### **Debiasing Word Embeddings**

Andrew Maurer

July 26, 2018

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## Cohomology Rings and Geometry

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### Plan

- Groups and Modules for Groups
- Cohomology
- Realizability and Consequences

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G-modules as defined above are really just modules for a certain ring denoted kG.

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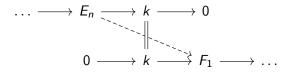
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Realizing cohomology groups  $H^n(G; k)$  in this way means we can multiply two sequences together.

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And the new sequence has the form

$$0 \to k \to \underbrace{E_1 \to \ldots \to E_n \to F_1 \to \ldots \to F_m}_{m+n} \to k \to 0$$

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Define a commutative, finitely generated ring:

$$H^{c}(G; k) = \begin{cases} H^{\bullet}(G; k) & \text{if } p = 2\\ \bigoplus H^{2n}(G; k) & \text{if } p > 2 \end{cases}$$

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$$I_M = \{x \in H^{\bullet}(G; k) \mid x. \operatorname{Ext}_{kG}^{\bullet}(M, M) = 0\}.$$

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For a G-module M,

$$\mathcal{V}_G(M) = \mathcal{Z}(I_M) \subseteq \mathcal{V}_G(k)$$

is the support variety of M.

#### Goal

Describe the representation theory of G in terms of the geometry of  $\mathcal{V}_G(k)$ 

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But these are statements about the geometry of  $V_G(k)$ . Can we figure out the representation theory of G by studying varieties  $V_G(M)$ ?

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$$\mathcal{V}_G(L_\zeta) = \mathcal{Z}(\zeta)$$

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And these two processes yield the same subvariety.

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$$X = \bigcap_{i=1}^n \mathcal{Z}(\zeta_i) = \bigcap_{i=1}^n \mathcal{V}_G(L_\zeta) = \mathcal{V}_G\left(\bigotimes_{i=1}^n L_\zeta\right).$$

### Partial converse to direct sum theorem

#### Theorem

If M is a G-module, and  $\mathcal{V}_G(M) = V_1 \cup V_2$ , with  $V_1 \cap V_2 = \{0\}$ , then there are modules with  $V_1 = \mathcal{V}_G(M_1)$  and  $V_2 = \mathcal{V}_G(M_2)$  such that  $M=M_1\oplus M_2$  and .

# Thank You