x^n$ that is the parent function to the given graph.

b. In order, outline the transformations that would be required on the graph of $y = x^n$ to make the graph of the given function. See Section 1.6, page 190.

c. Match the function with the graph of i–vi.

53. $g(x) = -\frac{1}{3}x^6 - 2$

54. $f(x) = -\frac{1}{2}(x - 3)^4$

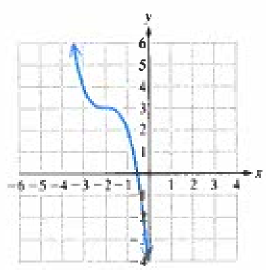
55. $k(x) = -(x + 2)^3 + 3$

56. $p(x) = 2(x + 4)^3 - 3$

57. $m(x) = (-x - 3)^5 + 1$

58. $n(x) = (-x + 3)^4 - 1$

i.



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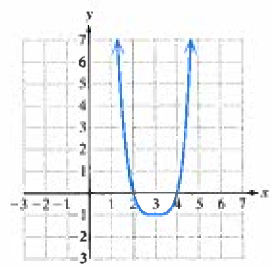

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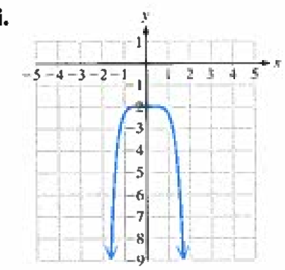

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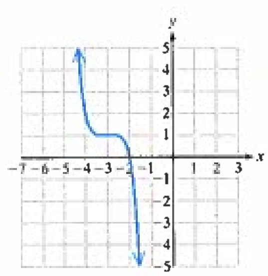

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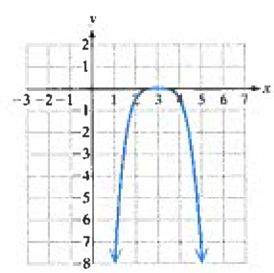

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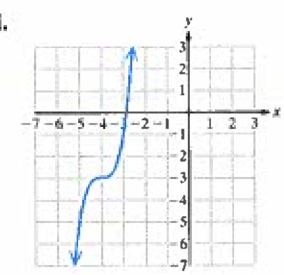


For Exercises 59–76, sketch the function. (See Examples 6–7)

59. $f(x) = x^3 - 5x^2$

60. $g(x) = x^5 - 2x^4$

61. $f(x) = \frac{1}{2}(x - 2)(x + 1)(x + 3)$

62. $h(x) = \frac{1}{4}(x - 1)(x - 4)(x + 2)$

63. $k(x) = x^4 + 2x^3 - 8x^2$

64. $h(x) = x^4 - x^3 - 6x^2$

65. $k(x) = 0.2(x + 2)^2(x - 4)^3$

66. $m(x) = 0.1(x - 3)^2(x + 1)^3$

67. $p(x) = 9x^5 + 9x^4 - 25x^3 - 25x^2$

68. $q(x) = 9x^5 + 18x^4 - 4x^3 - 8x^2$

69. $t(x) = -x^4 + 11x^2 - 28$

70. $v(x) = -x^4 + 15x^2 - 44$

71. $g(x) = -x^4 + 5x^2 - 4$

72. $h(x) = -x^4 + 10x^2 - 9$

73. $c(x) = 0.1x(x - 2)^4(x + 2)^3$

74. $d(x) = 0.05x(x - 2)^4(x + 3)^2$

75. $m(x) = -\frac{1}{10}(x + 3)(x - 3)(x + 1)^3$

76. $f(x) = -\frac{1}{10}(x - 1)(x + 3)(x - 4)^2$

Mixed Exercises

For Exercises 77–88, determine if the statement is true or false. If a statement is false, explain why.

77. The function defined by $f(x) = (x + 1)^5(x - 5)^2$ crosses the x-axis at 5.

78. The function defined by $g(x) = -3(x + 4)(2x - 3)^4$ touches but does not cross the x-axis at $\left(\frac{3}{2}, 0\right)$.

79. A third-degree polynomial has three turning points.

80. A third-degree polynomial has two turning points.

81. There is more than one polynomial function with zeros of 1, 2, and 6.

82. There is exactly one polynomial with integer coefficients with zeros of 2, 4, and 6.

83. The graph of a polynomial function with leading term of even degree is up to the far left and up to the far right.

84. If c is a real zero of an even polynomial function, then $-c$ is also a zero of the function.

85. The graph of $f(x) = x^3 - 27$ has three x-intercepts.

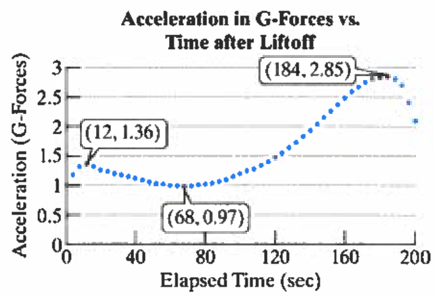
86. The graph of $f(x) = 3x^2(x - 4)^4$ has no points in Quadrants III or IV.

87. The graph of $p(x) = -5x^4(x + 1)^2$ has no points in Quadrants I or II.

88. A fourth-degree polynomial has exactly two relative minima and two relative maxima.

89. A rocket will carry a communications satellite into low Earth orbit. Suppose that the thrust during the first 200 sec of flight is provided by solid rocket boosters at different points during liftoff.

The graph shows the acceleration in G-forces (that is, acceleration in 9.8-m/sec^2 increments) versus time after launch.



a. Approximate the interval(s) over which the acceleration is increasing.

b. Approximate the interval(s) over which the acceleration is decreasing.

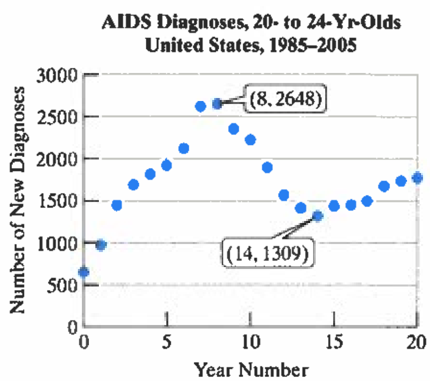
c. How many turning points does the graph show?

d. Based on the number of turning points, what is the minimum degree of a polynomial function that could be used to model acceleration versus time? Would the leading coefficient be positive or negative?

e. Approximate the time when the acceleration was the greatest.

f. Approximate the value of the maximum acceleration.

90. Data from a 20-yr study show the number of new AIDS cases diagnosed among 20- to 24-yr-olds in the United States x years after the study began.



a. Approximate the interval(s) over which the number of new AIDS cases among 20- to 24-yr-olds increased.

b. Approximate the interval(s) over which the number of new AIDS cases among 20- to 24-yr-olds decreased.

c. How many turning points does the graph show?

d. Based on the number of turning points, what is the minimum degree of a polynomial function that could be used to model the data? Would the leading coefficient be positive or negative?

e. How many years after the study began was the number of new AIDS cases among 20- to 24-yr-olds the greatest?

f. What was the maximum number of new cases diagnosed in a single year?

Write About It

91. Given a polynomial function defined by $y = f(x)$, explain how to find the x-intercepts.

92. Given a polynomial function, explain how to determine whether an x-intercept is a touch point or a cross point.

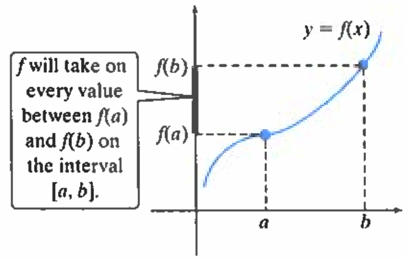
93. Write an informal explanation of what it means for a function to be continuous.

94. Write an informal explanation of the intermediate value theorem.

Expanding Your Skills

The intermediate value theorem given on page 260 is actually a special case of a broader statement of the theorem. Consider the following:

Let f be a polynomial function. For $a < b$, if $f(a) \neq f(b)$, then f takes on every value between $f(a)$ and $f(b)$ on the interval $[a, b]$.



Use this broader statement of the intermediate value theorem for Exercises 95–96.

95. Given $f(x) = x^2 - 3x + 2$,

a. Evaluate $f(3)$ and $f(4)$.

b. Use the intermediate value theorem to show that there exists at least one value of x for which $f(x) = 4$ on the interval $[3, 4]$.

c. Find the value(s) of x for which $f(x) = 4$ on the interval $[3, 4]$.

96. Given $f(x) = -x^2 - 4x + 3$,

a. Evaluate $f(-4)$ and $f(-3)$.

b. Use the intermediate value theorem to show that there exists at least one value of x for which $f(x) = 5$ on the interval $[-4, -3]$.

c. Find the value(s) of x for which $f(x) = 5$ on the interval $[-4, -3]$.

Technology Connections

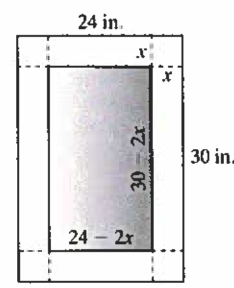
97. For a certain individual, the volume (in liters) of air in the lungs during a 4.5-sec respiratory cycle is shown in the table for 0.5-sec intervals. Graph the points and then find a third-degree polynomial function to model the volume $V(t)$ for t between 0 sec and 4.5 sec. (*Hint*: Use a CubicReg option or polynomial degree 3 option on a graphing utility.)

Time (sec)	Volume (L)
0.0	0.00
0.5	0.11
1.0	0.29
1.5	0.47
2.0	0.63
2.5	0.76
3.0	0.81
3.5	0.75
4.0	0.56
4.5	0.20

98. The torque (in ft-lb) produced by a certain automobile engine turning at x thousand revolutions per minute is shown in the table. Graph the points and then find a third-degree polynomial function to model the torque $T(x)$ for $1 \leq x \leq 5$.

Engine speed (1000 rpm)	Torque (ft-lb)
1.0	165
1.5	180
2.0	188
2.5	190
3.0	186
3.5	176
4.0	161
4.5	142
5.0	120

99. A solar oven is to be made from an open box with reflective sides. Each box is made from a 30-in. by 24-in. rectangular sheet of aluminum with squares of length x (in inches) removed from each corner. Then the flaps are folded up to form an open box.



a. Show that the volume of the box is given by

$$V(x) = 4x^3 - 108x^2 + 720x \quad \text{for } 0 < x < 12.$$

b. Graph the function from part (a) and use a "Maximum" feature on a graphing utility to approximate the length of the sides of the squares that should be removed to maximize the volume. Round to the nearest tenth of an inch.

c. Approximate the maximum volume. Round to the nearest cubic inch.

For Exercises 100–101, two viewing windows are given for the graph of $y = f(x)$. Choose the window that best shows the key features of the graph.

100. $f(x) = 2(x - 0.5)(x - 0.1)(x + 0.2)$

a. $[-10, 10, 1]$ by $[-10, 10, 1]$

b. $[-1, 1, 0.1]$ by $[-0.05, 0.05, 0.01]$

101. $g(x) = 0.08(x - 16)(x + 2)(x - 3)$

a. $[-10, 10, 1]$ by $[-10, 10, 1]$

b. $[-5, 20, 5]$ by $[-50, 30, 10]$

For Exercises 102–103, graph the function defined by $y = f(x)$ on an appropriate viewing window.

102. $k(x) = \dfrac{1}{100}(x - 20)(x + 1)(x + 8)(x - 6)$

103. $p(x) = (x - 0.4)(x + 0.5)(x + 0.1)(x - 0.8)$

SECTION 2.3 Division of Polynomials and the Remainder and Factor Theorems

OBJECTIVES

1. Divide Polynomials Using Long Division
2. Divide Polynomials Using Synthetic Division
3. Apply the Remainder and Factor Theorems

1. Divide Polynomials Using Long Division

In this section, we use the notation $f(x)$, $g(x)$, and so on to represent polynomials in x. We also present two types of polynomial division: long division and synthetic division. Polynomial division can be used to factor a polynomial, solve a polynomial equation, and find the zeros of a polynomial.

When dividing polynomials, if the divisor has two or more terms we can use a long division process similar to the division of real numbers. This is demonstrated in Examples 1–3.

TIP Take a minute to review long division of whole numbers: 2273 ÷ 5

$$\begin{array}{r} 454 \leftarrow \text{Quotient} \\ 5\overline{)2273} \\ \underline{-20} \\ 27 \\ \underline{-25} \\ 23 \\ \underline{-20} \\ 3 \leftarrow \text{Remainder} \end{array}$$

Answer: $454 + \frac{3}{5}$ or $454\frac{3}{5}$

EXAMPLE 1 Dividing Polynomials Using Long Division

Use long division to divide. $(6x^3 - 5x^2 - 3) \div (3x + 2)$

Solution:

First note that the dividend can be written as $6x^3 - 5x^2 + 0x - 3$. The term $0x$ is used as a place holder for the missing power of x. The place holder is helpful to keep the powers of x lined up. We also set up long division with both the dividend and divisor written in descending order.

$$3x + 2\overline{)6x^3 - 5x^2 + 0x - 3}$$

Divide the leading term in the dividend by the leading term in the divisor.

$\boxed{\frac{6x^3}{3x} = 2x^2}$ This is the first term in the quotient.

$$\begin{array}{r} 2x^2 \\ 3x + 2\overline{)6x^3 - 5x^2 + 0x - 3} \\ -(6x^3 + 4x^2) \end{array}$$

$\boxed{\text{Subtract.}}$

Multiply the divisor by $2x^2$: $2x^2(3x + 2) = 6x^3 + 4x^2$, and subtract the result.

$$\begin{array}{r} 2x^2 \\ 3x + 2\overline{)6x^3 - 5x^2 + 0x - 3} \\ \underline{-(6x^3 + 4x^2)} \\ -9x^2 + 0x \end{array}$$

$\boxed{\text{Bring down the next term from the dividend and repeat the process.}}$

$$\begin{array}{r} 2x^2 - 3x \\ 3x + 2\overline{)6x^3 - 5x^2 + 0x - 3} \\ \underline{-(6x^3 + 4x^2)} \\ -9x^2 + 0x \\ -(-9x^2 - 6x) \end{array}$$

Divide $-9x^2$ by the first term in the divisor.

$\boxed{\frac{-9x^2}{3x} = -3x}$

$\boxed{\text{Subtract.}}$

Multiply the divisor by $-3x$: $-3x(3x + 2) = -9x^2 - 6x$, and subtract the result.

$$\begin{array}{r} 2x^2 - 3x + 2 \\ 3x + 2\overline{)6x^3 - 5x^2 + 0x - 3} \\ \underline{-(6x^3 + 4x^2)} \\ -9x^2 + 0x \\ \underline{-(-9x^2 - 6x)} \\ 6x - 3 \\ \underline{-(6x + 4)} \\ -7 \end{array}$$

$\boxed{\text{Bring down the next term from the dividend and repeat the process.}}$

Divide $6x$ by the first term in the divisor. $\frac{6x}{3x} = 2$. This is the next term in the quotient.

Multiply the divisor by 2: $2(3x + 2) = 6x + 4$, and subtract the result.

$\boxed{\text{The remainder is } -7.}$

Long division is complete when the remainder is either zero or has degree less than the degree of the divisor.

The quotient is $2x^2 - 3x + 2$.
The remainder is -7.
The divisor is $3x + 2$.
The dividend is $6x^3 - 5x^2 - 3$.

The result of a long division problem is usually written as the quotient plus the remainder divided by the divisor.

$$\frac{\boxed{\text{Dividend}}\; 6x^3 - 5x^2 - 3}{\boxed{\text{Divisor}}\; 3x + 2} = \overbrace{2x^2 - 3x + 2}^{\boxed{\text{Quotient}}} + \frac{-7 \;\boxed{\text{Remainder}}}{3x + 2 \;\boxed{\text{Divisor}}}$$

Skill Practice 1 Use long division to divide $(4x^3 - 23x + 3) \div (2x - 5)$.

By clearing fractions, the result of Example 1 can be checked by multiplication.

$$\begin{aligned} \text{Dividend} &= (\text{Divisor})(\text{Quotient}) \qquad + \text{Remainder} \\ 6x^3 - 5x^2 - 3 &\stackrel{?}{=} (3x + 2)(2x^2 - 3x + 2) + (-7) \\ &\stackrel{?}{=} 6x^3 - 5x^2 + 4 + (-7) \\ &\stackrel{?}{=} 6x^3 - 5x^2 - 3 \checkmark \end{aligned}$$

This result illustrates the division algorithm.

Division Algorithm

Suppose that $f(x)$ and $d(x)$ are polynomials where $d(x) \neq 0$ and the degree of $d(x)$ is less than or equal to the degree of $f(x)$. Then there exists unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x) \cdot q(x) + r(x)$$

where either the degree of $r(x)$ is less than $d(x)$, or $r(x)$ is the zero polynomial.

Note: The polynomial $f(x)$ is the **dividend,** $d(x)$ is the **divisor,** $q(x)$ is the **quotient,** and $r(x)$ is the **remainder.**

EXAMPLE 2 Dividing Polynomials Using Long Division

Use long division to divide $(-5 + x + 4x^2 + 2x^3 + 3x^4) \div (x^2 + 2)$.

Solution:

Write the dividend and divisor in descending order and insert place holders for missing powers of x. $\quad (3x^4 + 2x^3 + 4x^2 + x - 5) \div (x^2 + 0x + 2)$

$$\begin{array}{r} 3x^2 + 2x - 2 \\ x^2 + 0x + 2 \overline{\smash{\big)}\, 3x^4 + 2x^3 + 4x^2 + x - 5} \\ \underline{-(3x^4 + 0x^3 + 6x^2)} \qquad\qquad \\ 2x^3 - 2x^2 + x \qquad \\ \underline{-(2x^3 + 0x^2 + 4x)} \qquad \\ -2x^2 - 3x - 5 \\ \underline{-(-2x^2 + 0x - 4)} \\ -3x - 1 \end{array}$$

To begin, divide the leading term in the dividend by the leading term in the divisor. $\dfrac{3x^4}{x^2} = 3x^2$

Multiply the divisor by $3x^2$ and subtract the result.

Bring down the next term from the dividend and repeat the process.

The process is complete when the remainder is either 0 or has degree less than the degree of the divisor.

The result is $3x^2 + 2x - 2 + \dfrac{-3x - 1}{x^2 + 2}$.

Answer

1. $2x^2 + 5x + 1 + \dfrac{8}{2x - 5}$

Check by using the division algorithm.

$$\begin{aligned}3x^4 + 2x^3 + 4x^2 + x - 5 &\stackrel{?}{=} (x^2 + 2)(3x^2 + 2x - 2) + (-3x - 1)\\ &\stackrel{?}{=} 3x^4 + 2x^3 - 2x^2 + 6x^2 + 4x - 4 + (-3x - 1)\\ &\stackrel{?}{=} 3x^4 + 2x^3 + 4x^2 + x - 5 \checkmark\end{aligned}$$

Skill Practice 2 Use long division to divide.
$(1 - 7x + 5x^2 - 3x^3 + 2x^4) \div (x^2 + 3)$

In Example 3, we discuss the implications of obtaining a remainder of zero when performing division of polynomials.

EXAMPLE 3 **Dividing Polynomials Using Long Division**

Use long division to divide. $\dfrac{2x^2 + 3x - 14}{x - 2}$

Solution:

$$\begin{array}{r l} & 2x + 7 \\ x - 2 & \overline{)\,2x^2 + 3x - 14} \\ & \underline{-(2x^2 - 4x)} \\ & 7x - 14 \\ & \underline{-(7x - 14)} \\ & 0 \end{array}$$

To begin, divide the leading term in the dividend by the leading term in the divisor.
$\dfrac{2x^2}{x} = 2x$

Multiply the divisor by $2x$ and subtract the result.

Bring down the next term from the dividend and repeat the process.

The process is complete when the remainder is either 0 or has degree less than the degree of the divisor.

$$\frac{2x^2 + 3x - 14}{x - 2} = 2x + 7$$

The remainder is zero. This implies that the divisor divides evenly into the dividend. Therefore, both the divisor and quotient are factors of the dividend. This is easily verified by the division algorithm.

$$\begin{aligned}\underbrace{2x^2 + 3x - 14}_{\text{Dividend}} &\stackrel{?}{=} \underbrace{(x - 2)}_{\text{Divisor}}\underbrace{(2x + 7)}_{\text{Quotient}} + \underbrace{0}_{\text{Remainder}}\\ &\stackrel{?}{=} \underbrace{(x - 2)(2x + 7)}_{\text{Factored form of } 2x^2 + 3x - 14}\end{aligned}$$

Skill Practice 3 Use long division to divide.
$(3x^2 - 14x + 15) \div (x - 3)$

2. Divide Polynomials Using Synthetic Division

When dividing polynomials where the divisor is a binomial of the form $(x - c)$ and c is a constant, we can use synthetic division. Synthetic division enables us to find the quotient and remainder more quickly than long division. It uses an algorithm that manipulates the coefficients of the dividend, divisor, and quotient without the accompanying variable factors.

The division of polynomials from Example 3 is shown at the top left of page 273. The equivalent synthetic division is shown on the right. Notice that the same coefficients are used in both cases.

Answers

2. $2x^2 - 3x - 1 + \dfrac{2x + 4}{x^2 + 3}$

3. $3x - 5$

$$\begin{array}{r}
2x + 7 \\
x - 2\overline{\big)\,2x^2 + 3x - 14} \\
\underline{-(2x^2 - 4x)} \\
7x - 14 \\
\underline{-(7x - 14)} \\
0
\end{array}
\qquad\qquad
\begin{array}{r|rrr}
2 & 2 & 3 & -14 \\
 & & 4 & 14 \\
\hline
 & 2 & 7 & \boxed{0}
\end{array}$$

Coefficients of dividend

Remainder

Coefficients of quotient

In Example 4, we demonstrate the process to divide polynomials by synthetic division.

EXAMPLE 4 Dividing Polynomials Using Synthetic Division

Use synthetic division to divide. $(-10x^2 + 2x^3 - 5) \div (x - 4)$

Solution:

As with long division, the terms of the dividend and divisor must be written in descending order with place holders for missing powers of x.

$$(2x^3 - 10x^2 + 0x - 5) \div (x - 4)$$

To use synthetic division, the divisor must be in the form $x - c$. In this case, $c = 4$.

Step 1: Write the value of c in a box.

Step 2: Write the coefficients of the dividend to the right of the box.

Step 3: Skip a line and draw a horizontal line below the list of coefficients.

Step 4: Bring down the leading coefficient from the dividend and write it below the line.

$$\begin{array}{r|rrrr}
4 & 2 & -10 & 0 & -5 \\
 & & & & \\
\hline
 & 2 & & &
\end{array}$$

Step 5: Multiply the value of c by the number below the line ($4 \times 2 = 8$). Write the result in the next column above the line.

Step 6: Add the numbers in the column above the line ($-10 + 8 = -2$), and write the result below the line.

$$\begin{array}{r|rrrr}
4 & 2 & -10 & 0 & -5 \\
 & & 8 & & \\
\hline
 & 2 & -2 & &
\end{array}$$

Repeat steps 5 and 6 until all columns have been completed.

$$\begin{array}{r|rrrr}
4 & 2 & -10 & 0 & -5 \\
 & & 8 & -8 & -32 \\
\hline
 & 2 & -2 & -8 & \boxed{-37} \\
 & x^2 & x & \text{constant} &
\end{array}$$

A box is often drawn around the remainder.

The rightmost number below the line is the remainder. The other numbers below the line are the coefficients of the quotient in order by the degree of the term.

Since the divisor is linear (first degree), the degree of the quotient is 1 less than the degree of the dividend. In this case, the dividend is of degree 3. Therefore, the quotient will be of degree 2.

The quotient is $2x^2 - 2x - 8$ and the remainder is -37. Therefore,

$$\frac{2x^3 - 10x^2 - 5}{x - 4} = 2x^2 - 2x - 8 + \frac{-37}{x - 4}$$

Avoiding Mistakes

It is important to check that the divisor is in the form $(x - c)$ before applying synthetic division. The variable x in the divisor must be of first degree, and its coefficient must be 1.

Skill Practice 4 Use synthetic division to divide.

$(4x^3 - 28x - 7) \div (x - 3)$

Answer

4. $4x^2 + 12x + 8 + \dfrac{17}{x - 3}$

TIP Given a divisor of the form $(x - c)$, we can determine the value of c by setting the divisor equal to zero and solving for x. In Example 5, we have $x + 2 = 0$, which implies that $x = -2$. The value of c is -2.

EXAMPLE 5 Dividing Polynomials Using Synthetic Division

Use synthetic division to divide. $(-2x + 4x^3 + 18 + x^4) \div (x + 2)$

Solution:

Write the dividend and divisor in descending order and insert place holders for missing powers of x. $(x^4 + 4x^3 + 0x^2 - 2x + 18) \div (x + 2)$

To use synthetic division, the divisor must be of the form $x - c$. In this case, we have $x + 2 = x - (-2)$. Therefore, $c = -2$.

$$\begin{array}{r|rrrrr} -2 & 1 & 4 & 0 & -2 & 18 \\ & \downarrow & -2 & -4 & 8 & -12 \\ \hline & 1 & 2 & -4 & 6 & \boxed{6} \end{array}$$

Value of c

Coefficients from the dividend

Draw a horizontal line. Bring down the first coefficient.

Remainder

Coefficients of quotient

The dividend is a fourth-degree polynomial and the divisor is a first-degree polynomial. Therefore, the quotient is a third-degree polynomial. The coefficients of the quotient are found below the line: 1, 2, -4, 6. The quotient is $x^3 + 2x^2 - 4x + 6$, and the remainder is 6.

$$\frac{x^4 + 4x^3 - 2x + 18}{x + 2} = x^3 + 2x^2 - 4x + 6 + \frac{6}{x + 2}$$

Skill Practice 5 Use synthetic division to divide.
$(-3x + 7x^3 + 5 + 2x^4) \div (x + 1)$

3. Apply the Remainder and Factor Theorems

Consider the special case of the division algorithm where $f(x)$ is the dividend and $(x - c)$ is the divisor.

$$f(x) = (x - c) \cdot q(x) + r$$

The remainder r is constant because its degree must be one less than the degree of $x - c$.

Now evaluate $f(c)$:

$$f(c) = (c - c) \cdot q(c) + r$$
$$f(c) = 0 \cdot q(c) + r$$
$$f(c) = r$$

This result is stated formally as the remainder theorem.

Remainder Theorem

If a polynomial $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

Note: The remainder theorem tells us that the value of $f(c)$ is the same as the remainder we get from dividing $f(x)$ by $x - c$.

Answer

5. $2x^3 + 5x^2 - 5x + 2 + \dfrac{3}{x + 1}$

EXAMPLE 6 Using the Remainder Theorem to Evaluate a Polynomial

Given $f(x) = x^4 + 6x^3 - 12x^2 - 30x + 35$, use the remainder theorem to evaluate

a. $f(2)$ **b.** $f(-7)$

Solution:

a. If $f(x)$ is divided by $x - 2$, then the remainder is $f(2)$.

$$\begin{array}{r|rrrrr} 2 & 1 & 6 & -12 & -30 & 35 \\ & & 2 & 16 & 8 & -44 \\ \hline & 1 & 8 & 4 & -22 & \boxed{-9} \end{array}$$

By the remainder theorem, $f(2) = -9$.

b. If $f(x)$ is divided by $x - (-7)$ or equivalently $x + 7$, then the remainder is $f(-7)$.

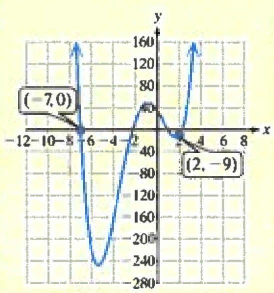
$$\begin{array}{r|rrrrr} -7 & 1 & 6 & -12 & -30 & 35 \\ & & -7 & 7 & 35 & -35 \\ \hline & 1 & -1 & -5 & 5 & \boxed{0} \end{array}$$

By the remainder theorem, $f(-7) = 0$.

The results can be checked by direct substitution.

$$f(2) = (2)^4 + 6(2)^3 - 12(2)^2 - 30(2) + 35 = -9 \checkmark$$
$$f(-7) = (-7)^4 + 6(-7)^3 - 12(-7)^2 - 30(-7) + 35 = 0 \checkmark$$

TIP From Example 6, the values $f(2) = -9$ and $f(-7) = 0$, imply that $(2, -9)$ and $(-7, 0)$ are on the graph of $y = f(x)$.



Skill Practice 6 Given $f(x) = x^4 + x^3 - 6x^2 - 5x - 15$, use the remainder theorem to evaluate

a. $f(5)$ **b.** $f(-3)$

The division algorithm and remainder theorem can be extended over the set of complex numbers. The definition of a polynomial was given in Section R.3.

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$ and the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_0$ are real numbers. We now extend our discussion to **complex polynomials.** These are polynomials with complex coefficients.

We will also evaluate polynomials over the set of complex numbers rather than restricting x to the set of real numbers. A complex number $a + bi$ is a zero of a polynomial $f(x)$ if $f(a + bi) = 0$. For example, given $f(x) = x - (5 + 2i)$, we see that the imaginary number $5 + 2i$ is a zero of $f(x)$.

TIP Polynomials with complex coefficients include polynomials with real coefficients and with imaginary coefficients. The following are complex polynomials.

$$f(x) = (2 + 3i)x^2 + 4i$$
$$g(x) = \sqrt{2}x^2 + 3x + 4i$$
$$h(x) = 2x^2 + 3x + 4$$

EXAMPLE 7 Using the Remainder Theorem to Identify Zeros of a Polynomial

Use the remainder theorem to determine if the given number c is a zero of the polynomial.

a. $f(x) = 2x^3 - 4x^2 - 13x - 9$; $c = 4$
b. $f(x) = x^3 + x^2 - 3x - 3$; $c = \sqrt{3}$
c. $f(x) = x^3 + x + 10$; $c = 1 + 2i$

Answers
6. a. 560 **b.** 0

Solution:

In each case, divide $f(x)$ by $x - c$ to determine the remainder. If the remainder is 0, then the value c is a zero of the polynomial.

a. Divide $f(x) = 2x^3 - 4x^2 - 13x - 9$ by $x - 4$.

$$\begin{array}{r|rrrr} 4 & 2 & -4 & -13 & -9 \\ & & 8 & 16 & 12 \\ \hline & 2 & 4 & 3 & \boxed{3} \end{array}$$

By the remainder theorem, $f(4) = 3$. Since $f(4) \neq 0$, 4 is not a zero of $f(x)$.

b. Divide $f(x) = x^3 + x^2 - 3x - 3$ by $x - \sqrt{3}$.

$$\begin{array}{r|cccc} \sqrt{3} & 1 & 1 & -3 & -3 \\ & & \sqrt{3} & 3 + \sqrt{3} & 3 \\ \hline & 1 & 1 + \sqrt{3} & \sqrt{3} & \boxed{0} \end{array}$$

By the remainder theorem, $f(\sqrt{3}) = 0$. Therefore, $\sqrt{3}$ is a zero of $f(x)$.

c. Divide $f(x) = x^3 + x + 10$ by $x - (1 + 2i)$

$$\begin{array}{r|cccc} 1 + 2i & 1 & 0 & 1 & 10 \\ & & 1 + 2i & -3 + 4i & -10 \\ \hline & 1 & 1 + 2i & -2 + 4i & \boxed{0} \end{array}$$

Note that $(1 + 2i)(1 + 2i)$
$= 1 + 2i + 2i + 4i^2$
$= 1 + 4i + 4(-1)$ Recall that $i^2 = -1$.
$= -3 + 4i$

Note that $(1 + 2i)(-2 + 4i)$
$= -2 + 4i - 4i + 8i^2$
$= -2 - 8$
$= -10$

By the remainder theorem, $f(1 + 2i) = 0$. Therefore, $1 + 2i$ is a zero of $f(x)$.

Skill Practice 7 Use the remainder theorem to determine if the given number, c, is a zero of the function.

a. $f(x) = 2x^4 - 3x^2 + 5x - 11$; $c = 2$
b. $f(x) = 2x^3 + 5x^2 - 14x - 35$; $c = \sqrt{7}$
c. $f(x) = x^3 - 7x^2 + 16x - 10$; $c = 3 + i$

Suppose that we again apply the division algorithm to a dividend of $f(x)$ and a divisor of $x - c$, where c is a complex number.

$$f(x) = (x - c) \cdot q(x) + r$$

By the remainder theorem, $r = f(c)$.

$$= (x - c) \cdot q(x) + f(c)$$

If $f(c) = 0$, then $f(x) = (x - c) \cdot q(x)$

This tells us that if $f(c)$ is a zero of $f(x)$, then $(x - c)$ is a factor of $f(x)$.

Now suppose that $x - c$ is a factor of $f(x)$. Then for some polynomial $q(x)$,

$$f(x) = (x - c) \cdot q(x)$$
$$f(c) = (c - c) \cdot q(x)$$
$$= 0$$

This tells us that if $(x - c)$ is a factor of $f(x)$, then c is a zero of $f(x)$.

These results can be summarized in the factor theorem.

Factor Theorem

Let $f(x)$ be a polynomial.

1. If $f(c) = 0$, then $(x - c)$ is a factor of $f(x)$.
2. If $(x - c)$ is a factor of $f(x)$, then $f(c) = 0$.

Answers

7. a. No b. Yes c. Yes

EXAMPLE 8 Identifying Factors of a Polynomial

Use the factor theorem to determine if the given polynomials are factors of $f(x) = x^4 - x^3 - 11x^2 + 11x + 12$.

a. $x - 3$ **b.** $x + 2$

Solution:

a. If $f(3) = 0$, then $x - 3$ is a factor of $f(x)$. Using synthetic division we have:

$$\begin{array}{r|rrrrr} 3 & 1 & -1 & -11 & 11 & 12 \\ & & 3 & 6 & -15 & -12 \\ \hline & 1 & 2 & -5 & -4 & \boxed{0} \end{array}$$

$\boxed{f(3) = 0}$

By the factor theorem, since $f(3) = 0$, $x - 3$ is a factor of $f(x)$.

b. If $f(-2) = 0$, then $x + 2$ is a factor of $f(x)$. Using synthetic division we have:

$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -11 & 11 & 12 \\ & & -2 & 6 & 10 & -42 \\ \hline & 1 & -3 & -5 & 21 & \boxed{-30} \end{array}$$

By the factor theorem, since $f(-2) \neq 0$, $x + 2$ is not a factor of $f(x)$. $\boxed{f(-2) = -30}$

Skill Practice 8 Use the factor theorem to determine if the given polynomials are factors of $f(x) = 2x^4 - 13x^3 + 10x^2 - 25x + 6$.

a. $x - 6$ **b.** $x + 3$

In Example 9, we illustrate the relationship between the zeros of a polynomial and the solutions (roots) of a polynomial equation.

EXAMPLE 9 Factoring a Polynomial Given a Known Zero

a. Factor $f(x) = 3x^3 + 25x^2 + 42x - 40$, given that -5 is a zero of $f(x)$.

b. Solve the equation. $3x^3 + 25x^2 + 42x - 40 = 0$

Solution:

a. The value -5 is a zero of $f(x)$, which means that $f(-5) = 0$. By the factor theorem, $x - (-5)$ or equivalently $x + 5$ is a factor of $f(x)$. Using synthetic division, we have

$$\begin{array}{r|rrrr} -5 & 3 & 25 & 42 & -40 \\ & & -15 & -50 & 40 \\ \hline & 3 & 10 & -8 & \boxed{0} \end{array}$$

This means that $3x^3 + 25x^2 + 42x - 40 = \overset{\text{divisor}}{(x + 5)}\underbrace{\overset{\text{quotient}}{(3x^2 + 10x - 8)}}_{\text{factors as }(3x-2)(x+4)} + \overset{\text{remainder}}{0}$

Therefore, $f(x) = (x + 5)(3x - 2)(x + 4)$.

b. $3x^3 + 25x^2 + 42x - 40 = 0$ To solve the equation, set one side equal to zero.

$(x + 5)(3x - 2)(x + 4) = 0$ Factor the left side.

$x = -5,\ x = \frac{2}{3},\ x = -4$ Set each factor equal to zero and solve for x.

The solution set is $\left\{-5, \frac{2}{3}, -4\right\}$.

Skill Practice 9

a. Factor $f(x) = 2x^3 + 7x^2 - 14x - 40$, given that -4 is a zero of f.

b. Solve the equation. $2x^3 + 7x^2 - 14x - 40 = 0$

Answers

8. a. Yes **b.** No

9. a. $f(x) = (x + 4)(x + 2)(2x - 5)$

b. $\left\{-4, -2, \frac{5}{2}\right\}$

EXAMPLE 10 Using the Factor Theorem to Build a Polynomial

Write a polynomial $f(x)$ of degree 3 that has the zeros $\frac{1}{2}$, $\sqrt{6}$, and $-\sqrt{6}$.

Solution:

By the factor theorem, if $\frac{1}{2}$, $\sqrt{6}$, and $-\sqrt{6}$ are zeros of a polynomial $f(x)$, then $\left(x - \frac{1}{2}\right)$, $\left(x - \sqrt{6}\right)$, and $\left(x + \sqrt{6}\right)$ are factors of $f(x)$. Therefore, $f(x) = \left(x - \frac{1}{2}\right)\left(x - \sqrt{6}\right)\left(x + \sqrt{6}\right)$ is a third-degree polynomial with the given zeros.

$$f(x) = \left(x - \frac{1}{2}\right)(x^2 - 6) \qquad \text{Multiply conjugates.}$$

$$f(x) = x^3 - \frac{1}{2}x^2 - 6x + 3$$

Skill Practice 10 Write a polynomial $f(x)$ of degree 3 that has the zeros $\frac{1}{3}$, $\sqrt{3}$, and $-\sqrt{3}$.

In Example 10, the polynomial $f(x)$ is not unique. If we multiply $f(x)$ by any nonzero constant a, the polynomial will still have the desired factors and zeros.

$$g(x) = a\left(x - \frac{1}{2}\right)\left(x - \sqrt{6}\right)\left(x + \sqrt{6}\right) \qquad \text{The zeros are still } \frac{1}{2}, \sqrt{6}, \text{ and } -\sqrt{6}.$$

If a is any nonzero multiple of 2, then the polynomial will have integer coefficients. For example:

$$\begin{aligned} g(x) &= 2\left(x - \frac{1}{2}\right)\left(x - \sqrt{6}\right)\left(x + \sqrt{6}\right) \\ &= 2\left(x^3 - \frac{1}{2}x^2 - 6x + 3\right) \\ &= 2x^3 - x^2 - 12x + 6 \end{aligned}$$

The zeros of $f(x)$ and $g(x)$ are real numbers and correspond to the x-intercepts of the graphs of the related functions. The graphs of $y = f(x)$ and $y = g(x)$ are shown in Figure 2-15. Notice that the graphs have the same x-intercepts and differ only by a vertical stretch.

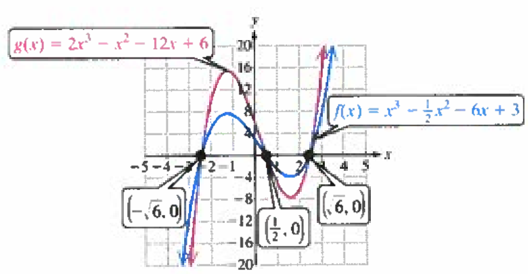


Figure 2-15

Answer

10. $f(x) = x^3 - \frac{1}{3}x^2 - 3x + 1$

SECTION 2.3 Practice Exercises

Prerequisite Review

R.1. Evaluate $(3 + 4i)^2$ and write the answer in standard form, $a + bi$.

R.2. Verify by substitution that the given values of x are solutions to $x^2 - 12x + 39 = 0$.

a. $6 - i\sqrt{3}$ **b.** $6 + i\sqrt{3}$

R.3. Solve by using the quadratic formula.

$z^2 + 8z + 19 = 0$

Concept Connections

1. Given the division algorithm, identify the polynomials representing the dividend, divisor, quotient, and remainder.

$f(x) = d(x) \cdot q(x) + r(x)$

2. Given $\dfrac{2x^3 - 5x^2 - 6x + 1}{x - 3} = 2x^2 + x - 3 + \dfrac{-8}{x - 3}$, use the division algorithm to check the result.

3. The remainder theorem indicates that if a polynomial $f(x)$ is divided by $x - c$, then the remainder is ______________.

4. Given a polynomial $f(x)$, the factor theorem indicates that if $f(c) = 0$, then $x - c$ is a ______________ of $f(x)$. Furthermore, if $x - c$ is a factor of $f(x)$, then $f(c) =$ ______________.

5. Answer true or false. If $\sqrt{5}$ is a zero of a polynomial, then $(x - \sqrt{5})$ is a factor of the polynomial.

6. Answer true or false. If $(x + 3)$ is a factor of a polynomial, then 3 is a zero of the polynomial.

Objective 1: Divide Polynomials Using Long Division

For Exercises 7–8, (See Example 1)

a. Use long division to divide.

b. Identify the dividend, divisor, quotient, and remainder.

c. Check the result from part (a) with the division algorithm.

7. $(6x^2 + 9x + 5) \div (2x - 5)$

8. $(12x^2 + 10x + 3) \div (3x + 4)$

For Exercises 9–22, use long division to divide. (See Examples 1–3)

9. $(3x^3 - 11x^2 - 10) \div (x - 4)$

10. $(2x^3 - 7x^2 - 65) \div (x - 5)$

11. $(8 + 30x - 27x^2 - 12x^3 + 4x^4) \div (x + 2)$

12. $(-48 - 28x + 20x^2 + 17x^3 + 3x^4) \div (x + 3)$

13. $(-20x^2 + 6x^4 - 16) \div (2x + 4)$

14. $(-60x^2 + 8x^4 - 108) \div (2x - 6)$

15. $(x^5 + 4x^4 + 18x^2 - 20x - 10) \div (x^2 + 5)$

16. $(x^5 - 2x^4 + x^3 - 8x + 18) \div (x^2 - 3)$

17. $\dfrac{6x^4 + 3x^3 - 7x^2 + 6x - 5}{2x^2 + x - 3}$

18. $\dfrac{12x^4 - 4x^3 + 13x^2 + 2x + 1}{3x^2 - x + 4}$

19. $\dfrac{x^3 - 27}{x - 3}$

20. $\dfrac{x^3 + 64}{x + 4}$

21. $(5x^3 - 2x^2 + 3) \div (2x - 1)$

22. $(2x^3 + x^2 + 1) \div (3x + 1)$

Objective 2: Divide Polynomials Using Synthetic Division

For Exercises 23–26, consider the division of two polynomials: $f(x) \div (x - c)$. The result of the synthetic division process is shown here. Write the polynomials representing the

a. Dividend. **b.** Divisor. **c.** Quotient. **d.** Remainder.

23.
$$\begin{array}{r|rrrrr} 3 & 2 & -5 & -5 & -4 & 29 \\ & & 6 & 3 & -6 & -30 \\ \hline & 2 & 1 & -2 & -10 & \boxed{-1} \end{array}$$

24.
$$\begin{array}{r|rrrrr} 2 & 1 & -5 & 2 & -1 & 20 \\ & & 2 & -6 & -8 & -18 \\ \hline & 1 & -3 & -4 & -9 & \boxed{2} \end{array}$$

25. $\begin{array}{r|rrrr} -4 & 1 & -2 & -25 & -4 \\ & & -4 & 24 & 4 \\ \hline & 1 & -6 & -1 & \boxed{0} \end{array}$

26. $\begin{array}{r|rrrr} -5 & 3 & 13 & -14 & -20 \\ & & -15 & 10 & 20 \\ \hline & 3 & -2 & -4 & \boxed{0} \end{array}$

For Exercises 27–38, use synthetic division to divide the polynomials. (See Examples 4–5)

27. $(4x^2 + 15x + 1) \div (x + 6)$

28. $(6x^2 + 25x - 19) \div (x + 5)$

29. $(5x^2 - 17x - 12) \div (x - 4)$

30. $(2x^2 + x - 21) \div (x - 3)$

31. $(4 - 8x - 3x^2 - 5x^4) \div (x + 2)$

32. $(-5 + 2x + 5x^3 - 2x^4) \div (x + 1)$

33. $\dfrac{4x^5 - 25x^4 - 58x^3 + 232x^2 + 198x - 63}{x - 3}$

34. $\dfrac{2x^5 + 13x^4 - 3x^3 - 58x^2 - 20x + 24}{x - 2}$

35. $\dfrac{x^5 + 32}{x + 2}$

36. $\dfrac{x^4 - 81}{x + 3}$

37. $(2x^4 - 7x^3 - 56x^2 + 37x + 84) \div \left(x - \dfrac{3}{2}\right)$

38. $(-5x^4 - 18x^3 + 63x^2 + 128x - 60) \div \left(x - \dfrac{2}{5}\right)$

Objective 3: Apply the Remainder and Factor Theorems

39. The value $f(-6) = 39$ for a polynomial $f(x)$. What can be concluded about the remainder or quotient of $\dfrac{f(x)}{x + 6}$?

40. Given a polynomial $f(x)$, the quotient $\dfrac{f(x)}{x - 2}$ has a remainder of 12. What is the value of $f(2)$?

41. Given $f(x) = 2x^4 - 5x^3 + x^2 - 7$,
   a. Evaluate $f(4)$.
   b. Determine the remainder when $f(x)$ is divided by $(x - 4)$.

42. Given $g(x) = -3x^5 + 2x^4 + 6x^2 - x + 4$,
   a. Evaluate $g(2)$.
   b. Determine the remainder when $g(x)$ is divided by $(x - 2)$.

For Exercises 43–46, use the remainder theorem to evaluate the polynomial for the given values of x. (See Example 6)

43. $f(x) = 2x^4 + x^3 - 49x^2 + 79x + 15$
   a. $f(-1)$ **b.** $f(3)$ **c.** $f(4)$ **d.** $f\left(\dfrac{5}{2}\right)$

44. $g(x) = 3x^4 - 22x^3 + 51x^2 - 42x + 8$
   a. $g(-1)$ **b.** $g(2)$ **c.** $g(1)$ **d.** $g\left(\dfrac{4}{3}\right)$

45. $h(x) = 5x^3 - 4x^2 - 15x + 12$
   a. $h(1)$ **b.** $h\left(\dfrac{4}{5}\right)$ **c.** $h(\sqrt{3})$ **d.** $h(-1)$

46. $k(x) = 2x^3 - x^2 - 14x + 7$
   a. $k(2)$ **b.** $k\left(\dfrac{1}{2}\right)$ **c.** $k(\sqrt{7})$ **d.** $k(-2)$

For Exercises 47–54, use the remainder theorem to determine if the given number c is a zero of the polynomial. (See Example 7)

47. $f(x) = x^4 + 3x^3 - 7x^2 + 13x - 10$
   a. $c = 2$ **b.** $c = -5$

48. $g(x) = 2x^4 + 13x^3 - 10x^2 - 19x + 14$
   a. $c = -2$ **b.** $c = -7$

49. $p(x) = 2x^3 + 3x^2 - 22x - 33$
   a. $c = -2$ **b.** $c = -\sqrt{11}$

50. $q(x) = 3x^3 + x^2 - 30x - 10$
   a. $c = -3$ **b.** $c = -\sqrt{10}$

51. $m(x) = x^3 - 2x^2 + 25x - 50$
   a. $c = 5i$ **b.** $c = -5i$

52. $n(x) = x^3 + 4x^2 + 9x + 36$
   a. $c = 3i$ **b.** $c = -3i$

53. $g(x) = x^3 - 11x^2 + 25x + 37$
   a. $c = 6 + i$ **b.** $c = 6 - i$

54. $f(x) = 2x^3 - 5x^2 + 54x - 26$
   a. $c = 1 + 5i$ **b.** $c = 1 - 5i$

For Exercises 55–60, use the factor theorem to determine if the given binomial is a factor of $f(x)$. (See Example 8)

55. $f(x) = x^4 + 11x^3 + 41x^2 + 61x + 30$
   a. $x + 5$ **b.** $x - 2$

56. $g(x) = x^4 - 10x^3 + 35x^2 - 50x + 24$
   a. $x - 4$ **b.** $x + 1$

57. $f(x) = x^3 + 64$
   a. $x - 4$ **b.** $x + 4$

58. $f(x) = x^4 - 81$
   a. $x - 3$ **b.** $x + 3$

59. $f(x) = 2x^3 + x^2 - 16x - 8$
   a. $x - 1$ **b.** $x - 2\sqrt{2}$

60. $f(x) = 3x^3 - x^2 - 54x + 18$
   a. $x - 2$ **c.** $x - 3\sqrt{2}$

61. Given $g(x) = x^4 - 14x^2 + 45$,
   a. Evaluate $g(\sqrt{5})$.
   b. Evaluate $g(-\sqrt{5})$.
   c. Solve $g(x) = 0$.

62. Given $h(x) = x^4 - 15x^2 + 44$,
   a. Evaluate $h(\sqrt{11})$.
   b. Evaluate $h(-\sqrt{11})$.
   c. Solve $h(x) = 0$.

63. **a.** Use synthetic division and the factor theorem to determine if $[x - (2 + 5i)]$ is a factor of $f(x) = x^2 - 4x + 29$.
   b. Use synthetic division and the factor theorem to determine if $[x - (2 - 5i)]$ is a factor of $f(x) = x^2 - 4x + 29$.
   c. Use the quadratic formula to solve the equation. $x^2 - 4x + 29 = 0$
   d. Find the zeros of the polynomial $f(x) = x^2 - 4x + 29$.

64. **a.** Use synthetic division and the factor theorem to determine if $[x - (3 + 4i)]$ is a factor of $f(x) = x^2 - 6x + 25$.
   b. Use synthetic division and the factor theorem to determine if $[x - (3 - 4i)]$ is a factor of $f(x) = x^2 - 6x + 25$.
   c. Use the quadratic formula to solve the equation. $x^2 - 6x + 25 = 0$
   d. Find the zeros of the polynomial $f(x) = x^2 - 6x + 25$.

65. **a.** Factor $f(x) = 2x^3 + x^2 - 37x - 36$, given that -1 is a zero. **(See Example 9)**
   b. Solve. $2x^3 + x^2 - 37x - 36 = 0$

66. **a.** Factor $f(x) = 3x^3 + 16x^2 - 5x - 50$, given that -2 is a zero.
   b. Solve. $3x^3 + 16x^2 - 5x - 50 = 0$

67. **a.** Factor $f(x) = 20x^3 + 39x^2 - 3x - 2$, given that $\frac{1}{4}$ is a zero.
   b. Solve. $20x^3 + 39x^2 - 3x - 2 = 0$

68. **a.** Factor $f(x) = 8x^3 - 18x^2 - 11x + 15$, given that $\frac{3}{4}$ is a zero.
   b. Solve. $8x^3 - 18x^2 - 11x + 15 = 0$

69. **a.** Factor $f(x) = 9x^3 - 33x^2 + 19x - 3$, given that 3 is a zero.
   b. Solve. $9x^3 - 33x^2 + 19x - 3 = 0$

70. **a.** Factor $f(x) = 4x^3 - 20x^2 + 33x - 18$, given that 2 is a zero.
   b. Solve. $4x^3 - 20x^2 + 33x - 18 = 0$

For Exercises 71–82, write a polynomial $f(x)$ that meets the given conditions. Answers may vary. (See Example 10)

71. Degree 3 polynomial with zeros 2, 3, and -4.

72. Degree 3 polynomial with zeros 1, -6, and 3.

73. Degree 4 polynomial with zeros 1, $\frac{3}{2}$ (each with multiplicity 1), and 0 (with multiplicity 2).

74. Degree 5 polynomial with zeros 2, $\frac{5}{2}$ (each with multiplicity 1), and 0 (with multiplicity 3).

75. Degree 2 polynomial with zeros $2\sqrt{11}$ and $-2\sqrt{11}$.

76. Degree 2 polynomial with zeros $5\sqrt{2}$ and $-5\sqrt{2}$.

77. Degree 3 polynomial with zeros -2, $3i$, and $-3i$.

78. Degree 3 polynomial with zeros 4, $2i$, and $-2i$.

79. Degree 3 polynomial with integer coefficients and zeros of $-\frac{7}{3}$, $\frac{1}{2}$, and 4.

80. Degree 3 polynomial with integer coefficients and zeros of $-\frac{2}{5}$, $\frac{3}{2}$, and 6.

81. Degree 2 polynomial with zeros of $7 + 8i$ and $7 - 8i$.

82. Degree 2 polynomial with zeros of $5 + 6i$ and $5 - 6i$.

Mixed Exercises

83. Given $p(x) = 2x^{452} - 4x^{92}$, is it easier to evaluate $p(1)$ by using synthetic division or by direct substitution? Find the value of $p(1)$.

84. Given $q(x) = 5x^{721} - 2x^{450}$, is it easier to evaluate $q(-1)$ by using synthetic division or by direct substitution? Find the value of $q(-1)$.

85. **a.** Is $(x - 1)$ a factor of $x^{100} - 1$?
   b. Is $(x + 1)$ a factor of $x^{100} - 1$?
   c. Is $(x - 1)$ a factor of $x^{99} - 1$?
   d. Is $(x + 1)$ a factor of $x^{99} - 1$?
   e. If n is a positive even integer, is $(x - 1)$ a factor of $x^n - 1$?
   f. If n is a positive odd integer, is $(x + 1)$ a factor of $x^n - 1$?

86. If a fifth-degree polynomial is divided by a second-degree polynomial, the quotient is a ______ -degree polynomial.

87. Determine if the statement is true or false: Zero is a zero of the polynomial $3x^5 - 7x^4 - 2x^3 - 14$.

88. Determine if the statement is true or false: Zero is a zero of the polynomial $-2x^4 + 5x^3 + 6x$.

89. Find m so that $x + 4$ is a factor of $4x^3 + 13x^2 - 5x + m$.

90. Find m so that $x + 5$ is a factor of $-3x^4 - 10x^3 + 20x^2 - 22x + m$.

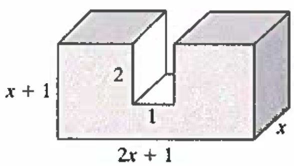
91. Find m so that $x + 2$ is a factor of $4x^3 + 5x^2 + mx + 2$.

92. Find m so that $x - 3$ is a factor of $2x^3 - 7x^2 + mx + 6$.

93. For what value of r is the statement an identity?
$\dfrac{x^2 - x - 12}{x - 4} = x + 3 + \dfrac{r}{x - 4}$ provided that $x \neq 4$

94. For what value of r is the statement an identity?
$\dfrac{x^2 - 5x - 8}{x - 2} = x - 3 + \dfrac{r}{x - 2}$ provided that $x \neq 2$

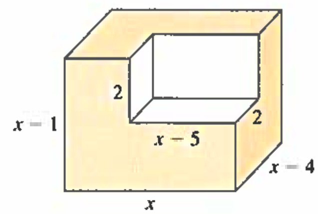
95. A metal block is formed from a rectangular solid with a rectangular piece cut out.



a. Write a polynomial $V(x)$ that represents the volume of the block. All distances in the figure are in centimeters.

b. Use synthetic division to evaluate the volume if x is 6 cm.

96. A wedge is cut from a rectangular solid.



a. Write a polynomial $V(x)$ that represents the volume of the remaining part of the solid. All distances in the figure are in feet.

b. Use synthetic division to evaluate the volume if x is 10 ft.

Write About It

97. Under what circumstances can synthetic division be used to divide polynomials?

98. How can the division algorithm be used to check the result of polynomial division?

99. Given a polynomial $f(x)$ and a constant c, state two methods by which the value $f(c)$ can be computed.

100. Write an informal explanation of the factor theorem.

Expanding Your Skills

101. a. Factor $f(x) = x^3 - 5x^2 + x - 5$ into factors of the form $(x - c)$, given that 5 is a zero.
b. Solve. $x^3 - 5x^2 + x - 5 = 0$

102. a. Factor $f(x) = x^3 - 3x^2 + 100x - 300$ into factors of the form $(x - c)$, given that 3 is a zero.
b. Solve. $x^3 - 3x^2 + 100x - 300 = 0$

103. a. Factor $f(x) = x^4 + 2x^3 - 2x^2 - 6x - 3$ into factors of the form $(x - c)$, given that -1 is a zero.
b. Solve. $x^4 + 2x^3 - 2x^2 - 6x - 3 = 0$

104. a. Factor $f(x) = x^4 + 4x^3 - x^2 - 20x - 20$ into factors of the form $(x - c)$, given that -2 is a zero.
b. Solve. $x^4 + 4x^3 - x^2 - 20x - 20 = 0$

Technology Connections

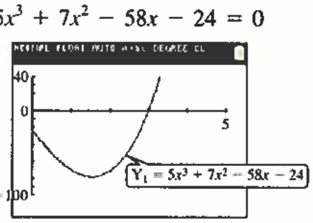
For Exercises 105–106,

a. Use the graph to determine a solution to the given equation.

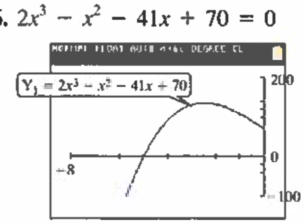
b. Verify your answer from part (a) using the remainder theorem.

c. Find the remaining solutions to the equation.

105. $5x^3 + 7x^2 - 58x - 24 = 0$



106. $2x^3 - x^2 - 41x + 70 = 0$



SECTION 3.1 Inverse Functions

OBJECTIVES

1. Identify One-to-One Functions
2. Determine Whether Two Functions Are Inverses
3. Find the Inverse of a Function

1. Identify One-to-One Functions

Throughout our study of algebra, we have made use of the fact that the operations of addition and subtraction are inverse operations. For example, adding 5 to a number and then subtracting 5 from the result gives us the original number. Likewise, multiplication and division are inverse operations. We now look at the concept of an inverse function.

Changing currency is an important consideration when traveling abroad. For example, traveling between the United States and several countries in Europe would involve changing American dollars to Euros and then changing back for the return trip. Fortunately, we can use a function to change from one currency to the other, and then use the function's *inverse* to change back again.

Suppose that \$1 (American dollar) can be exchanged for 0.8 € (Euro). Then,

$f(x) = 0.8x$ gives the number of Euros $f(x)$ that can be bought from x dollars.

$g(x) = \dfrac{x}{0.8}$ gives the number of dollars $g(x)$ that can be bought from x Euros.

Tables 3-1 and 3-2 show the values of $f(x)$ and $g(x)$ for several values of x.

Table 3-1

x (Dollars)	$f(x) = 0.8x$ (Euros)
100	80
150	120
200	160
250	200

Table 3-2

x (Euros)	$g(x) = \dfrac{x}{0.8}$ (Dollars)
80	100
120	150
160	200
200	250

In this example, functions f and g are inverses of each other, and we observe several interesting characteristics about inverse functions.

- By listing the ordered pairs from Tables 3-1 and 3-2, notice that the x and y values are reversed.

$$f:\ \{(100, 80), (150, 120), (200, 160), (250, 200)\}$$

$$g:\ \{(80, 100), (120, 150), (160, 200), (200, 250)\}$$

- For a function and its inverse, the values of x and y are interchanged. This tells us that the domain of a function is the same as the range of its inverse and vice versa.
- From the graphs of f and g (Figure 3-1), we see that the corresponding points on f and g are symmetric with respect to the line $y = x$.
- When we compose functions f and g in both directions, the result is the input value x. In a sense, what function f does to x, function g "undoes" and vice versa.

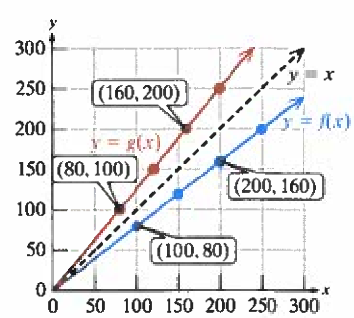


Figure 3-1

$$(f \circ g)(x) = f[g(x)] = 0.8\left(\frac{x}{0.8}\right) = x$$

$$(g \circ f)(x) = g[f(x)] = \frac{(0.8x)}{0.8} = x$$

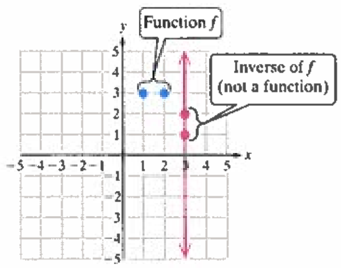


Figure 3-2

The inverse of any relation is found by interchanging the values of x and y in the relation. However, the inverse of a function may itself not be a function. For example, consider $f = \{(1, 3), (2, 3)\}$ shown in blue in Figure 3-2. The inverse is the set of ordered pairs $\{(3, 1), (3, 2)\}$ shown in red. Notice that the relation defining the inverse of f is not a function because it fails the vertical line test.

The function $f = \{(1, 3), (2, 3)\}$ has two points that are aligned horizontally. When the x and y values are reversed to form the inverse, the resulting points will be aligned vertically and will fail the vertical line test. Thus, a function will have an inverse function only if no points in the original function are aligned horizontally. That is, no two distinct points on the function may have the same y value. In such a case, the function is said to be one-to-one.

Definition of a One-to-One Function

A function f is a **one-to-one function,** if for a and b in the domain of f,

if $a \neq b$, then $f(a) \neq f(b)$, or equivalently, if $f(a) = f(b)$, then $a = b$.

The definition of a one-to-one function tells us that each y value in the range is associated with only one x value in the domain. This implies that the graph of a one-to-one function will have no two points aligned horizontally.

Horizontal Line Test for a One-to-One Function

A function defined by $y = f(x)$ is a one-to-one function if no horizontal line intersects the graph in more than one point.

EXAMPLE 1 Determining Whether a Function Is One-to-One

Determine whether the function is one-to-one.

a. $f = \{(1, 4), (2, 3), (-2, 4)\}$ **b.** $g = \{(-3, 4), (1, -1), (2, 0)\}$

Solution:

same y value

a. $f = \{(1, 4), (2, 3), (-2, 4)\}$

different x value

f is not a one-to-one function.

The ordered pairs $(1, 4)$ and $(-2, 4)$ have the same y value but different x values. That is, $f(1) = f(-2)$, but $1 \neq -2$.

A horizontal line passes through the function in more than one point (Figure 3-3).

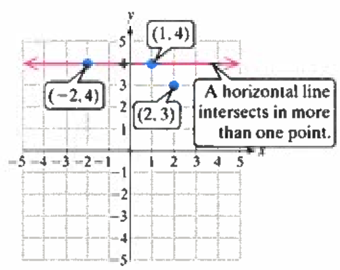


Figure 3-3

All points have different y-values.

b. $g = \{(-3, 4), (1, -1), (2, 0)\}$

g is a one-to-one function.

Each unique ordered pair has a different y value, so the function is one-to-one.

No horizontal line passes through the function in more than one point (Figure 3-4).

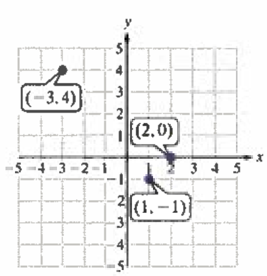


Figure 3-4

Skill Practice 1 Determine whether the function is one-to-one.

a. $h = \{(4, -5), (6, 1), (2, 4), (0, -3)\}$ **b.** $k = \{(1, 0), (3, 0), (4, -5)\}$

Answers

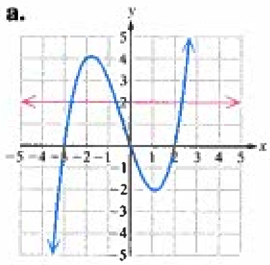
1. a. Yes **b.** No

EXAMPLE 2 Using the Horizontal Line Test

Use the horizontal line test to determine if the graph in blue defines y as a one-to-one function of x.

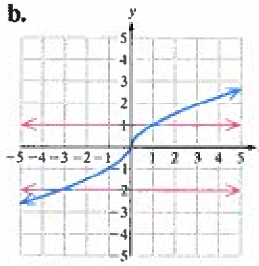
Solution:

a.



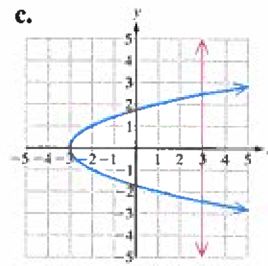
The graph does not define y as a one-to-one function of x because a horizontal line intersects the graph in more than one point.

b.



The graph does define y as a one-to-one function of x because no horizontal line intersects the graph in more than one point.

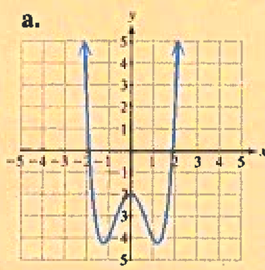
c.



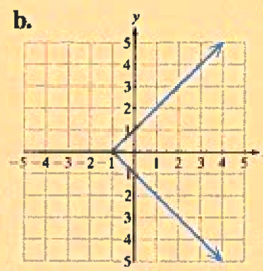
The relation does not define y as a function of x, because it fails the vertical line test. If the relation is not a function, it is not a one-to-one function.

Skill Practice 2 Use the horizontal line test to determine if the graph defines y as a one-to-one function of x.

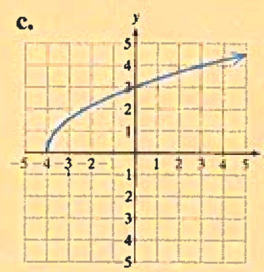
a.



b.



c.



In Example 3, we use algebraic methods to determine whether a function is one-to-one.

EXAMPLE 3 Determining Whether a Function Is One-to-One

Use the definition of a one-to-one function to determine whether the function is one-to-one.

a. $f(x) = 2x - 3$ **b.** $f(x) = x^2 + 1$

Solution:

a. We must show that if $f(a) = f(b)$, then $a = b$.

Assume that $f(a) = f(b)$. That is,

$$2a - 3 = 2b - 3$$
$$2a - 3 + 3 = 2b - 3 + 3$$
$$\frac{2a}{2} = \frac{2b}{2}$$
$$a = b$$

Since $f(a) = f(b)$ implies that $a = b$, then f is one-to-one.

The logic of this algebraic proof begins with the assumption that $f(a) = f(b)$, that is, that two y values are equal. For a one-to-one function, this can happen only if the x values (in this case a and b) are the same.

Otherwise, if $a \neq b$, we would have the same y value with two different x values and f would not be one-to-one.

Answers

2. a. No **b.** No **c.** Yes

TIP To show that a function is not one-to-one, we only need one counterexample. That is, any pair of distinct points on the function that have different y values is sufficient to show that the function is not one-to-one.

b. $f(x) = x^2 + 1$

Assume that $f(a) = f(b)$.

$$a^2 + 1 = b^2 + 1$$
$$a^2 = b^2$$
$$a = \pm b$$

For nonzero values of b, $f(a) = f(b)$ does not necessarily imply that $a = b$. Therefore, f is not one-to-one.

From the graph of $f(x) = x^2 + 1$, we see that f is not one-to-one (Figure 3-5). We can also show this algebraically by finding two ordered pairs with the same y value but different x values. From the graph, we have arbitrarily selected $(-2, 5)$ and $(2, 5)$.

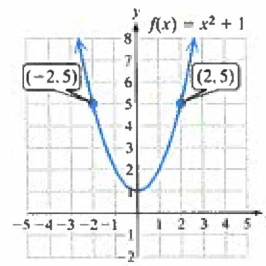


Figure 3-5

If $a = 2$ and $b = -2$, we have:

$$f(a) = f(2) = (2)^2 + 1 = 5$$
$$f(b) = f(-2) = (-2)^2 + 1 = 5$$

Same y value but different x values

We have that $f(a) = f(b)$, but $a \neq b$.

Therefore, f fails to be a one-to-one function.

Skill Practice 3 Determine whether the function is one-to-one.

a. $f(x) = -4x + 1$ **b.** $f(x) = |x| - 3$

2. Determine Whether Two Functions Are Inverses

We now have enough background to define an inverse function.

Definition of an Inverse Function

Let f be a one-to-one function. Then g is the **inverse of f** if the following conditions are both true.

1. $(f \circ g)(x) = x$ for all x in the domain of g.
2. $(g \circ f)(x) = x$ for all x in the domain of f.

Avoiding Mistakes

Do not confuse inverse notation f^{-1} with exponential notation. The notation f^{-1} does not mean $\frac{1}{f}$.

We should also note that if g is the inverse of f, then f is the inverse of g. Furthermore, given a function f, we often denote its inverse as f^{-1}. So given a function f and its inverse f^{-1}, the definition implies that

$$(f \circ f^{-1})(x) = x \text{ and } (f^{-1} \circ f)(x) = x$$

EXAMPLE 4 Determining Whether Two Functions Are Inverses

Determine whether the functions are inverses.

a. $f(x) = 100 + 12x$ and $g(x) = \dfrac{x - 100}{12}$

b. $h(x) = \sqrt[3]{x - 1}$ and $k(x) = -1 + x^3$

Answers

3. a. Yes b. No

Solution:

a. $(f \circ g)(x) = f(g(x))$

$$= f\left(\frac{x-100}{12}\right)$$

$$= 100 + 12\left(\frac{x-100}{12}\right)$$

$$= 100 + (x - 100)$$

$$= x \checkmark$$

$(g \circ f)(x) = g(f(x))$

$$= g(100 + 12x)$$

$$= \frac{(100 + 12x) - 100}{12}$$

$$= \frac{12x}{12}$$

$$= x \checkmark$$

Since $(f \circ g)(x) = (g \circ f)(x) = x$, f and g are inverses.

b. $(h \circ k)(x) = h(k(x))$

$$= \sqrt[3]{(-1 + x^3) - 1}$$

$$= \sqrt[3]{x^3 - 2} \neq x$$

Since $(h \circ k)(x) \neq x$, h and k are not inverses.

If either $(h \circ k)(x) \neq x$ or $(k \circ h)(x) \neq x$, then h and k are *not* inverses.

Skill Practice 4 Determine whether the functions are inverses.

a. $f(x) = \dfrac{x+6}{2}$ and $g(x) = 2(x - 6)$ **b.** $m(x) = \dfrac{5}{x-2}$ and $n(x) = \dfrac{2x+5}{x}$

3. Find the Inverse of a Function

For a one-to-one function defined by $y = f(x)$, the inverse is a function $y = f^{-1}(x)$ that performs the inverse operations in the reverse order. The function given by $f(x) = 100 + 12x$ multiplies x by 12 first, and then adds 100 to the result. Therefore, the inverse function must *subtract* 100 from x first and then *divide* by 12.

$$f^{-1}(x) = \frac{x - 100}{12}$$

To facilitate the process of finding an equation of the inverse of a one-to-one function, we offer the following steps.

Procedure to Find an Equation of an Inverse of a Function

For a one-to-one function defined by $y = f(x)$, the equation of the inverse can be found as follows.

Step 1 Replace $f(x)$ by y.
Step 2 Interchange x and y.
Step 3 Solve for y.
Step 4 Replace y by $f^{-1}(x)$.

EXAMPLE 5 Finding an Equation of an Inverse Function

Write an equation for the inverse function for $f(x) = 3x - 1$.

Answers
4. **a.** No **b.** Yes

Solution:

Function f is a linear function, and its graph is a nonvertical line. Therefore, f is a one-to-one function.

$$f(x) = 3x - 1$$

$$y = 3x - 1 \qquad \textbf{Step 1: } \text{Replace } f(x) \text{ by } y.$$

$$x = 3y - 1 \qquad \textbf{Step 2: } \text{Interchange } x \text{ and } y.$$

$$x + 1 = 3y \qquad \textbf{Step 3: } \text{Solve for } y. \text{ Add 1 to both sides and divide by 3.}$$

$$\frac{x+1}{3} = y$$

$$f^{-1}(x) = \frac{x+1}{3} \qquad \textbf{Step 4: } \text{Replace } y \text{ by } f^{-1}(x).$$

To check the result, verify that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

$$(f \circ f^{-1})(x) = 3\left(\frac{x+1}{3}\right) - 1 = x \checkmark \quad \text{and} \quad (f^{-1} \circ f)(x) = \frac{(3x-1)+1}{3} = x \checkmark$$

Skill Practice 5 Write an equation for the inverse function for $f(x) = 4x + 3$.

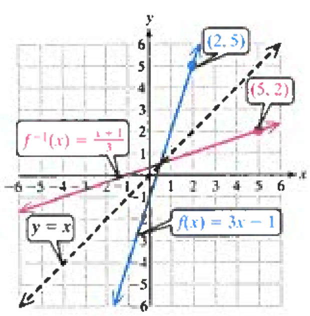
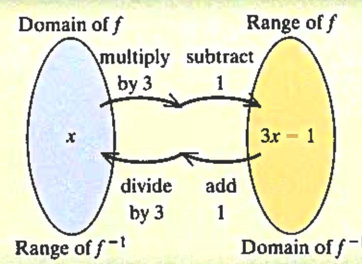


Figure 3-6

TIP We can sometimes find an equation of an inverse function by mentally reversing the operations given in the original function. In Example 5, the function f multiplies x by 3 and then subtracts 1. Therefore, f^{-1} must add 1 to x and then divide by 3.



The key step in determining the equation of the inverse of a function is to interchange x and y. By so doing, a point (a, b) on f corresponds to a point (b, a) on f^{-1}. This is why the graphs of f and f^{-1} are symmetric with respect to the line $y = x$. From Example 5, notice that the point (2, 5) on the graph of f corresponds to the point (5, 2) on the graph of f^{-1} (Figure 3-6).

EXAMPLE 6 Finding an Equation of an Inverse Function

Write an equation for the inverse function for the one-to-one function defined by $f(x) = \dfrac{3-x}{x+3}$.

Solution:

$$f(x) = \frac{3-x}{x+3}$$

$$y = \frac{3-x}{x+3} \qquad \textbf{Step 1: } \text{Replace } f(x) \text{ by } y.$$

$$x = \frac{3-y}{y+3} \qquad \textbf{Step 2: } \text{Interchange } x \text{ and } y.$$

TIP In Example 6, we can show that f is a one-to-one function by graphing the function (see Section 2.5). Or we can show that $f(a) = f(b)$ implies that $a = b$ by solving the equation $\dfrac{3-a}{a+3} = \dfrac{3-b}{b+3}$ for a or b to show that $a = b$.

Answer

5. $f^{-1}(x) = \dfrac{x-3}{4}$

$x(y + 3) = 3 - y$	**Step 3:** Solve for y. Clear fractions (multiply both sides by $y + 3$).
$xy + 3x = 3 - y$	Apply the distributive property.
$xy + y = 3 - 3x$	Collect the y terms on one side.
$y(x + 1) = 3 - 3x$	Factor out y as the greatest common factor.
$y = \dfrac{3 - 3x}{x + 1}$	Divide both sides by $x + 1$.
$f^{-1}(x) = \dfrac{3 - 3x}{x + 1}$	**Step 4:** Replace y by $f^{-1}(x)$.

Skill Practice 6 Write an equation for the inverse function for the one-to-one function defined by $f(x) = \dfrac{x - 2}{x + 2}$.

For a function that is not one-to-one, sometimes we restrict its domain to create a new function that is one-to-one. This is demonstrated in Example 7.

EXAMPLE 7 Finding an Equation of an Inverse Function

Given $m(x) = x^2 + 4$ for $x \geq 0$, write an equation of the inverse.

Solution:

The graph of $y = x^2 + 4$ is a parabola with vertex (0, 4). See Figure 3-7. The function is not one-to-one. However, with the restriction on the domain that $x \geq 0$, the graph consists of only the right branch of the parabola (Figure 3-8). This *is* a one-to-one function.

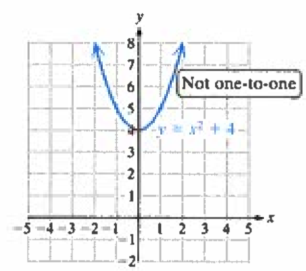


Figure 3-7

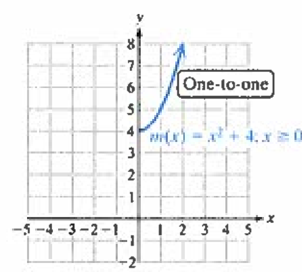


Figure 3-8

To find the inverse, we have

$m(x) = x^2 + 4; \quad x \geq 0$	
$y = x^2 + 4; \quad x \geq 0$	**Step 1:** Replace $m(x)$ by y.
$x = y^2 + 4 \quad y \geq 0$	**Step 2:** Interchange x and y. Notice that the restriction $x \geq 0$ becomes $y \geq 0$.
$x - 4 = y^2$	**Step 3:** Solve for y by subtracting 4 from both sides.
$y = \pm\sqrt{x - 4}$	Apply the square root property.
$y = +\sqrt{x - 4}$	Choose the positive square root of $(x - 4)$ because of the restriction $y \geq 0$.
$m^{-1}(x) = \sqrt{x - 4}$	**Step 4:** Replace y by $m^{-1}(x)$.

The graphs of m and m^{-1} are symmetric with respect to the line $y = x$ as expected (Figure 3-9).

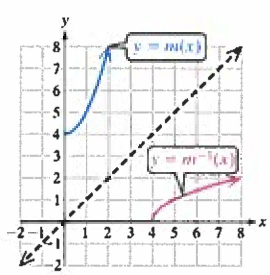


Figure 3-9

Answer

6. $f^{-1}(x) = -\dfrac{2x + 2}{x - 1}$

Skill Practice 7 Given $n(x) = x^2 + 1$ for $x \leq 0$, write an equation of the inverse.

EXAMPLE 8 Finding an Equation of an Inverse Function

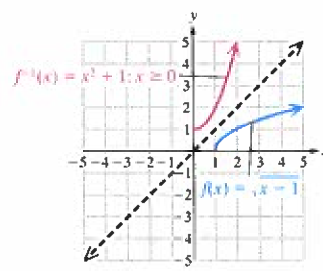
Given $f(x) = \sqrt{x - 1}$, find an equation of the inverse.

Solution:

The function f is a one-to-one function and the graph is the same as the graph of $y = \sqrt{x}$ with a shift 1 unit to the right. The domain of f is $\{x \mid x \geq 1\}$ and the range is $\{y \mid y \geq 0\}$. When defining the inverse, we will have the conditions that $x \geq 0$ and $y \geq 1$.

TIP When finding the inverse of a function, the key step of interchanging x and y has the effect of interchanging the domain and range between the function and its inverse.

$$f(x) = \sqrt{x - 1} \qquad \text{Note that } x \geq 1 \text{ and } y \geq 0.$$
$$y = \sqrt{x - 1}$$
$$x = \sqrt{y - 1} \qquad \text{Interchange } x \text{ and } y. \text{ Note that } y \geq 1 \text{ and } x \geq 0.$$
$$x^2 = y - 1 \qquad \text{Square both sides.}$$
$$y = x^2 + 1$$



$$f^{-1}(x) = x^2 + 1, \quad x \geq 0$$

The restriction $x \geq 0$ on f^{-1} is necessary because f has the restriction that $y \geq 0$. Furthermore, $y = x^2 + 1$ is not a one-to-one function without a restricted domain.

Skill Practice 8 Given $g(x) = \sqrt{x + 2}$, find an equation of the inverse.

Answers

7. $n^{-1}(x) = -\sqrt{x - 1}$

8. $g^{-1}(x) = x^2 - 2; x \geq 0$

SECTION 3.1 Practice Exercises

Prerequisite Review

For Exercises R.1–R.3, find the domain. Write the answer in interval notation.

R.1. $f(x) = \dfrac{x + 4}{x + 1}$ **R.2.** $k(p) = \dfrac{p + 1}{p^2 + 4}$ **R.3.** $m(x) = \sqrt{2 - 8x}$

For Exercises R.4–R.6, refer to functions n and p to evaluate the function.

$n(x) = x + 1 \quad p(x) = x^2 - 3x$

R.4. $(n \circ p)(x)$ **R.5.** $(p \circ n)(x)$ **R.6.** $(p \circ p)(x)$

Concept Connections

1. Given the function $f = \{(1, 2), (2, 3), (3, 4)\}$ write the set of ordered pairs representing f^{-1}.
2. The graph of a function and its inverse are symmetric with respect to the line ________.
3. If no horizontal line intersects the graph of a function f in more than one point, then f is a ________ - ________ - ________ function.

4. Given a one-to-one function f, if $f(a) = f(b)$, then a __________ b.

5. Let f be a one-to-one function and let g be the inverse of f. Then $(f \circ g)(x) =$ __________ and $(g \circ f)(x) =$ __________.

6. If (a, b) is a point on the graph of a one-to-one function f, then the corresponding ordered pair __________ is a point on the graph of f^{-1}.

Objective 1: Identify One-to-One Functions

For Exercises 7–12, a relation in x and y is given. Determine if the relation defines y as a one-to-one function of x. (See Example 1)

7. $\{(6, -5), (4, 2), (3, 1), (8, 4)\}$

8. $\{(-14, 1), (-2, 3), (7, 4), (-9, -2)\}$

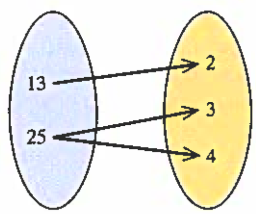
9.

x	y
0.6	1.8
1	−1.1
0.5	1.8
2.4	0.7

10.

x	y
12.5	3.21
5.75	−4.5
2.34	7.25
−12.7	3.21

11.



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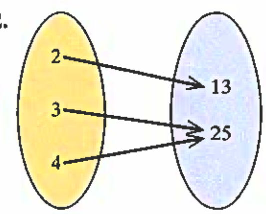
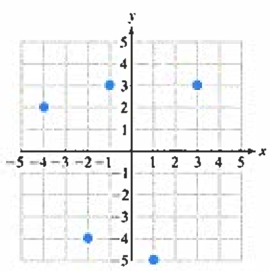


For Exercises 13–22, determine if the relation defines y as a one-to-one function of x. (See Example 2)

13.



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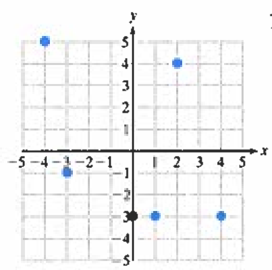

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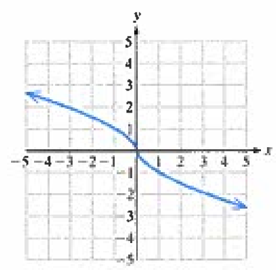

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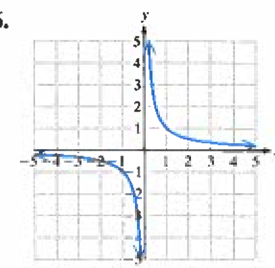

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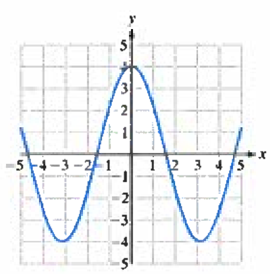

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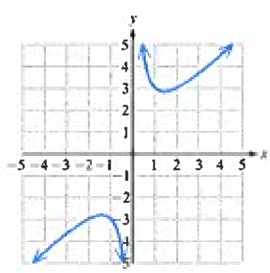

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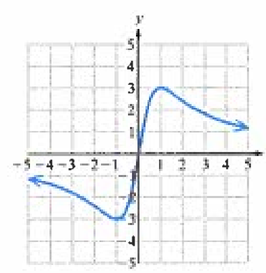

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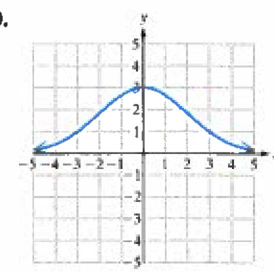

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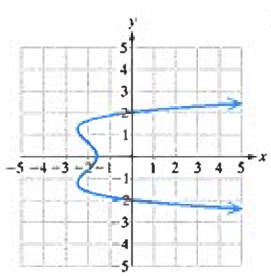

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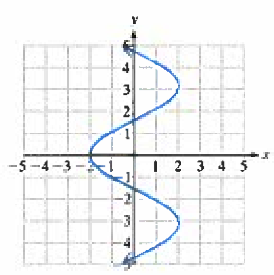


For Exercises 23–30, use the definition of a one-to-one function to determine if the function is one-to-one. (See Example 3)

23. $f(x) = 4x - 7$

24. $h(x) = -3x + 2$

25. $g(x) = x^3 + 8$

26. $k(x) = x^3 - 27$

27. $m(x) = x^2 - 4$

28. $n(x) = x^2 + 1$

29. $p(x) = |x + 1|$

30. $q(x) = |x - 3|$

Objective 2: Determine Whether Two Functions Are Inverses

For Exercises 31–36, determine whether the two functions are inverses. (See Example 4)

31. $f(x) = 5x + 4$ and $g(x) = \dfrac{x - 4}{5}$

32. $h(x) = 7x - 3$ and $k(x) = \dfrac{x + 3}{7}$

33. $m(x) = \dfrac{-2 + x}{6}$ and $n(x) = 6x - 2$

34. $p(x) = \dfrac{-3 + x}{4}$ and $q(x) = 4x - 3$

35. $t(x) = \dfrac{4}{x - 1}$ and $v(x) = \dfrac{x + 4}{x}$

36. $w(x) = \dfrac{6}{x + 2}$ and $z(x) = \dfrac{6 - 2x}{x}$

37. There were 2000 applicants for enrollment to the freshman class at a small college in the year 2010. The number of applications has risen linearly by roughly 150 per year. The number of applications $f(x)$ is given by $f(x) = 2000 + 150x$, where x is the number of years since 2010.

a. Determine if the function $g(x) = \dfrac{x - 2000}{150}$ is the inverse of f.

b. Interpret the meaning of function g in the context of this problem.

38. The monthly sales for January for a whole foods market was \$60,000 and has increased linearly by \$2500 per month. The amount in sales $f(x)$ (in \$) is given by $f(x) = 60{,}000 + 2500x$, where x is the number of months since January.

a. Determine if the function $g(x) = \dfrac{x - 60{,}000}{2500}$ is the inverse of f.

b. Interpret the meaning of function g in the context of this problem.

Objective 3: Find the Inverse of a Function

39. **a.** Show that $f(x) = 2x - 3$ defines a one-to-one function.

b. Write an equation for $f^{-1}(x)$.

c. Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system.

40. **a.** Show that $f(x) = 4x + 4$ defines a one-to-one function.

b. Write an equation for $f^{-1}(x)$.

c. Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system.

For Exercises 41–52, a one-to-one function is given. Write an equation for the inverse function. (See Examples 5–6)

41. $f(x) = \dfrac{4 - x}{9}$

42. $g(x) = \dfrac{8 - x}{3}$

43. $h(x) = \sqrt[3]{x - 5}$

44. $k(x) = \sqrt[3]{x + 8}$

45. $m(x) = 4x^3 + 2$

46. $n(x) = 2x^3 - 5$

47. $c(x) = \dfrac{5}{x + 2}$

48. $s(x) = \dfrac{2}{x - 3}$

49. $t(x) = \dfrac{x - 4}{x + 2}$

50. $v(x) = \dfrac{x - 5}{x + 1}$

51. $f(x) = \dfrac{(x - a)^3}{b} - c$

52. $g(x) = b(x + a)^3 + c$

53. **a.** Graph $f(x) = x^2 - 3;\ x \le 0$. **(See Example 7)**

b. From the graph of f, is f a one-to-one function?

c. Write the domain of f in interval notation.

d. Write the range of f in interval notation.

e. Write an equation for $f^{-1}(x)$.

f. Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system.

g. Write the domain of f^{-1} in interval notation.

h. Write the range of f^{-1} in interval notation.

54. **a.** Graph $f(x) = x^2 + 1;\ x \le 0$.

b. From the graph of f, is f a one-to-one function?

c. Write the domain of f in interval notation.

d. Write the range of f in interval notation.

e. Write an equation for $f^{-1}(x)$.

f. Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system.

g. Write the domain of f^{-1} in interval notation.

h. Write the range of f^{-1} in interval notation.

55. **a.** Graph $f(x) = \sqrt{x + 1}$. **(See Example 8)**
b. From the graph of f, is f a one-to-one function?
c. Write the domain of f in interval notation.
d. Write the range of f in interval notation.
e. Write an equation for $f^{-1}(x)$.
f. Explain why the restriction $x \geq 0$ is placed on f^{-1}.
g. Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system.
h. Write the domain of f^{-1} in interval notation.
i. Write the range of f^{-1} in interval notation.

56. **a.** Graph $f(x) = \sqrt{x - 2}$.
b. From the graph of f, is f a one-to-one function?
c. Write the domain of f in interval notation.
d. Write the range of f in interval notation.
e. Write an equation for $f^{-1}(x)$.
f. Explain why the restriction $x \geq 0$ is placed on f^{-1}.
g. Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system.
h. Write the domain of f^{-1} in interval notation.
i. Write the range of f^{-1} in interval notation.

57. Given that the domain of a one-to-one function f is $[0, \infty)$ and the range of f is $[0, 4)$, state the domain and range of f^{-1}.

58. Given that the domain of a one-to-one function f is $[-3, 5)$ and the range of f is $(-2, \infty)$, state the domain and range of f^{-1}.

59. Given $f(x) = |x| + 3$; $x \leq 0$, write an equation for f^{-1}. (*Hint*: Sketch $f(x)$ and note the domain and range.)

60. Given $f(x) = |x| - 3$; $x \geq 0$, write an equation for f^{-1}. (*Hint*: Sketch $f(x)$ and note the domain and range.)

For Exercises 61–66, fill in the blanks and determine an equation for $f^{-1}(x)$ mentally.

61. If function f adds 6 to x, then f^{-1} __________ 6 from x. Function f is defined by $f(x) = x + 6$, and function f^{-1} is defined by $f^{-1}(x) =$ __________.

62. If function f multiplies x by 2, then f^{-1} __________ x by 2. Function f is defined by $f(x) = 2x$, and function f^{-1} is defined by $f^{-1}(x) =$ __________.

63. Suppose that function f multiplies x by 7 and subtracts 4. Write an equation for $f^{-1}(x)$.

64. Suppose that function f divides x by 3 and adds 11. Write an equation for $f^{-1}(x)$.

65. Suppose that function f cubes x and adds 20. Write an equation for $f^{-1}(x)$.

66. Suppose that function f takes the cube root of x and subtracts 10. Write an equation for $f^{-1}(x)$.

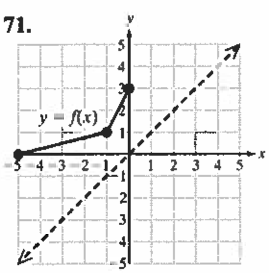
For Exercises 67–70, find the inverse mentally.

67. $f(x) = 8x + 1$ **68.** $p(x) = 2x - 10$ **69.** $q(x) = \sqrt[5]{x - 4} + 1$ **70.** $m(x) = \sqrt[3]{4x} + 3$

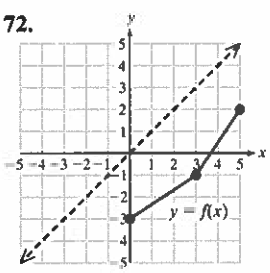
Mixed Exercises

For Exercises 71–74, the graph of a function is given. Graph the inverse function.

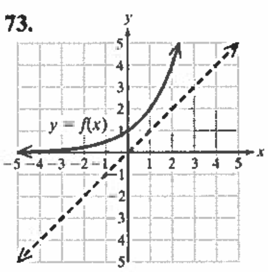
71.



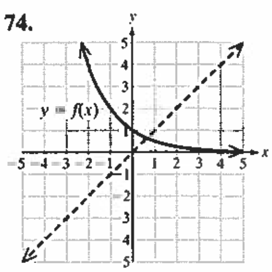
72.



73.

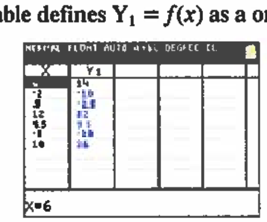


74.

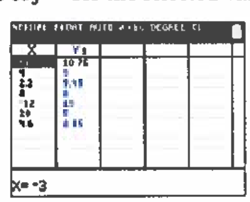


For Exercises 75–76, the table defines $Y_1 = f(x)$ as a one-to-one function of x. Find the values of f^{-1} for the selected values of x.

75. **a.** $f^{-1}(32)$
b. $f^{-1}(-2.5)$
c. $f^{-1}(26)$



76. **a.** $f^{-1}(5)$
b. $f^{-1}(9.45)$
c. $f^{-1}(8)$



For Exercises 77–80, determine if the statement is true or false. If a statement is false, explain why.

77. All linear functions with a nonzero slope have an inverse function.

78. The domain of any one-to-one function is the same as the domain of its inverse function.

79. The range of a one-to-one function is the same as the range of its inverse function.

80. No quadratic function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is one-to-one.

81. Based on data from Hurricane Katrina, the function defined by $w(x) = -1.17x + 1220$ gives the wind speed $w(x)$ (in mph) based on the barometric pressure x (in millibars, mb).
   a. Approximate the wind speed for a hurricane with a barometric pressure of 1000 mb.
   b. Write a function representing the inverse of w and interpret its meaning in context.
   c. Approximate the barometric pressure for a hurricane with wind speed 100 mph. Round to the nearest mb.

82. The function defined by $F(x) = \frac{9}{5}x + 32$ gives the temperature $F(x)$ (in degrees Fahrenheit) based on the temperature x (in Celsius).
   a. Determine the temperature in Fahrenheit if the temperature in Celsius is 25°C.
   b. Write a function representing the inverse of F and interpret its meaning in context.
   c. Determine the temperature in Celsius if the temperature in Fahrenheit is 5°F.

83. Suppose that during normal respiration, the volume of air inhaled per breath (called "tidal volume") by a mammal of any size is 6.33 mL per kilogram of body mass.
   a. Write a function representing the tidal volume $T(x)$ (in mL) of a mammal of mass x (in kg).
   b. Write an equation for $T^{-1}(x)$.
   c. What does the inverse function represent in the context of this problem?
   d. Find $T^{-1}(170)$ and interpret its meaning in context. Round to the nearest whole unit.

84. At a cruising altitude of 35,000 ft, a certain airplane travels 555 mph.
   a. Write a function representing the distance $d(x)$ (in mi) for x hours at cruising altitude.
   b. Write an equation for $d^{-1}(x)$.
   c. What does the inverse function represent in the context of this problem?
   d. Evaluate $d^{-1}(2553)$ and interpret its meaning in context.

85. The millage rate is the amount of property tax per \$1000 of the taxable value of a home. For a certain county the millage rate is 24 mil (\$24 in tax per \$1000 of taxable value of the home). A city within the county also imposes a flat fee of \$108 per home.
   a. Write a function representing the total amount of property tax $T(x)$ (in \$) for a home with a taxable value of x thousand dollars.
   b. Write an equation for $T^{-1}(x)$.
   c. What does the inverse function represent in the context of this problem?
   d. Evaluate $T^{-1}(2988)$ and interpret its meaning in context.

86. Beginning on January 1, park rangers in Everglades National Park began recording the water level for one particularly dry area of the park. The water level was initially 2.5 ft and decreased by approximately 0.015 ft/day.
   a. Write a function representing the water level $L(x)$ (in ft), x days after January 1.
   b. Write an equation for $L^{-1}(x)$.
   c. What does the inverse function represent in the context of this problem?
   d. Evaluate $L^{-1}(1.9)$ and interpret its meaning in context.

Write About It

87. Explain the relationship between the domain and range of a one-to-one function f and its inverse f^{-1}.

88. Write an informal definition of a one-to-one function.

89. Explain why if a horizontal line intersects the graph of a function in more than one point, then the function is not one-to-one.

90. Explain why the domain of $f(x) = x^2 + k$ must be restricted to find an inverse function.

Expanding Your Skills

91. Consider a function defined as follows. Given x, the value $f(x)$ is the exponent above the base of 2 that produces x. For example, $f(16) = 4$ because $2^4 = 16$. Evaluate

a. $f(8)$ **b.** $f(32)$
c. $f(2)$ **d.** $f(\frac{1}{8})$

92. Consider a function defined as follows. Given x, the value $f(x)$ is the exponent above the base of 3 that produces x. For example, $f(9) = 2$ because $3^2 = 9$. Evaluate

a. $f(27)$ **b.** $f(81)$
c. $f(3)$ **d.** $f(\frac{1}{9})$

93. Show that every increasing function is one-to-one.

94. A function is said to be periodic if there exists some nonzero real number p, called the period, such that $f(x + p) = f(x)$ for all real numbers x in the domain of f. Explain why no periodic function is one-to-one.

SECTION 3.2 Exponential Functions

OBJECTIVES

1. Graph Exponential Functions
2. Evaluate the Exponential Function Base *e*
3. Use Exponential Functions to Compute Compound Interest
4. Use Exponential Functions in Applications

1. Graph Exponential Functions

The concept of a function was first introduced in Section 1.3. Since then we have learned to recognize several categories of functions. In this section and the next, we will define two new types of functions called exponential functions and logarithmic functions.

To introduce exponential functions, consider two salary plans for a new job. Plan A pays \$1 million for 1 month's work. Plan B starts with 2¢ on the first day, and every day thereafter the salary is doubled. At first glance, the million-dollar plan appears to be more favorable. However, Table 3-3 shows otherwise. The daily payments for 30 days are listed for Plan B.

Table 3-3

Day	Payment	Day	Payment	Day	Payment
1	2¢	11	\$20.48	21	\$20,971.52
2	4¢	12	\$40.96	22	\$41,943.04
3	8¢	13	\$81.92	23	\$83,886.08
4	16¢	14	\$163.84	24	\$167,772.16
5	32¢	15	\$327.68	25	\$335,554.32
6	64¢	16	\$655.36	26	\$671,088.64
7	\$1.28	17	\$1310.72	27	\$1,342,177.28
8	\$2.56	18	\$2621.44	28	\$2,684,354.56
9	\$5.12	19	\$5242.88	29	\$5,368,709.12
10	\$10.24	20	\$10,485.76	30	\$10,737,418.24

TIP Consider the pattern involved for the payment for day $x = 1, 2, 3, 4, 5, \ldots$

2^1 ¢ = 2¢
2^2 ¢ = 4¢
2^3 ¢ = 8¢
2^4 ¢ = 16¢
2^5 ¢ = 32¢

The salary for the 30th day for Plan B is over \$10 million. Taking the sum of the payments, we see that the total salary for the 30-day period is \$21,474,836.46.

The daily salary $S(x)$ (in ¢) for Plan B can be represented by the function $S(x) = 2^x$, where x is the number of days on the job. An interesting characteristic of this function is that for every positive 1-unit change in x, the function value doubles. The function $S(x) = 2^x$ is called an exponential function.

Definition of an Exponential Function

Let b be a constant real number such that $b > 0$ and $b \neq 1$. Then for any real number x, a function of the form $f(x) = b^x$ is called an **exponential function of base b.**

An exponential function is recognized as a function with a constant base (positive and not equal to 1) with a variable exponent, x.

Exponential Functions	**Not Exponential Functions**	
$f(x) = 3^x$	$m(x) = x^2$	base is not constant
$g(x) = \left(\frac{1}{3}\right)^x$	$n(x) = \left(-\frac{1}{3}\right)^x$	base is negative
$h(x) = (\sqrt{2})^x$	$p(x) = 1^x$	base is 1

Avoiding Mistakes

- The base of an exponential function must not be negative to avoid situations where the function values are not real numbers. For example, $f(x) = (-4)^x$ is not defined for $x = \frac{1}{2}$ because $\sqrt{-4}$ is not a real number.
- The base of an exponential function must not equal 1 because $f(x) = 1^x = 1$ for all real numbers x. This is a constant function, not an exponential function.

At this point in the text, we have evaluated exponential expressions with integer exponents and with rational exponents. For example,

$$4^2 = 16 \qquad 4^{1/2} = \sqrt{4} = 2$$
$$4^{-1} = \frac{1}{4} \qquad 4^{10/23} = \sqrt[23]{4^{10}} \approx 1.827112184$$

However, how do we evaluate an exponential expression with an *irrational* exponent such as 4^{π}? In such a case, the exponent is a nonterminating, nonrepeating decimal. We define an exponential expression raised to an irrational exponent as a sequence of approximations using rational exponents. For example:

$$4^{3.14} \approx 77.7084726$$
$$4^{3.141} \approx 77.81627412$$
$$4^{3.1415} \approx 77.87023095$$
$$\ldots$$
$$4^{\pi} \approx 77.88023365$$

With this definition of a base raised to an irrational exponent, we can define an exponential function over the entire set of real numbers. In Example 1, we graph two exponential functions by plotting points.

EXAMPLE 1 Graphing Exponential Functions

Graph the functions.

a. $f(x) = 2^x$ **b.** $g(x) = \left(\frac{1}{2}\right)^x$

Solution:

Table 3-4 shows several function values $f(x)$ and $g(x)$ for both positive and negative values of x.

Table 3-4

x	$f(x) = 2^x$	$g(x) = \left(\frac{1}{2}\right)^x$
-3	$\frac{1}{8}$	8
-2	$\frac{1}{4}$	4
-1	$\frac{1}{2}$	2
0	1	1
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	8	$\frac{1}{8}$

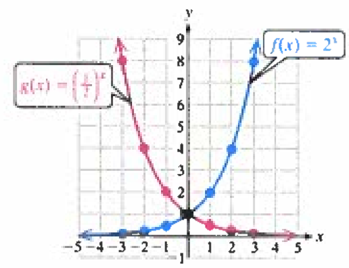


Figure 3-10

TIP The values of $f(x)$ become closer and closer to 0 as $x \to -\infty$. This means that the x-axis is a horizontal asymptote.

Likewise, the values of $g(x)$ become closer to 0 as $x \to \infty$. The x-axis is a horizontal asymptote.

Notice that $g(x) = \left(\frac{1}{2}\right)^x$ is equivalent to $g(x) = 2^{-x}$. Therefore, the graph of $g(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$ is the same as the graph of $f(x) = 2^x$ with a reflection across the y-axis (Figure 3-10).

Skill Practice 1 Graph the functions.

a. $f(x) = 5^x$ **b.** $g(x) = \left(\frac{1}{5}\right)^x$

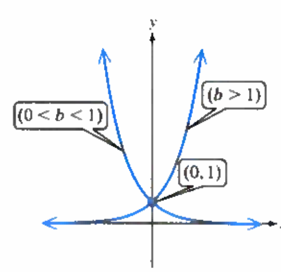
The graphs in Figure 3-10 illustrate several important features of exponential functions.

Graphs of $f(x) = b^x$

The graph of an exponential function defined by $f(x) = b^x$ ($b > 0$ and $b \neq 1$) has the following properties.

1. If $b > 1$, f is an *increasing* exponential function, sometimes called an **exponential growth function.**

   If $0 < b < 1$, f is a *decreasing* exponential function, sometimes called an **exponential decay function.**
2. The domain is the set of all real numbers, $(-\infty, \infty)$.
3. The range is $(0, \infty)$.
4. The line $y = 0$ (x-axis) is a horizontal asymptote.
5. The function passes through the point $(0, 1)$ because $f(0) = b^0 = 1$.



These properties indicate that the graph of an exponential function is an increasing function if the base is greater than 1. Furthermore, the base affects the rate of increase. Consider the graphs of $f(x) = 2^x$ and $k(x) = 5^x$ (Figure 3-11). For every positive 1-unit change in x, $f(x) = 2^x$ is 2 times as great and $k(x) = 5^x$ is 5 times as great (Table 3-5).

Answer

1.

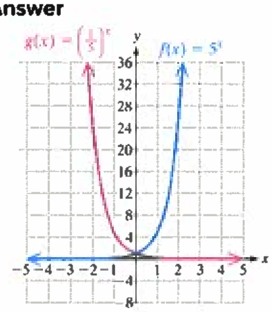


Table 3-5

x	$f(x) = 2^x$	$k(x) = 5^x$
-3	$\frac{1}{8}$	$\frac{1}{125}$
-2	$\frac{1}{4}$	$\frac{1}{25}$
-1	$\frac{1}{2}$	$\frac{1}{5}$
0	1	1
1	2	5
2	4	25
3	8	125

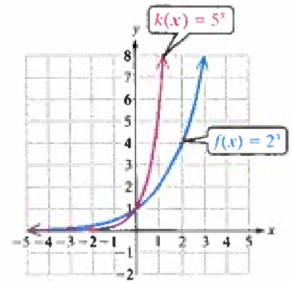


Figure 3-11

In Example 2, we use the transformations of functions learned in Section 1.6 to graph an exponential function.

If $h > 0$, shift to the right.
If $h < 0$, shift to the left.

$$f(x) = ab^{x-h} + k$$

If $a < 0$, reflect across the x-axis.
Shrink vertically if $0 < |a| < 1$.
Stretch vertically if $|a| > 1$.

If $k > 0$, shift upward.
If $k < 0$, shift downward.

EXAMPLE 2 Graphing an Exponential Function

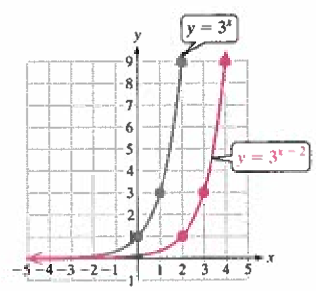
Graph. $f(x) = 3^{x-2} + 4$

Solution:

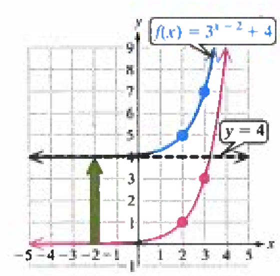
The graph of f is the graph of the parent function $y = 3^x$ shifted 2 units to the right and 4 units upward.

The parent function $y = 3^x$ is an increasing exponential function. We can plot a few points on the graph of $y = 3^x$ and use these points and the horizontal asymptote to form the outline of the transformed graph.

x	$y = 3^x$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9



Shift 2 units to the right. For example, the point $(0, 1)$ on $y = 3^x$ corresponds to $(2, 1)$ on $y = 3^{x-2}$.

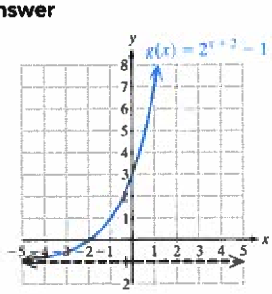


Shift the graph of $y = 3^{x-2}$ up 4 units. Notice that with the vertical shift, the new horizontal asymptote is $y = 4$.

Skill Practice 2 Graph. $g(x) = 2^{x+2} - 1$

Answer

2.



2. Evaluate the Exponential Function Base *e*

We now introduce an important exponential function whose base is an irrational number called *e*. Consider the expression $\left(1 + \frac{1}{x}\right)^x$. The value of the expression for increasingly large values of x approaches a constant (Table 3-6).

As $x \to \infty$, the expression $\left(1 + \frac{1}{x}\right)^x$ approaches a constant value that we call e. From Table 3-6, this value is approximately 2.718281828.

$$e \approx 2.718281828$$

The value of e is an irrational number (a nonterminating, nonrepeating decimal) and like the number π, it is a universal constant. The function defined by $f(x) = e^x$ is called the exponential function base e or the **natural exponential function.**

Table 3-6

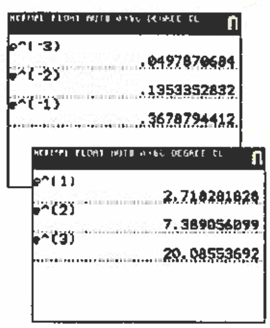
x	$\left(1 + \frac{1}{x}\right)^x$
100	2.70481382942
1000	2.71692393224
10,000	2.71814592683
100,000	2.71826823717
1,000,000	2.71828046932
1,000,000,000	2.71828182710

EXAMPLE 3 Graphing $f(x) = e^x$

Graph the function. $f(x) = e^x$

Solution:

Because the base e is greater than 1 ($e \approx 2.718281828$), the graph is an increasing exponential function. We can use a calculator to evaluate $f(x) = e^x$ at several values of x. On many calculators, the exponential function, base e, is invoked by selecting 2ND LN or by accessing e^x on the keyboard.



x	$f(x) = e^x$
−3	0.050
−2	0.135
−1	0.368
0	1.000
1	2.718
2	7.389
3	20.086

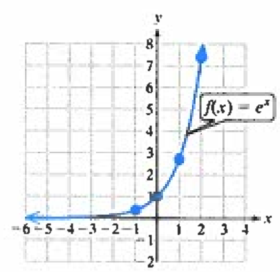


Figure 3-12

The graph of $f(x) = e^x$ is shown in Figure 3-12.

Skill Practice 3 Explain how the graph of $f(x) = -e^{x-1}$ is related to the graph of $y = e^x$.

TIP In Section 3.3, we will see that the exponential function base *e* is the inverse of the natural logarithmic function, $y = \ln x$. This is why the exponential function base *e* is accessed with the 2ND LN keys.

3. Use Exponential Functions to Compute Compound Interest

Recall that simple interest is interest computed on the principal amount invested (or borrowed). Compound interest is interest computed on both the original principal and the interest already accrued.

Answer

3. The graph of $f(x) = -e^{x-1}$ is the graph of $y = e^x$ with a shift to the right 1 unit and a reflection across the x-axis.

Suppose that interest is compounded annually (one time per year) on an investment of P dollars at an annual interest rate r for t years. Then the amount A (in \$) in the account after 1, 2, and 3 yr is computed as follows.

After 1 yr: $\begin{pmatrix}\text{Total}\\ \text{amount}\end{pmatrix} = \begin{pmatrix}\text{Initial}\\ \text{principal}\end{pmatrix} + (\text{Interest})$

$A = P + Pr$ — The interest is given by $I = Prt$, where $t = 1$ yr. So $I = Pr$.

$= P(1 + r)$ Factor out P.

After 2 yr: $\begin{pmatrix}\text{Total}\\ \text{amount}\end{pmatrix} = \begin{pmatrix}\text{Year 1}\\ \text{balance}\end{pmatrix} + \begin{pmatrix}\text{Interest on}\\ \text{Year 1 balance}\end{pmatrix}$

$A = P(1 + r) + [P(1 + r)]r$

$= P(1 + r)(1 + r)$ Factor out $P(1 + r)$.

$= P(1 + r)^2$

After 3 yr: $\begin{pmatrix}\text{Total}\\ \text{amount}\end{pmatrix} = \begin{pmatrix}\text{Year 2}\\ \text{balance}\end{pmatrix} + \begin{pmatrix}\text{Interest on}\\ \text{Year 2 balance}\end{pmatrix}$

$A = P(1 + r)^2 + [P(1 + r)^2]r$

$= P(1 + r)^2(1 + r)$ Factor out $P(1 + r)^2$.

$= P(1 + r)^3$

...

After t years: $A = P(1 + r)^t$ — Amount in an account with interest compounded annually.

Compound interest is often computed more frequently during the course of 1 yr. Let n represent the number of compounding periods per year. For example:

$n = 1$	for interest compounded annually
$n = 4$	for interest compounded quarterly
$n = 12$	for interest compounded monthly
$n = 365$	for interest compounded daily

Each compounding period represents a fraction of a year and the interest rate is scaled accordingly for each compounding period as $\frac{1}{n} \cdot r$ or $\frac{r}{n}$. The number of compounding periods over the course of the investment is nt. Therefore, to determine the amount in an account where interest is compounded n times per year we have

replace t by nt

$A = P(1 + r)^t$ $\qquad$ $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$ — Amount in an account with interest compounded n times per year.

replace r by $\frac{r}{n}$

Now suppose it were possible to compute interest continuously, that is, for $n \to \infty$. If we use the substitution $x = \frac{n}{r}$ (which implies that $n = xr$) the formula for compound interest becomes

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \xrightarrow{\text{Substitute } x = \frac{n}{r}} P\left(1 + \frac{1}{x}\right)^{xrt} = P\left[\left(1 + \frac{1}{x}\right)^{x}\right]^{rt}$$

For a fixed interest rate r, as n approaches infinity, x also approaches infinity. Since the expression $\left(1 + \frac{1}{x}\right)^x$ approaches e as $x \to \infty$, we have

$A = Pe^{rt}$ — Amount in an account with interest compounded continuously.

Summary of Formulas Relating to Simple and Compound Interest

Suppose that P dollars in principal is invested (or borrowed) at an annual interest rate r for t years. Then

- $I = Prt$ — Amount of simple interest I (in \$).
- $A = P\left(1 + \frac{r}{n}\right)^{nt}$ — The future value A (in \$) of the account after t years with n compounding periods per year.
- $A = Pe^{rt}$ — The future value A (in \$) of the account after t years under continuous compounding.

In Example 4, we compare the value of an investment after 10 yr under several different compounding options.

EXAMPLE 4 Computing the Balance on an Account

Suppose that \$5000 is invested and pays 6.5% per year under the following compounding options.

a. Compounded annually
b. Compounded quarterly
c. Compounded monthly
d. Compounded daily
e. Compounded continuously

Determine the total amount in the account after 10 yr with each option.

Solution:

Using $A = P\left(1 + \frac{r}{n}\right)^{nt}$ and $A = Pe^{rt}$, we have

Compounding Option	n Value	Formula	Result
Annually	$n = 1$	$A = 5000\left(1 + \frac{0.065}{1}\right)^{(1 \cdot 10)}$	\$9385.69
Quarterly	$n = 4$	$A = 5000\left(1 + \frac{0.065}{4}\right)^{(4 \cdot 10)}$	\$9527.79
Monthly	$n = 12$	$A = 5000\left(1 + \frac{0.065}{12}\right)^{(12 \cdot 10)}$	\$9560.92
Daily	$n = 365$	$A = 5000\left(1 + \frac{0.065}{365}\right)^{(365 \cdot 10)}$	\$9577.15
Continuously	Not applicable	$A = 5000e^{(0.065 \cdot 10)}$	\$9577.70

Notice that there is a \$192.01 difference in the account balance between annual compounding and continuous compounding. The table also supports our finding that

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{converges to} \quad A = Pe^{rt} \quad \text{as } n \to \infty.$$

Skill Practice 4 Suppose that \$8000 is invested and pays 4.5% per year under the following compounding options.

a. Compounded annually
b. Compounded quarterly
c. Compounded monthly
d. Compounded daily
e. Compounded continuously

Determine the total amount in the account after 5 yr with each option.

Answers

4. **a.** \$9969.46 **b.** \$10,006.00 **c.** \$10,014.37 **d.** \$10,018.44 **e.** \$10,018.58

4. Use Exponential Functions in Applications

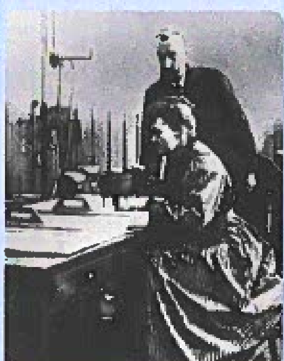
Increasing and decreasing exponential functions can be used in a variety of real-world applications. For example:

- Population growth can often be modeled by an exponential function.
- The growth of an investment under compound interest increases exponentially.
- The mass of a radioactive substance decreases exponentially with time.
- The temperature of a cup of coffee decreases exponentially as it approaches room temperature.

A substance that undergoes radioactive decay is said to be radioactive. The **half-life** of a radioactive substance is the amount of time it takes for one-half of the original amount of the substance to change into something else. That is, after each half-life, the amount of the original substance decreases by one-half.

Point of Interest

In 1898, Marie Curie discovered the highly radioactive element radium. She shared the 1903 Nobel Prize in physics for her research on radioactivity and was awarded the 1911 Nobel Prize in chemistry for her discovery of radium and polonium. Marie Curie died in 1934 from complications of excessive exposure to radiation.



Marie and Pierre Curie

EXAMPLE 5 Using an Exponential Function in an Application

The half-life of radium 226 is 1620 yr. In a sample originally having 1 g of radium 226, the amount $A(t)$ (in grams) of radium 226 present after t years is given by $A(t) = \left(\frac{1}{2}\right)^{t/1620}$ where t is the time in years after the start of the experiment. How much radium will be present after

a. 1620 yr? **b.** 3240 yr? **c.** 4860 yr?

Solution:

a. $A(t) = \left(\frac{1}{2}\right)^{t/1620}$

$$A(1620) = \left(\frac{1}{2}\right)^{1620/1620} = \left(\frac{1}{2}\right)^{1} = 0.5$$

b. $A(t) = \left(\frac{1}{2}\right)^{t/1620}$

$$A(3240) = \left(\frac{1}{2}\right)^{3240/1620} = \left(\frac{1}{2}\right)^{2} = 0.25$$

c. $A(t) = \left(\frac{1}{2}\right)^{t/1620}$

$$A(4860) = \left(\frac{1}{2}\right)^{4860/1620} = \left(\frac{1}{2}\right)^{3} = 0.125$$

The half-life of radium is 1620 yr. Therefore, we can interpret these results as follows.

After 1620 yr (1 half-life), 0.5 g remains ($\frac{1}{2}$ of the original amount remains).

After 3240 yr (2 half-lives), 0.25 g remains ($\frac{1}{4}$ of the original amount remains).

After 4860 yr (3 half-lives), 0.125 g remains ($\frac{1}{8}$ of the original amount remains).

Skill Practice 5 Cesium-137 is a radioactive metal with a short half-life of 30 yr. In a sample originally having 2 g of cesium-137, the amount $A(t)$ (in grams) of cesium-137 present after t years is given by $A(t) = 2\left(\frac{1}{2}\right)^{t/30}$. How much cesium-137 will be present after

a. 30 yr? **b.** 60 yr? **c.** 90 yr?

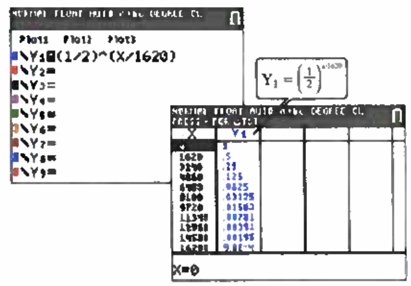
Answers

5. a. 1 g **b.** 0.5 g **c.** 0.25 g

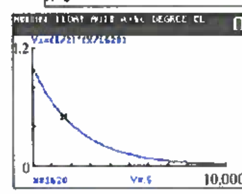
TECHNOLOGY CONNECTIONS

Graphing an Exponential Function

A graphing utility can be used to graph and analyze exponential functions. The table shows several values of $A(x) = \left(\frac{1}{2}\right)^{x/1620}$ for selected values of x.



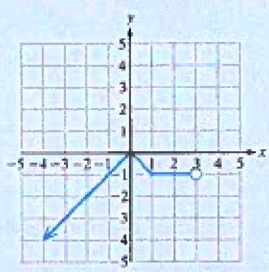
The graph of $A(x) = \left(\frac{1}{2}\right)^{x/1620}$ is shown on the viewing window [0, 10,000, 1000] by [0, 1.2, 0.2]. Notice that the graph is a decreasing exponential function because the base is between 0 and 1.



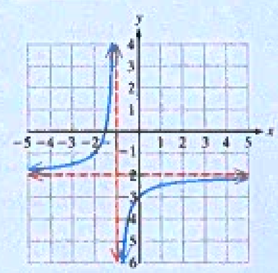
SECTION 3.2 Practice Exercises

Prerequisite Review

R.1. Determine the domain and range.



R.2. Refer to the graph of the function and complete the statement.



a. As $x \to -\infty, f(x) \to$ __________

b. As $x \to -1^+, f(x) \to$ __________

c. As $x \to \infty, f(x) \to$ __________

d. As $x \to -1^-, f(x) \to$ __________

e. The graph is decreasing over the interval(s) __________.

f. The graph is increasing over the interval(s) __________.

R.3. Use transformations to graph $q(x) = -(x - 3)^2 + 4$.

Concept Connections

1. The function defined by $y = x^3$ (is/is not) an exponential function, whereas the function defined by $y = 3^x$ (is/is not) an exponential function.

2. The graph of $f(x) = \left(\frac{5}{3}\right)^x$ is (increasing/decreasing) over its domain.

3. The graph of $f(x) = \left(\frac{3}{5}\right)^x$ is (increasing/decreasing) over its domain.

4. The domain of an exponential function $f(x) = b^x$ is __________.

5. The range of an exponential function $f(x) = b^x$ is __________.

6. All exponential functions $f(x) = b^x$ pass through the point __________.

7. The horizontal asymptote of an exponential function $f(x) = b^x$ is the line __________.

8. As $x \to \infty$, the value of $\left(1 + \frac{1}{x}\right)^x$ approaches __________.

Objective 1: Graph Exponential Functions

For Exercises 9–12, evaluate the functions at the given values of x. Round to 4 decimal places if necessary.

9. $f(x) = 5^x$
   a. $f(-1)$
   b. $f(4.8)$
   c. $f(\sqrt{2})$
   d. $f(\pi)$

10. $g(x) = 7^x$
   a. $g(-2)$
   b. $g(5.9)$
   c. $g(\sqrt{11})$
   d. $g(e)$

11. $h(x) = \left(\frac{1}{4}\right)^x$
   a. $h(-3)$
   b. $h(1.4)$
   c. $h(\sqrt{3})$
   d. $h(0.5e)$

12. $k(x) = \left(\frac{1}{6}\right)^x$
   a. $k(-3)$
   b. $k(1.4)$
   c. $k(\sqrt{0.5})$
   d. $h(0.5\pi)$

13. Which functions are exponential functions?
   a. $f(x) = 4.2^x$ b. $g(x) = x^{4.2}$ c. $h(x) = 4.2x$
   d. $k(x) = (\sqrt{4.2})^x$ e. $m(x) = (-4.2)^x$

14. Which functions are exponential functions?
   a. $v(x) = (-\pi)^x$ b. $t(x) = \pi^x$ c. $w(x) = \pi x$
   d. $n(x) = (\sqrt{\pi})^x$ e. $p(x) = x^\pi$

For Exercises 15–22, graph the functions and write the domain and range in interval notation. (See Example 1)

15. $f(x) = 3^x$
16. $g(x) = 4^x$
17. $h(x) = \left(\frac{1}{3}\right)^x$
18. $k(x) = \left(\frac{1}{4}\right)^x$
19. $m(x) = \left(\frac{3}{2}\right)^x$
20. $n(x) = \left(\frac{5}{4}\right)^x$
21. $b(x) = \left(\frac{2}{3}\right)^x$
22. $c(x) = \left(\frac{4}{5}\right)^x$

For Exercises 23–32,

a. Use transformations of the graphs of $y = 3^x$ (see Exercise 15) and $y = 4^x$ (see Exercise 16) to graph the given function. **(See Example 2)**

b. Write the domain and range in interval notation.

c. Write an equation of the asymptote.

23. $f(x) = 3^x + 2$
24. $g(x) = 4^x - 3$
25. $m(x) = 3^{x+2}$
26. $n(x) = 4^{x-3}$
27. $p(x) = 3^{x-4} - 1$
28. $q(x) = 4^{x+1} + 2$
29. $k(x) = -3^x$
30. $h(x) = -4^x$
31. $t(x) = 3^{-x}$
32. $v(x) = 4^{-x}$

For Exercises 33–36,

a. Use transformations of the graphs of $y = \left(\frac{1}{3}\right)^x$ (see Exercise 17) and $y = \left(\frac{1}{4}\right)^x$ (see Exercise 18) to graph the given function. **(See Example 2)**

b. Write the domain and range in interval notation.

c. Write an equation of the asymptote.

33. $f(x) = \left(\frac{1}{3}\right)^{x+1} - 3$
34. $g(x) = \left(\frac{1}{4}\right)^{x-2} + 1$
35. $k(x) = -\left(\frac{1}{3}\right)^x + 2$
36. $h(x) = -\left(\frac{1}{4}\right)^x - 2$

Objective 2: Evaluate the Exponential Function Base e

For Exercises 37–38, evaluate the functions for the given values of x. Round to 4 decimal places.

37. $f(x) = e^x$
- **a.** $f(4)$
- **b.** $f(-3.2)$
- **c.** $f(\sqrt{13})$
- **d.** $f(\pi)$

38. $f(x) = e^x$
- **a.** $f(-3)$
- **b.** $f(6.8)$
- **c.** $f(\sqrt{7})$
- **d.** $f(e)$

For Exercises 39–44,

a. Use transformations of the graph of $y = e^x$ to graph the given function. (**See Example 3**)

b. Write the domain and range in interval notation.

c. Write an equation of the asymptote.

39. $f(x) = e^{x-4}$ **40.** $g(x) = e^{x-2}$ **41.** $h(x) = e^x + 2$

42. $k(x) = e^x - 1$ **43.** $m(x) = -e^x - 3$ **44.** $n(x) = -e^x + 4$

Objective 3: Use Exponential Functions to Compute Compound Interest

For Exercises 45–46, complete the table to determine the effect of the number of compounding periods when computing interest. (See Example 4)

45. Suppose that \$10,000 is invested at 4% interest for 5 yr under the following compounding options. Complete the table.

	Compounding Option	n Value	Result
a.	Annually		
b.	Quarterly		
c.	Monthly		
d.	Daily		
e.	Continuously		

46. Suppose that \$8000 is invested at 3.5% interest for 20 yr under the following compounding options. Complete the table.

	Compounding Option	n Value	Result
a.	Annually		
b.	Quarterly		
c.	Monthly		
d.	Daily		
e.	Continuously		

For Exercises 47–48, suppose that P dollars in principal is invested for t years at the given interest rates with continuous compounding. Determine the amount that the investment is worth at the end of the given time period.

47. $P = \$20{,}000$, $t = 10$ yr
- **a.** 3% interest
- **b.** 4% interest
- **c.** 5.5% interest

48. $P = \$6000$, $t = 12$ yr
- **a.** 1% interest
- **b.** 2% interest
- **c.** 4.5% interest

49. Bethany needs to borrow \$10,000. She can borrow the money at 5.5% simple interest for 4 yr or she can borrow at 5% with interest compounded continuously for 4 yr.
- **a.** How much total interest would Bethany pay at 5.5% simple interest?
- **b.** How much total interest would Bethany pay at 5% interest compounded continuously?
- **c.** Which option results in less total interest?

50. Al needs to borrow \$15,000 to buy a car. He can borrow the money at 6.7% simple interest for 5 yr or he can borrow at 6.4% interest compounded continuously for 5 yr.
- **a.** How much total interest would Al pay at 6.7% simple interest?
- **b.** How much total interest would Al pay at 6.4% interest compounded continuously?
- **c.** Which option results in less total interest?

51. Jerome wants to invest \$25,000 as part of his retirement plan. He can invest the money at 5.2% simple interest for 30 yr, or he can invest at 3.8% interest compounded continuously for 30 yr. Which option results in more total interest?

52. Heather wants to invest \$35,000 of her retirement. She can invest at 4.8% simple interest for 20 yr, or she can choose an option with 3.6% interest compounded continuously for 20 yr. Which option results in more total interest?

Objective 4: Use Exponential Functions in Applications

53. Strontium-90 (^{90}Sr) is a by-product of nuclear fission with a half-life of approximately 28.9 yr. After the Chernobyl nuclear reactor accident in 1986, large areas surrounding the site were contaminated with ^{90}Sr. If 10 μg (micrograms) of ^{90}Sr is present in a sample, the function $A(t) = 10\left(\frac{1}{2}\right)^{t/28.9}$ gives the amount $A(t)$ (in μg) present after t years. Evaluate the function for the given values of t and interpret the meaning in context. Round to 3 decimal places if necessary. **(See Example 5)**

a. $A(28.9)$ **b.** $A(57.8)$ **c.** $A(100)$

54. In 2006, the murder of Alexander Litvinenko, a Russian dissident, was thought to be by poisoning from the rare and highly radioactive element polonium-210 (^{210}Po). The half-life of ^{210}Po is 138.4 yr. If 0.1 mg of ^{210}Po is present in a sample then $A(t) = 0.1\left(\frac{1}{2}\right)^{t/138.4}$ gives the amount $A(t)$ (in mg) present after t years. Evaluate the function for the given values of t and interpret the meaning in context. Round to 3 decimal places if necessary.

a. $A(138.4)$ **b.** $A(276.8)$ **c.** $A(500)$

55. According to the CIA's *World Fact Book*, in 2010, the population of the United States was approximately 310 million with a 0.97% annual growth rate. (*Source*: www.cia.gov) At this rate, the population $P(t)$ (in millions) can be approximated by $P(t) = 310(1.0097)^t$, where t is the time in years since 2010.

a. Is the graph of P an increasing or decreasing exponential function?

b. Evaluate $P(0)$ and interpret its meaning in the context of this problem.

c. Evaluate $P(10)$ and interpret its meaning in the context of this problem. Round the population value to the nearest million.

d. Evaluate $P(20)$ and $P(30)$.

e. Evaluate $P(200)$ and use this result to determine if it is reasonable to expect this model to continue indefinitely.

56. The population of Canada in 2010 was approximately 34 million with an annual growth rate of 0.804%. At this rate, the population $P(t)$ (in millions) can be approximated by $P(t) = 34(1.00804)^t$, where t is the time in years since 2010. (*Source*: www.cia.gov)

a. Is the graph of P an increasing or decreasing exponential function?

b. Evaluate $P(0)$ and interpret its meaning in the context of this problem.

c. Evaluate $P(5)$ and interpret its meaning in the context of this problem. Round the population value to the nearest million.

d. Evaluate $P(15)$ and $P(25)$.

57. The atmospheric pressure on an object decreases as altitude increases. If a is the height (in km) above sea level, then the pressure $P(a)$ (in mmHg) is approximated by $P(a) = 760e^{-0.13a}$.

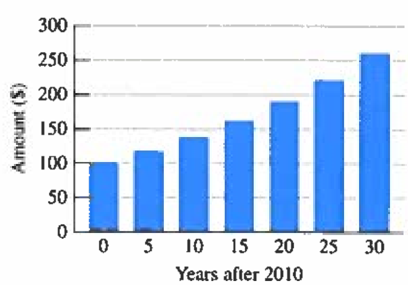
a. Find the atmospheric pressure at sea level.

b. Determine the atmospheric pressure at 8.848 km (the altitude of Mt. Everest). Round to the nearest whole unit.

58. The function defined by $A(t) = 100e^{0.0318t}$ approximates the equivalent amount of money needed t years after the year 2010 to equal \$100 of buying power in the year 2010. The value 0.0318 is related to the average rate of inflation.

a. Evaluate $A(15)$ and interpret its meaning in the context of this problem.

b. Verify that by the year 2032, more than \$200 will be needed to have the same buying power as \$100 in 2010.



Newton's law of cooling indicates that the temperature of a warm object, such as a cake coming out of the oven, will decrease exponentially with time and will approach the temperature of the surrounding air. The temperature $T(t)$ is modeled by $T(t) = T_a + (T_0 - T_a)e^{-kt}$. In this model, T_a represents the temperature of the surrounding air, T_0 represents the initial temperature of the object, and t is the time after the object starts cooling. The value of k is a constant of proportion relating the temperature of the object to its rate of temperature change. Use this model for Exercises 59–60.

59. A cake comes out of the oven at 350°F and is placed on a cooling rack in a 78°F kitchen. After checking the temperature several minutes later, the value of k is measured as 0.046.

a. Write a function that models the temperature $T(t)$ (in °F) of the cake t minutes after being removed from the oven.

b. What is the temperature of the cake 10 min after coming out of the oven? Round to the nearest degree.

c. It is recommended that the cake should not be frosted until it has cooled to under 100°F. If Jessica waits 1 hr to frost the cake, will the cake be cool enough to frost?

60. Water in a water heater is originally 122°F. The water heater is shut off and the water cools to the temperature of the surrounding air, which is 60°F. The water cools slowly because of the insulation inside the heater, and the value of k is measured as 0.00351.

a. Write a function that models the temperature $T(t)$ (in °F) of the water t hours after the water heater is shut off.

b. What is the temperature of the water 12 hr after the heater is shut off? Round to the nearest degree.

c. Dominic does not like to shower with water less than 115°F. If Dominic waits 24 hr, will the water still be warm enough for a shower?

61. A farmer depreciates a \$120,000 tractor. He estimates that the resale value $V(t)$ (in \$1000) of the tractor t years after purchase is 80% of its value from the previous year. Therefore, the resale value can be approximated by $V(t) = 120(0.8)^t$.

a. Find the resale value 5 yr after purchase. Round to the nearest \$1000.

b. The farmer estimates that the cost to run the tractor is \$18/hr in labor, \$36/hr in fuel, and \$22/hr in overhead costs (for maintenance and repair). Estimate the farmer's cost to run the tractor for the first year if he runs the tractor for a total of 800 hr. Include hourly costs and depreciation.

62. A veterinarian depreciates a \$10,000 X-ray machine. He estimates that the resale value $V(t)$ (in \$) after t years is 90% of its value from the previous year. Therefore, the resale value can be approximated by $V(t) = 10{,}000(0.9)^t$.

a. Find the resale value after 4 yr.

b. If the veterinarian wants to sell his practice 8 yr after the X-ray machine was purchased, how much is the machine worth? Round to the nearest \$100.

Mixed Exercises

For Exercises 63–64, solve the equations in parts (a)–(c) by inspection. Then estimate the solutions to parts (d) and (e) between two consecutive integers.

63. a. $2^x = 4$
b. $2^x = 8$
c. $2^x = 16$
d. $2^x = 7$
e. $2^x = 10$

64. a. $3^x = 3$
b. $3^x = 9$
c. $3^x = 27$
d. $3^x = 7$
e. $3^x = 10$

65. a. Graph $f(x) = 2^x$. **(See Example 1)**
b. Is f a one-to-one function?
c. Write the domain and range of f in interval notation.
d. Graph f^{-1} on the same coordinate system as f.
e. Write the domain and range of f^{-1} in interval notation.
f. From the graph evaluate $f^{-1}(1), f^{-1}(2)$, and $f^{-1}(4)$.

66. a. Graph $g(x) = 3^x$ (see Exercise 15).
b. Is g a one-to-one function?
c. Write the domain and range of g in interval notation.
d. Graph g^{-1} on the same coordinate system as g.
e. Write the domain and range of g^{-1} in interval notation.
f. From the graph evaluate $g^{-1}(1), g^{-1}(3)$, and $g^{-1}(\frac{1}{3})$.

67. Refer to the graphs of $f(x) = 2^x$ and the inverse function, $y = f^{-1}(x)$ from Exercise 65. Fill in the blanks.

a. As $x \to \infty$, $f(x) \to$ ________.
b. As $x \to -\infty$, $f(x) \to$ ________.
c. As $x \to \infty$, $f^{-1}(x) \to$ ________.
d. As $x \to 0^+$, $f^{-1}(x) \to$ ________.

68. Refer to the graphs of $g(x) = 3^x$ and the inverse function, $y = g^{-1}(x)$ from Exercise 66. Fill in the blanks.

a. As $x \to \infty$, $g(x) \to$ ________.
b. As $x \to -\infty$, $g(x) \to$ ________.
c. As $x \to \infty$, $g^{-1}(x) \to$ ________.
d. As $x \to 0^+$, $g^{-1}(x) \to$ ________.

Write About It

69. Explain why the equation $2^x = -2$ has no solution.

70. Explain why the $f(x) = x^2$ is not an exponential function.

Expanding Your Skills

For Exercises 71–72, find the real solutions to the equation.

71. $3x^2e^{-x} - 6xe^{-x} = 0$

72. $x^2e^x - e^x = 0$

73. Use the properties of exponents to simplify.

a. e^xe^h **b.** $(e^x)^2$ **c.** $\dfrac{e^x}{e^h}$

d. $e^x \cdot e^{-x}$ **e.** e^{-2x}

74. Factor.

a. $e^{x+h} - e^x$

b. $e^{4x} - e^{2x}$

75. Multiply. $(e^x + e^{-x})^2$

76. Multiply. $(e^x - e^{-x})^2$

77. Show that $\left(\dfrac{e^x + e^{-x}}{2}\right)^2 - \left(\dfrac{e^x - e^{-x}}{2}\right)^2 = 1.$

78. Show that $2\left(\dfrac{e^x - e^{-x}}{2}\right)\left(\dfrac{e^x + e^{-x}}{2}\right) = \dfrac{e^{2x} - e^{-2x}}{2}.$

For Exercises 79–80, find the difference quotient $\dfrac{f(x + h) - f(x)}{h}$. Write the answers in factored form.

79. $f(x) = e^x$

80. $f(x) = 2^x$

Technology Connections

81. Graph the following functions on the window $[-3, 3, 1]$ by $[-1, 8, 1]$ and comment on the behavior of the graphs near $x = 0$.

$$Y_1 = e^x$$

$$Y_2 = 1 + x + \frac{x^2}{2}$$

$$Y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

SECTION 3.3 Logarithmic Functions

OBJECTIVES

1. Convert Between Logarithmic and Exponential Forms
2. Evaluate Logarithmic Expressions
3. Apply Basic Properties of Logarithms
4. Graph Logarithmic Functions
5. Use Logarithmic Functions in Applications

1. Convert Between Logarithmic and Exponential Forms

Consider the following equations in which the variable is located in the exponent of an expression. In some cases, the solution can be found by inspection.

Equation	Solution
$5^x = 5$	$x = 1$
$5^x = 20$	$x = ?$
$5^x = 25$	$x = 2$
$5^x = 60$	$x = ?$
$5^x = 125$	$x = 3$

The equation $5^x = 20$ cannot be solved by inspection. However, we suspect that x is between 1 and 2 because $5^1 = 5$ and $5^2 = 25$. To solve for x explicitly, we must isolate x by performing the inverse operation of 5^x. Fortunately, all exponential functions $y = b^x$ ($b > 0$, $b \neq 1$) are one-to-one and therefore have inverse functions. The inverse of an exponential function, base b, is the *logarithmic* function base b which we define here.

Definition of a Logarithmic Function

If x and b are positive real numbers such that $b \neq 1$, then $y = \log_b x$ is called the **logarithmic function base *b*,** where

$$y = \log_b x \text{ is equivalent to } b^y = x$$

Notes:

- Given $y = \log_b x$, the value y is the exponent to which b must be raised to obtain x.
- The value of y is called the **logarithm,** b is called the **base,** and x is called the **argument.**
- The equations $y = \log_b x$ and $b^y = x$ both define the same relationship between x and y. The expression $y = \log_b x$ is called the **logarithmic form,** and $b^y = x$ is called the **exponential form.**

The logarithmic function base b is defined as the inverse of the exponential function base b.

exponential function	$f(x) = b^x$	First replace $f(x)$ by y.
	$y = b^x$	Next, interchange x and y.
inverse of exponential function	$x = b^y$	This equation provides an implicit relationship between x and y. To solve for y explicitly (that is, to isolate y), we must use logarithmic notation.
logarithmic function	$y = \log_b x$	

To be able to solve equations involving logarithms, it is often advantageous to write a logarithmic expression in its exponential form.

EXAMPLE 1 **Writing Logarithmic Form and Exponential Form**

Write each equation in exponential form.

a. $\log_2 16 = 4$ **b.** $\log_{10}\left(\frac{1}{100}\right) = -2$ **c.** $\log_7 1 = 0$

Solution:

Logarithmic form $y = \log_b x$		**Exponential form** $b^y = x$
a. $\log_2 16 = 4$	$\Leftrightarrow$	$2^4 = 16$
b. $\log_{10}\left(\frac{1}{100}\right) = -2$	$\Leftrightarrow$	$10^{-2} = \frac{1}{100}$
c. $\log_7 1 = 0$	$\Leftrightarrow$	$7^0 = 1$

The logarithm is the exponent to which the base is raised to obtain x.

Skill Practice 1 Write each equation in exponential form.

a. $\log_3 9 = 2$ **b.** $\log_{10}\left(\frac{1}{1000}\right) = -3$ **c.** $\log_6 1 = 0$

In Example 2 we reverse this process and write an exponential equation in its logarithmic form.

Answers

1. a. $3^2 = 9$ **b.** $10^{-3} = \frac{1}{1000}$

c. $6^0 = 1$

EXAMPLE 2 Writing Exponential Form and Logarithmic Form

Write each equation in logarithmic form.

a. $3^4 = 81$ **b.** $10^6 = 1{,}000{,}000$ **c.** $\left(\frac{1}{5}\right)^{-1} = 5$

Solution:

Exponential form $b^y = x$		Logarithmic form $\log_b x = y$
a. $\underbrace{3}_{\text{base}}^{\overbrace{4}^{\text{logarithm}}} = \underbrace{81}_{\text{argument}}$	$\Leftrightarrow$	$\log_{\underbrace{3}_{\text{base}}} \underbrace{81}_{\text{argument}} = \overbrace{4}^{\text{logarithm (power)}}$
b. $10^6 = 1{,}000{,}000$	$\Leftrightarrow$	$\log_{10} 1{,}000{,}000 = 6$
c. $\left(\frac{1}{5}\right)^{-1} = 5$	$\Leftrightarrow$	$\log_{1/5} 5 = -1$

Skill Practice 2 Write each equation in logarithmic form.

a. $2^5 = 32$ **b.** $10^4 = 10{,}000$ **c.** $\left(\frac{1}{8}\right)^{-2} = 64$

2. Evaluate Logarithmic Expressions

To evaluate a logarithmic expression, we can write the expression in exponential form. Then we make use of the equivalence property of exponential expressions. This states that if two exponential expressions of the same base are equal, then their exponents must be equal.

Equivalence Property of Exponential Expressions

If b, x, and y are real numbers, with $b > 0$ and $b \neq 1$, then

$$b^x = b^y \text{ implies that } x = y.$$

In Example 3, we evaluate several logarithmic expressions.

EXAMPLE 3 Evaluating a Logarithmic Expression

Evaluate each expression.

a. $\log_4 16$ **b.** $\log_2 8$ **c.** $\log_{1/2} 8$

Solution:

Let y represent the value of the logarithm.

a. $\log_4 16$ is the exponent to which 4 must be raised to equal 16. That is, $4^{\square} = 16$.

$\log_4 16 = y$

$4^y = 16$ or equivalently $4^y = 4^2$ Write the equivalent exponential form.

$y = 2$

Therefore, $\log_4 16 = 2$. Check: $4^2 = 16$ ✓

Answers

2. a. $\log_2 32 = 5$
b. $\log_{10} 10{,}000 = 4$
c. $\log_{1/8} 64 = -2$

TIP Once you become comfortable with the concept of a logarithm, you can take fewer steps to evaluate a logarithm.

To evaluate the expression $\log_4 16$ we ask $4^{\square} = 16$. The exponent is 2, so $\log_4 16 = 2$.

Likewise, to evaluate $\log_2 8$ we ask $2^{\square} = 8$. So $\log_2 8 = 3$.

b. $\log_2 8$ is the exponent to which 2 must be raised to obtain 8. That is, $2^{\square} = 8$.

$\log_2 8 = y$

$2^y = 8$ or equivalently $2^y = 2^3$ — Write the equivalent exponential form.

$y = 3$

Therefore, $\log_2 8 = 3$. — Check: $2^3 = 8$ ✓

c. $\log_{1/2} 8 = y$

$\left(\frac{1}{2}\right)^y = 8$ or equivalently $\left(\frac{1}{2}\right)^y = \left(\frac{1}{2}\right)^{-3}$ — Write the equivalent exponential form.

$y = -3$

Therefore, $\log_{1/2} 8 = -3$. — Check: $\left(\frac{1}{2}\right)^{-3} = 8$ ✓

Skill Practice 3 Evaluate each expression.

a. $\log_5 125$ **b.** $\log_3 81$ **c.** $\log_4\left(\frac{1}{64}\right)$

The statement $y = \log_b x$ represents a family of logarithmic functions where the base is any positive real number except 1. Two specific logarithmic functions that come up often in applications are the logarithmic functions base 10 and base e.

TIP To help you remember the notation $y = \ln x$, think of "ln" as "log natural."

Definition of Common and Natural Logarithmic Functions

- The logarithmic function base 10 is called the **common logarithmic function.** The common logarithmic function is denoted by $y = \log x$. Notice that the base 10 is not explicitly written; that is, $y = \log_{10} x$ is written simply as $y = \log x$.
- The logarithmic function base e is called the **natural logarithmic function.** The natural logarithmic function is denoted by $y = \ln x$; that is, $y = \log_e x$ is written as $y = \ln x$.

EXAMPLE 4 Evaluating Common and Natural Logarithms

Evaluate.

a. $\log 100{,}000$ **b.** $\log 0.001$ **c.** $\ln e^4$ **d.** $\ln\left(\frac{1}{e}\right)$

Solution:

Let y represent the value of the logarithm.

a. $\log 100{,}000 = y$

$10^y = 100{,}000$ or equivalently $10^y = 10^5$ — Write the exponential form.

$y = 5$

Thus, $\log 100{,}000 = 5$ because $10^5 = 100{,}000$.

b. $\log 0.001 = y$

$10^y = 0.001$ or equivalently $10^y = 10^{-3}$ — Write the exponential form.

$y = -3$

Thus, $\log 0.001 = -3$ because $10^{-3} = 0.001$.

Answers

3. a. 3 **b.** 4 **c.** -3

TIP In Example 4, to evaluate:

a. log 100,000 we ask $10^{\square} = 100{,}000$.

b. log 0.001 we ask $10^{\square} = 0.001$.

c. $\ln e^4$ we ask $e^{\square} = e^4$.

d. $\ln\left(\frac{1}{e}\right)$ we ask $e^{\square} = e^{-1}$.

c. $\ln e^4 = y$

$e^y = e^4$ Write the equivalent exponential form.

$y = 4$

Therefore, $\ln e^4 = 4$.

d. $\ln\left(\frac{1}{e}\right) = y$

$e^y = \left(\frac{1}{e}\right)$ or equivalently $e^y = e^{-1}$ Write the equivalent exponential form.

$y = -1$

Therefore, $\ln\left(\frac{1}{e}\right) = -1$.

Skill Practice 4 Evaluate.

a. log 10,000,000 **b.** log 0.1 **c.** $\ln e^5$ **d.** $\ln e$

Most scientific calculators have a key for the common logarithmic function LOG and a key for the natural logarithmic function LN. We demonstrate their use in Example 5.

EXAMPLE 5 Approximating Common and Natural Logarithms

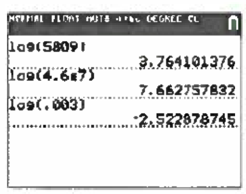
Approximate the logarithms.

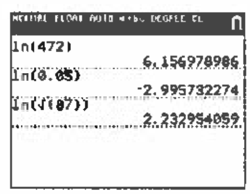
a. log 5809 **b.** $\log(4.6 \times 10^7)$ **c.** log 0.003

d. ln 472 **e.** ln 0.05 **f.** $\ln\sqrt{87}$

Solution:

For parts (a)–(c), use the LOG key. For parts (d)–(f), use the LN key.





TIP There are an infinite number of logarithmic functions. The calculator has keys for the base 10 and base e logarithmic functions only.

To approximate a logarithm of different base, we need to use the change-of-base formula presented in Section 3.4.

When using a calculator, there is always potential for user-input error. Therefore, it is good practice to estimate values when possible to confirm the reasonableness of an answer from a calculator. For example,

For part (a), $10^3 < 5809 < 10^4$. Therefore, $3 < \log 5809 < 4$.

For part (b), $10^7 < 4.6 \times 10^7 < 10^8$. Therefore, $7 < \log(4.6 \times 10^7) < 8$.

For part (c), $10^{-3} < 0.003 < 10^{-2}$. Therefore, $-3 < \log 0.003 < -2$.

Skill Practice 5 Approximate the logarithms. Round to 4 decimal places.

a. log 229 **b.** $\log(3.76 \times 10^{12})$ **c.** log 0.0216

d. ln 87 **e.** ln 0.0032 **f.** $\ln \pi$

Answers

4. a. 7 **b.** −1 **c.** 5 **d.** 1

5. a. 2.3598 **b.** 12.5752

c. −1.6655 **d.** 4.4659

e. −5.7446 **f.** 1.1447

3. Apply Basic Properties of Logarithms

From the definition of a logarithmic function, we have the following basic properties.

Basic Properties of Logarithms

Property	Example
1. $\log_b 1 = 0$ because $b^0 = 1$	$\log_5 1 = 0$ because $5^0 = 1$
2. $\log_b b = 1$ because $b^1 = b$	$\log_3 3 = 1$ because $3^1 = 3$
3. $\log_b b^x = x$ because $b^x = b^x$	$\log_2 2^x = x$ because $2^x = 2^x$
4. $b^{\log_b x} = x$ because $\log_b x = \log_b x$	$7^{\log_7 x} = x$ because $\log_7 x = \log_7 x$

Properties 3 and 4 follow from the fact that a logarithmic function is the inverse of an exponential function of the same base. Given $f(x) = b^x$ and $f^{-1}(x) = \log_b x$,

$$(f \circ f^{-1})(x) = b^{(\log_b x)} = x \qquad \text{(Property 4)}$$

$$(f^{-1} \circ f)(x) = \log_b(b^x) = x \qquad \text{(Property 3)}$$

EXAMPLE 6 Applying the Properties of Logarithms

Simplify.

a. $\log_3 3^{10}$ **b.** $\ln e^2$ **c.** $\log_{11} 11$ **d.** $\log 10$

e. $\log_{\sqrt{7}} 1$ **f.** $\ln 1$ **g.** $5^{\log_5(c^2+4)}$ **h.** $10^{\log(a^2+b^2)}$

Solution:

a. $\log_3 3^{10} = 10$	Property 3	**b.** $\ln e^2 = \log_e e^2 = 2$	Property 3
c. $\log_{11} 11 = 1$	Property 2	**d.** $\log 10 = \log_{10} 10 = 1$	Property 2
e. $\log_{\sqrt{7}} 1 = 0$	Property 1	**f.** $\ln 1 = 0$	Property 1
g. $5^{\log_5(c^2+4)} = c^2 + 4$	Property 4	**h.** $10^{\log(a^2+b^2)} = a^2 + b^2$	Property 4

Skill Practice 6 Simplify.

a. $\log_{13} 13$ **b.** $\ln e$ **c.** $a^{\log_a 3}$ **d.** $e^{\ln 6}$

e. $\log_\pi 1$ **f.** $\log 1$ **g.** $\log_9 9^{\sqrt{2}}$ **h.** $\log 10^e$

4. Graph Logarithmic Functions

Since a logarithmic function $y = \log_b x$ is the inverse of the corresponding exponential function $y = b^x$, their graphs must be symmetric with respect to the line $y = x$. See Figures 3-13 and 3-14.

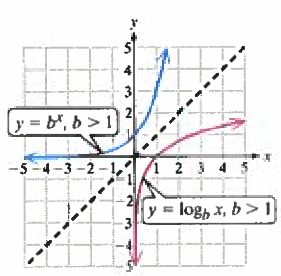


Figure 3-13

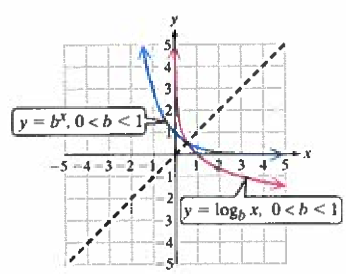


Figure 3-14

From Figures 3-13 and 3-14, the range of $y = b^x$ is the set of positive real numbers. As expected, the domain of its inverse function $y = \log_b x$ is the set of positive real numbers.

Answers

6. a. 1 **b.** 1 **c.** 3 **d.** 6
e. 0 **f.** 0 **g.** $\sqrt{2}$ **h.** e

EXAMPLE 7 Graphing Logarithmic Functions

Graph the functions.

a. $y = \log_2 x$ **b.** $y = \log_{1/4} x$

Solution:

To find points on a logarithmic function, we can interchange the x- and y-coordinates of the ordered pairs on the corresponding exponential function.

a. To graph $y = \log_2 x$, interchange the x- and y-coordinates of the ordered pairs from its inverse function $y = 2^x$. The graph of $y = \log_2 x$ is shown in Figure 3-15.

Exponential Function

x	$y = 2^x$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

Logarithmic Function

x	$y = \log_2 x$
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

Switch x and y.

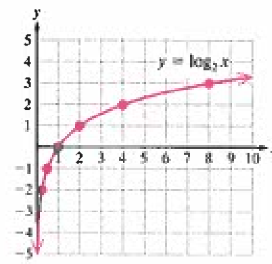


Figure 3-15

b. To graph $y = \log_{1/4} x$, interchange the x- and y-coordinates of the ordered pairs from its inverse function $y = (\frac{1}{4})^x$. See Figure 3-16.

Exponential Function

x	$y = \left(\frac{1}{4}\right)^x$
-3	64
-2	16
-1	4
0	1
1	$\frac{1}{4}$
2	$\frac{1}{16}$
3	$\frac{1}{64}$

Logarithmic Function

x	$y = \log_{1/4} x$
64	-3
16	-2
4	-1
1	0
$\frac{1}{4}$	1
$\frac{1}{16}$	2
$\frac{1}{64}$	3

Switch x and y.

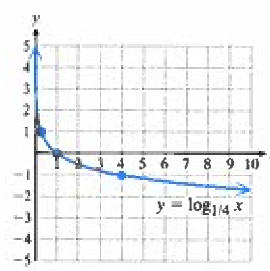


Figure 3-16

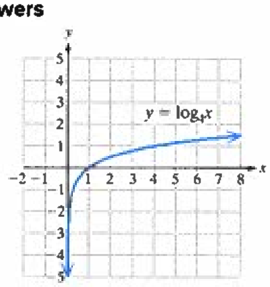
Skill Practice 7 Graph the functions.

a. $y = \log_4 x$ **b.** $y = \log_{1/2} x$

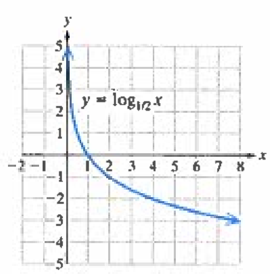
Based on the graphs in Example 7 and our knowledge of exponential functions, we offer the following summary of the characteristics of logarithmic and exponential functions.

Answers

7. a.

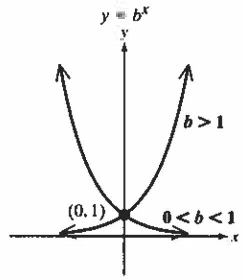


b.



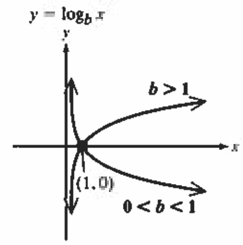
Graphs of Exponential and Logarithmic Functions

Exponential Functions



Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Horizontal asymptote: $y = 0$
Passes through $(0, 1)$
If $b > 1$, the function is increasing.
If $0 < b < 1$, the function is decreasing.

Logarithmic Functions



Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Vertical asymptote: $x = 0$
Passes through $(1, 0)$
If $b > 1$, the function is increasing.
If $0 < b < 1$, the function is decreasing.

The roles of x and y are reversed between a function and its inverse. Therefore, it is not surprising that the domain and range are reversed between exponential and logarithmic functions. Furthermore, an exponential function passes through $(0, 1)$, whereas a logarithmic function passes through $(1, 0)$. An exponential function has a horizontal asymptote of $y = 0$, whereas a logarithmic function has a vertical asymptote of $x = 0$.

In Example 8 we use the transformations of functions learned in Section 1.6 to graph a logarithmic function.

If $h > 0$, shift to the right.
If $h < 0$, shift to the left.

If $k > 0$, shift upward.
If $k < 0$, shift downward.

$$f(x) = a \log_b(x - h) + k$$

If $a < 0$, reflect across the x-axis.
Shrink vertically if $0 < |a| < 1$.
Stretch vertically if $|a| > 1$.

EXAMPLE 8 Using Transformations to Graph Logarithmic Functions

Graph the function. Identify the vertical asymptote and write the domain in interval notation.

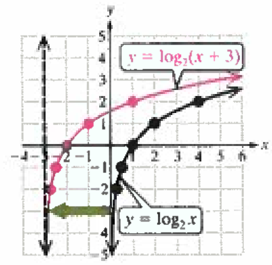
$$f(x) = \log_2(x + 3) - 2$$

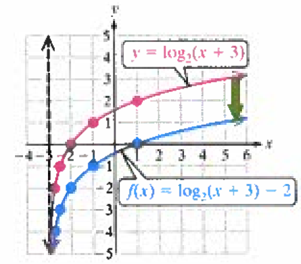
Solution:

The graph of the "parent" function $y = \log_2 x$ was presented in Example 7. The graph of $f(x) = \log_2(x + 3) - 2$ is the graph of $y = \log_2 x$ shifted to the left 3 units and down 2 units.

We can plot a few points on the graph of $y = \log_2 x$ and use these points and the vertical asymptote to form an outline of the transformed graph.

x	$y = \log_2 x$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2





The graph of $f(x) = \log_2(x + 3) - 2$ is shown in blue.
The vertical asymptote is $x = -3$. The domain is $(-3, \infty)$.

Skill Practice 8 Graph the function. Identify the vertical asymptote and write the domain in interval notation. $g(x) = \log_3(x - 4) + 1$

The domain of $f(x) = \log_b x$ is restricted to $x > 0$. In Example 8, this graph was shifted to the left 3 units, restricting the domain of $f(x) = \log_2(x + 3) - 2$ to $x > -3$. The domain of a logarithmic function is the set of real numbers that make the argument positive.

EXAMPLE 9 Identifying the Domain of a Logarithmic Function

Write the domain in interval notation.

a. $f(x) = \log_2(2x + 4)$ **b.** $g(x) = \ln(5 - x)$ **c.** $h(x) = \log(x^2 - 9)$

Solution:

a. $f(x) = \log_2(2x + 4)$

$2x + 4 > 0$ Set the argument greater than zero.

$2x > -4$ Solve for x.

$x > -2$

The domain is $(-2, \infty)$.
The graph of f is shown in Figure 3-17.
The vertical asymptote is $x = -2$.

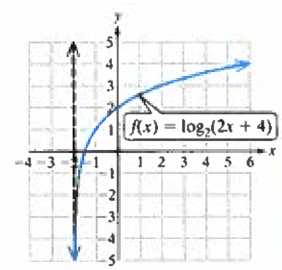


Figure 3-17

b. $g(x) = \ln(5 - x)$

$5 - x > 0$ Set the argument greater than zero.

$-x > -5$ Subtract 5 and divide by -1

$x < 5$ (reverse the inequality sign).

The domain is $(-\infty, 5)$.
The graph of g is shown in Figure 3-18.
The vertical asymptote is $x = 5$.

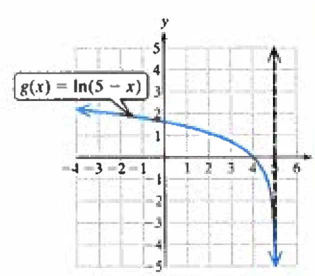
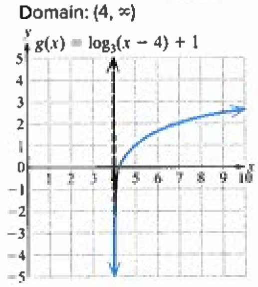


Figure 3-18

Answer

8. Vertical asymptote: $x = 4$
Domain: $(4, \infty)$



c. $h(x) = \log(x^2 - 9)$

$x^2 - 9 > 0$ — Set the argument greater than zero. The result is a polynomial inequality (Section 2.6).

$(x - 3)(x + 3) = 0$ — Solve the related equation by setting one side equal to zero and factoring the other side.

Sign of $(x - 3)$:	$-$	$-$	$+$
Sign of $(x + 3)$:	$-$	$+$	$+$
Sign of $(x - 3)(x + 3)$:	$+$	$-$	$+$
	-3	3	

The boundary points for the solution set are the solutions to the equation: $x = 3$ and $x = -3$.

The domain is $(-\infty, -3) \cup (3, \infty)$.

The graph of h is shown in Figure 3-19.

The vertical asymptotes are $x = -3$ and $x = 3$.

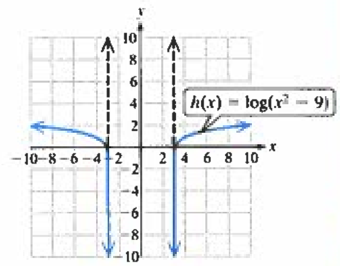


Figure 3-19

Skill Practice 9 Write the domain in interval notation.

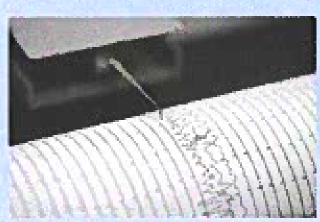
a. $\log_4(1 - 3x)$ **b.** $\log(2 + x)$ **c.** $m(x) = \ln(64 - x^2)$

5. Use Logarithmic Functions in Applications

When physical quantities vary over a large range, it is often convenient to take a logarithm of the quantity to have a more manageable set of numbers. For example, suppose a set of data values consists of 10, 100, 1000, and 10,000. The corresponding common logarithms are 1, 2, 3, and 4. The latter list of numbers is easier to manipulate and to visualize on a graph. For this reason, logarithmic scales are used in applications such as

- measuring pH (representing hydrogen ion concentration from 10^{-14} to 1 moles per liter).
- measuring wave energy from an earthquake (often ranging from 10^6 J to 10^{17} J).
- measuring loudness of sound on the decibel scale (representing sound intensity from 10^{-12} to 10^2 Watts per square meter).

Point of Interest



In 1935, American geologist Charles Richter developed the local magnitude (M_L) scale, or Richter scale, for measuring the intensity of moderate-sized earthquakes ($3 < M_L < 7$) in southern California. Today, seismologists no longer follow Richter's methodology because it does not give reliable results for earthquakes of higher magnitude. The magnitudes of modern earthquakes are based on a variety of data types from numerous seismic stations. However, both the Richter scale and modern magnitude scales use a base 10 logarithmic scale to compare amplitudes of waves on a seismogram. This means that an increase of 1 unit in magnitude represents a 10-fold increase in the amplitude of the waves on a seismogram.

Answers

9. a. $\left(-\infty, \frac{1}{3}\right)$ **b.** $(-2, \infty)$

c. $(-8, 8)$

EXAMPLE 10 Using a Logarithmic Function in an Application

The intensity I of an earthquake is measured by a seismograph—a device that measures amplitudes of shock waves. I_0 is a minimum reference intensity of a "zero-level" earthquake against which the intensities of other earthquakes may be compared. The magnitude M of an earthquake of intensity I is given by $M = \log\left(\frac{I}{I_0}\right)$.

a. Determine the magnitude of the earthquake that devastated Haiti on January 12, 2010, if the intensity was approximately $10^{7.0}$ times I_0.

b. Determine the magnitude of the earthquake that occurred near Washington, D.C., on August 23, 2011, if the intensity was approximately $10^{5.8}$ times I_0.

c. How many times more intense was the earthquake that hit Haiti than the earthquake that hit Washington, D.C.? Round to the nearest whole unit.

Solution:

a.
$$M = \log\left(\frac{I}{I_0}\right)$$
$$M = \log\left(\frac{10^{7.0} \cdot I_0}{I_0}\right)$$
$$= \log 10^{7.0}$$
$$= 7.0$$

b.
$$M = \log\left(\frac{I}{I_0}\right)$$
$$M = \log\left(\frac{10^{5.8} \cdot I_0}{I_0}\right)$$
$$= \log 10^{5.8}$$
$$= 5.8$$

c. Using the intensities given in parts (a) and (b) we have

$$\frac{10^{7.0}\, I_0}{10^{5.8}\, I_0} = 10^{7.0-5.8} = 10^{1.2} \approx 16$$

The earthquake in Haiti was approximately 16 times more intense.

Skill Practice 10

a. Determine the magnitude of an earthquake that is $10^{5.2}$ times I_0.

b. Determine the magnitude of an earthquake that is $10^{4.2}$ times I_0.

c. How many times more intense is a 5.2-magnitude earthquake than a 4.2-magnitude earthquake?

Answers

10. a. 5.2 b. 4.2
c. 10 times more intense

SECTION 3.3 Practice Exercises

Prerequisite Review

For Exercises R.1–R.4, simplify each expression.

R.1. $9^{-3/2}$ **R.2.** $64^{2/3}$ **R.3.** $(-8)^{2/3}$ **R.4.** $\dfrac{1}{25^{-1/2}}$

For Exercises R.5–R.6, solve the polynomial inequality. Write the answer in interval notation.

R.5. $y^2 + 20y \geq -75$ **R.6.** $x^2 - 8x + 16 > 0$

Concept Connections

1. Given positive real numbers x and b such that $b \neq 1$, $y = \log_b x$ is the __________ function base b and is equivalent to $b^y = x$.

2. Given $y = \log_b x$, the value y is called the __________, b is called the __________, and x is called the __________.

3. The logarithmic function base 10 is called the __________ logarithmic function, and the logarithmic function base e is called the __________ logarithmic function.

4. Given $y = \log x$, the base is understood to be __________. Given $y = \ln x$, the base is understood to be __________.

5. $\log_b 1 =$ __________ because $b^{\square} = 1$.

6. $\log_b b =$ __________ because $b^{\square} = b$.

7. $f(x) = \log_b x$ and $g(x) = b^x$ are inverse functions. Therefore, $\log_b b^x =$ __________ and $b^{\log_b x} =$ __________.

8. The graph of $y = \log_b x$ passes through the point $(1, 0)$ and the line __________ is a (horizontal/vertical) asymptote.

For the exercises in this set, assume that all variable expressions represent positive real numbers.

Objective 1: Convert Between Logarithmic and Exponential Forms

For Exercises 9–16, write the equation in exponential form. (See Example 1)

9. $\log_8 64 = 2$

10. $\log_9 81 = 2$

11. $\log\left(\frac{1}{10{,}000}\right) = -4$

12. $\log\left(\frac{1}{1{,}000{,}000}\right) = -6$

13. $\ln 1 = 0$

14. $\log_8 1 = 0$

15. $\log_a b = c$

16. $\log_x M = N$

For Exercises 17–24, write the equation in logarithmic form. (See Example 2)

17. $5^3 = 125$

18. $2^5 = 32$

19. $\left(\frac{1}{5}\right)^{-3} = 125$

20. $\left(\frac{1}{2}\right)^{-5} = 32$

21. $10^9 = 1{,}000{,}000{,}000$

22. $e^1 = e$

23. $a^7 = b$

24. $M^3 = N$

Objective 2: Evaluate Logarithmic Expressions

For Exercises 25–50, simplify the expression without using a calculator. (See Examples 3–4)

25. $\log_3 9$

26. $\log_2 16$

27. $\log_5 5$

28. $\log_6 6$

29. $\log 100{,}000{,}000$

30. $\log 10{,}000{,}000$

31. $\log_2\left(\frac{1}{16}\right)$

32. $\log_3\left(\frac{1}{9}\right)$

33. $\log\left(\frac{1}{10}\right)$

34. $\log\left(\frac{1}{10{,}000}\right)$

35. $\ln e^6$

36. $\ln e^{10}$

37. $\ln\left(\frac{1}{e^3}\right)$

38. $\ln\left(\frac{1}{e^8}\right)$

39. $\log_{1/7} 49$

40. $\log_{1/4} 16$

41. $\log_{1/2}\left(\frac{1}{32}\right)$

42. $\log_{1/6}\left(\frac{1}{36}\right)$

43. $\log 0.00001$

44. $\log 0.0001$

45. $\log_{3/2}\frac{4}{9}$

46. $\log_{3/2}\frac{9}{4}$

47. $\log_3 \sqrt[5]{3}$

48. $\log_2 \sqrt[3]{2}$

49. $\log_5\sqrt{\frac{1}{5}}$

50. $\log\sqrt{\frac{1}{1000}}$

For Exercises 51–52, estimate the value of each logarithm between two consecutive integers. Then use a calculator to approximate the value to 4 decimal places. For example, log 8970 is between 3 and 4 because $10^3 < 8970 < 10^4$. (See Example 5)

51. a. $\log 46{,}832$
    b. $\log 1{,}247{,}310$
    c. $\log 0.24$
    d. $\log 0.0000032$
    e. $\log(5.6 \times 10^5)$
    f. $\log(5.1 \times 10^{-3})$

52. a. $\log 293{,}416$
    b. $\log 897$
    c. $\log 0.038$
    d. $\log 0.00061$
    e. $\log(9.1 \times 10^8)$
    f. $\log(8.2 \times 10^{-2})$

For Exercises 53–54, approximate $f(x) = \ln x$ for the given values of x. Round to 4 decimal places. (See Example 5)

53. **a.** $f(94)$
b. $f(0.182)$
c. $f(\sqrt{155})$
d. $f(4\pi)$
e. $f(3.9 \times 10^9)$
f. $f(7.1 \times 10^{-4})$

54. **a.** $f(1860)$
b. $f(0.0694)$
c. $f(\sqrt{87})$
d. $f(2\pi)$
e. $f(1.3 \times 10^{12})$
f. $f(8.5 \times 10^{-17})$

Objective 3: Apply Basic Properties of Logarithms

For Exercises 55–64, simplify the expression without using a calculator. (See Example 6)

55. $\log_4 4^{11}$
56. $\log_6 6^7$
57. $\log_c c$
58. $\log_d d$
59. $5^{\log_5(x+y)}$
60. $4^{\log_4(a-c)}$
61. $\ln e^{a+b}$
62. $\ln e^{x^2+1}$
63. $\log_{\sqrt{5}} 1$
64. $\log_\pi 1$

Objective 4: Graph Logarithmic Functions

For Exercises 65–70, graph the function. (See Example 7)

65. $y = \log_3 x$
66. $y = \log_5 x$
67. $y = \log_{1/3} x$
68. $y = \log_{1/5} x$
69. $y = \ln x$
70. $y = \log x$

For Exercises 71–78, (See Example 8)

a. Use transformations of the graphs of $y = \log_2 x$ (see Example 7) and $y = \log_3 x$ (see Exercise 65) to graph the given functions.

b. Write the domain and range in interval notation.

c. Write an equation of the asymptote.

71. $y = \log_3(x + 2)$
72. $y = \log_2(x + 3)$
73. $y = 2 + \log_3 x$
74. $y = 3 + \log_2 x$
75. $y = \log_3(x - 1) - 3$
76. $y = \log_2(x - 2) - 1$
77. $y = -\log_3 x$
78. $y = -\log_2 x$

For Exercises 79–92, write the domain in interval notation. (See Example 9)

79. $f(x) = \log(8 - x)$
80. $g(x) = \log(3 - x)$
81. $h(x) = \log_2(6x + 7)$
82. $k(x) = \log_3(5x + 6)$
83. $m(x) = \ln(x^2 + 14)$
84. $n(x) = \ln(x^2 + 11)$
85. $f(x) = \log_4(x^2 - 16)$
86. $g(x) = \log_7(x^2 - 49)$
87. $m(x) = 3 + \ln\dfrac{1}{\sqrt{11 - x}}$
88. $n(x) = 4 - \log\dfrac{1}{\sqrt{x + 5}}$
89. $p(x) = \log(x^2 - x - 12)$
90. $q(x) = \log(x^2 + 10x + 9)$
91. $r(x) = \log_3(4 - x)^2$
92. $s(x) = \log_5(3 - x)^2$

Objective 5: Use Logarithmic Functions in Applications

93. In 1989, the Loma Prieta earthquake damaged the city of San Francisco with an intensity of approximately $10^{6.9} I_0$. Film footage of the 1989 earthquake was captured on a number of video cameras including a broadcast of Game 3 of the World Series played at Candlestick Park. **(See Example 10)**

a. Determine the magnitude of the Loma Prieta earthquake.

b. Smaller earthquakes occur daily in the San Francisco area and most are not detectable without a seismograph. Determine the magnitude of an earthquake with an intensity of $10^{3.2} I_0$.

c. How many times more intense was the Loma Prieta earthquake than an earthquake with a magnitude of 3.2? Round to the nearest whole unit.

94. The intensities of earthquakes are measured with seismographs all over the world at different distances from the epicenter. Suppose that the intensity of a medium earthquake is originally reported as $10^{5.4}$ times I_0. Later this value is revised as $10^{5.8}$ times I_0.

a. Determine the magnitude of the earthquake using the original estimate for intensity.

b. Determine the magnitude using the revised estimate for intensity.

c. How many times more intense was the earthquake than originally thought? Round to 1 decimal place.

Sounds are produced when vibrating objects create pressure waves in some medium such as air. When these variations in pressure reach the human eardrum, it causes the eardrum to vibrate in a similar manner and the ear detects sound. The intensity of sound is measured as power per unit area. The threshold for hearing (minimum sound detectable by a young, healthy ear) is defined to be $I_0 = 10^{-12}$ W/m^2 (watts per square meter). The sound level L, or "loudness" of sound, is measured in decibels (dB) as $L = 10 \log\left(\frac{I}{I_0}\right)$, where I is the intensity of the given sound. Use this formula for Exercises 95–96.

95. **a.** Find the sound level of a jet plane taking off if its intensity is 10^{15} times the intensity of I_0.

b. Find the sound level of the noise from city traffic if its intensity is 10^9 times I_0.

c. How many times more intense is the sound of a jet plane taking off than noise from city traffic?

96. **a.** Find the sound level of a motorcycle if its intensity is 10^{10} times I_0.

b. Find the sound level of a vacuum cleaner if its intensity is 10^7 times I_0.

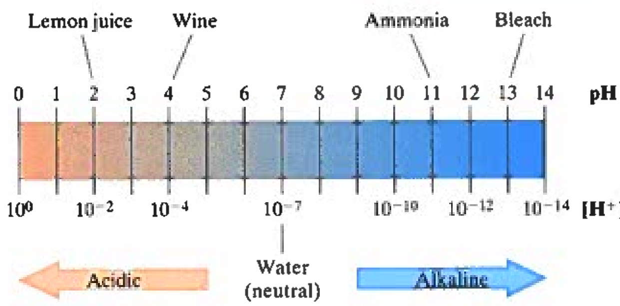
c. How many times more intense is the sound of a motorcycle than a vacuum cleaner?

Scientists use the pH scale to represent the level of acidity or alkalinity of a liquid. This is based on the molar concentration of hydrogen ions, $[H^+]$. Since the values of $[H^+]$ vary over a large range, 1×10^0 mole per liter to 1×10^{-14} mole per liter (mol/L), a logarithmic scale is used to compute pH. The formula

$$pH = -\log[H^+]$$

represents the pH of a liquid as a function of its concentration of hydrogen ions, $[H^+]$.

The pH scale ranges from 0 to 14. Pure water is taken as neutral having a pH of 7. A pH less than 7 is acidic. A pH greater than 7 is alkaline (or basic). For Exercises 97–98, use the formula for pH. Round pH values to 1 decimal place.



97. Vinegar and lemon juice are both acids. Their $[H^+]$ values are 5.0×10^{-3} mol/L and 1×10^{-2} mol/L, respectively.

a. Find the pH for vinegar.

b. Find the pH for lemon juice.

c. Which substance is more acidic?

98. Bleach and milk of magnesia are both bases. Their $[H^+]$ values are 2.0×10^{-13} mol/L and 4.1×10^{-10} mol/L, respectively.

a. Find the pH for bleach.

b. Find the pH for milk of magnesia.

c. Which substance is more basic?

Mixed Exercises

For Exercises 99–102,

a. Write the equation in exponential form.

b. Solve the equation from part (a).

c. Verify that the solution checks in the original equation.

99. $\log_3(x + 1) = 4$

100. $\log_2(x - 5) = 4$

101. $\log_4(7x - 6) = 3$

102. $\log_5(9x - 11) = 2$

For Exercises 103–106, evaluate the expressions.

103. $\log_3(\log_4 64)$

104. $\log_2\left[\log_{1/2}\left(\frac{1}{4}\right)\right]$

105. $\log_{16}(\log_{81} 3)$

106. $\log_4(\log_{16} 4)$

107. **a.** Evaluate $\log_2 2 + \log_2 4$

b. Evaluate $\log_2(2 \cdot 4)$

c. How do the values of the expressions in parts (a) and (b) compare?

108. **a.** Evaluate $\log_3 3 + \log_3 27$

b. Evaluate $\log_3(3 \cdot 27)$

c. How do the values of the expressions in parts (a) and (b) compare?

109. **a.** Evaluate $\log_4 64 - \log_4 4$

b. Evaluate $\log_4\left(\frac{64}{4}\right)$

c. How do the values of the expressions in parts (a) and (b) compare?

110. **a.** Evaluate $\log 100{,}000 - \log 100$

b. Evaluate $\log\left(\frac{100{,}000}{100}\right)$

c. How do the values of the expressions in parts (a) and (b) compare?

111. **a.** Evaluate $\log_2 2^5$
b. Evaluate $5 \cdot \log_2 2$
c. How do the values of the expressions in parts (a) and (b) compare?

112. **a.** Evaluate $\log_7 7^6$
b. Evaluate $6 \cdot \log_7 7$
c. How do the values of the expressions in parts (a) and (b) compare?

113. The time t (in years) required for an investment to double with interest compounded continuously depends on the interest rate r according to the function $t(r) = \frac{\ln 2}{r}$.
a. If an interest rate of 3.5% is secured, determine the length of time needed for an initial investment to double. Round to 1 decimal place.
b. Evaluate $t(0.04)$, $t(0.06)$, and $t(0.08)$.

114. The number n of monthly payments of P dollars each required to pay off a loan of A dollars in its entirety at interest rate r is given by

$$n = -\frac{\log\left(1 - \frac{Ar}{12P}\right)}{\log\left(1 + \frac{r}{12}\right)}$$

a. A college student wants to buy a car and realizes that he can only afford payments of \$200 per month. If he borrows \$3000 and pays it off at 6% interest, how many months will it take him to retire the loan? Round to the nearest month.
b. Determine the number of monthly payments of \$611.09 that would be required to pay off a home loan of \$128,000 at 4% interest.

For Exercises 115–116, use a calculator to approximate the given logarithms to 4 decimal places.

115. **a.** Avogadro's number is 6.022×10^{23}. Approximate $\log(6.022 \times 10^{23})$.
b. Planck's constant is 6.626×10^{-34} J · sec. Approximate $\log(6.626 \times 10^{-34})$.
c. Compare the value of the common logarithm to the power of 10 used in scientific notation.

116. **a.** The speed of light is 2.9979×10^8 m/sec. Approximate $\log(2.9979 \times 10^8)$.
b. An elementary charge is 1.602×10^{-19} C. Approximate $\log(1.602 \times 10^{-19})$.
c. Compare the value of the common logarithm to the power of 10 used in scientific notation.

Expanding Your Skills

For Exercises 117–122, write the domain in interval notation.

117. $t(x) = \log_4\left(\frac{x-1}{x-3}\right)$

118. $r(x) = \log_5\left(\frac{x+2}{x-4}\right)$

119. $s(x) = \ln\left(\sqrt{x+5} - 1\right)$

120. $v(x) = \ln\left(\sqrt{x-8} - 1\right)$

121. $c(x) = \log\left(\frac{1}{\sqrt{x-6}}\right)$

122. $d(x) = \log\left(\frac{1}{\sqrt{x+8}}\right)$

Technology Connections

123. **a.** Graph $f(x) = \ln x$ and
$g(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$
on the viewing window $[-2, 4, 1]$ by $[-5, 2, 1]$. How do the graphs compare on the interval $(0, 2)$?
b. Use function g to approximate $\ln 1.5$. Round to 4 decimal places.

124. Compare the graphs of $Y_1 = \frac{e^x - e^{-x}}{2}$, $Y_2 = \ln\left(x + \sqrt{x^2+1}\right)$, and $Y_3 = x$ on the viewing window $[-16.1, 16.1, 1]$ by $[-10, 10, 1]$. Based on the graphs, how do you suspect that the functions are related?

125. Compare the graphs of the functions.
$Y_1 = \ln(2x)$ and $Y_2 = \ln 2 + \ln x$

126. Compare the graphs of the functions.
$Y_1 = \ln\left(\frac{x}{2}\right)$ and $Y_2 = \ln x - \ln 2$

PROBLEM RECOGNITION EXERCISES

Analyzing Functions

For Exercises 1–14,

a. Write the domain. **b.** Write the range. **c.** Find the x-intercept(s). **d.** Find the y-intercept.
e. Determine the asymptotes if applicable. **f.** Determine the intervals over which the function is increasing.
g. Determine the intervals over which the function is decreasing. **h.** Match the function with its graph.

1. $f(x) = 3$

2. $g(x) = 2x - 3$

3. $d(x) = (x - 3)^2 - 4$

4. $h(x) = \sqrt[3]{x - 2}$

5. $k(x) = \dfrac{2}{x - 1}$

6. $z(x) = \dfrac{3x}{x + 2}$

7. $p(x) = \left(\dfrac{4}{3}\right)^x$

8. $q(x) = -x^2 - 6x - 9$

9. $m(x) = |x - 4| - 1$

10. $n(x) = -|x| + 3$

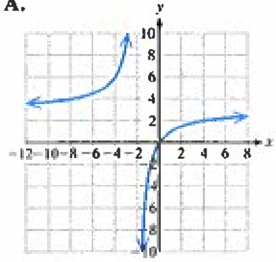
11. $r(x) = \sqrt{3 - x}$

12. $s(x) = \sqrt{x - 3}$

13. $t(x) = e^x + 2$

14. $v(x) = \ln(x + 2)$

A.



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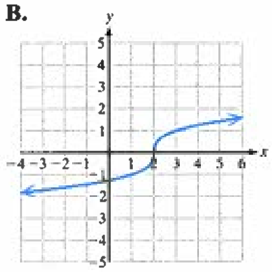

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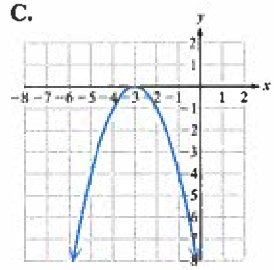

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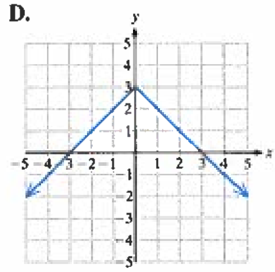

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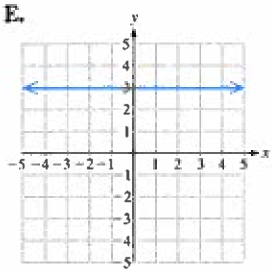

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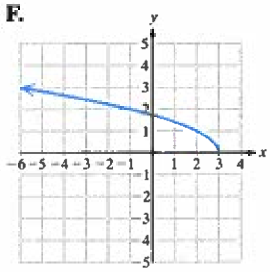

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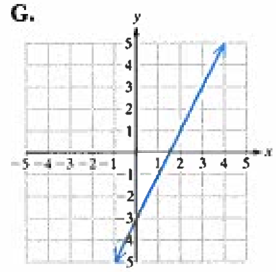

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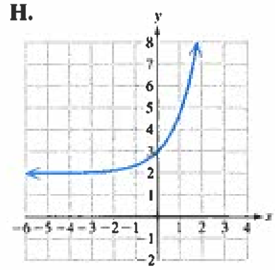

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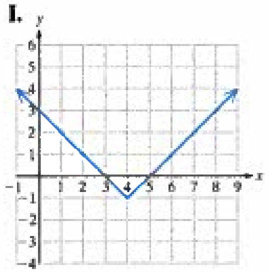

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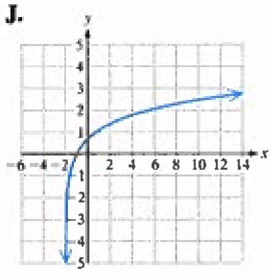

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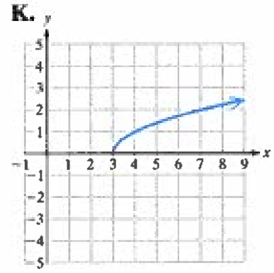

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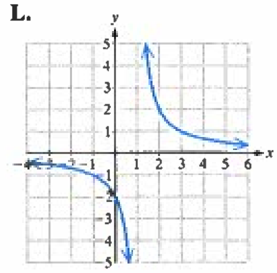

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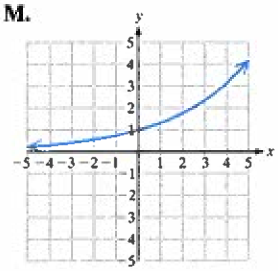

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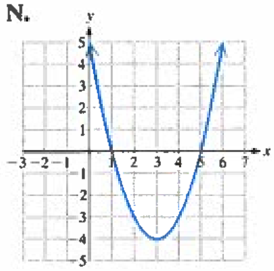


SECTION 3.4 Properties of Logarithms

OBJECTIVES

1. Apply the Product, Quotient, and Power Properties of Logarithms
2. Write a Logarithmic Expression in Expanded Form
3. Write a Logarithmic Expression as a Single Logarithm
4. Apply the Change-of-Base Formula

1. Apply the Product, Quotient, and Power Properties of Logarithms

By definition, $y = \log_b x$ is equivalent to $b^y = x$. Because a logarithm is an exponent, the properties of exponents can be applied to logarithms. The first is called the product property of logarithms.

Product Property of Logarithms

Let b, x, and y be positive real numbers where $b \neq 1$. Then

$$\log_b(xy) = \log_b x + \log_b y.$$

The logarithm of a product equals the sum of the logarithms of the factors.

TIP When two factors of the same base are multiplied, the base is unchanged and we add the exponents. This is the underlying principle for the product property of logarithms.

Proof:

Let $M = \log_b x$, which implies $b^M = x$.
Let $N = \log_b y$, which implies $b^N = y$.
Then $xy = b^M b^N = b^{M+N}$.

Writing the expression $xy = b^{M+N}$ in logarithmic form, we have,

$$\log_b(xy) = M + N$$
$$\log_b(xy) = \log_b x + \log_b y \checkmark$$

To demonstrate the product property of logarithms, simplify the following expressions by using the order of operations.

$$\log_3(3 \cdot 9) \stackrel{?}{=} \log_3 3 + \log_3 9$$
$$\log_3 27 \stackrel{?}{=} 1 + 2$$
$$3 \stackrel{?}{=} 3 \checkmark \text{ True}$$

EXAMPLE 1 Applying the Product Property of Logarithms

Write the logarithm as a sum and simplify if possible. Assume that x and y represent positive real numbers.

a. $\log_2(8x)$ **b.** $\ln(5xy)$

Solution:

a. $\log_2(8x) = \log_2 8 + \log_2 x$ Product property of logarithms
$= 3 + \log_2 x$ Simplify. $\log_2 8 = \log_2 2^3 = 3$

b. $\ln(5xy) = \ln 5 + \ln x + \ln y$

Skill Practice 1 Write the logarithm as a sum and simplify if possible. Assume that a, c, and d represent positive real numbers.

a. $\log_4(16a)$ **b.** $\log(12cd)$

Answers
1. **a.** $2 + \log_4 a$
b. $\log 12 + \log c + \log d$

The quotient rule of exponents tells us that $\frac{b^M}{b^N} = b^{M-N}$ for $b \neq 0$. This property can be applied to logarithms.

Quotient Property of Logarithms

Let b, x, and y be positive real numbers where $b \neq 1$. Then

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y.$$

The logarithm of a quotient equals the difference of the logarithm of the numerator and the logarithm of the denominator.

The proof of the quotient property for logarithms is similar to the proof of the product property (see Exercise 107). To demonstrate the quotient property for logarithms, simplify the following expressions by using the order of operations.

$$\log\left(\frac{1{,}000{,}000}{100}\right) \stackrel{?}{=} \log 1{,}000{,}000 - \log 100$$
$$\log 10{,}000 \stackrel{?}{=} 6 - 2$$
$$4 \stackrel{?}{=} 4 \checkmark \text{ True}$$

EXAMPLE 2 Applying the Quotient Property of Logarithms

Write the logarithm as the difference of logarithms and simplify if possible. Assume that the variables represent positive real numbers.

a. $\log_3\left(\frac{c}{d}\right)$ **b.** $\log\left(\frac{x}{1000}\right)$

Solution:

a. $\log_3\left(\frac{c}{d}\right) = \log_3 c - \log_3 d$ Quotient property of logarithms.

b. $\log\left(\frac{x}{1000}\right) = \log x - \log 1000$ Quotient property of logarithms.

$= \log x - 3$ Simplify. $\log 1000 = \log 10^3 = 3$

Skill Practice 2 Write the logarithm as the difference of logarithms and simplify if possible. Assume that t represents a positive real number.

a. $\log_6\left(\frac{8}{t}\right)$ **b.** $\ln\left(\frac{e}{12}\right)$

The last property we present here is the power property of logarithms. The power property of exponents tells us that $(b^M)^N = b^{MN}$. The same principle can be applied to logarithms.

Power Property of Logarithms

Let b and x be positive real numbers where $b \neq 1$. Let p be any real number. Then

$$\log_b x^p = p \log_b x.$$

The power property of logarithms is proved in Exercise 108.

Answers

2. a. $\log_6 8 - \log_6 t$ **b.** $1 - \ln 12$

EXAMPLE 3 Applying the Power Property of Logarithms

Apply the power property of logarithms. **a.** $\ln \sqrt[5]{x^2}$ **b.** $\log x^2$

Solution:

a. $\ln \sqrt[5]{x^2} = \ln x^{2/5}$ Write $\sqrt[5]{x^2}$ using rational exponents.

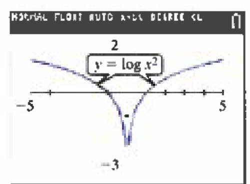
$= \frac{2}{5} \ln x$ provided that $x > 0$ Apply the power rule.

b. $\log x^2 = 2 \log x$ provided that $x > 0$ Apply the power rule.

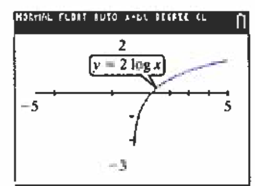
In both parts (a) and (b), the condition that $x > 0$ is mandatory. The properties of logarithms hold true only for values of the variable for which the logarithms are defined. That is, the arguments must be positive.

From the graphs of $y = \log x^2$ and $y = 2 \log x$, we see that the domains are different. Therefore, the statement $\log x^2 = 2 \log x$ is true only for $x > 0$.

$y = \log x^2$ Domain: $(-\infty, 0) \cup (0, \infty)$



$y = 2 \log x$ Domain: $(0, \infty)$



Skill Practice 3 Apply the power property of logarithms.

a. $\log_5 \sqrt[5]{x^4}$ **b.** $\ln x^4$

At this point, we have learned seven properties of logarithms. The properties hold true for values of the variable for which the logarithms are defined. **Therefore, in the examples and exercises, we will assume that the variable expressions within the logarithms represent positive real numbers.**

Properties of Logarithms

Let b, x, and y be positive real numbers where $b \neq 1$, and let p be a real number. Then the following properties of logarithms are true.

1. $\log_b 1 = 0$	**5.** $\log_b(xy) = \log_b x + \log_b y$	**Product property**
2. $\log_b b = 1$	**6.** $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$	**Quotient property**
3. $\log_b b^p = p$	**7.** $\log_b x^p = p \log_b x$	**Power property**
4. $b^{\log_b x} = x$		

2. Write a Logarithmic Expression in Expanded Form

Properties 5, 6, and 7 can be used in either direction. For example,

$$\log\left(\frac{ab}{c}\right) = \log a + \log b - \log c \quad \text{or} \quad \log a + \log b - \log c = \log\left(\frac{ab}{c}\right).$$

In some applications of algebra and calculus, the "condensed" form of the logarithm is preferred. In other applications, the "expanded" form is preferred. In Examples 4–6, we practice manipulating logarithmic expressions in both forms.

Answers

3. a. $\frac{4}{5} \log_5 x$ provided that $x > 0$

b. $4 \ln x$ provided that $x > 0$

EXAMPLE 4 Writing a Logarithmic Expression in Expanded Form

Write the expression as the sum or difference of logarithms.

a. $\log_2\left(\frac{z^3}{xy^5}\right)$ **b.** $\log\sqrt[3]{\frac{(x+y)^2}{10}}$

Solution:

a. $\log_2\left(\frac{z^3}{xy^5}\right) = \log_2 z^3 - \log_2(xy^5)$ Apply the quotient property.

$= \log_2 z^3 - (\log_2 x + \log_2 y^5)$ Apply the product property.

$= \log_2 z^3 - \log_2 x - \log_2 y^5$ Apply the distributive property.

$= 3\log_2 z - \log_2 x - 5\log_2 y$ Apply the power property.

b. $\log\sqrt[3]{\frac{(x+y)^2}{10}} = \log\left[\frac{(x+y)^2}{10}\right]^{1/3}$ Write the radical expression with rational exponents.

$= \frac{1}{3}\log\left[\frac{(x+y)^2}{10}\right]$ Apply the power property.

$= \frac{1}{3}[\log(x+y)^2 - \log 10]$ Apply the quotient property.

$= \frac{1}{3}[2\log(x+y) - 1]$ Apply the power property and simplify: $\log 10 = 1$.

$= \frac{2}{3}\log(x+y) - \frac{1}{3}$ Apply the distributive property.

Avoiding Mistakes

In Example 4(b) do not try to simplify $\log(x + y)$. The argument contains a sum, not a product.

$\log(\underbrace{x+y}_{\text{sum}})$ cannot be simplified.

Compare to the logarithm of a product which can be simplified.

$\log(\underbrace{xy}_{\text{product}}) = \log x + \log y$

Skill Practice 4 Write the expression as the sum or difference of logarithms.

a. $\ln\left(\frac{a^4b}{c^9}\right)$ **b.** $\log_5\sqrt[3]{\frac{25}{(a^2+b)^2}}$

3. Write a Logarithmic Expression as a Single Logarithm

In Examples 5 and 6, we demonstrate how to write a sum or difference of logarithms as a single logarithm. We apply Properties 5, 6, and 7 of logarithms in reverse.

EXAMPLE 5 Writing the Sum or Difference of Logarithms as a Single Logarithm

Write the expression as a single logarithm and simplify the result if possible.

$$\log_2 560 - \log_2 7 - \log_2 5$$

Solution:

$\log_2 560 - \log_2 7 - \log_2 5$

$= \log_2 560 - (\log_2 7 + \log_2 5)$ Factor out -1 from the last two terms.

$= \log_2 560 - \log_2(7 \cdot 5)$ Apply the product property.

$= \log_2\left(\frac{560}{7 \cdot 5}\right)$ Apply the quotient property.

$= \log_2 16$ Simplify within the argument.

$= 4$ Simplify. $\log_2 16 = \log_2 2^4 = 4$

Answers

4. a. $4\ln a + \ln b - 9\ln c$

b. $\frac{2}{3} - \frac{2}{3}\log_5(a^2 + b)$

Skill Practice 5 Write the expression as a single logarithm and simplify the result if possible. $\log_3 54 + \log_3 10 - \log_3 20$

EXAMPLE 6 Writing the Sum or Difference of Logarithms as a Single Logarithm

Write the expression as a single logarithm and simplify the result if possible.

a. $3\log a - \frac{1}{2}\log b - \frac{1}{2}\log c$ **b.** $\frac{1}{2}\ln x + \ln(x^2 - 1) - \ln(x + 1)$

Solution:

a. $3\log a - \frac{1}{2}\log b - \frac{1}{2}\log c$

$= 3\log a - \frac{1}{2}(\log b + \log c)$ Factor out $-\frac{1}{2}$ from the last two terms.

$= 3\log a - \frac{1}{2}\log(bc)$ Apply the product property.

$= \log a^3 - \log(bc)^{1/2}$

$= \log a^3 - \log\sqrt{bc}$ Apply the power property.

$= \log\left(\frac{a^3}{\sqrt{bc}}\right)$ Apply the quotient property.

Avoiding Mistakes

In all examples and exercises in which we manipulate logarithmic expressions, it is important to note that the equivalences are true only for the values of the variables that make the expressions defined. In Example 6(b) we have the restriction that $x > 1$.

b. $\frac{1}{2}\ln x + \ln(x^2 - 1) - \ln(x + 1)$

$= \ln x^{1/2} + \ln(x^2 - 1) - \ln(x + 1)$ Apply the power property.

$= \ln[x^{1/2}(x^2 - 1)] - \ln(x + 1)$ Apply the product property.

$= \ln\left[\frac{\sqrt{x}(x^2 - 1)}{x + 1}\right]$ Apply the quotient property.

$= \ln\left[\frac{\sqrt{x}(x + 1)(x - 1)}{x + 1}\right]$ Factor the numerator of the argument.

$= \ln\left[\sqrt{x}(x - 1)\right]$ Simplify the argument.

Skill Practice 6 Write the expression as a single logarithm and simplify the result if possible.

a. $3\log x - \frac{1}{3}\log y - \frac{2}{3}\log z$ **b.** $\frac{1}{3}\ln t + \ln(t^2 - 9) - \ln(t - 3)$

EXAMPLE 7 Applying Properties of Logarithms

Given that $\log_b 2 \approx 0.356$ and $\log_b 3 \approx 0.565$, approximate the value of $\log_b 36$.

Solution:

$\log_b 36 = \log_b(2 \cdot 3)^2$ Write the argument as a product of the factors 2 and 3.

$= 2\log_b(2 \cdot 3)$ Apply the power property of logarithms.

$= 2(\log_b 2 + \log_b 3)$ Apply the product property of logarithms.

$\approx 2(0.356 + 0.565)$ Simplify.

≈ 1.842

Answers

5. $\log_3 27 = 3$

6. a. $\log\left(\frac{x^3}{\sqrt[3]{yz^2}}\right)$

b. $\ln\left[\sqrt[3]{t}(t + 3)\right]$

Skill Practice 7 Given that $\log_b 2 \approx 0.356$ and $\log_b 3 \approx 0.565$, approximate the value of $\log_b 24$.

4. Apply the Change-of-Base Formula

A calculator can be used to approximate the value of a logarithm base 10 or base e by using the LOG key or the LN key, respectively. However, to use a calculator to evaluate a logarithmic expression with a different base, we must use the change-of-base formula.

Change-of-Base Formula

Let a and b be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then for any positive real number x,

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Note: The change-of-base formula converts a logarithm of one base to a ratio of logarithms of a different base. For the purpose of using a calculator, we often apply the change-of-base formula with base 10 or base e.

$$\log_b x = \frac{\log x}{\log b}$$

Original base is b. Ratio of base 10 logarithms

$$\log_b x = \frac{\ln x}{\ln b}$$

Original base is b. Ratio of base e logarithms

To derive the change-of-base formula, assume that a and b are positive real numbers with $a \neq 1$ and $b \neq 1$. Begin by letting $y = \log_b x$. If $y = \log_b x$, then

$b^y = x$	Write the original logarithm in exponential form.
$\log_a b^y = \log_a x$	Take the logarithm base a on both sides.
$y \cdot \log_a b = \log_a x$	Apply the power property of logarithms.
$y = \dfrac{\log_a x}{\log_a b}$	Solve for y.
$\log_b x = \dfrac{\log_a x}{\log_a b}$	Replace y by $\log_b x$. This is the change-of-base formula.

EXAMPLE 8 Applying the Change-of-Base Formula

a. Estimate $\log_4 153$ between two consecutive integers.

b. Use the change-of-base formula to approximate $\log_4 153$ by using base 10. Round to 4 decimal places.

c. Use the change-of-base formula to approximate $\log_4 153$ by using base e.

d. Check the result by using the related exponential form.

Solution:

a. $64 < 153 < 256$

$4^3 < 153 < 4^4$

$3 < \log_4 153 < 4$ $\quad$ $\log_4 153$ is between 3 and 4.

Answer

7. 1.633

TIP Although the numerators and denominators in parts (b) and (c) are different, their ratios are the same.

b. $\log_4 153 = \dfrac{\log 153}{\log 4} \approx \dfrac{2.184691431}{0.6020599913} \approx 3.6287$

c. $\log_4 153 = \dfrac{\ln 153}{\ln 4} \approx \dfrac{5.030437921}{1.386294361} \approx 3.6287$

d. Check: $4^{3.6287} \approx 153$ ✓

Skill Practice 8

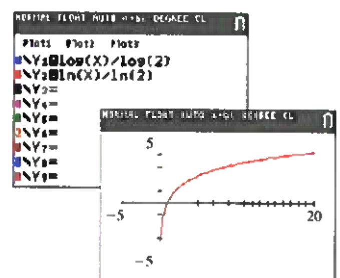
a. Estimate $\log_6 23$ between two consecutive integers.
b. Use the change-of-base formula to evaluate $\log_6 23$ by using base 10. Round to 4 decimal places.
c. Use the change-of-base formula to evaluate $\log_6 23$ by using base e. Round to 4 decimal places.
d. Check the result by using the related exponential form.

TECHNOLOGY CONNECTIONS

Using the Change-of-Base Formula to Graph a Logarithmic Function

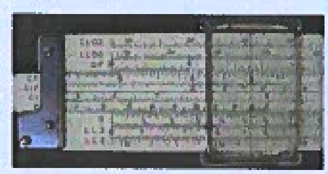
The change-of-base formula can be used to graph logarithmic functions using a graphing utility. For example, to graph $Y_1 = \log_2 x$, enter the function as

$$Y = \log(x)/\log(2) \quad \text{or} \quad Y = \ln(x)/\ln(2)$$



Point of Interest

The slide rule, first built in England in the early 17th century, is a mechanical computing device that uses logarithmic scales to perform operations involving multiplication, division, roots, logarithms, exponentials, and trigonometry. Amazingly, slide rules were used into the space age by engineers in the 1960's to help send astronauts to the moon. It was only with the invention of the pocket calculator that slide rules were replaced by modern computing devices.



Answers

8. a. Between 1 and 2
   b. 1.7500
   c. 1.7500
   d. $6^{1.7500} \approx 23$

SECTION 3.4 Practice Exercises

Prerequisite Review

For Exercises R.1–R.4, use the properties of exponents to simplify the expression.

R.1. $x^{-3} \cdot x^5 \cdot x^7$ **R.2.** $\dfrac{y^{-2}y^{10}}{y^3}$ **R.3.** $(4w^{-3}z^4)^2$ **R.4.** $\left(\dfrac{7k^4}{n}\right)^{-3}$

Concept Connections

1. The product property of logarithms states that $\log_b(xy) =$ ______________ for positive real numbers b, x, and y, where $b \neq 1$.

2. The ______________ property of logarithms states that $\log_b\left(\frac{x}{y}\right) =$ ______________ for positive real numbers b, x, and y, where $b \neq 1$.

3. The power property of logarithms states that for any real number p, $\log_b x^p =$ ______________ for positive real numbers b, x, and y, where $b \neq 1$.

4. The change-of-base formula states that $\log_b x$ can be written as a ratio of logarithms with base a as

$$\log_b x = \frac{\square}{\square}$$

5. The change-of-base formula is often used to convert a logarithm to a ratio of logarithms with base ______________ or base ______________ so that a calculator can be used to approximate the logarithm.

6. To use a graphing utility to graph the function defined by $y = \log_5 x$, use the change-of-base formula to write the function as $y =$ ______________ or $y =$ ______________.

For the exercises in this set, assume that all variable expressions represent positive real numbers.

Objective 1: Apply the Product, Quotient, and Power Properties of Logarithms

For Exercises 7–12, use the product property of logarithms to write the logarithm as a sum of logarithms. Then simplify if possible. (See Example 1)

7. $\log_5(125z)$
8. $\log_7(49k)$
9. $\log(8cd)$
10. $\log(24vw)$
11. $\log_2[(x + y) \cdot z]$
12. $\log_3[(a + b) \cdot c]$

For Exercises 13–18, use the quotient property of logarithms to write the logarithm as a difference of logarithms. Then simplify if possible. (See Example 2)

13. $\log_{12}\left(\frac{p}{q}\right)$
14. $\log_9\left(\frac{m}{n}\right)$
15. $\ln\left(\frac{e}{5}\right)$
16. $\ln\left(\frac{x}{e}\right)$
17. $\log\left(\frac{m^2 + n}{100}\right)$
18. $\log\left(\frac{1000}{c^2 + 1}\right)$

For Exercises 19–24, apply the power property of logarithms. (See Example 3)

19. $\log(2x - 3)^4$
20. $\log(8t - 3)^2$
21. $\log_6 \sqrt[7]{x^3}$
22. $\log_8 \sqrt[4]{x^3}$
23. $\ln 2^{kt}$
24. $\ln(0.5)^{rt}$

Objective 2: Write a Logarithmic Expression in Expanded Form

For Exercises 25–44, write the logarithm as a sum or difference of logarithms. Simplify each term as much as possible. (See Example 4)

25. $\log_4(7yz)$
26. $\log_2(5ab)$
27. $\log_7\left(\frac{1}{7}mn^2\right)$
28. $\log_4\left(\frac{1}{16}t^3v\right)$
29. $\log_2\left(\frac{x^{10}}{yz}\right)$
30. $\log_5\left(\frac{p^5}{mn}\right)$
31. $\log_6\left(\frac{p^5}{qt^3}\right)$
32. $\log_8\left(\frac{a^4}{b^9c}\right)$
33. $\log\left(\frac{10}{\sqrt{a^2 + b^2}}\right)$
34. $\log\left(\frac{\sqrt{d^2 + 1}}{10{,}000}\right)$
35. $\ln\left(\frac{\sqrt[3]{xy}}{wz^2}\right)$
36. $\ln\left(\frac{\sqrt[4]{pq}}{t^3m}\right)$
37. $\ln\sqrt[4]{\frac{a^2 + 4}{e^3}}$
38. $\ln\sqrt[5]{\frac{e^2}{c^2 + 5}}$
39. $\log\left[\frac{2x(x^2 + 3)^8}{\sqrt{4 - 3x}}\right]$
40. $\log\left[\frac{5y(4x + 1)^7}{\sqrt[3]{2 - 7x}}\right]$
41. $\log_5 \sqrt[3]{x\sqrt{5}}$
42. $\log_2 \sqrt[4]{y\sqrt{2}}$
43. $\log_2\left[\frac{4a^2\sqrt{3 - b}}{c(b + 4)^2}\right]$
44. $\log_3\left[\frac{27x^3\sqrt{y^2 - 1}}{y(x - 1)^2}\right]$

Objective 3: Write a Logarithmic Expression as a Single Logarithm

For Exercises 45–68, write the logarithmic expression as a single logarithm with coefficient 1, and simplify as much as possible. (See Examples 5–6)

45. $\ln y + \ln 4$

46. $\log 5 + \log p$

47. $\log_{15} 3 + \log_{15} 5$

48. $\log_{12} 8 + \log_{12} 18$

49. $\log_7 98 - \log_7 2$

50. $\log_6 144 - \log_6 4$

51. $\log 150 - \log 3 - \log 5$

52. $\log_3 693 - \log_3 33 - \log_3 7$

53. $2\log_2 x + \log_2 t$

54. $5\log_4 y + \log_4 w$

55. $4\log_8 m - 3\log_8 n - 2\log_8 p$

56. $8\log_3 x - 2\log_3 z - 7\log_3 y$

57. $3[\ln x - \ln(x+3) - \ln(x-3)]$

58. $2[\log(p-4) - \log(p-1) - \log(p+4)]$

59. $\frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x-1)$

60. $\frac{1}{3}\ln(x^2+1) - \frac{1}{3}\ln(x+1)$

61. $6\log x - \frac{1}{3}\log y - \frac{2}{3}\log z$

62. $15\log c - \frac{1}{4}\log d - \frac{3}{4}\log k$

63. $\frac{1}{3}\log_4 p + \log_4(q^2-16) - \log_4(q-4)$

64. $\frac{1}{4}\log_2 w + \log_2(w^2-100) - \log_2(w+10)$

65. $\frac{1}{2}[6\ln(x+2) + \ln x - \ln x^2]$

66. $\frac{1}{3}[12\ln(x-5) + \ln x - \ln x^3]$

67. $\log(8y^2 - 7y) + \log y^{-1}$

68. $\log(9t^3 - 5t) + \log t^{-1}$

For Exercises 69–78, use $\log_b 2 \approx 0.356$, $\log_b 3 \approx 0.565$, and $\log_b 5 \approx 0.827$ to approximate the value of the given logarithms. (See Example 7)

69. $\log_b 15$

70. $\log_b 10$

71. $\log_b 81$

72. $\log_b 125$

73. $\log_b 50$

74. $\log_b 12$

75. $\log_b\left(\frac{15}{2}\right)$

76. $\log_b\left(\frac{6}{5}\right)$

77. $\log_b 100$

78. $\log_b 225$

Objective 4: Apply the Change-of-Base Formula

For Exercises 79–84, (See Example 8)

a. Estimate the value of the logarithm between two consecutive integers. For example, $\log_2 7$ is between 2 and 3 because $2^2 < 7 < 2^3$.

b. Use the change-of-base formula and a calculator to approximate the logarithm to 4 decimal places.

c. Check the result by using the related exponential form.

79. $\log_2 15$

80. $\log_3 15$

81. $\log_5 3$

82. $\log_8 5$

83. $\log_2 0.3$

84. $\log_2 0.2$

For Exercises 85–88, use the change-of-base formula and a calculator to approximate the given logarithms. Round to 4 decimal places. Then check the answer by using the related exponential form. (See Example 8)

85. $\log_2(4.68 \times 10^7)$

86. $\log_2(2.54 \times 10^{10})$

87. $\log_4(5.68 \times 10^{-6})$

88. $\log_4(9.84 \times 10^{-5})$

Mixed Exercises

For Exercises 89–98, determine if the statement is true or false. For each false statement, provide a counterexample. For example, $\log(x+y) \neq \log x + \log y$ because $\log(2+8) \neq \log 2 + \log 8$ (the left side is 1 and the right side is approximately 1.204).

89. $\log e = \frac{1}{\ln 10}$

90. $\ln 10 = \frac{1}{\log e}$

91. $\log_5\left(\frac{1}{x}\right) = \frac{1}{\log_5 x}$

92. $\log_6\left(\frac{1}{t}\right) = \frac{1}{\log_6 t}$

93. $\log_4\left(\frac{1}{p}\right) = -\log_4 p$

94. $\log_8\left(\frac{1}{w}\right) = -\log_8 w$

95. $\log(xy) = (\log x)(\log y)$

96. $\log\left(\frac{x}{y}\right) = \frac{\log x}{\log y}$

97. $\log_2(7y) + \log_2 1 = \log_2(7y)$

98. $\log_4(3d) + \log_4 1 = \log_4(3d)$

Write About It

99. Explain why the product property of logarithms does not apply to the following statement.

$\log_5(-5) + \log_5(-25)$
$= \log_5[(-5)(-25)]$
$= \log_5 125 = 3$

100. Explain how to use the change-of-base formula and explain why it is important.

Expanding Your Skills

101. a. Write the difference quotient for $f(x) = \ln x$.

b. Show that the difference quotient from part (a) can be written as $\ln\left(\frac{x+h}{x}\right)^{1/h}$.

102. Show that
$-\ln\left(x - \sqrt{x^2 - 1}\right) = \ln\left(x + \sqrt{x^2 - 1}\right)$

103. Show that
$\log\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) + \log\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$
$= \log c - \log a$

104. Show that
$\ln\left(\frac{c + \sqrt{c^2 - x^2}}{c - \sqrt{c^2 - x^2}}\right) = 2\ln\left(c + \sqrt{c^2 - x^2}\right) - 2\ln x$

105. Use the change-of-base formula to write $(\log_2 5)(\log_5 9)$ as a single logarithm.

106. Use the change-of-base formula to write $(\log_3 11)(\log_{11} 4)$ as a single logarithm.

107. Prove the quotient property of logarithms:

$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y.$

(*Hint*: Modify the proof of the product property given on page 397.)

108. Prove the power property of logarithms:
$\log_b x^p = p \log_b x.$

Technology Connections

For Exercises 109–112, graph the function.

109. $f(x) = \log_5(x + 4)$

110. $g(x) = \log_7(x - 3)$

111. $k(x) = -3 + \log_{1/2} x$

112. $h(x) = 4 + \log_{1/3} x$

113. a. Graph $Y_1 = \log|x|$ and $Y_2 = \frac{1}{2}\log x^2$. How are the graphs related?

b. Show algebraically that $\frac{1}{2}\log x^2 = \log|x|$.

114. Graph $Y_1 = \ln(0.1x)$, $Y_2 = \ln(0.5x)$, $Y_3 = \ln x$, and $Y_4 = \ln(2x)$. How are the graphs related? Support your answer algebraically.

SECTION 3.5 Exponential and Logarithmic Equations and Applications

OBJECTIVES

1. Solve Exponential Equations
2. Solve Logarithmic Equations
3. Use Exponential and Logarithmic Equations in Applications

1. Solve Exponential Equations

A couple invests \$8000 in a bond fund. The expected yield is 4.5% and the earnings are reinvested monthly. The growth of the investment is modeled by

$$A = 8000\left(1 + \frac{0.045}{12}\right)^{12t} \quad \text{where } A \text{ is the amount in the account after } t \text{ years.}$$

If the couple wants to know how long it will take for the investment to double, they would solve the equation:

$$16{,}000 = 8000\left(1 + \frac{0.045}{12}\right)^{12t} \quad \text{(See Example 11.)}$$

This equation is called an **exponential equation** because the equation contains a variable in the exponent. To solve an exponential equation first note that all exponential functions are one-to-one. Therefore, $b^x = b^y$ implies that $x = y$. This is called the equivalence property of exponential expressions.

TIP The equivalence property tells us that if two exponential expressions with the same base are equal, then their exponents must be equal.

Equivalence Property of Exponential Expressions

If b, x, and y are real numbers with $b > 0$ and $b \neq 1$, then

$$b^x = b^y \quad \text{implies that } x = y.$$

EXAMPLE 1 **Solving Exponential Equations Using the Equivalence Property**

Solve. **a.** $3^{2x-6} = 81$ **b.** $25^{4-t} = \left(\frac{1}{5}\right)^{3t+1}$

Solution:

a.
$$\begin{aligned} 3^{2x-6} &= 81 \\ 3^{2x-6} &= 3^4 && \text{Write 81 as an exponential expression with a base of 3.} \\ 2x - 6 &= 4 && \text{Equate the exponents.} \\ x &= 5 \end{aligned}$$

Check: $3^{2x-6} = 81$

$3^{2(5)-6} \stackrel{?}{=} 81$

$3^4 \stackrel{?}{=} 81$ ✓

The solution set is $\{5\}$.

b.
$$\begin{aligned} 25^{4-t} &= \left(\frac{1}{5}\right)^{3t+1} \\ (5^2)^{4-t} &= (5^{-1})^{3t+1} && \text{Express both 25 and } \tfrac{1}{5} \text{ as integer powers of 5.} \\ 5^{2(4-t)} &= 5^{-1(3t+1)} && \text{Apply the power property of exponents: } (b^m)^n = b^{m\cdot n}. \\ 5^{8-2t} &= 5^{-3t-1} && \text{Apply the distributive property within the exponents.} \\ 8 - 2t &= -3t - 1 && \text{Equate the exponents.} \\ t &= -9 && \text{The solution checks in the original equation.} \end{aligned}$$

The solution set is $\{-9\}$.

Avoiding Mistakes

When writing the expression $(5^2)^{4-t}$ as $5^{2(4-t)}$, it is important to use parentheses around the quantity $(4 - t)$. The exponent of 2 must be multiplied by the entire quantity $(4 - t)$. Likewise, parentheses are used around $(3t + 1)$ in the expression $5^{-1(3t+1)}$.

Skill Practice 1 Solve. **a.** $4^{2x-3} = 64$ **b.** $27^{2w+5} = \left(\frac{1}{3}\right)^{2-5w}$

In Example 1, we were able to write the left and right sides of the equation with a common base. However, most exponential equations cannot be written in this form by inspection. For example:

$7^x = 60$ — 60 is not a recognizable power of 7.

$7^x = 7^?$

To solve such an equation, we can take a logarithm of the same base on each side of the equation, and then apply the power property of logarithms. This is demonstrated in Examples 2–4.

Answers

1. **a.** $\{3\}$ **b.** $\{-17\}$

Steps to Solve Exponential Equations by Using Logarithms

1. Isolate the exponential expression on one side of the equation.
2. Take a logarithm of the same base on both sides of the equation.
3. Use the power property of logarithms to "bring down" the exponent.
4. Solve the resulting equation.

EXAMPLE 2 Solving an Exponential Equation Using Logarithms

Solve. $7^x = 60$

Solution:

$$7^x = 60$$ The exponential expression 7^x is isolated.

$$\log 7^x = \log 60$$ Take a logarithm of the same base on both sides of the equation. In this case, we have chosen base 10.

$$x \log 7 = \log 60$$ Apply the power property of logarithms.

This equation is now linear.

$$x = \frac{\log 60}{\log 7} \approx 2.1041$$ Divide both sides by log 7.

It is important to note that the exact solution to this equation is $\frac{\log 60}{\log 7}$ or equivalently by the change-of-base formula, $\log_7 60$. The value 2.1041 is merely an approximation.

The solution set is $\left\{\frac{\log 60}{\log 7}\right\}$ or $\{\log_7 60\}$.

Avoiding Mistakes

While 2.1041 is only an approximation, it is useful to check the result.

$$7^{2.1041} \approx 60$$

Skill Practice 2 Solve. $5^x = 83$

To solve the equation from Example 2, we can take a logarithm of any base. For example:

$$7^x = 60$$
$$\log_7 7^x = \log_7 60$$
$$x = \log_7 60 \text{ (solution)}$$

Take the logarithm base 7 on both sides.

$$7^x = 60$$
$$\ln 7^x = \ln 60$$
$$x \ln 7 = \ln 60$$
$$x = \frac{\ln 60}{\ln 7} \text{ (solution)}$$

Take the natural logarithm on both sides.

The values $\log_7 60$, $\frac{\log 60}{\log 7}$, and $\frac{\ln 60}{\ln 7}$ are all equivalent. However, common logarithms and natural logarithms are often used to express the solution to an exponential equation so that the solution can be approximated on a calculator.

Answer

2. $\left\{\frac{\log 83}{\log 5}\right\}$ or $\{\log_5 83\}$

EXAMPLE 3 Solving Exponential Equations Using Logarithms

Solve. **a.** $10^{5+2x} + 820 = 49{,}600$ **b.** $2000 = 18{,}000e^{-0.4t}$

Solution:

a.

$$10^{5+2x} + 820 = 49{,}600$$

$$10^{5+2x} = 48{,}780$$

Isolate the exponential expression on the left by subtracting 820 on both sides.

$$\log 10^{5+2x} = \log 48{,}780$$

Since the exponential expression on the left has a base of 10, take the log base 10 on both sides.

$$5 + 2x = \log 48{,}780$$

On the left, $\log 10^{5+2x} = 5 + 2x$.

$$2x = \log 48{,}780 - 5$$

Solve the linear equation by subtracting 5 and dividing by 2.

$$x = \frac{\log 48{,}780 - 5}{2} \approx -0.1559$$

The solution checks in the original equation.

The solution set is $\left\{\dfrac{\log 48{,}780 - 5}{2}\right\}$.

b.

$$2000 = 18{,}000e^{-0.4t}$$

Isolate the exponential expression on the right by dividing both sides by 18,000.

$$\frac{1}{9} = e^{-0.4t}$$

Since the exponential expression on the left has a base of e, take the log base e on both sides.

$$\ln\left(\frac{1}{9}\right) = \ln e^{-0.4t}$$

On the right, $\ln e^{-0.4t} = -0.4t$.

$$\ln\left(\frac{1}{9}\right) = -0.4t$$

linear equation

Solve the linear equation by dividing by -0.4.

$$\frac{\ln\left(\frac{1}{9}\right)}{-0.4} = t$$

The exact solution to the equation can be written in a variety of forms by applying the properties of logarithms:

$$\frac{\ln\left(\frac{1}{9}\right)}{-0.4} = \frac{\ln 1 - \ln 9}{-0.4} = \frac{0 - \ln 9}{-0.4} = \frac{\ln 9}{0.4} \approx 5.4931$$

Alternatively, $\dfrac{\ln 9}{0.4} = \dfrac{\ln 9}{\frac{2}{5}} = \dfrac{5\ln 9}{2} \approx 5.4931$

The solution set is $\left\{\dfrac{\ln 9}{0.4}\right\}$ or $\left\{\dfrac{5\ln 9}{2}\right\}$.

Skill Practice 3 Solve.

a. $400 + 10^{4x-1} = 63{,}000$ **b.** $100 = 700e^{-0.2k}$

Answers

3. a. $\left\{\dfrac{\log 62{,}600 + 1}{4}\right\}$

b. $\left\{\dfrac{\ln 7}{0.2}\right\}$ or $\{5 \ln 7\}$

In Example 4, we have an equation with two exponential expressions involving different bases.

EXAMPLE 4 **Solving an Exponential Equation**

Solve. $4^{2x-7} = 5^{3x+1}$

Solution:

$4^{2x-7} = 5^{3x+1}$	
$\ln 4^{2x-7} = \ln 5^{3x+1}$	Take a logarithm of the same base on both sides.
$(2x-7)\ln 4 = (3x+1)\ln 5$	Apply the power property of logarithms.
$2x\ln 4 - 7\ln 4 = 3x\ln 5 + \ln 5$	Apply the distributive property.
$2x\ln 4 - 3x\ln 5 = \ln 5 + 7\ln 4$	Collect x terms on one side of the equation.
$x(2\ln 4 - 3\ln 5) = \ln 5 + 7\ln 4$	Factor out x on the left.
$x = \dfrac{\ln 5 + 7\ln 4}{2\ln 4 - 3\ln 5} \approx -5.5034$	Divide by $(2\ln 4 - 3\ln 5)$.
The solution set is $\left\{\dfrac{\ln 5 + 7\ln 4}{2\ln 4 - 3\ln 5}\right\}$.	The solution checks in the original equation.

Skill Practice 4 Solve. $3^{5x-6} = 2^{4x+1}$

In Example 5, we look at an exponential equation in quadratic form.

EXAMPLE 5 **Solving an Exponential Equation in Quadratic Form**

Solve. $e^{2x} + 5e^x - 36 = 0$

Solution:

$e^{2x} + 5e^x - 36 = 0$	
$(e^x)^2 + 5(e^x) - 36 = 0$	Note that $e^{2x} = (e^x)^2$.
$u^2 + 5u - 36 = 0$	The equation is in quadratic form. Let $u = e^x$.
$(u-4)(u+9) = 0$	Factor.
$u = 4$ or $u = -9$	
$e^x = 4$ or $e^x = -9$	Back substitute. The second equation $e^x = -9$ has no solution.
$\ln e^x = \ln 4$	
$x = \ln 4 \approx 1.3863$	No solution to this equation because $\ln(-9)$ is undefined.
The solution set is $\{\ln 4\}$.	The solution checks in the original equation.

Skill Practice 5 Solve. $e^{2x} - 5e^x - 14 = 0$

Avoiding Mistakes

Recall that the range of $f(x) = e^x$ is the set of positive real numbers. Therefore, $e^x \neq -9$.

Answers

4. $\left\{\dfrac{\ln 2 + 6\ln 3}{5\ln 3 - 4\ln 2}\right\}$

5. $\{\ln 7\}$

2. Solve Logarithmic Equations

An equation containing a variable within a logarithmic expression is called a **logarithmic equation.** For example:

$\log_2(3x - 4) = \log_2(x + 2)$ and $\ln(x + 4) = 7$ are logarithmic equations.

Given an equation in which two logarithms of the same base are equated, we can apply the equivalence property of logarithms. Since all logarithmic functions are one-to-one, $\log_b x = \log_b y$ implies that $x = y$.

TIP The equivalence property tells us that if two logarithmic expressions with the same base are equal, then their arguments must be equal.

Equivalence Property of Logarithmic Expressions

If b, x, and y are positive real numbers with $b \neq 1$, then

$$\log_b x = \log_b y \quad \text{implies that } x = y.$$

EXAMPLE 6 Solving a Logarithmic Equation Using the Equivalence Property

Solve. $\log_2(3x - 4) = \log_2(x + 2)$

Solution:

$\log_2(3x - 4) = \log_2(x + 2)$	Two logarithms of the same base are equated.
$3x - 4 = x + 2$	Equate the arguments.
$2x = 6$	Solve for x.
$x = 3$	Because the domain of a logarithmic function is restricted, it is mandatory that we check all potential solutions to a logarithmic equation.

Check: $\log_2(3x - 4) = \log_2(x + 2)$

$\log_2[3(3) - 4] \stackrel{?}{=} \log_2[(3) + 2]$

$\log_2 5 \stackrel{?}{=} \log_2 5$ ✓

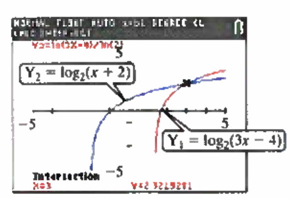
The solution set is $\{3\}$.

Skill Practice 6 Solve. $\log_2(7x - 4) = \log_2(2x + 1)$

TECHNOLOGY CONNECTIONS

Using a Calculator to View the Potential Solutions to a Logarithmic Equation

The solution to the equation in Example 6 is the x-coordinate of the point of intersection of $Y_1 = \log_2(3x - 4)$ and $Y_2 = \log_2(x + 2)$. The domain of $Y_1 = \log_2(3x - 4)$ is $\{x \mid x > \frac{4}{3}\}$ and the domain of $Y_2 = \log_2(x + 2)$ is $\{x \mid x > -2\}$. The solution to the equation $Y_1 = Y_2$ may not lie outside the domain of either function. This is why it is mandatory to check all potential solutions to a logarithmic equation.



Answer

6. {1}

In Example 7, we encounter a logarithmic equation in which one or more solutions does not check.

EXAMPLE 7 **Solving a Logarithmic Equation**

Solve. $\ln(x - 4) = \ln(x + 6) - \ln x$

Solution:

$$\ln(x - 4) = \ln(x + 6) - \ln x$$

$$\ln(x - 4) = \ln\left(\frac{x + 6}{x}\right)$$ Combine the two logarithmic terms on the right.

$$x - 4 = \frac{x + 6}{x}$$ Apply the equivalence property of logarithms.

$$x^2 - 4x = x + 6$$ Clear fractions by multiplying both sides by x.

$$x^2 - 5x - 6 = 0$$ The resulting equation is quadratic.

$$(x - 6)(x + 1) = 0$$

$$x = 6 \quad \text{or} \quad x = -1$$ The potential solutions are 6 and -1.

Check:

$\ln(x - 4) = \ln(x + 6) - \ln x$
$\ln(6 - 4) \stackrel{?}{=} \ln(6 + 6) - \ln 6$
$\ln 2 \stackrel{?}{=} \ln 12 - \ln 6$
$\ln 2 \stackrel{?}{=} \ln\left(\frac{12}{6}\right)$ ✓

$\ln(x - 4) = \ln(x + 6) - \ln x$
$\ln(-1 - 4) \stackrel{?}{=} \ln(-1 + 6) - \ln(-1)$
$\ln(-5) \stackrel{?}{=} \ln 5 - \ln(-1)$
undefined undefined

The only solution that checks is 6.
The solution set is $\{6\}$.

Skill Practice 7 Solve. $\ln x + \ln(x - 8) = \ln(x - 20)$

Many logarithmic equations, such as $4 \log_3 (2t - 7) = 8$ and $\log_2 x = 3 - \log_2 (x - 2)$, involve logarithmic terms and constant terms. In such a case, we can apply the properties of logarithms to write the equation in the form $\log_b x = k$, where k is a constant. At this point, we can solve for x by writing the equation in its equivalent exponential form $x = b^k$.

Solving Logarithmic Equations by Using Exponential Form

Step 1 Given a logarithmic equation, isolate the logarithms on one side of the equation.
Step 2 Use the properties of logarithms to write the equation in the form $\log_b x = k$, where k is a constant.
Step 3 Write the equation in exponential form.
Step 4 Solve the equation from step 3.
Step 5 Check the potential solution(s) in the original equation.

Answer
7. { }; The values 4 and 5 do not check.

EXAMPLE 8 Solving a Logarithmic Equation

Solve. $4\log_3(2t - 7) = 8$

Solution:

$4\log_3(2t - 7) = 8$

$\log_3(2t - 7) = 2$ Isolate the logarithm by dividing both sides by 4.

The equation is in the form $\log_b x = k$, where $x = 2t - 7$.

$2t - 7 = 3^2$ Write the equation in exponential form.

$2t - 7 = 9$

$t = 8$

Check: $4\log_3(2t - 7) = 8$

$4\log_3[2(8) - 7] \stackrel{?}{=} 8$

$4\log_3 9 \stackrel{?}{=} 8$

$4 \cdot 2 \stackrel{?}{=} 8$ ✓

The solution set is $\{8\}$.

Skill Practice 8 Solve. $8\log_4(w + 6) = 24$

EXAMPLE 9 Solving a Logarithmic Equation

Solve. $\log(w + 47) = 2.6$

Solution:

$\log(w + 47) = 2.6$ The equation is in the form $\log_b x = k$ where $x = w + 47$ and $b = 10$.

$w + 47 = 10^{2.6}$ Write the equation in exponential form.

$w = 10^{2.6} - 47 \approx 351.1072$ Solve the resulting linear equation.

Check: $\log(w + 47) = 2.6$

$\log[(10^{2.6} - 47) + 47] \stackrel{?}{=} 2.6$

$\log 10^{2.6} \stackrel{?}{=} 2.6$ ✓

The solution set is $\{10^{2.6} - 47\}$.

Skill Practice 9 Solve. $\log(t - 18) = 1.4$

Example 10 contains multiple logarithmic terms and a constant term. We apply the strategy of collecting the logarithmic terms on one side and the constant term on the other side. Then after combining the logarithmic terms, we write the equation in exponential form.

Answers

8. $\{58\}$

9. $\{10^{1.4} + 18\}$

EXAMPLE 10 Solving a Logarithmic Equation

Solve. $\log_2 x = 3 - \log_2(x - 2)$

Solution:

$\log_2 x = 3 - \log_2(x - 2)$

$\log_2 x + \log_2(x - 2) = 3$ Isolate the logarithms on one side of the equation.

$\log_2[x(x - 2)] = 3$ Use the product property of logarithms to write a single logarithm.

$x(x - 2) = 2^3$ Write the equation in exponential form.

$x^2 - 2x = 8$

$x^2 - 2x - 8 = 0$ Set one side equal to zero.

$(x - 4)(x + 2) = 0$

$x = 4 \quad x = -2$ Check:

$\log_2 x = 3 - \log_2(x - 2)$	$\log_2 x = 3 - \log_2(x - 2)$
$\log_2 4 \stackrel{?}{=} 3 - \log_2(4 - 2)$	$\log_2(-2) \stackrel{?}{=} 3 - \log_2(-2 - 2)$
$\log_2 4 \stackrel{?}{=} 3 - \log_2 2$	$\log_2(-2) \stackrel{?}{=} 3 - \log_2(-4)$
$2 \stackrel{?}{=} 3 - 1$ ✓	undefined undefined

The only solution that checks is $x = 4$.
The solution set is $\{4\}$.

Skill Practice 10 Solve. $2 - \log_7 x = \log_7(x - 48)$

3. Use Exponential and Logarithmic Equations in Applications

In Examples 11 and 12, we solve applications involving exponential and logarithmic equations.

EXAMPLE 11 Using an Exponential Equation in a Finance Application

A couple invests \$8000 in a bond fund. The expected yield is 4.5% and the earnings are reinvested monthly.

a. Use $A = P\left(1 + \frac{r}{n}\right)^{nt}$ to write a model representing the amount A (in \$) in the account after t years. The value r is the interest rate and n is the number of times interest is compounded per year.

b. Determine how long it will take the initial investment to double. Round to 1 decimal place.

TIP Recall that monthly compounding indicates that interest is computed $n = 12$ times per year.

Solution:

a. $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$A = 8000\left(1 + \frac{0.045}{12}\right)^{12t}$ Substitute $P = 8000$, $r = 0.045$, and $n = 12$.

Answer
10. {49}; The value -1 does not check.

b. $16{,}000 = 8000\left(1 + \frac{0.045}{12}\right)^{12t}$ — The couple wants to double their money from \$8000 to \$16,000. Substitute $A = 16{,}000$ and solve for t.

$2 = \left(1 + \frac{0.045}{12}\right)^{12t}$ — Isolate the exponential expression by dividing both sides by 8000.

$\ln 2 = \ln\left(1 + \frac{0.045}{12}\right)^{12t}$ — Take a logarithm of the same base on both sides. We have chosen to use the natural logarithm.

$\ln 2 = 12t \ln\left(1 + \frac{0.045}{12}\right)$ — Apply the power property of logarithms. The equation is now linear in the variable t.

$\frac{\ln 2}{12 \ln\left(1 + \frac{0.045}{12}\right)} = t$ — Divide both sides by $12 \ln\left(1 + \frac{0.045}{12}\right)$.

$t \approx 15.4$

It will take approximately 15.4 yr for the investment to double.

Skill Practice 11 Determine how long it will take \$8000 compounded monthly at 6% to double. Round to 1 decimal place.

In Example 12, we use a logarithmic equation in an application.

EXAMPLE 12 Using a Logarithmic Equation in a Medical Application

Suppose that the sound at a rock concert measures 124 dB (decibels).

a. Use the formula $L = 10 \log\left(\frac{I}{I_0}\right)$ to find the intensity of sound I (in W/m^2). The variable L represents the loudness of sound (in dB) and $I_0 = 10^{-12}$ W/m^2.

b. If the threshold at which sounds become painful is 1 W/m^2, will the music at this concert be physically painful? (Ignore the quality of the music.)

Solution:

a. $L = 10 \log\left(\frac{I}{I_0}\right)$

$124 = 10 \log\left(\frac{I}{10^{-12}}\right)$ — Substitute 124 for L and 10^{-12} for I_0.

$12.4 = \log\left(\frac{I}{10^{-12}}\right)$ — Divide both sides by 10. The logarithm is now isolated.

$10^{12.4} = \frac{I}{10^{-12}}$ — Write the equation in exponential form.

$10^{12.4} \cdot 10^{-12} = I$ — Multiply both sides by 10^{-12}.

$I = 10^{0.4} \approx 2.5$ W/m^2 — Simplify.

b. The intensity of sound at the rock concert is approximately 2.5 W/m^2. This is above the threshold for pain.

Skill Practice 12

a. Find the intensity of sound from a leaf blower if the decibel level is 115 dB.

b. Is the intensity of sound from a leaf blower above the threshold for pain?

Answers

11. 11.6 yr

12. a. $10^{-0.5}$ W/m^2 $\approx$ 0.3 W/m^2

b. No

TECHNOLOGY CONNECTIONS

Using a Calculator to Approximate the Solutions to Exponential and Logarithmic Equations

There are many situations in which analytical methods fail to give a solution to a logarithmic or exponential equation. To find solutions graphically,

Enter the left side of the equation as Y_1.

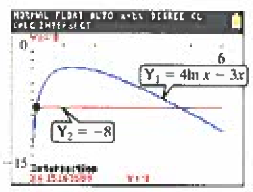
Enter the right side of the equation as Y_2.

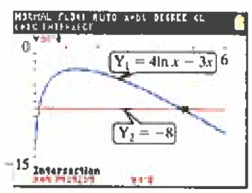
Then determine the point(s) of intersection of the graphs.

Example: $4 \ln x - 3x = -8$

$Y_1 = 4 \ln x - 3x$

$Y_2 = -8$ Solutions: $x \approx 0.1516$ and $x \approx 4.7419$





SECTION 3.5 Practice Exercises

Prerequisite Review

For Exercises R.1–R.6, solve the equation.

R.1. $7 = 5 - 2(4q - 1)$

R.2. $15n(n + 3) = 14n - 10$

R.3. $y^2 - 5y - 9 = 0$

R.4. $\dfrac{20}{n^2 - 2n} + 5 = \dfrac{10}{n - 2}$

R.5. $\sqrt{a + 18} + 2 = a$

R.6. $36d^{-2} - 5d^{-1} - 1 = 0$

Concept Connections

1. An equation such as $4^x = 9$ is called an __________ equation because the equation contains a variable in the exponent.

2. The equivalence property of exponential expressions states that if $b^x = b^y$, then __________ = __________.

3. The equivalence property of logarithmic expressions states that if $\log_b x = \log_b y$, then __________ = __________.

4. An equation containing a variable within a logarithmic expression is called a __________ equation.

Objective 1: Solve Exponential Equations

For Exercises 5–16, solve the equation. (See Example 1)

5. $3^x = 81$

6. $2^x = 32$

7. $\sqrt[3]{5} = 5^t$

8. $\sqrt{3} = 3^w$

9. $2^{-3y+1} = 16$

10. $5^{2z+2} = 625$

11. $11^{3c+1} = \left(\dfrac{1}{11}\right)^{c-5}$

12. $7^{2x-3} = \left(\dfrac{1}{49}\right)^{x+1}$

13. $8^{2x-5} = 32^{x-6}$

14. $27^{x-4} = 9^{2x+1}$

15. $100^{3t-5} = 1000^{3-t}$

16. $100{,}000^{2w+1} = 10{,}000^{4-w}$

For Exercises 17–34, solve the equation. Write the solution set with the exact values given in terms of common or natural logarithms. Also give approximate solutions to 4 decimal places. (See Examples 2–5)

17. $6^t = 87$

18. $2^z = 70$

19. $1024 = 19^x + 4$

20. $801 = 23^y + 6$

21. $10^{3+4x} - 8100 = 120{,}000$

22. $10^{5+8x} + 4200 = 84{,}000$

23. $21{,}000 = 63{,}000e^{-0.2t}$

24. $80 = 320e^{-0.5t}$

25. $4e^{2n-5} + 3 = 11$

26. $5e^{4m-3} - 7 = 13$

27. $3^{6x+5} = 5^{2x}$

28. $7^{4x-1} = 3^{5x}$

29. $2^{1-6x} = 7^{3x+4}$

30. $11^{1-8x} = 9^{2x+3}$

31. $e^{2x} - 9e^x - 22 = 0$

32. $e^{2x} - 6e^x - 16 = 0$

33. $e^{2x} = -9e^x$

34. $e^{2x} = -7e^x$

Objective 2: Solve Logarithmic Equations

For Exercises 35–36, determine if the given value of x is a solution to the logarithmic equation.

35. $\log_2(x - 31) = 5 - \log_2 x$

- **a.** $x = 16$
- **b.** $x = 32$
- **c.** $x = -1$

36. $\log_4 x = 3 - \log_4(x - 63)$

- **a.** $x = 64$
- **b.** $x = -1$
- **c.** $x = 32$

For Exercises 37–60, solve the equation. Write the solution set with the exact solutions. Also give approximate solutions to 4 decimal places if necessary. (See Examples 6–10)

37. $\log_4(3w + 11) = \log_4(3 - w)$

38. $\log_7(12 - t) = \log_7(t + 6)$

39. $\log(x^2 + 7x) = \log 18$

40. $\log(p^2 + 6p) = \log 7$

41. $6\log_5(4p - 3) - 2 = 16$

42. $5\log_6(7w + 1) + 3 = 13$

43. $2\log_8(3y - 5) + 20 = 24$

44. $5\log_3(7 - 5z) + 2 = 17$

45. $\log(p + 17) = 4.1$

46. $\log(q - 6) = 3.5$

47. $2\ln(4 - 3t) + 1 = 7$

48. $4\ln(6 - 5t) + 2 = 22$

49. $\log_2 w - 3 = -\log_2(w + 2)$

50. $\log_3 y + \log_3(y + 6) = 3$

51. $\log_6(7x - 2) = 1 + \log_6(x + 5)$

52. $\log_4(5x - 13) = 1 + \log_4(x - 2)$

53. $\log_5 z = 3 - \log_5(z - 20)$

54. $\log_2 x = 4 - \log_2(x - 6)$

55. $\ln x + \ln(x - 4) = \ln(3x - 10)$

56. $\ln x + \ln(x - 3) = \ln(5x - 7)$

57. $\log x + \log(x - 7) = \log(x - 15)$

58. $\log x + \log(x - 10) = \log(x - 18)$

59. $\log_8(6 - m) + \log_8(-m - 1) = 1$

60. $\log_3(n - 5) + \log_3(n + 3) = 2$

Objective 3: Use Exponential and Logarithmic Equations in Applications

For Exercises 61–70, use the model $A = Pe^{rt}$ or $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where A is the future value of P dollars invested at interest rate r compounded continuously or n times per year for t years. (See Example 11)

61. If \$10,000 is invested in an account earning 5.5% interest compounded continuously, determine how long it will take the money to triple. Round to the nearest year.

62. If a couple has \$80,000 in a retirement account, how long will it take the money to grow to \$1,000,000 if it grows by 6% compounded continuously? Round to the nearest year.

63. A \$2500 bond grows to \$3729.56 in 10 yr under continuous compounding. Find the interest rate. Round to the nearest whole percent.

64. \$5000 grows to \$5438.10 in 2 yr under continuous compounding. Find the interest rate. Round to the nearest tenth of a percent.

65. An \$8000 investment grows to \$9289.50 at 3% interest compounded quarterly. For how long was the money invested? Round to the nearest year.

66. \$20,000 is invested at 3.5% interest compounded monthly. How long will it take for the investment to double? Round to the nearest tenth of a year.

67. A \$25,000 inheritance is invested for 15 yr compounded quarterly and grows to \$52,680. Find the interest rate. Round to the nearest percent.

68. A \$10,000 investment grows to \$11,273 in 4 yr compounded monthly. Find the interest rate. Round to the nearest percent.

69. If \$4000 is put aside in a money market account with interest compounded continuously at 2.2%, find the time required for the account to *earn* \$1000. Round to the nearest month.

70. Victor puts aside \$10,000 in an account with interest compounded continuously at 2.7%. How long will it take for him to *earn* \$2000? Round to the nearest month.

71. Physicians often treat thyroid cancer with a radioactive form of iodine called iodine-131 (^{131}I). The radiological half-life of ^{131}I is approximately 8 days, but the biological half-life for most individuals is 4.2 days. The biological half-life is shorter because in addition to ^{131}I being lost to decay, the iodine is also excreted from the body in urine, sweat, and saliva.

For a patient treated with 100 mCi (millicuries) of ^{131}I, the radioactivity level R (in mCi) after t days is given by $R = 100(2)^{-t/4.2}$.

a. State law mandates that the patient stay in an isolated hospital room for 2 days after treatment with ^{131}I. Determine the radioactivity level at the end of 2 days. Round to the nearest whole unit.

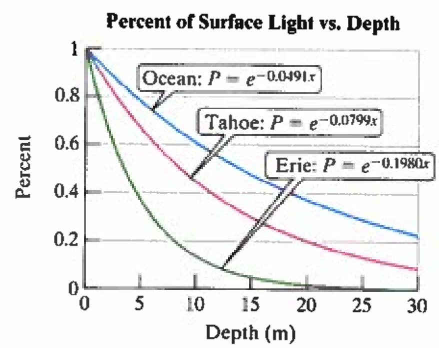
b. After the patient is released from the hospital, the patient is directed to avoid direct human contact until the radioactivity level drops below 30 mCi. For how many days *after* leaving the hospital will the patient need to stay in isolation? Round to the nearest tenth of a day.

72. Caffeine occurs naturally in a variety of food products such as coffee, tea, and chocolate. The kidneys filter the blood and remove caffeine and other drugs through urine. The biological half-life of caffeine is approximately 6 hr. If one cup of coffee has 80 mg of caffeine, then the amount of caffeine C (in mg) remaining after t hours is given by $C = 80(2)^{-t/6}$.

a. How long will it take for the amount of caffeine to drop below 60 mg? Round to 1 decimal place.

b. Laura has trouble sleeping if she has more than 30 mg of caffeine in her bloodstream. How many hours after drinking a cup of coffee would Laura have to wait so that the coffee would not disrupt her sleep? Round to 1 decimal place.

Sunlight is absorbed in water, and as a result the light intensity in oceans, lakes, and ponds decreases exponentially with depth. The percentage of visible light, P (in decimal form), at a depth of x meters is given by $P = e^{-kx}$, where k is a constant related to the clarity and other physical properties of the water. The graph shows models for the open ocean, Lake Tahoe, and Lake Erie for data taken under similar conditions. Use these models for Exercises 73–76.



73. Determine the depth at which the light intensity is half the value from the surface for each body of water given. Round to the nearest tenth of a meter.

74. Determine the depth at which the light intensity is 20% of the value from the surface for each body of water given. Round to the nearest tenth of a meter.

75. The *euphotic* depth is the depth at which light intensity falls to 1% of the value at the surface. This depth is of interest to scientists because no appreciable photosynthesis takes place. Find the euphotic depth for the open ocean. Round to the nearest tenth of a meter.

76. Refer to Exercise 75, and find the euphotic depth for Lake Tahoe and for Lake Erie. Round to the nearest tenth of a meter.

77. Forge welding is a process in which two pieces of steel are joined together by heating the pieces of steel and hammering them together. A welder takes a piece of steel from a forge at 1600°F and places it on an anvil where the outdoor temperature is 50°F. The temperature of the steel T (in °F) can be modeled by $T = 50 + 1550e^{-0.05t}$, where t is the time in minutes after the steel is removed from the forge. How long will it take for the steel to reach a temperature of 100°F so that it can be handled without heat protection? Round to the nearest minute.

78. A pie comes out of the oven at 325°F and is placed to cool in a 70°F kitchen. The temperature of the pie T (in °F) after t minutes is given by $T = 70 + 255e^{-0.017t}$. The pie is cool enough to cut when the temperature reaches 110°F. How long will this take? Round to the nearest minute.

For Exercises 79–80, the formula $L = 10 \log\left(\frac{I}{I_0}\right)$ gives the loudness of sound L (in dB) based on the intensity of sound I (in W/m^2). The value $I_0 = 10^{-12}$ W/m^2 is the minimal threshold for hearing for midfrequency sounds. Hearing impairment is often measured according to the minimal sound level (in dB) detected by an individual for sounds at various frequencies. For one frequency, the table depicts the level of hearing impairment.

Category	Loudness (dB)
Mild	$26 \le L \le 40$
Moderate	$41 \le L \le 55$
Moderately severe	$56 \le L \le 70$
Severe	$71 \le L \le 90$
Profound	$L > 90$

79. **a.** If the minimum intensity heard by an individual is 3.4×10^{-8} W/m^2, determine if the individual has a hearing impairment.

b. If the minimum loudness of sound detected by an individual is 30 dB, determine the corresponding intensity of sound. **(See Example 12)**

80. Determine the range that represents the intensity of sound that can be heard by an individual with severe hearing impairment.

For Exercises 81–82, use the formula $\text{pH} = -\log[\text{H}^+]$. The variable pH represents the level of acidity or alkalinity of a liquid on the pH scale, and H^+ is the concentration of hydronium ions in the solution. Determine the value of H^+ (in mol/L) for the following liquids, given their pH values.

81. **a.** Seawater pH = 8.5

b. Acid rain pH = 2.3

82. **a.** Milk pH = 6.2

b. Sodium bicarbonate pH = 8.4

83. A new teaching method to teach vocabulary to sixth-graders involves having students work in groups on an assignment to learn new words. After the lesson was completed, the students were tested at 1-month intervals. The average score for the class $S(t)$ can be modeled by

$$S(t) = 94 - 18\ln(t + 1)$$

where t is the time in months after completing the assignment. If the average score is 65, how many months had passed since the students completed the assignment? Round to the nearest month.

84. A company spends x hundred dollars on an advertising campaign. The amount of money in sales $S(x)$ (in \$1000) for the 4-month period after the advertising campaign can be modeled by

$$S(x) = 5 + 7\ln(x + 1)$$

If the sales total \$19,100, how much was spent on advertising? Round to the nearest dollar.

85. Radiated seismic energy from an earthquake is estimated by $\log E = 4.4 + 1.5M$, where E is the energy in Joules (J) and M is surface wave magnitude.

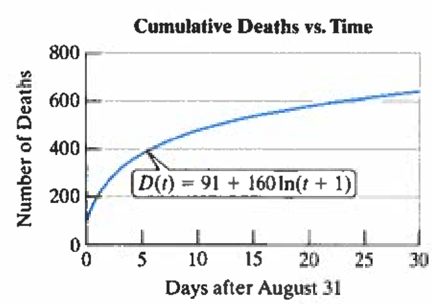
a. How many times more energy does an 8.2-magnitude earthquake have than a 5.5-magnitude earthquake? Round to the nearest thousand.

b. How many times more energy does a 7-magnitude earthquake have than a 6-magnitude earthquake? Round to the nearest whole number.

86. On August 31, 1854, an epidemic of cholera was discovered in London, England, resulting from a contaminated community water pump. By the end of September, more than 600 citizens who drank water from the pump had died. The cumulative number of deaths $D(t)$ at a time t days after August 31 is given by $D(t) = 91 + 160\ln(t + 1)$.

a. Determine the cumulative number of deaths by September 15. Round to the nearest whole unit.

b. Approximately how many days after August 31 did the cumulative number of deaths reach 600?



Mixed Exercises

For Exercises 87–94, find an equation for the inverse function.

87. $f(x) = 2^x - 7$

88. $f(x) = 5^x + 6$

89. $f(x) = \ln(x + 5)$

90. $f(x) = \ln(x - 7)$

91. $f(x) = 10^{x-3} + 1$

92. $f(x) = 10^{x+2} - 4$

93. $f(x) = \log(x + 7) - 9$

94. $f(x) = \log(x - 11) + 8$

For Exercises 95–112, solve the equation. Write the solution set with exact solutions. Also give approximate solutions to 4 decimal places if necessary.

95. $5^{|x|} - 3 = 122$

96. $11^{|x|} + 9 = 130$

97. $\log x - 2\log 3 = 2$

98. $\log y - 3\log 5 = 3$

99. $6^{x^2-2} = 36$

100. $8^{y^2-7} = 64$

101. $\log_9|x + 4| = \log_9 6$

102. $\log_8|3 - x| = \log_8 5$

103. $x^2e^x = 9e^x$

104. $x^26^x = 6^x$

105. $\log_3(\log_3 x) = 0$

106. $\log_5(\log_5 x) = 1$

107. $3|\ln x| - 12 = 0$

108. $7|\ln x| - 14 = 0$

109. $\log_3 x - \log_3(2x + 6) = \frac{1}{2}\log_3 4$

110. $\log_5 x - \log_5(x + 1) = \frac{1}{3}\log_5 8$

111. $2e^x(e^x - 3) = 3e^x - 4$

112. $3e^x(e^x - 6) = 4e^x - 7$

Write About It

113. Explain the process to solve the equation $4^x = 11$.

114. Explain the process to solve the equation $\log_b 5 + \log_b(x - 3) = 4$.

Expanding Your Skills

For Exercises 115–126, solve the equation.

115. $\dfrac{10^x - 13 \cdot 10^{-x}}{3} = 4$

116. $\dfrac{e^x - 9e^{-x}}{2} = 4$

117. $(\ln x)^2 - \ln x^5 = -4$

118. $(\ln x)^2 + \ln x^3 = -2$

119. $(\log x)^2 = \log x^2$

120. $(\log x)^2 = \log x^3$

121. $\log w + 4\sqrt{\log w} - 12 = 0$

122. $\ln x + 3\sqrt{\ln x} - 10 = 0$

123. $e^{2x} - 8e^x + 6 = 0$

124. $e^{2x} - 6e^x + 4 = 0$

125. $\log_5\sqrt{6c + 5} + \log_5\sqrt{c} = 1$

126. $\log_3\sqrt{x - 8} + \log_3\sqrt{x} = 1$

Technology Connections

For Exercises 127–130, an equation is given in the form $Y_1(x) = Y_2(x)$. Graph Y_1 and Y_2 on a graphing utility on the window [10, 10, 1] by [10, 10, 1]. Then approximate the point(s) of intersection to approximate the solution(s) to the equation. Round to 4 decimal places.

127. $4x - e^x + 6 = 0$

128. $x^3 - e^{2x} + 4 = 0$

129. $x^2 + 5\log x = 6$

130. $x^2 - 0.05\ln x = 4$

SECTION 3.6 Modeling with Exponential and Logarithmic Functions

OBJECTIVES

1. Solve Literal Equations for a Specified Variable
2. Create Models for Exponential Growth and Decay
3. Apply Logistic Growth Models
4. Create Exponential and Logarithmic Models Using Regression

1. Solve Literal Equations for a Specified Variable

A short-term model to predict the U.S. population P is $P = 310e^{0.00965t}$, where t is the number of years since 2010. If we solve this equation for t, we have

$$t = \frac{\ln\left(\frac{P}{310}\right)}{0.00965} \quad \text{or equivalently} \quad t = \frac{\ln P - \ln 310}{0.00965}.$$

This is a model that predicts the time required for the U.S. population to reach a value P. Manipulating an equation for a specified variable was first introduced in Section R.5. In Example 1, we revisit this skill using exponential and logarithmic equations.

EXAMPLE 1 Solving an Equation for a Specified Variable

a. Given $P = 100e^{kx} - 100$, solve for x. (Used in geology)
b. Given $L = 8.8 + 5.1 \log D$, solve for D. (Used in astronomy)

Solution:

a. $P = 100e^{kx} - 100$

$$P + 100 = 100e^{kx}$$ Add 100 to both sides to isolate the x term.

$$\frac{P + 100}{100} = e^{kx}$$ Divide by 100.

$$\ln\left(\frac{P + 100}{100}\right) = \ln e^{kx}$$ Take the natural logarithm of both sides.

$$\ln\left(\frac{P + 100}{100}\right) = kx$$ Simplify: $\ln e^{kx} = kx$

$$x = \frac{\ln\left(\frac{P + 100}{100}\right)}{k}$$ Divide by k.

$$x = \frac{\ln\left(\frac{P + 100}{100}\right)}{k} \text{ or equivalently } x = \frac{\ln(P + 100) - \ln 100}{k}$$

b. $L = 8.8 + 5.1 \log D$

$$\frac{L - 8.8}{5.1} = \log D$$ Subtract 8.8 from both sides and divide by 5.1.

$$D = 10^{(L-8.8)/5.1}$$ Write the equation in exponential form.

Skill Practice 1

a. Given $T = 78 + 272e^{-kt}$, solve for k.
b. Given $S = 90 - 20\ln(t + 1)$, solve for t.

2. Create Models for Exponential Growth and Decay

In Section 3.2, we defined an exponential function as $y = b^x$, where $b > 0$ and $b \neq 1$. Throughout the chapter, we have used transformations of basic exponential functions to solve a variety of applications. The following variation of the general exponential form is used to solve applications involving exponential growth and decay.

Answers

1. a. $k = -\dfrac{\ln\left(\frac{T - 78}{272}\right)}{t}$ or $k = \dfrac{\ln 272 - \ln(T - 78)}{t}$

b. $e^{(90-S)/20} - 1$

Exponential Growth and Decay Models

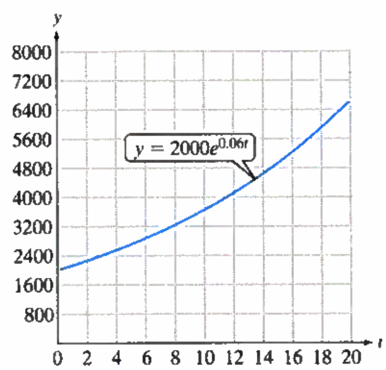
Let y be a variable changing exponentially with respect to t, and let y_0 represent the initial value of y when $t = 0$. Then for a constant k:

If $k > 0$, then $y = y_0e^{kt}$ is a model for exponential growth.

Example:

$y = 2000e^{0.06t}$ represents the value of a \$2000 investment after t years with interest compounded continuously.

(*Note*: $k = 0.06 > 0$)

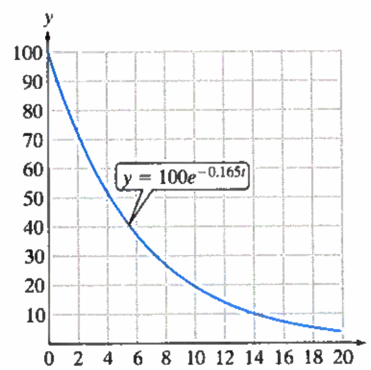


If $k < 0$, then $y = y_0e^{kt}$ is a model for exponential decay.

Example:

$y = 100e^{-0.165t}$ represents the radioactivity level t hours after a patient is treated for thyroid cancer with 100 mCi of radioactive iodine.

(*Note*: $k = -0.165 < 0$)



The model $y = y_0e^{kt}$ is often presented with different letters or symbols in place of y, y_0, k, and t to convey their meaning in the context of the application. For example, to compute the value of an investment under continuous compounding, we have

$A = Pe^{rt}$
- P (for principal) is used in place of y_0.
- r (for the annual interest rate) is used in place of k.
- A (for the future value of the investment) is used in place of y.

We can also use function notation when expressing a model for exponential growth or decay. For example, consider the model for population growth.

$P(t) = P_0e^{kt}$
- P_0 (for initial population) is used in place of y_0.
- $P(t)$ represents the population as a function of time and is used in place of y.

EXAMPLE 2 Creating a Model for Growth of an Investment

Suppose that \$15,000 is invested and at the end of 3 yr, the value of the account is \$19,356.92. Use the model $A = Pe^{rt}$ to determine the average rate of return r under continuous compounding.

Solution:

$$A = Pe^{rt}$$ Begin with an appropriate model.

$$A = 15{,}000e^{rt}$$ P represents the initial value of the account (initial principal). Substitute 15,000 for P.

$$19{,}356.92 = 15{,}000e^{r(3)}$$ We have a known data point where $A = 19{,}356.92$ when $t = 3$. Substituting these values into the formula enables us to solve for r.

$$\frac{19{,}356.92}{15{,}000} = e^{3r}$$ Divide both sides by 15,000.

$$\ln\left(\frac{19{,}356.92}{15{,}000}\right) = \ln(e^{3r})$$ Take the natural logarithm of both sides.

$$\ln\left(\frac{19{,}356.92}{15{,}000}\right) = 3r$$ Simplify: $\ln e^{3r} = 3r$

$$r = \frac{\ln\left(\frac{19{,}356.92}{15{,}000}\right)}{3}$$ Divide by 3 to isolate r.

$$r \approx 0.085$$

The average rate of return is approximately 8.5%.

Skill Practice 2 Suppose that \$10,000 is invested and at the end of 5 yr, the value of the account is \$13,771.28. Use the model $A = Pe^{rt}$ to determine the average rate of return r under continuous compounding.

In Example 3, we build a model to predict short-term population growth.

EXAMPLE 3 Creating a Model for Population Growth

On January 1, 2000, the population of California was approximately 34 million. On January 1, 2010, the population was 37.3 million.

a. Write a function of the form $P(t) = P_0e^{kt}$ to represent the population of California $P(t)$ (in millions), t years after January 1, 2000. Round k to 5 decimal places.

b. Use the function in part (a) to predict the population on January 1, 2018. Round to 1 decimal place.

c. Use the function from part (a) to determine the year during which the population of California will be twice the value from the year 2000.

Solution:

a.

$$P(t) = P_0e^{kt}$$ Begin with an appropriate model.

$$P(t) = 34e^{kt}$$ The initial population is $P_0 = 34$ million.

$$37.3 = 34e^{k(10)}$$ We have a known data point $P(10) = 37.3$. Substituting these values into the function enables us to solve for k.

$$\frac{37.3}{34} = e^{k(10)}$$ Divide both sides by 34.

$$\ln\left(\frac{37.3}{34}\right) = 10k$$ Take the natural logarithm of both sides.

$$k = \frac{\ln\left(\frac{37.3}{34}\right)}{10} \approx 0.00926$$ Divide by 10 to isolate k.

$$P(t) = 34e^{0.00926t}$$ This model gives the population as a function of time.

TIP The value of k in the model $P(t) = P_0e^{kt}$ is called a parameter and is related to the growth rate of the population being studied. The value of k will be different for different populations.

Answer

2. 6.4%

b. $P(t) = 34e^{0.00926t}$

$P(18) = 34e^{0.00926(18)}$ Substitute 18 for t.

$= 40.2$

The population in California on January 1, 2018, will be approximately 40.2 million if this trend continues.

c. $P(t) = 34e^{0.00926t}$

$68 = 34e^{0.00926t}$ Substitute 68 for $P(t)$.

$\dfrac{68}{34} = e^{0.00926t}$ Divide both sides by 34.

$\ln 2 = 0.00926t$ Take the natural logarithm of both sides.

$t = \dfrac{\ln 2}{0.00926} \approx 74.85$ Divide by 0.00926 to isolate t.

The population of California will reach 68 million toward the end of the year 2074 if this trend continues.

Skill Practice 3 On January 1, 2000, the population of Texas was approximately 21 million. On January 1, 2010, the population was 25.2 million.

a. Write a function of the form $P(t) = P_0e^{kt}$ to represent the population $P(t)$ of Texas t years after January 1, 2000. Round k to 5 decimal places.

b. Use the function in part (a) to predict the population on January 1, 2020. Round to 1 decimal place.

c. Use the function in part (a) to determine the year during which the population of Texas will reach 40 million if this trend continues.

An exponential model can be presented with a base other than base e. For example, suppose that a culture of bacteria begins with 5000 organisms and the population doubles every 4 hr. Then the population $P(t)$ can be modeled by

$P(t) = 5000(2)^{t/4}$, where t is the time in hours after the culture was started.

Notice that this function is defined using base 2. It is important to realize that any exponential function of one base can be rewritten in terms of an exponential function of another base. In particular we are interested in expressing the function with base e.

Writing an Exponential Expression Using Base e

Let t and b be real numbers, where $b > 0$ and $b \neq 1$. Then,

$$b^t \text{ is equivalent to } e^{(\ln b)t}.$$

To show that $e^{(\ln b)t} = b^t$, use the power property of exponents; that is,

$$e^{(\ln b)t} = (e^{\ln b})^t = b^t$$

EXAMPLE 4 Writing an Exponential Function with Base e

a. The population $P(t)$ of a culture of bacteria is given by $P(t) = 5000(2)^{t/4}$, where t is the time in hours after the culture was started. Write the rule for this function using base e.

b. Find the population after 12 hr using both forms of the function from part (a).

Answers

3. **a.** $P(t) = 21e^{0.01823t}$
b. 30.2 million
c. 2035

Solution:

a. $P(t) = 5000(2)^{t/4}$

Note that $2^{t/4} = (2^t)^{1/4}$

$= \left[e^{(\ln 2)t}\right]^{1/4}$ Apply the property that $e^{(\ln b)t} = b^t$.

$= e^{[(\ln 2)/4]t}$ Apply the power rule of exponents.

Therefore, $P(t) = 5000(2)^{t/4}$

$= 5000e^{[(\ln 2)/4]t}$

$\approx 5000e^{0.17329t}$

b. $P(t) = 5000(2)^{t/4}$ $\quad$ $P(t) \approx 5000e^{0.17329t}$

$P(12) = 5000(2)^{(12)/4}$ $\quad$ $P(12) \approx 5000e^{0.17329(12)}$

$= 40{,}000$ $\quad$ $\approx 40{,}000$

Skill Practice 4

a. Given $P(t) = 10{,}000(2)^{-0.4t}$, write the rule for this function using base e.

b. Find the function value for $t = 10$ for both forms of the function from part (a).

In Example 5, we apply an exponential decay function to determine the age of a bone through radiocarbon dating. Animals ingest carbon through respiration and through the food they eat. Most of the carbon is carbon-12 (^{12}C), an abundant and stable form of carbon. However, a small percentage of carbon is the radioactive isotope, carbon-14 (^{14}C). The ratio of carbon-12 to carbon-14 is constant for all living things. When an organism dies, it no longer takes in carbon from the environment. Therefore, as the carbon-14 decays, the ratio of carbon-12 to carbon-14 changes. Scientists know that the half-life of ^{14}C is 5730 years and from this, they can build a model to represent the amount of ^{14}C remaining t years after death. This is illustrated in Example 5.

EXAMPLE 5 Creating a Model for Exponential Decay

a. Carbon-14 has a half-life of 5730 yr. Write a model of the form $Q(t) = Q_0e^{-kt}$ to represent the amount $Q(t)$ of carbon-14 remaining after t years if no additional carbon is ingested.

b. An archeologist uncovers human remains at an ancient Roman burial site and finds that 76.6% of the carbon-14 still remains in the bone. How old is the bone? Round to the nearest hundred years.

TIP Given the half-life of a radioactive substance, we can also write an exponential model using base $\frac{1}{2}$. The format is

$$Q(t) = Q_0\left(\frac{1}{2}\right)^{t/h}$$

where h is the half-life of the substance.

In Example 5, we have

$$Q(t) = Q_0\left(\frac{1}{2}\right)^{t/5730}$$

Solution:

a. $Q(t) = Q_0e^{-kt}$ Begin with a general exponential decay model.

$0.5Q_0 = Q_0e^{-k(5730)}$ Substitute the known data value. One-half of the original quantity Q_0 is present after 5730 yr.

$0.5 = e^{-k(5730)}$ Divide by Q_0 on both sides.

$\ln 0.5 = -5730k$ Take the natural logarithm of both sides.

$k = \dfrac{\ln 0.5}{-5730}$ Divide by -5730.

≈ 0.000121

$Q(t) = Q_0e^{-0.000121t}$

Answers

4. a. $P(t) = 10{,}000e^{-0.27726t}$

b. 625

b. $0.766Q_0 = Q_0e^{-0.000121t}$ — The quantity $Q(t)$ of carbon-14 in the bone is 76.6% of Q_0.

$0.766 = e^{-0.000121t}$ — Divide by Q_0 on both sides.

$\ln 0.766 = -0.000121t$ — Take the natural logarithm of both sides.

$t = \dfrac{\ln 0.766}{-0.000121} \approx 2200$ — Divide by -0.000121 to isolate t.

The bone is approximately 2200 years old.

Skill Practice 5 Use the function $Q(t) = Q_0e^{-0.000121t}$ to determine the age of a piece of wood that has 42% of its carbon-14 remaining. Round to the nearest 10 yr.

3. Apply Logistic Growth Models

In Examples 3 and 4, we used a model of the form $P(t) = P_0e^{kt}$ to predict population as an exponential function of time. However, unbounded population growth is not possible due to limited resources. A growth model that addresses this problem is called logistic growth. In particular, a logistic growth model imposes a limiting value on the dependent variable.

Logistic Growth Model

A logistic growth model is a function written in the form

$$y = \frac{c}{1 + ae^{-bt}}$$

where a, b, and c are positive constants.

The general logistic growth equation can be written with a complex fraction.

$$y = \frac{c}{1 + \dfrac{a}{e^{bt}}}$$

This term approaches 0 as t approaches ∞.

In this form, we can see that for large values of t, the term $\frac{a}{e^{bt}}$ approaches 0, and the function value y approaches $\frac{c}{1}$.

The line $y = c$ is a horizontal asymptote of the graph, and c represents the limiting value of the function (Figure 3-20).

Notice that the graph of a logistic curve is increasing over its entire domain. However, the *rate* of increase begins to decrease as the function levels off and approaches the horizontal asymptote $y = c$.

In Example 3 we created a function to approximate the population of California assuming unlimited growth. In Example 6, we use a logistic growth model.

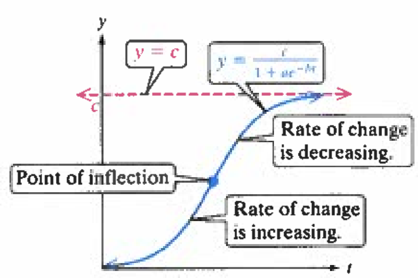


Figure 3-20

TIP The rate of increase of a logistic curve changes from increasing to decreasing to the left and right of a point called the *point of inflection*.

Answer
5. 7170 yr

EXAMPLE 6 **Using Logistic Growth to Model Population**

The population of California $P(t)$ (in millions) can be approximated by the logistic growth function

$$P(t) = \frac{95.2}{1 + 1.8e^{-0.018t}}, \text{ where } t \text{ is the number of years since the year 2000.}$$

a. Determine the population in the year 2000.

b. Use this function to determine the time required for the population of California to double from its value in 2000. Compare this with the result from Example 3(c).

c. What is the limiting value of the population of California under this model?

Solution:

a. $P(t) = \dfrac{95.2}{1 + 1.8e^{-0.018t}}$

$$P(0) = \frac{95.2}{1 + 1.8e^{-0.018(0)}} = \frac{95.2}{1 + 1.8(1)} = 34$$ Substitute 0 for t. Recall that $e^0 = 1$.

The population was approximately 34 million in the year 2000.

b. $68 = \dfrac{95.2}{1 + 1.8e^{-0.018t}}$ Substitute 68 for $P(t)$.

$68(1 + 1.8e^{-0.018t}) = 95.2$ Multiply both sides by $(1 + 1.8e^{-0.018t})$.

$1 + 1.8e^{-0.018t} = 1.4$ Divide by 68 on both sides.

$1.8e^{-0.018t} = 1.4 - 1$ Subtract 1 from both sides.

$e^{-0.018t} = \dfrac{0.4}{1.8}$ Divide by 1.8 on both sides.

$-0.018t = \ln\left(\dfrac{0.4}{1.8}\right)$ Take the natural logarithm of both sides.

$t = \dfrac{\ln\left(\dfrac{0.4}{1.8}\right)}{-0.018} \approx 83.6$ Divide by -0.018 on both sides.

The population will double in approximately 83.6 yr. This is 9 yr later than the predicted value from Example 3(c).

The graphs of $P(t) = \dfrac{95.2}{1 + 1.8e^{-0.018t}}$ and $P(t) = 34e^{0.00926t}$ are shown in Figure 3-21. Notice that the two models agree relatively closely for short-term population growth (out to about 2060). However, in the long term, the unbounded exponential model breaks down. The logistic growth model approaches a limiting population, which is reasonable due to the limited resources to sustain a large human population.

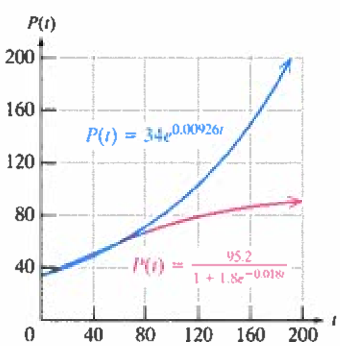


Figure 3-21

c. $P(t) = \dfrac{95.2}{1 + 1.8e^{-0.018t}} = \dfrac{95.2}{1 + \dfrac{1.18}{e^{0.018t}}}$

As $t \to \infty$, the term $\dfrac{1.18}{e^{0.018t}} \to 0$.

As t becomes large, the denominator of $\dfrac{1.18}{e^{0.018t}}$ also becomes large. This causes the quotient to approach zero. Therefore, as t approaches infinity, $P(t)$ approaches 95.2. Under this model, the limiting value for the population of California is 95.2 million.

Skill Practice 6 The score on a test of dexterity is given by $P(t) = \dfrac{100}{1 + 19e^{-0.354x}}$, where x is the number of times the test is taken.

a. Determine the initial score.

b. Use the function to determine the minimum number of times required for the score to exceed 90.

c. What is the limiting value of the scores?

4. Create Exponential and Logarithmic Models Using Regression

In Examples 7 and 8, we use a graphing utility and regression techniques to find an exponential model or logarithmic model based on observed data.

EXAMPLE 7 Creating an Exponential Model from Observed Data

The amount of sunlight y [in langleys (Ly)—a unit used to measure solar energy in calories/cm^2] is measured for six different depths x (in meters) in Lake Lyndon B. Johnson in Texas.

x (m)	1	3	5	7	9	11
y (Ly)	300	161	89	50	27	15

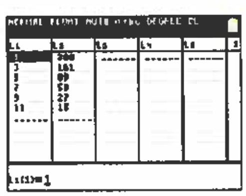
a. Graph the data.

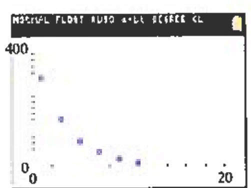
b. From visual inspection of the graph, which model would best represent the data? Choose from $y = mx + b$ (linear), $y = ab^x$ (exponential), or $y = a + b\ln x$ (logarithmic).

c. Use a graphing utility to find a regression equation that fits the data.

Solution:

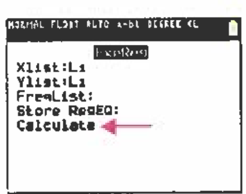
a. Enter the data in two lists.

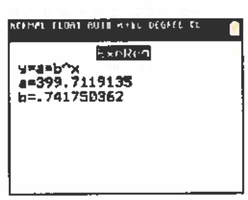


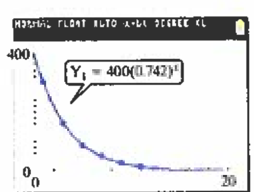


b. Note that for large depths, the amount of sunlight approaches 0. Therefore, the curve is asymptotic to the x-axis. This is consistent with a decreasing exponential model. The exponential model $y = ab^x$ appears to fit.

c. Under the STAT menu, choose CALC, ExpReg, and then Calculate.







The equation $y = ab^x$ is $y = 400(0.742)^x$.

Answers

6. a. 5 **b.** 15 **c.** 100

Skill Practice 7 For the given data,

x	1	3	5	7	9	11
y	2.9	5.6	11.1	22.4	43.0	85.0

a. Graph the data points.

b. Use a graphing utility to find a model of the form $y = ab^x$ to fit the data.

EXAMPLE 8 Creating a Logarithmic Model from Observed Data

The diameter x (in mm) of a sugar maple tree, along with the corresponding age y (in yr) of the tree is given for six different trees.

x (mm)	1	50	100	200	300	400
y (yr)	4	60	72	82	89	94

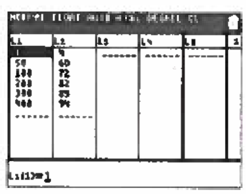
a. Graph the data.

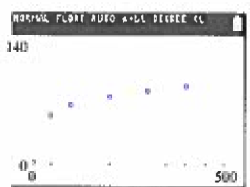
b. From visual inspection of the graph, which model would best represent the data? Choose from $y = mx + b$ (linear), $y = ab^x$ (exponential), or $y = a + b \ln x$ (logarithmic).

c. Use a graphing utility to find a regression equation that fits the data.

Solution:

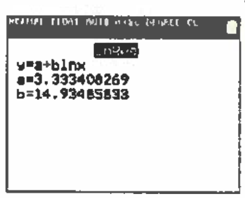
a. Enter the data into two lists.

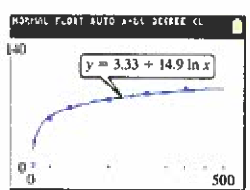




b. By inspection of the graph, the logarithmic model $y = a + b \ln x$ appears to fit.

c. Under the STAT menu, choose CALC, and then LnReg.





Skill Practice 8 For the given data,

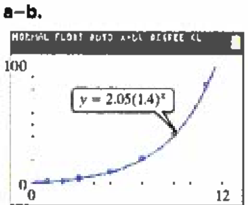
x	1	5	9	13	17	21
y	11.9	19.3	21.9	23.5	24.7	25.7

a. Graph the data points.

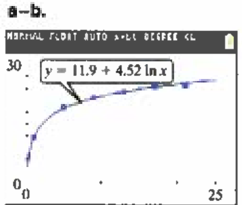
b. Use a graphing utility to find a model of the form $y = a + b \ln x$ to fit the data.

Answers

7. a–b.



8. a–b.



SECTION 3.6 Practice Exercises

Prerequisite Review

For Exercises R.1–R.3, solve for the indicated variable.

R.1. $P = \frac{1}{3}Ln$ for L

R.2. $Q = \frac{3}{4}m^3$ for m, $m > 0$

R.3. $9 + \sqrt{a^2 - b^2} = c$ for b

R.4. Given $f(x) = 5x^2 + 2x$, evaluate

a. $f(-2)$ **b.** $f(-1)$ **c.** $f(0)$ **d.** $f(1)$

Concept Connections

1. If $k > 0$, the equation $y = y_0e^{kt}$ is a model for exponential (growth/decay), whereas if $k < 0$, the equation is a model for exponential (growth/decay).

2. A function defined by $y = ab^x$ can be written in terms of an exponential function base e as ________.

3. A function defined by $y = \frac{c}{1 + ae^{-bt}}$ is called a ________ growth model and imposes a limiting value on y.

4. Given a logistic growth function $y = \frac{c}{1 + ae^{-bt}}$, the limiting value of y is ________.

Objective 1: Solve Literal Equations for a Specified Variable

For Exercises 5–14, solve for the indicated variable. (See Example 1)

5. $Q = Q_0e^{-kt}$ for k (used in chemistry)

6. $N = N_0e^{-0.025t}$ for t (used in chemistry)

7. $M = 8.8 + 5.1 \log D$ for D (used in astronomy)

8. $\log E - 12.2 = 1.44M$ for E (used in geology)

9. $\text{pH} = -\log[\text{H}^+]$ for $[\text{H}^+]$ (used in chemistry)

10. $L = 10 \log\left(\frac{I}{I_0}\right)$ for I (used in medicine)

11. $A = P(1 + r)^t$ for t (used in finance)

12. $A = Pe^{rt}$ for r (used in finance)

13. $\ln\left(\frac{k}{A}\right) = \frac{-E}{RT}$ for k (used in chemistry)

14. $-\frac{1}{k}\ln\left(\frac{P}{14.7}\right) = A$ for P (used in meteorology)

Objective 2: Create Models for Exponential Growth and Decay

15. Suppose that \$12,000 is invested in a bond fund and the account grows to \$14,309.26 in 4 yr. **(See Example 2)**

a. Use the model $A = Pe^{rt}$ to determine the average rate of return under continuous compounding. Round to the nearest tenth of a percent.

b. How long will it take the investment to reach \$20,000 if the rate of return continues? Round to the nearest tenth of a year.

16. Suppose that \$50,000 from a retirement account is invested in a large cap stock fund. After 20 yr, the value is \$194,809.67.

a. Use the model $A = Pe^{rt}$ to determine the average rate of return under continuous compounding. Round to the nearest tenth of a percent.

b. How long will it take the investment to reach one-quarter million dollars? Round to the nearest tenth of a year.

17. Suppose that P dollars in principal is invested in an account earning 3.2% interest compounded continuously. At the end of 3 yr, the amount in the account has earned \$806.07 in interest.

a. Find the original principal. Round to the nearest dollar. (*Hint:* Use the model $A = Pe^{rt}$ and substitute $P + 806.07$ for A.)

b. Using the original principal from part (a) and the model $A = Pe^{rt}$, determine the time required for the investment to reach \$10,000. Round to the nearest year.

18. Suppose that P dollars in principal is invested in an account earning 2.1% interest compounded continuously. At the end of 2 yr, the amount in the account has earned \$193.03 in interest.

a. Find the original principal. Round to the nearest dollar. (*Hint:* Use the model $A = Pe^{rt}$ and substitute $P + 193.03$ for A.)

b. Using the original principal from part (a) and the model $A = Pe^{rt}$, determine the time required for the investment to reach \$6000. Round to the nearest tenth of a year.

19. The populations of two countries are given for January 1, 2000, and for January 1, 2010.

a. Write a function of the form $P(t) = P_0e^{kt}$ to model each population $P(t)$ (in millions) t years after January 1, 2000. **(See Example 3)**

Country	Population in 2000 (millions)	Population in 2010 (millions)	$P(t) = P_0e^{kt}$
Australia	19.0	22.6	
Taiwan	22.9	23.7	

b. Use the models from part (a) to approximate the population on January 1, 2020, for each country. Round to the nearest hundred thousand.

c. Australia had fewer people than Taiwan in the year 2000, yet from the result of part (b), Australia would have more people in the year 2020? Why?

d. Use the models from part (a) to predict the year during which each population would reach 30 million if this trend continues.

20. The populations of two countries are given for January 1, 2000, and for January 1, 2010.

a. Write a function of the form $P(t) = P_0e^{kt}$ to model each population $P(t)$ (in millions) t years after January 1, 2000.

Country	Population in 2000 (millions)	Population in 2010 (millions)	$P(t) = P_0e^{kt}$
Switzerland	7.3	7.8	
Israel	6.7	7.7	

b. Use the models from part (a) to approximate the population on January 1, 2020, for each country. Round to the nearest hundred thousand.

c. Israel had fewer people than Switzerland in the year 2000, yet from the result of part (b), Israel would have more people in the year 2020? Why?

d. Use the models from part (a) to predict the year during which each population would reach 10 million if this trend continues.

21. A function of the form $P(t) = ab^t$ represents the population (in millions) of the given country t years after January 1, 2000. **(See Example 4)**

a. Write an equivalent function using base e; that is, write a function of the form $P(t) = P_0e^{kt}$. Also, determine the population of each country for the year 2000.

Country	$P(t) = ab^t$	$P(t) = P_0e^{kt}$	Population in 2000
Costa Rica	$P(t) = 4.3(1.0135)^t$		
Norway	$P(t) = 4.6(1.0062)^t$		

b. The population of the two given countries is very close for the year 2000, but their growth rates are different. Use the model to approximate the year during which the population of each country reached 5 million.

c. Costa Rica had fewer people in the year 2000 than Norway. Why would Costa Rica reach a population of 5 million sooner than Norway?

22. A function of the form $P(t) = ab^t$ represents the population (in millions) of the given country t years after January 1, 2000.

a. Write an equivalent function using base e; that is, write a function of the form $P(t) = P_0e^{kt}$. Also, determine the population of each country for the year 2000.

Country	$P(t) = ab^t$	$P(t) = P_0e^{kt}$	Population in 2000
Haiti	$P(t) = 8.5(1.0158)^t$		
Sweden	$P(t) = 9.0(1.0048)^t$		

b. The population of the two given countries is very close for the year 2000, but their growth rates are different. Use the model to approximate the year during which the population of each country would reach 10.5 million.

c. Haiti had fewer people in the year 2000 than Sweden. Why would Haiti reach a population of 10.5 million sooner?

For Exercises 23–24, refer to the model $Q(t) = Q_0e^{-0.000121t}$ used in Example 5 for radiocarbon dating.

23. A sample from a mummified bull was taken from a pyramid in Dashur, Egypt. The sample shows that 78% of the carbon-14 still remains. How old is the sample? Round to the nearest year. **(See Example 5)**

24. At the "Marmes Man" archeological site in southeastern Washington State, scientists uncovered the oldest human remains yet to be found in Washington State. A sample from a human bone taken from the site showed that 29.4% of the carbon-14 still remained. How old is the sample? Round to the nearest year.

25. The isotope of plutonium ^{238}Pu is used to make thermoelectric power sources for spacecraft. Suppose that a space probe was launched in 2012 with 2.0 kg of ^{238}Pu.

a. If the half-life of ^{238}Pu is 87.7 yr, write a function of the form $Q(t) = Q_0e^{-kt}$ to model the quantity $Q(t)$ of ^{238}Pu left after t years.

b. If 1.6 kg of ^{238}Pu is required to power the spacecraft's data transmitter, for how long after launch would scientists be able to receive data? Round to the nearest year.

26. Technetium-99 (^{99m}Tc) is a radionuclide used widely in nuclear medicine. ^{99m}Tc is combined with another substance that is readily absorbed by a targeted body organ. Then, special cameras sensitive to the gamma rays emitted by the technetium are used to record pictures of the organ. Suppose that a technician prepares a sample of ^{99m}Tc-pyrophosphate to image the heart of a patient suspected of having had a mild heart attack.

a. At noon, the patient is given 10 mCi (millicuries) of ^{99m}Tc. If the half-life of ^{99m}Tc is 6 hr, write a function of the form $Q(t) = Q_0e^{-kt}$ to model the radioactivity level $Q(t)$ after t hours.

b. At what time will the level of radioactivity reach 3 mCi? Round to the nearest tenth of an hour.

27. Fluorodeoxyglucose is a derivative of glucose that contains the radionuclide fluorine-18 (^{18}F). A patient is given a sample of this material containing 300 MBq of ^{18}F (a megabecquerel is a unit of radioactivity). The patient then undergoes a PET scan (positron emission tomography) to detect areas of metabolic activity indicative of cancer. After 174 min, one-third of the original dose remains in the body.

a. Write a function of the form $Q(t) = Q_0e^{-kt}$ to model the radioactivity level $Q(t)$ of fluorine-18 at a time t minutes after the initial dose.

b. What is the half-life of ^{18}F? Round to the nearest minute.

28. Painful bone metastases are common in advanced prostate cancer. Physicians often order treatment with strontium-89 (^{89}Sr), a radionuclide with a strong affinity for bone tissue. A patient is given a sample containing 4 mCi of ^{89}Sr.

a. If 20% of the ^{89}Sr remains in the body after 90 days, write a function of the form $Q(t) = Q_0e^{-kt}$ to model the amount $Q(t)$ of radioactivity in the body t days after the initial dose.

b. What is the biological half-life of ^{89}Sr under this treatment? Round to the nearest tenth of a day.

29. Two million *E. coli* bacteria are present in a laboratory culture. An antibacterial agent is introduced and the population of bacteria $P(t)$ decreases by half every 6 hr. The population can be represented by $P(t) = 2{,}000{,}000\left(\frac{1}{2}\right)^{t/6}$.

a. Convert this to an exponential function using base e.

b. Verify that the original function and the result from part (a) yield the same result for $P(0)$, $P(6)$, $P(12)$, and $P(60)$. (*Note*: There may be round-off error.)

30. The half-life of radium-226 is 1620 yr. Given a sample of 1 g of radium-226, the quantity left $Q(t)$ (in g) after t years is given by $Q(t) = \left(\frac{1}{2}\right)^{t/1620}$.

a. Convert this to an exponential function using base e.

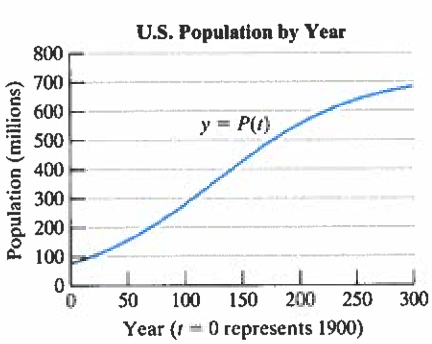
b. Verify that the original function and the result from part (a) yield the same result for $Q(0)$, $Q(1620)$, and $Q(3240)$. (*Note*: There may be round-off error.)

Objective 3: Apply Logistic Growth Models

31. The population of the United States $P(t)$ (in millions) since January 1, 1900, can be approximated by

$$P(t) = \frac{725}{1 + 8.295e^{-0.0165t}}$$

where t is the number of years since January 1, 1900. (**See Example 6**)



a. Evaluate $P(0)$ and interpret its meaning in the context of this problem.

b. Use the function to approximate the U.S. population on January 1, 2020. Round to the nearest million.

c. Use the function to approximate the U.S. population on January 1, 2050.

d. From the model, during which year would the U.S. population reach 500 million?

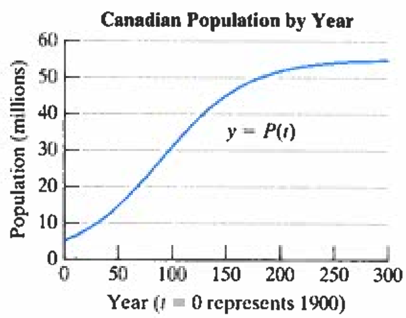
e. What value will the term $\dfrac{8.295}{e^{0.0165t}}$ approach as $t \to \infty$?

f. Determine the limiting value of $P(t)$.

32. The population of Canada $P(t)$ (in millions) since January 1, 1900, can be approximated by

$$P(t) = \frac{55.1}{1 + 9.6e^{-0.02515t}}$$

where t is the number of years since January 1, 1900.



a. Evaluate $P(0)$ and interpret its meaning in the context of this problem.

b. Use the function to approximate the Canadian population on January 1, 2015. Round to the nearest tenth of a million.

c. Use the function to approximate the Canadian population on January 1, 2040.

d. From the model, during which year would the Canadian population reach 45 million?

e. What value will the term $\dfrac{9.6}{e^{0.02515t}}$ approach as $t \to \infty$?

f. Determine the limiting value of $P(t)$.

33. The number of computers $N(t)$ (in millions) infected by a computer virus can be approximated by

$$N(t) = \frac{2.4}{1 + 15e^{-0.72t}}$$

where t is the time in months after the virus was first detected.

a. Determine the number of computers initially infected when the virus was first detected.

b. How many computers were infected after 6 months? Round to the nearest hundred thousand.

c. Determine the amount of time required after initial detection for the virus to affect 1 million computers. Round to the nearest tenth of a month.

d. What is the limiting value of the number of computers infected according to this model?

34. After a new product is launched the cumulative sales $S(t)$ (in \$1000) t weeks after launch is given by

$$S(t) = \frac{72}{1 + 9e^{-0.36t}}$$

a. Determine the cumulative amount in sales 3 weeks after launch. Round to the nearest thousand.

b. Determine the amount of time required for the cumulative sales to reach \$70,000.

c. What is the limiting value in sales?

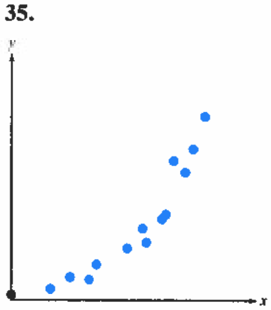
Objective 4: Create Exponential and Logarithmic Models Using Regression

For Exercises 35–38, a graph of data is given. From visual inspection, which model would best fit the data? Choose from

$y = mx + b$ **(linear)** $\quad$ $y = ab^x$ **(exponential)**

$y = a + b\ln x$ **(logarithmic)** $\quad$ $y = \dfrac{c}{1 + ae^{-bx}}$ **(logistic)**

35.



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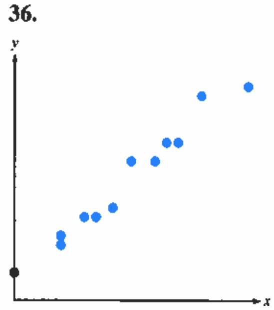

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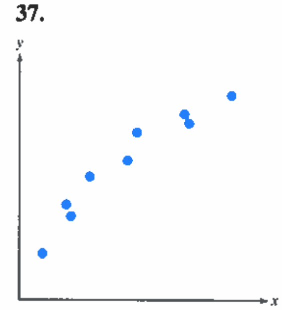

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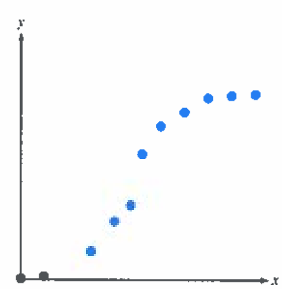


For Exercises 39–46, a table of data is given.

a. Graph the points and from visual inspection, select the model that would best fit the data. Choose from

$y = mx + b$ (linear) $y = ab^x$ (exponential)

$y = a + b \ln x$ (logarithmic) $y = \dfrac{c}{1 + ae^{-bx}}$ (logistic)

b. Use a graphing utility to find a function that fits the data. (*Hint:* For a logistic model, go to STAT, CALC, Logistic.)

39.

x	y
0	2.3
4	3.6
8	5.7
12	9.1
16	14
20	22

40.

x	y
0	52
1	67
2	87
3	114
4	147
5	195

41.

x	y
3	2.7
7	12.2
13	25.7
15	30
17	34
21	44.4

42.

x	y
0	640
20	530
40	430
50	360
80	210
100	90

43.

x	y
10	43.3
20	50
30	53
40	56.8
50	58.8
60	60.8

44.

x	y
5	29
10	40
15	45.6
20	50
25	53.3
30	56

45.

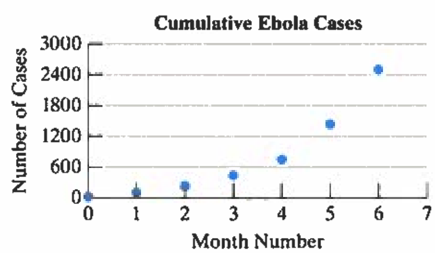
x	y
2	0.326
4	2.57
6	10.8
8	16.8
10	17.9
5	6
7	14.8

46.

x	y
0	0.05
2	0.45
4	2.94
5	5.8
6	8.8
7	10.6
8	11.5
10	11.9

47. During a recent outbreak of Ebola in western Africa, the cumulative number of cases y was reported t months after April 1. **(See Example 7)**

Month Number (t)	Cumulative Cases (y)
0	18
1	105
2	230
3	438
4	752
5	1437
6	2502



a. Use a graphing utility to find a model of the form $y = ab^t$. Round a to 1 decimal place and b to 3 decimal places.

b. Write the function from part (a) as an exponential function of the form $y = ae^{bt}$.

c. Use either model to predict the number of Ebola cases 8 months after April 1 if this trend continues. Round to the nearest thousand.

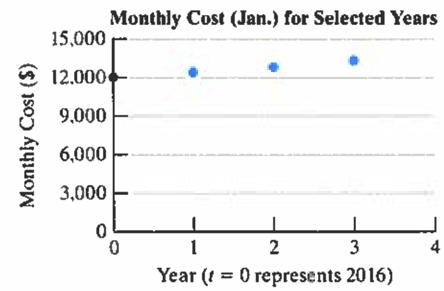
d. Would it seem reasonable for this trend to continue indefinitely?

e. Use a graphing utility to find a logistic model $y = \dfrac{c}{1 + ae^{-bt}}$. Round a and c to the nearest whole number and b to 2 decimal places.

f. Use the logistic model from part (e) to predict the number of Ebola cases 8 months after April 1. Round to the nearest thousand.

48. The monthly costs for a small company to do business has been increasing over time due in part to inflation. The table gives the monthly cost y (in \$) for the month of January for selected years. The variable t represents the number of years since 2016.

Year (t = 0 is 2016)	Monthly Costs (\$) y
0	12,000
1	12,400
2	12,800
3	13,300

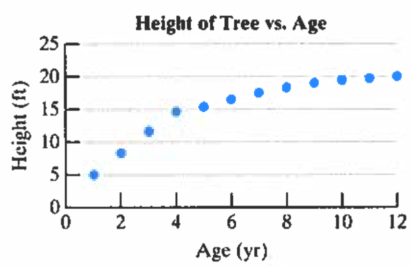


a. Use a graphing utility to find a model of the form $y = ab^t$. Round a to the nearest whole unit and b to 3 decimal places.

b. Write the function from part (a) as an exponential function with base e.

c. Use either model to predict the monthly cost for January in the year 2023 if this trend continues. Round to the nearest hundred dollars.

49. The age of a tree t (in yr) and its corresponding height $H(t)$ are given in the table. **(See Example 8)**



Age of Tree (yr) t	Height (ft) $H(t)$
1	5
2	8.3
3	11.6
4	14.6
5	15.4
6	16.5
7	17.5
8	18.3
9	19
10	19.4
11	19.7
12	20

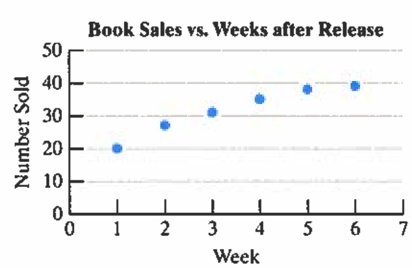
a. Write a model of the form $H(t) = a + b \ln t$. Round a and b to 2 decimal places.

b. Use the model to predict the age of a tree if it is 25 ft high. Round to the nearest year.

c. Is it reasonable to assume that this logarithmic trend will continue indefinitely? Why or why not?

50. The sales of a book tend to increase over the short-term as word-of-mouth makes the book "catch on." The number of books sold $N(t)$ for a new novel t weeks after release at a certain book store is given in the table for the first 6 weeks.

Weeks t	Number Sold $N(t)$
1	20
2	27
3	31
4	35
5	38
6	39



a. Find a model of the form $N(t) = a + b \ln t$. Round a and b to 1 decimal place.

b. Use the model to predict the sales in week 7. Round to the nearest whole unit.

c. Is it reasonable to assume that this logarithmic trend will continue? Why or why not?

Mixed Exercises

51. A van is purchased new for \$29,200.

a. Write a linear function of the form $y = mt + b$ to represent the value y of the vehicle t years after purchase. Assume that the vehicle is depreciated by \$2920 per year.

b. Suppose that the vehicle is depreciated so that it holds only 80% of its value from the previous year. Write an exponential function of the form $y = V_0b^t$, where V_0 is the initial value and t is the number of years after purchase.

c. To the nearest dollar, determine the value of the vehicle after 5 yr and after 10 yr using the linear model.

d. To the nearest dollar, determine the value of the vehicle after 5 yr and after 10 yr using the exponential model.

52. A delivery truck is purchased new for \$54,000.

a. Write a linear function of the form $y = mt + b$ to represent the value y of the vehicle t years after purchase. Assume that the vehicle is depreciated by \$6750 per year.

b. Suppose that the vehicle is depreciated so that it holds 70% of its value from the previous year. Write an exponential function of the form $y = V_0b^t$, where V_0 is the initial value and t is the number of years after purchase.

c. To the nearest dollar, determine the value of the vehicle after 4 yr and after 8 yr using the linear model.

d. To the nearest dollar, determine the value of the vehicle after 4 yr and after 8 yr using the exponential model.

Write About It

53. Why is it important to graph a set of data before trying to find an equation or function to model the data.

54. How does the average rate of change differ for a linear function versus an increasing exponential function?

55. Explain the difference between an exponential growth model and a logistic growth model.

56. Explain how to convert an exponential expression b^t to an exponential expression base e.

Expanding Your Skills

57. The monthly payment P (in \$) to pay off a loan of amount A (in \$) at an interest rate r in t years is given by

$$P = \frac{\frac{Ar}{12}}{1 - \left(1 + \frac{r}{12}\right)^{-12t}}.$$

a. Solve for t (note that there are numerous equivalent algebraic forms for the result).

b. Interpret the meaning of the resulting relationship.

58. Suppose that a population follows a logistic growth pattern, with a limiting population N. If the initial population is denoted by P_0, and t is the amount of time elapsed, then the population P can be represented by

$$P = \frac{P_0N}{P_0 + (N - P_0)e^{-kt}},$$

where k is a constant related to the growth rate.

a. Solve for t (note that there are numerous equivalent algebraic forms for the result).

b. Interpret the meaning of the resulting relationship.

CHAPTER 3 KEY CONCEPTS

SECTION 3.1 Inverse Functions	Reference
A function f is **one-to-one** if for a and b in the domain of f, if $a \neq b$, then $f(a) \neq f(b)$, or equivalently, if $f(a) = f(b)$, then $a = b$.	p. 357
Horizontal line test: A function defined by $y = f(x)$ is one-to-one if no horizontal line intersects the graph in more than one point.	p. 357
Function **g is the inverse of f** if $(f \circ g)(x) = x$ for all x in the domain of g and $(g \circ f)(x) = x$ for all x in the domain of f.	p. 359
Procedure to find $f^{-1}(x)$: 1. Replace $f(x)$ by y. 2. Interchange x and y. 3. Solve for y. 4. Replace y by $f^{-1}(x)$.	p. 360

SECTION 3.2 Exponential Functions	Reference
Let b be a real number with $b > 0$ and $b \neq 1$. Then for any real number x, a function of the form $f(x) = b^x$ is an **exponential function of base b.**	p. 369
For the graph of an exponential function $f(x) = b^x$, • If $b > 1$, f is an increasing function. • If $0 < b < 1$, f is a decreasing function. • The domain is $(-\infty, \infty)$. • The range is $(0, \infty)$. • The line $y = 0$ is a horizontal asymptote. • The function passes through $(0, 1)$.	p. 370
The irrational number e is the limiting value of the expression $\left(1 + \frac{1}{x}\right)^x$ as x approaches ∞. $e \approx 2.71828$	p. 372
If P dollars in principal is invested or borrowed at an annual interest rate r for t years, then $I = Prt$ Simple interest $A = P\left(1 + \frac{r}{n}\right)^{nt}$ Future value A with interest compounded n times per year. $A = Pe^{rt}$ Future value A with interest compounded continuously.	p. 374

SECTION 3.3 Logarithmic Functions	Reference
If x and b are positive real numbers such that $b \neq 1$, then $y = \log_b x$ is called the **logarithmic function** base b, where $y = \log_b x$ is equivalent to $b^y = x$. logarithmic form exponential form	p. 382
The functions $f(x) = \log_b x$ and $g(x) = b^x$ are inverses.	p. 382
Basic properties of logarithms: 1. $\log_b 1 = 0$ 2. $\log_b b = 1$ 3. $\log_b b^x = x$ 4. $b^{\log_b x} = x$	p. 386
$y = \log_{10} x$ is written as $y = \log x$ and is called the **common logarithmic function.** $y = \log_e x$ is written as $y = \ln x$ and is called the **natural logarithmic function.**	p. 384
Given $f(x) = \log_b x$, • If $b > 1$, f is an increasing function. • If $0 < b < 1$, f is a decreasing function. • The domain is $(0, \infty)$. • The range is $(-\infty, \infty)$. • The line $x = 0$ is a vertical asymptote. • The function passes through $(1, 0)$.	p. 388
The domain of $f(x) = \log_b x$ is $\{x \mid x > 0\}$.	p. 389

SECTION 3.4 Properties of Logarithms	Reference
Let b, x, and y be positive real numbers with $b \neq 1$. Then, $\log_b(xy) = \log_b x + \log_b y$ (Product property) $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$ (Quotient property) $\log_b x^p = p \log_b x$ (Power property)	p. 399
Change-of-base formula: For positive real numbers a and b, where $a \neq 1$ and $b \neq 1$, $\log_b x = \frac{\log_a x}{\log_a b}$.	p. 402

SECTION 3.5 Exponential and Logarithmic Equations and Applications	Reference
Equivalence property of exponential expressions: If b, x, and y are real numbers with $b > 0$ and $b \neq 1$, then $b^x = b^y$ implies that $x = y$.	p. 407
Equivalence property of logarithmic expressions: If b, x, and y are positive real numbers and $b \neq 1$, then $\log_b x = \log_b y$ implies that $x = y$.	p. 411
Steps to solve exponential equations by using logarithms: 1. Isolate the exponential expression on one side of the equation. 2. Take a logarithm of the same base on both sides of the equation. 3. Use the power property of logarithms to "bring down" the exponent. 4. Solve the resulting equation.	p. 408
Guidelines to solve a logarithmic equation: 1. Isolate the logarithms on one side of the equation. 2. Use the properties of logarithms to write the equation in the form $\log_b x = k$, where k is a constant. 3. Write the equation in exponential form. 4. Solve the equation from step 3. 5. Check the potential solution(s) in the original equation.	p. 412

SECTION 3.6 Modeling with Exponential and Logarithmic Functions	Reference
The function defined by $y = y_0e^{kt}$ represents exponential growth if $k > 0$ and exponential decay if $k < 0$.	p. 422
An exponential expression can be rewritten as an expression of a different base. In particular, to convert to base e, we have b^t is equivalent to $e^{(\ln b)t}$.	p. 424
A **logistic growth function** is a function of the form $y = \frac{c}{1 + ae^{-bt}}$. A logistic growth function imposes a limiting value on the dependent variable.	p. 426

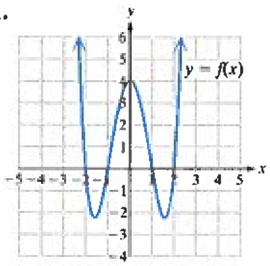
Expanded Chapter Summary available at www.mhhe.com/millerprecalculus.

CHAPTER 3 Review Exercises

SECTION 3.1

For Exercises 1–2, determine if the relation defines y as a one-to-one function of x.

1.



2.

x	y
5	7
−3	1
−4	−2
6	0

For Exercises 3–4, use the definition of a one-to-one function to determine if the function is one-to-one. Recall that f is one-to-one if $a \neq b$ implies that $f(a) \neq f(b)$, or equivalently, if $f(a) = f(b)$, then $a = b$.

3. $f(x) = x^3 - 1$

4. $f(x) = x^2 - 1$

For Exercises 5–6, determine if the functions are inverses.

5. $f(x) = 4x - 3$ and $g(x) = \dfrac{x+3}{4}$

6. $m(x) = \sqrt[3]{x+1}$ and $n(x) = (x-1)^3$

For Exercises 7–8, a one-to-one function is given. Write an equation for the inverse function.

7. $f(x) = 2x^3 - 5$

8. $f(x) = \dfrac{2}{x+7}$

9. a. Graph $f(x) = x^2 - 9, x \le 0$.
   b. Is f a one-to-one function?
   c. Write the domain of f in interval notation.
   d. Write the range of f in interval notation.
   e. Find an equation for f^{-1}.
   f. Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system.
   g. Write the domain of f^{-1} in interval notation.
   h. Write the range of f^{-1} in interval notation.

10. a. Graph $g(x) = \sqrt{x+4}$.
   b. Is g a one-to-one function?
   c. Write the domain of g in interval notation.
   d. Write the range of g in interval notation.
   e. Find an equation for g^{-1}.
   f. Graph $y = g(x)$ and $y = g^{-1}(x)$ on the same coordinate system.
   g. Write the domain of g^{-1} in interval notation.
   h. Write the range of g^{-1} in interval notation.

11. The function $f(x) = 5280x$ provides the conversion from x miles to $f(x)$ feet.
   a. Write an equation for f^{-1}.
   b. What does the inverse function represent in the context of this problem?
   c. Determine the number of miles represented by 22,176 ft.

SECTION 3.2

12. Which of the functions is an exponential function?
   a. $f(x) = x^4$
   b. $h(x) = 4^{-x}$
   c. $g(x) = \left(\dfrac{4}{3}\right)^x$
   d. $k(x) = \dfrac{4x}{3}$
   e. $n(x) = \dfrac{4}{3x}$
   f. $r(x) = \left(-\dfrac{4}{3}\right)^x$

For Exercises 13–16,

a. Graph the function.
b. Write the domain in interval notation.
c. Write the range in interval notation.
d. Write an equation of the asymptote.

13. $f(x) = \left(\dfrac{5}{2}\right)^x$

14. $g(x) = \left(\dfrac{5}{2}\right)^{-x}$

15. $k(x) = -3^x + 1$

16. $h(x) = 2^{x-3} - 4$

17. Is the graph of $y = e^x$ an increasing or decreasing exponential function?

For Exercises 18–19, use the formulas on page 374.

18. Suppose that \$24,000 is invested at the given interest rates and compounding options. Determine the amount that the investment is worth at the end of t years.
   a. 5% interest compounded monthly for 10 yr
   b. 4.5% interest compounded continuously for 30 yr

19. Jorge needs to borrow \$12,000 to buy a car. He can borrow the money at 7.2% simple interest for 4 yr or he can borrow at 6.5% interest compounded continuously for 4 yr.
   a. How much total interest would Jorge pay at 7.2% simple interest?
   b. How much total interest would Jorge pay at 6.5% interest compounded continuously?
   c. Which option results in less total interest?

20. A patient is treated with 128 mCi (millicuries) of iodine-131 (^{131}I). The radioactivity level $R(t)$ (in mCi) after t days is given by $R(t) = 128(2)^{-t/4.2}$. (In this model, the value 4.2 is related to the biological half-life of radioactive iodine in the body.)
   a. Determine the radioactivity level of ^{131}I in the body after 6 days. Round to the nearest whole unit.

b. Evaluate $R(4.2)$ and interpret its meaning in the context of this problem.

c. After how many half-lives will the radioactivity level be 16 mCi?

SECTION 3.3

For Exercises 21–22, write the equation in exponential form.

21. $\log_b(x^2 + y^2) = 4$ **22.** $\ln x = (c + d)$

For Exercises 23–24, write the equation in logarithmic form.

23. $10^6 = 1{,}000{,}000$ **24.** $8^{-1/3} = \frac{1}{2}$

For Exercises 25–32, simplify the logarithmic expression without using a calculator.

25. $\log_3 81$ **26.** $\log 100{,}000$

27. $\log_2\left(\frac{1}{64}\right)$ **28.** $\log_{1/4}(16)$

29. $\log_{11} 1$ **30.** $\log_5 5$

31. $4^{\log_4 7}$ **32.** $\ln e^{11}$

For Exercises 33–37, write the domain of the function in interval notation.

33. $f(x) = \log(x - 4)$ **34.** $g(x) = \ln(3 - 2x)$

35. $h(x) = \log_2(x^2 + 4)$ **36.** $k(x) = \log_2(x^2 - 4)$

37. $m(x) = \log_2(x - 4)^2$

For Exercises 38–39,

a. Graph the function.

b. Write the domain in interval notation.

c. Write the range in interval notation.

d. Write an equation of the asymptote.

38. $f(x) = \log_2(x - 3)$ **39.** $g(x) = 2 + \ln x$

For Exercise 40–41, use the formula $\text{pH} = -\log[H^+]$ to compute the pH of a liquid as a function of its concentration of hydronium ions, $[H^+]$ in mol/L. If the pH is less than 7, then the substance is acidic. If the pH is greater than 7, then the substance is alkaline (or basic).

a. Find the pH. Round to 1 decimal place.

b. Determine whether the substance is acidic or alkaline.

40. Baking soda: $[H^+] = 5.0 \times 10^{-9}$ mol/L

41. Tomatoes: $[H^+] = 3.16 \times 10^{-5}$ mol/L

SECTION 3.4

For Exercises 42–48, fill in the blanks to state the basic properties of logarithms. Assume that x, y, and b are positive real numbers with $b \neq 1$.

42. $\log_b 1 =$ ________ **43.** $\log_b b =$ ________

44. $\log_b b^p =$ ________ **45.** $b^{\log_b x} =$ ________

46. $\log_b(xy) =$ ________ **47.** $\log_b\left(\frac{x}{y}\right) =$ ________

48. $\log_b x^p =$ ________

For Exercises 49–52, write the logarithm as a sum or difference of logarithms. Simplify each term as much as possible.

49. $\log\left(\frac{100}{\sqrt{c^2 + 10}}\right)$ **50.** $\log_2\left(\frac{1}{8}a^2b\right)$

51. $\ln\left(\frac{\sqrt[3]{ab^2}}{cd^5}\right)$ **52.** $\log\left(\frac{x^2(2x + 1)^5}{\sqrt{1 - x}}\right)$

For Exercises 53–55, write the logarithmic expression as a single logarithm with coefficient 1, and simplify as much as possible.

53. $4\log_5 y - 3\log_5 x + \frac{1}{2}\log_5 z$

54. $\log 250 + \log 2 - \log 5$

55. $\frac{1}{4}\ln(x^2 - 9) - \frac{1}{4}\ln(x - 3)$

For Exercises 56–58, use $\log_b 2 \approx 0.289$, $\log_b 3 \approx 0.458$, and $\log_b 5 \approx 0.671$ to approximate the value of the given logarithms.

56. $\log_b 8$ **57.** $\log_b 45$ **58.** $\log_b\left(\frac{1}{9}\right)$

For Exercises 59–60, use the change-of-base formula and a calculator to approximate the given logarithms. Round to 4 decimal places. Then check the answer by using the related exponential form.

59. $\log_7 596$ **60.** $\log_4 0.982$

SECTION 3.5

For Exercises 61–80, solve the equation. Write the solution set with exact values and give approximate solutions to 4 decimal places.

61. $4^{2y-7} = 64$ **62.** $1000^{2x+1} = \left(\frac{1}{100}\right)^{x-4}$

63. $7^x = 51$ **64.** $516 = 11^w - 21$

65. $3^{2x+1} = 4^{3x}$ **66.** $2^{c+3} = 7^{2c+5}$

67. $400e^{-2t} = 2.989$ **68.** $2 \cdot 10^{1.2t} = 58$

69. $e^{2x} - 3e^x - 40 = 0$ **70.** $e^{2x} = -10e^x$

71. $\log_5(4p + 7) = \log_5(2 - p)$

72. $\log_2(m^2 + 10m) = \log_2 11$

73. $2\log_6(4 - 8y) + 6 = 10$

74. $5 = -4\log_3(2 - 5x) + 1$

75. $3\ln(n - 8) = 6.3$

76. $-4 + \log_2 x = -\log_2(x + 6)$

77. $\log_6(3x + 2) = \log_6(x + 4) + 1$

78. $\ln x + \ln(x + 2) = \ln(x + 6)$

79. $\log_5(\log_2 x) = 1$

80. $(\log x)^2 - \log x^2 = 35$

For Exercises 81–82, find the inverse of the function.

81. $f(x) = 4^x$

82. $g(x) = \log(x - 5) - 1$

83. The percentage of visible light P (in decimal form) at a depth of x meters for Long Island Sound can be approximated by $P = e^{-0.5x}$.

a. Determine the depth at which the light intensity is half the value from the surface. Round to the nearest hundredth of a meter. Based on your answer, would you say that Long Island Sound is murky or clear water?

b. Determine the euphotic depth for Long Island Sound. That is, find the depth at which the light intensity falls below 1%. Round to the nearest tenth of a meter.

SECTION 3.6

For Exercises 84–85, solve for the indicated variable.

84. $\log B - 1.7 = 2.3M$ for B

85. $T = T_f + T_0e^{-kt}$ for t

86. Suppose that $18,000 is invested in a bond fund and the account grows to $23,344.74 in 5 yr.

a. Use the model $A = Pe^{rt}$ to determine the average rate of return under continuous compounding. Round to the nearest tenth of a percent.

b. How long will it take the investment to reach $30,000 if the rate of return continues? Round to the nearest tenth of a year.

87. The population of Germany in 2011 was approximately 85.5 million. The model $P = 85.5e^{-0.00208t}$ represents a short-term model for the population, t years after 2011.

a. Based on this model, is the population of Germany increasing or decreasing?

b. Determine the number of years after 2011 at which the population of Germany would decrease to 80 million if this trend continues. Round to the nearest year.

88. The population of Chile was approximately 16.9 million in the year 2011, with an annual growth rate of 0.836%. The population $P(t)$ (in millions) can be modeled by $P(t) = 16.9(1.00836)^t$, where t is the number of years since 2011.

a. Write a function of the form $P(t) = P_0e^{kt}$ to model the population.

b. Determine the amount of time required for the population to grow to 20 million if this trend continues. Round to the nearest year.

89. A sample from human remains found near Stonehenge in England shows that 71.2% of the carbon-14 still remains. Use the model $Q(t) = Q_0e^{-0.000121t}$ to determine the age of the sample. In this model, $Q(t)$ represents the amount of carbon-14 remaining t years after death, and Q_0 represents the initial amount of carbon-14 at the time of death. Round to the nearest 100 yr.

90. A lake is stocked with bass by the U.S. Park Service. The population of bass is given by $P(t) = \dfrac{3000}{1 + 2e^{-0.37t}}$, where t is the time in years after the lake was stocked.

a. Evaluate $P(0)$ and interpret its meaning in the context of this problem.

b. Use the function to predict the bass population 2 yr after being stocked. Round to the nearest whole unit.

c. Use the function to predict the bass population 4 yr after being stocked.

d. Determine the number of years required for the bass population to reach 2800. Round to the nearest year.

e. What value will the term $\dfrac{2}{e^{0.37t}}$ approach as $t \to \infty$.

f. Determine the limiting value of $P(t)$.

91. For the given data,

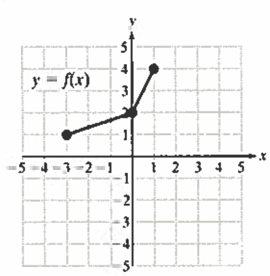
a. Use a graphing utility to find an exponential function $Y_1 = ab^x$ that fits the data.

b. Graph the data and the function from part (a) on the same coordinate system.

x	y
0	2.4
1	3.5
2	5.5
3	8.1
4	12.0
5	18.4

CHAPTER 3 Test

1. Given $f(x) = 4x^3 - 1$,
   a. Write an equation for $f^{-1}(x)$.
   b. Verify that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.
2. The graph of f is given.
   a. Is f a one-to-one function?
   b. If f is a one-to-one function, graph f^{-1} on the same coordinate system as f.



3. Given $f(x) = \dfrac{x + 3}{x - 4}$, write an equation for the inverse function.

For Exercises 4–7,

a. Write the domain and range of f in interval notation.
b. Write an equation of the inverse function.
c. Write the domain and range of f^{-1} in interval notation.

4. $f(x) = -x^2 + 1, x \leq 0$
5. $f(x) = \log x$
6. $f(x) = 3^x + 1$
7. $f(x) = \sqrt{x + 5}$

For Exercises 8–11,

a. Graph the function.

b. Write the domain in interval notation.

c. Write the range in interval notation.

d. Write an equation of the asymptote.

8. $f(x) = \left(\frac{1}{3}\right)^x + 2$ **9.** $g(x) = 2^{x-4}$

10. $h(x) = -\ln x$ **11.** $k(x) = \log_2(x + 1) - 3$

12. Write the statement in exponential form. $\ln(x + y) = a$

For Exercises 13–18, evaluate the logarithmic expression without using a calculator.

13. $\log_9 \frac{1}{81}$ **14.** $\log_6 216$ **15.** $\ln e^8$

16. $\log 10^{-4}$ **17.** $10^{\log(a^2+b^2)}$ **18.** $\log_{1/2} 1$

For Exercises 19–20, write the domain of the function in interval notation.

19. $f(x) = \log(7 - 2x)$ **20.** $g(x) = \log_4(x^2 - 25)$

For Exercises 21–22, write the logarithm as a sum or difference of logarithms. Simplify each term as much as possible.

21. $\ln\left(\frac{x^5y^2}{w\sqrt[3]{z}}\right)$ **22.** $\log\left(\frac{\sqrt{a^2+b^2}}{10^4}\right)$

For Exercises 23–24, write the logarithmic expression as a single logarithm with coefficient 1, and simplify as much as possible.

23. $6 \log_2 a - 4 \log_2 b + \frac{2}{3}\log_2 c$

24. $\frac{1}{2}\ln(x^2 - x - 12) - \frac{1}{2}\ln(x - 4)$

For Exercises 25–26, use $\log_b 2 \approx 0.289$, $\log_b 3 \approx 0.458$, and $\log_b 5 \approx 0.671$ to approximate the value of the given logarithms.

25. $\log_b 72$ **26.** $\log_b\left(\frac{1}{125}\right)$

For Exercises 27–36, solve the equation. Write the solution set with exact values and give approximate solutions to 4 decimal places.

27. $2^{5y+1} = 4^{y-3}$ **28.** $5^{x+3} + 3 = 56$

29. $2^{c+7} = 3^{2c+3}$ **30.** $7e^{4x} - 2 = 12$

31. $e^{2x} + 7e^x - 8 = 0$

32. $\log_5(3 - x) = \log_5(x + 1)$

33. $5 \ln(x + 2) + 1 = 16$

34. $\log x + \log(x - 1) = \log 12$

35. $-3 + \log_4 x = -\log_4(x + 30)$

36. $\log 3 + \log(x + 3) = \log(4x + 5)$

For Exercises 37–38, solve for the indicated variable.

37. $S = 92 - k \ln(t + 1)$ for t

38. $A = P\left(1 + \frac{r}{n}\right)^{nt}$ for t

39. Suppose that \$10,000 is invested and the account grows to \$13,566.25 in 5 yr.

a. Use the model $A = Pe^{rt}$ to determine the average rate of return under continuous compounding. Round to the nearest tenth of a percent.

b. Using the interest rate from part (a), how long will it take the investment to reach \$50,000? Round to the nearest tenth of a year.

40. The number of bacteria in a culture begins with approximately 10,000 organisms at the start of an experiment. If the bacteria doubles every 5 hr, the model $P(t) = 10{,}000(2)^{t/5}$ represents the population $P(t)$ after t hours.

a. Write a function of the form $P(t) = P_0e^{kt}$ to model the population.

b. Determine the amount of time required for the population to grow to 5 million. Round to the nearest hour.

41. The population $P(t)$ of a herd of deer on an island can be modeled by $P(t) = \frac{1200}{1 + 2e^{-0.12t}}$, where t represents the number of years since the park service has been tracking the herd.

a. Evaluate $P(0)$ and interpret its meaning in the context of this problem.

b. Use the function to predict the deer population after 4 yr. Round to the nearest whole unit.

c. Use the function to predict the deer population after 8 yr.

d. Determine the number of years required for the deer population to reach 900. Round to the nearest year.

e. What value will the term $\frac{2}{e^{0.12t}}$ approach as $t \to \infty$.

f. Determine the limiting value of $P(t)$.

42. The number N of visitors to a new website is given in the table t weeks after the website was launched.

t	0	1	2	3	4
N	24	50	121	270	640

a. Use a graphing utility to find an equation of the form $N = ab^t$ to model the data. Round a to 1 decimal place and b to 3 decimal places.

b. Use a graphing utility to graph the data and the model from part (a).

c. Use the model to predict the number of visitors to the website 10 weeks after launch. Round to the nearest thousand.

CHAPTER 3 Cumulative Review Exercises

For Exercises 1–2, simplify the expression.

1. $\dfrac{3x^{-1} - 6x^{-2}}{2x^{-2} - x^{-1}}$

2. $\dfrac{5}{\sqrt[3]{2x^2}}$

3. Factor. $a^3 - b^3 - a + b$

4. Perform the operations and write the answer in scientific notation. $\dfrac{(3.0 \times 10^7)(8.2 \times 10^{-3})}{1.23 \times 10^{-5}}$

For Exercises 5–13, solve the equations and inequalities. Write the solution sets to the inequalities in interval notation.

5. $5 \leq 3 + |2x - 7|$

6. $3x(x - 1) = x + 6$

7. $\sqrt{t + 3} + 4 = t + 1$

8. $9^{2m-3} = 27^{m+1}$

9. $-x^3 - 5x^2 + 4x + 20 < 0$

10. $|5x - 1| = |3 - 4x|$

11. $(x^2 - 9)^2 - 2(x^2 - 9) - 35 = 0$

12. $\log_2(3x - 1) = \log_2(x + 1) + 3$

13. $\dfrac{x - 4}{x + 2} \leq 0$

14. Find all the zeros of $f(x) = x^4 + 10x^3 + 10x^2 + 10x + 9$

15. Given $f(x) = x^2 - 16x + 55$,
    a. Does the graph of the parabola open upward or downward?
    b. Find the vertex of the parabola.
    c. Identify the maximum or minimum point.
    d. Identify the maximum or minimum value of the function.
    e. Identify the x-intercept(s).
    f. Identify the y-intercept.
    g. Write an equation for the axis of symmetry.
    h. Write the domain in interval notation.
    i. Write the range in interval notation.

16. Graph. $f(x) = -1.5x^2(x - 2)^3(x + 1)$

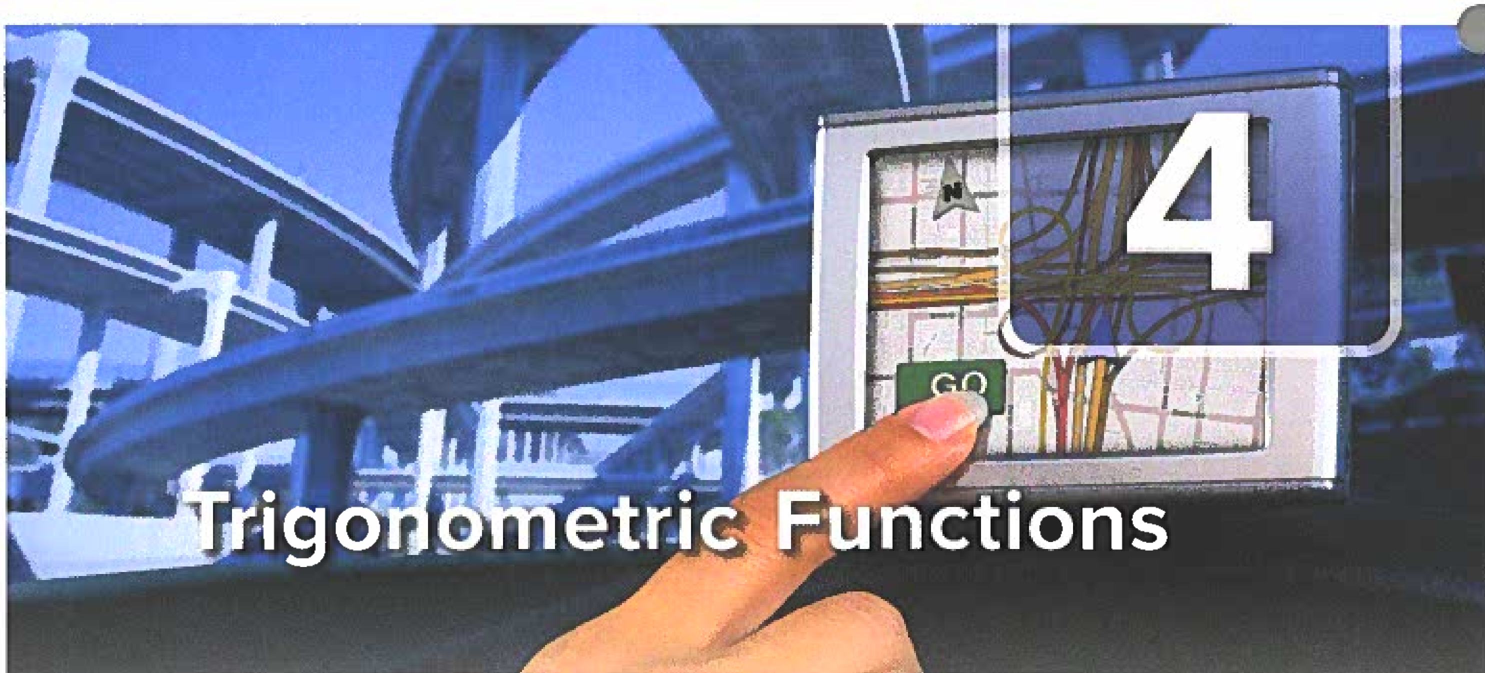
17. Given $f(x) = \dfrac{3x + 6}{x - 2}$,
    a. Write an equation of the vertical asymptote(s).
    b. Write an equation of the horizontal or slant asymptote.
    c. Graph the function.

18. Given $f(x) = 2^{x+2} - 3$,
    a. Write an equation of the asymptote.
    b. Write the domain in interval notation.
    c. Write the range in interval notation.

19. Write the expression as a single logarithm and simplify. $\log 40 + \log 50 - \log 2$

20. Given the one-to-one function defined by $f(x) = \sqrt[3]{x - 4} + 1$, write an equation for $f^{-1}(x)$.

Trigonometric Functions



4 Trigonometric Functions

Chapter Outline

With the luxury of the modern world, we have become accustomed to having GPS navigation systems in our cars, boats, aircraft, and even smart phones and cameras. But how do these systems work? And how did people in earlier times navigate across continents and oceans or approximate large distances? The answer is by using a branch of mathematics called trigonometry.

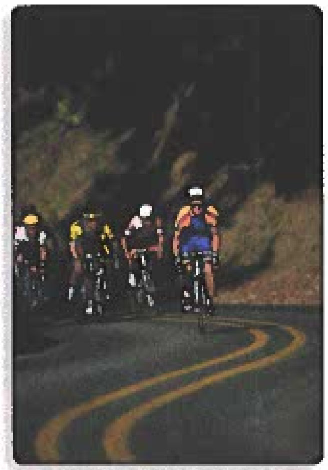
Trigonometry, from the Greek roots "trigon" (three-sided) and "metron" (measure), is the study of the relationships among the sides and angles of a triangle. The building blocks of trigonometry were developed in antiquity, going back as far as the early Egyptians and Babylonians who studied the ratios of lengths of sides of similar triangles. Today, trigonometry is used in a wide variety of fields, including navigation. The Global Positioning System (GPS) is a network of 24 satellites and their ground stations. Using trigonometry and the known location of GPS satellites in their precise orbits enables mathematicians to locate points on Earth accurate to within a few meters.

SECTION 4.1 Angles and Their Measure

OBJECTIVES

1. Find Degree Measure
2. Find Radian Measure
3. Determine Coterminal Angles
4. Compute Arc Length of a Sector of a Circle
5. Compute Linear and Angular Speed
6. Compute the Area of a Sector of a Circle

1. Find Degree Measure



The Tour de France is the most famous bicycle race in the world spanning 3500 km (2175 mi) in 21 days through France and two mountain ranges. The speed of a racer depends on a number of variables including the gear ratio and the cadence. The gear ratio determines the number of times the rear wheel turns with each pedal stroke, and the cadence is the number of revolutions of the pedals per minute. In this section, we study angles and their measure, and the relationship between angular and linear speeds to study such applications as the speed of a bicycle.

A **ray** is a part of a line that consists of an endpoint and all points on the line to one side of the endpoint. In Figure 4-1, ray $\overrightarrow{PQ}$ is named by using the endpoint P and another point Q on the ray. Notice that the rays $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ are different because the initial points are different and the rays extend in opposite directions.

TIP A ray has only one endpoint, which is always written first when naming the ray.

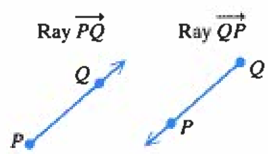


Figure 4-1

An **angle** is formed by rotating a ray about its endpoint. The starting position of the ray is called the **initial side** of the angle, and the final position of the ray is called the **terminal side.** The common endpoint is called the **vertex** of the angle and the vertex is often named by a capital letter such as A (Figure 4-2).

Angle A in Figure 4-2 can be denoted by $\angle A$ (read as "angle A") or by $\angle BAC$, where B is a point on the initial side of the angle, A represents the vertex, and C is a point on the terminal side. Alternatively, Greek letters such as θ (theta), α (alpha), β (beta), and γ (gamma) are often used to denote angles.

TIP When denoting an angle such as $\angle BAC$, the vertex is the middle letter.

An angle is in **standard position** if its vertex is at the origin in the xy-plane, and its initial side is the positive x-axis. In Figure 4-3, angle α is drawn in standard position.

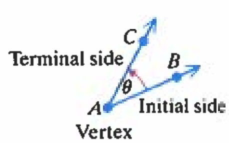


Figure 4-2

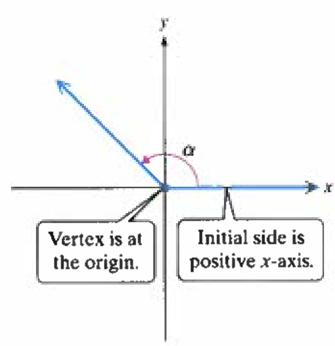
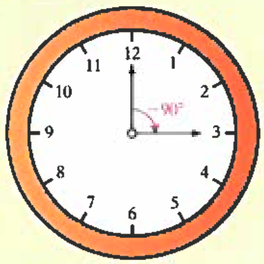


Figure 4-3

TIP The hands of a clock each resemble a ray. When the minute hand moves from 12:00 to 12:15, the rotation is $-90°$.



The **measure** of an angle quantifies the direction and amount of rotation from the initial side to the terminal side. The measure of an angle is *positive* if the rotation is counterclockwise, and the measure is *negative* if the rotation is clockwise. One unit with which to measure an angle is the **degree.** One full rotation of a ray about its endpoint is 360 degrees, denoted 360°. Therefore, 1° is $\frac{1}{360}$ of a full rotation. Figure 4-4 shows a variety of angles and their measures.

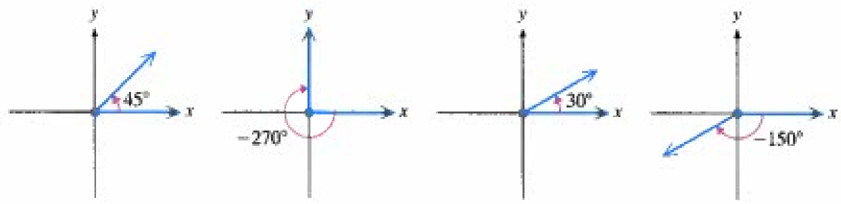


Figure 4-4

If the measure of an angle θ is 30°, we denote the measure of θ as $m(\theta) = 30°$ or simply $\theta = 30°$. We may also refer to θ as a 30° angle, rather than using the more formal, but cumbersome language "an angle whose measure is 30°." We also introduce several key terms associated with the measure of an angle (Figure 4-5).

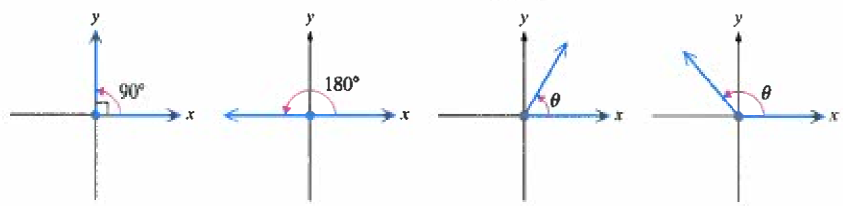


Figure 4-5

If the sum of the measures of two angles is 90°, we say that the angles are **complementary** (for example, the *complement* of a 20° angle is a 70° angle and vice versa). If the sum of the measures of two angles is 180°, we say that the angles are **supplementary** (for example, the supplement of a 20° angle is a 160° angle and vice versa).

A degree can be divided into 60 equal parts called **minutes** (min or ′), and each minute is divided into 60 equal parts called **seconds** (sec or ″).

- $1\text{ min} = \left(\frac{1}{60}\right)^\circ$ or $1' = \left(\frac{1}{60}\right)^\circ$
- $1\text{ sec} = \left(\frac{1}{60}\right)' = \left(\frac{1}{3600}\right)^\circ$ or $1'' = \left(\frac{1}{60}\right)' = \left(\frac{1}{3600}\right)^\circ$

For example, 74 degrees, 42 minutes, 15 seconds is denoted 74°42′15″.

EXAMPLE 1 Converting from Degrees, Minutes, Seconds (DMS) to Degree Decimal Form

Convert 74°42′15″ to decimal degrees. Round to 4 decimal places.

Solution:

Convert the minute and second portions of the angle to degrees. Choose the conversion factors so that the original units (minutes and seconds) "cancel," leaving the measurement in degrees.

$$74°42'15'' = 74° + (42\ \cancel{\text{min}}) \cdot \left(\frac{1°}{60\ \cancel{\text{min}}}\right) + (15\ \cancel{\text{sec}}) \cdot \left(\frac{1°}{3600\ \cancel{\text{sec}}}\right)$$
$$= 74° + 0.7° + 0.0041\overline{6}°$$
$$\approx 74.7042°$$

Skill Practice 1 Convert 131°12′33″ to decimal degrees. Round to 4 decimal places.

EXAMPLE 2 Converting from Decimal Degrees to Degrees, Minutes, Seconds (DMS) Form

Convert 159.26° to degree, minute, second form.

Solution:

$159.26° = 159° + 0.26°$ Write the decimal as a whole number part plus a fractional part. The fractional part of 1° needs to be converted from degrees to minutes and seconds.

$= 159° + 0.26° \cdot \left(\frac{60'}{1°}\right)$ Use the conversion factor 60′ = 1° to convert to minutes.

$= 159° + 15.6'$ Now convert the fractional part of 1 minute to seconds.

$= 159° + 15' + 0.6'$ Write 15.6′ as a whole number part plus a fractional part.

$= 159° + 15' + 0.6' \cdot \left(\frac{60''}{1'}\right)$ Use the conversion factor 60″ = 1′ to convert to seconds.

$= 159° + 15' + 36''$

$= 159°15'36''$

Skill Practice 2 Convert 26.48° to degree, minute, second form.

2. Find Radian Measure

Degree measure is used extensively in many applications of engineering, surveying, and navigation. Another type of angular measure that is better suited for applications in trigonometry and calculus is **radian measure.** To begin, we define a **central angle** as an angle with the vertex at the center of a circle.

Definition of One Radian

A central angle that intercepts an arc on the circle with length equal to the radius of the circle has a measure of **1 radian** (Figure 4-6).

Note: One radian may be denoted as 1 radian, 1 rad, or simply 1. That is, radian measure carries no units.

Answers
1. 131.2092°
2. 26°28′48″

TIP When two lines or rays cross a circle, the part of the circle between the intersection points is called the *intercepted arc* and is often denoted by *s*.

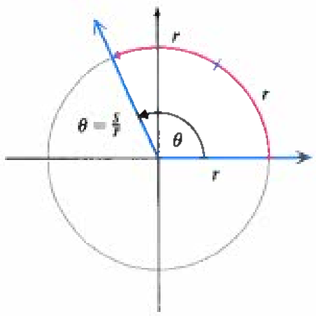


Figure 4-7

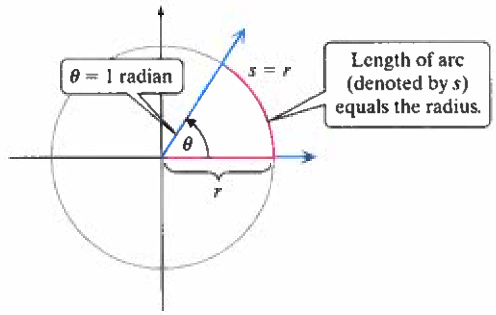


Figure 4-6

Any central angle can be measured in radians by dividing the length s of the intercepted arc by the radius r. For example, in Figure 4-7, the length of the red arc is $2r$ (twice the radius). Therefore, the measure of angle θ is given by

$$\theta = \frac{s}{r} = \frac{2r}{r} = 2 \quad \text{(2 radians)}$$

Definition of Radian Measure of an Angle

The radian measure of a central angle θ subtended by an arc of length s on a circle of radius r is given by $\theta = \frac{s}{r}$.

TIP Radian measure carries no units because it is measured as a ratio of two lengths with the same units (the units associated with s/r "cancel"). Thus, 2π rad is simply written as 2π. It is universally understood that the measure is in radians. Sometimes the notation "rad" is included for emphasis, but is not necessary.

You may have an intuitive feel for angles measured in degrees (for example, 90° is one-quarter of a full rotation). However, radian measure is unfamiliar. From Figure 4-6, notice that an angle of 1 radian (1 rad) is approximately 57.3°. If we divide the circumference of a circle into arcs of length r (Figure 4-8), we see that there are just over 6 rad in one full rotation. In fact, we can show that there are exactly 2π rad in one full rotation.

Recall that π is defined as the ratio of the circumference of a circle to its diameter d. Therefore, the circumference C is given by $C = \pi d$ or equivalently $C = 2\pi r$, where r is the radius of the circle. The circumference is the arc length of a full circle and dividing this by the radius gives the number of radians in one revolution.

One revolution
$= 2\pi$ radians
≈ 6.28 rad

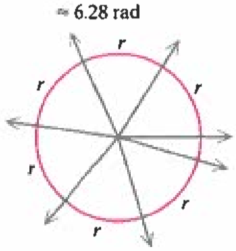


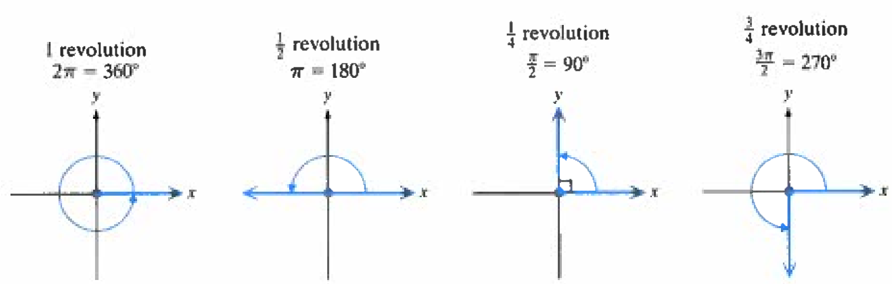
Figure 4-8

Number of radians in one revolution

Arc length of one revolution

$$\theta = \frac{s}{r} = \frac{\text{circumference}}{\text{radius}} = \frac{2\pi r}{r} = 2\pi \approx 6.28$$

The angular measure of one full rotation is 2π (2π rad). Therefore, we have the following relationships.



TIP Some angles are used frequently in the study of trigonometry. Their degree measures and equivalent radian measures are worth memorizing.

$30° = \frac{\pi}{6}$ $45° = \frac{\pi}{4}$

$60° = \frac{\pi}{3}$ $90° = \frac{\pi}{2}$

The statement $\pi = 180°$ gives us a conversion factor to convert between degree measure and angular measure.

Converting Between Degree and Radian Measure

- To convert from degrees to radians, multiply the degree measure by $\frac{\pi}{180°}$.
- To convert from radians to degrees, multiply the radian measure by $\frac{180°}{\pi}$.

EXAMPLE 3 Converting from Degrees to Radians

Convert from degrees to radians.

a. 210° **b.** −135°

Solution:

a. $210° \cdot \left(\frac{\pi \text{ rad}}{180°}\right) = \frac{\overset{7}{\cancel{210}}\pi}{\underset{6}{\cancel{180}}}\text{rad} = \frac{7\pi}{6}\text{rad} = \frac{7\pi}{6}$

The units of degrees "cancel" in the numerator and denominator in the first step, leaving units of radians.

b. $-135° \cdot \left(\frac{\pi}{180°}\right) = -\frac{\overset{3}{\cancel{135}}\pi}{\underset{4}{\cancel{180}}} = -\frac{3\pi}{4}$

The units of "rad" are implied in the numerator of the conversion factor $\frac{\pi}{180°}$.

Skill Practice 3 Convert from degrees to radians.

a. 300° **b.** −70°

EXAMPLE 4 Converting from Radians to Degrees

Convert from radians to degrees.

a. $\frac{\pi}{12}$ **b.** $-\frac{4\pi}{3}$

Solution:

a. $\frac{\pi}{12}\text{rad} \cdot \left(\frac{180°}{\pi \text{ rad}}\right) = \left(\frac{\overset{15}{\cancel{180}}\pi}{\underset{1}{\cancel{12}}\pi}\right)^{\circ} = 15°$

The units of rad "cancel" in the numerator and denominator in the first step, leaving units of degrees.

b. $-\frac{4\pi}{3} \cdot \left(\frac{180°}{\pi}\right) = \left(-\frac{\overset{240}{\cancel{720}}\pi}{\underset{1}{\cancel{3}}\pi}\right)^{\circ} = -240°$

The units of "rad" are implied in the denominator of the conversion factor $\frac{180°}{\pi}$.

TIP To help determine which conversion factor to use, remember that you want the original units to "cancel," leaving the new unit of measurement in the numerator.

Skill Practice 4 Convert from radians to degrees.

a. $\frac{\pi}{18}$ **b.** $-\frac{7\pi}{4}$

3. Determine Coterminal Angles

Two angles in standard position with the same initial side and same terminal side are called **coterminal angles.** Figure 4-9 illustrates three angles in standard position that are coterminal to 30°. Notice that each angle is 30° plus or minus some number of full revolutions clockwise or counterclockwise.

Answers

3. a. $\frac{5\pi}{3}$ **b.** $-\frac{7\pi}{18}$

4. a. 10° **b.** −315°

Point of Interest

Aerial snowboarding is a winter sport in which competitors perform aerial tricks after launching from the sides of a halfpipe. The degree of difficulty of a move is measured in part by the number of full rotations that a competitor completes through the air. Ulrik Badertscher of Norway was the first to rotate through the the air an amazing 4.5 full rotations, an angle of 1620°.

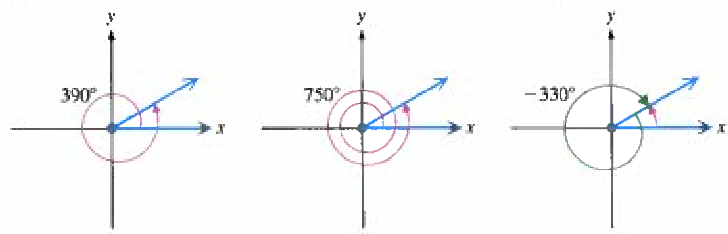


Figure 4-9 Coterminal Angles

EXAMPLE 5 Finding Coterminal Angles

Find an angle coterminal to θ between 0° and 360°.

a. $\theta = 960°$ **b.** $\theta = -225°$

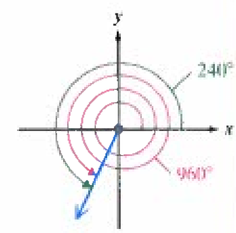
Solution:

a. $\theta = 960°$ is more than 360°. Therefore, we will subtract some multiple of 360° to get an angle coterminal to θ between 0° and 360°.

- One revolution measures 360°.
- Two revolutions measure 720°.
- Three revolutions measure 1080°.

360°(2) is subtracted from 960° because θ is between 2 and 3 revolutions. Therefore,

$$960° - (360°)(2) = 960° - 720° = 240°$$



TIP The number of revolutions contained in 960° can be found by dividing 960° by 360°.

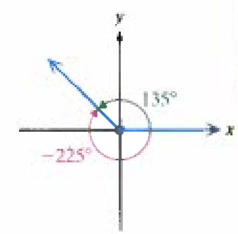
$$\begin{array}{r} 2 \\ 360\overline{)960} \\ \underline{720} \\ 240 \end{array}$$

2 revolutions; remainder

$\theta = 960°$ is two full revolutions plus 240°.

b. An angle of $-225°$ is less than one revolution. Therefore, we will add 360°(1) to obtain a positive angle between 0° and 360° and coterminal to $-225°$.

$$-225° + 360° = 135°$$



Skill Practice 5 Find an angle coterminal to θ between 0° and 360°.

a. $\theta = 1230°$ **b.** $\theta = -315°$

Two angles in standard position are coterminal if their measures differ by a multiple of 360° or 2π rad. The angles shown in Figure 4-10 are coterminal to an angle of $\frac{\pi}{6}$ rad (or 30°).

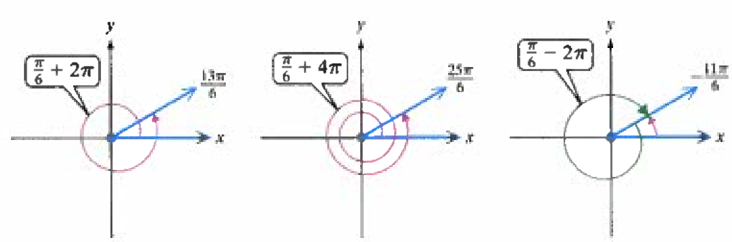


Figure 4-10 Coterminal Angles

Answers

5. a. 150° b. 45°

EXAMPLE 6 Finding Coterminal Angles

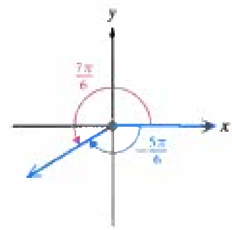
Find an angle coterminal to θ on the interval $[0, 2\pi)$.

a. $\theta = -\dfrac{5\pi}{6}$ b. $\theta = \dfrac{13\pi}{2}$

Solution:

a. One revolution is 2π rad or equivalently $\dfrac{12\pi}{6}$.

Therefore, adding any multiple of $\dfrac{12\pi}{6}$ to $-\dfrac{5\pi}{6}$ results in an angle coterminal to $-\dfrac{5\pi}{6}$.

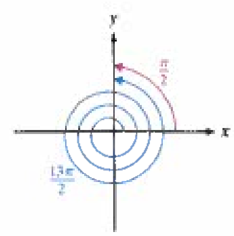


$$-\frac{5\pi}{6} + \frac{12\pi}{6} = \frac{7\pi}{6}$$

b. One revolution is 2π rad, or equivalently $\dfrac{4\pi}{2}$.

We can divide $\dfrac{13\pi}{2} \div \dfrac{4\pi}{2} = \dfrac{13\pi}{2} \cdot \dfrac{2}{4\pi} = \dfrac{13}{4} = 3\dfrac{1}{4}$.

Therefore, $\dfrac{13\pi}{2}$ is three full revolutions plus $\dfrac{1}{4}$ of a revolution. Subtracting three revolutions results in an angle coterminal to $\dfrac{13\pi}{2}$.



Subtract 3 revolutions.

Fractional part of a revolution.

$$\frac{13\pi}{2} - 3\left(\frac{4\pi}{2}\right) = \frac{13\pi}{2} - \frac{12\pi}{2} = \frac{\pi}{2}$$

Skill Practice 6 Find an angle coterminal to θ on the interval $[0, 2\pi)$.

a. $\theta = -\dfrac{\pi}{8}$ b. $\theta = \dfrac{19\pi}{4}$

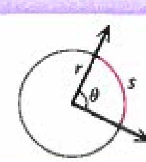
4. Compute Arc Length of a Sector of a Circle

By definition, the measure of a central angle θ in radians equals the length s of the intercepted arc divided by the radius r, that is, $\theta = \dfrac{s}{r}$. Solving this relationship for s gives $s = r\theta$, which enables us to compute arc length if the measure of the central angle and radius are known.

Arc Length

Given a circle of radius r, the length s of an arc intercepted by a central angle θ (in radians) is given by

$$s = r\theta$$



Answers

6. a. $\dfrac{15\pi}{8}$ b. $\dfrac{3\pi}{4}$

EXAMPLE 7 Determining Arc Length

Find the length of the arc made by an angle of 105° on a circle of radius 15 cm. Give the exact arc length and round to the nearest tenth of a centimeter.

Solution:

$\theta = 105° \cdot \dfrac{\pi}{180°} = \dfrac{7\pi}{12}$ Convert θ to radians.

$s = r\theta = (15\text{ cm}) \cdot \dfrac{7\pi}{12} = 8.75\pi\,\text{cm}$ Apply the arc length formula.

The arc is $8.75\pi \approx 27.5$ cm.

Skill Practice 7 Find the length of the arc made by an angle of 220° on a circle of radius 9 in.

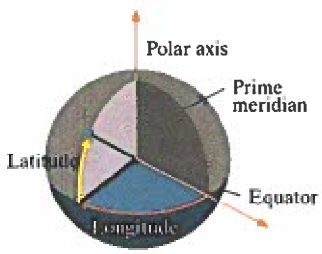
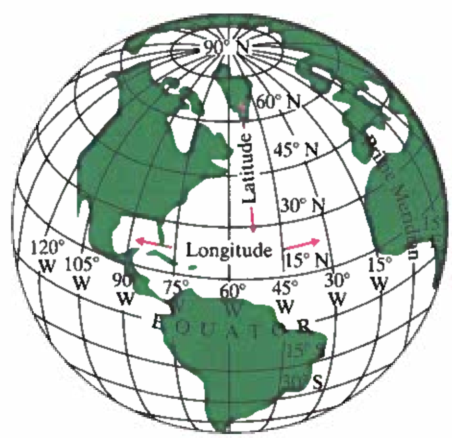


Figure 4-11

The Earth is approximately spherical and the most common way to locate points on the surface is by using *latitude* and *longitude*. These coordinates are measured in degrees and represent central angles measured from the center of the Earth.

The **equator** is an imaginary circle around the Earth equidistant between the north and south poles. **Latitude** is the angular measure of a central angle measuring north (N) or south (S) from the equator. There are 90° of latitude measured north from the equator and 90° of latitude measured south from the equator. The equator has a latitude of 0°, the north pole has a latitude of 90°N, and the south pole has a latitude of 90°S (Figure 4-11).

Lines of **longitude,** called meridians, are circles that pass through both poles and run perpendicular to the equator. By international agreement, 0° longitude is taken to be the meridian line through Greenwich, England. This is called the **prime meridian.** Thus, longitude is the angular measure of a central angle east (E) or west (W) of the prime meridian. The Earth is divided into 360° of longitude. There are 180° east (E) of the prime meridian and 180° west (W) of the prime meridian.



For example, New York City is located at 40.7°N, 74.0°W. This means that New York City is located 40.7° north of the equator and 74.0° west of the prime meridian.

EXAMPLE 8 Determining the Distance Between Cities

Seattle, Washington, is located at 47.6°N, 122.4°W, and San Francisco, California, is located at 37.8°N, 122.3°W. Since the longitudes are nearly the same, the cities are roughly due north-south of each other. Using the difference in latitude, approximate the distance between the cities assuming that the radius of the earth is 3960 mi. Round to the nearest mile.

Answer

7. $11\pi \approx 34.6$ in.

Avoiding Mistakes

When applying the formula $s = r\theta$, always use radian measure for θ.

Solution:

$47.6° - 37.8° = 9.8°$

$\theta = 9.8° \cdot \dfrac{\pi}{180°} \approx 0.17104$

$s = r\theta$

$s = (3960)(0.17104) \approx 677$ mi

Skill Practice 8 Lincoln, Nebraska, is located at 40.8°N, 96.7°W and Dallas, Texas, is located at 32.8°N, 96.7°W. Since the longitudes are the same, the cities are north-south of each other. Using the difference in latitudes, approximate the distance between the cities assuming that the radius of the Earth is 3960 mi. Round to the nearest mile.

5. Compute Linear and Angular Speed

Point of Interest

The Earth rotates at a constant angular speed of 360° in 24 hr or equivalently 15° in 1 hr. Therefore, each time zone is approximately 15° of longitude in width (with local variations) and is 1 hr earlier than the zone immediately to the east.

Consider a ceiling fan rotating at a constant rate. For a small increment of time, suppose that point A on the tip of a blade travels a distance s_1. In the same amount of time, point B will travel a shorter distance s_2 (Figure 4-12). Therefore, point A on the tip has a greater linear speed v than point B. However, each point has the same *angular* speed ω (omega) because they move through the same angle for a given unit of time.

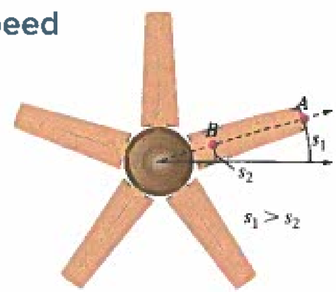


Figure 4-12

TIP In an application, if a rate is given in degrees, radians, or revolutions per unit time, it is an angular speed. If a rate is given in linear units (such as feet) per unit time, it is a linear speed.

Angular and Linear Speed

If a point on a circle of radius r moves through an angle of θ radians in time t, the angular and linear speeds of the point are

$$\text{angular speed: } \omega = \frac{\theta}{t}$$

$$\text{linear speed: } \quad v = \frac{s}{t} \quad \text{or} \quad v = \frac{r\theta}{t} \quad \text{or} \quad v = r\omega$$

EXAMPLE 9 Finding Linear and Angular Speed

A ceiling fan rotates at 90 rpm (revolutions per minute). For a point at the tip of a 2-ft blade,

a. Find the angular speed.

b. Find the linear speed. Round to the nearest whole unit.

Solution:

a. For each revolution of the blade, the point moves through an angle of 2π radians.

$$\omega = \frac{\theta}{t} = \frac{90\ \cancel{\text{rev}}}{\text{min}} \cdot \frac{2\pi\ \text{rad}}{\cancel{\text{rev}}} = 180\pi\ \text{rad/min} = 180\pi/\text{min}$$

b. $v = r\omega$

$$v = (2\ \text{ft})\left(\frac{180\pi}{\text{min}}\right) = 360\pi\ \text{ft/min} \approx 1131\ \text{ft/min}$$

TIP Converting the result from Example 9 to miles per hour, we have

$$\frac{1131\ \text{ft}}{\cancel{\text{min}}} \cdot \frac{1\ \text{mi}}{5280\ \text{ft}} \cdot \frac{60\ \cancel{\text{min}}}{1\ \text{hr}} \approx 12.9\ \text{mph.}$$

Answer

8. 553 mi

Skill Practice 9 A bicycle wheel rotates at 2 revolutions per second.

a. Find the angular speed.

b. How fast does the bicycle travel (in ft/sec) if the wheel is 2.2 ft in diameter? Round to the nearest tenth.

6. Compute the Area of a Sector of a Circle

A sector of a circle is a "pie-shaped" wedge of a circle bounded by the sides of a central angle and the intercepted arc (Figure 4-13).

The area of a sector of a circle is proportional to the measure of the central angle. The expression $\dfrac{\theta}{2\pi}$ is the fractional amount of a full rotation represented by angle θ (in radians). So the area of a sector formed by θ is

$$\underbrace{\frac{\theta}{2\pi}}_{\text{Fractional amount of full circle}} \cdot \underbrace{(\pi r^2)}_{\text{Area of full circle}} = \underbrace{\frac{1}{2}r^2\theta}_{\text{Area of sector}}$$

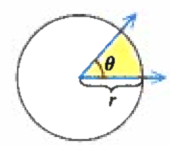


Figure 4-13

Area of a Sector

The area A of a sector of a circle of radius r with central angle θ (in radians) is given by

$$A = \frac{1}{2}r^2\theta$$

EXAMPLE 10 Determining the Area of a Sector

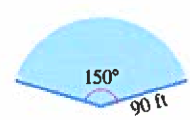
A crop sprinkler rotates through an angle of 150° and sprays water a distance of 90 ft. Find the amount of area watered. Round to the nearest whole unit.

Solution:

To use the formula $A = \frac{1}{2}r^2\theta$, we need to convert θ to radians.

$$150° \cdot \frac{\pi}{180°} = \frac{5\pi}{6}$$

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}(90 \text{ ft})^2\left(\frac{5\pi}{6}\right) = 3375\pi \text{ ft}^2 \approx 10{,}603 \text{ ft}^2$$



Skill Practice 10 A sprinkler rotates through an angle of 120° and sprays water a distance of 30 ft. Find the amount of area watered. Round to the nearest whole unit.

Answers

9. a. 4π/sec

b. ≈ 13.8 ft/sec (≈ 9.4 mph)

10. $300\pi \text{ ft}^2 \approx 942 \text{ ft}^2$

SECTION 4.1 Practice Exercises

Prerequisite Review

R.1. Solve for the specified variable.

$p = hz$ for z

R.2. If a plane travels 280 mph for 3.5 hr, find the distance traveled.

For Exercises R.3–R.5, convert the unit of time.

R.3. 210 min = ________ hr

R.4. 120 sec = ________ min

R.5. 7200 sec = ________ hr

R.6. Find the circumference of a circle with a radius of $2\frac{1}{2}$ m.

a. Give the exact answer in terms of π.

b. Approximate the answer by using 3.14 for π. Round to 1 decimal place.

R.7. Determine the area of a circle with a diameter of 40 ft. Use 3.14 for π. Round to the nearest whole unit.

Concept Connections

1. The measure of an angle is (positive/negative) ________ if its rotation from the initial side to the terminal side is clockwise. If the rotation is counterclockwise, then the measure is ________.

2. An angle with its vertex at the origin of an xy-coordinate plane and with initial side on the positive x-axis is in ________ position.

3. Two common units used to measure angles are ________ and ________.

4. One degree is what fractional amount of a full rotation?

5. An angle that measures 360° has a measure of ________ radians.

6. Two angles are called ________ if the sum of their measures is 90°. Two angles are called ________ if the sum of their measures is 180°.

7. An angle with measure ________° or ________ radians is a right angle. A straight angle has a measure of ________° or ________ radians.

8. A(n) ________ angle has a measure between 0° and 90°, whereas a(n) ________ angle has a measure between 90° and 180°.

9. One degree is equally divided into 60 parts called ________.

10. One minute is equally divided into 60 parts called ________.

11. 1° = ________′ = ________″

12. An angle with its vertex at the center of a circle is called a(n) ________ angle.

13. A central angle of a circle that intercepts an arc equal in length to the radius of the circle has a measure of ________.

14. Which angle has a greater measure, 2° or 2 radians?

15. To convert from radians to degrees, multiply by ________. To convert from degrees to radians, multiply by ________.

16. Two angles are ________ if they have the same initial side and same terminal side.

17. The measure of all angles coterminal to $\frac{7\pi}{4}$ differ from $\frac{7\pi}{4}$ by a multiple of ________.

18. The measure of all angles coterminal to 112° differ from 112° by a multiple of ________.

19. The length s of an arc made by an angle θ on a circle of radius r is given by the formula ________, where θ is measured in ________.

20. To locate points on the surface of the Earth that are north or south of the equator, we measure the ________ of the point. To locate points that are east or west of the prime meridian, we measure the ________ of the point.

21. The relationship $v = \frac{\text{arc length}}{\text{time}}$ represents the ________ speed of a point traveling in a circular path.

22. The symbol ω is typically used to denote ______________ speed and represents the number of radians per unit time that an object rotates.

23. A wedge of a circle, similar in shape to a slice of pie, is called a(n) ______________ of the circle.

24. The area A of a sector of a circle of radius r with central angle θ is given by the formula ______________, where θ is measured in ______________.

Objective 1: Find Degree Measure

For Exercises 25–26, sketch the angles in standard position.

25. a. $60°$ b. $225°$ c. $-210°$ d. $-86°$

26. a. $30°$ b. $120°$ c. $-135°$ d. $-73°$

For Exercises 27–30, convert the given angle to decimal degrees. Round to 4 decimal places. (See Example 1)

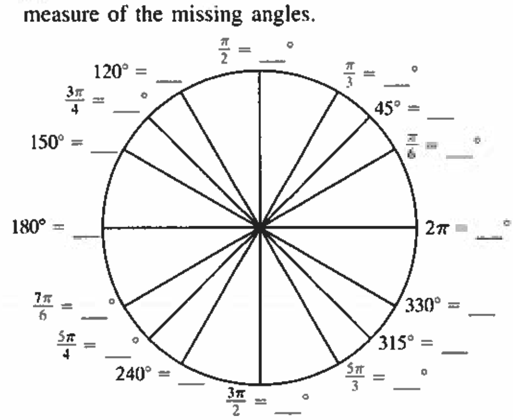
27. $17°34'$ 28. $215°47'$ 29. $54°36'55''$ 30. $23°42'48''$

For Exercises 31–34, convert the given angle to DMS (degree-minute-second) form. Round to the nearest second if necessary. (See Example 2)

31. $46.418°$ 32. $82.074°$ 33. $-84.64°$ 34. $-61.46°$

Objective 2: Find Radian Measure

35. Use the conversion factor $\pi = 180°$ along with the symmetry of the circle to complete the degree or radian measure of the missing angles.



36. Insert the appropriate symbol $<$, $>$, or $=$ in the blank.

a. $\frac{5\pi}{6} \square 120°$

b. $-\frac{4\pi}{3} \square -270°$

For Exercises 37–40, convert from degrees to radians. Give the answers in exact form in terms of π. (See Example 3)

37. $75°$ 38. $240°$ 39. $-210°$ 40. $-195°$

For Exercises 41–44, convert from degrees to radians. Round to 4 decimal places.

41. $-64.6°$ 42. $-312.4°$ 43. $12°6'36''$ 44. $108°42'9''$

For Exercises 45–56, convert from radians to decimal degrees. Round to 1 decimal place if necessary. (See Example 4)

45. $\frac{\pi}{4}$ 46. $\frac{11\pi}{6}$ 47. $-\frac{5\pi}{3}$ 48. $-\frac{7\pi}{6}$

49. $\frac{5\pi}{18}$ 50. $\frac{7\pi}{9}$ 51. $-\frac{2\pi}{5}$ 52. $-\frac{3\pi}{8}$

53. 2.7 54. 5.3 55. $\frac{9\pi}{2}$ 56. 7π

Objective 3: Determine Coterminal Angles

For Exercises 57–64, find a positive angle and a negative angle that is coterminal to the given angle. (See Examples 5 and 6)

57. $57°$ 58. $313°$ 59. $-105°$ 60. $-12°$

61. $\frac{5\pi}{6}$ 62. $\frac{3\pi}{4}$ 63. $-\frac{3\pi}{2}$ 64. $-\pi$

For Exercises 65–70, find an angle between 0° and 360° or between 0 and 2π that is coterminal to the given angle. (See Examples 5 and 6)

65. $1521°$

66. $-603°$

67. $\dfrac{17\pi}{4}$

68. $\dfrac{11\pi}{3}$

69. $-\dfrac{7\pi}{18}$

70. $-\dfrac{5\pi}{9}$

Objective 4: Compute Arc Length of a Sector of a Circle

For Exercises 71–74, find the exact length of the arc intercepted by a central angle θ on a circle of radius r. Then round to the nearest tenth of a unit. (See Example 7)

71. $\theta = \dfrac{\pi}{3}$, $r = 12$ cm

72. $\theta = \dfrac{5\pi}{6}$, $r = 4$ m

73. $\theta = 135°$, $r = 10$ in.

74. $\theta = 315°$, $r = 2$ yd

75. A 6-ft pendulum swings through an angle of 40°36′. What is the length of the arc that the tip of the pendulum travels? Round to the nearest hundredth of a foot.

76. A gear with a 1.2-cm radius moves through an angle of 220°15′. What distance does a point on the edge of the gear move? Round to the nearest tenth of a centimeter.

77. Which location would best fit the coordinates 12°S, 77°W?

a. Paris, France
b. Lima, Peru
c. Miami, Florida
d. Moscow, Russia

78. a. What is the geographical relationship between two points that have the same latitude?

b. What is the geographical relationship between two points that have the same longitude?

For Exercises 79–82, assume that the Earth is approximately spherical with radius 3960 mi. Approximate the distances to the nearest mile. (See Example 8)

79. Barrow, Alaska (71.3°N, 156.8°W), and Kailua, Hawaii (19.7°N, 156.1°W), have approximately the same longitude, which means that they are roughly due north-south of each other. Use the difference in latitude to approximate the distance between the cities.

80. Rochester, New York (43.2°N, 77.6°W), and Richmond, Virginia (37.5°N, 77.5°W), have approximately the same longitude, which means that they are roughly due north-south of each other. Use the difference in latitude to approximate the distance between the cities.

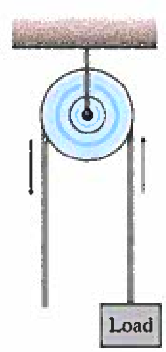
81. Raleigh, North Carolina (35.8°N, 78.6°W), is located north of the equator, and Quito, Ecuador (0.3°S, 78.6°W), is located south of the equator. The longitudes are the same indicating that the cities are due north-south of each other. Use the difference in latitude to approximate the distance between the cities.

82. Trenton, New Jersey (40.2°N, 74.8°W), is located north of the equator, and Ayacucho, Peru (13.2°S, 74.2°W), is located south of the equator. The longitudes are nearly the same indicating that the cities are roughly due north-south of each other. Use the difference in latitude to approximate the distance between the cities.

83. A pulley is 16 cm in diameter.

a. Find the distance the load will rise if the pulley is rotated 1350°. Find the exact distance in terms of π and then round to the nearest centimeter.

b. Through how many degrees should the pulley rotate to lift the load 100 cm? Round to the nearest degree.

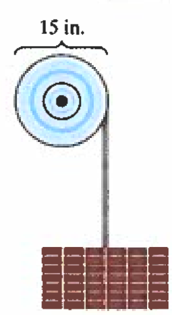


84. A pulley is 1.2 ft. in diameter.

a. Find the distance the load will rise if the pulley is rotated 630°. Find the exact distance in terms of π and then round to the nearest tenth of a foot.

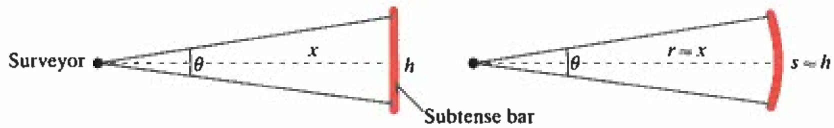
b. Through how many degrees should the pulley rotate to lift the load 24 ft? Round to the nearest degree.

85. A hoist is used to lift a palette of bricks. The drum on the hoist is 15 in. in diameter. How many degrees should the drum be rotated to lift the palette a distance of 6 ft? Round to the nearest degree.



86. A winch on a sailboat is 8 in. in diameter and is used to pull in the "sheets" (ropes used to control the corners of a sail). To the nearest degree, how far should the winch be turned to pull in 2 ft of rope?

Before the widespread introduction of electronic devices to measure distances, surveyors used a subtense bar to measure a distance x that is not directly measurable. A subtense bar is a bar of known length h with marks or "targets" at either end. The surveyor measures the angle θ formed by the location of the surveyor's scope and the top and bottom of the bar (this is the angle *subtended* by the bar). Since the angle and height of the bar are known, right triangle trigonometry can be used to find the horizontal distance. Alternatively, if the distance from the surveyor to the bar is large, then the distance can be approximated by the radius r of the arc s intercepted by the bar. Use this information for Exercises 87–88.



87. A surveyor uses a subtense bar to find the distance across a river. If the angle of sight between the bottom and top marks on a 2-m bar is 57′18″, approximate the distance across the river between the surveyor and the bar. Round to the nearest meter.

88. A surveyor uses a subtense bar to find the distance across a canyon. If the angle of sight between the bottom and top marks on a 2-m bar is 24′33″, approximate the distance across the river between the surveyor and the bar. Round to the nearest meter.

Objective 5: Compute Linear and Angular Speed

89. A circular paddle wheel of radius 3 ft is lowered into a flowing river. The current causes the wheel to rotate at a speed of 12 rpm. To 1 decimal place,
   a. What is the angular speed? **(See Example 9)**
   b. Find the speed of the current in ft/min.
   c. Find the speed of the current in mph.

90. An energy-efficient hard drive has a 2.5-in. diameter and spins at 4200 rpm.
   a. What is the angular speed?
   b. How fast in in./min does a point on the edge of the hard drive spin? Give the exact speed and the speed rounded to the nearest in./min.

91. A $7\frac{1}{4}$-in.-diameter circular saw has 24 teeth and spins at 5800 rpm.
   a. What is the angular speed?
   b. What is the linear speed of one of the "teeth" on the outer edge of the blade? Round to the nearest inch per minute.

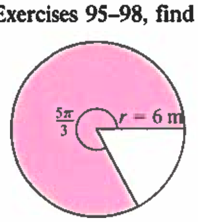
92. On a weed-cutting device, a thick nylon line rotates on a spindle at 3000 rpm.
   a. Determine the angular speed.
   b. Determine the linear speed (to the nearest inch per minute) of a point on the tip of the line if the line is 5 in.

93. A truck has 2.5-ft tires (in diameter).
   a. What distance will the truck travel with one rotation of the wheels? Give the exact distance and an approximation to the nearest tenth of a foot.
   b. How far will the truck travel with 10,000 rotations of the wheels? Give the exact distance and an approximation to the nearest foot.
   c. If the wheels turn at 672 rpm, what is the angular speed?
   d. If the wheels turn at 672 rpm, what is the linear speed in feet per minute? Give the exact distance and an approximation to the nearest whole unit.
   e. If the wheels turn at 672 rpm, what is the linear speed in miles per hour? Round to the nearest mile per hour. (*Hint*: 1 mi = 5280 ft and 1 hr = 60 min.)

94. A bicycle has 25-in. wheels (in diameter).
   a. What distance will the bicycle travel with one rotation of the wheels? Give the exact distance and an approximation to the nearest tenth of an inch.
   b. How far will the bicycle travel with 200 rotations of the wheels? Give the exact distance and approximations to the nearest inch and nearest foot.
   c. If the wheels turn at 80 rpm, what is the angular speed?
   d. If the wheels turn at 80 rpm, what is the linear speed in inches per minute? Give the exact speed and an approximation to the nearest inch per minute.
   e. If the wheels turn at 80 rpm, what is the linear speed in miles per hour? Round to the nearest mile per hour. (*Hint*: 1 ft = 12 in., 1 mi = 5280 ft, and 1 hr = 60 min.)

Objective 6: Compute the Area of a Sector of a Circle

For Exercises 95–98, find the exact area of the sector. Then round the result to the nearest tenth of a unit. (See Example 10)

95.


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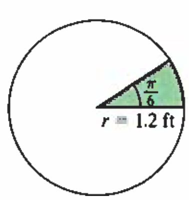

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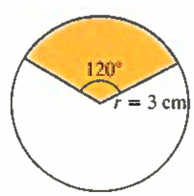

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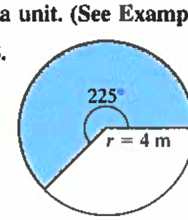


99. A slice of a circular pizza 12 in. in diameter is cut into a wedge with a 45° angle. Find the area and round to the nearest tenth of a square inch. **(See Example 10)**

100. A circular cheesecake 9 in. in diameter is cut into a slice with a 20° angle. Find the area and round to the nearest tenth of a square inch.

101. The back wiper blade on an SUV extends 3 in. from the pivot point to a distance of 17 in. from the pivot point. If the blade rotates through an angle of 175°, how much area does it cover? Round to the nearest square inch.

102. A robotic arm rotates through an angle of 160°. It sprays paint between a distance of 0.5 ft and 3 ft from the pivot point. Determine the amount of area that the arm makes. Round to the nearest square foot.

Mixed Exercises

For Exercises 103–108, find the (a) complement and (b) supplement of the given angle.

103. 16.21° **104.** 49.87° **105.** 18°13′37″

106. 22°9′54″ **107.** 9°42′7″ **108.** 82°15′3″

109. The second hand of a clock moves from 12:10 to 12:30.

a. How many degrees does it move during this time?

b. How many radians does it move during this time?

c. If the second hand is 10 in. in length, determine the exact distance that the tip of the second hand travels during this time.

d. Determine the exact angular speed of the second hand in radians per second.

e. What is the exact linear speed (in inches per second) of the tip of the second hand?

f. What is the amount of area that the second hand sweeps out during this time? Give the exact area in terms of π and then approximate to the nearest square inch.

110. The minute hand of a clock moves from 12:10 to 12:15.

a. How many degrees does it move during this time?

b. How many radians does it move during this time?

c. If the minute hand is 9 in. in length, determine the exact distance that the tip of the minute hand travels during this time.

d. Determine the exact angular speed of the minute hand in radians per minute.

e. What is the exact linear speed (in inches per minute) of the tip of the minute hand?

f. What is the amount of area that the minute hand sweeps out during this time? Give the exact area in terms of π and then approximate to the nearest square inch.

111. The Earth's orbit around the Sun is elliptical (oval shaped). However, the elongation is small, and for our discussion here, we take the orbit to be circular with a radius of approximately 93,000,000 mi.

a. Find the linear speed (in mph) of the Earth through its orbit around the Sun. Round to the nearest hundred miles per hour.

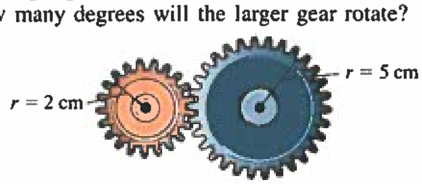
b. How far does the Earth travel in its orbit in one day? Round to the nearest thousand miles.

112. The Earth completes one full rotation around its axis (poles) each day.

a. Determine the angular speed (in radians per hour) of the Earth during its rotation around its axis.

b. The Earth is nearly spherical with a radius of approximately 3960 mi. Find the linear speed of a point on the surface of the Earth rounded to the nearest mile per hour.

113. Two gears are calibrated so that the smaller gear drives the larger gear. For each rotation of the smaller gear, how many degrees will the larger gear rotate?

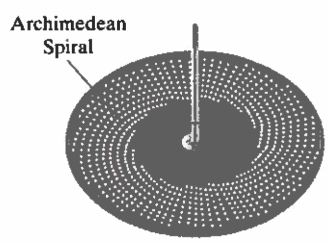


114. Two gears are calibrated so that the larger gear drives the smaller gear. The larger gear has a 6-in. radius, and the smaller gear has a 1.5-in. radius. For each rotation of the larger gear, by how many degrees will the smaller gear rotate?

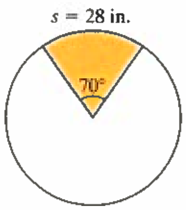
115. A spinning-disc confocal microscope contains a rotating disk with multiple small holes arranged in a series of nested Archimedean spirals. An intense beam of light is projected through the holes, enabling biomedical researchers to obtain detailed video images of live cells. The spinning disk has a diameter of 55 mm and rotates at a rate of 1800 rpm. At the edge of the disk,

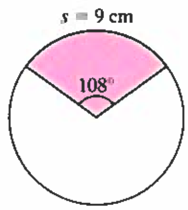
a. Find the angular speed.

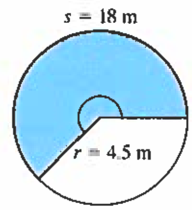
b. Find the linear speed. Round to the nearest whole unit.

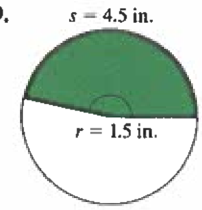


For Exercises 116–119, approximate the area of the shaded region to 1 decimal place. In the figure, *s* represents arc length, and *r* represents the radius of the circle.

116. 

117. 

118. 

119. 

Write About It

120. Explain what is meant by 1 radian. Explain what is meant by 1°.

121. For an angle drawn in standard position, explain how to determine in which quadrant the terminal side lies.

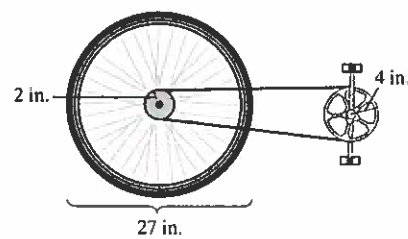
122. As the fan rotates (see figure), which point A or B has a greater angular speed? Which point has a greater linear speed? Why?

123. If an angle of a sector is held constant, but the radius is doubled, how will the arc length of the sector and area of the sector be affected?

124. If an angle of a sector is doubled, but the radius is held constant, how will the arc length of the sector and the area of the sector be affected?

Expanding Your Skills

125. When a person pedals a bicycle, the front sprocket moves a chain that drives the back wheel and propels the bicycle forward. For each rotation of the front sprocket, the chain moves a distance equal to the circumference of the front sprocket. The back sprocket is smaller, so it will simultaneously move through a greater rotation. Furthermore, since the back sprocket is rigidly connected to the back wheel, each rotation of the back sprocket generates a rotation of the wheel.



Suppose that the front sprocket of a bicycle has a 4-in. radius and the back sprocket has a 2-in. radius.

a. How much chain will move with one rotation of the pedals (front sprocket)?

b. How many times will the back sprocket turn with one rotation of the pedals?

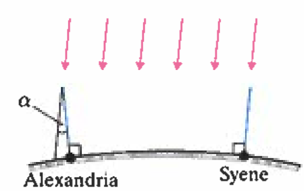
c. How many times will the wheels turn with one rotation of the pedals?

d. If the wheels are 27 in. in diameter, how far will the bicycle travel with one rotation of the pedals?

e. If the bicyclist pedals 80 rpm, what is the linear speed (in ft/min) of the bicycle?

f. If the bicyclist pedals 80 rpm, what is the linear speed (in mph) of the bicycle? (*Hint*: 1 mi = 5280 ft, and 1 hr = 60 min)

126. In the third century B.C., the Greek astronomer Eratosthenes approximated the Earth's circumference. On the summer solstice at noon in Alexandria, Egypt, Eratosthenes measured the angle α of the Sun relative to a line perpendicular to the ground. At the same time in Syene (now Aswan), located on the Tropic of Cancer, the Sun was directly overhead.



a. If $\alpha = \frac{1}{50}$ of a circle, find the measure of α in degrees. (In Eratosthenes' time, the degree measure had not yet been defined.)

b. If the distance between Alexandria and Syene is 5000 stadia, find the circumference of the Earth measured in stadia.

c. If 10 stadia = 1 mi, find Eratosthenes' approximation of the circumference of the Earth in miles (the modern-day approximation at the equator is 24,900 mi).

127. The space shuttle program involved 135 manned space flights in 30 yr. In addition to supplying and transporting astronauts to the International Space Station, space shuttle missions serviced the Hubble Space Telescope and deployed satellites. For a particular mission, a space shuttle orbited the Earth in 1.5 hr at an altitude of 200 mi.

a. Determine the angular speed (in radians per hour) of the shuttle.

b. Determine the linear speed of the shuttle in miles per hour. Assume that the Earth's radius is 3960 mi. Round to the nearest hundred miles per hour.

128. What is the first time (to the nearest second) after 12:00 midnight for which the minute hand and hour hand of a clock make a 120° angle?

Technology Connections

For Exercises 129–130, use the functions in the ANGLE menu on your calculator to

a. Convert from decimal degrees to DMS (degree-minute-second) form and

b. Convert from DMS form to decimal degrees.

On some calculators, the ″ symbol is accessed by hitting ALPHA **followed by** +.

129. a. $-216.479°$

b. $42°13'5.9''$

130. a. $-14.908°$

b. $71°19'4.7''$

For Exercises 131–132, use a calculator to convert from degrees to radians.

131. a. $147°26'9''$

b. $-228.459°$

132. a. $36°4'47''$

b. $-25.716°$

For Exercises 133–134, use a calculator to convert from radians to degrees.

133. a. $\dfrac{4\pi}{9}$

b. -5.718

134. a. $\dfrac{11\pi}{18}$

b. -1.356

SECTION 4.2 Trigonometric Functions Defined on the Unit Circle

OBJECTIVES

1. **Evaluate Trigonometric Functions Using the Unit Circle**
2. **Identify the Domains of the Trigonometric Functions**
3. **Use Fundamental Trigonometric Identities**
4. **Apply the Periodic and Even and Odd Function Properties of Trigonometric Functions**
5. **Approximate Trigonometric Functions on a Calculator**

1. Evaluate Trigonometric Functions Using the Unit Circle

The functions we have studied thus far have been used to model phenomena such as exponential growth and decay, projectile motion, and profit and cost, to name a few. However, none of the functions in our repertoire represent cyclical behavior such as the orbits of the moon and planets, the variation in air pressure that produces sound, the back-and-forth oscillations of a stretched spring, and so on. To model these behaviors, we introduce six new functions called *trigonometric functions*.

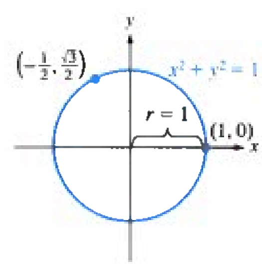


Figure 4-14

Historically, the study of trigonometry arose from the need to study relationships among the angles and sides of a triangle. An alternative approach is to define trigonometric functions as *circular functions*. To begin, we define the **unit circle** as the circle of radius 1 unit, centered at the origin. The unit circle consists of all points (x, y) that satisfy the equation $x^2 + y^2 = 1$. For example, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is a point on the unit circle because $\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$ (Figure 4-14).

TIP The circumference of the unit circle is $2\pi r = 2\pi(1) = 2\pi$. Thus, the value $t = \pi$ represents one-half of a revolution and falls on the point $(-1, 0)$. Likewise the value $t = \frac{\pi}{2}$ represents one-quarter of a revolution and falls on the point $(0, 1)$.

Now suppose we wrap the real number line around the unit circle by placing the point $t = 0$ from the number line on the point $(1, 0)$ on the circle. Positive real numbers from the number line ($t > 0$) wrap onto the unit circle in the counterclockwise direction. Negative numbers from the number line ($t < 0$) wrap onto the unit circle in the clockwise direction (Figure 4-15). For a value of t greater than 2π (or less than -2π) more than one revolution around the unit circle is required.

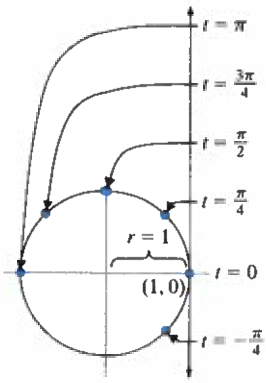


Figure 4-15

The purpose of wrapping the real number line around the unit circle is to associate each real number t with a unique point (x, y) on the unit circle.

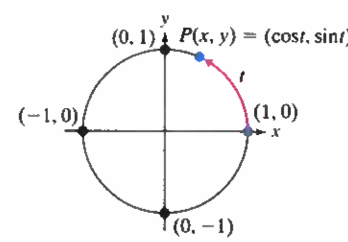
For a real number t corresponding to a point $P(x, y)$ on the unit circle, we can use the coordinates of P to define six trigonometric functions of t (Table 4-1). Notice that rather than using letters such as f, g, h, and so on, trigonometric functions are given the word names sine, cosine, tangent, cosecant, secant, and cotangent. These functions are abbreviated as "sin," "cos," "tan," "csc," "sec," and "cot," respectively. The value t is the input value or **argument** of each function.

Table 4-1

Unit Circle Definitions of the Trigonometric Functions

Let $P(x, y)$ be the point associated with a real number t measured along the circumference of the unit circle from the point $(1, 0)$.

Function Name	Definition
sine	$\sin t = y$
cosine	$\cos t = x$
tangent	$\tan t = \frac{y}{x}\ (x \neq 0)$
cosecant	$\csc t = \frac{1}{y}\ (y \neq 0)$
secant	$\sec t = \frac{1}{x}\ (x \neq 0)$
cotangent	$\cot t = \frac{x}{y}\ (y \neq 0)$



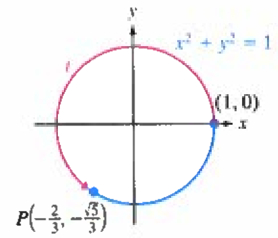
- If P is on the y-axis [either $(0, 1)$ or $(0, -1)$], then $x = 0$ and the tangent and secant functions are undefined.
- If P is on the x-axis [either $(1, 0)$ or $(-1, 0)$], then $y = 0$ and the cotangent and cosecant functions are undefined.

EXAMPLE 1 Evaluating Trigonometric Functions

Suppose that the real number t corresponds to the point $P\left(-\frac{2}{3}, -\frac{\sqrt{5}}{3}\right)$ on the unit circle. Evaluate the six trigonometric functions of t.

TIP In the figure in Example 1, t was arbitrarily taken to be positive as noted by the red arc wrapping counterclockwise. However, we can just as easily use a negative value of t, terminating at the same point in Quadrant III.

Solution:



$$\sin t = y = -\frac{\sqrt{5}}{3} \qquad \cos t = x = -\frac{2}{3}$$

$$\tan t = \frac{y}{x} = \frac{-\frac{\sqrt{5}}{3}}{-\frac{2}{3}} = -\frac{\sqrt{5}}{3} \cdot \left(-\frac{3}{2}\right) = \frac{\sqrt{5}}{2}$$

$$\csc t = \frac{1}{y} = \frac{1}{-\frac{\sqrt{5}}{3}} = -\frac{3}{\sqrt{5}} = -\frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$$

$$\sec t = \frac{1}{x} = \frac{1}{-\frac{2}{3}} = -\frac{3}{2}$$

$$\cot t = \frac{x}{y} = \frac{-\frac{2}{3}}{-\frac{\sqrt{5}}{3}} = -\frac{2}{3} \cdot \left(-\frac{3}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

Skill Practice 1 Suppose that the real number t corresponds to the point $P\left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$ on the unit circle. Evaluate the six trigonometric functions of t.

Let $P(x, y)$ be the point on the unit circle associated with a real number $t \geq 0$. Let $\theta \geq 0$ be the central angle in standard position measured in radians with terminal side through P and arc length t (Figure 4-16). Because the radius of the unit circle is 1, the arc length formula $s = r\theta$ becomes $t = 1 \cdot \theta$ or simply $t = \theta$.

A similar argument can be made for $t < 0$ and $\theta < 0$. The arc length formula is $s = r(-\theta)$, or equivalently $-t = r(-\theta)$, which also implies that $t = \theta$ (Figure 4-17).

Avoiding Mistakes

If θ is negative, then the opposite of θ is used in the arc length formula to ensure that the length of the arc is positive.

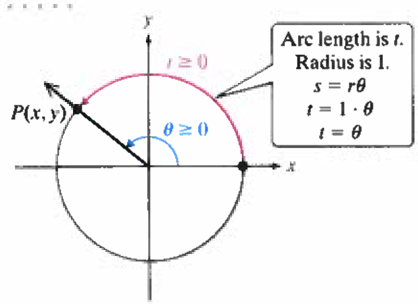


Figure 4-16

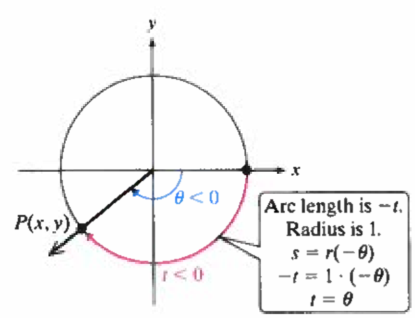


Figure 4-17

From this discussion, we have the following important result. The real number t taken along the circumference of the unit circle gives the radian measure of the corresponding central angle. That is, $\theta = t$ radians. Furthermore, this one-to-one correspondence between the real number t and the radian measure of the central angle θ means that the trigonometric functions can also be defined as functions of θ.

Answer

1. $\sin t = \frac{2\sqrt{5}}{5}$, $\cos t = \frac{\sqrt{5}}{5}$, $\tan t = 2$, $\csc t = \frac{\sqrt{5}}{2}$, $\sec t = \sqrt{5}$, $\cot t = \frac{1}{2}$

Trigonometric Functions of Real Numbers and Angles

If $\theta = t$ radians, then

$\sin t = \sin\theta$	$\cos t = \cos\theta$	$\tan t = \tan\theta$
$\csc t = \csc\theta$	$\sec t = \sec\theta$	$\cot t = \cot\theta$

We now want to determine the values of the trigonometric functions for several "special" values of t corresponding to central angles θ that are integer multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. Consider the point $P(x, y)$ on the unit circle corresponding to $t = \frac{\pi}{4}$ (Figure 4-18). Point P lies on the line $y = x$, and the coordinates of P can be written as (x, x). Substituting the coordinates of (x, x) into the equation $x^2 + y^2 = 1$, we have

$$x^2 + x^2 = 1$$
$$2x^2 = 1$$
$$x^2 = \frac{1}{2}$$
$$x = \sqrt{\frac{1}{2}}$$
$$x = \frac{1}{\sqrt{2}}$$

Choose x positive for a first quadrant point.

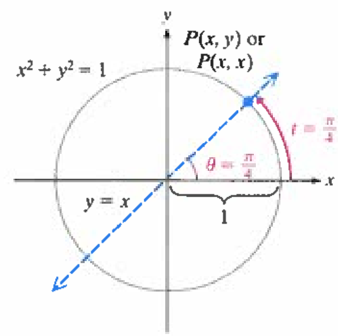


Figure 4-18

Rationalizing the denominator, we have

$$x = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Since $y = x$, point P has coordinates

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ or } \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$$

From the symmetry of the circle, the points corresponding to $t = \frac{3\pi}{4}$, $t = \frac{5\pi}{4}$, and $t = \frac{7\pi}{4}$ have coordinates $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, and $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, respectively (Figure 4-19).

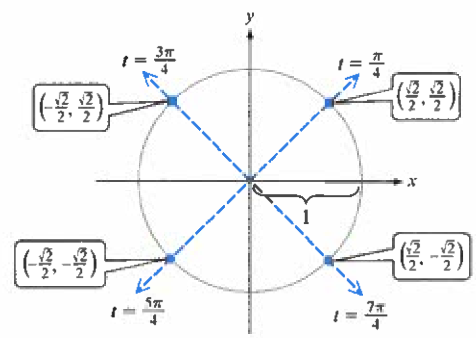


Figure 4-19

Now consider the point $Q(x, y)$ on the unit circle corresponding to $t = \frac{\pi}{6}$ (Figure 4-20). Dropping a line segment from Q perpendicular to the x-axis at point A, we construct a right triangle (ΔOAQ). The acute angles in ΔOAQ are 30° and 60°, and the hypotenuse of the triangle is 1 unit. Placing two such triangles adjacent to one another on opposite sides of the x-axis we have an equilateral triangle (ΔOBQ) with sides of 1 unit and angles of 60° (Figure 4-21).

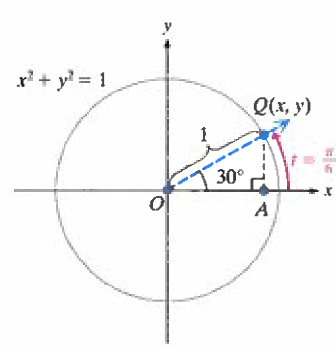


Figure 4-20

From ΔOBQ we have $2y = 1$. Thus, $y = \frac{1}{2}$. Since $Q(x, y)$ is a point on the unit circle, we can substitute $y = \frac{1}{2}$ into the equation $x^2 + y^2 = 1$ and solve for x.

$$x^2 + y^2 = 1$$
$$x^2 + \left(\frac{1}{2}\right)^2 = 1$$
$$x^2 + \frac{1}{4} = 1$$
$$x^2 = \frac{3}{4}$$
$$x = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

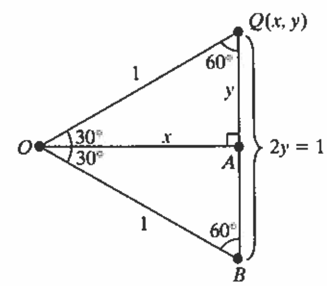
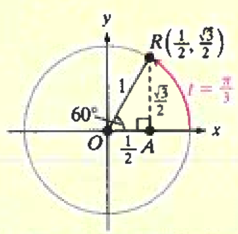


Figure 4-21

TIP The value $t = \frac{\pi}{3}$ corresponds to the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the unit circle. ΔOAR is congruent to the triangle shown in Figure 4-20, but is oriented with the 60° angle in standard position rather than the 30° angle in standard position. As a result, notice that the x- and y-coordinates of points Q and R are reversed.



Thus, Q has coordinates $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Using symmetry, we can also find coordinates of the points on the unit circle corresponding to $t = \frac{5\pi}{6}$, $t = \frac{7\pi}{6}$, and $t = \frac{11\pi}{6}$ (Figure 4-22).

Using similar reasoning, we can show that the point $R(x, y)$ corresponding to $t = \frac{\pi}{3}$ has coordinates $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

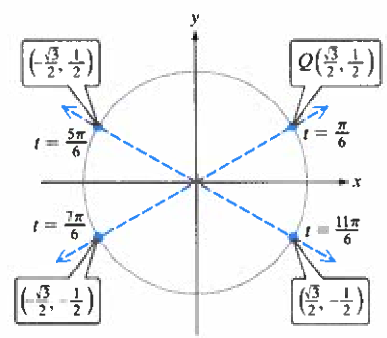


Figure 4-22

We summarize our findings in Figure 4-23 for selected values of t and the corresponding central angle θ.

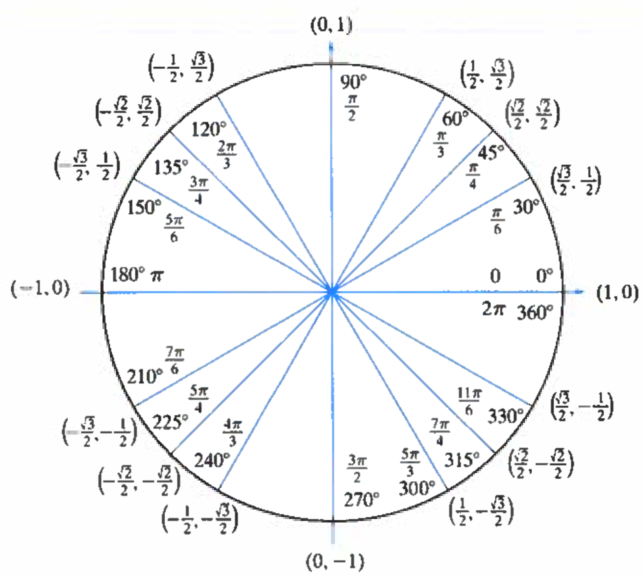


Figure 4-23

Figure 4-23 looks daunting, but using the symmetry of the circle and recognizing several patterns can make the coordinates of these special points easy to determine. First we recommend memorizing the coordinates of the three special points in Quadrant I.

- Note that for $t = \frac{\pi}{4}$, the x- and y-coordinates are both $\frac{\sqrt{2}}{2}$, or equivalently $\frac{1}{\sqrt{2}}$.
- For $t = \frac{\pi}{6}$ and $t = \frac{\pi}{3}$, the x- and y-coordinates are reversed.
- The value $\frac{\sqrt{3}}{2} \approx 0.866$ is greater than the value $\frac{1}{2} = 0.5$. For $t = \frac{\pi}{6}$, the x-coordinate is greater than the y-coordinate. Therefore, x must be $\frac{\sqrt{3}}{2}$. For $t = \frac{\pi}{3}$, the x-coordinate is less than the y-coordinate. Therefore, x must be $\frac{1}{2}$.
- For the special points in Quadrants II, III, and IV, use the values $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, and $\frac{\sqrt{2}}{2}$ with the appropriate signs attached.

EXAMPLE 2 Evaluating Trigonometric Functions

Evaluate the six trigonometric functions of the real number t.

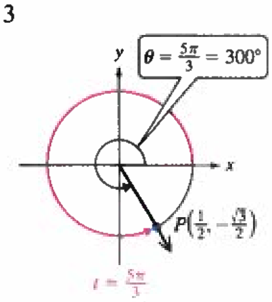
a. $t = \frac{5\pi}{3}$ **b.** $t = -\frac{5\pi}{4}$

Solution:

a. $t = \frac{5\pi}{3}$ corresponds to the point $(x, y) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ on the unit circle.

$$\sin\frac{5\pi}{3} = y = -\frac{\sqrt{3}}{2}$$

$$\csc\frac{5\pi}{3} = \frac{1}{y} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$



$$\cos\frac{5\pi}{3} = x = \frac{1}{2}$$

$$\sec\frac{5\pi}{3} = \frac{1}{x} = \frac{1}{\frac{1}{2}} = 2$$

$$\tan\frac{5\pi}{3} = \frac{y}{x} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2} \cdot \frac{2}{1} = -\sqrt{3}$$

$$\cot\frac{5\pi}{3} = \frac{x}{y} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2} \cdot \left(-\frac{2}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

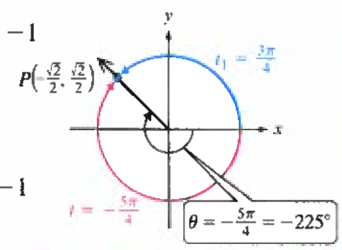
b. Since the circumference of the unit circle is 2π, the values $t = -\dfrac{5\pi}{4}$ and $t_1 = -\dfrac{5\pi}{4} + 2\pi = \dfrac{3\pi}{4}$ have the same location in Quadrant II on the unit circle. Both correspond to the point $(x, y) = \left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$, or equivalently $\left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$. Thus,

$$\sin\left(-\frac{5\pi}{4}\right) = y = \frac{\sqrt{2}}{2} \qquad \csc\left(-\frac{5\pi}{4}\right) = \frac{1}{y} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

$$\cos\left(-\frac{5\pi}{4}\right) = x = -\frac{\sqrt{2}}{2} \qquad \sec\left(-\frac{5\pi}{4}\right) = \frac{1}{x} = \frac{1}{-\frac{1}{\sqrt{2}}} = -\sqrt{2}$$

$$\tan\left(-\frac{5\pi}{4}\right) = \frac{y}{x} = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2}\cdot\left(-\frac{2}{\sqrt{2}}\right) = -1$$

$$\cot\left(-\frac{5\pi}{4}\right) = \frac{x}{y} = \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -\frac{\sqrt{2}}{2}\cdot\frac{2}{\sqrt{2}} = -1$$



Skill Practice 2 Evaluate the six trigonometric functions of the real number t.

a. $t = \dfrac{5\pi}{6}$ **b.** $t = -\dfrac{3\pi}{4}$

Answers

2. a. $\sin\frac{5\pi}{6} = \frac{1}{2}$, $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$, $\tan\frac{5\pi}{6} = -\frac{\sqrt{3}}{3}$, $\csc\frac{5\pi}{6} = 2$, $\sec\frac{5\pi}{6} = -\frac{2\sqrt{3}}{3}$, $\cot\frac{5\pi}{6} = -\sqrt{3}$

b. $\sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\cos\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\tan\left(-\frac{3\pi}{4}\right) = 1$, $\csc\left(-\frac{3\pi}{4}\right) = -\sqrt{2}$, $\sec\left(-\frac{3\pi}{4}\right) = -\sqrt{2}$, $\cot\left(-\frac{3\pi}{4}\right) = 1$

2. Identify the Domains of the Trigonometric Functions

From Table 4-1, $\sin t = y$ and $\cos t = x$ have no restrictions on their domain. However, $\tan t = \dfrac{y}{x}$ and $\sec t = \dfrac{1}{x}$ are undefined for all values of t corresponding to the points (0, 1) and (0, −1) on the unit circle. These are $t = \dfrac{\pi}{2}$, $t = \dfrac{3\pi}{2}$, and all other *odd* multiples of $\dfrac{\pi}{2}$ (Figure 4-24).

- $\tan t$ and $\sec t$ are undefined for $\ldots -\dfrac{3\pi}{2}, -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots, \dfrac{(2n+1)\pi}{2}$, for all integers n.

Likewise, $\cot t = \dfrac{x}{y}$ and $\csc t = \dfrac{1}{y}$ are undefined for all values of t corresponding to the points (1, 0) and (−1, 0) on the unit circle. These are $t = 0$, $t = \pi$, $t = 2\pi$, and all other multiples of π (Figure 4-24).

- $\cot t$ and $\csc t$ are undefined for $\ldots -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots, n\pi$ for all integers n.

TIP For integer values of n, the formula $\frac{(2n+1)\pi}{2}$ generates odd multiples of $\frac{\pi}{2}$. For example, if $n = 6$, $\frac{[2(6)+1]\pi}{2} = \frac{13\pi}{2}$.
The formula $n\pi$ generates multiples of π. For example, if $n = 6$, $n\pi = (6)\pi = 6\pi$.

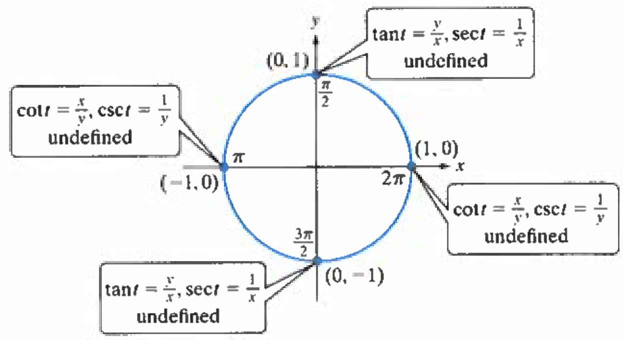


Figure 4-24

Domains of the Trigonometric Functions

Function	Domain*	Notes
$f(t) = \sin t$	All real numbers	No restrictions
$f(t) = \cos t$	All real numbers	No restrictions
$f(t) = \tan t$	$\left\{t \,\middle\vert\, t \neq \frac{(2n+1)\pi}{2} \text{ for all integers } n\right\}$	Exclude odd multiples of $\frac{\pi}{2}$.
$f(t) = \cot t$	$\{t \mid t \neq n\pi \text{ for all integers } n\}$	Exclude multiples of π.
$f(t) = \sec t$	$\left\{t \,\middle\vert\, t \neq \frac{(2n+1)\pi}{2} \text{ for all integers } n\right\}$	Exclude odd multiples of $\frac{\pi}{2}$.
$f(t) = \csc t$	$\{t \mid t \neq n\pi \text{ for all integers } n\}$	Exclude multiples of π.

*The range of each trigonometric function will be discussed in Sections 4.5 and 4.6 when we cover the graphs of the functions.

EXAMPLE 3 Evaluate the Trigonometric Functions for Given Values of t

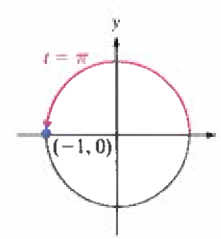
Evaluate the six trigonometric functions of the real number t.

a. $t = \pi$ **b.** $t = \frac{5\pi}{2}$

Solution:

a. The value $t = \pi$ corresponds to $(-1, 0)$ on the unit circle.

$\sin \pi = y = 0$ $\quad\cos \pi = x = -1$ $\quad\tan \pi = \frac{y}{x} = \frac{0}{-1} = 0$



$\csc \pi = \frac{1}{y}$ is undefined because $\frac{1}{0}$ is undefined.

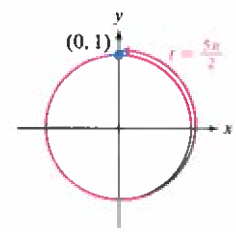
$\sec \pi = \frac{1}{x} = \frac{1}{-1} = -1$

$\cot \pi = \frac{x}{y}$ is undefined because $\frac{-1}{0}$ is undefined.

TIP Determining the location of a real number t wrapped onto the unit circle is equivalent to finding the point where the terminal side of the central angle θ intercepts the unit circle. This is true for θ measured in radians and drawn in standard position. In Example 3(b), the angle $\theta = \frac{5\pi}{2}$ is coterminal to $\frac{\pi}{2}$, which intercepts the unit circle at (0, 1).

b. The circumference of the unit circle is 2π. Therefore, since $t = \dfrac{5\pi}{2} = 2\pi + \dfrac{\pi}{2}$, the value $t = \dfrac{5\pi}{2}$ corresponds to the point (0, 1) on the unit circle.

$$\sin\frac{5\pi}{2} = y = 1 \qquad \cos\frac{5\pi}{2} = x = 0$$



$$\tan\frac{5\pi}{2} = \frac{y}{x} \text{ is undefined because } \frac{1}{0} \text{ is undefined.}$$

$$\csc\frac{5\pi}{2} = \frac{1}{y} = \frac{1}{1} = 1$$

$$\sec\frac{5\pi}{2} = \frac{1}{x} \text{ is undefined because } \frac{1}{0} \text{ is undefined.}$$

$$\cot\frac{5\pi}{2} = \frac{x}{y} = \frac{0}{1} = 0$$

Skill Practice 3 Evaluate the six trigonometric functions of the real number t.

a. $t = -2\pi$ **b.** $\dfrac{3\pi}{2}$

3. Use Fundamental Trigonometric Identities

You may have already noticed several relationships among the six trigonometric functions that follow directly from their definitions. For example, for a real number t corresponding to a point (x, y) on the unit circle, $\sin t = y$ and $\csc t = \dfrac{1}{y}$. Thus, the sine and cosecant functions are reciprocals for all t in their common domain. Table 4-2 summarizes the reciprocal relationships among the trigonometric functions along with two important quotient properties. These relationships should be committed to memory.

Avoiding Mistakes

It is important to note that the argument *must* be included when denoting the value of a trigonometric function. For example,

$$\tan t = \frac{\sin t}{\cos t}$$

$$\text{not } \tan = \frac{\sin}{\cos}$$

Table 4-2

Reciprocal and Quotient Identities	
$\csc t = \dfrac{1}{\sin t}$ or $\sin t = \dfrac{1}{\csc t}$	$\sin t$ and $\csc t$ are reciprocals.
$\sec t = \dfrac{1}{\cos t}$ or $\cos t = \dfrac{1}{\sec t}$	$\cos t$ and $\sec t$ are reciprocals.
$\cot t = \dfrac{1}{\tan t}$ or $\tan t = \dfrac{1}{\cot t}$	$\tan t$ and $\cot t$ are reciprocals.
$\tan t = \dfrac{\sin t}{\cos t}$	$\tan t$ is the ratio of $\sin t$ and $\cos t$.
$\cot t = \dfrac{\cos t}{\sin t}$	$\cot t$ is the ratio of $\cos t$ and $\sin t$.

Answers

3. a. $\sin(-2\pi) = 0$, $\cos(-2\pi) = 1$, $\tan(-2\pi) = 0$, $\csc(-2\pi)$ is undefined, $\sec(-2\pi) = 1$, $\cot(-2\pi)$ is undefined

b. $\sin\frac{3\pi}{2} = -1$, $\cos\frac{3\pi}{2} = 0$, $\tan\frac{3\pi}{2}$ is undefined, $\csc\frac{3\pi}{2} = -1$, $\sec\frac{3\pi}{2}$ is undefined, $\cot\frac{3\pi}{2} = 0$

EXAMPLE 4 Using the Reciprocal and Quotient Identities

Given that $\sin t = \frac{5}{8}$ and $\cos t = \frac{\sqrt{39}}{8}$, use the reciprocal and quotient identities to find the values of the other trigonometric functions of t.

Solution:

Given the values of $\sin t$ and $\cos t$, we can use the quotient identities to find $\tan t$ and $\cot t$.

$$\tan t = \frac{\sin t}{\cos t} = \frac{\frac{5}{8}}{\frac{\sqrt{39}}{8}} = \frac{5}{8} \cdot \frac{8}{\sqrt{39}} = \frac{5}{\sqrt{39}} = \frac{5}{\sqrt{39}} \cdot \frac{\sqrt{39}}{\sqrt{39}} = \frac{5\sqrt{39}}{39}$$

$$\cot t = \frac{\cos t}{\sin t} = \frac{\frac{\sqrt{39}}{8}}{\frac{5}{8}} = \frac{\sqrt{39}}{8} \cdot \frac{8}{5} = \frac{\sqrt{39}}{5}$$

The remaining two functions can be found by using the reciprocal identities.

$$\csc t = \frac{1}{\sin t} = \frac{1}{\frac{5}{8}} = \frac{8}{5}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{\sqrt{39}}{8}} = \frac{8}{\sqrt{39}} = \frac{8}{\sqrt{39}} \cdot \frac{\sqrt{39}}{\sqrt{39}} = \frac{8\sqrt{39}}{39}$$

Skill Practice 4 Given $\sin t = \frac{3}{7}$ and $\cos t = \frac{2\sqrt{10}}{7}$ use the reciprocal and quotient identities to find the values of the other trigonometric functions of t.

Consider a real number t corresponding to the point $P(x, y)$ on the unit circle. Since $\cos t = x$ and $\sin t = y$, point P can be labeled $P(\cos t, \sin t)$. See Figure 4-25. Furthermore, since P is on the unit circle, it satisfies the equation $x^2 + y^2 = 1$. Thus,

$$(\cos t)^2 + (\sin t)^2 = 1 \text{ or } (\sin t)^2 + (\cos t)^2 = 1$$

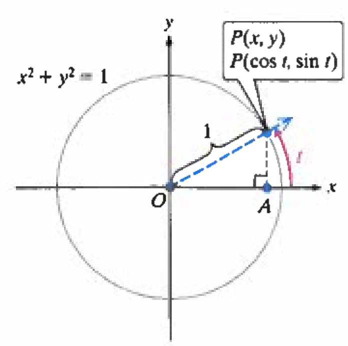


Figure 4-25

It is universally preferred to write the exponent associated with a trigonometric function between the name of the function and its argument. For example, $(\sin t)^2$ is most often written as $\sin^2 t$. Thus, the relationship $(\sin t)^2 + (\cos t)^2 = 1$ is written as $\sin^2 t + \cos^2 t = 1$. This relationship is called a "Pythagorean identity" because $\sin^2 t + \cos^2 t = 1$ is a statement of the Pythagorean theorem for ΔOAP.

Table 4-3 summarizes three Pythagorean identities. The second and third relationships can be derived by dividing both sides of the equation $\sin^2 t + \cos^2 t = 1$ by $\cos^2 t$ and $\sin^2 t$, respectively (see Exercises 47 and 48).

Answer

4. $\tan t = \frac{3\sqrt{10}}{20}$, $\cot t = \frac{2\sqrt{10}}{3}$, $\csc t = \frac{7}{3}$, $\sec t = \frac{7\sqrt{10}}{20}$

Table 4-3

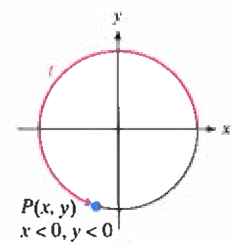
Pythagorean Identities		
$\sin^2 t + \cos^2 t = 1$	$\tan^2 t + 1 = \sec^2 t$	$1 + \cot^2 t = \csc^2 t$

EXAMPLE 5 Using the Pythagorean Identities

Given that $\tan t = \frac{12}{5}$ for $\pi < t < \frac{3\pi}{2}$, use an appropriate identity to find the value of $\sec t$.

Solution:

The real number t corresponds to a point $P(x, y)$ on the unit circle in Quadrant III. Since the value of $\tan t$ is known, we can use the relationship $\tan^2 t + 1 = \sec^2 t$ to find the value of $\sec t$.



$\tan^2 t + 1 = \sec^2 t$

$\sec t = \pm\sqrt{\tan^2 t + 1}$

$\sec t = -\sqrt{\tan^2 t + 1}$ Since t is between π and $\frac{3\pi}{2}$, it corresponds to a point $P(x, y)$ on the unit circle in Quadrant III. Since P is in Quadrant III, the values of both x and y are negative. Hence $\sec t = \frac{1}{x}$ is negative.

$$\sec t = -\sqrt{\left(\frac{12}{5}\right)^2 + 1} = -\sqrt{\frac{144}{25} + \frac{25}{25}} = -\sqrt{\frac{169}{25}} = -\frac{13}{5}$$

Skill Practice 5 Given that $\csc t = \frac{5}{4}$ for $\frac{\pi}{2} < t < \pi$, use an appropriate identity to find the value of $\cot t$.

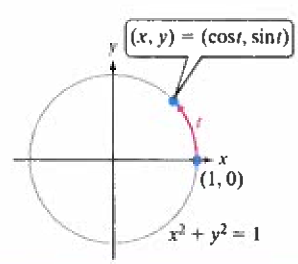
Sometimes it is beneficial to express one trigonometric function in terms of another. This is demonstrated in Example 6.

EXAMPLE 6 Expressing a Trigonometric Function in Terms of Another Trigonometric Function

For a given real number t, express $\sin t$ in terms of $\cos t$.

Solution:

Let $P(x, y) = (\cos t, \sin t)$ be the point on the unit circle determined by t.



$x^2 + y^2 = 1$ Equation of unit circle

$\cos^2 t + \sin^2 t = 1$ Substitute $x = \cos t$ and $y = \sin t$.

$\sin^2 t = 1 - \cos^2 t$

$\sin t = \pm\sqrt{1 - \cos^2 t}$ The sign is determined by the quadrant in which P lies.

Skill Practice 6 For a given real number t, express $\tan t$ in terms of $\sec t$.

Answers

5. $\cot t = -\frac{3}{4}$

6. $\tan t = \pm\sqrt{\sec^2 t - 1}$

4. Apply the Periodic and Even and Odd Function Properties of Trigonometric Functions

Many cyclical patterns occur in nature such as the changes of the seasons, the rise and fall of the tides, and the phases of the moon. In many cases, we can predict these behaviors by determining the period of one complete cycle. For example, an observer on the Earth would note that the moon changes from a new moon to a full moon and back again to a new moon in approximately 29.5 days. If $m(t)$ represents the percentage of the moon seen on day t, then

$$m(t + 29.5) = m(t) \text{ for all } t \text{ in the domain of } m.$$

We say that function m is *periodic* because it repeats at regular intervals. In this example, the period is 29.5 days because this is the shortest time required to complete one full cycle.

Definition of a Periodic Function

A function f is **periodic** if $f(t + p) = f(t)$ for some constant p.
The smallest positive value p for which f is periodic is called the **period** of f.

The values of the six trigonometric functions of t are determined by the corresponding point $P(x, y)$ on the unit circle. Since the circumference of the unit circle is 2π, adding (or subtracting) 2π to t results in the same terminal point (x, y). Consequently, the values of the trigonometric functions are the same for t and $t + 2n\pi$.

In Exercises 121 and 122, we show that the sine and cosine functions are periodic with period 2π. Likewise, their reciprocal functions, cosecant and secant, are periodic with period 2π. However, the period of the tangent and cotangent functions is π, which we show in Exercises 119 and 120.

Periodic Properties of Trigonometric Functions

Function	Period	Property
Sine	2π	$\sin(t + 2\pi) = \sin t$
Cosine	2π	$\cos(t + 2\pi) = \cos t$
Cosecant	2π	$\csc(t + 2\pi) = \csc t$
Secant	2π	$\sec(t + 2\pi) = \sec t$
Tangent	π	$\tan(t + \pi) = \tan t$
Cotangent	π	$\cot(t + \pi) = \cot t$

Given a periodic function f, if the period is p, then $f(t + p) = f(t)$. It is also true that $f(t + np) = f(t)$ for any integer n. That is, adding any integer multiple of the period to a domain element of a periodic function results in the same function value. This is demonstrated in Example 7.

EXAMPLE 7 Applying the Periodic Properties of the Trigonometric Functions

Given $\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$, determine the value of $\sin\frac{49\pi}{12}$.

Solution:

The period of the sine function is 2π or equivalently $\frac{24\pi}{12}$. Therefore, adding or subtracting any multiple of $\frac{24\pi}{12}$ to the argument $\frac{\pi}{12}$ results in the same value of the sine function.

$$\sin\frac{49\pi}{12} = \sin\left(\frac{\pi}{12} + \frac{48\pi}{12}\right) = \sin\left[\frac{\pi}{12} + 2\left(\frac{24\pi}{12}\right)\right] = \sin\left(\frac{\pi}{12} + 2(2\pi)\right)$$

$$= \sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Skill Practice 7 Given $\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$, determine the value of $\sin\left(-\frac{15\pi}{8}\right)$.

Recall that a function f is even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$. Suppose that t is a real number associated with a point $P(x, y)$ on the unit circle. Then $-t$ is associated with the point $Q(x, -y)$. See Figure 4-26.

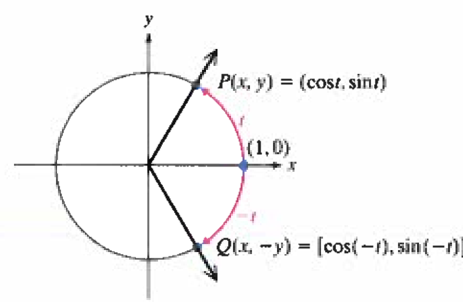


Figure 4-26

Notice that $\cos(-t) = \cos t = x$, so the cosine function is an even function. The sine function, however, is an odd function because $\sin t = y$, but $\sin(-t) = -y$. Thus, $\sin(-t) = -\sin t$. The tangent function is also an odd function because $\tan(-t) = \frac{-y}{x}$ and $\tan t = \frac{y}{x}$. Therefore, $\tan(-t) = -\tan t$. The functions secant, cosecant, and cotangent carry the same even and odd properties as their reciprocals: cosine, sine, and tangent, respectively.

Even and Odd Properties of Trigonometric Functions

Function	Evaluate at t and $-t$	Property
Sine	$\sin t = y$ and $\sin(-t) = -y$	$\sin(-t) = -\sin t$ (odd function)
Cosine	$\cos t = x$ and $\cos(-t) = x$	$\cos(-t) = \cos t$ (even function)
Cosecant	$\csc t = \frac{1}{y}$ and $\csc(-t) = \frac{1}{-y}$	$\csc(-t) = -\csc t$ (odd function)
Secant	$\sec t = \frac{1}{x}$ and $\sec(-t) = \frac{1}{x}$	$\sec(-t) = \sec t$ (even function)
Tangent	$\tan t = \frac{y}{x}$ and $\tan(-t) = \frac{-y}{x}$	$\tan(-t) = -\tan t$ (odd function)
Cotangent	$\cot t = \frac{x}{y}$ and $\cot(-t) = \frac{x}{-y}$	$\cot(-t) = -\cot t$ (odd function)

Answer

7. $\frac{\sqrt{2-\sqrt{2}}}{2}$

Avoiding Mistakes

A factor from the argument of a function cannot be factored out in front of the function. For example,

$f(2x) \neq 2f(x)$

In step 2 of Example 8(b), the value -1 that appears in front of the first term was not factored out, but rather is a result of the odd function property of the tangent function.

EXAMPLE 8 Simplifying Expressions Using the Even-Odd Properties of Trigonometric Functions

Use the properties of the trigonometric functions to simplify.

a. $4\cos t + \cos(-t)$ **b.** $\tan(-3t) - \tan(-3t + \pi)$

Solution:

a. $4\cos t + \cos(-t)$

$= 4\cos t + \cos t$ The cosine function is even. $\cos(-t) = \cos t$

$= 5\cos t$

b. $\tan(-3t) - \tan(3t + \pi)$ The tangent function is odd. $\tan(-3t) = -\tan 3t$

$= -\tan 3t - \tan 3t$ The period of the tangent function is π. Therefore, $\tan(3t + \pi) = \tan 3t$.

$= -2\tan 3t$

Skill Practice 8 Use the properties of the trigonometric functions to simplify.

a. $\sec t - 2\sec(-t)$ **b.** $\sin(2t + 2\pi) - \sin(-2t)$

5. Approximate Trigonometric Functions on a Calculator

From Figure 4-23, we display the coordinates of the points on the unit circle that correspond to t values in multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. From this information we can find the values of the six trigonometric functions for these real numbers. However, it is often the case that the exact value of a trigonometric function for a real number t cannot be found analytically. In such a case, we can use a calculator to approximate the function values.

EXAMPLE 9 Approximating Trigonometric Functions Using a Calculator

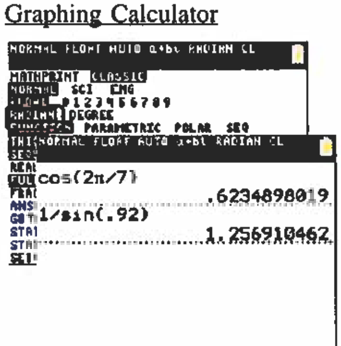
Use a calculator to approximate the function values. Round to 4 decimal places.

a. $\cos\frac{2\pi}{7}$ **b.** $\csc 0.92$

Solution:

The real number t corresponds to a point $P(x, y)$ on the unit circle. The value of t is the same as the radian measure of the central angle θ formed by the positive x-axis and the ray from the origin to point P. Therefore, we must first be sure that the calculator is in radian mode.

Graphing Calculator



Scientific Calculator (radian mode)	Rounded
a. $2 \times \pi \div 7 =$ COS	0.6235
b. 0.92 SIN x^{-1}	1.2569

This takes the reciprocal of the sine function (cosecant).

Answers

8. a. $-\sec t$ **b.** $2\sin 2t$

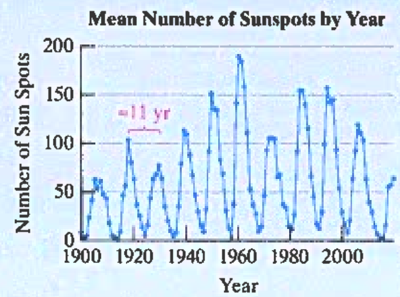
Skill Practice 9 Use a calculator to approximate the function values. Round to 4 decimal places.

a. $\tan 1.4$ **b.** $\cot\frac{\pi}{8}$

Point of Interest (Cyclical Data)

In September 1859, a gigantic *coronal mass ejection* erupted from the Sun, sending a huge amount of electromagnetic radiation toward the Earth. This geomagnetic storm was the largest on record to have struck the Earth, causing telegraph lines to short out and compasses to go haywire. The aurora borealis or "northern lights" following the event could be seen as far south as Cuba.

Solar storms are natural phenomena that occur as a result of the rise and fall of the Sun's magnetic activity. Although the cause of solar storms is not completely understood, they seem to be cyclical. Scientists have established a correlation between solar storms and the number of sun spots that cycle over a period of approximately 11 yr. The effects of a massive solar storm today could bring potentially devastating disruption to power grids, satellite communication, and air travel. As a result, solar activity and "space weather" are studied intensely by NASA and NOAA.



(*Source*: SILSO data/image, Royal Observatory of Belgium, Brussels)

Answers

9. a. 5.7979 **b.** 2.4142

SECTION 4.2 Practice Exercises

Prerequisite Review

For Exercises R.1–R.3, determine if the function is even, odd, or neither.

R.1. $k(x) = 11x^3 + 12x$ **R.2.** $r(x) = \sqrt{25 - (x+2)^2}$ **R.3.** $q(x) = \sqrt{23 + x^2}$

For Exercises R.4–R.5, write the domain of the function in set-builder notation.

R.4. $p(v) = \dfrac{v+3}{v+4}$ **R.5.** $k(q) = \dfrac{q^2}{2q^2 + 3q - 27}$

Concept Connections

1. The graph of $x^2 + y^2 = 1$ is known as the __________ circle. It has a radius of length __________ and center at the __________.
2. The circumference of the unit circle is __________. Thus, the value $t = \frac{\pi}{2}$ represents __________ of a revolution.
3. When $\tan t = 0$, the value of $\cot t$ is __________.
4. The value of $\sec t = \dfrac{1}{\cos t}$. Therefore, when $\cos t = 0$, the value of $\sec t$ is __________.
5. On the interval $[0, 2\pi)$, $\sin t = 0$ for $t =$ __________ and $t =$ __________. Since $\csc t = \dfrac{1}{\sin t}$, the value of $\csc t$ is __________ for these values of t.
6. The domain of the trigonometric functions __________ and __________ is all real numbers.
7. Which trigonometric functions have domain $\left\{t \,\middle|\, t \neq \dfrac{(2n+1)\pi}{2} \text{ for all integers } n\right\}$?

8. A function f is ______________ if $f(t + p) = f(t)$ for some constant p.

9. The period of the tangent and cotangent functions is ______________. The period of the sine, cosine, cosecant, and secant functions is ______________.

10. The cosine function is an ______________ function because $\cos(-t) = \cos t$. The sine function is an ______________ function because $\sin(-t) = -\sin t$.

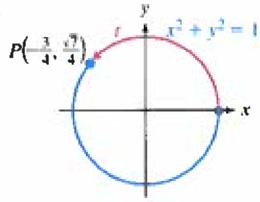
Objective 1: Evaluate Trigonometric Functions Using the Unit Circle

For Exercises 11–14, determine if the point lies on the unit circle.

11. $\left(\dfrac{\sqrt{10}}{5}, \dfrac{\sqrt{15}}{5}\right)$ **12.** $\left(\dfrac{\sqrt{61}}{8}, -\dfrac{\sqrt{2}}{8}\right)$ **13.** $\left(-\dfrac{\sqrt{7}}{10}, -\dfrac{2\sqrt{23}}{10}\right)$ **14.** $\left(-\dfrac{2\sqrt{34}}{17}, \dfrac{3\sqrt{17}}{17}\right)$

For Exercises 15–18, the real number t corresponds to the point P on the unit circle. Evaluate the six trigonometric functions of t. (See Example 1)

15.



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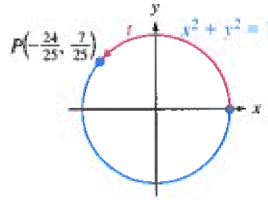

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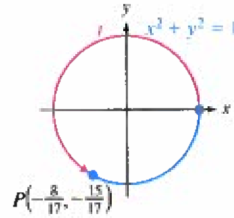

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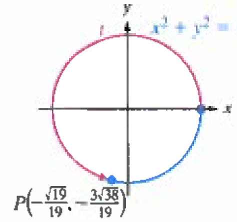
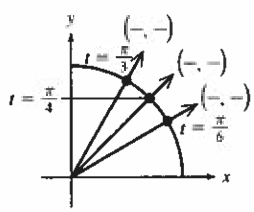
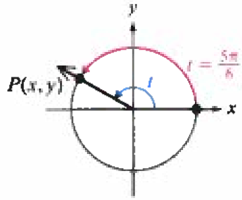


19. Fill in the ordered pairs on the unit circle corresponding to each real number t.



For Exercises 20–23, identify the coordinates of point P. Then evaluate the six trigonometric functions of t. (See Example 2)

20.



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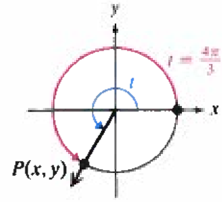

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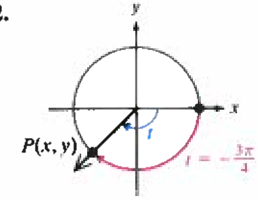

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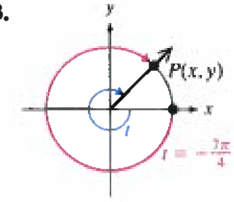


For Exercises 24–26, identify the ordered pairs on the unit circle corresponding to each real number t.

24. a. $t = \dfrac{2\pi}{3}$ **b.** $t = -\dfrac{5\pi}{4}$ **c.** $t = \dfrac{5\pi}{6}$

25. a. $t = \dfrac{7\pi}{6}$ **b.** $t = -\dfrac{3\pi}{4}$ **c.** $t = \dfrac{4\pi}{3}$

26. a. $t = \dfrac{5\pi}{3}$ **b.** $t = \dfrac{7\pi}{4}$ **c.** $t = -\dfrac{\pi}{6}$

For Exercises 27–32, evaluate the trigonometric function at the given real number.

27. $f(t) = \cos t;\ t = \dfrac{2\pi}{3}$ **28.** $g(t) = \sin t;\ t = \dfrac{5\pi}{4}$ **29.** $h(t) = \cot t;\ t = \dfrac{11\pi}{6}$

30. $s(t) = \sec t;\ t = \dfrac{5\pi}{3}$ **31.** $z(t) = \csc t;\ t = \dfrac{7\pi}{4}$ **32.** $r(t) = \tan t;\ t = \dfrac{5\pi}{6}$

Objective 2: Identify the Domains of the Trigonometric Functions

For Exercises 33–38, select the domain of the trigonometric function.

a. All real numbers **b.** $\left\{t \,\middle|\, t \neq \frac{(2n+1)\pi}{2} \text{ for all integers } n\right\}$ **c.** $\{t \mid t \neq n\pi \text{ for all integers } n\}$

33. $f(t) = \sin t$ **34.** $f(t) = \tan t$ **35.** $f(t) = \cot t$
36. $f(t) = \cos t$ **37.** $f(t) = \sec t$ **38.** $f(t) = \csc t$

For Exercises 39–42, evaluate the function if possible. (See Example 3)

39. a. $\sin 0$ **b.** $\cot \pi$ **c.** $\tan 3\pi$
d. $\sec \pi$ **e.** $\csc 0$ **f.** $\cos \pi$

40. a. $\cos\left(-\frac{\pi}{2}\right)$ **b.** $\csc\frac{\pi}{2}$ **c.** $\cot\frac{5\pi}{2}$
d. $\tan\frac{\pi}{2}$ **e.** $\sec\frac{3\pi}{2}$ **f.** $\sin\frac{\pi}{2}$

41. a. $\sin\frac{3\pi}{2}$ **b.** $\cos\frac{7\pi}{2}$ **c.** $\tan\frac{3\pi}{2}$
d. $\csc\left(-\frac{\pi}{2}\right)$ **e.** $\sec 1.5\pi$ **f.** $\cot\frac{\pi}{2}$

42. a. $\cot 0$ **b.** $\cos 2\pi$ **c.** $\csc \pi$
d. $\tan 0$ **e.** $\sin(-3\pi)$ **f.** $\sec 0$

Objective 3: Use Fundamental Trigonometric Identities

For Exercises 43–46, given the values for $\sin t$ and $\cos t$, use the reciprocal and quotient identities to find the values of the other trigonometric functions of t. (See Example 4)

43. $\sin t = \frac{\sqrt{5}}{3}$ and $\cos t = \frac{2}{3}$

44. $\sin t = \frac{3}{4}$ and $\cos t = \frac{\sqrt{7}}{4}$

45. $\sin t = -\frac{\sqrt{39}}{8}$ and $\cos t = -\frac{5}{8}$

46. $\sin t = \frac{28}{53}$ and $\cos t = -\frac{45}{53}$

For Exercises 47–48, derive the given identity from the Pythagorean identity, $\sin^2 t + \cos^2 t = 1$.

47. $\tan^2 t + 1 = \sec^2 t$ **48.** $1 + \cot^2 t = \csc^2 t$

For Exercises 49–54, use an appropriate Pythagorean identity to find the indicated value. (See Example 5)

49. Given $\cos t = -\frac{7}{25}$ for $\frac{\pi}{2} < t < \pi$, find the value of $\sin t$.

50. Given $\sin t = -\frac{8}{17}$ for $\pi < t < \frac{3\pi}{2}$, find the value of $\cos t$.

51. Given $\cot t = \frac{45}{28}$ for $\pi < t < \frac{3\pi}{2}$, find the value of $\csc t$.

52. Given $\csc t = -\frac{41}{40}$ for $\frac{3\pi}{2} < t < 2\pi$, find the value of $\cot t$.

53. Given $\tan t = -\frac{11}{60}$ for $\frac{3\pi}{2} < t < 2\pi$, find the value of $\sec t$.

54. Given $\sec t = -\frac{37}{35}$ for $\frac{\pi}{2} < t < \pi$, find the value of $\tan t$.

55. Write $\sin t$ in terms of $\cos t$ for
a. t in Quadrant I. **(See Example 6)**
b. t in Quadrant III.

56. Write $\tan t$ in terms of $\sec t$ for
a. t in Quadrant II.
b. t in Quadrant IV.

57. Write $\cot t$ in terms of $\csc t$ for
 a. t in Quadrant I.
 b. t in Quadrant III.

58. Write $\cos t$ in terms of $\sin t$ for
 a. t in Quadrant II.
 b. t in Quadrant IV.

Objective 4: Apply the Periodic and Even and Odd Function Properties of Trigonometric Functions

59. Given that $\cos\frac{29\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$, determine the value of $\cos\frac{5\pi}{12}$. **(See Example 7)**

60. Given that $\sec\frac{11\pi}{12} = \sqrt{2} - \sqrt{6}$, determine the value of $\sec\left(-\frac{13\pi}{12}\right)$.

61. Given $\tan\left(-\frac{\pi}{8}\right) = 1 - \sqrt{2}$, determine the value of $\cot\frac{7\pi}{8}$.

62. Given that $\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$, determine the value of $\csc\frac{21\pi}{10}$.

For exercises 63–66, use the periodic properties of the trigonometric functions to simplify each expression to a single function of t.

63. $\sin(t + 2\pi) \cdot \cot(t + \pi)$
64. $\sin(t + 2\pi) \cdot \sec(t + 2\pi)$
65. $\csc(t + 2\pi) \cdot \cos(t + 2\pi)$
66. $\tan(t + \pi) \cdot \csc(t + 2\pi)$

For Exercises 67–74, use the even-odd and periodic properties of the trigonometric functions to simplify. (See Example 8)

67. $\csc t - 4\csc(-t)$
68. $\tan(-t) - 3\tan t$
69. $-\cot(-t + \pi) - \cot t$
70. $\sec(t + 2\pi) - \sec(-t)$
71. $-2\sin(3t + 2\pi) - 3\sin(-3t)$
72. $\cot(-3t) - 3\cot(3t + \pi)$
73. $\cos(-2t) - \cos 2t$
74. $\sec(-2t) + 3\sec 2t$

Objective 5: Approximate Trigonometric Functions on a Calculator

Use a calculator to approximate the function values. Round to 4 decimal places. (See Example 9)

75. a. $\sin(-0.15)$ **b.** $\cos\frac{2\pi}{5}$
76. a. $\cos\left(-\frac{7\pi}{11}\right)$ **b.** $\sin 0.96$
77. a. $\cot\frac{12\pi}{7}$ **b.** $\sec 5.43$
78. a. $\csc 7.58$ **b.** $\tan\frac{3\pi}{8}$

Mixed Exercises

For Exercises 79–80, evaluate $\sin t$, $\cos t$, and $\tan t$ for the real number t.

79. a. $t = \frac{2\pi}{3}$ **b.** $t = -\frac{4\pi}{3}$
80. a. $t = -\frac{11\pi}{6}$ **b.** $t = \frac{\pi}{6}$

For Exercises 81–86, identify the values of t on the interval $[0, 2\pi]$ that make the function undefined (if any).

81. $y = \sin t$
82. $y = \cot t$
83. $y = \tan t$
84. $y = \cos t$
85. $y = \csc t$
86. $y = \sec t$

For Exercises 87–92, select all properties that apply to the trigonometric function.

a. The function is even. **b.** The function is odd.
c. The period is 2π. **d.** The period is π.
e. The domain is all real numbers.
f. The domain is all real numbers excluding odd multiples of $\frac{\pi}{2}$.
g. The domain is all real numbers excluding multiples of π.

87. $f(t) = \sin(t)$
88. $f(t) = \tan(t)$
89. $f(t) = \sec(t)$
90. $f(t) = \cot(t)$
91. $f(t) = \csc(t)$
92. $f(t) = \cos(t)$

For Exercises 93–98, simplify using properties of trigonometric functions.

93. $\sin^2(t + 2\pi) + \cos^2 t + \tan^2(t + \pi)$

94. $\cot(-t) \cdot \sin(t + 2\pi) + \cos^2 t \cdot \sec(-t + 2\pi)$

95. $\sec^2\left(-\frac{\pi}{3}\right) + \tan^2\left(-\frac{\pi}{3}\right)$

96. $\sin^2\left(\frac{\pi}{6}\right) - \cos^2\left(-\frac{\pi}{6}\right)$

97. $\sec^2\left(-\frac{\pi}{4}\right) - \csc^2\left(\frac{7\pi}{6}\right)$

98. $\tan^2\left(\frac{2\pi}{3}\right) + \csc^2\left(-\frac{5\pi}{4}\right)$

99. In a small coastal town, the monthly revenue received from tourists rises and falls throughout the year. The tourist revenue peaks during the months when the town holds seafood cooking competitions. The function

$$f(t) = 5.6\cos\left(\frac{\pi}{3}t\right) + 11.2$$

represents the monthly revenue $f(t)$, in tens of thousands of dollars, for a month t, where $t = 0$ represents April.

a. Complete the table and give the period of the function. Round to 1 decimal place.

t	0	1	2	3	4	5
$f(t)$						

t	6	7	8	9	10	11
$f(t)$						

b. What are the months of peak revenue and what is the revenue for those months?

100. The fluctuating brightness of a distant star is given by the function

$$f(d) = 3.8 + 0.25\sin\left(\frac{2\pi}{3}d\right)$$

where d is the number of days and $f(d)$ is the apparent brightness.

Complete the table and give the period of the function. Round to 2 decimal places.

d	0	1	2	3	4	5
$f(d)$						

For Exercises 101–104, identify each function as even, odd, or neither.

101. $f(t) = t^2\sin t$

102. $g(t) = t\cos t$

103. $z(t) = t^3\tan t$

104. $h(t) = t^3 + \sec t$

Write About It

105. Describe the changes in $\sin t$ and $\cos t$ as t increases from 0 to $\frac{\pi}{2}$.

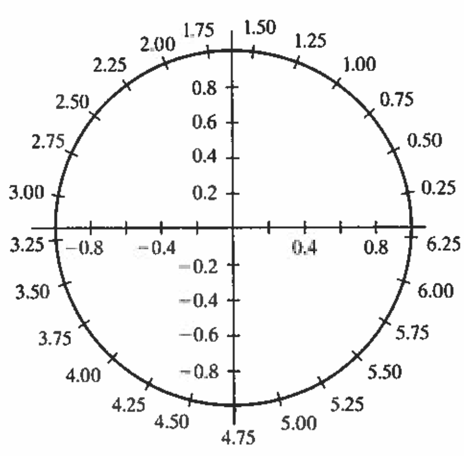
106. Describe the changes in $\cos t$ and $\sec t$ as t increases from 0 to $\frac{\pi}{2}$.

107. Explain why $-1 \le \cos t \le 1$ and $-1 \le \sin t \le 1$ for all real numbers t.

108. Do the trigonometric functions satisfy the definition of a function learned in Chapter 1? Explain your answer.

Expanding Your Skills

For Exercises 109–114, use the figure to estimate the value of (a) sin t and (b) cos t for the given value of t.



109. $t = 0.5$

110. $t = 1.25$

111. $t = 3.75$

112. $t = 2.75$

113. $t = 5$

114. $t = 3$

For Exercises 115–120, use the figure from Exercises 109–114 to approximate the solutions to the equation over the interval $[0, 2\pi)$.

115. $\sin t = 0.2$

116. $\sin t = -0.4$

117. $\cos t = -0.4$

118. $\cos t = 0.6$

119. Show that $\tan(t + \pi) = \tan t$.

120. Show that $\cot(t + \pi) = \cot t$

121. Prove that the period of $f(t) = \sin t$ is 2π.

122. Prove that the period of $f(t) = \cos t$ is 2π.

Technology Connections

123. a. Complete the table to show that $\sin t \approx t$ for values of t close to zero. Round to 5 decimal places.

t	$\sin t$	$\dfrac{\sin t}{t}$
0.2		
0.1		
0.01		
0.001		

b. What value does the ratio $\dfrac{\sin t}{t}$ seem to approach as $t \to 0$?

124. Consider the expression $\dfrac{1 - \cos t}{t}$.

a. What is the value of the numerator for $t = 0$?

b. What is the value of the denominator for $t = 0$?

c. The expression $\dfrac{1 - \cos t}{t}$ is undefined at $t = 0$. Complete the table to investigate the value of the expression *close* to $t = 0$. Round to 5 decimal places.

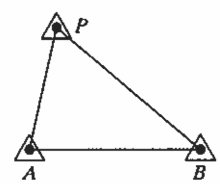
t	$\dfrac{1 - \cos t}{t}$
0.2	
0.1	
0.01	
0.001	

SECTION 4.3 Right Triangle Trigonometry

OBJECTIVES

1. Evaluate Trigonometric Functions of Acute Angles
2. Use Fundamental Trigonometric Identities
3. Use Trigonometric Functions in Applications

1. Evaluate Trigonometric Functions of Acute Angles



The science of land surveying encompasses the measurement and mapping of land using mathematics and specialized equipment and technology. Surveyors provide data relevant to the shape and contour of the Earth's surface for engineering, map-making, and construction projects.

One technique used by surveyors to map a landscape is called *triangulation*. The surveyor first measures the distance between two fixed points, A and B. Then the surveyor measures the bearing from A and the bearing from B to a third point P. From this information, the surveyor can compute the measures of the angles of the triangle formed by the points, and can use trigonometry to find the lengths of the unknown sides.

To use trigonometry in such applications, we present alternative definitions of the trigonometric functions using right triangles. Consider a right triangle with an acute angle θ. The longest side in the triangle is the hypotenuse ("hyp") and is opposite the right angle (Figure 4-27). The two legs of the triangle will be distinguished by their relative positions to θ. The leg that lies on one ray of angle θ is called the adjacent leg ("adj") and the leg that lies across the triangle from θ is called the opposite leg ("opp").

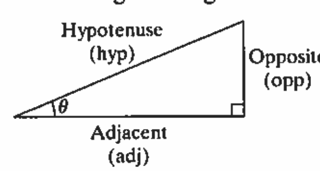


Figure 4-27

We now define the six trigonometric functions as functions whose input (or **argument**) is an acute angle θ. The output value for each function will be one of the six possible ratios of the lengths of the sides of the triangle (Table 4-4).

The definitions of the six trigonometric functions of acute angles are given in Table 4-4 along with examples based on Figure 4-28.

TIP The mnemonic device "SOH-CAH-TOA" may help you remember the ratios for $\sin\theta$, $\cos\theta$, and $\tan\theta$, respectively.

Sine: Opp over Hyp
Cosine: Adj over Hyp
Tangent: Opp over Adj

Table 4-4

Definition of Trigonometric Functions of Acute Angles

Function Name	Definition	Example
sine	$\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$	$\sin\theta = \dfrac{3\text{ ft}}{5\text{ ft}} = \dfrac{3}{5}$
cosine	$\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$	$\cos\theta = \dfrac{4\text{ ft}}{5\text{ ft}} = \dfrac{4}{5}$
tangent	$\tan\theta = \dfrac{\text{opp}}{\text{adj}}$	$\tan\theta = \dfrac{3\text{ ft}}{4\text{ ft}} = \dfrac{3}{4}$
cosecant	$\csc\theta = \dfrac{\text{hyp}}{\text{opp}}$	$\csc\theta = \dfrac{5\text{ ft}}{3\text{ ft}} = \dfrac{5}{3}$
secant	$\sec\theta = \dfrac{\text{hyp}}{\text{adj}}$	$\sec\theta = \dfrac{5\text{ ft}}{4\text{ ft}} = \dfrac{5}{4}$
cotangent	$\cot\theta = \dfrac{\text{adj}}{\text{opp}}$	$\cot\theta = \dfrac{4\text{ ft}}{3\text{ ft}} = \dfrac{4}{3}$

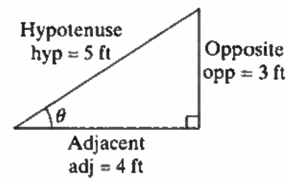


Figure 4-28

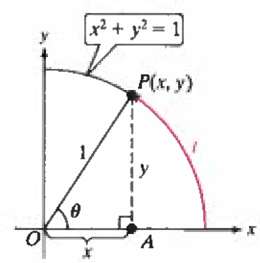


Figure 4-29

The notation $\sin\theta$ is read as "sine of theta" (likewise, $\cos\theta$ is read as "cosine of theta" and so on). Also notice that the output value of a trigonometric function is unitless because the common units of length "cancel" within each ratio.

The definitions of the trigonometric functions given in Table 4-4 are consistent with the definitions based on the unit circle from Section 4.2. For example, consider a real number $0 < t < \dfrac{\pi}{2}$ that corresponds to the point $P(x, y)$ on the unit circle (Figure 4-29). Suppose that we drop a perpendicular line segment from P to the x-axis at point A. Then ΔOAP is a right triangle. The lengths of the legs are x and y, and the hypotenuse is 1 unit. Furthermore, the radian measure of $\angle AOP$ is equal to t. If we let θ represent the degree measure of $\angle AOP$, we have

<u>**Unit circle definition**</u>	<u>**Right triangle definition**</u>
$\sin t = y$	$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{y}{1} = y$
$\cos t = x$	$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{x}{1} = x$
$\tan t = \dfrac{y}{x}$	$\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{y}{x}$

It is also very important to note that the values of the trigonometric functions depend only on the measure of the angle, not the size of the triangle. For any given acute angle θ, a series of right triangles can be formed by drawing a line segment perpendicular to the initial side of θ with endpoints on the initial and terminal sides of θ (Figure 4-30). Triangles ΔABC and ΔADE are similar triangles with common angle θ. Therefore, the corresponding sides are proportional, and the value of each trigonometric function of θ will be the same regardless of which triangle is used. For example, $\sin\theta = \frac{x}{b} = \frac{y}{d}$.

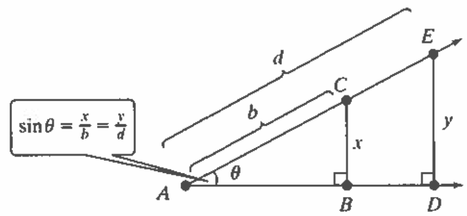


Figure 4-30

In Example 1, we find the values of the six trigonometric functions of an acute angle within a right triangle where the lengths of two legs are given.

EXAMPLE 1 Evaluating Trigonometric Functions by First Applying the Pythagorean Theorem

Suppose that a right triangle has legs of length 4 cm and 7 cm. Evaluate the six trigonometric functions for the smaller acute angle.

Solution:

A right triangle ΔABC meeting the conditions of this example is drawn in Figure 4-31. Since $\angle B$ is opposite the shorter leg, $\angle B$ is the smaller acute angle.

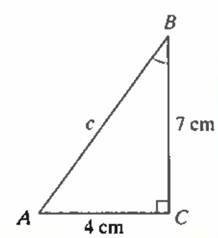


Figure 4-31

To find the values of all six trigonometric functions, we also need to know the length of the hypotenuse. The hypotenuse is always opposite the right angle.

$c^2 = 4^2 + 7^2$ Apply the Pythagorean theorem.

$c^2 = 16 + 49$ Simplify terms with exponents.

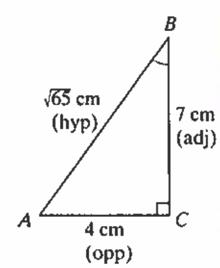
$c^2 = 65$ Since c represents the length of a side of a triangle, take the positive square root of 65.

$c = \sqrt{65}$

Avoiding Mistakes

In Example 1, the opposite (opp) and adjacent (adj) legs are labeled in the triangle relative to angle B.

Relative to angle B, the 4-cm side is the opposite side. The 7-cm side is the adjacent leg, and the hypotenuse is $c = \sqrt{65}$.



$$\sin B = \frac{\text{opp}}{\text{hyp}} = \frac{4}{\sqrt{65}} = \frac{4}{\sqrt{65}} \cdot \frac{\sqrt{65}}{\sqrt{65}} = \frac{4\sqrt{65}}{65}$$

$$\cos B = \frac{\text{adj}}{\text{hyp}} = \frac{7}{\sqrt{65}} = \frac{7}{\sqrt{65}} \cdot \frac{\sqrt{65}}{\sqrt{65}} = \frac{7\sqrt{65}}{65}$$

$$\tan B = \frac{\text{opp}}{\text{adj}} = \frac{4}{7} \qquad \cot B = \frac{\text{adj}}{\text{opp}} = \frac{7}{4}$$

$$\csc B = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{65}}{4} \qquad \sec B = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{65}}{7}$$

Skill Practice 1 Suppose that a right triangle ΔABC has legs of length 6 cm and 3 cm. Evaluate the six trigonometric functions for angle B, where angle B is the larger acute angle.

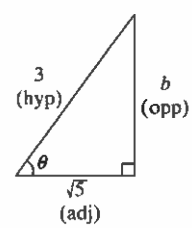
In Example 1, we were given the lengths of the sides of a right triangle and asked to find the values of the six trigonometric functions. This process can be reversed. Given the value of one trigonometric function of an acute angle, we can find the lengths of the sides of a representative triangle, and thus the values of the other trigonometric functions.

EXAMPLE 2 Using a Known Value of a Trigonometric Function to Determine Other Function Values

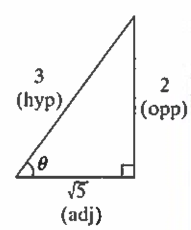
Suppose that $\cos\theta = \dfrac{\sqrt{5}}{3}$ for the acute angle θ. Evaluate $\tan\theta$.

Solution:

Since $\cos\theta$ is the ratio of the length of the leg adjacent to θ and the length of the hypotenuse, it is convenient to construct a triangle with adjacent leg $\sqrt{5}$ units and hypotenuse 3 units. Using the Pythagorean theorem, we can find the length of the opposite leg.



$(\sqrt{5})^2 + b^2 = (3)^2$	Apply the Pythagorean theorem.
$5 + b^2 = 9$	Simplify terms with exponents.
$b^2 = 4$	
$b = 2$	Take the positive square root of 4.
$\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{2}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{2\sqrt{5}}{5}$	Once the proper ratio is identified for $\tan\theta$, rationalize the denominator.



Skill Practice 2 Suppose that $\sin\theta = \dfrac{3}{4}$ for the acute angle θ. Evaluate $\sec\theta$.

In Example 3, we find the sine, cosine, and tangent of 45° by using a right triangle approach.

EXAMPLE 3 Determine the Trigonometric Function Values of a 45° Angle

Evaluate $\sin 45°$, $\cos 45°$, and $\tan 45°$.

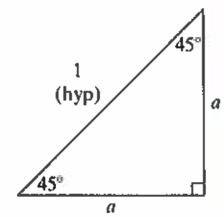
Answers

1. $\sin B = \dfrac{2\sqrt{5}}{5}$, $\cos B = \dfrac{\sqrt{5}}{5}$, $\tan B = 2$, $\csc B = \dfrac{\sqrt{5}}{2}$, $\sec B = \sqrt{5}$, $\cot B = \dfrac{1}{2}$

2. $\sec\theta = \dfrac{4\sqrt{7}}{7}$

Solution:

A right triangle with an acute angle of 45° must have a second acute angle of 45° because the sum of the angles must equal 180°. This is called an isoceles right triangle or a 45°-45°-90° triangle. Since all such triangles are similar, we can choose one of the sides to be of arbitrary length. We have chosen the hypotenuse to be 1 unit. The sides opposite the 45° angles are equal in length and have been labeled a.



$$a^2 + a^2 = 1^2$$ Apply the Pythagorean theorem.

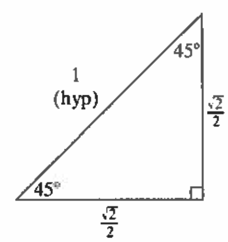
$$2a^2 = 1$$

$$a^2 = \frac{1}{2}$$

$$a = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$ Take the positive square root of $\frac{1}{2}$. Rationalize the denominator.

TIP The values of $\sin 45°$, $\cos 45°$, and $\tan 45°$ found in Example 3 are consistent with the values of $\sin\frac{\pi}{4}$, $\cos\frac{\pi}{4}$, and $\tan\frac{\pi}{4}$ found by using the unit circle.

$$\sin 45° = \frac{\text{opp}}{\text{hyp}} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$$



$$\cos 45° = \frac{\text{adj}}{\text{hyp}} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{\text{opp}}{\text{adj}} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

Skill Practice 3 Evaluate $\csc 45°$, $\sec 45°$, and $\cot 45°$.

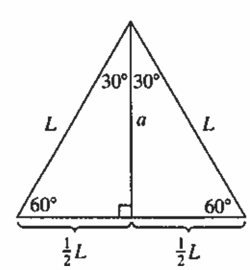
A right triangle with a 30° angle will also have a 60° angle so that the sum of the angular measures is 180°. Such a triangle is called a 30°-60°-90° triangle. Interestingly, as we will show in Example 4, the length of the shorter leg of a 30°-60°-90° triangle is always one-half the length of the hypotenuse.

EXAMPLE 4 Determine the Trigonometric Function Values of a 60° Angle

Evaluate $\sin 60°$, $\cos 60°$, and $\tan 60°$.

Solution:

Suppose that we start with an equilateral triangle with sides of length L. All angles in the triangle are equal to 60°. Any altitude a of this triangle bisects a 60° angle as well as the opposite side. Thus, we can create two congruent right triangles with legs of length a and $\frac{1}{2}L$ and hypotenuse of length L.



Answer

3. $\csc 45° = \sqrt{2}$, $\sec 45° = \sqrt{2}$, and $\cot 45° = 1$

Use the Pythagorean theorem to find the altitude a in terms of L.

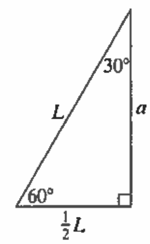
$$\left(\frac{1}{2}L\right)^2 + a^2 = L^2 \qquad \text{Apply the Pythagorean theorem.}$$

$$\frac{1}{4}L^2 + a^2 = L^2 \qquad \text{Simplify terms with exponents.}$$

$$a^2 = \frac{3}{4}L^2 \qquad \text{Combine like terms.}$$

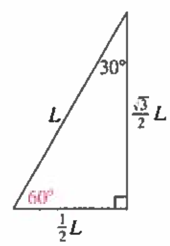
$$a = \sqrt{\frac{3}{4}}L \qquad \text{Take the positive square root.}$$

$$a = \frac{\sqrt{3}}{2}L \qquad \text{Simplify the radical.}$$



$$\sin 60° = \frac{\text{opp}}{\text{hyp}} = \frac{\frac{\sqrt{3}}{2}L}{L} = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{\text{adj}}{\text{hyp}} = \frac{\frac{1}{2}L}{L} = \frac{1}{2}$$

$$\tan 60° = \frac{\text{opp}}{\text{adj}} = \frac{\frac{\sqrt{3}}{2}L}{\frac{1}{2}L} = \left(\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{2}{1}\right) = \sqrt{3}$$



Skill Practice 4 Evaluate $\sin 30°$, $\cos 30°$, and $\tan 30°$.

TIP As a memory device, note the 1-2-3 pattern in the numerator for the sine function for the special angles.

$$\sin 30° = \frac{\sqrt{1}}{2}$$
$$\sin 45° = \frac{\sqrt{2}}{2}$$
$$\sin 60° = \frac{\sqrt{3}}{2}$$

For cosine, the order is reversed, 3-2-1.

$$\cos 30° = \frac{\sqrt{3}}{2}$$
$$\cos 45° = \frac{\sqrt{2}}{2}$$
$$\cos 60° = \frac{\sqrt{1}}{2}$$

From Examples 3 and 4, we can summarize the trigonometric function values of the "special" angles 30°, 45°, and 60°, or equivalently $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ radians (Table 4-5).

Table 4-5

Trigonometric Function Values of Special Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$30° = \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$45° = \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60° = \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

To help you remember the values of the trigonometric functions for 30°, 45°, and 60°, you can refer to the first quadrant of the unit circle. The values $t = \frac{\pi}{6}$, $t = \frac{\pi}{4}$, and $t = \frac{\pi}{3}$ correspond to the central angles $\theta = 30°$, $\theta = 45°$, and $\theta = 60°$, respectively (Figure 4-32).

Answer

4. $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$, and $\tan 30° = \frac{\sqrt{3}}{3}$

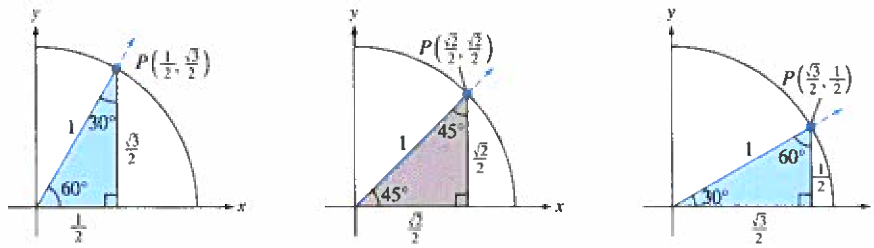


Figure 4-32

EXAMPLE 5 **Simplifying Expressions Involving Trigonometric Functions of Special Angles**

Simplify the expressions.

a. $\tan 60° - \tan 30°$ **b.** $2\sin\frac{\pi}{6}\cos\frac{\pi}{6}$

Solution:

a. $\tan 60° - \tan 30° = \sqrt{3} - \frac{\sqrt{3}}{3}$ Evaluate $\tan 60°$ and $\tan 30°$.

$= \frac{3}{3} \cdot \frac{\sqrt{3}}{1} - \frac{\sqrt{3}}{3}$ Write the first term with a common denominator of 3.

$= \frac{2\sqrt{3}}{3}$ Combine like terms.

b. $2\sin\frac{\pi}{6}\cos\frac{\pi}{6} = 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$ Evaluate $\sin\frac{\pi}{6}$ and $\cos\frac{\pi}{6}$.

$= \frac{\sqrt{3}}{2}$ Multiply.

Skill Practice 5 Simplify the expressions.

a. $\cot 60° - \cot 30°$ **b.** $\sin\frac{\pi}{3}\cos\frac{\pi}{6} + \cos\frac{\pi}{3}\sin\frac{\pi}{6}$

2. Use Fundamental Trigonometric Identities

In Section 4.2, we observed several relationships among the trigonometric functions that follow directly from the definitions of the functions. The reciprocal and quotient identities also follow from the right triangle defintions of trigonometric functions of acute angles (Table 4-6). For example, since $\sin\theta = \frac{\text{opp}}{\text{hyp}}$ and $\csc\theta = \frac{\text{hyp}}{\text{opp}}$ for an acute angle θ, we know that $\sin\theta$ and $\csc\theta$ are reciprocals.

Answers

5. a. $-\frac{2\sqrt{3}}{3}$ **b.** 1

Table 4-6

Reciprocal and Quotient Identities	
$\csc\theta = \dfrac{1}{\sin\theta}$ or $\sin\theta = \dfrac{1}{\csc\theta}$	$\sin\theta$ and $\csc\theta$ are reciprocals.
$\sec\theta = \dfrac{1}{\cos\theta}$ or $\cos\theta = \dfrac{1}{\sec\theta}$	$\cos\theta$ and $\sec\theta$ are reciprocals.
$\cot\theta = \dfrac{1}{\tan\theta}$ or $\tan\theta = \dfrac{1}{\cot\theta}$	$\tan\theta$ and $\cot\theta$ are reciprocals.
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\tan\theta$ is the ratio of $\sin\theta$ and $\cos\theta$.
$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	$\cot\theta$ is the ratio of $\cos\theta$ and $\sin\theta$.

We can also show that the Pythagorean identities follow from the right triangle definitions of the trigonometric functions of an acute angle θ. Consider a right triangle with legs of length a and b and hypotenuse c (Figure 4-33). From the Pythagorean theorem, we know that $a^2 + b^2 = c^2$. Dividing both sides by the positive real number c^2 we have

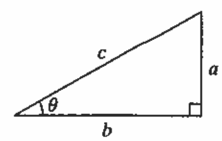


Figure 4-33

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2} \qquad \text{Divide by } c^2.$$

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \left(\frac{c}{c}\right)^2 \qquad \text{Rewrite each term as the power of a quotient.}$$

$$(\sin\theta)^2 + (\cos\theta)^2 = 1 \qquad \text{From Figure 4-33, } \sin\theta = \frac{a}{c} \text{ and } \cos\theta = \frac{b}{c}.$$

$$\sin^2\theta + \cos^2\theta = 1$$

Table 4-7 summarizes this and two other Pythagorean identities. The second and third relationships can be derived by dividing both sides of the equation $a^2 + b^2 = c^2$ by b^2 and a^2, respectively.

Table 4-7

Pythagorean Identities		
$\sin^2\theta + \cos^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$	$1 + \cot^2\theta = \csc^2\theta$

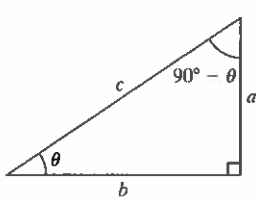


Figure 4-34

Visualing the ratios of the lengths of the sides of a right triangle can also help us understand the cofunction identities.

The two acute angles in a right triangle are complementary because the sum of their measures is 90°. Symbolically, the complement of angle θ is $(90° - \theta)$. In Figure 4-34, notice that side a is opposite angle θ but is the adjacent leg to angle $(90° - \theta)$. Likewise, side b is adjacent to angle θ but opposite angle $(90° - \theta)$. From these relationships, we have

$$\sin\theta = \cos(90° - \theta) = \frac{a}{c}$$

$$\tan\theta = \cot(90° - \theta) = \frac{a}{b}$$

$$\sec\theta = \csc(90° - \theta) = \frac{c}{b}$$

Notice that the sine of angle θ equals the cosine of the complement $(90° - \theta)$. For this reason, the sine and cosine functions are called *cofunctions*. More generally, for an acute angle θ, two trigonometric functions f and g are **cofunctions** if

$$f(\theta) = g(90° - \theta) \text{ and } g(\theta) = f(90° - \theta)$$

TIP Cosine means "complement's sine."
Cotangent means "complement's tangent."
Cosecant means "complement's secant."

Cofunction Identities

Cofunctions of complementary angles are equal.

$\sin\theta = \cos(90° - \theta)$	$\cos\theta = \sin(90° - \theta)$	Sine and cosine are cofunctions.
$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$	
$\tan\theta = \cot(90° - \theta)$	$\cot\theta = \tan(90° - \theta)$	Tangent and cotangent are cofunctions.
$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$	
$\sec\theta = \csc(90° - \theta)$	$\csc\theta = \sec(90° - \theta)$	Secant and cosecant are cofunctions.
$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$	

EXAMPLE 6 Using the Cofunction Identities

For each function value, find a cofunction with the same value.

a. $\cot 15° = 2 + \sqrt{3}$ **b.** $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Solution:

a. The complement of 15° is $(90° - 15°) = 75°$.
Cotangent and tangent are cofunctions.
Therefore, $\cot 15° = \tan 75° = 2 + \sqrt{3}$.

b. The complement of $\frac{\pi}{6}$ is $\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \left(\frac{3\pi}{6} - \frac{\pi}{6}\right) = \frac{2\pi}{6} = \frac{\pi}{3}$.
Cosine and sine are cofunctions.
Therefore, $\cos\frac{\pi}{6} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Skill Practice 6 For each function value, find a cofunction with the same value.

a. $\tan 22.5° = \sqrt{2} - 1$ **b.** $\sec\frac{\pi}{3} = 2$

3. Use Trigonometric Functions in Applications

In many applications of right triangle trigonometry, angles are measured relative to an imaginary horizontal line of reference. An angle of elevation is an angle measured upward from a horizontal line of reference. An angle of depression is an angle measured downward from a horizontal line of reference (Figure 4-35).

Answers

6. a. $\cot 67.5° = \sqrt{2} - 1$

b. $\csc\frac{\pi}{6} = 2$

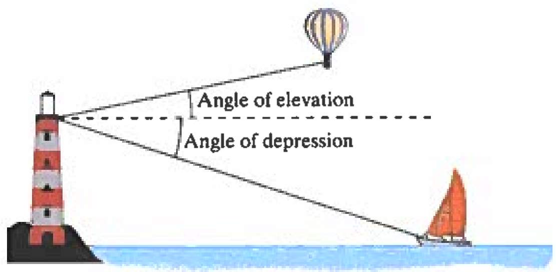
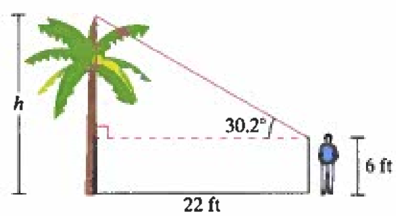


Figure 4-35

EXAMPLE 7 Using Trigonometry to Find the Height of a Tree

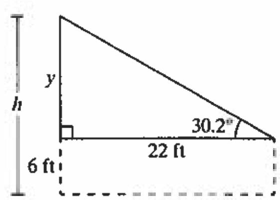
Palm trees are easy to transplant relative to similarly sized broad-leaved trees. At one tree farm, palm trees are harvested once they reach a height of 20 ft. Suppose a farm worker determines that the distance along the ground from his position to the base of a palm tree is 22 ft. He then uses an instrument called a clinometer held at an eye level of 6 ft to measure the angle of elevation to the top of the tree as 30.2°. Is the tree tall enough to harvest?



Solution:

To find the height of the tree, we will use the right triangle to find the vertical leg y and then add 6 ft.

Relative to the 30.2° angle, we know the adjacent side is 22 ft and we want to know the length of the opposite side y. The two trigonometric functions that are defined by the opposite and adjacent legs are tangent and cotangent. To solve for y, we can use the relationship



$\tan 30.2° = \frac{y}{22}$ or $\cot 30.2° = \frac{22}{y}$. Using the tangent function we have

$\tan 30.2° = \frac{y}{22}$	We use the tangent function because it is easily approximated on a calculator.
$y = 22\tan 30.2°$	Multiply both sides by 22.
$y \approx 12.8$ ft	Use a calculator to approximate $22\tan 30.2°$.
$h \approx 12.8 \text{ ft} + 6 \text{ ft}$	To find the height of the tree, add 6 ft.
≈ 18.8 ft	The tree is approximately 18.8 ft and is not ready to harvest.

Skill Practice 7 Suppose that a farm worker determines that the distance along the ground from her position to the base of a palm tree is 16 ft. She measures the angle of elevation from an eye level of 5 ft to the top of the tree as 45.9°. Is the tree tall enough to harvest (at least 20 ft tall)?

Point of Interest

A variety of instruments are used to measure angles. Historically, quadrants and sextants were used for early celestial navigation. Theodolites (or transits) are used for surveying, and clinometers are used in construction to measure slope.

Nineteenth-century sextant, courtesy of Geoff Miller

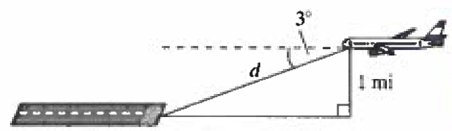
TIP To approximate tan30.2° on a calculator, be sure that the calculator is in degree mode.

Answer

7. Yes; the tree is approximately 21.5 ft.

EXAMPLE 8 Using Trigonometry in Aeronautical Science

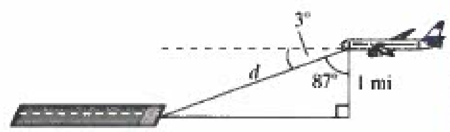
A pilot flying an airplane at an altitude of 1 mi sights a point at the end of a runway. The angle of depression is 3°. What is the distance d from the plane to the point on the runway? Round to the nearest tenth of a mile.



Solution:

The complement of the 3° angle of depression is the acute angle 87° within the right triangle.

Relative to the 87° angle, we know that the adjacent side is 1 mi and we want to find the length of the hypotenuse. The two trigonometric functions that are defined by the adjacent leg and hypotenuse are the cosine and secant functions.



Using the relationship $\cos 87° = \frac{1}{d}$, we have $d = \frac{1}{\cos 87°} \approx 19.1$.

Therefore, the plane is approximately 19.1 mi from the end of the runway.

Skill Practice 8 If a 15-ft ladder is leaning against a wall at an angle of 62° with the ground, how high up the wall will the ladder reach? Round to the nearest tenth of a foot.

Answer
8. 13.2 ft

SECTION 4.3 Practice Exercises

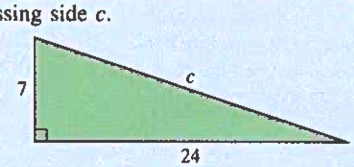
Prerequisite Review

R.1. Simplify the radical. $\sqrt{45}$

R.2. Rationalize the denominator. $\frac{14}{\sqrt{7}}$

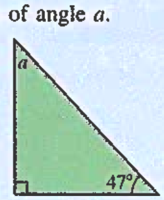
R.3. Rationalize the denominator. $\frac{9}{\sqrt{18}}$

R.4. Use the Pythagorean theorem to find the length of the missing side c.



R.5. In a right triangle, one leg measures 24 ft and the hypotenuse measures 26 ft. Find the length of the missing side.

R.6. Find the measure of angle a.



Concept Connections

1. In a right triangle with an acute angle θ, the longest side in the triangle is the ______________ and is opposite the ______________ angle.

2. The leg of a right triangle that lies on one ray of angle θ is called the ______________ leg, and the leg that lies across the triangle from θ is called the ______________ leg.

3. The mnemonic device "SOH-CAH-TOA" stands for the ratios

$$\sin\theta = \frac{\square}{\square},\ \cos\theta = \frac{\square}{\square},\ \text{and } \tan\theta = \frac{\square}{\square}.$$

4. Complete the reciprocal and quotient identities.

$$\csc\theta = \frac{1}{\square},\ \sec\theta = \frac{1}{\square},\ \cot\theta = \frac{1}{\square} \text{ or } \frac{\square}{\square}$$

5. Given the lengths of two sides of a right triangle, we can find the length of the third side by using the ______________ theorem.

6. An ______________ right triangle is a right triangle in which the two legs are of equal length. The two acute angles in this triangle each measure ______________°.

7. The length of the shorter leg of a 30°–60°–90° triangle is always ______________ the length of the hypotenuse.

8. For the six trigonometric functions $\sin\theta$, $\cos\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$, and $\cot\theta$, identify the three reciprocal pairs.

9. $\tan\theta$ is the ______________ of $\sin\theta$ and $\cos\theta$.

10. Complete the Pythagorean identities.

$$\sin^2\theta + \_\_\_\_ = 1,\ \_\_\_\_ + 1 = \sec^2\theta,\ 1 + \cot^2\theta = \_\_\_\_$$

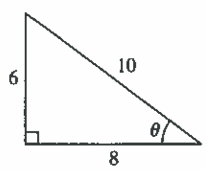
11. The two acute angles in a right triangle are complementary because the sum of their measures is ______________°.

12. The sine of angle θ equals the cosine of ______________. For this reason, the sine and cosine functions are called ______________.

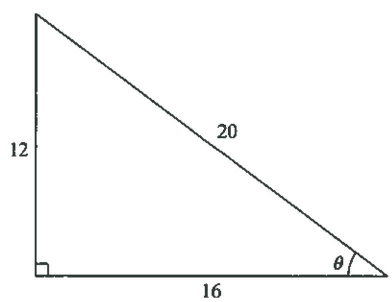
Objective 1: Evaluate Trigonometric Functions of Acute Angles

For Exercises 13–14, find the exact values of the six trigonometric functions for angle θ.

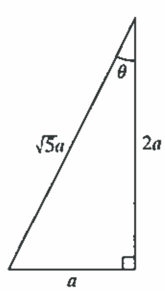
13. a.



b.



14. a.



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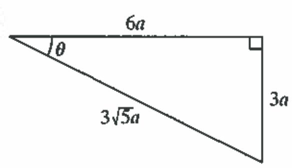
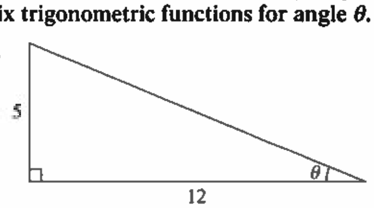
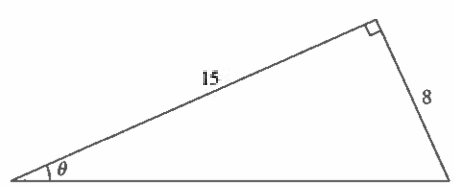


For Exercises 15–18, first use the Pythagorean theorem to find the length of the missing side. Then find the exact values of the six trigonometric functions for angle θ. (See Example 1)

15.



16.



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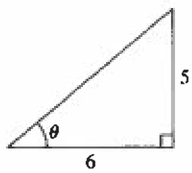

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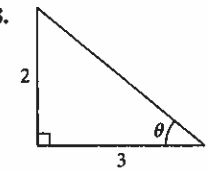


For Exercises 19–22, first use the Pythagorean theorem to find the length of the missing side of the right triangle. Then find the exact values of the six trigonometric functions for the angle θ opposite the shortest side. (See Example 1)

19. Leg $= 2$ cm, leg $= 6$ cm

20. Leg $= 3$ cm, leg $= 15$ cm

21. Leg $= 2\sqrt{7}$ m, hypotenuse $= 2\sqrt{11}$ m

22. Leg $= 5\sqrt{3}$ in., hypotenuse $= 2\sqrt{21}$ in.

In Exercises 23–24, given the value of one trigonometric function of an acute angle θ, find the values of the remaining five trigonometric functions of θ. (See Example 2)

23. $\tan\theta = \dfrac{4}{7}$

24. $\cos\theta = \dfrac{\sqrt{10}}{10}$

For Exercises 25–30, assume that θ is an acute angle. (See Example 2)

25. If $\cos\theta = \dfrac{\sqrt{21}}{7}$, find $\csc\theta$.

26. If $\sin\theta = \dfrac{\sqrt{17}}{17}$, find $\cot\theta$.

27. If $\sec\theta = \dfrac{3}{2}$, find $\sin\theta$.

28. If $\csc\theta = 3$, find $\cos\theta$.

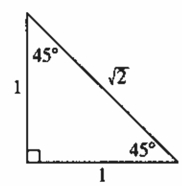
29. If $\tan\theta = \dfrac{\sqrt{15}}{9}$, find $\cos\theta$.

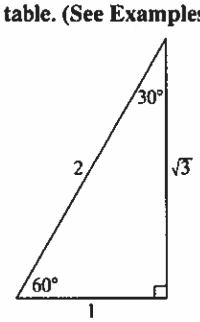
30. If $\cot\theta = \dfrac{\sqrt{3}}{2}$, find $\cos\theta$.

For Exercise 31, use the isosceles right triangle and the 30°–60°–90° triangle to complete the table. (See Examples 3–4)

31.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$30° = \dfrac{\pi}{6}$						
$45° = \dfrac{\pi}{4}$						
$60° = \dfrac{\pi}{3}$						





32. a. Evaluate $\sin 60°$.

b. Evaluate $\sin 30° + \sin 30°$.

c. Are the values in parts (a) and (b) the same?

For Exercises 33–38, find the exact value of each expression without the use of a calculator. (See Example 5)

33. $\sin\dfrac{\pi}{4} \cdot \cot\dfrac{\pi}{3}$

34. $\tan\dfrac{\pi}{6} \cdot \csc\dfrac{\pi}{4}$

35. $3\cos\dfrac{\pi}{3} + 4\sin\dfrac{\pi}{6}$

36. $2\cos\dfrac{\pi}{6} - 5\sin\dfrac{\pi}{3}$

37. $\csc^2 60° - \sin^2 45°$

38. $\cos^2 45° + \tan^2 60°$

Objective 2: Use Fundamental Trigonometric Identities

For Exercises 39–44, determine whether the statement is true or false for an acute angle θ by using the fundamental identities. If the statement is false, provide a counterexample by using a special angle: $\dfrac{\pi}{3}, \dfrac{\pi}{4}$, or $\dfrac{\pi}{6}$.

39. $\sin\theta \cdot \tan\theta = 1$

40. $\cos^2\theta \cdot \tan^2\theta = \sin^2\theta$

41. $\sin^2\theta + \tan^2\theta + \cos^2\theta = \sec^2\theta$

42. $\csc\theta \cdot \cot\theta = \sec\theta$

43. $\dfrac{1}{\tan\theta} \cdot \cot\theta + 1 = \csc^2\theta$

44. $\sin\theta \cdot \cos\theta \cdot \tan\theta + 1 = \cos^2\theta$

For Exercises 45–50, given the function value, find a cofunction of another angle with the same value. (See Example 6)

45. $\tan 75° = 2 + \sqrt{3}$

46. $\sec\dfrac{\pi}{12} = \sqrt{6} - \sqrt{3}$

47. $\csc\dfrac{\pi}{3} = \dfrac{2\sqrt{3}}{3}$

48. $\sin 15° = \dfrac{\sqrt{6} - \sqrt{2}}{4}$

49. $\cos\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$

50. $\cot\dfrac{\pi}{6} = \sqrt{3}$

For Exercises 51–54, use a calculator to approximate the function values to 4 decimal places. Be sure that your calculator is in the correct mode.

51. a. $\cos 48.2°$ **b.** $\sin 2°55'42''$ **c.** $\tan\frac{3\pi}{8}$

52. a. $\sin 12.6°$ **b.** $\tan 19°36'18''$ **c.** $\cos\left(\frac{5\pi}{22}\right)$

53. a. $\csc 39.84°$ **b.** $\sec\frac{\pi}{18}$ **c.** $\cot 0.8$

54. a. $\cot 18.46°$ **b.** $\csc\frac{2\pi}{9}$ **c.** $\sec 1.25$

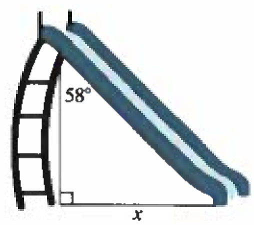
Objective 3: Use Trigonometric Functions in Applications

55. An observer at the top of a 462-ft cliff measures the angle of depression from the top of a cliff to a point on the ground to be 5°. What is the distance from the base of the cliff to the point on the ground? Round to the nearest foot. **(See Example 7)**

56. A lamppost casts a shadow of 18 ft when the angle of elevation of the Sun is 33.7°. How high is the lamppost? Round to the nearest foot.

57. A 30-ft boat ramp makes a 7° angle with the water. What is the height of the ramp above the water at the ramp's highest point? Round to the nearest tenth of a foot. **(See Example 8)**

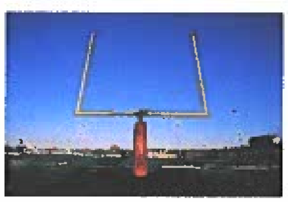
58. A backyard slide is designed for a child's playground. If the top of the 10-ft slide makes an angle of 58° from the vertical, how far out from the base of the steps will the slide extend? Round to the nearest tenth of a foot.



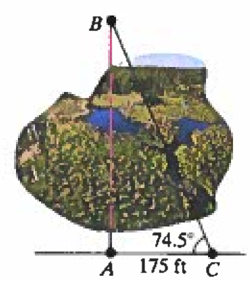
59. The Lookout Mountain Incline Railway, located in Chattanooga, Tennessee, is 4972 ft long and runs up the side of the mountain at an average incline of 17°. What is the gain in altitude? Round to the nearest foot.

60. A 12-ft ladder leaning against a house makes a 64° angle with the ground. Will the ladder reach a window sill that is 10.5 ft up from the base of the house?

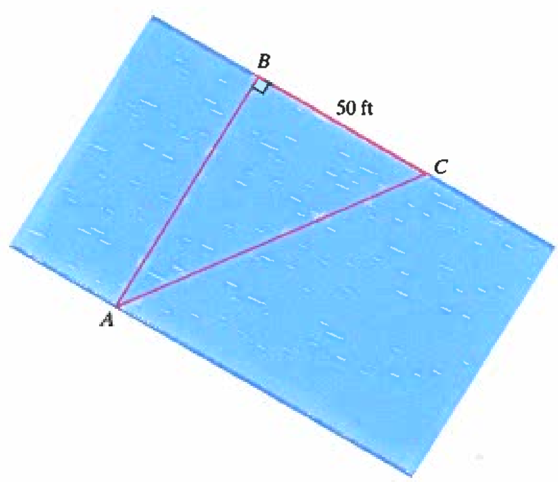
61. According to National Football League (NFL) rules, all crossbars on goalposts must be 10 ft from the ground. However, teams are allowed some freedom on how high the vertical posts on each end may extend, as long as they measure at least 30 ft. A measurement on an NFL field taken 100 yd from the goalposts yields an angle of 7.8° from the ground to the top of the posts. If the crossbar is 10 ft from the ground, do the goalposts satisfy the NFL rules?



62. A zip line is to be built between two towers labeled A and B across a wetland area. To approximate the distance of the zip line, a surveyor marks a third point C, a distance of 175 ft from one end of the zip line and perpendicular to the zip line. The measure of $\angle ACB$ is 74.5°. How long is the zip line? Round to the nearest foot.



63. To determine the width of a river from point A to point B, a surveyor walks downriver 50 ft along a line perpendicular to $\overline{AB}$ to a new position at point C. The surveyor determines that the measure of $\angle ACB$ is 60°. Find the exact width of the river from point A to point B.



Mixed Exercises

For Exercises 64–68, use the fundamental trigonometric identities as needed.

64. Given that $\sin x^\circ \approx 0.3746$, approximate the given function values. Round to 4 decimal places.

a. $\cos(90 - x)^\circ$ **b.** $\cos x^\circ$ **c.** $\tan x^\circ$

d. $\sin(90 - x)^\circ$ **e.** $\cot(90 - x)^\circ$ **f.** $\csc x^\circ$

65. Given that $\cos x \approx 0.6691$, approximate the given function values. Round to 4 decimal places.

a. $\sin x$ **b.** $\sin\left(\frac{\pi}{2} - x\right)$ **c.** $\tan x$

d. $\cos\left(\frac{\pi}{2} - x\right)$ **e.** $\sec x$ **f.** $\cot\left(\frac{\pi}{2} - x\right)$

66. Given that $\cos\frac{\pi}{12} = \frac{\sqrt{2} + \sqrt{6}}{4}$, give the exact function values.

a. $\sin\frac{5\pi}{12}$ **b.** $\sin\frac{\pi}{12}$ **c.** $\sec\frac{\pi}{12}$

67. Given that $\tan 36^\circ = \sqrt{5 - 2\sqrt{5}}$, give the exact function values.

a. $\sec 36^\circ$ **b.** $\csc 54^\circ$ **c.** $\cot 54^\circ$

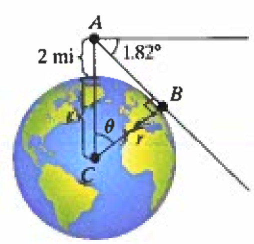
68. Simplify each expression to a single trigonometric function.

a. $\sec\theta \cdot \tan\left(\frac{\pi}{2} - \theta\right)$ **b.** $\cot^2\theta \cdot \csc(90^\circ - \theta) \cdot \sin\theta$

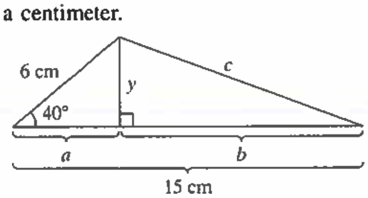
69. A scenic overlook along the Pacific Coast Highway in Big Sur, California, is 280 ft above sea level. A 6-ft-tall hiker standing at the overlook sees a sailboat and estimates the angle of depression to be 30°. Approximately how far off the coast is the sailboat? Round to the nearest foot.

70. An airplane traveling 400 mph at a cruising altitude of 6.6 mi begins its descent. If the angle of descent is 2° from the horizontal, determine the new altitude after 15 min. Round to the nearest tenth of a mile.

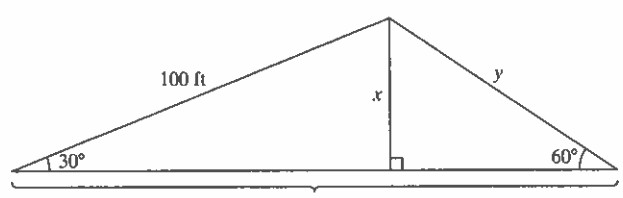
71. A scientist standing at the top of a mountain 2 mi above sea level measures the angle of depression to the ocean horizon to be 1.82°. Use this information to approximate the radius of the Earth to the nearest mile. (*Hint*: The line of sight $\overleftrightarrow{AB}$ is tangent to the Earth and forms a right angle with the radius at the point of tangency.)



72. Find the lengths a, b, y, and c. Round to the nearest tenth of a centimeter.

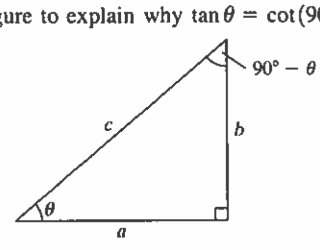


73. Find the exact lengths x, y, and z.



Write About It

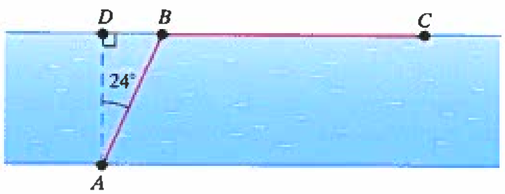
74. Use the figure to explain why $\tan\theta = \cot(90^\circ - \theta)$.



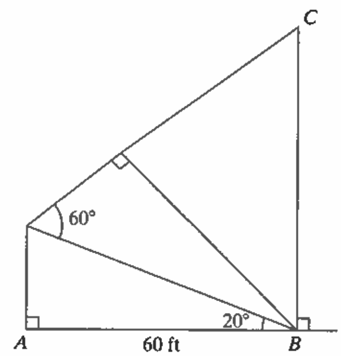
75. Explain the difference between an angle of elevation and an angle of depression.

Expanding Your Skills

76. An athlete is in a boat at point A, $\frac{1}{4}$ mi from the nearest point D on a straight shoreline. She can row at a speed of 3 mph and run at a speed of 6 mph. Her planned workout is to row to point D and then run to point C farther down the shoreline. However, the current pushes her at an angle of 24° from her original path so that she comes ashore at point B, 2 mi from her final destination at point C. How many minutes will her trip take? Round to the nearest minute.



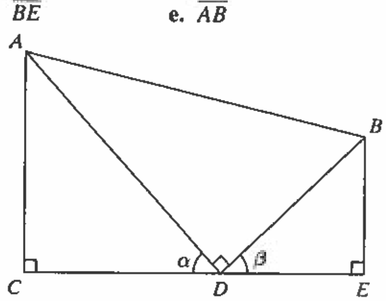
77. Given that the distance from A to B is 60 ft, find the distance from B to C. Round to the nearest foot.



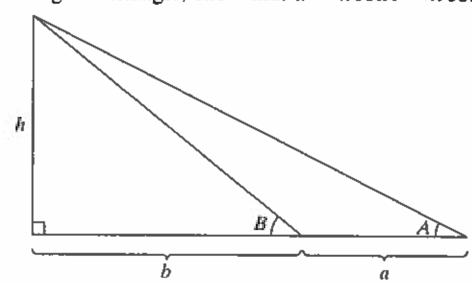
78. In the figure, $\overline{CD} = 15$, $\overline{DE} = 8$, $\tan\alpha = \dfrac{4}{3}$, and $\sin\beta = \dfrac{3}{5}$. Find the lengths of

a. $\overline{AC}$ **b.** $\overline{AD}$ **c.** $\overline{DB}$

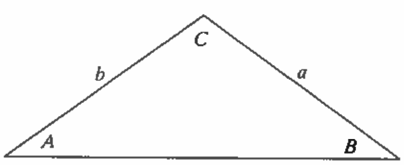
d. $\overline{BE}$ **e.** $\overline{AB}$



79. For the given triangle, show that $a = h\cot A - h\cot B$.



80. Show that the area A of the triangle is given by $A = \dfrac{1}{2}bc\sin A$.



81. Use a cofunction relationship to show that the product $(\tan 1°)(\tan 2°)(\tan 3°) \cdot \cdots \cdot (\tan 87°)(\tan 88°)(\tan 89°)$ is equal to 1.

Technology Connections

For Exercises 82–85, use a calculator to approximate the values of the left- and right-hand sides of each statement for $A = 30°$ and $B = 45°$. Based on the approximations from your calculator, determine if the statement appears to be true or false.

82. a. $\sin(A + B) = \sin A + \sin B$

b. $\sin(A + B) = \sin A\cos B + \cos A\sin B$

83. a. $\tan(A - B) = \tan A - \tan B$

b. $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

84. a. $\cos\left(\dfrac{A}{2}\right) = \sqrt{\dfrac{1 + \cos A}{2}}$

b. $\cos\left(\dfrac{A}{2}\right) = \dfrac{1}{2}\cos A$

85. a. $\tan\left(\dfrac{B}{2}\right) = \dfrac{1 - \cos B}{\sin B}$

b. $\tan\left(\dfrac{B}{2}\right) = \dfrac{\sin B}{1 + \cos B}$

SECTION 4.4 Trigonometric Functions of Any Angle

OBJECTIVES

1. Evaluate Trigonometric Functions of Any Angle
2. Determine Reference Angles
3. Evaluate Trigonometric Functions Using Reference Angles

1. Evaluate Trigonometric Functions of Any Angle

In many applications we encounter angles that are not acute. For example, a robotic arm may have a range of motion of 360°, or an object may rotate in a repeated circular pattern through all possible angles. For these situations, we need to extend the definition of trigonometric functions to any angle.

In Figure 4-36, the real number t corresponds to a point $Q(a, b)$ on the unit circle. Angle θ is a second quadrant angle passing through point $P(x, y)$ with terminal side passing through Q. Suppose that we drop perpendicular line segments from points P and Q to the x-axis to form two similar right triangles. For ΔORP, the hypotenuse r is given by $r = \sqrt{x^2 + y^2}$ and the lengths of the two legs are $|x|$ and y. For ΔOSQ, the hypotenuse is 1 unit and the lengths of the legs are $|a|$ and b.

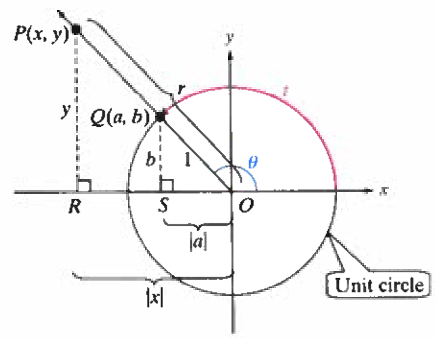


Figure 4-36

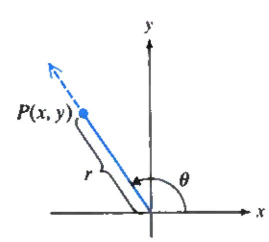
Since ΔORP and ΔOSQ are similar triangles, the ratios of their corresponding sides are proportional, which implies that $\frac{b}{1} = \frac{y}{r}$. Furthermore, since $\sin t = b$, we have $\sin t = \frac{y}{r}$. We also know that there is a one-to-one correspondence between the real number t and the radian measure of the central angle θ in standard position with terminal side through P. From this correspondence, we define $\sin\theta = \sin t = \frac{y}{r}$. Similar logic can be used to define the other five trigonometric functions for a point P in any quadrant or on the x- or y-axes. The results are summarized in Table 4-8.

Table 4-8

Trigonometric Functions of Any Angle

Let θ be an angle in standard position with point $P(x, y)$ on the terminal side, and let $r = \sqrt{x^2 + y^2} \neq 0$ represent the distance from $P(x, y)$ to $(0, 0)$. Then,

$\sin\theta = \frac{y}{r}$	$\csc\theta = \frac{r}{y}\ (y \neq 0)$
$\cos\theta = \frac{x}{r}$	$\sec\theta = \frac{r}{x}\ (x \neq 0)$
$\tan\theta = \frac{y}{x}\ (x \neq 0)$	$\cot\theta = \frac{x}{y}\ (y \neq 0)$



Point P is on the terminal side of angle θ and not at the origin, which means that $r = \sqrt{x^2 + y^2} \neq 0$. Therefore, the signs of the values of the trigonometric functions depend solely on the signs of x and y. For an acute angle, the values of the trigonometric functions are positive because x and y are both positive in the first quadrant. For angles in the other quadrants, we determine the signs of the trigonometric functions by analyzing the signs of x and y (Figure 4-37). Note that the reciprocal functions cosecant, secant, and cotangent will have the same signs as sine, cosine, and tangent, respectively for a given value of θ.

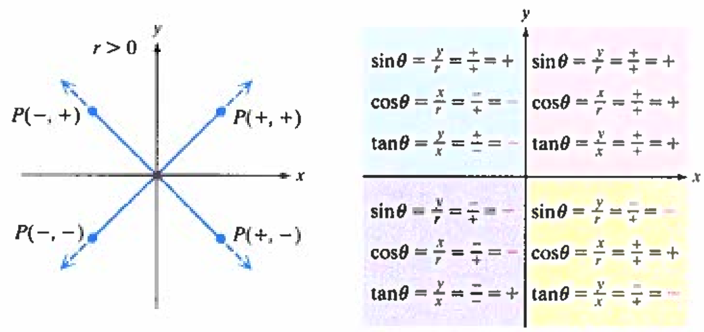


Figure 4-37

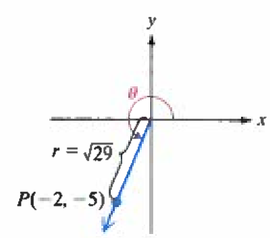
EXAMPLE 1 Evaluating Trigonometric Functions

Let $P(-2, -5)$ be a point on the terminal side of angle θ drawn in standard position. Find the values of the six trigonometric functions of θ.

Solution:

We first draw θ in standard position and label point P.

The distance between $P(-2, -5)$ and $(0, 0)$ is



$$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (-5)^2} = \sqrt{4 + 25} = \sqrt{29}.$$

We have $x = -2$, $y = -5$, and $r = \sqrt{29}$.

$$\sin\theta = \frac{y}{r} = \frac{-5}{\sqrt{29}} = \frac{-5}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = -\frac{5\sqrt{29}}{29} \qquad \csc\theta = \frac{r}{y} = \frac{\sqrt{29}}{-5} = -\frac{\sqrt{29}}{5}$$

$$\cos\theta = \frac{x}{r} = \frac{-2}{\sqrt{29}} = \frac{-2}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = -\frac{2\sqrt{29}}{29} \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{29}}{-2} = -\frac{\sqrt{29}}{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-5}{-2} = \frac{5}{2} \qquad \cot\theta = \frac{x}{y} = \frac{-2}{-5} = \frac{2}{5}$$

Skill Practice 1 Let $P(-5, -7)$ be a point on the terminal side of angle θ drawn in standard position. Find the values of the six trigonometric functions of θ.

Answer

1. $\sin\theta = -\frac{7\sqrt{74}}{74}$, $\cos\theta = -\frac{5\sqrt{74}}{74}$, $\tan\theta = \frac{7}{5}$, $\csc\theta = -\frac{\sqrt{74}}{7}$, $\sec\theta = -\frac{\sqrt{74}}{5}$, $\cot\theta = \frac{5}{7}$

2. Determine Reference Angles

For nonacute angles θ, we often find values of the six trigonometric functions by using the related *reference angle.*

Definition of a Reference Angle

Let θ be an angle in standard position. The **reference angle** for θ is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

Figure 4-38 shows the reference angles θ' for angles on the interval $[0, 2\pi)$ drawn in standard position for each of the four quadrants.

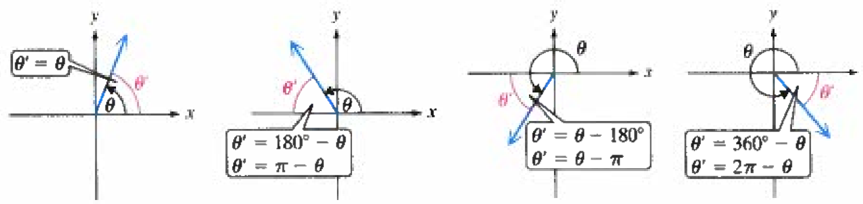


Figure 4-38

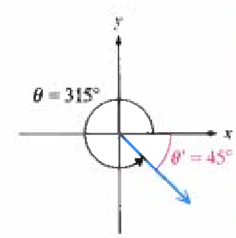
EXAMPLE 2 Finding Reference Angles

Find the reference angle θ'.

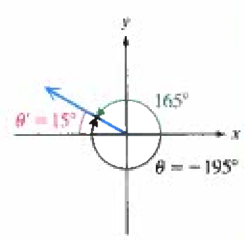
a. $\theta = 315°$ **b.** $\theta = -195°$ **c.** $\theta = 3.5$ **d.** $\theta = \dfrac{25\pi}{4}$

Solution:

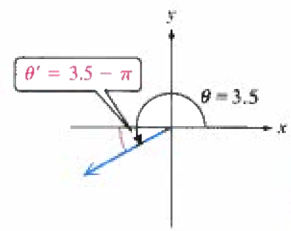
a. $\theta = 315°$ is a fourth quadrant angle. The acute angle it makes with the x-axis is $\theta' = 360° - 315° = 45°$.



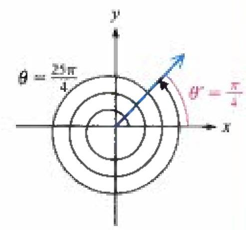
b. $\theta = -195°$ is a second quadrant angle coterminal to $165°$. The actue angle it makes with the x-axis is $\theta' = 180° - 165° = 15°$.



c. $\theta = 3.5$ is measured in radians. Since $\pi \approx 3.14$ and $\frac{3\pi}{2} \approx 4.71$, we know that $\pi < 3.5 < \frac{3\pi}{2}$ implying that 3.5 is in the third quadrant. The reference angle is $\theta' = 3.5 - \pi \approx 0.3584$.



d. $\theta = \frac{25\pi}{4}$ is a first quadrant angle coterminal to $\frac{\pi}{4}$. The angle $\frac{\pi}{4}$ is an acute angle and is its own reference angle. Therefore, $\theta' = \frac{\pi}{4}$.



Skill Practice 2 Find the reference angle θ'.

a. $\theta = 150°$ **b.** $\theta = -157.5°$ **c.** $\theta = 5$ **d.** $\theta = \frac{13\pi}{3}$

3. Evaluate Trigonometric Functions Using Reference Angles

In Figure 4-39, an angle θ is drawn in standard position along with its accompanying reference angle θ'. A right triangle can be formed by dropping a perpendicular line segment from a point $P(x, y)$ on the terminal side of θ to the x-axis. The length of the vertical leg of the triangle is $|y|$, the length of the horizontal leg is $|x|$, and the hypotenuse is r.

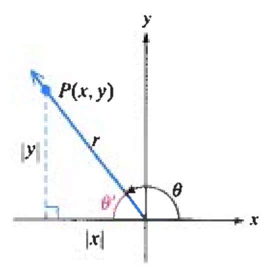


Figure 4-39

Now suppose we compare the values of the cosine function of θ and θ'.

$$\cos\theta = \frac{x}{r} \quad \text{and} \quad \cos\theta' = \frac{\text{adj}}{\text{hyp}} = \frac{|x|}{r}$$

Notice that the ratio for $\cos\theta$ has x in the numerator and the ratio for $\cos\theta'$ has $|x|$. It follows that $\cos\theta$ and $\cos\theta'$ are equal except possibly for their signs. A similar argument holds for the other five trigonometric functions. This leads to the following procedure to evaluate a trigonometric function based on reference angles.

Evaluating Trigonometric Functions Using Reference Angles

To find the value of a trigonometric function of a given angle θ,

1. Determine the function value of the reference angle θ'.
2. Affix the appropriate sign based on the quadrant in which θ lies.

Before we present examples of this process, take a minute to review the values of the trigonometric functions of 30°, 45°, and 60° (Table 4-9).

Answers

2. a. $\theta' = 30°$
 b. $\theta' = 22.5°$
 c. $\theta' = 2\pi - 5$
 d. $\theta' = \frac{\pi}{3}$

Table 4-9

Trigonometric Function Values of Special Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$30° = \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$45° = \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60° = \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

EXAMPLE 3 Using Reference Angles to Evaluate Functions

Evaluate the functions. **a.** $\sin 240°$ **b.** $\tan(-225°)$ **c.** $\sec\frac{11\pi}{6}$

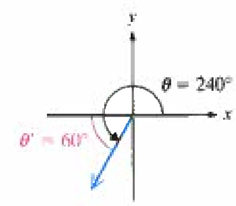
Solution:

a. $\theta = 240°$ is in Quadrant III. The reference

angle is $240° - 180° = 60°$

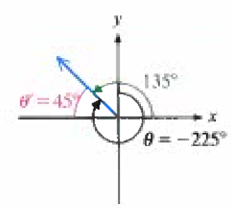
Since $\sin\theta$ is negative in Quadrant III,

$$\sin 240° = -\sin 60° = -\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}$$



b. $\theta = -225°$ is an angle in Quadrant II, coterminal to $135°$. The reference angle is $180° - 135° = 45°$. Since $\tan\theta$ is negative in Quadrant II,

$$\tan(-225°) = -\tan 45° = -(1) = -1$$

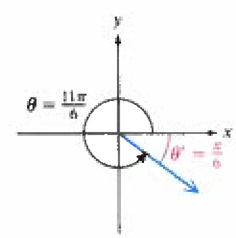


c. $\theta = \frac{11\pi}{6}$ is in Quadrant IV. The reference angle is

$$2\pi - \frac{11\pi}{6} = \frac{12\pi}{6} - \frac{11\pi}{6} = \frac{\pi}{6}$$

Since $\cos\theta$ and its reciprocal $\sec\theta$ are positive in Quadrant IV,

$$\sec\frac{11\pi}{6} = \sec\frac{\pi}{6} = \frac{2\sqrt{3}}{3}$$



Skill Practice 3 Evaluate the functions.

a. $\cos\frac{5\pi}{6}$ **b.** $\cot(-120°)$ **c.** $\csc\frac{7\pi}{4}$

Answers

3. a. $-\frac{\sqrt{3}}{2}$ **b.** $\frac{\sqrt{3}}{3}$ **c.** $-\sqrt{2}$

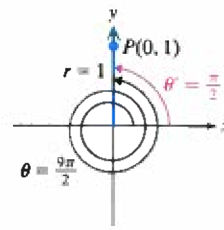
EXAMPLE 4 Evaluate Trigonometric Functions

Evaluate the functions. **a.** $\sec\dfrac{9\pi}{2}$ **b.** $\sin(-510°)$

Solution:

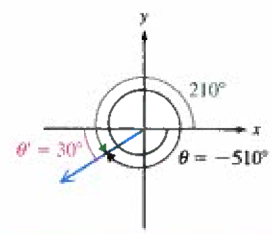
a. $\dfrac{9\pi}{2}$ is coterminal to $\dfrac{\pi}{2}$. The terminal side of $\dfrac{\pi}{2}$ is on the positive y-axis, where we have selected the arbitrary point $(0, 1)$.

$$\sec\frac{9\pi}{2} = \sec\frac{\pi}{2} = \frac{r}{x} \rightarrow \frac{1}{0} \text{ is undefined.}$$



b. $-510°$ is coterminal to $210°$, which is a third-quadrant angle. The reference angle is $210° - 180° = 30°$.
Since $\sin\theta$ is negative in Quadrant III,

$$\sin(-510°) = -\sin 30° = -\left(\frac{1}{2}\right) = -\frac{1}{2}$$



Skill Practice 4 Evaluate the functions.
a. $\csc(11\pi)$ **b.** $\cos(-600°)$

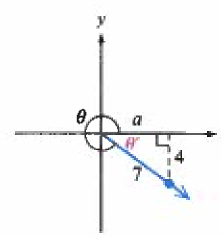
In Example 5, we use a reference angle to determine the values of trigonometric functions based on given information about θ.

EXAMPLE 5 Evaluating Trigonometric Functions

Given $\sin\theta = -\dfrac{4}{7}$ and $\cos\theta > 0$, find $\cos\theta$ and $\tan\theta$.

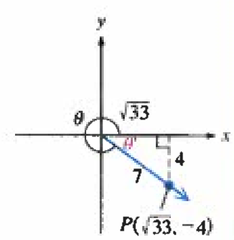
Solution:

First note that $\sin\theta < 0$ and $\cos\theta > 0$ in Quadrant IV. We can label the reference angle θ' and then draw a representative triangle with opposite leg of length 4 units and hypotenuse of length 7 units.



Using the Pythagorean theorem, we can determine the length of the adjacent leg.

$$a^2 + (4)^2 = (7)^2$$
$$a^2 + 16 = 49$$
$$a^2 = 33$$
$$a = \sqrt{33}$$ Choose the positive square root for the length of a side of a triangle.



Answers

4. a. Undefined **b.** $-\dfrac{1}{2}$

TIP With the reference angle and representative triangle drawn in Quadrant IV, we can find point $P(\sqrt{33}, -4)$ on the terminal side of θ.

$$\cos\theta = \frac{x}{r} = \frac{\sqrt{33}}{7}$$

$$\tan\theta = \frac{y}{x} = \frac{-4}{\sqrt{33}}$$

$$\cos\theta = \cos\theta' = \frac{\sqrt{33}}{7}$$ cos θ is positive in Quadrant IV.

$$\tan\theta = -\tan\theta' = -\frac{4}{\sqrt{33}} = -\frac{4}{\sqrt{33}} \cdot \frac{\sqrt{33}}{\sqrt{33}} = -\frac{4\sqrt{33}}{33}$$ tan θ is negative in Quadrant IV.

Skill Practice 5 Given $\cos\theta = -\frac{3}{8}$ and $\sin\theta < 0$, find $\sin\theta$ and $\tan\theta$.

The identities involving trigonometric functions of acute angles presented in Section 4.3 are also true for trigonometric functions of non-acute angles provided that the functions are well defined at θ.

Pythagorean Identities	Reciprocal Identities	Quotient Identities
$\sin^2\theta + \cos^2\theta = 1$	$\csc\theta = \frac{1}{\sin\theta}$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$\tan^2\theta + 1 = \sec^2\theta$	$\sec\theta = \frac{1}{\cos\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$
$1 + \cot^2\theta = \csc^2\theta$	$\cot\theta = \frac{1}{\tan\theta}$	

In Example 6, we are given information about an angle θ and will find the values of the trigonometric functions by applying the fundamental identities and by using reference angles.

EXAMPLE 6 Using Fundamental Identities

Given $\cos\theta = -\frac{3}{5}$ for θ in Quadrant II, find $\sin\theta$ and $\tan\theta$.

Solution:

Using Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta + \left(-\frac{3}{5}\right)^2 = 1$$

$$\sin^2\theta + \frac{9}{25} = 1$$

$$\sin^2\theta = 1 - \frac{9}{25}$$

$$\sin^2\theta = \frac{25}{25} - \frac{9}{25}$$

$$\sin^2\theta = \frac{16}{25}$$

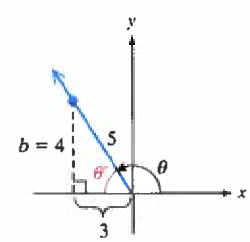
$$\sin\theta = \pm\frac{4}{5}$$

Alternative Approach

Label the reference angle θ' and then draw a representative triangle with adjacent leg of length 3 units and hypotenuse of length 5 units.

Using the Pythagorean theorem, we can determine the length of the opposite leg.

$$(3)^2 + b^2 = (5)^2$$
$$9 + b^2 = 25$$
$$b^2 = 16$$
$$b = 4$$



TIP With the reference angle and representative triangle drawn in Quadrant II, we can find point $P(-3, 4)$ on the terminal side of θ. Therefore, $x = -3$, $y = 4$, and $r = 5$.

$$\sin\theta = \frac{y}{r} = \frac{4}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{4}{-3}$$

Answer

5. $\sin\theta = -\frac{\sqrt{55}}{8}$ and $\tan\theta = \frac{\sqrt{55}}{3}$

In Quadrant II, $\sin\theta > 0$.

Therefore, choose $\sin\theta = \frac{4}{5}$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$$

In Quadrant II, $\sin\theta > 0$.

Therefore, $\sin\theta = \sin\theta' = \frac{4}{5}$.

In Quadrant II, $\tan\theta < 0$.

Therefore, $\tan\theta = -\tan\theta' = -\frac{4}{3}$.

Skill Practice 6 Given $\sin\theta = -\frac{5}{13}$ for θ in Quadrant IV, find $\cos\theta$ and $\tan\theta$.

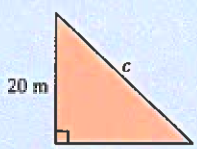
Answer

6. $\cos\theta = \frac{12}{13}$, $\tan\theta = -\frac{5}{12}$

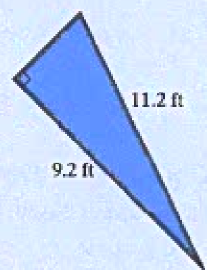
SECTION 4.4 Practice Exercises

Prerequisite Review

R.1. Find the length of the missing side of the right triangle.



R.2. Find the length of the third side using the Pythagorean theorem. Round the answer to the nearest tenth of a foot.



R.3. Use the distance formula to find the distance from (2, 5) to (−3, 10).

Concept Connections

1. The distance from the origin to a point $P(x, y)$ is given by ______________.

2. Angles with terminal sides on the coordinate axes are referred to as ______________ angles.

3. If θ is an angle in standard position, the ______________ angle for θ is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

4. The values $\tan\theta$ and $\sec\theta$ are undefined for odd multiples of ______________.

5. The values $\cot\theta$ and $\csc\theta$ are undefined for multiples of ______________.

6. For what values of θ is $\sin\theta$ greater than 1?

Objective 1: Evaluate Trigonometric Functions of Any Angle

7. Let $P(x, y)$ be a point on the terminal side of an angle θ drawn in standard position and let r be the distance from P to the origin. Fill in the boxes to form the ratios defining the six trigonometric functions.

a. $\sin\theta = \frac{\square}{\square}$ **b.** $\cos\theta = \frac{\square}{\square}$ **c.** $\tan\theta = \frac{\square}{\square}$

d. $\csc\theta = \frac{\square}{\square}$ **e.** $\sec\theta = \frac{\square}{\square}$ **f.** $\cot\theta = \frac{\square}{\square}$

8. Fill in the cells in the table with the appropriate sign for each trigonometric function for θ in Quadrants I, II, III, and IV. The signs for the sine function are done for you.

	Quadrant			
	I	II	III	IV
$\sin\theta$	+	+	−	−
$\cos\theta$				
$\tan\theta$				
$\csc\theta$				
$\sec\theta$				
$\cot\theta$				

For Exercises 9–14, given the stated conditions, identify the quadrant in which θ lies.

9. $\sin\theta < 0$ and $\tan\theta > 0$
10. $\csc\theta > 0$ and $\cot\theta < 0$
11. $\sec\theta < 0$ and $\tan\theta < 0$
12. $\cot\theta < 0$ and $\sin\theta < 0$
13. $\cos\theta > 0$ and $\cot\theta < 0$
14. $\cot\theta < 0$ and $\sec\theta < 0$

For Exercises 15–20, a point is given on the terminal side of an angle θ drawn in standard position. Find the values of the six trigonometric functions of θ. (See Example 1)

15. $(5, -12)$
16. $(-8, 15)$
17. $(-3, -5)$
18. $(2, -3)$
19. $\left(-\frac{3}{2}, 2\right)$
20. $\left(5, -\frac{8}{3}\right)$

21. Complete the table for the given angles.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$0° = 0$						
$90° = \frac{\pi}{2}$						
$180° = \pi$						
$270° = \frac{3\pi}{2}$						
$360° = 2\pi$						

22. Evaluate the expression for $A = 90°$, $B = 180°$, and $C = 270°$.

$$\frac{\sin A + 2\cos B}{\sec B - 3\csc C}$$

Objective 2: Determine Reference Angles

For Exercises 23–30, find the reference angle for the given angle. (See Example 2)

23. a. $135°$ b. $-330°$ c. $660°$ d. $-690°$
24. a. $-120°$ b. $225°$ c. $-1035°$ d. $510°$
25. a. $\frac{2\pi}{3}$ b. $-\frac{5\pi}{6}$ c. $\frac{13\pi}{4}$ d. $-\frac{10\pi}{3}$
26. a. $-\frac{5\pi}{4}$ b. $\frac{11\pi}{6}$ c. $\frac{17\pi}{3}$ d. $\frac{19\pi}{6}$
27. a. $\frac{20\pi}{17}$ b. $-\frac{99\pi}{20}$ c. $110°$ d. $-422°$
28. a. $\frac{25\pi}{11}$ b. $-\frac{27\pi}{14}$ c. $-512°$ d. $1280°$
29. a. 1.8 b. 1.8π c. 5.1 d. $5.1°$
30. a. 0.6 b. 0.6π c. 100 d. $100°$

Objective 3: Evaluate Trigonometric Functions Using Reference Angles

For Exercises 31–54, use reference angles to find the exact value. (See Examples 3 and 4)

31. $\sin 120°$
32. $\cos 225°$
33. $\cos\frac{4\pi}{3}$
34. $\sin\frac{5\pi}{6}$
35. $\sec(-330°)$
36. $\csc(-225°)$
37. $\sec\frac{13\pi}{3}$
38. $\csc\frac{5\pi}{3}$
39. $\cot 240°$
40. $\tan(-150°)$
41. $\tan\frac{5\pi}{4}$
42. $\cot\left(-\frac{3\pi}{4}\right)$
43. $\cos(-630°)$
44. $\sin 630°$
45. $\sin\frac{17\pi}{6}$
46. $\cos\left(-\frac{11\pi}{4}\right)$
47. $\sec 1170°$
48. $\csc 750°$
49. $\csc(-5\pi)$
50. $\sec 5\pi$
51. $\tan(-2400°)$
52. $\cot 900°$
53. $\cot\frac{42\pi}{8}$
54. $\tan\frac{18\pi}{4}$

For Exercises 55–58, find two angles between 0° and 360° for the given condition.

55. $\sin\theta = \frac{1}{2}$ **56.** $\cos\theta = -\frac{\sqrt{2}}{2}$ **57.** $\cot\theta = -\sqrt{3}$ **58.** $\tan\theta = \sqrt{3}$

For Exercises 59–62, find two angles between 0 and 2π for the given condition.

59. $\sec\theta = \sqrt{2}$ **60.** $\csc\theta = -\frac{2\sqrt{3}}{3}$ **61.** $\tan\theta = -\frac{\sqrt{3}}{3}$ **62.** $\cot\theta = 1$

In Exercises 63–68, find the values of the trigonometric functions from the given information. (See Example 5)

63. Given $\tan\theta = -\frac{20}{21}$ and $\cos\theta < 0$, find $\sin\theta$ and $\cos\theta$.

64. Given $\cot\theta = \frac{11}{60}$ and $\sin\theta < 0$, find $\cos\theta$ and $\sin\theta$.

65. Given $\sin\theta = -\frac{3}{10}$ and $\tan\theta > 0$, find $\cos\theta$ and $\cot\theta$.

66. Given $\cos\theta = -\frac{5}{8}$ and $\csc\theta > 0$, find $\sin\theta$ and $\tan\theta$.

67. Given $\sec\theta = \frac{\sqrt{58}}{7}$ and $\cot\theta < 0$, find $\csc\theta$ and $\cos\theta$.

68. Given $\csc\theta = -\frac{\sqrt{26}}{5}$ and $\cos\theta < 0$, find $\sin\theta$ and $\cot\theta$.

For Exercises 69–74, use fundamental trigonometric identities to find the values of the functions. (See Example 6)

69. Given $\cos\theta = \frac{20}{29}$ for θ in Quadrant IV, find $\sin\theta$ and $\tan\theta$.

70. Given $\sin\theta = -\frac{8}{17}$ for θ in Quadrant III, find $\cos\theta$ and $\cot\theta$.

71. Given $\tan\theta = -4$ for θ in Quadrant II, find $\sec\theta$ and $\cot\theta$.

72. Given $\sec\theta = 5$ for θ in Quadrant IV, find $\csc\theta$ and $\cos\theta$.

73. Given $\cot\theta = \frac{5}{4}$ for θ in Quadrant III, find $\csc\theta$ and $\sin\theta$.

74. Given $\csc\theta = \frac{7}{3}$ for θ in Quadrant II, find $\cot\theta$ and $\cos\theta$.

Mixed Exercises

For Exercises 75–76, find the sign of the expression for θ in each quadrant.

75.

		I	II	III	IV
a.	$\sin\theta\cos\theta$				
b.	$\frac{\tan\theta}{\cos\theta}$				

76.

		I	II	III	IV
a.	$\frac{\cot\theta}{\sin\theta}$				
b.	$\tan\theta\sec\theta$				

For Exercises 77–84, suppose that θ is an acute angle. Identify each statement as true or false. If the statement is false, rewrite the statement to give the correct answer for the right side.

77. $\cos(180° - \theta) = -\cos\theta$ **78.** $\tan(180° - \theta) = \tan\theta$ **79.** $\tan(180° + \theta) = \tan\theta$

80. $\sin(180° + \theta) = -\sin\theta$ **81.** $\csc(\pi - \theta) = -\csc\theta$ **82.** $\sec(\pi + \theta) = \sec\theta$

83. $\cos(\pi + \theta) = -\cos\theta$ **84.** $\sin(\pi + \theta) = -\sin\theta$

For Exercises 85–90, find the value of each expression.

85. $\sin 30° \cdot \cos 150° \cdot \sec 60° \cdot \csc 120°$ **86.** $\cos 45° \cdot \sin 240° \cdot \tan 135° \cdot \cot 60°$ **87.** $\cos^2\frac{5\pi}{4} - \sin^2\frac{2\pi}{3}$

88. $\sin^2\frac{11\pi}{6} + \cos^2\frac{4\pi}{3}$ **89.** $\dfrac{2\tan\frac{11\pi}{6}}{1 - \tan^2\frac{11\pi}{6}}$ **90.** $\dfrac{\cot^2\frac{4\pi}{3} - 1}{2\cot\frac{4\pi}{3}}$

For Exercises 91–94, verify the statement for the given values.

91. $\sin(A - B) = \sin A\cos B - \cos A\sin B$; $A = 240°$, $B = 120°$

92. $\cos(B - A) = \cos B\cos A + \sin B\sin A$; $A = 330°$, $B = 120°$

93. $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$; $A = 210°$, $B = 120°$

94. $\cot(A + B) = \dfrac{\cot A\cot B - 1}{\cot A + \cot B}$; $A = 300°$, $B = 150°$

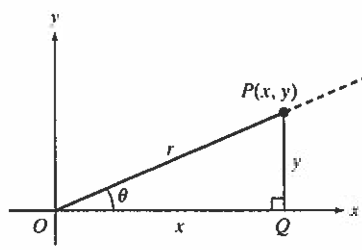
For Exercises 95–96, give the exact values if possible. Otherwise, use a calculator and approximate the result to 4 decimal places.

95. a. $\sin 30°$ **b.** $\sin(30\pi)$ **c.** $\sin 30$

96. a. $\cos(0.25\pi)$ **b.** $\dfrac{\cos\pi}{0.25}$ **c.** $\cos(25°)$

Write About It

97. Explain why neither $\sin\theta$ nor $\cos\theta$ can be greater than 1. Refer to the figure for your explanation.



98. Explain why $\tan\theta$ is undefined at $\theta = \dfrac{\pi}{2}$ but $\cot\theta$ is defined at $\theta = \dfrac{\pi}{2}$.

Expanding Your Skills

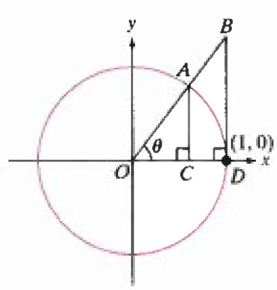
99. a. Graph the point (3, 4) on a rectangular coordinate system and draw a line segment connecting the point to the origin. Find the slope of the line segment.

b. Draw another line segment from the point (3, 4) to meet the x-axis at a right angle, thus forming a right triangle with the x-axis as one side. Find the tangent of the acute angle that has the x-axis as its initial side.

c. Compare the results in part (a) and part (b).

100. The circle shown is centered at the origin with a radius of 1. The segment $\overline{BD}$ is tangent to the circle at D. Match the length of each segment with the appropriate trigonometric function.

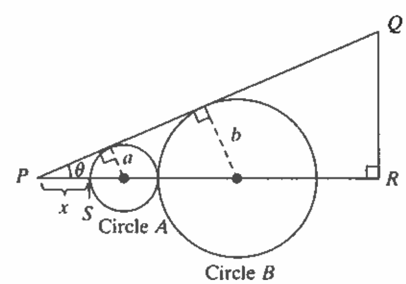
a. $\overline{AC}$	i. $\tan\theta$
b. $\overline{BD}$	ii. $\cos\theta$
c. $\overline{OB}$	iii. $\sin\theta$
d. $\overline{OC}$	iv. $\sec\theta$



101. Circle A, with radius a, and circle B, with radius b, are tangent to each other and to $\overline{PQ}$ (see figure). $\overline{PR}$ passes through the center of each circle. Let x be the distance from point P to a point S where $\overline{PR}$ intersects circle A on the left. Let θ denote $\angle RPQ$.

a. Show that $\sin\theta = \dfrac{a}{x + a}$ and $\sin\theta = \dfrac{b}{x + 2a + b}$.

b. Use the results from part (a) to show that $\sin\theta = \dfrac{b - a}{b + a}$.



SECTION 4.5 Graphs of Sine and Cosine Functions

OBJECTIVES

1. Graph $y = \sin x$ and $y = \cos x$
2. Graph $y = A\sin x$ and $y = A\cos x$
3. Graph $y = A\sin Bx$ and $y = A\cos Bx$
4. Graph $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$
5. Model Sinusoidal Behavior

1. Graph $y = \sin x$ and $y = \cos x$

Throughout this text, we have been progressively introducing categories of functions and analyzing their properties and graphs. In each case, we begin with a fundamental "parent" function such as $y = x^2$ and then develop interesting variations on the graph by applying transformations. For example, $y = a(x - h)^2 + k$ is a family of quadratic functions whose properties help us model physical phenomena such as projectile motion.

In a similar manner we will graph the basic functions $y = \sin x$ and $y = \cos x$ and then analyze their variations. To begin, note that we will use variable x (rather than θ or t) to represent the independent variable.

To develop the graph of $y = \sin x$ we return to the notion of a number line wrapped onto the unit circle. The points on the number line correspond to the real numbers x in the domain of the sine function. The value of $\sin x$ is the y-coordinate of the corresponding point on the unit circle (Figure 4-40).

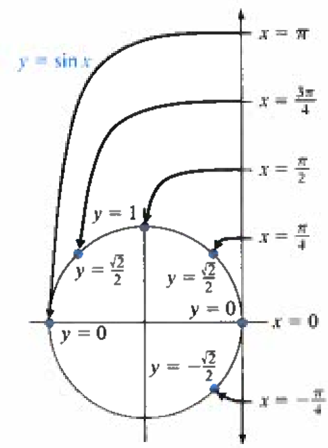


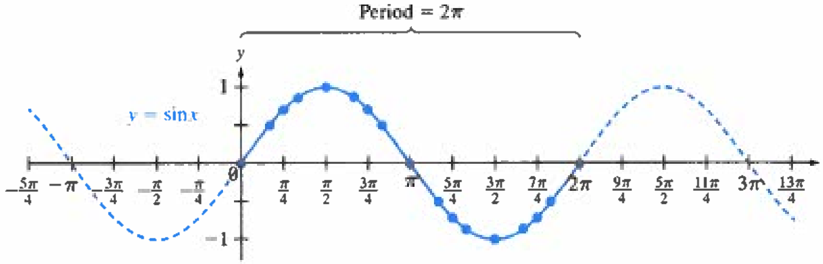
Figure 4-40

Table 4-10 gives several values of x and the corresponding values of $\sin x$. Plotting these as ordered pairs gives the graph of $y = \sin x$ in Figure 4-41.

Table 4-10

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0

TIP The decimal approximations of the coordinates involving irrational numbers can be used to sketch the graph. For example, $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) \approx (0.79, 0.71)$.



Domain: $(-\infty, \infty)$; Range: $[-1, 1]$

Figure 4-41

Recall that the sine function is periodic with period 2π. This means that the pattern shown between 0 and 2π repeats infinitely far to the left and right as denoted by the dashed portion of the curve (Figure 4-41). In Figure 4-42, one complete cycle of the graph of $y = \sin x$ is shown over the interval $[0, 2\pi]$. Notice the relationship between the y-coordinates on the unit circle and the graph of $y = \sin x$.

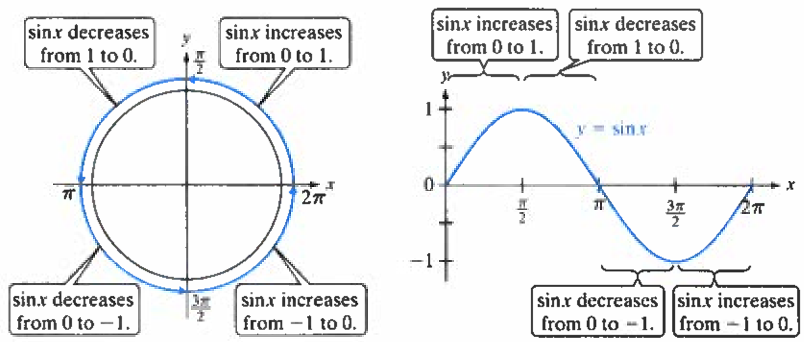


Figure 4-42

In a similar fashion, we can graph $y = \cos x$. Table 4-11 gives several values of x with the corresponding values of $\cos x$. Plotting these as ordered pairs gives the graph of $y = \cos x$ in Figure 4-43.

Table 4-11

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

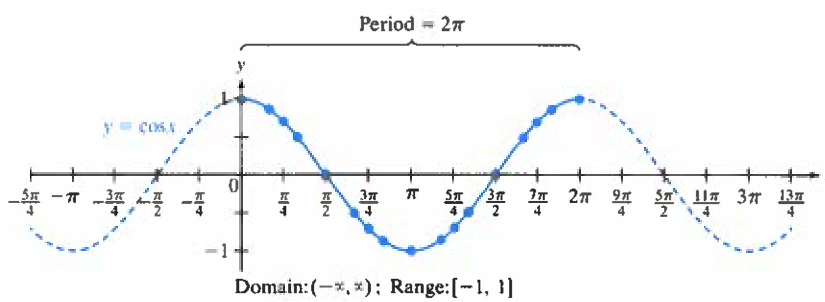


Figure 4-43

Avoiding Mistakes

When we define $\sin t = y$ and $\cos t = x$, the values of x and y refer to the coordinates of a point on the unit circle $x^2 + y^2 = 1$.

When we define $y = \sin x$ and $y = \cos x$, the variable x has taken the role of t (a real number on a number line wrapped onto the unit circle) and y is the corresponding sine or cosine value.

In Figure 4-44, the graphs of $y = \sin x$ and $y = \cos x$ are shown on the interval $[-2\pi, 2\pi]$ for comparison. From the graphs, we make the following observations.

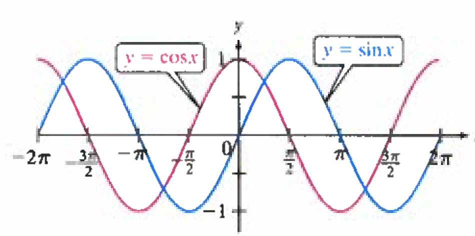


Figure 4-44

Characteristics of the Graphs of $y = \sin x$ and $y = \cos x$

- The domain is $(-\infty, \infty)$.
- The range is $[-1, 1]$.
- The period is 2π.
- The graph of $y = \sin x$ is symmetric with respect to the origin. $y = \sin x$ is an odd function.
- The graph of $y = \cos x$ is symmetric with respect to the y-axis. $y = \cos x$ is an even function.
- The graphs of $y = \sin x$ and $y = \cos x$ differ by a horizontal shift of $\frac{\pi}{2}$.

2. Graph $y = A\sin x$ and $y = A\cos x$

Figures 4-45 and 4-46 show one complete cycle of the graphs of $y = \sin x$ and $y = \cos x$. When we graph variations of these functions we want to graph the key points. These are the relative minima, relative maxima, and x-intercepts. Notice that the graphs of both the sine and cosine function alternate between a relative minimum or maximum and an x-intercept for each quarter of a period.

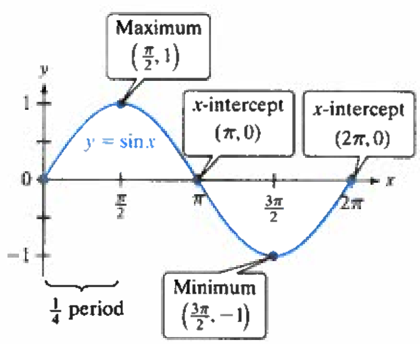


Figure 4-45

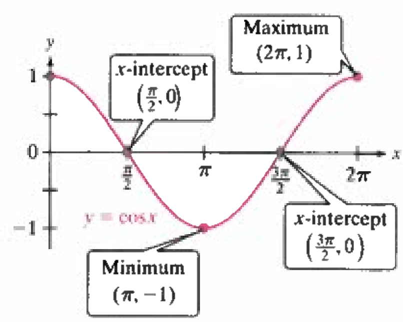


Figure 4-46

We begin analyzing variations of the sine and cosine graphs by graphing functions of the form $y = A\sin x$ and $y = A\cos x$.

EXAMPLE 1 **Graphing $y = A\sin x$ and $y = A\cos x$**

Graph the function and identify the key points on one full period.

a. $y = 3\sin x$ **b.** $y = -\frac{1}{2}\cos x$

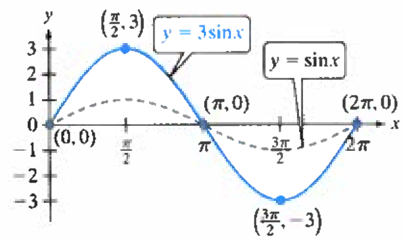
Solution:

Recall from Section 1.6 that the graph of $y = A \cdot f(x)$ is the graph of $y = f(x)$ with

- A vertical stretch if $|A| > 1$ or
- A vertical shrink if $0 < |A| < 1$.
- A reflection across the x-axis if $A < 0$.

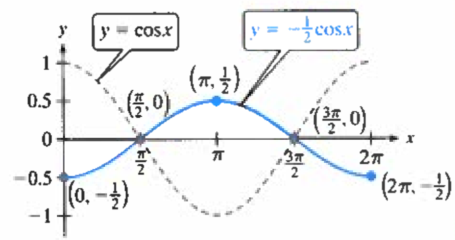
a. Given $y = 3\sin x$, the value of A is 3, so the graph of $y = 3\sin x$ is the graph of $y = \sin x$ with a vertical stretch by a factor of 3.

The dashed curve $y = \sin x$ is shown for comparison.



b. Given $y = -\frac{1}{2}\cos x$, the value of A is $-\frac{1}{2}$, so the graph of $y = -\frac{1}{2}\cos x$ is the graph of $y = \cos x$ with a vertical shrink and a reflection across the x-axis.

The dashed curve, $y = \cos x$, is shown for comparison.



Skill Practice 1 Graph the function and identify the key points on one full period.

a. $y = 2\cos x$ **b.** $y = -\dfrac{1}{3}\sin x$

Notice that the graph of $y = 3\sin x$ from Example 1(a) deviates 3 units above and below the x-axis. For the graphs of $y = |A|\sin x$ and $y = |A|\cos x$, the value of the vertical scaling factor $|A|$ is called the **amplitude** of the function. The amplitude represents the amount of deviation from the central position of the sine wave. The amplitude is half the distance between the maximum value of the function and the minimum value of the function. In Example 1(a), the amplitude is $|3| = 3$, and in Example 1(b), the amplitude is $\left|-\dfrac{1}{2}\right| = \dfrac{1}{2}$.

3. Graph $y = A\sin Bx$ and $y = A\cos Bx$

Recall from Section 1.6 that the graph of $y = f(Bx)$ is the graph of $y = f(x)$ with a horizontal shrink or stretch. This means that for $y = \sin x$ and $y = \cos x$, the factor B will affect the period of the graph. For example, Figures 4-47 and 4-48 show the graphs of $y = \sin 2x$ and $y = \sin\frac{1}{2}x$, respectively, as compared to the parent function $y = \sin x$.

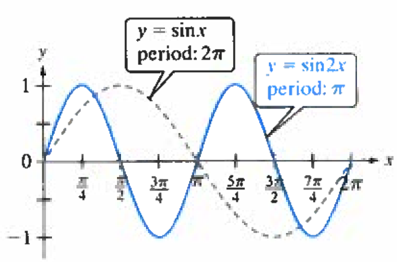


Figure 4-47

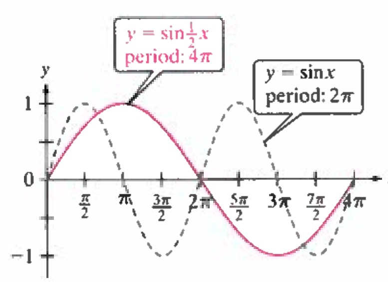
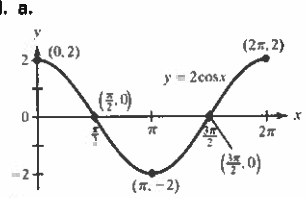


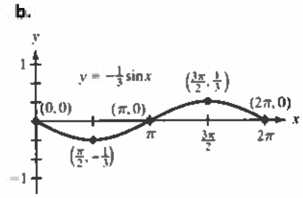
Figure 4-48

Answers

1. a.



b.



The period of the functions $y = \sin x$ and $y = \cos x$ is 2π. That is, the graphs of $y = \sin x$ and $y = \cos x$ show one complete cycle on the interval $0 \le x \le 2\pi$. To determine a comparable interval for one complete cycle of the graphs of $y = \sin Bx$ and $y = \cos Bx$ for $B > 0$, we can solve the following inequality.

$$0 \le Bx \le 2\pi$$

$$\frac{0}{B} \le \frac{Bx}{B} \le \frac{2\pi}{B}$$

"Starting" value for the cycle

$$0 \le x \le \frac{2\pi}{B}$$

"Ending" value for the cycle

This tells us that for $B > 0$, the period of $y = \sin Bx$ and $y = \cos Bx$ is $\frac{2\pi}{B}$.

Amplitude and Period of the Sine and Cosine Functions

For $y = A\sin Bx$ and $y = A\cos Bx$ and $B > 0$, the amplitude and period are

$$\text{Amplitude} = |A| \quad \text{and} \quad \text{Period} = \frac{2\pi}{B}$$

To analyze the period of a sine or cosine function with a negative coefficient on x, we would first rewrite $y = A\sin(-Bx)$ and $y = A\cos(-Bx)$ using the odd and even properties. That is, for $B > 0$,

Rewrite $y = \sin(-Bx)$ as $y = -\sin Bx$ because $y = \sin x$ is an odd function.
Rewrite $y = \cos(-Bx)$ as $y = \cos Bx$ because $y = \cos x$ is an even function.

EXAMPLE 2 Graphing $y = A\sin Bx$

Given $f(x) = 4\sin 3x$,

a. Identify the amplitude and period.

b. Graph the function and identify the key points on one full period.

Solution:

a. $f(x) = 4\sin 3x$ is in the form $f(x) = A\sin Bx$ with $A = 4$ and $B = 3$.
The amplitude is $|A| = |4| = 4$.
The period is $\frac{2\pi}{B} = \frac{2\pi}{3}$.

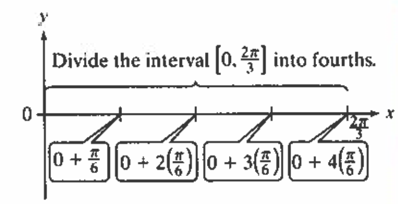
b. The period $\frac{2\pi}{3}$ is shorter than 2π, which tells us that the graph is compressed horizontally. For $y = \sin x$, one complete cycle can be graphed for x on the interval $0 \le x \le 2\pi$. For $y = 4\sin 3x$, one complete cycle can be graphed on the interval defined by $0 \le 3x \le 2\pi$.

$$0 \le 3x \le 2\pi$$

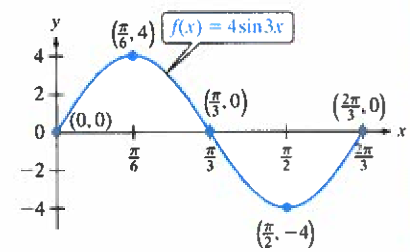
$$\frac{0}{3} \le \frac{3x}{3} \le \frac{2\pi}{3}$$

$$0 \le x \le \frac{2\pi}{3}$$

Dividing the period into fourths, we have increments of $\frac{1}{4}\left(\frac{2\pi}{3}\right) = \frac{\pi}{6}$.



Since the amplitude is 4, the maximum and minimum points have y-coordinates of 4 and -4.



Skill Practice 2 Given $f(x) = 2\sin 4x$,

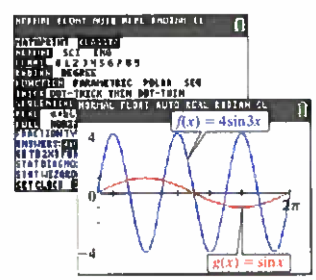
a. Identify the amplitude and period.

b. Graph the function and identify the key points on one full period.

TECHNOLOGY CONNECTIONS

Graphing a Trigonometric Function

To graph a trigonometric function, first be sure that the calculator is in radian mode. The graph of $f(x) = 4\sin 3x$ is shown along with the graph of $g(x) = \sin x$ for comparison. Notice that the graph of f deviates more from the x-axis than g because the amplitude is greater. The period of f is smaller and as a result, shows the sine wave "compressed" horizontally.



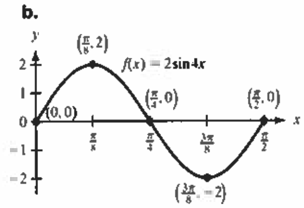
4. Graph $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$

Recall that the graph of $y = f(x - h)$ is the graph of $y = f(x)$ with a horizontal shift of $|h|$ units. If $h > 0$, the shift is to the right, and if $h < 0$, the shift is to the left. The graphs of $y = A\sin(Bx - C)$ and $y = A\cos(Bx - C)$ may have both a change in period and a horizontal shift. In Example 3, we illustrate how to graph such a function.

Answers

2. a. Amplitude: 2, Period: $\frac{\pi}{2}$

b.



EXAMPLE 3 Graphing $y = A\cos(Bx - C)$

Given $y = \cos\left(2x + \frac{\pi}{2}\right)$.

a. Identify the amplitude and period.

b. Graph the function and identify the key points on one full period.

Solution:

The function is of the form $y = A\cos(Bx - C)$, where $A = 1$, $B = 2$, and $C = -\frac{\pi}{2}$.

a. $y = \cos\left(2x + \frac{\pi}{2}\right)$ can be written as $y = 1 \cdot \cos\left(2x + \frac{\pi}{2}\right)$.

The amplitude is $|A| = |1| = 1$.

The period is $\frac{2\pi}{B} = \frac{2\pi}{2} = \pi$.

b. To find an interval over which this function completes one cycle, solve the inequality.

$$0 \le 2x + \frac{\pi}{2} \le 2\pi$$

$$-\frac{\pi}{2} \le 2x \le 2\pi - \frac{\pi}{2} \quad \text{Subtract } \frac{\pi}{2}.$$

$$-\frac{\pi}{2} \le 2x \le \frac{4\pi}{2} - \frac{\pi}{2} \quad \text{Write the right side with a common denominator. Then divide by 2.}$$

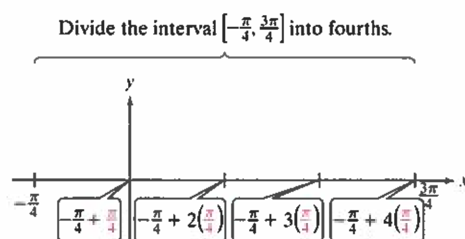
$$-\frac{\pi}{2} \le 2x \le \frac{3\pi}{2}$$

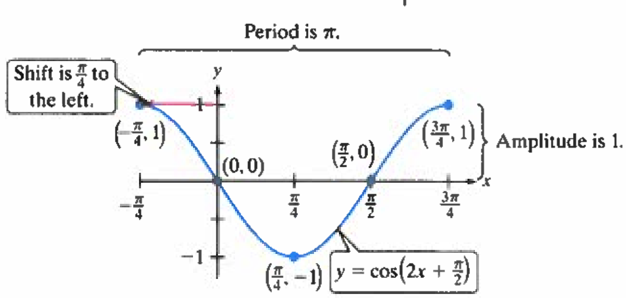
$$-\frac{\pi}{4} \le x \le \frac{3\pi}{4}$$

"Starting" value for the cycle

"Ending" value for the cycle

Dividing the period into fourths, we have increments of $\frac{1}{4}(\pi) = \frac{\pi}{4}$.





TIP Note that the distance between the "ending" point and "starting" point of the cycle shown is

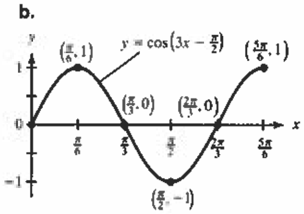
$$\frac{3\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{4\pi}{4} = \pi$$

which is the period of the function.

Answers

3. a. Amplitude: 1, Period: $\frac{2\pi}{3}$

b.



Skill Practice 3 Given $y = \cos\left(3x - \frac{\pi}{2}\right)$.

a. Identify the amplitude and period.

b. Graph the function and identify the key points on one full period.

In Example 3, the graph of $y = \cos\left(2x + \frac{\pi}{2}\right)$ is the graph of $y = \cos x$ compressed horizontally by a factor of 2 and shifted to the left $\frac{\pi}{4}$ units. The shift to the left is called the *phase shift*. In general, given $y = A\sin(Bx - C)$ and $y = A\cos(Bx - C)$ for $B > 0$, the horizontal transformations are controlled by the variables B and C. The value of B controls the horizontal shrink or stretch, and thus the period of the function. For $B > 0$, the phase shift can be determined by solving the following inequality.

$$0 \le Bx - C \le 2\pi$$

$$C \le Bx \le 2\pi + C \qquad \text{Add } C \text{ to each part.}$$

$$\frac{C}{B} \le \frac{Bx}{B} \le \frac{2\pi + C}{B} \qquad \text{Divide by } B\ (B > 0).$$

The phase shift is the left endpoint.

$$\frac{C}{B} \le x \le \frac{2\pi + C}{B} \qquad \text{The graph is shifted horizontally } \frac{C}{B} \text{ units. The phase shift is } \frac{C}{B}.$$

The phase shift is a horizontal shift of a trigonometric function. To find the vertical shift, recall that the graph of $y = f(x) + D$ is the graph of $y = f(x)$ shifted $|D|$ units upward for $D > 0$ and $|D|$ units downward if $D < 0$. We are now ready to summarize the properties of the graphs of $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$.

Properties of the General Sine and Cosine Functions

Consider the graphs of $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$ with $B > 0$.

1. The amplitude is $|A|$.
2. The period is $\frac{2\pi}{B}$.
3. The phase shift is $\frac{C}{B}$.
4. The vertical shift is D.
5. One full cycle is given on the interval $0 \le Bx - C \le 2\pi$.
6. The domain is $-\infty < x < \infty$.
7. The range is $-|A| + D \le y \le |A| + D$.

EXAMPLE 4 Graphing $y = A\cos(Bx - C) + D$

Given $y = 2\cos(4x - 3\pi) + 5$,

a. Identify the amplitude, period, phase shift, and vertical shift.

b. Graph the function and identify the key points on one full period.

Solution:

a. $y = 2\cos(4x - 3\pi) + 5$ has the form $y = A\cos(Bx - C) + D$, where $A = 2$, $B = 4$, $C = 3\pi$, and $D = 5$.

The amplitude is $|A| = |2| = 2$.

The period is $\frac{2\pi}{B} = \frac{2\pi}{4} = \frac{\pi}{2}$.

The phase shift is $\frac{C}{B} = \frac{3\pi}{4}$.

The vertical shift is D. Since $D = 5 > 0$, the shift is upward 5 units.

b. To find an interval over which this function completes one cycle, solve the inequality.

$$0 \le 4x - 3\pi \le 2\pi$$

$$3\pi \le 4x \le 2\pi + 3\pi \qquad \text{Add } 3\pi.$$

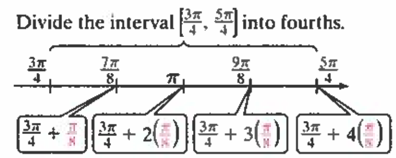
$$\frac{3\pi}{4} \le \frac{4x}{4} \le \frac{5\pi}{4} \qquad \text{Divide by 4.}$$

"Starting" value for the cycle

$$\frac{3\pi}{4} \le x \le \frac{5\pi}{4}$$

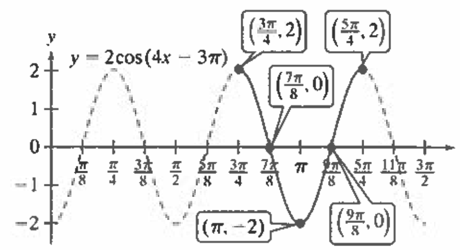
"Ending" value for the cycle

Dividing the period into fourths, we have increments of $\frac{1}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{8}$.

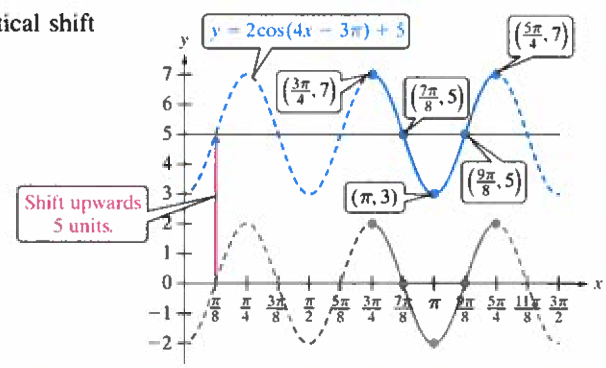


First sketch the function on the interval $\left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ without the vertical shift (solid gray curve).

The dashed curve is a continuation of this pattern.



Now apply the vertical shift upwards 5 units.



Skill Practice 4 Given $y = 2\cos(3x - \pi) + 3$,

a. Identify the amplitude, period, phase shift, and vertical shift.

b. Graph the function and identify the key points on one full period.

Answers

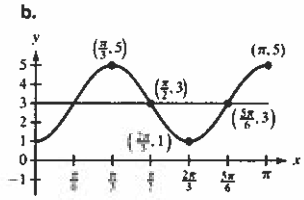
4. a. Amplitude: 2,

Period: $\frac{2\pi}{3}$,

Phase shift: $\frac{\pi}{3}$,

Vertical shift: 3

b.



In Example 4, notice that the x-intercepts of the graph before the vertical shift become the points where the graph intersects the line $y = 5$ after the vertical shift. The curve oscillates above and below the line $y = 5$ rather than the x-axis. In a sense, $y = 5$ is the "central" value or "equilibrium" value of the function. This line represents the midpoint of the range, and the amplitude tells us by how much the function deviates from this line.

EXAMPLE 5 Graphing $y = A\sin(Bx - C) + D$

Given $y = 3\sin\left(-\frac{\pi}{4}x - \frac{\pi}{2}\right) - 4$,

a. Identify the amplitude, period, phase shift, and vertical shift.

b. Graph the function and identify the key points on one full period.

Solution:

a. First note that the coefficient on x in the argument is not positive. We want to write the function in the form $y = A\sin(Bx - C) + D$, where $B > 0$.

$$y = 3\sin\left[-1\left(\frac{\pi}{4}x + \frac{\pi}{2}\right)\right] - 4$$ Factor out -1 from the argument.

$$y = -3\sin\left(\frac{\pi}{4}x + \frac{\pi}{2}\right) - 4$$ The sine function is an odd function. $\sin(-x) = -\sin(x)$

$$y = -3\sin\left[\frac{\pi}{4}x - \left(-\frac{\pi}{2}\right)\right] + (-4)$$ Write the equation in the form $y = A\sin(Bx - C) + D$, where $B > 0$.

$$A = -3,\ B = \frac{\pi}{4},\ C = -\frac{\pi}{2},\ \text{and } D = -4$$

The amplitude is $|A| = |-3| = 3$.

The period is $\dfrac{2\pi}{B} = \dfrac{2\pi}{\frac{\pi}{4}} = 2\pi \cdot \dfrac{4}{\pi} = 8$.

The phase shift is $\dfrac{C}{B} = \dfrac{-\frac{\pi}{2}}{\frac{\pi}{4}} = -\dfrac{\pi}{2} \cdot \dfrac{4}{\pi} = -2$.

The vertical shift is D. Since $D = -4 < 0$, the shift is downward 4 units.

b. To find an interval over which this function completes one cycle, solve the inequality.

$$0 \le \frac{\pi}{4}x - \left(-\frac{\pi}{2}\right) \le 2\pi$$

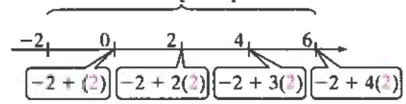
$$-\frac{\pi}{2} \le \frac{\pi}{4}x \le \frac{3\pi}{2}$$ Add $-\frac{\pi}{2}$.

$$\frac{4}{\pi}\cdot\left(-\frac{\pi}{2}\right) \le \frac{4}{\pi}\cdot\left(\frac{\pi}{4}x\right) \le \frac{4}{\pi}\cdot\left(\frac{3\pi}{2}\right)$$ Multiply by $\frac{4}{\pi}$.

"Starting" value for the cycle $\quad -2 \le x \le 6 \quad$ "Ending" value for the cycle

Dividing the period into fourths, we have increments of $\frac{1}{4}(8) = 2$.

Divide the interval $[-2, 6]$ into fourths.



Avoiding Mistakes

A factor from the argument of a function cannot be factored out in front of the function. For example,

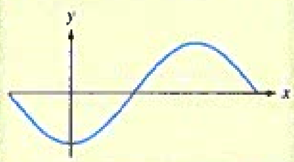
$$f(2x) \neq 2f(x)$$

In step 2 of Example 5(a), the value -1 that appears in front of the function was not factored out, but rather is a result of the odd function property of the sine function.

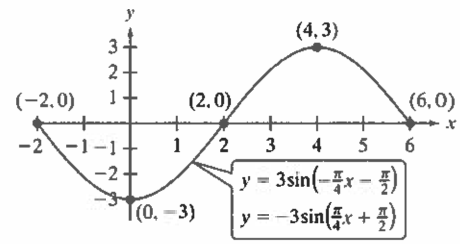
$$\sin(-x) = -\sin(x)$$

TIP The negative sign on the phase shift indicates that the shift is to the left.

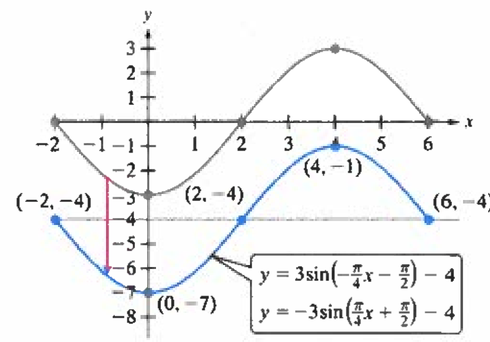
TIP For the equation $y = -3\sin\left(\frac{\pi}{4}x + \frac{\pi}{2}\right)$, since $A = -3$, the general shape of the curve "starting" at the phase shift is an inverted sine curve.



First sketch the function without the vertical shift.



Now apply the vertical shift downward 4 units.



Skill Practice 5 Given $y = -2\sin\left(-\frac{\pi}{6}x - \frac{\pi}{2}\right) + 1$,

a. Identify the amplitude, period, phase shift, and vertical shift.

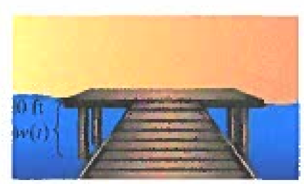
b. Graph the function and identify the key points on one full period.

5. Model Sinusoidal Behavior

To this point, we have taken an equation of a sine or cosine function and sketched its graph. Now we reverse the process. In Example 6, we take observed data that follow a "wavelike" pattern similar to a sine or cosine graph and build a model. When the graph of a data set is shaped like a sine or cosine graph, we say the graph is **sinusoidal.**

EXAMPLE 6 Modeling the Level of the Tide

The water level relative to the top of a boat dock varies with the tides. One particular day, low tide occurs at midnight and the water level is 7 ft below the dock. The first high tide of the day occurs at approximately 6:00 A.M., and the water level is 3 ft below the dock. The next low tide occurs at noon and the water level is again 7 ft below the dock.

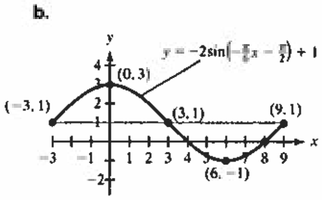


Assuming that this pattern continues indefinitely and behaves like a cosine wave, write a function of the form $w(t) = A\cos(Bt - C) + D$. The value $w(t)$ is the water level (in ft) relative to the top of the dock, t hours after midnight.

Answers

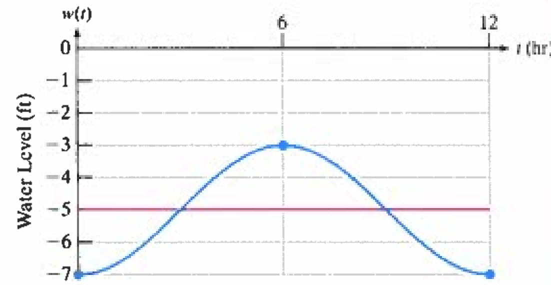
5. a. Amplitude: 2,
Period: 12,
Phase shift: −3,
Vertical shift: 1

b.



Solution:

Plotting the water level at midnight, 6:00 A.M., and noon helps us visualize the curve. By inspection, the curve behaves like a cosine reflected across the t-axis and shifted down 5 units.



$|A| = \frac{1}{2}[-3 - (-7)] = \frac{1}{2}(4) = 2$ — The amplitude of the curve is half the distance between the highest value and lowest value.

Vertical shift: $D = \dfrac{-7 + (-3)}{2} = -5$ — The midpoint of the range gives us the vertical shift.

$P = \dfrac{2\pi}{B}$, which implies that $12 = \dfrac{2\pi}{B}$. — One complete cycle takes place between $t = 0$ and $t = 12$ (from low tide to low tide). Therefore, the period P is $12 - 0 = 12$.

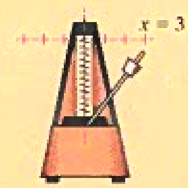
Therefore, $B = \dfrac{2\pi}{12} = \dfrac{\pi}{6}$.

Since a minimum value of the curve occurs at $t = 0$, there is no phase shift for this cosine function, implying that $C = 0$.

Finally, the amplitude is 2, however, we take A to be *negative* because the graph was reflected across the t-axis before being shifted downward.

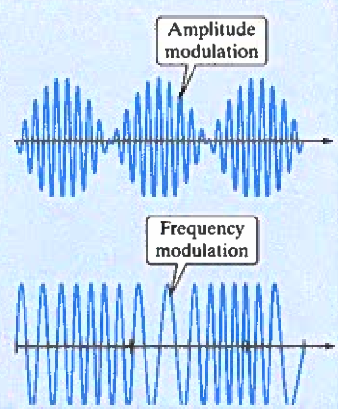
$w(t) = -2\cos\left(\frac{\pi}{6}t\right) - 5$ — Substitute $A = -2$, $B = \frac{\pi}{6}$, $C = 0$, and $D = -5$, into $w(t) = A\cos(Bt - C) + D$.

Skill Practice 6 A mechanical metronome uses an inverted pendulum that makes a regular, rhythmic click as it swings to the left and right. With each swing, the pendulum moves 3 in. to the left and right of the center position. The pendulum is initially pulled to the right 3 in. and then released. It returns to its starting position in 0.8 sec. Assuming that this pattern continues indefinitely and behaves like a cosine wave, write a function of the form $x(t) = A\cos(Bt - C) + D$. The value $x(t)$ is the horizontal position (in inches) relative to the center line of the pendulum.



Point of Interest

AM (amplitude modulation) and FM (frequency modulation) radio are ways of broadcasting radio signals by sending electromagnetic waves through space from a transmitter to a receiver. Each method transmits information in the form of electromagnetic waves. AM works by modulating (or varying) the amplitude of the signal while the frequency remains constant. FM works by varying the frequency and keeping the amplitude constant. In 1873, James Maxwell showed mathematically that electromagnetic waves could propagate through free space. Today the far-reaching applications of wireless radio technology include use in televisions, computers, cell phones, and even deep-space radio communications.



Answer

6. $x(t) = 3\cos\left(\frac{5\pi}{2}t\right)$

SECTION 4.5 Practice Exercises

Prerequisite Review

For Exercises R.1–R.5, use translations to graph the function.

R.1. $h(x) = |x| - 1$ **R.2.** $n(x) = |x + 1|$ **R.3.** $b(x) = 2|x|$ **R.4.** $h(x) = \frac{1}{3}|x|$ **R.5.** $p(x) = -|x|$

Concept Connections

1. The value of $\sin x$ (increases/decreases) ______ on $\left(0, \frac{\pi}{2}\right)$ and (increases/decreases) ______ on $\left(\frac{\pi}{2}, \pi\right)$.

2. The value of $\cos x$ (increases/decreases) ______ on $\left(0, \frac{\pi}{2}\right)$ and (increases/decreases) ______ on $\left(\frac{\pi}{2}, \pi\right)$.

3. The graph of $y = \sin x$ and $y = \cos x$ differ by a horizontal shift of ______ units.

4. Given $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$, for $B > 0$ the amplitude is ______, the period is ______, the phase shift is ______, and the vertical shift is ______.

5. The sine function is an (even/odd) ______ function because $\sin(-x) =$ ______. The cosine function is an (even/odd) ______ function because $\cos(-x) =$ ______.

6. Given $B > 0$, how would the equation $y = A\sin(-Bx - C) + D$ be rewritten to obtain a positive coefficient on x?

7. Given $B > 0$, how would the equation $y = A\cos(-Bx - C) + D$ be rewritten to obtain a positive coefficient on x?

8. Given $y = \sin(Bx)$ and $y = \cos(Bx)$, for $B > 1$, is the period less than or greater than 2π? If $0 < B < 1$, is the period less than or greater than 2π?

Objective 1: Graph $y = \sin x$ and $y = \cos x$

9. From memory, sketch $y = \sin x$ on the interval $[0, 2\pi]$.

10. From memory, sketch $y = \cos x$ on the interval $[0, 2\pi]$.

11. For $y = \cos x$,

a. The domain is ______.

b. The range is ______.

c. The amplitude is ______.

d. The period is ______.

e. The cosine function is symmetric to the ______ -axis.

f. On the interval $[0, 2\pi]$, the x-intercepts are ______.

g. On the interval $[0, 2\pi]$, the maximum points are ______ and ______, and the minimum point is ______.

12. For $y = \sin x$,

a. The domain is ______.

b. The range is ______.

c. The amplitude is ______.

d. The period is ______.

e. The sine function is symmetric to the ______.

f. On the interval $[0, 2\pi]$, the x-intercepts are ______.

g. On the interval $[0, 2\pi]$, the maximum point is ______ and the minimum point is ______.

13. a. Over what interval(s) taken between 0 and 2π is the graph of $y = \sin x$ increasing?

b. Over what interval(s) taken between 0 and 2π is the graph of $y = \sin x$ decreasing?

14. a. Over what interval(s) taken between 0 and 2π is the graph of $y = \cos x$ increasing?

b. Over what interval(s) taken between 0 and 2π is the graph of $y = \cos x$ decreasing?

Objective 2: Graph $y = A\sin x$ and $y = A\cos x$

For Exercises 15–16, identify the amplitude of the function.

15. a. $y = 7\sin x$

b. $y = \frac{1}{7}\sin x$

c. $y = -7\sin x$

16. a. $y = 2\cos x$

b. $y = \frac{1}{2}\cos x$

c. $y = -2\cos x$

17. By how many units does the graph of $y = \frac{1}{4}\cos x$ deviate from the x-axis?

18. By how many units does the graph of $y = -5\sin x$ deviate from the x-axis?

For Exercises 19–24, graph the function and identify the key points on one full period. (See Example 1)

19. $y = 5\cos x$ **20.** $y = 4\sin x$ **21.** $y = \frac{1}{2}\sin x$ **22.** $y = \frac{1}{4}\cos x$

23. $y = -2\cos x$ **24.** $y = -3\sin x$

Objective 3: Graph $y = A\sin Bx$ and $y = A\cos Bx$

For Exercises 25–26, identify the period.

25. a. $\sin 2x$ **b.** $\sin 2\pi x$ **c.** $\sin\left(-\frac{2}{3}x\right)$

26. a. $\cos\frac{1}{3}x$ **b.** $\cos(-3\pi x)$ **c.** $\cos\frac{1}{3}\pi x$

For Exercises 27–32

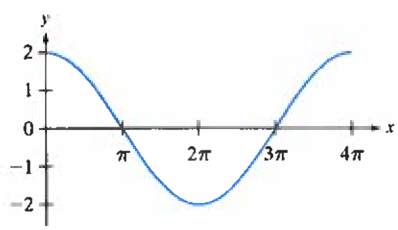
a. Identify the amplitude and period.

b. Graph the function and identify the key points on one full period. **(See Example 2)**

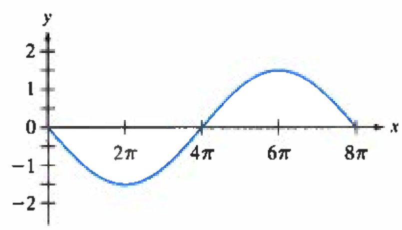
27. $y = 2\cos 3x$ **28.** $y = 6\sin 4x$ **29.** $y = 4\sin\frac{\pi}{3}x$ **30.** $y = 5\cos\frac{\pi}{6}x$

31. $y = \sin\left(-\frac{1}{3}x\right)$ **32.** $y = \cos\left(-\frac{1}{2}x\right)$

33. Write a function of the form $f(x) = A\cos Bx$ for the given graph.



34. Write a function of the form $g(x) = A\sin Bx$ for the given graph.

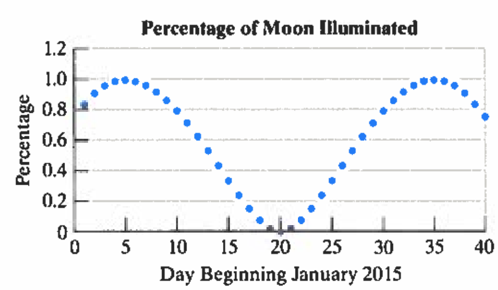


35. The graph shows the percentage in decimal form of the moon illuminated for the first 40 days of a recent year. (*Source*: Astronomical Applications Department, U.S. Naval Observatory: http://aa.usno.navy.mil)

a. On approximately which days during this time period did a full moon occur? (A full moon corresponds to 100% or 1.0.)

b. On which day was there a new moon (no illumination)?

c. From the graph, approximate the period of a synodic month. A synodic month is the period of one lunar cycle (full moon to full moon).



36. A respiratory cycle is defined as the beginning of one breath to the beginning of the next breath. The rate of air intake r (in L/sec) during a respiratory cycle for a physically fit male can be approximated by $r(t) = 0.9\sin\frac{\pi}{3.5}t$, where t is the number of seconds into the cycle. A positive value for r represents inhalation and a negative value represents exhalation.

a. How long is the respiratory cycle?
b. What is the maximum rate of air intake?
c. Graph one cycle of the function. On what interval does inhalation occur? On what interval does exhalation occur?

Objective 4: Graph $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$

For Exercises 37–38, identify the phase shift and indicate whether the shift is to the left or to the right.

37. a. $\cos\left(x - \frac{\pi}{3}\right)$ **b.** $\cos\left(2x - \frac{\pi}{3}\right)$ **c.** $\cos\left(3\pi x + \frac{\pi}{3}\right)$

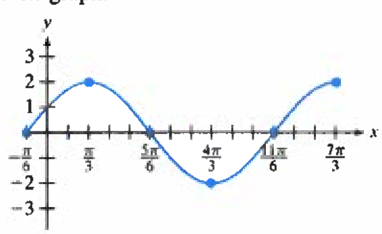
38. a. $\sin\left(x + \frac{\pi}{8}\right)$ **b.** $\sin\left(2\pi x - \frac{\pi}{8}\right)$ **c.** $\sin\left(4x - \frac{\pi}{8}\right)$

For Exercises 39–44,
a. Identify the amplitude, period, and phase shift.
b. Graph the function and identify the key points on one full period. (**See Example 3**)

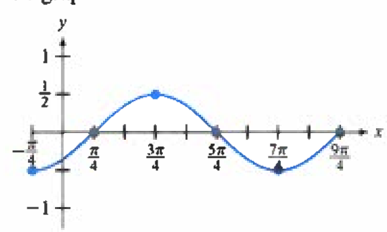
39. $y = 2\cos(x + \pi)$ **40.** $y = 4\sin\left(x + \frac{\pi}{2}\right)$ **41.** $y = \sin\left(2x - \frac{\pi}{3}\right)$

42. $y = \cos\left(3x - \frac{\pi}{4}\right)$ **43.** $y = -6\cos\left(\frac{1}{2}x + \frac{\pi}{4}\right)$ **44.** $y = -5\sin\left(\frac{1}{3}x + \frac{\pi}{6}\right)$

45. Write a function of the form $f(x) = A\cos(Bx - C)$ for the given graph.



46. Write a function of the form $f(x) = A\sin(Bx - C)$ for the given graph.



47. Given $y = -2\sin\left(2x - \frac{\pi}{6}\right) - 7$,

a. Is the period less than or greater than 2π?
b. Is the phase shift to the left or right?
c. Is the vertical shift upward or downward?

48. Given $y = \cos\left(\frac{1}{2}x + \pi\right) + 4$,

a. Is the period less than or greater than 2π?
b. Is the phase shift to the left or right?
c. Is the vertical shift upward or downward?

For Exercises 49–52, rewrite the equation so that the coefficient on x is positive.

49. $y = \cos\left(-2x + \frac{\pi}{6}\right) - 4$ **50.** $y = 4\cos(-3x - \pi) + 5$

51. $y = \sin\left(-2x + \frac{\pi}{6}\right) - 4$ **52.** $y = 4\sin(-3x - \pi) + 5$

53. Given $y = 2\sin\left(-\frac{\pi}{6}x + \frac{\pi}{2}\right) - 3$, is the phase shift to the right or left?

54. Given $y = -4\cos\left(-\frac{\pi}{6}x - \frac{\pi}{2}\right) + 1$, is the phase shift to the right or left?

For Exercises 55–68,

a. Identify the amplitude, period, phase shift, and vertical shift.

b. Graph the function and identify the key points on one full period. **(See Examples 4–5)**

55. $h(x) = 3\sin(4x - \pi) + 5$

56. $g(x) = 2\sin(3x - \pi) - 4$

57. $f(x) = 4\cos\left(3x - \frac{\pi}{2}\right) - 1$

58. $k(x) = 5\cos\left(2x - \frac{\pi}{2}\right) + 1$

59. $y = \frac{1}{2}\sin\left(-\frac{1}{3}x\right)$

60. $y = \frac{2}{3}\sin\left(-\frac{1}{2}x\right)$

61. $v(x) = 1.6\cos(-\pi x)$

62. $m(x) = 2.4\cos(-4\pi x)$

63. $y = 2\sin(-2x - \pi) + 5$

64. $y = 3\sin(-4x - \pi) - 7$

65. $p(x) = -\cos\left(-\frac{\pi}{2}x - \pi\right) + 2$

66. $q(x) = -\cos\left(-\frac{\pi}{3}x - \pi\right) - 2$

67. $y = \sin\left(-\frac{\pi}{4}x - \frac{\pi}{2}\right) - 3$

68. $y = \sin\left(-\frac{\pi}{3}x - \frac{\pi}{2}\right) + 4$

69. The temperature T (in °F) for Kansas City, Missouri, over a several day period in April can be approximated by $T(t) = -5.9\cos(0.262t - 1.245) + 48.2$, where t is the number of hours since midnight on day 1.

a. What is the period of the function? Round to the nearest hour.

b. What is the significance of the term 48.2 in this model?

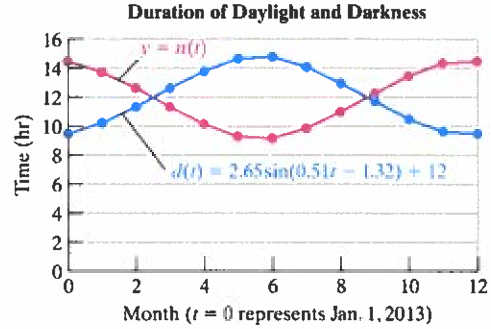
c. What is the significance of the factor 5.9 in this model?

d. What was the minimum temperature for the day? When did it occur?

e. What was the maximum temperature for the day? When did it occur?

70. The duration of daylight and darkness varies during the year due to the angle of the Sun in the sky. The model $d(t) = 2.65\sin(0.51t - 1.32) + 12$ approximates the amount of daylight $d(t)$ (in hours) for Sacramento, California, as a function of the time t (in months) after January 1 for a recent year; that is, $t = 0$ is January 1, $t = 1$ is February 1, and so on. The model $y = n(t)$ represents the amount of darkness as a function of t.

Duration of Daylight and Darkness



a. Describe the relationship between the graphs of the functions and the line $y = 12$.

b. Use the result of part (a) and a transformation of $y = d(t)$ to write an equation representing n as a function of t.

c. What do the points of intersection of the two graphs represent?

d. What do the relative minima and relative maxima of the graphs represent?

e. What does $T(t) = d(t) + n(t)$ represent?

Objective 5: Model Sinusoidal Behavior

71. The probability of precipitation in Modesto, California, varies from a peak of 0.34 (34%) in January to a low of 0.04 (4%) in July. Assume that the percentage of precipitation varies monthly and behaves like a cosine curve.

a. Write a function of the form $P(t) = A\cos(Bt - C) + D$ to model the precipitation probability. The value $P(t)$ is the probability of precipitation (as a decimal), for month t, with January as $t = 1$.

b. Graph the function from part (a) on the interval [0, 13] and plot the points (1, 0.34), (7, 0.04), and (13, 0.34) to check the accuracy of your model.

72. The monthly high temperature for Atlantic City, New Jersey, peaks at an average high of 86° in July and goes down to an average high of 64° in January. Assume that this pattern for monthly high temperatures continues indefinitely and behaves like a cosine wave.

a. Write a function of the form $H(t) = A\cos(Bt - C) + D$ to model the average high temperature. The value $H(t)$ is the average high temperature for month t, with January as $t = 0$.

b. Graph the function from part (a) on the interval [0, 13] and plot the points (0, 64), (6, 86), and (12, 64) to check the accuracy of your model.

73. An adult human at rest inhales and exhales approximately 500 mL of air (called the tidal volume) in approximately 5 sec. However, at the end of each exhalation, the lungs still contain a volume of air, called the functional residual capacity (FRC), which is approximately 2000 mL. **(See Example 6)**

a. What volume of air is in the lungs after inhalation?

b. What volume of air is in the lungs after exhalation?

c. What is the period of a complete respiratory cycle?

d. Write a function $V(t) = A\cos Bt + D$ to represent the volume of air in the lungs t seconds after the end of an inhalation.

e. What is the average amount of air in the lungs during one breathing cycle?

f. During hyperventilation, breathing is more rapid with deep inhalations and exhalations. What parts of the equation from part (d) change?

74. The times for high and low tides are given in the table for a recent day in Jacksonville Beach, Florida. The times are rounded to the nearest hour and the tide levels are measured relative to mean sea level (MSL).

a. Write a model $h(t) = A\cos(Bt - C)$ to represent the tide level $h(t)$ (in feet) in terms of the amount of time t elapsed since midnight.

b. Use the model from part (a) to estimate the tide level at 3:00 P.M.

	Time (hr after midnight)	Height Relative to MSL (ft)
High tide	0	3.4
Low tide	6	−3.4
High tide	12	3.4
Low tide	18	−3.4
High tide	24	3.4

75. The data in the table represent the monthly power bills (in dollars) for a homeowner in southern California.

a. Enter the data in a graphing utility and use the sinusoidal regression tool (SinReg) to find a model of the form $A(t) = a\sin(bt + c) + d$, where $A(t)$ represents the amount of the bill for month t ($t = 1$ represents January, $t = 2$ represents February, and so on).

b. Graph the data and the resulting function.

Month, t	1	2	3	4	5	6
Amount ($)	104.73	66.13	48.99	56.04	85.51	98.57

Month, t	7	8	9	10	11	12
Amount ($)	125.08	124.48	113.93	81.06	63.30	71.85

76. The data in the table represent the duration of daylight $d(t)$ (in hours) for Houston, Texas, for the first day of the month, t months after January 1 for a recent year. (*Source*: Astronomical Applications Department, U.S. Naval Observatory: http://aa.usno.navy.mil)

a. Enter the data in a graphing utility and use the sinusoidal regression tool (SinReg) to find a model of the form $d(t) = a\sin(bt + c) + d$.

b. Graph the data and the resulting function.

Month, t	0	1	2	3	4	5
Time (hr)	10.28	10.80	11.57	12.48	13.65	13.95

Month, t	6	7	8	9	10	11
Time (hr)	14.02	13.55	12.73	11.87	11.00	10.38

Mixed Exercises

For Exercises 77–78, write the range of the function in interval notation.

77. a. $y = 8\cos(2x - \pi) + 4$

b. $y = -3\cos\left(x + \frac{\pi}{3}\right) - 5$

78. a. $y = -6\sin\left(3x - \frac{\pi}{2}\right) - 2$

b. $y = 2\sin(3x + 2\pi) + 12$

79. Given $f(x) = \cos x$ and $h(x) = 3x + 2$, find $(h \circ f)(x)$ and for $(h \circ f)(x)$,

a. Find the amplitude.

b. Find the period.

c. Write the domain in interval notation.

d. Write the range in interval notation.

80. Given $g(x) = \sin x$ and $k(x) = 6x$, find $(g \circ k)(x)$ and for $(g \circ k)(x)$,

a. Find the amplitude.

b. Find the period.

c. Write the domain in interval notation.

d. Write the range in interval notation.

81. Write a function of the form $y = A\sin(Bx - C) + D$ that has period $\frac{\pi}{3}$, amplitude 4, phase shift $\frac{\pi}{2}$, and vertical shift 5.

82. Write a function of the form $y = A\cos(Bx - C) + D$ that has period $\frac{\pi}{4}$, amplitude 2, phase shift $-\frac{\pi}{3}$, and vertical shift 7.

83. Write a function of the form $y = A\cos(Bx - C) + D$ that has period 16, phase shift −4, and range $3 \le y \le 7$.

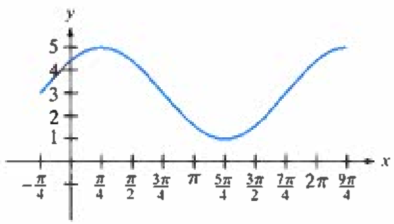
84. Write a function of the form $y = A\sin(Bx - C) + D$ that has period 8, phase shift −2, and range $-14 \le y \le -6$.

For Exercises 85–86,

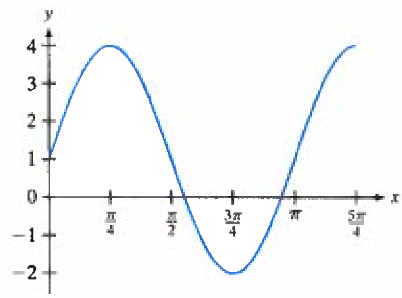
a. Write an equation of the form $y = A\cos(Bx - C) + D$ with $A > 0$ to model the graph.

b. Write an equation of the form $y = A\sin(Bx - C) + D$ with $A > 0$ to model the graph.

85.



86.



For Exercises 87–90, explain how to graph the given function by performing transformations on the "parent" graphs $y = \sin x$ and $y = \cos x$.

87. a. $y = \sin 2x$
b. $y = 2\sin x$

88. a. $y = \frac{1}{3}\cos x$
b. $y = \cos\frac{1}{3}x$

89. a. $y = \sin(x + 2)$
b. $y = \sin x + 2$

90. a. $y = \cos x - 4$
b. $y = \cos(x - 4)$

Write About It

91. Is $f(x) = \sin x$ one-to-one? Explain why or why not.

92. Is $f(x) = \cos x$ one-to-one? Explain why or why not.

93. If f and g are both periodic functions with period P, is $(f + g)(x)$ also periodic? Explain why or why not.

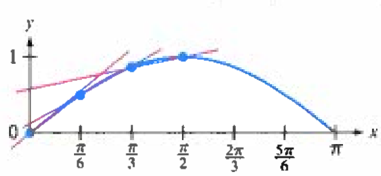
Expanding Your Skills

94. Explain why a function that is increasing on its entire domain cannot be periodic.

For Exercises 95–96, find the average rate of change on the given interval. Give the exact value and an approximation to 4 decimal places. Verify that your results are reasonable by comparing the results to the slopes of the lines given in the graph.

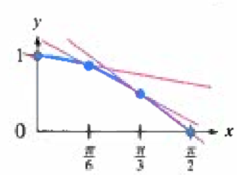
95. $f(x) = \sin x$

a. $\left[0, \frac{\pi}{6}\right]$ **b.** $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ **c.** $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$



96. $f(x) = \cos x$

a. $\left[0, \frac{\pi}{6}\right]$ **b.** $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ **c.** $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$



For Exercises 97–100, use your knowledge of the graphs of the sine function and linear functions to determine the number of solutions to the equation.

97. $\sin x = x - 2$

98. $\cos x = -x$

99. $\sin 2x = -2$

100. $2\sin 2x = -2$

For Exercises 101–102, graph the piecewise-defined function.

101. $g(x) = \begin{cases} \sin x & \text{for} \quad 0 \le x \le \pi \\ -\sin x & \text{for} \quad \pi < x \le 2\pi \end{cases}$

102. $f(x) = \begin{cases} \cos x & \text{for} \quad 0 \le x \le \frac{\pi}{4} \\ \sin x & \text{for} \quad \frac{\pi}{4} < x \le \frac{\pi}{2} \end{cases}$

Technology Connections

103. Functions a and m approximate the duration of daylight, respectively, for Albany, New York, and Miami, Florida, for a given year for day t. The value $t = 1$ represents January 1, $t = 2$ represents February 1, and so on.

$$a(t) = 12 + 3.1\sin\left[\frac{2\pi}{365}(t - 80)\right] \qquad m(t) = 12 + 1.6\sin\left[\frac{2\pi}{365}(t - 80)\right]$$

a. Graph the two functions with a graphing utility and comment on the difference between the two graphs.

b. Both functions have a constant term of 12. What does this represent graphically and in the context of this problem?

c. What do the factors 3.1 and 1.6 represent in the two functions?

d. What is the period of each function?

e. What does the horizontal shift of 80 units represent in the context of this problem.

f. Use the Intersect feature to approximate the points of intersection.

g. Interpret the meaning of the points of intersection.

For Exercises 104–105, we demonstrate that trigonometric functions can be approximated by polynomial functions over a given interval in the domain.

Graph functions f, g, h, and k on the viewing window $-4 \le x \le 4$, $-4 \le y \le 4$. Then use a Table feature on a graphing utility to evaluate each function for the given values of x. How do functions g, h, and k compare to function f for x values farther from 0? [*Hint*: For a given natural number n, the value $n!$, read as "n factorial," is defined as $n! = n(n-1)(n-2)\cdots 1$. For example, $3! = 3 \cdot 2 \cdot 1 = 6$.]

104.

Function	$x = 0.1$	$x = 0.5$	$x = 1.0$	$x = 1.5$
$f(x) = \cos x$				
$g(x) = 1 - \frac{x^2}{2!}$				
$h(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$				
$k(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$				

105.

Function	$x = 0.1$	$x = 0.5$	$x = 1.0$	$x = 1.5$
$f(x) = \sin x$				
$g(x) = x - \frac{x^3}{3!}$				
$h(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$				
$k(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$				

For Exercises 106–107, use a graph to solve the equation on the given interval.

106. $\cos\left(2x - \frac{\pi}{3}\right) = 0.5$ on $[0, \pi]$

Viewing window: $\left[0, \pi, \frac{\pi}{3}\right]$ by $\left[-1, 1, \frac{1}{2}\right]$

107. $\sin\left(2x + \frac{\pi}{4}\right) = 1$ on $[0, 2\pi]$

Viewing window: $\left[0, 2\pi, \frac{\pi}{8}\right]$ by $[-2, 2, 1]$

For Exercises 108–109, use a graph to solve the equation on the given interval. Round the answer to 2 decimal places.

108. $\sin\left(x - \frac{\pi}{4}\right) = -e^x$ on $[-\pi, \pi]$

Viewing window: $\left[-\pi, \pi, \frac{\pi}{2}\right]$ by $[-2, 2, 1]$

109. $6\cos\left(x + \frac{\pi}{6}\right) = \ln x$ on $[0, 2\pi]$

Viewing window: $\left[0, 2\pi, \frac{\pi}{2}\right]$ by $[-7, 7, 1]$

110. Graph the functions on the window provided.

a. $y = 2$ $\quad y = -2$ $\quad y = 2\sin x$

Viewing window: $\left[-2\pi, 2\pi, \frac{\pi}{2}\right]$ by $[-3, 3, 1]$

b. $y = 0.5x$ $\quad y = -0.5x$ $\quad y = 0.5x\sin x$

Viewing window: $[-5\pi, 5\pi, \pi]$ by $[-8, 8, 1]$

c. $y = \cos x$ $\quad y = -\cos x$ $\quad y = \cos x \cdot \sin 12x$

Viewing window: $[0, 2.5\pi, 0.25\pi]$ by $[-1, 1, 0.5]$

d. Explain the relationship among the three functions in parts (a), (b), and (c).

SECTION 4.7 Inverse Trigonometric Functions

OBJECTIVES

1. Evaluate the Inverse Sine Function
2. Evaluate the Inverse Cosine and Tangent Functions
3. Approximate Inverse Trigonometric Functions on a Calculator
4. Compose Trigonometric Functions and Inverse Trigonometric Functions
5. Apply Inverse Trigonometric Functions
6. Evaluate the Inverse Secant, Cosecant, and Cotangent Functions

1. Evaluate the Inverse Sine Function

Suppose that a yardstick casts a 4-ft shadow when the Sun is at an angle of elevation θ (Figure 4-55). It seems reasonable that we should be able to determine the angle of elevation from the relationship $\tan\theta = \frac{3}{4}$.

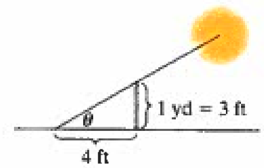


Figure 4-55

Until now, we have always been given an angle and then asked to find the sine, cosine, or tangent of the angle. However, finding the angle of elevation of the Sun from $\tan\theta = \frac{3}{4}$ requires that we reverse this process. Given the value of the tangent, we must find an angle that produced it. Therefore, we need to use the inverse of the tangent function.

We begin our study of the inverse trigonometric functions with the inverse of the sine function. First recall that a function must be one-to-one to have an inverse function. From the graph of $y = \sin x$ (Figure 4-56), we see that any horizontal line taken between $-1 \le y \le 1$ intersects the graph infinitely many times. Therefore, $y = \sin x$ is not a one-to-one function.

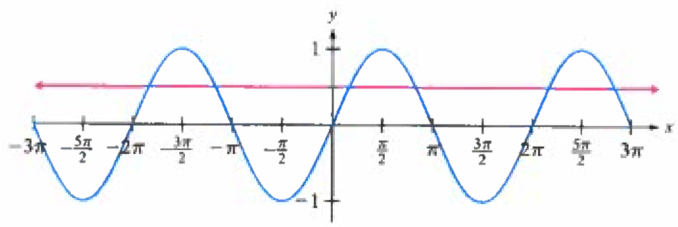


Figure 4-56

However, suppose we restrict the domain of $y = \sin x$ to the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ (shown in blue in Figure 4-57). The graph of the restricted sine function is one-to-one and contains the entire range of y values $-1 \le y \le 1$. The inverse of this restricted sine function is called the inverse sine function and is denoted by $\sin^{-1}$ or arcsin (shown in red in Figure 4-58). Recall that a point (a, b) on the graph of a function f corresponds to the point (b, a) on its inverse. Thus, points on the graph of $y = \sin x$ such as $\left(-\frac{\pi}{2}, -1\right)$ and $\left(\frac{\pi}{2}, 1\right)$ have their coordinates reversed on the graph of $y = \sin^{-1}x$. Also notice that the graphs of $y = \sin x$ and $y = \sin^{-1}x$ are symmetric with respect to the line $y = x$ as expected.

TIP The horizontal line test indicates that if no horizontal line intersects the graph of a function more than once, the function is one-to-one and has an inverse function. The restricted sine function is one-to-one (shown in blue in Figure 4-57).

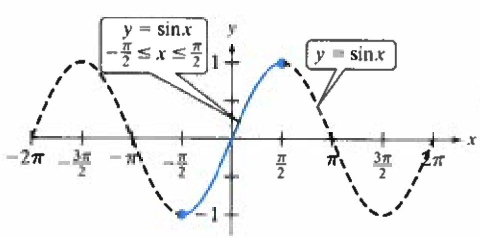


Figure 4-57

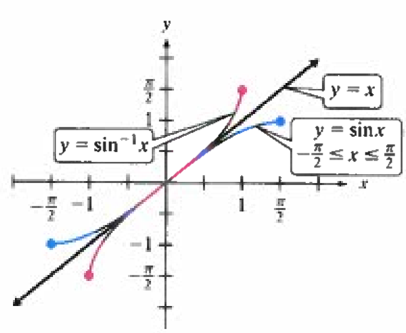


Figure 4-58

TIP The double arrow symbol $\Leftrightarrow$ means that the statements to the left and right of the arrow are logically equivalent. That is, one statement follows from the other and vice versa.

The Inverse Sine Function

The **inverse sine function** (or arcsine) denoted by $\sin^{-1}$ or arcsin is the inverse of the restricted sine function $y = \sin x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Therefore,

$$y = \sin^{-1} x \quad \Leftrightarrow \quad \sin y = x$$
$$y = \arcsin x \quad \Leftrightarrow \quad \sin y = x$$

where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Avoiding Mistakes

The notation $\sin^{-1} x$ represents the inverse of the sine function, not the reciprocal. That is, $\sin^{-1} x \ne \frac{1}{\sin x}$.

- $y = \sin^{-1} x$ is read as "y equals the inverse sine of x" and $y = \arcsin x$ is read as "y equals the arcsine of x."
- To evaluate $y = \sin^{-1} x$ or $y = \arcsin x$ means to find an angle y between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, inclusive, whose sine value is x.

EXAMPLE 1 Evaluating the Inverse Sine Function

Find the exact values or state that the expression is undefined.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **b.** $\arcsin \frac{1}{2}$ **c.** $\sin^{-1} 2$

Solution:

Avoiding Mistakes

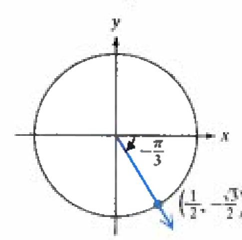
There are infinitely many values x for which $\sin x = -\frac{\sqrt{3}}{2}$, such as $x = \frac{4\pi}{3}$, $\frac{5\pi}{3}$, $-\frac{\pi}{3}$, and $-\frac{2\pi}{3}$ to name a few. However, the inverse sine function requires that x be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

a. Let $y = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

Then $\sin y = -\frac{\sqrt{3}}{2}$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

$y = -\frac{\pi}{3}$. Therefore $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$.

Find an angle y on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = -\frac{\sqrt{3}}{2}$.

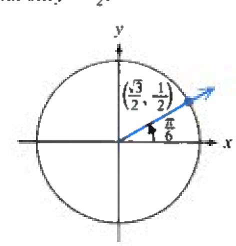


b. Let $y = \arcsin \frac{1}{2}$.

Then $\sin y = \frac{1}{2}$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

$y = \frac{\pi}{6}$. Therefore $\arcsin \frac{1}{2} = \frac{\pi}{6}$.

Find an angle y on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = \frac{1}{2}$.



c. Let $y = \sin^{-1} 2$.

To evaluate $y = \sin^{-1} 2$ would mean that we find an angle y such that $\sin y = 2$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. However, recall that $-1 \le \sin y \le 1$ for any angle y. Therefore $\sin^{-1} 2$ is undefined.

Skill Practice 1 Find the exact values or state that the expression is undefined.

a. $\sin^{-1}\dfrac{\sqrt{2}}{2}$ **b.** $\arcsin(-1)$ **c.** $\sin^{-1}(-3)$

2. Evaluate the Inverse Cosine and Tangent Functions

The inverse cosine function and the inverse tangent function are defined in a similar way. First, the domain of $y = \cos x$ and $y = \tan x$ must each be restricted to create a one-to-one function on an interval containing all values in the range.

The restricted cosine function is defined on $0 \le x \le \pi$ (Figure 4-59). The graph of the inverse cosine or "arccosine" (denoted by $\cos^{-1}$ or arccos) is shown in red in Figure 4-60.

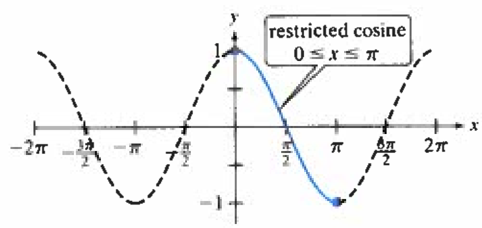


Figure 4-59

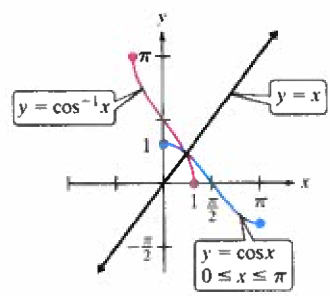


Figure 4-60

The restricted tangent function is defined on $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ (Figure 4-61). The graph of the inverse tangent or "arctangent" (denoted by $\tan^{-1}$ or arctan) is shown in red in Figure 4-62.

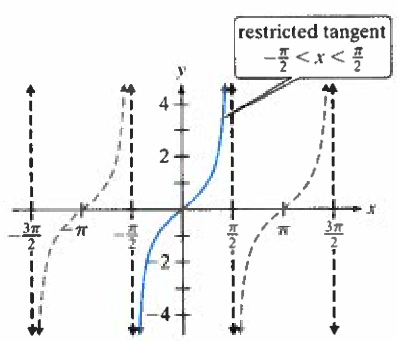


Figure 4-61

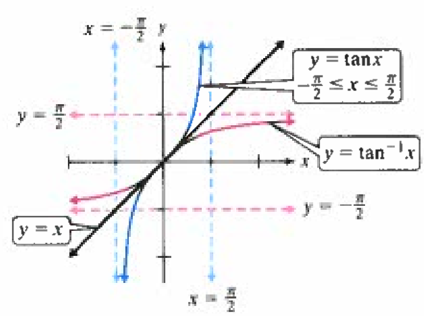


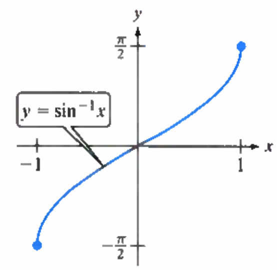
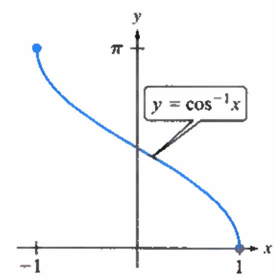
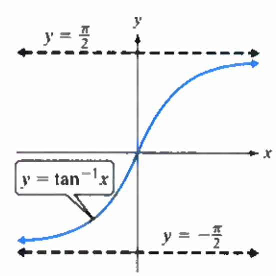
Figure 4-62

The restricted tangent function has vertical asymptotes at $x = -\dfrac{\pi}{2}$ and $x = \dfrac{\pi}{2}$. The inverse tangent function has horizontal asymptotes at $y = -\dfrac{\pi}{2}$ and $y = \dfrac{\pi}{2}$. We are now ready to summarize the definitions of the inverse functions for sine, cosine, and tangent.

Answers

1. a. $\dfrac{\pi}{4}$ b. $-\dfrac{\pi}{2}$ c. Undefined

Inverse Trigonometric Functions

Restricted Function	Inverse Function	Graph
$y = \sin x$ $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ $-1 \le y \le 1$	The **inverse sine function** (or arcsine), denoted by $\sin^{-1}$ or arcsin, is defined by $y = \sin^{-1} x \iff \sin y = x$ $y = \arcsin x \iff \sin y = x$ $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	
$y = \cos x$ $0 \le x \le \pi$ $-1 \le y \le 1$	The **inverse cosine function** (or arccosine), denoted by $\cos^{-1}$ or arccos, is defined by $y = \cos^{-1} x \iff \cos y = x$ $y = \arccos x \iff \cos y = x$ $-1 \le x \le 1$ and $0 \le y \le \pi$	
$y = \tan x$ $-\frac{\pi}{2} < x < \frac{\pi}{2}$ $y \in \mathbb{R}$ Vertical asymptotes: $x = -\frac{\pi}{2}, x = \frac{\pi}{2}$	The **inverse tangent function** (or arctangent), denoted by $\tan^{-1}$ or arctan, is defined by $y = \tan^{-1} x \iff \tan y = x$ $y = \arctan x \iff \tan y = x$ $x \in \mathbb{R}$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$ Horizontal asymptotes: $y = -\frac{\pi}{2}, y = \frac{\pi}{2}$	

EXAMPLE 2 Evaluating Inverse Trigonometric Functions

Find the exact values.

a. $\cos^{-1}\left(-\frac{1}{2}\right)$ **b.** $\tan^{-1}\sqrt{3}$ **c.** $\arctan(-1)$

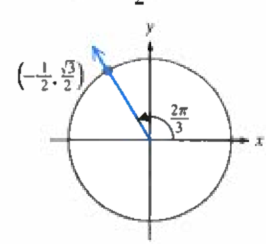
Solution:

a. Let $y = \cos^{-1}\left(-\frac{1}{2}\right)$.

Find an angle y on the interval $[0, \pi]$ such that $\cos y = -\frac{1}{2}$.

Then $\cos y = -\frac{1}{2}$ for $0 \le y \le \pi$.

$y = \frac{2\pi}{3}$. Therefore, $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.



Avoiding Mistakes

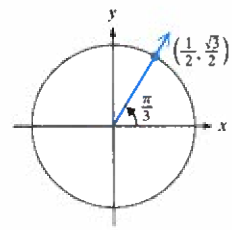
Perhaps the most common error when evaluating the inverse trigonometric functions is to fail to recognize the restrictions on the range. For instance, in Example 2(a), the result of the inverse cosine function must be an angle between 0 and π.

b. Let $y = \tan^{-1}\sqrt{3}$.

Then $\tan y = \sqrt{3}$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$y = \frac{\pi}{3}$. Therefore, $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$.

Find an angle y on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan y = \sqrt{3}$.

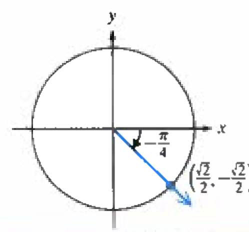


c. Let $y = \arctan(-1)$.

Then $\tan y = -1$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$y = -\frac{\pi}{4}$. Therefore, $\arctan(-1) = -\frac{\pi}{4}$

Find an angle y on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan y = -1$.



Skill Practice 2 Find the exact values.

a. $\cos^{-1}(-1)$ **b.** $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ **c.** $\arctan 0$

3. Approximate Inverse Trigonometric Functions on a Calculator

On a calculator we press the 2ND key followed by the SIN key, COS key, or TAN key to invoke $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$. By definition, the values of the inverse trigonometric functions are in radians. However, we often use the inverse functions in applications where the degree measure of an angle is desired. In Example 3, we approximate the values of several inverse trigonometric functions in both radians and degrees.

EXAMPLE 3 Approximating Values of Inverse Functions

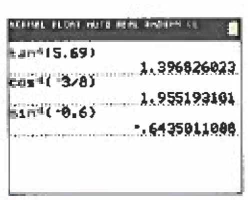
Use a calculator to approximate the function values in both radians and degrees.

a. $\tan^{-1}5.69$ **b.** $\cos^{-1}\left(-\frac{3}{8}\right)$ **c.** $\arcsin(-0.6)$

Solution:

Calculator in radian mode

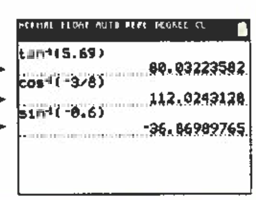
Calculator in degree mode



Press 2ND TAN

Press 2ND COS

Press 2ND SIN



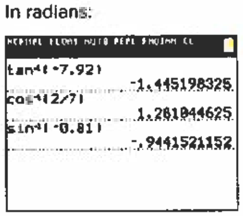
Skill Practice 3 Use a calculator to approximate the function values in both radians and degrees.

a. $\tan^{-1}(-7.92)$ **b.** $\arccos\frac{2}{7}$ **c.** $\sin^{-1}(-0.81)$

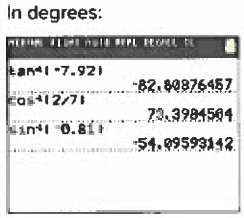
Answers

2. a. π **b.** $-\frac{\pi}{6}$ **c.** 0

3. In radians:



In degrees:



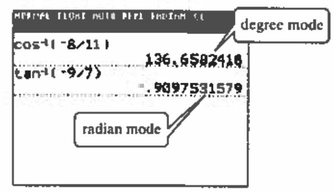
We sometimes encounter applications in which we use the inverse trigonometric functions to find the value of an angle where the desired angle is not within the range of the inverse function. In such cases, we need to adjust the output value from the calculator to obtain an angle in the desired quadrant. This is demonstrated in Example 4.

EXAMPLE 4 **Approximating Angles Based on Characteristics About the Angle**

Use a calculator to approximate the degree measure (to 1 decimal place) or radian measure (to 4 decimal places) of the angle θ subject to the given conditions.

a. $\cos\theta = -\frac{8}{11}$ and $180° \le \theta \le 270°$ **b.** $\tan\theta = -\frac{9}{7}$ and $\frac{\pi}{2} < \theta < \pi$

Solution:

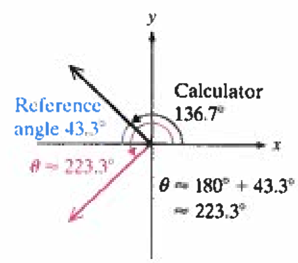


a. $\cos^{-1}\left(-\frac{8}{11}\right) \approx 136.7°$

The calculator returns a second quadrant angle. The reference angle is

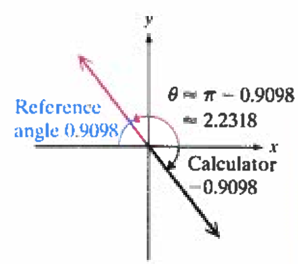
$180° - \cos^{-1}\left(-\frac{8}{11}\right) \approx 43.3°$.

The corresponding third quadrant angle is
$\theta \approx 180° + 43.3°$
$\approx 223.3°$.



b. $\tan^{-1}\left(-\frac{9}{7}\right) \approx -0.9098$ (radians)

The calculator returns a negative fourth quadrant angle. The reference angle is 0.9098. The corresponding angle in Quadrant II is $\theta \approx \pi - 0.9098 \approx 2.2318$.

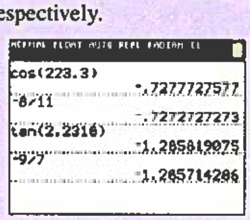


Skill Practice 4 Use a calculator to approximate the degree measure (to 1 decimal place) or radian measure (to 4 decimal places) of the angle θ subject to the given conditions.

a. $\sin\theta = -\frac{3}{7}$ and $180° \le \theta \le 270°$ **b.** $\tan\theta = -\frac{8}{3}$ and $\frac{\pi}{2} < \theta < \pi$

Avoiding Mistakes

The results of Example 4 can be checked on a calculator. Evaluate $\cos(223.3°)$ in degree mode and evaluate $\tan(2.2318)$ in radian mode. The results are approximately equal to $-8/11$ and $-9/7$, respectively.



4. Compose Trigonometric Functions and Inverse Trigonometric Functions

Recall from Section 3.1 that inverse functions are defined such that

$$(f \circ f^{-1})(x) = f[f^{-1}(x)] = x \text{ and } (f^{-1} \circ f)(x) = f^{-1}[f(x)] = x$$

Answers

4. a. 205.4° **b.** 1.9296

When composing a trigonometric function with its inverse and vice versa, particular care must be taken regarding their domains.

Composing Trigonometric Functions and Their Inverses

$\sin(\sin^{-1}x) = x$ for $-1 \le x \le 1$ $\qquad$ $\tan(\tan^{-1}x) = x$ for $x \in \mathbb{R}$

$\sin^{-1}(\sin x) = x$ for $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$ $\qquad$ $\tan^{-1}(\tan x) = x$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$\cos(\cos^{-1}x) = x$ for $-1 \le x \le 1$

$\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$

EXAMPLE 5 Composing Inverse Trigonometric Functions

Find the exact values.

a. $\sin(\sin^{-1} 1)$ $\qquad$ **b.** $\sin^{-1}\left(\sin\dfrac{7\pi}{6}\right)$

TIP As an alternative method, we can simplify the expressions from Example 5 by applying the order of operations.

$\sin[\sin^{-1}(1)] = \sin\dfrac{\pi}{2} = 1$

and

$\sin^{-1}\left(\sin\dfrac{7\pi}{6}\right) = \sin^{-1}\left(-\dfrac{1}{2}\right)$

$= -\dfrac{\pi}{6}$

Solution:

a. $\sin(\sin^{-1} 1) = 1$ $\qquad$ The value $x = 1$ lies in the domain of the inverse sine function. Therefore, we can apply the inverse property $\sin(\sin^{-1}x) = x$.

b. $\sin^{-1}\left(\sin\dfrac{7\pi}{6}\right)$ $\qquad$ Since $\frac{7\pi}{6}$ is not on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, it is not in the domain of the restricted sine function. Therefore, we cannot conclude that $\sin^{-1}(\sin x) = x$.

$= \sin^{-1}\left[\sin\left(-\dfrac{\pi}{6}\right)\right]$ $\qquad$ Rewrite $\sin\frac{7\pi}{6}$ as an equivalent expression with an angle on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$= -\dfrac{\pi}{6}$ $\qquad$ Apply the property $\sin^{-1}(\sin x) = x$.

Skill Practice 5 Find the exact values.

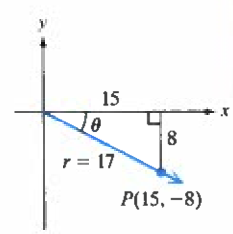
a. $\cos[\cos^{-1}(-1)]$ $\qquad$ **b.** $\cos^{-1}\left(\cos\dfrac{4\pi}{3}\right)$

EXAMPLE 6 Composing Trigonometric Functions and Inverse Trigonometric Functions

Find the exact value of $\cos\left[\tan^{-1}\left(-\dfrac{8}{15}\right)\right]$.

Solution:

Let $\theta = \tan^{-1}\left(-\dfrac{8}{15}\right)$. Since $-\dfrac{8}{15} < 0$, angle θ is on the interval $\left(-\dfrac{\pi}{2}, 0\right)$. Let $P(15, -8)$ be a point on the terminal side of θ. Then, $r = \sqrt{(15)^2 + (-8)^2} = \sqrt{289} = 17$.



$$\cos\left[\tan^{-1}\left(-\frac{8}{15}\right)\right] = \cos\theta = \frac{15}{17}$$

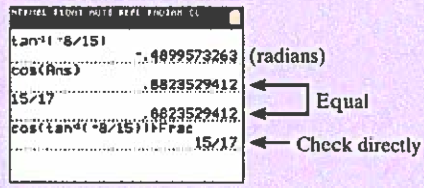
Answers

5. a. -1 $\qquad$ **b.** $\dfrac{2\pi}{3}$

Skill Practice 6 Find the exact value of $\sin\left[\tan^{-1}\left(\frac{12}{5}\right)\right]$.

Avoiding Mistakes

You can confirm your answer from Example 6 on a calculator by applying the order of operations.



EXAMPLE 7 Composing Trigonometric Functions and Inverse Trigonometric Functions

Find the exact value of $\sin\left[\cos^{-1}\left(-\frac{3}{7}\right)\right]$.

Solution:

Let $\theta = \cos^{-1}\left(-\frac{3}{7}\right)$. Since $-\frac{3}{7} < 0$, angle θ is on the interval $\left(\frac{\pi}{2}, \pi\right)$. Let $P(-3, y)$ be a point on the terminal side of θ. From Figure 4-63:

$$(-3)^2 + y^2 = 7^2$$
$$9 + y^2 = 49$$
$$y^2 = 40$$
$$y = \sqrt{40} = 2\sqrt{10}$$

Therefore, $\sin\left[\cos^{-1}\left(-\frac{3}{7}\right)\right] = \sin\theta = \frac{2\sqrt{10}}{7}$.

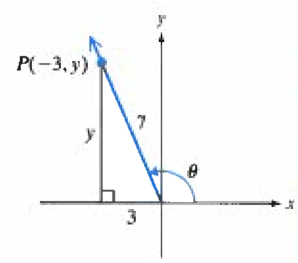
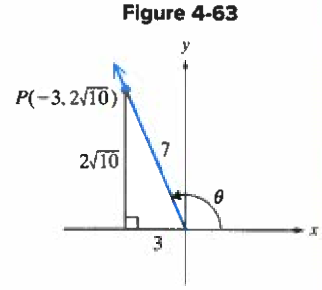


Figure 4-63



Skill Practice 7 Find the exact value of $\cos\left[\sin^{-1}\left(-\frac{2}{11}\right)\right]$.

In Example 8, we use a right triangle to visualize the composition of a trigonometric function and an inverse function, and then write an equivalent algebraic expression. This skill is used often in calculus.

EXAMPLE 8 Writing a Trigonometric Expression as an Algebraic Expression

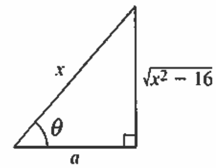
Write the expression $\tan\left(\sin^{-1}\frac{\sqrt{x^2-16}}{x}\right)$ as an algebraic expression for $x > 4$.

Answers

6. $\frac{12}{13}$

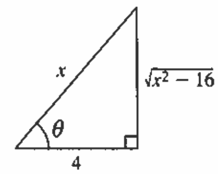
7. $\frac{\sqrt{117}}{11}$

Solution:



Let $\theta = \sin^{-1}\dfrac{\sqrt{x^2-16}}{x}$. Since $x > 4$, $\dfrac{\sqrt{x^2-16}}{x} > 0$, and we know that θ is an acute angle. We can set up a representative right triangle using the relationship

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{x^2-16}}{x}$$



To find an expression for the adjacent side a, apply the Pythagorean theorem.

$$\begin{aligned} a^2 + (\sqrt{x^2-16})^2 &= x^2 \\ a^2 + x^2 - 16 &= x^2 \\ a^2 &= 16 \\ a &= 4 \end{aligned}$$

Therefore, $\tan\left(\sin^{-1}\dfrac{\sqrt{x^2-16}}{x}\right) = \tan\theta = \dfrac{\sqrt{x^2-16}}{4}$.

Skill Practice 8 Write the expression $\tan\left(\sin^{-1}\dfrac{x}{\sqrt{x^2+9}}\right)$ as an algebraic expression for $x > 0$.

5. Apply Inverse Trigonometric Functions

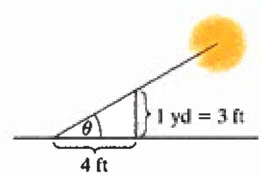
We now revisit the scenario that we used to introduce this section. Example 9 shows how inverse trigonometric functions can help us find the measure of an unknown angle in an application.

EXAMPLE 9 Applying an Inverse Trigonometric Function

A yardstick casts a 4-ft shadow when the Sun is at an angle of elevation θ. Determine the angle of elevation of the Sun to the nearest tenth of a degree.

Solution:

The tip of the shadow and the points at the top and bottom of the yardstick form a right triangle with θ representing the angle of elevation of the Sun.



$$\tan\theta = \frac{3}{4}$$

$\theta = \tan^{-1}\dfrac{3}{4}$ Use the inverse tangent to find the acute angle θ whose tangent is $\frac{3}{4}$.

$\theta \approx 36.9°$ Be sure that your calculator is in degree mode.

Skill Practice 9 For the construction of a house, a 16-ft by 6-ft wooden frame is made. Find the angle that the diagonal beam makes with the base of the frame. Round to the nearest tenth of a degree.

Answers

8. $\dfrac{x}{3}$ **9.** 20.6°

6. Evaluate the Inverse Secant, Cosecant, and Cotangent Functions

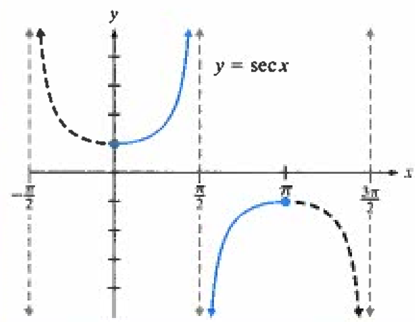
We need to establish intervals over which the secant, cosecant, and cotangent functions are one-to-one before we can define their corresponding inverse functions. These intervals are chosen to accommodate the entire range of function values and to make the restricted functions one-to-one. The graphs of the restricted secant, cosecant, and cotangent functions and their inverses are shown in Table 4-18.

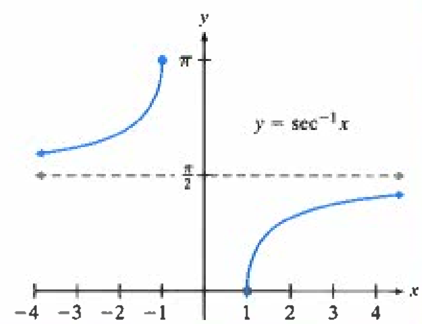
TIP It is important to note that there are infinitely many intervals over which the secant, cosecant, and cotangent functions can be restricted to define the inverse functions. These restrictions are not universally agreed upon.

Table 4-18

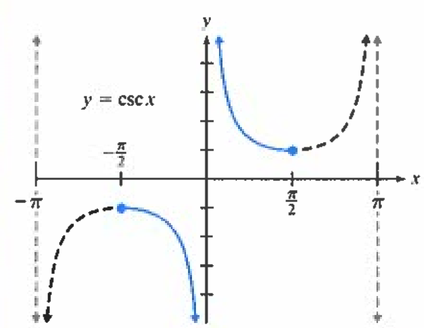
The Inverse Secant, Cosecant, and Cotangent Functions

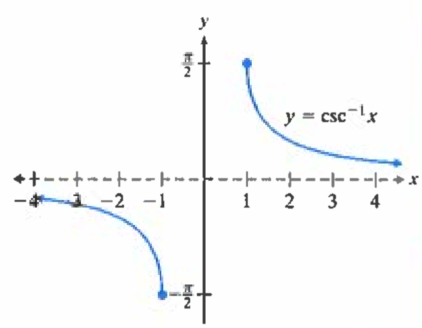
$y = \sec^{-1}x \quad \Leftrightarrow \quad \sec y = x$, where $|x| \geq 1$ and $0 \leq y < \frac{\pi}{2}$ or $\frac{\pi}{2} < y \leq \pi$.



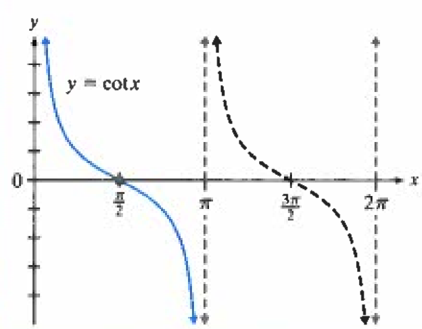


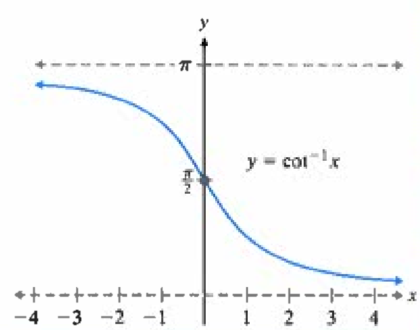
$y = \csc^{-1}x \quad \Leftrightarrow \quad \csc y = x$, where $|x| \geq 1$ and $-\frac{\pi}{2} \leq y < 0$ or $0 < y \leq \frac{\pi}{2}$.





$y = \cot^{-1}x \quad \Leftrightarrow \quad \cot y = x$, where x is any real number and $0 < y < \pi$.





A calculator does not have keys for inverse secant, cosecant, or cotangent. Therefore, to evaluate an inverse secant, cosecant, or cotangent on a calculator, we first rewrite the expression using inverse sine or cosine.

EXAMPLE 10 Approximating the Value of an Inverse Secant, Cosecant, or Cotangent

Approximate each expression in radians, rounded to 4 decimal places.

a. $\csc^{-1}3$ **b.** $\cot^{-1}\left(-\frac{3}{4}\right)$

Solution:

a. Let $\theta = \csc^{-1}3$. Then $\csc\theta = 3$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\theta \ne 0$.

Since $y = \sin^{-1}x$ has the same range as $y = \csc^{-1}x$ for $x \ne 0$, we have

$\csc\theta = 3$ implies that $\frac{1}{\sin\theta} = 3$ and that $\sin\theta = \frac{1}{3}$.

Therefore, $\theta = \sin^{-1}\frac{1}{3} \approx 0.3398$.

b. Let $\theta = \cot^{-1}\left(-\frac{3}{4}\right)$. Then $\cot\theta = -\frac{3}{4}$ for $0 < \theta < \pi$. Furthermore, since the argument $-\frac{3}{4}$ is negative, $\theta = \cot^{-1}\left(-\frac{3}{4}\right)$ must be a second quadrant angle. The related expression using the inverse tangent function will *not* return a second quadrant angle but rather an angle on the interval $-\frac{\pi}{2} < \theta < 0$. Therefore, we will rewrite the expression $\cot\theta = -\frac{3}{4}$ using the cosine function. From Figure 4-64, $\cos\theta = -\frac{3}{5}$. Therefore,

$\theta = \cos^{-1}\left(-\frac{3}{5}\right) \approx 2.2143$.

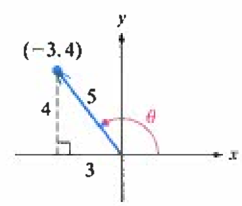


Figure 4-64

Skill Practice 10 Approximate each expression in radians, rounded to 4 decimal places.

a. $\sec^{-1}(-4)$ **b.** $\cot^{-1}\left(-\frac{5}{12}\right)$

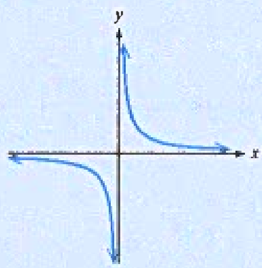
Answers

10. a. 1.8235 **b.** 1.9656

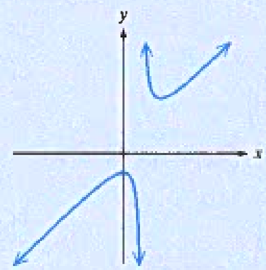
SECTION 4.7 Practice Exercises

Prerequisite Review

R.1. Determine if the relation defines y as a one-to-one function of x.



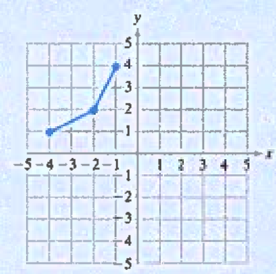
R.2. Determine if the relation defines y as a one-to-one function of x.



R.3. A one-to-one function is given. Write an equation for the inverse function.

$$g(x) = \frac{3 - x}{4}$$

R.4. The graph of a function is given. Graph the inverse function.



Concept Connections

1. A function must be ______________ on its entire domain to have an inverse function.

2. If $\left(\frac{2\pi}{3}, -\frac{1}{2}\right)$ is on the graph of $y = \cos x$, what is the related point on $y = \cos^{-1}x$?

3. The graph of $y = \tan^{-1}x$ has two ______________ (horizontal/vertical) asymptotes represented by the equations ______________ and ______________.

4. The domain of $y = \arctan x$ is ______________. The output is a real number (or angle in radians) between ______________ and ______________.

5. In interval notation, the domain of $y = \cos^{-1}x$ is ______________. The output is a real number (or angle in radians) between ______________ and ______________, inclusive.

6. In interval notation, the domain of $y = \sin^{-1}x$ is ______________. The output is a real number (or angle in radians) between ______________ and ______________, inclusive.

Objective 1: Evaluate the Inverse Sine Function

For Exercises 7–12, find the exact value or state that the expression is undefined. (See Example 1)

7. $\arcsin\frac{\sqrt{2}}{2}$

8. $\sin^{-1}\left(-\frac{1}{2}\right)$

9. $\sin^{-1}\pi$

10. $\sin^{-1}\frac{3}{2}$

11. $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

12. $\arcsin\frac{\sqrt{3}}{2}$

For Exercises 13–16, find the exact value.

13. $\sin^{-1}\frac{\sqrt{3}}{2} + \sin^{-1}\frac{1}{2}$

14. $\sin^{-1}\frac{\sqrt{2}}{2} - \sin^{-1}(-1)$

15. $2\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) - \frac{\pi}{3}$

16. $\frac{\pi}{2} + 3\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Objective 2: Evaluate the Inverse Cosine and Tangent Functions

For Exercises 17–28, find the exact value or state that the expression is undefined. (See Example 2)

17. $\arccos\left(-\frac{\sqrt{2}}{2}\right)$

18. $\tan^{-1}(-\sqrt{3})$

19. $\cos^{-1}0$

20. $\tan^{-1}0$

21. $\cos^{-1}(-2)$

22. $\arccos\frac{4}{3}$

23. $\arctan\frac{\sqrt{3}}{3}$

24. $\cos^{-1}\frac{\sqrt{2}}{2}$

25. $\tan^{-1}1$

26. $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

27. $\arccos\frac{\sqrt{3}}{2}$

28. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

For Exercises 29–32, find the exact value.

29. $\tan^{-1}(-1) + \tan^{-1}\sqrt{3}$

30. $\cos^{-1}\frac{\sqrt{2}}{2} + \cos^{-1}\frac{1}{2}$

31. $2\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \tan^{-1}\frac{\sqrt{3}}{3}$

32. $3\tan^{-1}1 + \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Objective 3: Approximate Inverse Trigonometric Functions on a Calculator

For Exercises 33–36, use a calculator to approximate the function values in both radians and degrees. (See Example 3)

33. a. $\cos^{-1}\frac{3}{8}$
b. $\tan^{-1}25$
c. $\arcsin 0.05$

34. a. $\sin^{-1}0.93$
b. $\arccos 0.17$
c. $\arctan\frac{7}{4}$

35. a. $\tan^{-1}(-28)$
b. $\arccos\frac{\sqrt{3}}{5}$
c. $\sin^{-1}(-0.14)$

36. a. $\cos^{-1}(-0.75)$
b. $\tan^{-1}\frac{8}{3}$
c. $\arcsin\frac{\pi}{7}$

For Exercises 37–46, use a calculator to approximate the degree measure (to 1 decimal place) or radian measure (to 4 decimal places) of the angle θ subject to the given conditions. (See Example 4)

37. $\cos\theta = -\frac{5}{6}$ and $180° < \theta < 270°$

38. $\sin\theta = -\frac{4}{5}$ and $180° < \theta < 270°$

39. $\tan\theta = -\frac{12}{5}$ and $90° < \theta < 180°$

40. $\cos\theta = -\frac{2}{13}$ and $180° < \theta < 270°$

41. $\sin\theta = \frac{12}{19}$ and $90° < \theta < 180°$

42. $\tan\theta = \frac{7}{15}$ and $180° < \theta < 270°$

43. $\cos\theta = -\frac{5}{8}$ and $\pi < \theta < \frac{3\pi}{2}$

44. $\tan\theta = -\frac{9}{5}$ and $\frac{\pi}{2} < \theta < \pi$

45. $\sin\theta = -\frac{17}{20}$ and $\pi < \theta < \frac{3\pi}{2}$

46. $\cos\theta = \frac{1}{17}$ and $\frac{3\pi}{2} < \theta < 2\pi$

Objective 4: Compose Trigonometric Functions and Inverse Trigonometric Functions

For Exercises 47–58, find the exact values. (See Example 5)

47. $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$

48. $\arcsin\left(\sin\frac{5\pi}{3}\right)$

49. $\sin^{-1}\left(\sin\frac{5\pi}{4}\right)$

50. $\sin\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

51. $\cos\left(\cos^{-1}\frac{2}{3}\right)$

52. $\arccos\left(\cos\frac{11\pi}{6}\right)$

53. $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$

54. $\cos\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$

55. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$

56. $\tan(\tan^{-1}2)$

57. $\tan[\arctan(-\pi)]$

58. $\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$

For Exercises 59–70, find the exact values. (See Examples 6–7)

59. $\cos\left(\tan^{-1}\frac{\sqrt{3}}{3}\right)$

60. $\sin\left(\cos^{-1}\frac{1}{2}\right)$

61. $\tan\left[\sin^{-1}\left(-\frac{2}{3}\right)\right]$

62. $\sin\left[\cos^{-1}\left(-\frac{2}{3}\right)\right]$

63. $\sin\left(\cos^{-1}\frac{3}{4}\right)$

64. $\sin\left(\tan^{-1}\frac{4}{3}\right)$

65. $\sin[\tan^{-1}(-1)]$

66. $\cos[\tan^{-1}(-1)]$

67. $\cos\left[\sin^{-1}\left(-\frac{2}{7}\right)\right]$

68. $\cos\left[\tan^{-1}\left(-\frac{5}{12}\right)\right]$

69. $\tan\left[\cos^{-1}\left(-\frac{5}{6}\right)\right]$

70. $\tan\left[\sin^{-1}\left(-\frac{\sqrt{5}}{3}\right)\right]$

For Exercises 71–76, write the given expression as an algebraic expression. It is not necessary to rationalize the denominator. (See Example 8)

71. $\cos\left(\sin^{-1}\frac{x}{\sqrt{25+x^2}}\right)$ for $x > 0$.

72. $\cot\left(\cos^{-1}\frac{\sqrt{x^2-1}}{x}\right)$ for $x > 1$.

73. $\sin(\tan^{-1}x)$ for $x > 0$.

74. $\tan(\sin^{-1}x)$ for $|x| < 1$.

75. $\tan\left(\cos^{-1}\frac{3}{x}\right)$ for $x > 3$.

76. $\sin\left(\cos^{-1}\frac{\sqrt{x^2-25}}{x}\right)$ for $x > 5$.

Objective 5: Apply Inverse Trigonometric Functions

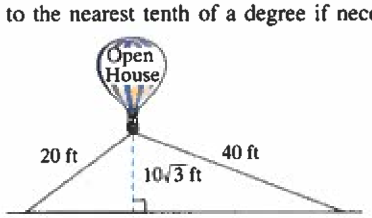
77. To meet the requirements of the Americans with Disabilities Act (ADA) a wheelchair ramp must have a slope of 1:12 or less. That is, for every 1 in. of "rise," there must be at least 12 in. of "run." **(See Example 9)**

a. If a wheelchair ramp is constructed with the maximum slope, what angle does the ramp make with the ground? Round to the nearest tenth of a degree.

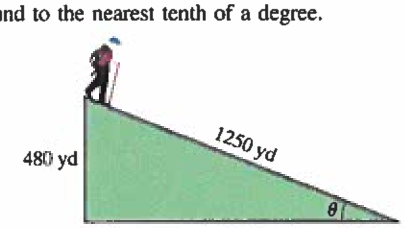
b. If the ramp is 22 ft long, how much elevation does the ramp provide? Round to the nearest tenth of a foot.

78. A student measures the length of the shadow of the Washington Monument to be 620 ft. If the Washington Monument is 555 ft tall, approximate the angle of elevation of the Sun to the nearest tenth of a degree.

79. A balloon advertising an open house is stabilized by two cables of lengths 20 ft and 40 ft tethered to the ground. If the perpendicular distance from the balloon to the ground is $10\sqrt{3}$ ft, what is the degree measure of the angle each cable makes with the ground? Round to the nearest tenth of a degree if necessary.



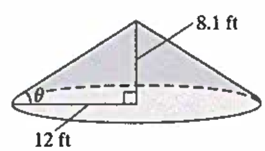
80. A group of campers hikes down a steep path. One member of the group has an altimeter on his watch to measure altitude. If the path is 1250 yd and the amount of altitude lost is 480 yd, what is the angle of incline? Round to the nearest tenth of a degree.



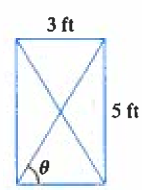
81. Navajo Tube Hill, a snow tubing hill in Utah, is 550 ft long and has a 75-ft vertical drop. Find the angle of incline of the hill. Round to the nearest tenth of a degree.

82. A ski run on Giant Steps Mountain in Utah is 1475 m long. The difference in altitude from the beginning to the end of the run is 350 m. Find the angle of the ski run. Round to the nearest tenth of a degree.

83. When granular material such as sand or gravel is poured onto a horizontal surface it forms a right circular cone. The angle that the surface of the cone makes with the horizontal is called the **angle of repose**. The angle of repose depends on a number of variables such as the shape of the particles and the amount of friction between them. "Stickier" particles have a greater angle of repose, and "slippery" particles have a smaller angle of repose. Find the angle of repose for the pile of dry sand. Round to the nearest degree.



84. For the construction of a bookcase, a 5-ft by 3-ft wooden frame is made with a cross-brace in the back for stability. Find the angle that the diagonal brace makes with the base of the frame. Round to the nearest tenth of a degree.



Objective 6: Evaluate the Inverse Secant, Cosecant, and Cotangent Functions

85. Show that $\sec^{-1}x = \cos^{-1}\frac{1}{x}$ for $x \geq 1$.

86. Show that $\csc^{-1}x = \sin^{-1}\frac{1}{x}$ for $x \geq 1$.

87. Show that $\sec^{-1}x + \csc^{-1}x = \frac{\pi}{2}$ for $x \geq 1$.

88. Complete the table, giving the domain and range in interval notation for each inverse function.

Inverse Function	Domain	Range
$y = \sin^{-1}x$		
$y = \csc^{-1}x$		
$y = \cos^{-1}x$		
$y = \sec^{-1}x$		
$y = \tan^{-1}x$		
$y = \cot^{-1}x$		

For Exercises 89–94, find the exact values.

89. $\sec^{-1}\frac{2\sqrt{3}}{3}$

90. $\sec^{-1}(-\sqrt{2})$

91. $\csc^{-1}(-1)$

92. $\csc^{-1}(2)$

93. $\cot^{-1}\sqrt{3}$

94. $\cot^{-1}(1)$

For Exercises 95–100, use a calculator to approximate each expression in radians, rounded to 4 decimal places. (See Example 10)

95. $\sec^{-1}\frac{7}{4}$

96. $\csc^{-1}\frac{6}{5}$

97. $\csc^{-1}(-8)$

98. $\sec^{-1}(-6)$

99. $\cot^{-1}\left(-\frac{8}{15}\right)$

100. $\cot^{-1}\left(-\frac{24}{7}\right)$

Mixed Exercises

For Exercises 101–104, find the exact value if possible. Otherwise find an approximation to 4 decimal places or state that the expression is undefined.

101. a. $\sin\frac{\pi}{4}$ **b.** $\sin^{-1}\frac{\pi}{4}$ **c.** $\sin^{-1}\frac{\sqrt{2}}{2}$

102. a. $\cos\frac{2\pi}{3}$ **b.** $\cos^{-1}\frac{2\pi}{3}$ **c.** $\cos^{-1}\left(-\frac{1}{2}\right)$

103. a. $\cos^{-1}\left(-\frac{\pi}{6}\right)$ **b.** $\cos\left(-\frac{\pi}{6}\right)$ **c.** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

104. a. $\sin^{-1}\frac{7\pi}{6}$ **b.** $\sin\frac{7\pi}{6}$ **c.** $\sin^{-1}\left(-\frac{1}{2}\right)$

For Exercises 105–108, find the inverse function and its domain and range.

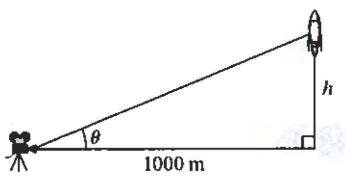
105. $f(x) = 3\sin x + 2$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

106. $g(x) = 6\cos x - 4$ for $0 \leq x \leq \pi$

107. $h(x) = \frac{\pi}{4} + \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

108. $k(x) = \pi + \sin x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

109. A video camera located at ground level follows the liftoff of an Atlas V Rocket from the Kennedy Space Center. Suppose that the camera is 1000 m from the launch pad.

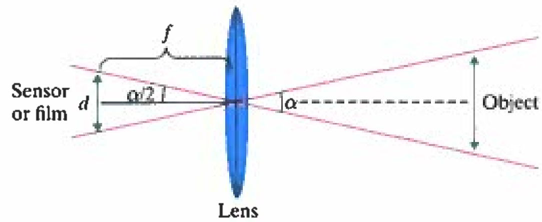


a. Write the angle of elevation θ from the camera to the rocket as a function of the rocket's height, h.

b. Without the use of a calculator, will the angle of elevation be less than 45° or greater than 45° when the rocket is 400 m high?

c. Use a calculator to find θ to the nearest tenth of a degree when the rocket's height is 400 m, 1500 m, and 3000 m.

110. The effective focal length f of a camera is the distance required for the lens to converge light to a single focal point. The angle of view α of a camera describes the angular range (either horizontally, vertically, or diagonally) that is imaged by a camera.

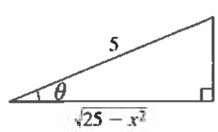


a. Show that $\alpha = 2\arctan\frac{d}{2f}$ where d is the dimension of the image sensor or film.

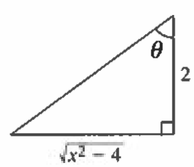
b. A typical 35-mm camera has image dimensions of 24 mm (vertically) by 36 mm (horizontally). If the focal length is 50 mm, find the vertical and horizontal viewing angles. Round to the nearest tenth of a degree.

For Exercises 111–114, use the relationship given in the right triangle and the inverse sine, cosine, and tangent functions to write θ as a function of x in three different ways. It is not necessary to rationalize the denominator.

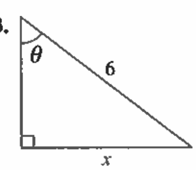
111.



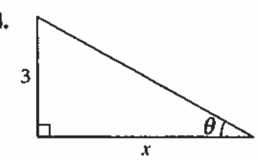
112.



113.



114.



For Exercises 115–120, find the exact solution to each equation.

115. $-2\sin^{-1}x - \pi = 0$

116. $3\cos^{-1}x - \pi = 0$

117. $6\cos^{-1}x - 3\pi = 0$

118. $4\sin^{-1}x + \pi = 0$

119. $4\tan^{-1}2x = \pi$

120. $6\tan^{-1}2x = 2\pi$

Write About It

121. Explain the difference between the reciprocal of a function and the inverse of a function.

122. Explain the flaw in the logic: $\cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. Therefore, $\cos^{-1}\frac{\sqrt{2}}{2} = -\frac{\pi}{4}$.

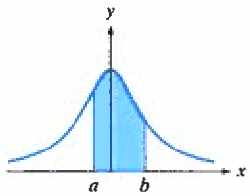
123. In terms of angles, explain what is meant when we find $\sin^{-1}\left(-\frac{1}{2}\right)$.

124. Why must the domains of the sine, cosine, and tangent functions be restricted in order to define their inverse functions?

Expanding Your Skills

125. In calculus, we can show that the area below the graph of $f(x) = \dfrac{1}{1+x^2}$, above the x-axis, and between the lines $x = a$ and $x = b$ for $a < b$, is given by

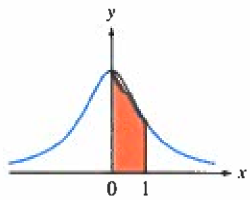
$$\tan^{-1}b - \tan^{-1}a$$



a. Find the area under the curve between $x = 0$ and $x = 1$.

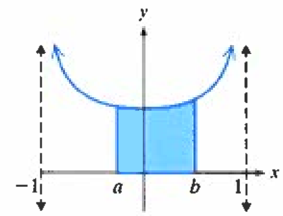
b. Evaluate $f(0)$ and $f(1)$.

c. Find the area of the trapezoid defined by the points $(0, 0)$, $(1, 0)$, $[0, f(0)]$, and $[1, f(1)]$ to confirm that your answer from part (a) is reasonable.



126. In calculus, we can show that the area below the graph of $f(x) = \dfrac{1}{\sqrt{1-x^2}}$, above the x-axis, and between the lines $x = a$ and $x = b$ for $a < b$, is given by

$$\sin^{-1}b - \sin^{-1}a$$

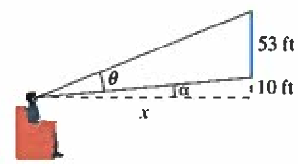


a. Find the area under the curve between $x = 0$ and $x = 0.5$.

b. Evaluate $f(0)$ and $f(0.5)$.

c. Find the area of the trapezoid defined by the points $(0, 0)$, $(1, 0)$, $[0, f(0)]$, and $[0.5, f(0.5)]$ to confirm that your answer from part (a) is reasonable.

127. The vertical viewing angle θ to a movie screen is the angle formed from the bottom of the screen to a viewer's eye to the top of the screen. Suppose that the viewer is sitting x horizontal feet from an IMAX screen 53 ft high and that the bottom of the screen is 10 vertical feet above the viewer's eye level. Let α be the angle of elevation to the bottom of the screen.



a. Write an expression for $\tan\alpha$.

b. Write an expression for $\tan(\alpha + \theta)$.

c. Using the relationships found in parts (a) and (b), show that $\theta = \tan^{-1}\dfrac{63}{x} - \tan^{-1}\dfrac{10}{x}$.

Technology Connections

128. Refer to the movie screen and observer in Exercise 127. The vertical viewing angle is given by $\theta = \tan^{-1}\dfrac{63}{x} - \tan^{-1}\dfrac{10}{x}$.

a. Find the vertical viewing angle (in radians) for an observer sitting 15, 25, and 35 ft away. Round to 2 decimal places.

b. Graph $y = \tan^{-1}\dfrac{63}{x} - \tan^{-1}\dfrac{10}{x}$ on a graphing utility on the window $[0, 100, 10]$ by $[0, 1, 0.1]$.

c. Use the Maximum feature on the graphing utility to estimate the distance that an observer should sit from the screen to produce the maximum viewing angle. Round to the nearest tenth of a foot.

129. a. Graph the functions $y = \tan^{-1}x$ and $y = x - \dfrac{x^3}{3} + \dfrac{x^5}{5}$ on the window $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{\pi}{4}\right]$ by $[-2, 2, 1]$.

b. Graph the functions $y = \tan^{-1}x$ and $y = \dfrac{x}{1 + \dfrac{x^2}{3 - x^2}}$ on the window $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{\pi}{4}\right]$ by $[-2, 2, 1]$.

c. How do the functions in parts (a) and (b) compare for values of x taken close to 0?

CHAPTER 4 KEY CONCEPTS

SECTION 4.1 Angles and Their Measure	Reference
The measure of an angle may be given in **degrees,** where $1°$ is $\frac{1}{360}$ of a full rotation. A measure of $1°$ can be further divided into 60 equal parts called **minutes.** Each minute can be divided into 60 equal parts called **seconds.**	p. 447
The measure of an angle may be given in **radians,** where 1 radian is the measure of the central angle that intercepts an arc equal in length to the radius of the circle.	p. 448

$\theta = \frac{s}{r}$ gives the measure of a central angle θ (in radians), where s is the length of the arc intercepted by θ and r is the radius of the circle.	p. 449
The relationship $180° = \pi$ radians provides the conversion factors to convert from radians to degrees and vice versa.	p. 450
Two angles with the same initial and terminal sides are called **coterminal angles.** Coterminal angles differ in measure by an integer multiple of $360°$ or 2π radians.	p. 450
Given a circle of radius r, the length s of an arc intercepted by a central angle θ (in radians) is given by $s = r\theta$.	p. 452
Points on the Earth are identified by their latitude and longitude. **Latitude** is the angular measure of a central angle measuring north or south from the equator. Lines of **longitude,** also called meridians, are circles passing through both poles and running perpendicular to the equator.	p. 453
Suppose that a point on a circle of radius r moves through an angle of θ radians in time t. Then the angular and linear speeds of the point are given by angular speed: $\omega = \frac{\theta}{t}$ linear speed: $v = \frac{s}{t}$ or $v = \frac{r\theta}{t}$ or $v = r\omega$	p. 454
The area A of a sector of a circle of radius r with central angle θ (in radians) is given by $A = \frac{1}{2}r^2\theta$.	p. 455

SECTION 4.2 Trigonometric Functions Defined on the Unit Circle	Reference
The **unit circle** is a circle of radius r with center at the origin of a rectangular coordinate system: $x^2 + y^2 = 1$.	p. 462
The six trigonometric functions can be defined as functions of real numbers by using the unit circle. Let $P(x, y)$ be the point associated with a real number t measured along the circumference of the unit circle from the point $(1, 0)$. $\sin t = y$ $\cos t = x$ $\tan t = \frac{y}{x}\ (x \neq 0)$ $\csc t = \frac{1}{y}\ (y \neq 0)$ $\sec t = \frac{1}{x}\ (x \neq 0)$ $\cot t = \frac{x}{y}\ (y \neq 0)$	p. 463
The real number t taken along the circumference of the unit circle gives the number of radians of the corresponding central angle θ; that is, $\theta = t$ radians.	p. 465
• The domain of both the sine function and cosine function is all real numbers. • The domain of both the tangent function and secant function excludes real numbers t that are odd multiples of $\frac{\pi}{2}$. • The domain of both the cotangent function and cosecant function excludes real numbers t that are multiples of π.	p. 469
A function f is **periodic** if $f(t + p) = f(t)$ for some constant p. The smallest positive value p for which f is periodic is called the **period** of f. • The period of the sine, cosine, secant, and cosecant functions is 2π. • The period of the tangent and cotangent functions is π.	p. 473
• The cosine and secant functions are even functions. $f(-x) = f(x)$. • The sine, cosecant, tangent, and cotangent functions are odd functions. $f(-x) = -f(x)$.	p. 474

SECTION 4.3 Right Triangle Trigonometry — Reference

Each trigonometric function of an acute angle θ is one of the six possible ratios of the sides of a right triangle containing θ. — p. 482

$$\sin\theta = \frac{\text{opp}}{\text{hyp}},\ \cos\theta = \frac{\text{adj}}{\text{hyp}},\ \tan\theta = \frac{\text{opp}}{\text{adj}},\ \csc\theta = \frac{\text{hyp}}{\text{opp}},\ \sec\theta = \frac{\text{hyp}}{\text{adj}},\ \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Reciprocal Identities: $\csc\theta = \dfrac{1}{\sin\theta},\ \sec\theta = \dfrac{1}{\cos\theta},\ \cot\theta = \dfrac{1}{\tan\theta}$ — p. 488

Quotient Identities: $\tan\theta = \dfrac{\sin\theta}{\cos\theta},\ \cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Pythagorean Identities: $\sin^2\theta + \cos^2\theta = 1,\ \tan^2\theta + 1 = \sec^2\theta,\ 1 + \cot^2\theta = \csc^2\theta$ — p. 488

The **cofunction identities** indicate that cofunctions of complementary angles are equal. — p. 489

An **angle of elevation** or **depression** from an observer to an object is used in many applications of trigonometric functions. — p. 490

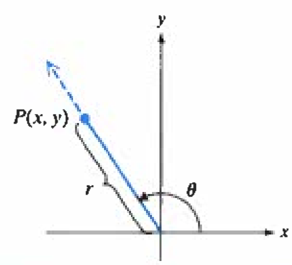
SECTION 4.4 Trigonometric Functions of Any Angle — Reference

$\sin\theta = \dfrac{y}{r}$ $\qquad\csc\theta = \dfrac{r}{y}\ (y \neq 0)$

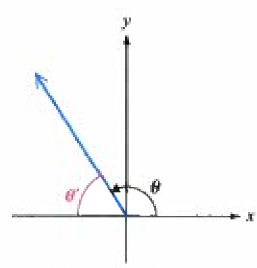
$\cos\theta = \dfrac{x}{r}$ $\qquad\sec\theta = \dfrac{r}{x}\ (x \neq 0)$

$\tan\theta = \dfrac{y}{x}\ (x \neq 0)$ $\qquad\cot\theta = \dfrac{x}{y}\ (y \neq 0)$

Let $P(x, y)$ be a point on the terminal side of angle θ drawn in standard position, and r be the distance from P to the origin. — p. 497



Let θ be an nonquadrantal angle in standard position. The **reference angle** for θ is the acute angle θ' formed by the terminal side of θ and the horizontal axis. — p. 499

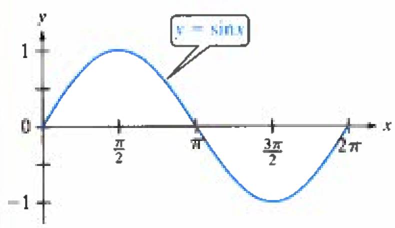


The value of each trigonometric function of θ is the same as the corresponding function of the reference angle θ' except possibly for the sign. — p. 500

SECTION 4.5 Graphs of Sine and Cosine Functions

Reference

The graph of $y = \sin x$ (for one period):

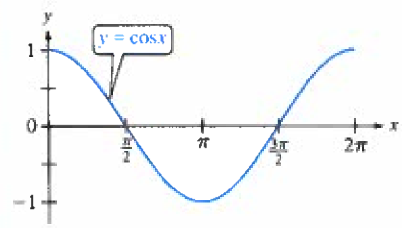


Period: 2π Amplitude: 1

Domain: $(-\infty, \infty)$ Range: $[-1, 1]$

Symmetric to the origin

The graph of $y = \cos x$ (for one period):



Period: 2π Amplitude: 1

Domain: $(-\infty, \infty)$ Range: $[-1, 1]$

Symmetric to the y-axis

p. 510

Characteristics of $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$ with $B > 0$:

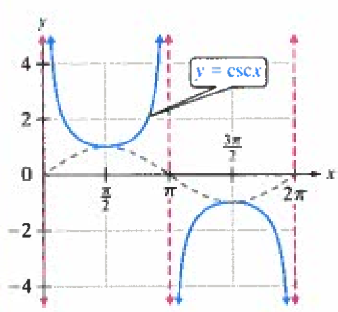
1. The amplitude is $|A|$.
2. The period is $\frac{2\pi}{B}$.
3. The phase shift is $\frac{C}{B}$.
4. The vertical shift is D.
5. One full cycle is given on the interval $0 \le Bx - C \le 2\pi$.
6. The domain is the set of real numbers.
7. The range is $-|A| + D \le y \le |A| + D$.

p. 515

SECTION 4.6 Graphs of Other Trigonometric Functions

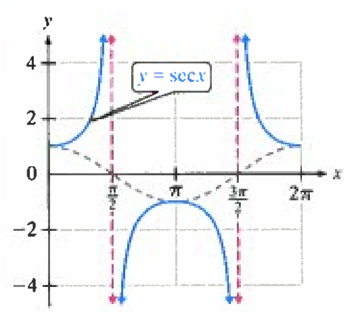
Reference

The graph of $y = \csc x = \frac{1}{\sin x}$ has vertical asymptotes where $\sin x = 0$.



Period: 2π
Amplitude: None
Domain: $\{x \mid x \neq n\pi$ for integers $n\}$
Range: $(-\infty, -1] \cup [1, \infty)$
Vertical asymptotes: $x = n\pi$
Symmetric to the origin (odd function)

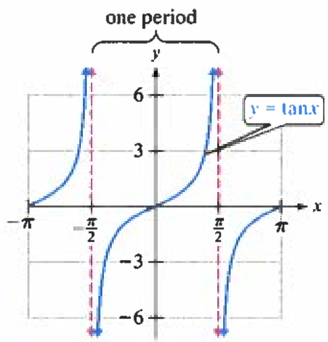
The graph of $y = \sec x = \frac{1}{\cos x}$ has vertical asymptotes where $\cos x = 0$.



Period: 2π
Amplitude: None
Domain: $\{x \mid x \neq \frac{(2n+1)\pi}{2}$ for integers $n\}$
Range: $(-\infty, -1] \cup [1, \infty)$
Vertical asymptotes: $x = \frac{(2n+1)\pi}{2}$
Symmetric to the y-axis (even function)

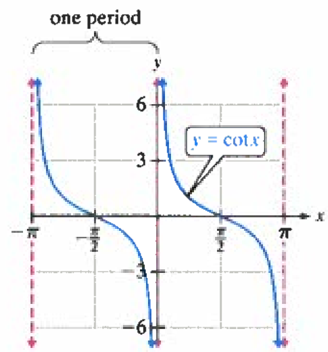
p. 528

The graph of $y = \tan x = \dfrac{\sin x}{\cos x}$ has vertical asymptotes where $\cos x = 0$.



Period: π
Amplitude: None
Domain: $\left\{x \mid x \neq \frac{(2n+1)\pi}{2} \text{ for integers } n\right\}$
Range: $(-\infty, \infty)$
Vertical asymptotes: $x = \frac{(2n+1)\pi}{2}$
Symmetric to the origin (odd function)

The graph of $y = \cot x = \dfrac{\cos x}{\sin x}$ has vertical asymptotes where $\sin x = 0$.



Period: π
Amplitude: None
Domain: $\{x \mid x \neq n\pi \text{ for integers } n\}$
Range: $(-\infty, \infty)$
Vertical asymptotes: $x = n\pi$
Symmetric to the origin (odd function)

p. 532

To graph variations of $y = \csc x$ or $y = \sec x$, use the graph of the related reciprocal function for reference. p. 529

To graph variations of $y = \tan x$ and $y = \cot x$, p. 533

1. First graph two consecutive asymptotes.
2. Plot an x-intercept halfway between the asymptotes.
3. Sketch the general shape of the "parent" function between the asymptotes.
4. Apply a vertical shift if applicable.
5. Sketch additional cycles to the right or left as desired.

SECTION 4.7 Inverse Trigonometric Functions — Reference

The **inverse sine function** (or arcsine), denoted by $\sin^{-1}$ or arcsin, is defined by p. 541

$$y = \sin^{-1}x \quad \Leftrightarrow \quad \sin y = x \text{ for } -1 \le x \le 1 \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

To compose the sine function and its inverse,

$$\sin(\sin^{-1}x) = x \text{ for } -1 \le x \le 1 \qquad \text{and } \sin^{-1}(\sin x) = x \text{ for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

The **inverse cosine function** (or arccosine), denoted by $\cos^{-1}$ or arccos, is defined by p. 543
$y = \cos^{-1}x \quad \Leftrightarrow \quad \cos y = x$ for $-1 \le x \le 1$ and $0 \le y \le \pi$.
To compose the cosine function and its inverse,
$\cos(\cos^{-1}x) = x$ for $-1 \le x \le 1$ and $\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$.

The **inverse tangent function** (or arctangent), denoted by $\tan^{-1}$ or arctan, is defined by p. 543

$$y = \tan^{-1}x \quad \Leftrightarrow \quad \tan y = x \text{ for } x \in \mathbb{R} \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

The graph of the inverse tangent function has horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$.

To compose the tangent function and its inverse,

$$\tan(\tan^{-1}x) = x \text{ for } x \in \mathbb{R} \text{ and } \tan^{-1}(\tan x) = x \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Expanded Chapter Summary available at www.mhhe.com/millerprecalculus.

CHAPTER 4 Review Exercises

SECTION 4.1

1. Convert 96°28′ to decimal degrees. Round to 2 decimal places.
2. Convert 225.24° to DMS (degree-minute-second) form. Round to the nearest second if necessary.
3. Convert 124° to radians. Give the answer in exact form in terms of π.
4. Convert −73.8° to radians. Round to 2 decimal places.
5. Convert $\frac{5\pi}{8}$ radians to decimal degrees. Round to 1 decimal place if necessary.

For Exercises 6–8, find a positive angle coterminal with the given angle.

6. 48° 7. −110° 8. $\frac{3\pi}{5}$

For Exercises 9–11, find a negative angle coterminal with the given angle.

9. $\frac{7\pi}{4}$ 10. $-\frac{5\pi}{6}$ 11. 126°

12. Find an angle between 0° and 360° that is coterminal to 745°.
13. Find an angle between 0 and 2π that is coterminal to $-\frac{19\pi}{6}$.
14. A unicycle with a wheel diameter of 24 in. moves through an angle of 140°. What distance does a point on the edge of the wheel move? Round the answer to the nearest tenth of an inch.
15. A pulley is 1.5 ft in diameter. Find the distance the load will move if the pulley is rotated 750°. Find the exact distance in terms of π and then round the answer to the nearest tenth of a foot.
16. Seattle, Washington (47.61°N, 122.33°W), and San Francisco, California (37.78°N, 122.42°W), have approximately the same longitude, which means that they are roughly due north-south of each other. Use the difference in latitude to approximate the distance between the cities assuming that the radius of the Earth is 3960 mi. Round the answer to the nearest mile.
17. A spinning disk has radius of 10 in. and rotates at 2800 rpm. For a point at the edge of the disk,
    a. Find the exact value of the angular speed.
    b. Find the linear speed. Round the answer to the nearest inch per minute.
18. A bicycle has wheels 26 inches in diameter. If the wheels turn at 90 rpm, what is the linear speed in inches per minute? Give the exact speed and an approximation to the nearest inch per minute.
19. A sprinkler rotates through an angle of 75° spraying water outward for a distance of 6 ft. Find the exact area watered, then round the result to the nearest tenth of a square foot.
20. A round pie 10 in. in diameter is cut into a slice with a 30° angle. Find the exact area of the slice, then round the result to the nearest tenth of a square inch.

SECTION 4.2

21. The real number t corresponds to the point $P\left(-\frac{3\sqrt{5}}{7}, \frac{2}{7}\right)$ on the unit circle. Evaluate each expression.
    a. $\sin t$ b. $\cos t$ c. $\tan t$
    d. $\csc t$ e. $\sec t$ f. $\cot t$
22. Identify the ordered pair on the unit circle corresponding to each real number t.
    a. $t = \frac{2\pi}{3}$ b. $t = \frac{5\pi}{4}$ c. $t = \frac{11\pi}{6}$
23. Identify the domain for each pair of functions.
    a. $f(t) = \sin t$, $g(t) = \cos t$
    b. $f(t) = \tan t$, $f(t) = \sec t$
    c. $f(t) = \cot t$, $f(t) = \csc t$

For Exercises 24–29, use the unit circle and the period of the function to evaluate the function or state that the function is undefined at the given value.

24. $\cos\frac{15\pi}{2}$ 25. $\cot 540°$
26. $\csc(-240°)$ 27. $\sin\left(-\frac{14\pi}{3}\right)$
28. $\tan\left(-\frac{23\pi}{6}\right)$ 29. $\sec(480°)$

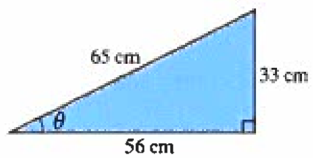
For Exercises 30–31, use the even-odd and periodic properties of the trigonometric functions to simplify.

30. $\tan(\pi - t) \cdot \cos(2\pi - t)$
31. $\sin(-t - \pi) + \sin(t + 2\pi)$
32. Write $\cos t$ in terms of $\sin t$ for t in Quadrant IV.
33. Write $\cot t$ in terms of $\csc t$ for t in Quadrant II.

SECTION 4.3

34. Suppose that a right triangle ΔABC has legs of length 5 cm and $\sqrt{2}$ cm. Evaluate the six trigonometric functions for angle θ, where angle θ is the larger acute angle.

35. Determine the values of the six trigonometric functions of θ for the given right triangle.



For Exercises 36–37, construct a right triangle to find the indicated values. Assume θ is an acute angle.

36. If $\cos\theta = \frac{5}{7}$, find $\csc\theta$ and $\tan\theta$.

37. If $\tan\theta = 3$, find $\sin\theta$ and $\sec\theta$.

For Exercises 38–39, evaluate the expression without the use of a calculator.

38. $\sin\frac{\pi}{3} + \tan\frac{\pi}{6}$ **39.** $\cos 45° \cdot \csc 60°$

40. Given $\sin\theta = \frac{99}{101}$ and $\cos\theta = \frac{20}{101}$, use the reciprocal and quotient identities to find the values of the other trigonometric functions of θ.

For Exercises 41–43, use an appropriate Pythagorean identity to find the indicated value for an acute angle θ.

41. Given $\cos\theta = \frac{40}{41}$, find the value of $\sin\theta$.

42. Given $\sec\theta = \frac{29}{20}$, find the value of $\tan\theta$.

43. Given $\cot\theta = \frac{13}{84}$, find the value of $\csc\theta$.

For Exercises 44–45, given the function value, find a cofunction of another angle with the same function value.

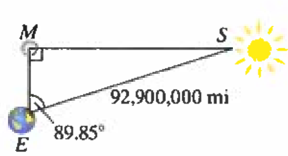
44. $\cos 15° = \frac{\sqrt{6} + \sqrt{2}}{4}$ **45.** $\tan\frac{\pi}{12} = 2 - \sqrt{3}$

46. Use a calculator to approximate the function values. Round to 4 decimal places.

a. $\tan 23.8°$ **b.** $\cos\frac{5}{8}$ **c.** $\sin\frac{\pi}{8}$

47. An observer at the top of a 48-ft building measures the angle of depression from the top of the building to a point on the ground to be 27°. What is the distance from the base of the building to the point on the ground? Round to the nearest foot.

48. During the first quarter moon, the Earth, Sun, and Moon form a right triangle. The distance between the Sun and the Earth is approximately 92,900,000 mi and the measure of $\angle SEM$ is approximately 89.85°. Determine the distance between the Earth and the Moon. Round to the nearest thousand miles.



SECTION 4.4

49. Let $P(3, -5)$ be a point on the terminal side of angle θ drawn in standard position. Find the values of the six trigonometric functions of θ.

For Exercises 50–55, find the reference angle for the given angle.

50. $\frac{5\pi}{6}$ **51.** $260°$ **52.** $-200°$

53. 5 **54.** $\frac{7\pi}{4}$ **55.** $750°$

For Exercises 56–61, use reference angles to find the exact value.

56. $\cos\frac{11\pi}{6}$ **57.** $\sin\left(-\frac{5\pi}{3}\right)$ **58.** $\tan\frac{17\pi}{6}$

59. $\cot\left(-\frac{3\pi}{4}\right)$ **60.** $\csc(-120°)$ **61.** $\sec 240°$

62. Identify which expressions are undefined.

a. $\sec 270°$ **b.** $\tan\frac{\pi}{2}$ **c.** $\cot 180°$ **d.** $\csc\left(-\frac{\pi}{2}\right)$

63. Given $\tan\theta = -\frac{2}{3}$ and $\sin\theta > 0$, find $\sec\theta$.

64. Given $\cos\theta = -\frac{3}{7}$ and $\cot\theta > 0$, find $\sin\theta$.

65. Given $\sin\theta = -\frac{60}{61}$ and θ in Quadrant III, find $\cot\theta$.

SECTION 4.5

For Exercises 66–69, determine the amplitude and period.

66. $y = 4\sin 2x$ **67.** $y = -2\cos\frac{x}{2}$

68. $y = \frac{1}{3}\cos\pi x$ **69.** $y = \sin 3\pi x$

For Exercises 70–72, graph one period of the function.

70. $y = \cos 3\pi x$

71. $y = 3\sin 2x$

72. $y = -2\cos\frac{x}{4}$

73. Determine the amplitude, period, and phase shift for each function.

a. $y = -\frac{1}{3}\cos(2x + \pi)$ **b.** $y = 5\sin(2\pi x - \pi)$

For Exercises 74–75, graph one period of the function.

74. $y = \frac{1}{3}\cos\left(x - \frac{\pi}{4}\right)$ **75.** $y = -2\sin\left(\pi x + \frac{\pi}{3}\right)$

For Exercises 76–77, determine the amplitude, period, phase shift, and vertical shift for each function.

76. $y = 4\sin(3x + \pi) - 2$ **77.** $y = -\frac{1}{4}\cos\left(\frac{\pi}{3}x - \frac{\pi}{2}\right) + 5$

For Exercises 78–79, graph one period of the function.

78. $y = \cos\left(x - \frac{\pi}{3}\right) + 2$ **79.** $y = 2\sin\left(2x + \frac{\pi}{4}\right) - 1$

80. The depth of water along a coastal inlet varies with the tides. On a summer day, the water depth at the end of a pier is 23 ft at 6:30 A.M., 21 ft at 12:30 P.M., and 23 ft at 6:30 P.M. Assuming that this pattern continues indefinitely and behaves like a cosine wave, write a function of the form $h(t) = A\cos(Bt - C) + D$. The value $h(t)$ is the water depth (in feet), t hours after 6:30 A.M.

81. The data in the table represent the percentage of the moon illuminated for selected days in January for a recent year. The value 0.0 = 0% represents a new moon and 1.0 = 100% represents a full moon. (*Source*: Astronomical Applications Department, U.S. Naval Observatory: http://aa.usno.navy.mil)

a. Enter the data in a graphing utility and use the sinusoidal regression tool (SinReg) to find a model of the form $y = a\sin(bx + c) + d$. Round a, b, c, and d to 1 decimal place.

b. Graph the data and the resulting function.

Day	1	3	5	7	9	11	13	15
Percent	0.00	0.05	0.20	0.40	0.61	0.79	0.92	0.99

Day	17	19	21	23	25	27	29	31
Percent	0.99	0.92	0.79	0.61	0.40	0.19	0.05	0.00

SECTION 4.6

For Exercises 82–85, give the period of the function and the equations of two asymptotes.

82. $y = \csc 3x$ **83.** $y = \tan 3x$

84. $y = \cot(\pi x + 2\pi)$ **85.** $y = -2\sec(x + \pi)$

For Exercises 86–89, graph one period of the function.

86. $y = 5\sec 3x$ **87.** $y = -2\csc\frac{x}{2}$

88. $y = 2\sec(\pi x + \pi)$ **89.** $y = 3\csc 4x + 2$

For Exercises 90–91, graph two periods of the function.

90. $y = \tan 4x$ **91.** $y = -2\tan\left(x - \frac{\pi}{3}\right)$

92. $y = \cot 2\pi x$ **93.** $y = \cot 2x + 3$

SECTION 4.7

For Exercises 94–109, evaluate the expression.

94. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ **95.** $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

96. $\arctan\left(\frac{\sqrt{3}}{3}\right)$ **97.** $\arccos\left(\frac{\sqrt{3}}{2}\right)$

98. $\arcsin\left(\frac{\sqrt{3}}{2}\right)$ **99.** $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

100. $\tan[\tan^{-1}(7)]$ **101.** $\cos(\cos^{-1}0.35)$

102. $\arcsin\left(\sin\frac{7\pi}{6}\right)$ **103.** $\sin\left[\sin^{-1}\left(-\frac{2}{3}\right)\right]$

104. $\cos\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$ **105.** $\sin\left[\arccos\left(-\frac{\sqrt{3}}{2}\right)\right]$

106. $\sin\left(\arctan\frac{3}{2}\right)$ **107.** $\cos\left(\sin^{-1}\frac{2}{5}\right)$

108. $\tan\left[\arccos\left(-\frac{3}{4}\right)\right]$ **109.** $\sin\left[\cos^{-1}\left(-\frac{1}{4}\right)\right]$

110. Use a calculator to approximate the function values in both radians and degrees.

a. $\sin^{-1}(-0.35)$ **b.** $\cos^{-1}\frac{\sqrt{2}}{5}$ **c.** $\arctan 10$

For Exercises 111–114, use a calculator to approximate the degree measure (to 1 decimal place) or radian measure (to 4 decimal places) of the angle θ subject to the given conditions.

111. $\tan\theta = \frac{4}{19}$ and $180° \le \theta \le 270°$

112. $\cos\theta = \frac{2}{15}$ and $270° \le \theta \le 360°$

113. $\sin\theta = -\frac{5}{11}$ and $\pi < \theta < \frac{3\pi}{2}$

114. $\tan\theta = 4$ and $\pi < \theta < \frac{3\pi}{2}$

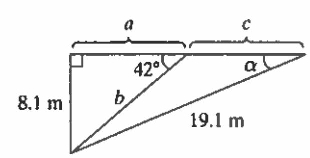
115. Write the given expression as an algebraic expression. It is not necessary to rationalize the denominator.

a. $\cos(\sin^{-1}x)$ for $|x| < 1$.

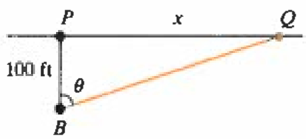
b. $\sin(\arctan x)$

c. $\tan\left(\arccos\frac{x}{\sqrt{x^2 + 9}}\right)$ for $x \ne 0$.

116. Find the lengths of sides a, b, and c to the nearest tenth of a meter and the measure of angle α to the nearest tenth of a degree.



117. The length of the perpendicular line segment $\overline{BP}$ from a rotating beacon to a straight shoreline is 100 ft. The beam of light emitted from the beacon strikes the shoreline at a point Q, a distance of x feet from point P. Let θ represent $\angle QBP$.

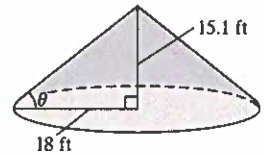


a. Write θ as a function of x.

b. Find θ for $x = 100$, 200, and 300 ft. Round to the nearest degree.

c. What happens to θ as $x \to \infty$?

118. Find the angle of repose θ for the pile of coarse gravel. Round to the nearest degree.



CHAPTER 4 Test

1. Convert 15.36° to DMS (degree-minute-second) form. Round to the nearest second if necessary.
2. Convert 130.3° to radians. Round to 2 decimal places.
3. Find the exact length of the arc intercepted by a central angle of 27° on a circle of radius 5 ft.
4. A skateboard designed for rough surfaces has wheels with diameter 60 mm. If the wheels turn at 2200 rpm, what is the linear speed in mm per minute?
5. A pulley is 20 in. in diameter. Through how many degrees should the pulley rotate to lift a load 3 ft? Round to the nearest degree.
6. A circle has a radius of 9 yd. Find the area of a sector with a central angle of 120°.
7. For an acute angle θ, if $\sin\theta = \frac{5}{6}$, evaluate $\cos\theta$ and $\tan\theta$.
8. Evaluate $\tan\frac{\pi}{6} - \cot\frac{\pi}{6}$ without the use of a calculator.
9. Given $\sin\theta = \frac{5}{13}$, use a Pythagorean identity to find $\cos\theta$.
10. Given $\sec 75° = \sqrt{2} + \sqrt{6}$, find a cofunction of another angle with the same function value.
11. Use a calculator to approximate $\sin\frac{2\pi}{11}$ to 4 decimal places.
12. A flag pole casts a shadow of 20 ft when the angle of elevation of the Sun is 40°. How tall is the flag pole? Round to the nearest foot.
13. A newly planted tree is anchored by a covered wire running from the top of the tree to a post in the ground 5 ft from the base of the tree. If the angle between the wire and the top of the tree is 20°, what is the length of the wire? Round to the nearest foot.

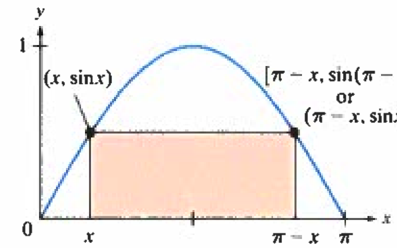
For Exercises 14–21, evaluate the function or state that the function is undefined at the given value.

14. $\sin\left(-\frac{3\pi}{4}\right)$
15. $\tan 930°$
16. $\sec\frac{11\pi}{6}$
17. $\csc(-150°)$
18. $\cot 20\pi$
19. $\cos 690°$
20. $\tan\frac{7\pi}{3}$
21. $\sec(-630°)$
22. Given $\sin\theta = \frac{5}{8}$ and $\cos\theta < 0$, find $\tan\theta$.
23. Given $\sec\theta = \frac{4}{3}$ and $\sin\theta < 0$, find $\csc\theta$.
24. Given $\tan\theta = -\frac{4}{3}$ and θ is in Quadrant IV, find $\sec\theta$.

For Exercises 25–28, select the trigonometric function, $f(t) = \sin t$, $g(t) = \cos t$, $h(t) = \tan t$ or $r(t) = \cot t$, with the given properties.

25. The function is odd, with period 2π, and domain of all real numbers.
26. The function is odd, with period π, and domain of all real numbers excluding odd multiples of $\frac{\pi}{2}$.
27. The function is even, with period 2π, and domain of all real numbers.
28. The function is odd, with period π, and domain of all real numbers excluding multiples of π.
29. Use the even-odd and periodic properties of the trigonometric functions to simplify $\cos(-\theta - 2\pi) - \sin(\theta - 2\pi)\cdot\cot(\theta + \pi)$

30. Suppose that a rectangle is bounded by the x-axis and the graph of $y = \sin x$ on the interval $[0, \pi]$.



a. Write a function that represents the area $A(x)$ of the rectangle for $0 < x < \frac{\pi}{2}$.

b. Determine the area of the rectangle for $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$.

For Exercises 31–32, determine the amplitude, period, phase shift, and vertical shift for each function.

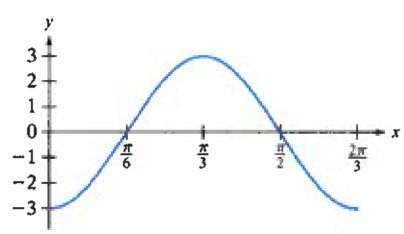
31. $y = \frac{3}{4}\sin\left(2\pi x - \frac{\pi}{6}\right)$ **32.** $y = -5\cos(5x + \pi) + 7$

For Exercises 33–36, graph one period of the function.

33. $y = -2\sin x$ **34.** $y = 3\cos 2x$

35. $y = \sin\left(2x - \frac{\pi}{4}\right)$ **36.** $y = 3 - 2\cos(2\pi x - \pi)$

37. Write a function of the form $f(x) = A\cos(Bx)$ for the given graph.



38. Identify each statement as true or false. If a statement is false, explain why.

a. The relative maxima of the graph of $y = \sin x$ correspond to the relative minima of the graph of $y = \csc x$.

b. The period of $y = \tan 2x$ is π.

c. The amplitude of $y = 2\cot x$ is 2.

d. The period of $y = \sec 2x$ is π.

e. The vertical asymptotes of the graph of $y = \cot x$ occur where the graph of $y = \cos x$ has x-intercepts.

For Exercises 39–40, graph one period of the function.

39. $y = -4\sec 2x$ **40.** $y = \csc\left(2x - \frac{\pi}{4}\right)$

For Exercises 41–42, graph two periods of the function.

41. $y = \tan 3x$ **42.** $y = -4\cot\left(x + \frac{\pi}{4}\right)$

43. Simplify each expression.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\frac{\sqrt{2}}{2}$

b. $\cos\left[\arctan\left(-\sqrt{3}\right)\right]$

44. Use a calculator to approximate the degree measure (to 1 decimal place) of the angle θ subject to the given conditions.

$$\sin\theta = \frac{3}{8} \text{ and } 90^\circ \le \theta \le 180^\circ$$

45. Find the exact value of $\sin^{-1}\left(\sin\frac{11\pi}{6}\right)$.

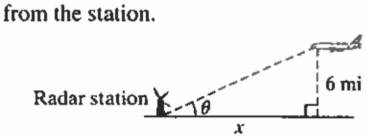
46. Find the exact value of $\tan\left(\cos^{-1}\frac{7}{8}\right)$.

47. Find the exact value of $\cos\left[\arcsin\left(-\frac{5}{6}\right)\right]$.

48. Write the expression $\tan\left(\cos^{-1}\frac{x}{\sqrt{x^2+25}}\right)$ for $x \ne 0$ as an algebraic expression.

49. A radar station tracks a plane flying at a constant altitude of 6 mi on a path directly over the station. Let θ be the angle of elevation from the radar station to the plane.

a. Write θ as a function of the plane's ground distance $x > 0$ from the station.



b. Without the use of a calculator, will the angle of elevation be less than 45° or greater than 45° when the plane's ground distance is 3.2 mi away?

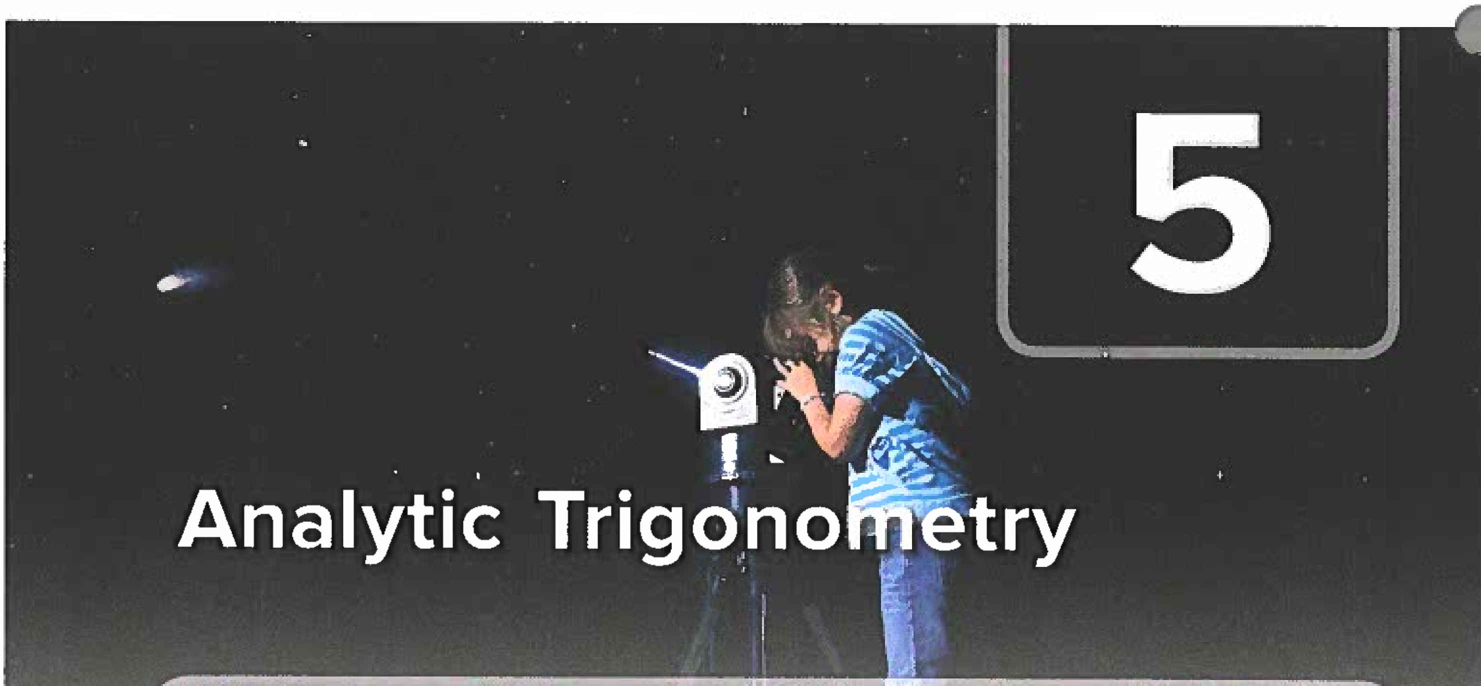
c. Use a calculator to find θ to the nearest degree for $x = 3.2$, 1.6, and 0.5 mi.

CHAPTER 4 Cumulative Review Exercises

For Exercises 1–9, solve the equations and inequalities. Write the solution set to inequalities in interval notation.

1. $|3x - 5| + 3 \geq 6$
2. $(x - 2)^2(x + 5) < 0$
3. $\dfrac{x + 2}{x - 3} \geq 2$
4. $3x^4 - x^2 - 10 = 0$
5. $\sqrt{x + 7} = 4 - \sqrt{x - 1}$
6. $2x^4 + 5x^3 - 29x^2 - 17x + 15 = 0$
7. $e^{3x} = 2$
8. $\log(3x + 2) - \log x = 3$
9. $\ln(x - 1) + \ln x = \ln 2$
10. Given $x^2 + y^2 - 4x + 6y + 9 = 0$
    a. Write the equation of the circle in standard form.
    b. Identify the center and radius.
    c. Write the domain and range in interval notation.
11. Given $y = -x^2 + 2x - 4$
    a. Identify the vertex.
    b. Write the domain and range in interval notation.
12. Given $f(x) = x^4 - 2x^3 - 4x^2 + 8x$,
    a. Find the x-intercepts of the graph of f.
    b. Determine the end behavior of the graph of f.
13. a. Graph $y = \dfrac{3x^2 + x - 5}{3x - 2}$.
    b. Identify the asymptotes.
14. Given $y = -2\sin(4x - \pi) - 6$, identify the
    a. Domain and range in interval notation.
    b. Amplitude.
    c. Period.
    d. Phase shift.
    e. Vertical shift.
15. Given $f(x) = \tan x$ and $g(x) = \dfrac{x}{2}$, evaluate
    a. $(f \circ g)\left(\dfrac{\pi}{2}\right)$
    b. $(g \circ f)(\pi)$
16. Evaluate $\cos(-450^\circ)$.
17. Evaluate $\tan\left[\arcsin\left(-\dfrac{3}{8}\right)\right]$.
18. Write the logarithm as the sum or difference of logarithms. Simplify as much as possible.
    $\log\left(\dfrac{x^2y}{100z}\right)$
19. Divide. Write the answer in standard form, $a + bi$.
    $\dfrac{2 - 8i}{3 - 5i}$
20. Suppose that y varies inversely as x and directly as z. If y is 12 when x is 8 and z is 3, find the constant of variation k.

Analytic Trigonometry



5

Analytic Trigonometry

Chapter Outline

Most applications of trigonometry require the use of technology for the computation of the six trigonometric functions of a given angle. However, ancient and medieval mathematicians and astronomers had no such luxury. Instead, they built a table of sines: a table giving the sine of every integer angle between 1° and 90°. The cofunction and reciprocal identities can then be used to calculate the values of the other five trigonometric functions from the value of the sine.

For certain angles, such as 30°, 45°, and 60°, the sine of the angle is easy to compute geometrically (see Section 4.3). The trigonometric identities presented in this chapter then give us the means to compute the sine of many other angles.* For example, if θ is the sum of two angles u and v with known sine values, we can use the identity $\sin(u + v) = \sin u \cos v + \cos u \sin v$ to find $\sin\theta$ (Section 5.2). For example,

$$\sin 75° = \sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°.$$

Table of Sines

θ	$\sin\theta$	θ	$\sin\theta$
1°	0.0174	31°	0.5150
2°	0.0349	32°	0.5299
3°	0.0523	33°	0.5446
4°	0.0698	34°	0.5592
5°	0.0872	35°	0.5736
6°	0.1045	36°	0.5878
...	...	...	...

In addition to evaluating trigonometric functions of certain angles, the trigonometric identities are very useful in simplifying complex trigonometric expressions into more manageable forms and for solving trigonometric equations.

*Not all integer angles between 1° and 90° can be found using the fundamental identities and the sine values of the special angles. In fact, only angles that are a multiple of 3° can be found in this way. For ancient astronomers and mathematicians, finding the value of sin 1° became a problem of critical importance and was addressed throughout the ages by using numerical methods of approximation.

SECTION 5.1 Fundamental Trigonometric Identities

OBJECTIVES

1. Simplify Trigonometric Expressions
2. Verify Trigonometric Identities
3. Write an Algebraic Expression as a Trigonometric Expression

1. Simplify Trigonometric Expressions

In Chapter 4 we presented the six trigonometric functions by using two different approaches. We used the lengths of the sides of right triangles, and we used coordinates taken from points on the unit circle. The resulting definitions yield consistent results and give rise to thousands of relationships among the functions. In this section, we use algebra skills to simplify trigonometric expressions and verify identities.

You have already been introduced to the fundamental trigonometric identities given in Table 5-1.

Table 5-1

Fundamental Identities			
Reciprocal Identities	$\csc x = \dfrac{1}{\sin x}$	$\sec x = \dfrac{1}{\cos x}$	$\cot x = \dfrac{1}{\tan x}$
	$\sin x = \dfrac{1}{\csc x}$	$\cos x = \dfrac{1}{\sec x}$	$\tan x = \dfrac{1}{\cot x}$
Quotient Identities	$\tan x = \dfrac{\sin x}{\cos x}$	$\cot x = \dfrac{\cos x}{\sin x}$	
Pythagorean Identities	$\sin^2 x + \cos^2 x = 1$	$\tan^2 x + 1 = \sec^2 x$	$1 + \cot^2 x = \csc^2 x$
Even and Odd Identities	$\sin(-x) = -\sin x$	$\cos(-x) = \cos x$	
	$\csc(-x) = -\csc x$	$\sec(-x) = \sec x$	
	$\tan(-x) = -\tan x$	$\cot(-x) = -\cot x$	

TIP It is often helpful to recognize alternative forms of the Pythagorean identities. For example, $\sin^2 x = 1 - \cos^2 x$ and $\cos^2 x = 1 - \sin^2 x$.

Each statement in Table 5-1 is true for every value of x for which the expressions within the statement are defined. For example,

$$\sin x = \frac{1}{\csc x} \text{ is true provided that } x \neq n\pi \text{ for integers } n.$$

The identities in Table 5-1 are tools to help us simplify trigonometric expressions and to verify other identities.

EXAMPLE 1 Simplifying an Expression Using the Quotient and Reciprocal Identities

Simplify the expression. Write the final form with no fractions. $\sec^2 x \cot x \cos x$

Solution:

$\sec^2 x \cot x \cos x$

$= \dfrac{1}{\cos^2 x} \cdot \dfrac{\cos x}{\sin x} \cdot \dfrac{\cos x}{1}$ Write the trigonometric functions in terms of sine and cosine.

$= \dfrac{1}{\cancel{\cos^2 x}} \cdot \dfrac{\cancel{\cos x}}{\sin x} \cdot \dfrac{\cancel{\cos x}}{1}$ Simplify common factors from the numerator and denominator.

$= \dfrac{1}{\sin x}$ Simplify.

$= \csc x$ Apply the reciprocal identity relating the sine and cosecant functions.

Skill Practice 1 Simplify. Write the final form with no fractions. $\tan x \cos^2 x \sec x$

Answer
1. $\sin x$

In Example 2, we simplify an expression by writing each term with a common denominator and adding the terms.

EXAMPLE 2 Simplifying by Adding Fractional Expressions

Simplify the expression. Write the final form with no fractions. $\dfrac{\cos\theta}{1+\sin\theta} + \tan\theta$

Solution:

$$\frac{\cos\theta}{1+\sin\theta} + \tan\theta$$

$$= \frac{\cos\theta}{1+\sin\theta} + \frac{\sin\theta}{\cos\theta}$$ Write the trigonometric functions in terms of sine and cosine.

$$= \frac{\cos\theta}{\cos\theta} \cdot \frac{\cos\theta}{(1+\sin\theta)} + \frac{\sin\theta}{\cos\theta} \cdot \frac{(1+\sin\theta)}{(1+\sin\theta)}$$ The least common denominator is $\cos\theta(1+\sin\theta)$.

$$= \frac{\cos^2\theta + \sin\theta + \sin^2\theta}{\cos\theta(1+\sin\theta)}$$ Add the fractions.

$$= \frac{\sin^2\theta + \cos^2\theta + \sin\theta}{\cos\theta(1+\sin\theta)}$$ Group the squared terms because we recognize the Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$.

$$= \frac{1 + \sin\theta}{\cos\theta(1+\sin\theta)}$$ Substitute 1 for $\sin^2\theta + \cos^2\theta$.

$$= \frac{\cancel{1+\sin\theta}}{\cos\theta\cancel{(1+\sin\theta)}}$$ Simplify common factors from the numerator and denominator.

$$= \frac{1}{\cos\theta}$$

$$= \sec\theta$$ Apply the reciprocal identity relating the cosine and secant functions.

Skill Practice 2 Simplify. Write the final form with no fractions.
$\dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{1+\cos\theta}$

TIP Recall that two fractions with different denominators b and d can be added as $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$ provided that $b \neq 0$ and $d \neq 0$.

In Example 3, we make use of factoring techniques to simplify a trigonometric expression.

EXAMPLE 3 Simplifying a Trigonometric Expression by Factoring

Simplify. Write the final form with no fractions. $\dfrac{\tan^2 t - 1}{\tan t \sin t + \sin t}$

Answer
2. $\csc\theta$

Solution:

$$\frac{\tan^2 t - 1}{\tan t \sin t + \sin t}$$

$= \dfrac{(\tan t - 1)(\tan t + 1)}{\sin t(\tan t + 1)}$ The numerator factors as a difference of squares. The terms in the denominator share a common factor of sin t.

$= \dfrac{(\tan t - 1)\cancel{(\tan t + 1)}}{\sin t\cancel{(\tan t + 1)}}$ Simplify common factors from the numerator and denominator.

$= \dfrac{\tan t - 1}{\sin t}$ Simplify.

$= \dfrac{\dfrac{\sin t}{\cos t} - 1}{\sin t}$ Write $\tan t$ as $\dfrac{\sin t}{\cos t}$.

$= \left(\dfrac{\sin t}{\cos t} - 1\right) \cdot \dfrac{1}{\sin t}$ Multiply the numerator of the complex fraction by the reciprocal of the denominator.

$= \dfrac{1}{\cos t} - \dfrac{1}{\sin t}$ Apply the distributive property.

$= \sec t - \csc t$ Apply the reciprocal identities.

Skill Practice 3 Simplify. Write the final form with no fractions.

$$\frac{1 - \sec^2 t}{\tan t - \tan t \sec t}$$

2. Verify Trigonometric Identities

In Example 1, we showed that:

$$\sec^2 x \cot x \cos x = \csc x \qquad (1)$$

Example 1 also illustrates that trigonometric expressions can be written in many different equivalent forms. Equation (1) is called an **identity** because it is true for all values of x for which the expressions on the left and right are defined.

One method to verify that an equation is an identity is to manipulate one side of the equation (usually the more complicated side) until it takes the form of the other side of the equation. This process is the same as simplifying an expression, except that we know the final form.

While there is no explicit procedure to verify a trigonometric identity, we recommend some helpful guidelines.

TIP Recall that a **conditional equation** is an equation that is true only for *some* values of the variable. For example, $\tan x = 1$ is true only for $x = \frac{\pi}{4} + n\pi$.

Guidelines for Verifying a Trigonometric Identity

1. Work with one side of the equation (usually the more complicated side) and keep the other side in mind as your final goal.
2. Look for opportunities to apply the fundamental identities.
   - If the expression is a product or quotient of factors, consider the reciprocal and quotient identities.
   - If squared terms are present, look to see if the terms can be grouped in one of the forms of a Pythagorean identity.
   - If an expression involves a negative argument, consider using the even or odd function identities.
3. Apply basic algebraic techniques such as factoring, multiplying terms, combining like terms, and writing fractions with a common denominator.
4. Consider writing expressions explicitly in terms of sine and cosine.

Answer
3. $\cot t + \csc t$

EXAMPLE 4 Verifying an Identity Containing Negative Arguments

Verify that the equation is an identity. $\dfrac{\cos(-x)\tan(-x)}{\sin x} = -1$

Solution:

$\dfrac{\cos(-x)\tan(-x)}{\sin x} = -1$ The left-hand side (LHS) is the more complicated side.

$= \dfrac{\cos x(-\tan x)}{\sin x}$ To simplify, we require the same arguments for each factor. The cosine function is an even function. Thus, $\cos(-x) = \cos x$. The tangent function is an odd function. Thus, $\tan(-x) = -\tan x$.

$= \dfrac{\cancel{\cos x}\left(-\dfrac{\sin x}{\cancel{\cos x}}\right)}{\sin x}$ Write tangent as the ratio of sine and cosine.

$= \dfrac{-\sin x}{\sin x}$

$= -1$ (RHS) This equals the right-hand side (RHS) of the original equation. Therefore, we have verified the identity.

Skill Practice 4 Verify that the equation is an identity. $\dfrac{\sin(-x)\cot(-x)}{\cos x} = 1$

When we verify an identity, we are trying to show that the expression on one side equals the expression on the other. Therefore, we cannot use the equality of the statement because we have not yet proved it. For this reason, we work with the left or right side of the equation, but not both. In Example 4, we worked only with the left side of the equation.

EXAMPLE 5 Verifying an Identity by Combining Fractions

Verify that the equation is an identity. $\dfrac{1}{1-\cos x} - \dfrac{1}{1+\cos x} = 2\cot x\csc x$

Solution:

$\dfrac{1}{1-\cos x} - \dfrac{1}{1+\cos x}$ The LHS has two terms and the RHS has one term. Therefore, a good strategy is to combine the two terms on the left by adding the fractions.

$= \dfrac{(1+\cos x)}{(1+\cos x)} \cdot \dfrac{1}{(1-\cos x)} - \dfrac{1}{(1+\cos x)} \cdot \dfrac{(1-\cos x)}{(1-\cos x)}$ The LCD is $(1+\cos x)(1-\cos x)$.

$= \dfrac{(1+\cos x)-(1-\cos x)}{(1+\cos x)(1-\cos x)}$ Subtract the fractions.

$= \dfrac{1+\cos x-1+\cos x}{1-\cos^2 x}$ Apply the distributive property in the numerator. Multiply conjugates in the denominator.

$= \dfrac{2\cos x}{\sin^2 x}$ In the denominator, from the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, we know that $\sin^2 x = 1 - \cos^2 x$.

$= 2 \cdot \dfrac{\cos x}{\sin x} \cdot \dfrac{1}{\sin x}$ The RHS of the original equation has factors of $\cot x$ and $\csc x$. Therefore, we group factors in the numerator and denominator conveniently to reach the desired result.

$= 2\cot x\csc x$ (RHS)

Answer

4. $\dfrac{\sin(-x)\cot(-x)}{\cos x} = \dfrac{-\sin x(-\cot x)}{\cos x} = \dfrac{\sin x\left(\dfrac{\cos x}{\sin x}\right)}{\cos x} = \dfrac{\cos x}{\cos x} = 1$

Skill Practice 5 Verify that the equation is an identity.
$$\frac{1}{1 - \sin x} - \frac{1}{1 + \sin x} = 2\tan x \sec x$$

Keep in mind that the Pythagorean identities have several alternative forms.

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$
$$1 - \cos^2 x = \sin^2 x \qquad \sec^2 x - 1 = \tan^2 x \qquad \csc^2 x - 1 = \cot^2 x$$
$$1 - \sin^2 x = \cos^2 x \qquad \sec^2 x - \tan^2 x = 1 \qquad \csc^2 x - \cot^2 x = 1$$

These alternative forms each involve a difference of squares that factors as a product of conjugates. This was illustrated in steps 2–5 of Example 5.

$$(1 + \cos x)(1 - \cos x) = 1 - \cos^2 x = \sin^2 x$$

Notice that upon simplification, the expression ultimately results in a single term. This pattern is sometimes helpful when we want to simplify a fraction with a denominator containing two terms as a fraction with a single-term denominator.

EXAMPLE 6 Verifying an Identity by Using Conjugates

Verify that the equation is an identity. $\dfrac{1 - \sin t}{1 + \sin t} = (\sec t - \tan t)^2$

Solution:

$$\frac{1 - \sin t}{1 + \sin t} = (\sec t - \tan t)^2$$

$$\frac{1 - \sin t}{1 + \sin t} = \frac{(1 - \sin t)}{(1 + \sin t)} \cdot \frac{(1 - \sin t)}{(1 - \sin t)}$$

Since the RHS has no fractions, one strategy might be to multiply the numerator and denominator of the LHS by the conjugate of the denominator. The purpose is to create the difference of squares $1 - \sin^2 t$ which then simplifies to the single term $\cos^2 t$.

$$= \frac{(1 - \sin t)^2}{1 - \sin^2 t}$$

The product of conjugates in the denominator results in a difference of squares.

$$= \frac{(1 - \sin t)^2}{\cos^2 t}$$

Substitute $\cos^2 t$ for $1 - \sin^2 t$.

$$= \left(\frac{1 - \sin t}{\cos t}\right)^2$$

Write the quotient of squares as the square of a quotient.

$$= \left(\frac{1}{\cos t} - \frac{\sin t}{\cos t}\right)^2$$

Use the property $\dfrac{a - b}{c} = \dfrac{a}{c} - \dfrac{b}{c}$.

$$= (\sec t - \tan t)^2 \quad \text{(RHS)}$$

Apply reciprocal and quotient identities.

Skill Practice 6 Verify that the equation is an identity. $1 - \dfrac{\sin^2 t}{1 + \cos t} = \cos t$

It is important to understand that there are often many paths to simplifying a trigonometric identity. A different set of algebraic steps can be used, or we can manipulate the other side of the expression. For example, the expression in Example 6 can also be simplified by manipulating the RHS.

Answers

5. See page SA-40.
6. See page SA-40.

$$(\sec t - \tan t)^2 = \left(\frac{1}{\cos t} - \frac{\sin t}{\cos t}\right)^2 = \left(\frac{1 - \sin t}{\cos t}\right)^2 = \frac{(1 - \sin t)^2}{\cos^2 t}$$

Write the expression in terms of $\sin t$ and $\cos t$. Combine terms on the left by subtracting fractions.

$$= \frac{(1 - \sin t)^2}{1 - \sin^2 t} = \frac{(1 - \sin t)^2}{(1 - \sin t)(1 + \sin t)} = \frac{1 - \sin t}{1 + \sin t} = \text{LHS}$$

Factor the denominator and simplify common factors in the numerator and denominator.

EXAMPLE 7 **Verifying an Identity Involving Logarithms**

Verify that the equation is an identity. $\ln|\sin x| + \ln|\sec x| = \ln|\tan x|$

Solution:

$\ln|\sin x| + \ln|\sec x|$ — The LHS is the more complicated side.

$= \ln|\sin x| + \ln\left|\frac{1}{\cos x}\right|$ — Write $\sec x$ as $\frac{1}{\cos x}$.

$= \ln\left(|\sin x| \cdot \left|\frac{1}{\cos x}\right|\right)$ — Apply the property $\ln x + \ln y = \ln(xy)$.

$= \ln\left|\frac{\sin x}{\cos x}\right|$ — Apply the property of absolute value $|a| \cdot |b| = |ab|$.

$= \ln|\tan x|$ (RHS) — Write $\frac{\sin x}{\cos x}$ as $\tan x$.

Skill Practice 7 Verify that the equation is an identity.
$\ln|\cot x| + \ln|\sin x| = \ln|\cos x|$

TIP Recall that
1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^p = p$
4. $b^{\log_b x} = x$
5. $\log_b(xy) = \log_b x + \log_b y$
6. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
7. $\log_b x^p = p\log_b x$

When verifying an identity, we generally work on one side of the equation only and manipulate the expression until it takes the form of the other side. It is logically equivalent, however, to verify an identity by transforming each side of the equation *independently* until we reach a common equivalent expression. This is demonstrated in Example 8.

EXAMPLE 8 **Verifying an Identity by Working Each Side Independently**

Verify that the equation is an identity by manipulating each side of the equation independently. $\frac{\sin x}{1 + \sin x} = \frac{\csc x - 1}{\cot^2 x}$

Solution:

LHS: $\frac{\sin x}{1 + \sin x}$

$$= \frac{(\sin x)}{(1 + \sin x)} \cdot \frac{(1 - \sin x)}{(1 - \sin x)}$$

$$= \frac{\sin x(1 - \sin x)}{1 - \sin^2 x}$$

$$= \frac{\sin x - \sin^2 x}{\cos^2 x}$$

RHS: $\frac{\csc x - 1}{\cot^2 x}$

$$= \frac{\frac{1}{\sin x} - 1}{\frac{\cos^2 x}{\sin^2 x}} = \frac{\frac{\sin^2 x}{1} \cdot \left(\frac{1}{\sin x} - 1\right)}{\frac{\sin^2 x}{1} \cdot \left(\frac{\cos^2 x}{\sin^2 x}\right)}$$

$$= \frac{\sin x - \sin^2 x}{\cos^2 x}$$

TIP On the left side of the identity, we multiplied numerator and denominator by the conjugate of the denominator. Recall that the product of conjugates results in a difference of squares. The motivation is to use the Pythagorean relationship
$1 - \sin^2 x = \cos^2 x$.

Answer
7. See page SA-40.

Since the left-hand side equals the right-hand side, we have verified that the equation is an identity.

Skill Practice 8 Verify the equation is an identity by manipulating each side of the equation independently. $\dfrac{1-\cos x}{\cos x} = \dfrac{\tan x \sin x}{1+\cos x}$

The process used in Example 8 was to manipulate each side of the equation independently into a third, equivalent expression. In a rigorous setting, this method is frowned upon. In Example 8, we could have manipulated only one side (say the LHS) by first reaching the intermediate step $\dfrac{\sin x - \sin^2 x}{\cos^2 x}$. The remaining steps to verify the identity would be to reverse the steps used to manipulate the RHS.

3. Write an Algebraic Expression as a Trigonometric Expression

In some applications of calculus, it is advantageous to write an algebraic expression of the form $\sqrt{u^2+a^2}$, $\sqrt{a^2-u^2}$, or $\sqrt{u^2-a^2}$ as a trigonometric expression by way of an appropriate substitution. Notice that each algebraic expression looks similar to an expression that might be used to find the length of a side of a right triangle using the Pythagorean theorem. Therefore, it seems reasonable that these algebraic expressions may be linked to trigonometric expressions.

EXAMPLE 9 Applying Trigonometric Substitution

Write $\sqrt{x^2+9}$ as a function of θ, where $0 < \theta < \frac{\pi}{2}$, by making the substitution $x = 3\tan\theta$.

Solution:

$\sqrt{x^2+9} = \sqrt{(3\tan\theta)^2+9}$	Substitute $3\tan\theta$ for x.
$= \sqrt{9\tan^2\theta+9}$	Square the expression $3\tan\theta$.
$= \sqrt{9(1+\tan^2\theta)}$	Factor.
$= \sqrt{9\sec^2\theta}$	Apply the Pythagorean identity $1+\tan^2\theta = \sec^2\theta$.
$= 3\sec\theta$	Take the positive square root, since $0 < \theta < \frac{\pi}{2}$.

Skill Practice 9 Write $\sqrt{x^2-121}$ as a function of θ, where $0 < \theta < \frac{\pi}{2}$, by making the substitution $x = 11\sec\theta$.

From Example 9, we can illustrate the link between the algebraic form $\sqrt{x^2+9}$ and the trigonometric form $3\sec\theta$ by using a right triangle. The expression $\sqrt{x^2+9}$ represents the hypotenuse of a right triangle with legs of lengths x and 3 (Figure 5-1).

From the triangle, $\sec\theta = \dfrac{\sqrt{x^2+9}}{3}$, so $\sqrt{x^2+9} = 3\sec\theta$.

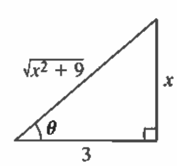


Figure 5-1

Answers

8. LHS:

$$\frac{1-\cos x}{\cos x} = \frac{(1-\cos x)}{\cos x}\cdot\frac{(1+\cos x)}{(1+\cos x)} = \frac{1-\cos^2 x}{\cos x(1+\cos x)} = \frac{\sin^2 x}{\cos x(1+\cos x)}$$

RHS:

$$\frac{\tan x\sin x}{1+\cos x} = \frac{\frac{\sin x}{\cos x}\cdot\sin x}{1+\cos x} = \frac{\sin^2 x}{\cos x(1+\cos x)}$$

9. $11\tan\theta$

SECTION 5.1 Practice Exercises

Prerequisite Review

R.1. Factor. $5r^2 + 33rs + 18s^2$

R.2. Find the least common denominator. $\dfrac{3}{(c-3)(c+3)}; \dfrac{-6}{(c-3)(c-1)}$

R.3. Subtract. $\dfrac{3a}{a+9} - \dfrac{2}{a-5}$

R.4. Add. $\dfrac{y-4}{y-5} + \dfrac{5y^2 - 38y + 55}{y^2 - 25}$

R.5. Simplify. $\dfrac{\dfrac{8x^2}{19y^2}}{\dfrac{3x}{xy^2}}$

R.6. Simplify. $\dfrac{\dfrac{8}{n} + \dfrac{3}{n^2}}{\dfrac{6}{n^2} - \dfrac{7}{n}}$

R.7. Assume that y represents a positive real number. What is the conjugate of $\sqrt{y} - 24$?

R.8. Rationalize the denominator. $\dfrac{5}{6 - \sqrt{3}}$

Concept Connections

1. The value $\tan\theta$ is the quotient of ______ and ______, and $\cot\theta$ is the quotient of ______ and ______.

2. The value $\csc\theta$ is the reciprocal of ______, $\cot\theta$ is the reciprocal of ______, and $\sec\theta$ is the reciprocal of ______.

3. Given the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, write two alternative forms that involve the difference of squares.

4. Given the Pythagorean identity $\tan^2 x + 1 = \sec^2 x$, write two alternative forms that involve the difference of squares.

5. $\sin(-x) =$ ______, $\cos(-x) =$ ______, and $\tan(-x) =$ ______.

6. An equation that is true for all values of the variable except where the individual expressions are not defined is called an ______.

Objective 1: Simplify Trigonometric Expressions

For Exercises 7–10, find the least common denominator. Then rewrite each expression with the new denominator.

7. a. $\dfrac{1}{a}; \dfrac{1}{b}$ **b.** $\dfrac{1}{\cos x}; \dfrac{1}{\sin x}$

8. a. $\dfrac{a}{b}; a$ **b.** $\dfrac{\sin x}{\cos x}; \sin x$

9. a. $\dfrac{1}{1+a}; \dfrac{a}{b}$ **b.** $\dfrac{1}{1+\sin x}; \dfrac{\sin x}{\cos x}$

10. a. $\dfrac{1}{a}; \dfrac{a}{1-a}$ **b.** $\dfrac{1}{\cos x}; \dfrac{\cos x}{1-\cos x}$

For Exercises 11–12, multiply.

11. a. $(a+b)^2$ **b.** $(\cos x + \cot x)^2$

12. a. $(a-b)^2$ **b.** $(\sin x - \tan x)^2$

For Exercises 13–20, factor each expression.

13. a. $a^2 - b^2$ **b.** $\sin^2 x - \cos^2 x$

14. a. $a^2 + b^2$ **b.** $\tan^2 x + \sec^2 x$

15. a. $a^3 + b^3$ **b.** $\sin^3 x + \sec^3 x$

16. a. $a^3 - b^3$ **b.** $\cos^3 x - \cot^3 x$

17. a. $a^2 + 2ab + b^2$ **b.** $\cos^2 x + 2\cos x \csc x + \csc^2 x$

18. a. $a^2 - 2ab + b^2$ **b.** $\sin^2 x - 2\sin x \tan x + \tan^2 x$

19. a. $3a^2 - 4ab - 4b^2$ **b.** $3\sin^2 x - 4\sin x\cos x - 4\cos^2 x$

20. a. $4a^2 - 12ab + 5b^2$ **b.** $4\cos^2 x - 12\cos x\sin x + 5\sin^2 x$

For Exercises 21–36, simplify the expression. Write the final form with no fractions. (See Examples 1–3)

21. $\sin x \sec x \cot x$

22. $\cos x \tan x \csc x$

23. $\dfrac{\sin x}{\sec x \tan x}$

24. $\dfrac{\sin x}{\sec x \cot x}$

25. $\tan^2 x \cot x \csc x$

26. $\csc^2 x \tan^2 x \sec x$

27. $\dfrac{\tan^2 x}{\sin x \sec^2 x}$

28. $\dfrac{\tan x}{\sin x \sec^2 x}$

29. $\dfrac{\cos^2 x + 1}{\sin x} + \sin x$

30. $\dfrac{\sin^2 x + 1}{\cos^2 x} + 1$

31. $\dfrac{\csc\theta - \sin\theta}{\csc\theta}$

32. $\dfrac{\cos\theta - \sec\theta}{\sec\theta}$

33. $\dfrac{\sin^2 x + 2\sin x + 1}{\sin^2 x + \sin x}$

34. $\dfrac{\sin x\sec^2 x - \sin x}{\sin x\sec x + \sin x}$

35. $\dfrac{\cos^2 x - 2\cos x + 1}{\sin x\cos x - \sin x}$

36. $\dfrac{\tan^2 x + 2\tan x + 1}{\sin x\tan x + \sin x}$

Objective 2: Verify Trigonometric Identities

For Exercises 37–40, verify that the equation is an identity. (See Example 4)

37. $\sin(-x) + \csc x = \cot x\cos x$

38. $\sin(-x) + \cot(-x)\cos(-x) = -\csc x$

39. $\cot x[\cot(-x) + \tan(-x)] = -\csc^2 x$

40. $\dfrac{\csc^2(-x)\tan x}{\cot(-x)} = -\sec^2 x$

For Exercises 41–46, verify that the equation is an identity. (See Example 5)

41. $\dfrac{\sec x}{\tan x} - \dfrac{\tan x}{\sec x} = \cos x\cot x$

42. $\dfrac{\cot x}{\csc x} - \dfrac{\csc x}{\cot x} = -\sin x\tan x$

43. $\dfrac{1}{1 + \sin t} + \dfrac{1}{1 - \sin t} = 2\sec^2 t$

44. $\dfrac{1}{\sec t - 1} - \dfrac{1}{\sec t + 1} = 2\cot^2 t$

45. $\dfrac{1}{1 + \sec x} - \dfrac{1}{1 - \sec x} = 2\cot x\csc x$

46. $\dfrac{1}{1 - \csc x} - \dfrac{1}{1 + \csc x} = -2\tan x\sec x$

For Exercises 47–52, verify that the equation is an identity. (See Example 6)

47. $\dfrac{\sin\theta}{\csc\theta - \cot\theta} = 1 + \cos\theta$

48. $\dfrac{\cos\theta}{\tan\theta + \sec\theta} = 1 - \sin\theta$

49. $\dfrac{1}{\cos x + \sin x\cos x} = \dfrac{1 - \sin x}{\cos^3 x}$

50. $\dfrac{1}{\sin x - \sin^2 x} = \dfrac{1 + \sin x}{\sin x\cos^2 x}$

51. $\dfrac{\sec x + 1}{\sec x - 1} = (\csc x + \cot x)^2$

52. $\dfrac{\sin x + \cos x}{\sin x - \cos x} = \dfrac{1 + 2\sin x\cos x}{1 - 2\cos^2 x}$

For Exercises 53–56, verify that the equation is an identity. (See Example 7)

53. $\ln|\cos t| - \ln|\cot t| = \ln|\sin t|$

54. $\ln|\cot t| - \ln|\tan t| = 2\ln|\cot t|$

55. $\ln|\sec\theta + \tan\theta| = -\ln|\sec\theta - \tan\theta|$

56. $\ln|\csc\theta + \cot\theta| = -\ln|\csc\theta - \cot\theta|$

For Exercises 57–60, simplify each side of the equation independently to reach a common equivalent expression. (See Example 8)

57. $\dfrac{\cos x}{1 - \sin x} = \sec x + \tan x$

58. $\dfrac{1 + \tan^2 x}{\tan x} = \dfrac{\cos x}{\sin x - \sin^3 x}$

59. $\cot^2 t - \cos^2 t = \csc^2 t\cos^2 t - \cos^2 t$

60. $\tan^2 t + \sin^2 t = \sec^2 t - \cos^2 t$

Objective 3: Write an Algebraic Expression as a Trigonometric Expression

For Exercises 61–68, write the given algebraic expression as a function of θ, where $0 < \theta < \dfrac{\pi}{2}$, by making the given substitution. (See Example 9)

61. $\sqrt{x^2 + 25}$; Substitute $x = 5\tan\theta$.

62. $\sqrt{49 - x^2}$; Substitute $x = 7\sin\theta$.

63. $\sqrt{16 - (x - 1)^2}$; Substitute $x = 4\sin\theta + 1$.

64. $\sqrt{(x + 2)^2 - 36}$; Substitute $x = 6\sec\theta - 2$.

65. $\dfrac{\sqrt{9 - x^2}}{x}$; Substitute $x = 3\sin\theta$.

66. $\dfrac{\sqrt{x^2 + 16}}{x}$; Substitute $x = 4\tan\theta$.

67. $\dfrac{1}{(49x^2 - 1)^{3/2}}$; Substitute $x = \dfrac{1}{7}\sec\theta$.

68. $\dfrac{1}{(1 - 100x^2)^{3/2}}$; Substitute $x = \dfrac{1}{10}\sin\theta$.

Mixed Exercises

For Exercises 69–72, factor the expression and simplify factors using fundamental identities if possible.

69. $2\sin^2 x + 5\sin x - 3$

70. $2\cos^2 x - \cos x - 1$

71. $\sin^4 x - \cos^4 x$

72. $1 - \cot^4 x$

For Exercises 73–104, verify that the equation is an identity.

73. $(\cos^2\theta - 1)(\csc^2\theta - 1) = -\cos^2\theta$

74. $(1 - \sin^2\theta)(1 + \tan^2\theta) = 1$

75. $\sec x \cdot \dfrac{\tan x}{\sin x} = \sec^2 x$

76. $\sec x \cdot \dfrac{\cot x}{\sin x} = \csc^2 x$

77. $\dfrac{\sec x \sin x}{\csc x \cos x} = \tan^2 x$

78. $\dfrac{\sin x \cot x}{\cos x \tan x} = \cot x$

79. $\dfrac{2 + \sin^2 t - 3\sin^4 t}{\cos^2 t} = 2 + 3\sin^2 t$

80. $\dfrac{2\tan^4 x + 7\tan^2 x + 5}{\sec^2 x} = 2\tan^2 x + 5$

81. $\dfrac{\tan x \sin x}{\tan x - \sin x} = \dfrac{\sin x}{1 - \cos x}$

82. $\dfrac{\csc x \cos x}{\csc x - \sin x} = \sec x$

83. $(\sec\theta + \tan\theta)^2 = \dfrac{1 + \sin\theta}{1 - \sin\theta}$

84. $(\csc t + \cot t)^2 = \dfrac{1 + \cos t}{1 - \cos t}$

85. $(1 + \sin x)[1 + \sin(-x)] = \cos^2 x$

86. $[1 + \csc(-x)](1 + \csc x) = -\cot^2 x$

87. $\dfrac{\sin x}{\cos x + 1} = \dfrac{1 - \cos x}{\sin x}$

88. $\dfrac{\tan x}{\sec x - 1} = \dfrac{\sec x + 1}{\tan x}$

89. $\dfrac{1 + \sin\theta}{\tan\theta} = \dfrac{\csc\theta + 1}{\sec\theta}$

90. $\dfrac{\sec\theta + 1}{\csc\theta} = \dfrac{1 + \cos\theta}{\cot\theta}$

91. $\dfrac{\sin^3 t - \cos^3 t}{\cos t \sin t - \cos^2 t} = \sec t + \sin t$

92. $\dfrac{\cos^3 t + \sin^3 t}{\cos t \sin t + \sin^2 t} = \csc t - \cos t$

93. $\dfrac{\tan^4\theta - 4}{\sec^2\theta + 1} = \sec^2\theta - 3$

94. $\dfrac{\sin^4\theta - 9}{\cos^2\theta + 2} = \cos^2\theta - 4$

95. $\dfrac{\cot^3 z - 1}{\cot z - 1} - \cot z = \csc^2 z$

96. $\dfrac{\tan^3 z + 1}{\tan z + 1} + \tan z = \sec^2 z$

97. $\cos x \tan x - \sec x \cot x = -\cot x \cos x$

98. $\sin x \cot x + \cos x \tan^2 x = \sec x$

99. $\log|\cos x| - \log|\sin x| + \log|\sec x| = \log|\csc x|$

100. $\ln|\cot x| + \ln|\sec x| + \ln|\sin x| = 0$

101. $e^{\ln(\sin^2 x + \cos^2 x)} = 1$

102. $\log 100^{\sin x} = 2\sin x$

103. $\dfrac{1}{\sin x + \cos x} + \dfrac{1}{\sin x - \cos x} = \dfrac{2\sin x}{1 - 2\cos^2 x}$

104. $\dfrac{1}{\sin x + \cos x} - \dfrac{\sin x}{(\sin x + \cos x)^2} = \dfrac{\cos x}{1 + 2\sin x \cos x}$

105. Given $f(x) = \sqrt{1 - x^2}$ and $g(x) = \cos x$, show that $(f \circ g)(x) = |\sin x|$.

106. Given $f(x) = \dfrac{1}{\sqrt{x^2 + 1}}$ and $g(x) = \tan x$, show that $(f \circ g)(x) = |\cos x|$.

Write About It

107. Determine if the following logic is correct and explain why or why not.

$$\sin x \sec x = 1 \text{ because } \sin\frac{\pi}{4}\sec\frac{\pi}{4} = \frac{\sqrt{2}}{2} \cdot \sqrt{2} = \frac{2}{2} = 1$$

108. Determine if the following procedure to verify an identity is correct and explain why or why not.

$$\frac{\csc\theta - 1}{\cot\theta} = \frac{\cot\theta}{\csc\theta + 1}$$

Multiply both sides by the LCD: $\cot\theta(\csc\theta + 1)$

$$\cot\theta(\csc\theta + 1)\left(\frac{\csc\theta - 1}{\cot\theta}\right) = \left(\frac{\cot\theta}{\csc\theta + 1}\right)\cot\theta(\csc\theta + 1)$$

$$(\csc\theta + 1)(\csc\theta - 1) = \cot^2\theta$$

$$\csc^2\theta - 1 = \cot^2\theta \quad ✓$$

109. Explain how to verify a trigonometric identity.

Expanding Your Skills

For Exercises 110–116, verify that the equation is an identity.

110. $\sqrt{\dfrac{1 + \cos x}{1 - \cos x}} = \csc x + \cot x$, for $0 < x < \dfrac{\pi}{2}$.

111. $\sqrt{\dfrac{\csc x - \cot x}{\csc x + \cot x}} = \dfrac{1 - \cos x}{\sin x}$, for $0 < x < \dfrac{\pi}{2}$.

112. $(\sin x + \cos y)^2 + (\sin x - \cos y)(\sin x + \cos y) = 2\sin x(\sin x + \cos y)$

113. $(\tan x + \tan y)(\tan x - \tan y) + 2\tan y(\tan y - \tan x) = (\tan x - \tan y)^2$

114. $(A\cot\theta - B)(A\cot\theta + B) - (B\cot\theta - A)(B\cot\theta + A) = (A^2 - B^2)\csc^2\theta$

115. $\dfrac{\sec^2 x\tan x - 25\tan x}{\sec^3 x + 5\sec^2 x - \sec x - 5} = \csc x - 5\cot x$

116. $\sin^6 x - \cos^6 x = (\sin^2 x - \cos^2 x)(1 - \sin^2 x\cos^2 x)$

Technology Connections

117. An equation of the form $f(x) = g(x)$ is given. Use a graphing utility to graph $y = f(x)$ and $y = g(x)$ on the same screen on the interval $[0, 2\pi]$ (be sure to use radian mode). Make a conjecture as to whether the equation is an identity, a conditional equation, or a contradiction.

a. $\sin 2x = 2\sin x$

b. $\cos 2x = 2\cos^2 x - 1$

c. $\dfrac{\sec x\sin x}{\tan x} = 2$

118. Graph the function in the standard viewing window to determine the constant value it might be equivalent to. Confirm your conjecture algebraically.

$$y = (2\cos x - \sin x)^2 + (\cos x + 2\sin x)^2$$

SECTION 5.2 Sum and Difference Formulas

OBJECTIVES

1. Apply the Sum and Difference Formulas for Sine and Cosine
2. Apply the Sum and Difference Formulas for Tangent
3. Use Sum and Difference Formulas to Verify Identities
4. Write a Sum of $A\sin x$ and $B\cos x$ as a Single Term

1. Apply the Sum and Difference Formulas for Sine and Cosine

In Exercise 32 from Section 4.3, we evaluated the expressions $\sin 60°$ and $\sin 30° + \sin 30°$ and found that they are not equal. That is,

$$\sin 60° = \frac{\sqrt{3}}{2}, \text{ whereas } \sin 30° + \sin 30° = \frac{1}{2} + \frac{1}{2} = 1.$$

So the question arises, can we express $\sin(u \pm v)$ and $\cos(u \pm v)$ in terms of trigonometric functions of u and v alone?

We begin by investigating $\cos(u - v)$. Let points P and Q be the points on the unit circle associated with the real numbers u and v, measured from the point $S(1, 0)$. For simplicity, we show the case where u and v are positive and $u - v > 0$ (Figure 5-2). Let R be the point on the unit circle associated with the real number $u - v$ (Figure 5-3).

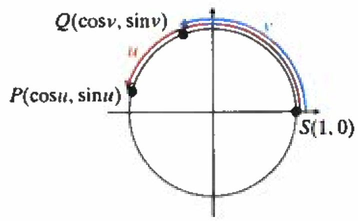


Figure 5-2

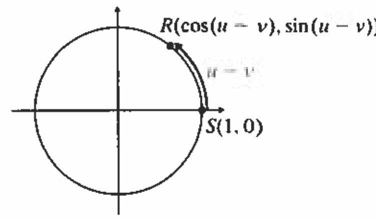


Figure 5-3

TIP A chord of a circle is a line segment with endpoints on the circle.

The lengths of the arcs between P and Q and between R and S are the same. Consequently, the lengths of the chords $\overline{PQ}$ and $\overline{RS}$ are also equal. By the distance formula we have:

$$\sqrt{(\cos u - \cos v)^2 + (\sin u - \sin v)^2} = \sqrt{[\cos(u - v) - 1]^2 + [\sin(u - v) - 0]^2}$$

Squaring both sides and expanding the squared expressions yields

$$
\begin{aligned}
\text{LHS: } & \cos^2 u - 2\cos u \cos v + \cos^2 v + \sin^2 u - 2\sin u \sin v + \sin^2 v \\
&= \cos^2 u + \sin^2 u - 2\cos u \cos v + \cos^2 v + \sin^2 v - 2\sin u \sin v && \text{Group the squared terms.} \\
&= 1 - 2\cos u \cos v + 1 - 2\sin u \sin v && \text{Substitute } \sin^2 u + \cos^2 u = 1,\ \sin^2 v + \cos^2 v = 1. \\
&= 2 - 2\cos u \cos v - 2\sin u \sin v
\end{aligned}
$$

$$
\begin{aligned}
\text{RHS: } & \cos^2(u - v) - 2\cos(u - v) + 1 + \sin^2(u - v) \\
&= \cos^2(u - v) + \sin^2(u - v) - 2\cos(u - v) + 1 && \text{Group the squared terms.} \\
&= 1 - 2\cos(u - v) + 1 && \text{Substitute } \cos^2(u - v) + \sin^2(u - v) = 1. \\
&= 2 - 2\cos(u - v)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
2 - 2\cos u \cos v - 2\sin u \sin v &= 2 - 2\cos(u - v) \\
-2\cos u \cos v - 2\sin u \sin v &= -2\cos(u - v) && \text{Subtract 2.} \\
\cos u \cos v + \sin u \sin v &= \cos(u - v) && \text{Divide by } -2.
\end{aligned}
$$

Similar identities for $\cos(u + v)$, $\sin(u \pm v)$, and $\tan(u \pm v)$ are given in Table 5-2. The derivations for these identities are addressed in Exercises 73–76.

Table 5-2

Sum and Difference Formulas	
Sine Formulas	$\sin(u + v) = \sin u \cos v + \cos u \sin v$ $\sin(u - v) = \sin u \cos v - \cos u \sin v$
Cosine Formulas	$\cos(u + v) = \cos u \cos v - \sin u \sin v$ $\cos(u - v) = \cos u \cos v + \sin u \sin v$
Tangent Formulas	$\tan(u + v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$ $\tan(u - v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

If an angle θ is the sum or difference of two angles for which the trigonometric function values are known, we can use the identities from Table 5-2 to find the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$. For example, consider the angles 15°, 105°, and $\frac{7\pi}{12}$ as a sum or difference of other "special" angles.

$$15° = 45° - 30° \qquad 105° = 135° - 30° \qquad \frac{7\pi}{12} = \frac{3\pi}{12} + \frac{4\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$$

EXAMPLE 1 Applying the Addition and Subtraction Formulas

Find the exact values. **a.** $\cos 15°$ **b.** $\sin\frac{11\pi}{12}$

Solution:

a. $\cos 15° = \cos(45° - 30°)$ — Write 15° as the sum or difference of angles for which the sine and cosine functions are known (that is, integer multiples of 30° or 45°).

$= \cos 45°\cos 30° + \sin 45°\sin 30°$ — Apply $\cos(u - v) = \cos u\cos v + \sin u\sin v$ with $u = 45°$ and $v = 30°$.

$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$ — Substitute the known function values.

$= \frac{\sqrt{6} + \sqrt{2}}{4}$ — Multiply radicals and write the result over the common denominator.

TIP The combination $45° - 30°$ is not the only way to make a sum or difference of 15° from our "special" angles. For example, the same result is obtained from

$\cos(-45° + 60°)$ and $\cos(135° - 120°)$.

b. First note that $\frac{11\pi}{12} = \frac{3\pi}{12} + \frac{8\pi}{12} = \frac{\pi}{4} + \frac{2\pi}{3}$. — Look for a combination of angles that are integer multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$.

$\sin\frac{11\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)$

$= \sin\frac{\pi}{4}\cos\frac{2\pi}{3} + \cos\frac{\pi}{4}\sin\frac{2\pi}{3}$ — Apply $\sin(u + v) = \sin u\cos v + \cos u\sin v$ with $u = \frac{\pi}{4}$ and $v = \frac{2\pi}{3}$.

$= \frac{\sqrt{2}}{2}\cdot\left(-\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)$ — Substitute the known function values.

$= -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$

$= \frac{\sqrt{6} - \sqrt{2}}{4}$ — Add the fractions.

Skill Practice 1 Find the exact values.

a. $\sin 195°$ **b.** $\cos\frac{5\pi}{12}$

Answers

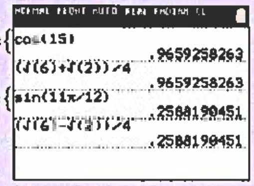
1. **a.** $\frac{\sqrt{2} - \sqrt{6}}{4}$ **b.** $\frac{\sqrt{6} - \sqrt{2}}{4}$

Avoiding Mistakes

You can support your results from Example 1 by using a calculator. Confirm that the decimal approximations agree to the level of accuracy provided by the calculator.

Degree mode

Radian mode



EXAMPLE 2 Applying the Sum Formula for Sine

Find the exact value of the expression. $\sin 25^\circ \cos 35^\circ + \cos 25^\circ \sin 35^\circ$

Solution:

$\sin 25^\circ \cos 35^\circ + \cos 25^\circ \sin 35^\circ$ — Recognize that this expression is in the form $\sin u \cos v + \cos u \sin v$ for $u = 25^\circ$ and $v = 35^\circ$.

$= \sin(25^\circ + 35^\circ)$ — Apply the formula $\sin(u + v) = \sin u \cos v + \cos u \sin v$.

$= \sin 60^\circ$

$= \dfrac{\sqrt{3}}{2}$

Skill Practice 2 Find the exact value of the expression.
$\cos 99^\circ \cos 36^\circ - \sin 99^\circ \sin 36^\circ$

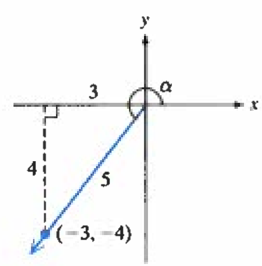
EXAMPLE 3 Applying the Difference Formula for Cosine

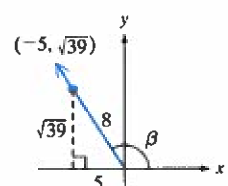
Find the exact value of $\cos(\alpha - \beta)$ given that $\sin\alpha = -\frac{4}{5}$ and $\cos\beta = -\frac{5}{8}$ for α in Quadrant III and β in Quadrant II.

Solution:

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

To find the value of $\cos(\alpha - \beta)$, we need to know the values of the factors on the right side of the identity. We can set up a representative triangle for each angle.





$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ — Apply the difference formula for cosine.

$= -\frac{3}{5}\cdot\left(-\frac{5}{8}\right) + \left(-\frac{4}{5}\right)\left(\frac{\sqrt{39}}{8}\right)$ — Substitute values for the sine and cosine of α and β.

$= \dfrac{15 - 4\sqrt{39}}{40}$ — Simplify.

Answer

2. $-\dfrac{\sqrt{2}}{2}$

Skill Practice 3 Find the exact value of $\cos(\alpha + \beta)$ given that $\sin\alpha = \frac{5}{13}$ and $\cos\beta = \frac{5}{6}$ for α in Quadrant II and β in Quadrant IV.

EXAMPLE 4 Evaluating the Sine of the Sum of Two Angles Given as Inverse Functions

Find the exact value of $\sin\left[\tan^{-1}\left(-\frac{9}{40}\right) + \cos^{-1}\frac{8}{17}\right]$.

Solution:

Let $u = \tan^{-1}\left(-\frac{9}{40}\right)$ and $v = \cos^{-1}\frac{8}{17}$.

Draw angles u and v in standard position and set up representative triangles with the hypotenuse on the terminal sides of u and v (Figures 5-4 and 5-5).

It follows that $\tan u = -\frac{9}{40}$ and $\cos v = \frac{8}{17}$.

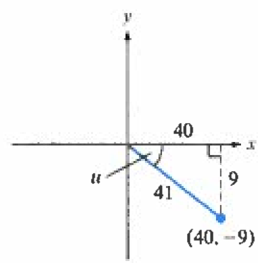


Figure 5-4

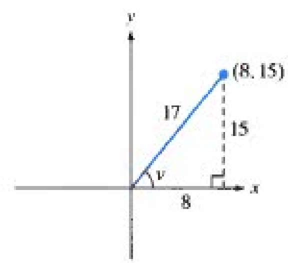


Figure 5-5

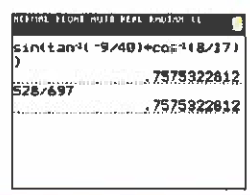
$$\sin\left[\tan^{-1}\left(-\frac{9}{40}\right) + \cos^{-1}\frac{8}{17}\right]$$

$$= \sin(u + v) = \sin u\cos v + \cos u \sin v$$

$$= -\frac{9}{41}\cdot\frac{8}{17} + \frac{40}{41}\cdot\frac{15}{17}$$

$$= \frac{528}{697}$$

Apply the formula for the sine of a sum. The result checks on a calculator.



TIP The values of $\sin u$, $\cos u$, $\sin v$, and $\cos v$ can also be found by using trigonometric identities.

Because $\tan u = -\frac{9}{40}$ for u in Quadrant IV, $\sec^2 u = 1 + \tan^2 u$ and

$$\sec u = \sqrt{1 + \left(-\tfrac{9}{40}\right)^2}$$
$$= \tfrac{41}{40}$$

Thus, $\cos u = \frac{40}{41}$.

Using $\tan u = \frac{\sin u}{\cos u}$, we have $\sin u = \tan u \cos u$ and $\sin u = -\frac{9}{40}\cdot\frac{40}{41}$

$$= -\tfrac{9}{41}$$

We also know that $\cos v = \frac{8}{17}$ for v in Quadrant I. Using the Pythagorean identity $\sin^2 v + \cos^2 v = 1$, we find that $\sin v = \frac{15}{17}$.

Skill Practice 4 Find the exact value of $\cos\left[\sin^{-1}\left(-\frac{12}{37}\right) + \tan^{-1}\left(\frac{5}{12}\right)\right]$.

2. Apply the Sum and Difference Formulas for Tangent

We can use the identities for $\sin(u \pm v)$ and $\cos(u \pm v)$ to derive similar identities for $\tan(u \pm v)$.

Answers

3. $\frac{-60 + 5\sqrt{11}}{78}$

4. $\frac{480}{481}$

$$\tan(u+v) = \frac{\sin(u+v)}{\cos(u+v)} = \frac{\sin u\cos v + \cos u\sin v}{\cos u\cos v - \sin u\sin v}$$

To write this identity in terms of tan u and tan v, multiply numerator and denominator by $\frac{1}{\cos u\cos v}$.

$$= \frac{\frac{1}{\cos u\cos v}\cdot(\sin u\cos v + \cos u\sin v)}{\frac{1}{\cos u\cos v}\cdot(\cos u\cos v - \sin u\sin v)}$$

$$= \frac{\frac{\sin u\cancel{\cos v}}{\cos u\cancel{\cos v}} + \frac{\cancel{\cos u}\sin v}{\cancel{\cos u}\cos v}}{\frac{\cancel{\cos u\cos v}}{\cancel{\cos u\cos v}} - \frac{\sin u\sin v}{\cos u\cos v}}$$

Simplify common factors within each term of the complex fraction.

$$= \frac{\tan u + \tan v}{1 - \tan u\tan v}$$

Thus, $\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u\tan v}$.

Applying this result to $\tan(u-v) = \tan[u+(-v)]$ and applying the odd-function identity $\tan(-x) = -\tan x$, we can derive the identity $\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u\tan v}$.

EXAMPLE 5 **Applying the Subtraction Formula for Tangent**

Find the exact value of $\tan 255°$.

Solution:

$\tan 255° = \tan(300° - 45°)$ — Write 255° as the sum or difference of integer multiples of 30° or 45°.

$= \frac{\tan 300° - \tan 45°}{1 + \tan 300°\tan 45°}$ — Apply the formula $\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u\tan v}$ with $u = 300°$ and $v = 45°$.

$= \frac{-\sqrt{3} - 1}{1 + (-\sqrt{3})(1)}$ — Substitute known values for $\tan 300°$ and $\tan 45°$.

$= -\frac{(\sqrt{3}+1)}{(1-\sqrt{3})}\cdot\frac{(1+\sqrt{3})}{(1+\sqrt{3})}$ — Rationalize the denominator by multiplying numerator and denominator by the conjugate of the denominator.

$= -\frac{4 + 2\sqrt{3}}{1 - 3}$ — Multiply.

$= -\frac{2(2+\sqrt{3})}{-2}$ — Factor the numerator and simplify the denominator.

$= 2 + \sqrt{3}$ — Simplify common factors.

Skill Practice 5 Find the exact value of $\tan 165°$.

3. Use Sum and Difference Formulas to Verify Identities

The sum and difference formulas for sine, cosine, and tangent can be used to verify the cofunction identities and periodic identities of the six trigonometric functions.

Answer
5. $\sqrt{3} - 2$

Cofunction Identities*		Periodic Identities*
$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$	$\sin(\theta + 2\pi) = \sin\theta$ $\cos(\theta + 2\pi) = \cos\theta$
$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$	$\tan(\theta + \pi) = \tan\theta$ $\cot(\theta + \pi) = \cot\theta$
$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$	$\csc(\theta + 2\pi) = \csc\theta$ $\sec(\theta + 2\pi) = \sec\theta$

*All statements can be made using 90° for $\frac{\pi}{2}$, 180° for π, and 360° for 2π.

EXAMPLE 6 Verifying a Cofunction Identity

Verify the cofunction identity $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$.

Solution:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \sin\frac{\pi}{2}\cos\theta - \cos\frac{\pi}{2}\sin\theta$$
$$= (1)\cos\theta - (0)\sin\theta$$
$$= \cos\theta$$

Apply $\sin(u - v) = \sin u\cos v - \cos u\sin v$ with $u = \frac{\pi}{2}$ and $v = \theta$.

Skill Practice 6 Verify the identity. $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$

EXAMPLE 7 Verifying an Identity

Verify the identity. $\cos(x + y)\cos(x - y) = \cos^2 x - \sin^2 y$

Solution:

$$\cos(x + y)\cos(x - y)$$
$$= (\cos x\cos y - \sin x\sin y)(\cos x\cos y + \sin x\sin y)$$

Apply the sum and difference formulas for cosine.

$$= \cos^2 x\cos^2 y + \cos x\cos y\sin x\sin y - \sin x\sin y\cos x\cos y - \sin^2 x\sin^2 y$$
$$= \cos^2 x\cos^2 y - \sin^2 x\sin^2 y$$
$$= \cos^2 x(1 - \sin^2 y) - (1 - \cos^2 x)\sin^2 y$$

$\cos^2 y = 1 - \sin^2 y$ and $\sin^2 x = 1 - \cos^2 x$.

$$= \cos^2 x - \cos^2 x\sin^2 y - \sin^2 y + \cos^2 x\sin^2 y$$
$$= \cos^2 x - \sin^2 y$$

Apply the distributive property and simplify.

Skill Practice 7 Verify the identity. $\sin(x + y) - \sin(x - y) = 2\cos x\sin y$

Answers

6. $\cos\left(\frac{3\pi}{2} - \theta\right)$
$= \cos\frac{3\pi}{2}\cos\theta + \sin\frac{3\pi}{2}\sin\theta$
$= (0)\cos\theta + (-1)\sin\theta = -\sin\theta$

7. $\sin(x + y) - \sin(x - y)$
$= \sin x\cos y + \cos x\sin y$
$\quad - (\sin x\cos y - \cos x\sin y)$
$= \sin x\cos y + \cos x\sin y$
$\quad - \sin x\cos y + \cos x\sin y$
$= 2\cos x\sin y$

4. Write a Sum of $A\sin x$ and $B\cos x$ as a Single Term

Consider an expression of the form $A\sin x + B\cos x$ for real numbers A and B. We can write this in terms of a single trigonometric function of the form $k\sin(x + \alpha)$, where k is a positive real number. To begin, apply the sum formula for the expression $\sin(x + \alpha)$.

$$\sin(x + \alpha) = \sin x\cos\alpha + \cos x\sin\alpha$$

Let $\cos\alpha = \frac{A}{k}$ and $\sin\alpha = \frac{B}{k}$ for some positive real number k. Figure 5-6 shows the case where $0 \le \alpha \le \frac{\pi}{2}$. Then,

$$\sin(x + \alpha) = \sin x\left(\frac{A}{k}\right) + \cos x\left(\frac{B}{k}\right).$$

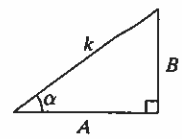


Figure 5-6

Multiplying both sides by k gives

$$k\sin(x + \alpha) = A\sin x + B\cos x \qquad (1)$$

To write the constant k in terms of A and B, use a Pythagorean identity.

$$\begin{aligned} \sin^2\alpha &+ \cos^2\alpha &= 1 \\ \left(\frac{B}{k}\right)^2 &+ \left(\frac{A}{k}\right)^2 &= 1 \\ B^2 &+ A^2 &= k^2 \end{aligned}$$

Since k is assumed to be positive, we have $k = \sqrt{A^2 + B^2}$.
Thus, equation (1) becomes $\sqrt{A^2 + B^2}\sin(x + \alpha) = A\sin x + B\cos x$.

Sum of $A\sin x$ and $B\cos x$

For the real numbers A, B, and x,

$$A\sin x + B\cos x = k\sin(x + \alpha)$$

where $k = \sqrt{A^2 + B^2}$, and α satisfies $\cos\alpha = \frac{A}{k}$ and $\sin\alpha = \frac{B}{k}$.

The formula presented here to simplify a sum of $A\sin x$ and $B\cos x$ is also called a **reduction formula** because it "reduces" the sum of two trigonometric functions into one term.

EXAMPLE 8 Writing a Sum of Sine and Cosine as a Single Term

Write $5\sin x - 12\cos x$ in the form $k\sin(x + \alpha)$.

Solution:

$5\sin x - 12\cos x$ is in the form $A\sin x + B\cos x$ with $A = 5$ and $B = -12$.

$$k = \sqrt{A^2 + B^2} = \sqrt{(5)^2 + (-12)^2} = \sqrt{169} = 13$$

$$\cos\alpha = \frac{A}{k} = \frac{5}{13} \text{ and } \sin\alpha = \frac{B}{k} = \frac{-12}{13} = -\frac{12}{13}$$

Since $\cos\alpha > 0$ and $\sin\alpha < 0$, α is in Quadrant IV. Taking the inverse cosine, we have the first quadrant angle

$$\cos^{-1}\left(\frac{5}{13}\right) \approx 1.1760 \text{ or } \approx 67.4°.$$

To find a fourth quadrant angle, subtract from 2π (or 360°).

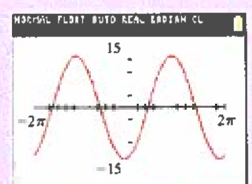
$$\alpha = 2\pi - \cos^{-1}\frac{5}{13} \approx 5.1072 \text{ or } \alpha = 360° - \cos^{-1}\frac{5}{13} \approx 292.6°$$

Therefore, $5\sin x - 12\cos x \approx 13\sin(x + 5.1072)$ or
$5\sin x - 12\cos x \approx 13\sin(x + 292.6°)$.

Skill Practice 8 Write $-4\sin x + 3\cos x$ in the form $k\sin(x + \alpha)$.

Avoiding Mistakes

You can support your results from Example 8 by using a calculator to graph $y = 5\sin x - 12\cos x$ and $y = 13\sin(x + 5.1072)$. Confirm that the two graphs coincide.



Answer

8. $-4\sin x + 3\cos x \approx 5\sin(x + 2.498)$ or $-4\sin x + 3\cos x \approx 5\sin(x + 143.1°)$

SECTION 5.2 Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, use reference angles to find the exact value.

R.1. $\sin 240°$

R.2. $\cos\frac{3\pi}{4}$

R.3. Given that $\sin\theta < 0$ and $\cos\theta > 0$, identify the quadrant in which θ lies.

R.4. Find the difference quotient and simplify.
$f(x) = -5x^2 - 8x + 3$

Concept Connections

1. Fill in the blanks with the correct operation $+$ or $-$.

$\sin(u + v) = \sin u\cos v$ _____ $\cos u\sin v$

$\cos(u + v) = \cos u\cos v$ _____ $\sin u\sin v$

2. Fill in the blanks to complete each identity.

$\sin(u - v) =$ ________ $-$ ________

$\cos(u - v) =$ ________ $+$ ________

3. Fill in the boxes to complete each identity.

$$\tan(u + v) = \frac{\boxed{}}{1 - \tan u\tan v};\ \tan(u - v) = \frac{\tan u - \tan v}{\boxed{}}$$

4. If $0° < \alpha < 90°$ and $90° < \beta < 180°$, then ____________ $< \alpha + \beta <$ ____________ and ____________ $< \beta - \alpha <$ ____________.

5. Write $\frac{13\pi}{12}$ as a sum or difference of angles that are multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$.

6. Write $285°$ as a sum or difference of angles that are multiples of $30°$ or $45°$.

Objectives 1–2: Apply the Sum and Difference Formulas for Sine, Cosine, and Tangent.

For Exercises 7–18, use an addition or subtraction formula to find the exact value. (See Examples 1 and 5)

7. $\cos 165°$

8. $\sin\frac{5\pi}{12}$

9. $\sin\left(-\frac{\pi}{12}\right)$

10. $\tan\frac{19\pi}{12}$

11. $\cos\left(-\frac{19\pi}{12}\right)$

12. $\sin(-15°)$

13. $\tan\frac{7\pi}{12}$

14. $\cos\frac{7\pi}{12}$

15. $\sin 105°$

16. $\tan(-105°)$

17. $\tan 15°$

18. $\cos(-195°)$

For Exercises 19–26, use an addition or subtraction formula to find the exact value. (See Examples 2 and 5)

19. $\sin 140°\cos 20° - \cos 140°\sin 20°$

20. $\cos\frac{35\pi}{18}\cos\frac{5\pi}{18} + \sin\frac{35\pi}{18}\sin\frac{5\pi}{18}$

21. $\dfrac{\tan\frac{5\pi}{4} + \tan\frac{\pi}{12}}{1 - \tan\frac{5\pi}{4}\tan\frac{\pi}{12}}$

22. $\sin\frac{35\pi}{36}\cos\frac{13\pi}{18} - \cos\frac{35\pi}{36}\sin\frac{13\pi}{18}$

23. $\cos\frac{2\pi}{3}\cos\frac{7\pi}{6} - \sin\frac{2\pi}{3}\sin\frac{7\pi}{6}$

24. $\dfrac{\tan 15° - \tan 45°}{1 + \tan 15°\tan 45°}$

25. $\cos 200°\cos 25° - \sin 200°\sin 25°$

26. $\sin\frac{10\pi}{9}\cos\frac{7\pi}{18} + \cos\frac{10\pi}{9}\sin\frac{7\pi}{18}$

For Exercises 27–32, find the exact value for the expression under the given conditions. (See Examples 3 and 5)

27. $\cos(\alpha + \beta)$; $\sin\alpha = -\frac{3}{5}$ for α in Quadrant III and $\cos\beta = -\frac{3}{4}$ for β in Quadrant II.

28. $\sin(\alpha - \beta)$; $\sin\alpha = \frac{3}{8}$ for α in Quadrant II and $\cos\beta = \frac{12}{13}$ for β in Quadrant IV.

29. $\tan(\alpha - \beta)$; $\sin\alpha = \frac{8}{17}$ for α in Quadrant II and $\cos\beta = -\frac{9}{41}$ for β in Quadrant III.

30. $\cos(\alpha - \beta)$; $\sin\alpha = \frac{2}{3}$ for α in Quadrant II and $\cos\beta = -\frac{1}{4}$ for β in Quadrant III.

31. $\sin(\alpha + \beta)$; $\cos\alpha = \frac{3}{7}$ for α in Quadrant IV and $\sin\beta = \frac{7}{25}$ for β in Quadrant II.

32. $\tan(\alpha + \beta)$; $\cos\alpha = -\frac{13}{85}$ for α in Quadrant III and $\sin\beta = -\frac{63}{65}$ for β in Quadrant IV.

For Exercises 33–40, find the exact value. (See Example 4)

33. $\sin\left(\arcsin\frac{1}{2} - \arccos\frac{\sqrt{2}}{2}\right)$

34. $\cos\left(\tan^{-1}\sqrt{3} - \sin^{-1}\frac{\sqrt{3}}{2}\right)$

35. $\tan\left[\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

36. $\sin\left[\arccos\left(-\frac{1}{2}\right) + \arcsin\left(-\frac{\sqrt{2}}{2}\right)\right]$

37. $\cos\left(\tan^{-1}\frac{8}{15} - \cos^{-1}\frac{3}{5}\right)$

38. $\sin\left(\cos^{-1}\frac{12}{13} + \tan^{-1}\frac{4}{3}\right)$

39. $\cos\left(\sin^{-1}\frac{4}{5} + \tan^{-1}2\right)$

40. $\tan\left(\arccos\frac{1}{5} - \arcsin\frac{3}{5}\right)$

Objective 3: Use Sum and Difference Formulas to Verify Identities

For Exercises 41–62, verify the identity. (See Examples 6–7)

41. $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$

42. $\sin\left(\frac{\pi}{2} + x\right) = \cos x$

43. $\tan(\pi + x) = \tan x$

44. $\cos(x - \pi) = -\cos x$

45. $\sin\left(\frac{3\pi}{2} + x\right) = -\cos x$

46. $\tan(\pi - x) = -\tan x$

47. $\sin(x + y) + \sin(x - y) = 2\sin x\cos y$

48. $\cos(x - y) - \cos(x + y) = 2\sin x\sin y$

49. $\frac{\cos(x + y)}{\sin x\cos y} = \cot x - \tan y$

50. $\frac{\sin(x + y)}{\cos x\sin y} = \tan x\cot y + 1$

51. $\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right) = \sqrt{2}\sin x$

52. $\cos\left(x + \frac{\pi}{4}\right) - \cos\left(x - \frac{\pi}{4}\right) = -\sqrt{2}\sin x$

53. $\tan\left(x - \frac{\pi}{4}\right) = \frac{\sin x - \cos x}{\sin x + \cos x}$

54. $\tan\left(x + \frac{\pi}{4}\right) = \frac{\cos x + \sin x}{\cos x - \sin x}$

55. $\frac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)} = \frac{\cot\alpha\cot\beta + 1}{\cot\alpha\cot\beta - 1}$

56. $\frac{\sin(\alpha - \beta)}{\sin(\alpha + \beta)} = \frac{\tan\alpha - \tan\beta}{\tan\alpha + \tan\beta}$

57. $\csc(x + y) = \frac{\sec x\sec y}{\tan x + \tan y}$

58. $\sec(x - y) = \frac{\csc x\sec y}{\cot x + \tan y}$

59. $\cos(A + B)\cos(A - B) = \cos^2 A + \cos^2 B - 1$

60. $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$

61. $\tan(x + y) - \tan(x - y) = \frac{2\tan y\sec^2 x}{1 - \tan^2 x\tan^2 y}$

62. $\tan(x + y) + \tan(x - y) = \frac{2\tan x\sec^2 y}{1 - \tan^2 x\tan^2 y}$

Objective 4: Write a Sum of $A\sin x$ and $B\cos x$ as a Single Term

For Exercises 63–66,

a. Write the given expression in the form $k\sin(x + \alpha)$ for $0 \le \alpha < 2\pi$. Round α to 3 decimal places. **(See Example 8)**

b. Verify the result from part (a) by applying the sum formula for sine.

63. $8\sin x - 6\cos x$

64. $-15\sin x + 8\cos x$

65. $-\sin x - 3\cos x$

66. $2\sin x + 9\cos x$

Often graphing a function of the form $y = A\sin x + B\cos x$ is easier by using its reduction formula $y = k\sin(x + \alpha)$. For Exercises 67–70,

a. Use the reduction formula to write the given function as a sine function.

b. Graph the function.

67. $y = \sqrt{3}\sin x + \cos x$

68. $y = \sqrt{2}\sin x - \sqrt{2}\cos x$

69. $y = -\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x$

70. $y = -\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x$

Mixed Exercises

71. a. Is it true that $\sin(2 \cdot 30°) = 2\sin 30°$?

b. Expand $\sin(2 \cdot 30°)$ as $\sin(30° + 30°)$ using the sum formula for sine. Then simplify the result.

72. a. Is it true that $\cos(2 \cdot 45°) = 2\cos 45°$?

b. Expand $\cos(2 \cdot 45°)$ as $\cos(45° + 45°)$ using the sum formula for cosine. Then simplify the result.

73. Derive $\cos(u + v) = \cos u \cos v - \sin u \sin v$ by using the identity for $\cos(u - v)$ and the odd and even function identities for sine and cosine.

74. Derive $\tan(u - v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$ by using the identity for $\tan(u + v)$ and the odd function identity for tangent.

75. Derive $\sin(u + v) = \sin u \cos v + \cos u \sin v$. {*Hint*: Write $\sin(u + v) = \cos\left[\dfrac{\pi}{2} - (u + v)\right]$ and apply the cofunction identities.}

76. Derive $\sin(u - v) = \sin u \cos v - \cos u \sin v$.

77. For $f(x) = \sin x$, show that

$$\frac{f(x + h) - f(x)}{h} = -\sin x\left(\frac{1 - \cos h}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$$

78. For $f(x) = \cos x$, show that

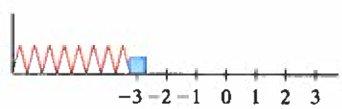
$$\frac{f(x + h) - f(x)}{h} = -\cos x\left(\frac{1 - \cos h}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)$$

79. A human ear detects sound as a result of fluctuations in air pressure against the eardrum, which are in turn transmitted into nerve impulses that the brain translates as sound. Suppose that the pressure $P(t)$ of a sound wave on a person's eardrum from a source 20 ft away is given by $P(t) = 0.02\cos(1000t - 10\pi)$. In this model $P(t)$ is in pounds per square foot and t is the time in seconds after the sound begins. Simplify the right side by applying the difference formula for cosine.

80. The raised part of a sundial, called a *gnomon*, casts a shadow of length l when the angle of elevation of the Sun is θ. The length of the shadow is given by $l = \dfrac{h\sin(90° - \theta)}{\sin\theta}$, where h is the height of the gnomon. Simplify the right side of this equation.

For Exercises 81–82, consider a 1-kg object oscillating at the end of a horizontal spring. The horizontal position $x(t)$ of the object is given by

$$x(t) = \frac{v_0}{\omega}\sin(\omega t) + x_0\cos(\omega t)$$



where v_0 is the initial velocity, x_0 is the initial position, and ω is the number of back-and-forth cycles that the object makes per unit time t.

81. At time $t = 0$ sec, the object is moved 3 ft to the left of the equilibrium position and then given a velocity of 4 ft/sec to the right ($v_0 = 4$ ft/sec).

a. If the object completes 1 cycle in 1 sec ($\omega = 1$), write a model of the form $x(t) = \dfrac{v_0}{\omega}\sin(\omega t) + x_0\cos(\omega t)$ to represent the horizontal motion of the spring.

b. Use the reduction formula to write the function in part (a) in the form $x(t) = k\sin(t + \alpha)$. Round α to 2 decimal places.

c. What is the maximum displacement of the object from its equilibrium position?

82. At time $t = 0$ sec, the object is moved 2 ft to the right of the equilibrium position and then given a velocity of 3 ft/sec to the left ($v_0 = -3$ ft/sec).

a. If the block completes 1 cycle in 1 sec ($\omega = 1$), write a model of the form $x(t) = \dfrac{v_0}{\omega}\sin(\omega t) + x_0\cos(\omega t)$ to represent the horizontal motion of the spring.

b. Use the reduction formula to write the function in part (a) in the form $x(t) = k\sin(t + \alpha)$. Round α to 2 decimal places.

c. What is the maximum displacement of the object from its equilibrium position? Round to 2 decimal places.

Write About It

83. Explain why we cannot use $\tan\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{\tan\frac{\pi}{2} - \tan\theta}{1 - \tan\frac{\pi}{2} \cdot \tan\theta}$ to prove the cofunction identity $\cot\theta = \tan\left(\dfrac{\pi}{2} - \theta\right)$.

84. Describe the pattern for the expansions of $\cos(u + v)$ and $\cos(u - v)$.

85. Describe the pattern for the expansions of $\sin(u + v)$ and $\sin(u - v)$.

86. The sum and difference formulas for cosine can be written in a single statement as $\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$. Explain why the symbol $\mp$ is used on the right.

Expanding Your Skills

For Exercises 87–90, write the trigonometric expression as an algebraic expression in x and y. Assume that x and y are Quadrant I angles.

87. $\cos(\sin^{-1}x + \tan^{-1}y)$

88. $\sin(\cos^{-1}x + \tan^{-1}y)$

89. $\sin(\sin^{-1}x + \sin^{-1}y)$

90. $\cos(\sin^{-1}x - \cos^{-1}y)$

For Exercises 91–96, verify the identity.

91. $\cos(a + b + c) = \cos a\cos b\cos c - \sin a\sin b\cos c - \sin a\cos b\sin c - \cos a\sin b\sin c$

92. $\sin(a + b + c) = \sin a\cos b\cos c + \cos a\sin b\cos c + \cos a\cos b\sin c - \sin a\sin b\sin c$

93. $\cos(x + y)\cos y + \sin(x + y)\sin y = \cos x$

94. $\cos(x - y)\sin y + \sin(x - y)\cos y = \sin x$

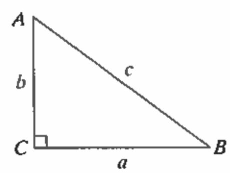
95. $\sec(x + y) = \dfrac{\cos(x - y)}{\cos^2 x - \sin^2 y}$

96. $\sec(x - y) = \dfrac{\cos(x + y)}{\cos^2 x - \sin^2 y}$

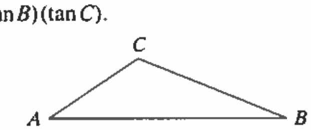
97. Suppose that ΔABC is a right triangle.

Show that

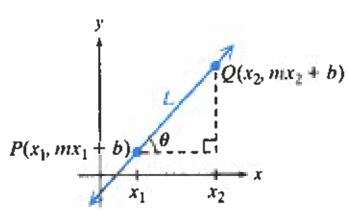
$\sin(A - B) = \dfrac{a^2 - b^2}{c^2}$ and $\cos(A - B) = \dfrac{2ab}{c^2}$



98. Suppose that ΔABC contains no right angle. Show that $\tan A + \tan B + \tan C = (\tan A)(\tan B)(\tan C)$.
(*Hint*: $A + B + C = 180°$ which implies that $A + B = 180° - C$. Then take the tangent of both sides of the equation.)



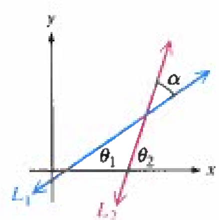
99. Let L be a line defined by $y = mx + b$ with a positive slope, and let θ be the acute angle formed by L and the horizontal. Let $P(x_1, mx_1 + b)$ and $Q(x_2, mx_2 + b)$ be arbitrary points on L. Show that $m = \tan\theta$.



100. Let L_1 and L_2 be nonparallel lines with positive slopes m_1 and m_2, respectively, where $m_2 > m_1$.

a. Show that the acute angle α formed by L_1 and L_2 must satisfy $\tan\alpha = \dfrac{m_2 - m_1}{1 + m_2 m_1}$.

b. Find the measure of the acute angle formed by the lines $y = \frac{3}{4}x + 2$ and $y = 2x - 3$. Round to the nearest tenth of a degree.



Technology Connections

101. a. Graph $y = \cos\left(x + \dfrac{\pi}{2}\right) - \cos\left(x - \dfrac{\pi}{2}\right)$ on the interval $[0, 2\pi]$. What simpler function $y = f(x)$ does this graph appear to represent.

b. Simplify $\cos\left(x + \dfrac{\pi}{2}\right) - \cos\left(x - \dfrac{\pi}{2}\right)$ to confirm your response to part (a).

102. a. Graph $y = \sin 3x\cos 2x - \cos 3x\sin 2x$ on the interval $[0, 2\pi]$. What simpler function $y = f(x)$ does this graph appear to represent.

b. Simplify $\sin 3x\cos 2x - \cos 3x\sin 2x$ to confirm your response to part (a).