Debiasing Word Embeddings

Andrew Maurer

July 26, 2018

~~Debiasing Word Embeddings~~

Andrew Maurer

July 26, 2018

Cohomology Rings and Geometry

Andrew Maurer

July 26, 2018

Plan

1. Groups and Modules for Groups
2. Cohomology
3. Realizability and Consequences

Finite groups

Finite groups

Recall that a finite group is a finite set G with a multiplication rule satisfying some axioms that make it behave like symmetry.

Finite groups

Recall that a finite group is a finite set G with a multiplication rule satisfying some axioms that make it behave like symmetry.

- $\mathrm{GL}(V) =$ all linear isomorphisms of a vector space $V/\mathbb{F}_q$.

Finite groups

Recall that a finite group is a finite set G with a multiplication rule satisfying some axioms that make it behave like symmetry.

- $\mathrm{GL}(V) =$ all linear isomorphisms of a vector space $V/\mathbb{F}_q$.
- $S_n =$ all bijections $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Finite groups

Recall that a finite group is a finite set G with a multiplication rule satisfying some axioms that make it behave like symmetry.

- $\mathrm{GL}(V) =$ all linear isomorphisms of a vector space $V/\mathbb{F}_q$.
- $S_n =$ all bijections $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.
- $(\mathbb{Z}/p)^r =$ elementary Abelian group of rank r.

Finite groups

Recall that a finite group is a finite set G with a multiplication rule satisfying some axioms that make it behave like symmetry.

- $\mathrm{GL}(V) =$ all linear isomorphisms of a vector space $V/\mathbb{F}_q$.
- $S_n =$ all bijections $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.
- $(\mathbb{Z}/p)^r =$ elementary Abelian group of rank r.
- Others

Modules

Modules

Groups are modeled on symmetries. A G-**module** is just a vector space where G acts as symmetries, i.e., linear automorphisms.

Modules

Groups are modeled on symmetries. A G-**module** is just a vector space where G acts as symmetries, i.e., linear automorphisms.

Namely, there's a map

$$\varphi : G \to \mathrm{GL}(V), g \mapsto \varphi_g$$

Modules

Groups are modeled on symmetries. A G-**module** is just a vector space where G acts as symmetries, i.e., linear automorphisms.

Namely, there's a map

$$\varphi : G \to \mathrm{GL}(V), g \mapsto \varphi_g$$

Examples:

Modules

Groups are modeled on symmetries. A G-**module** is just a vector space where G acts as symmetries, i.e., linear automorphisms.

Namely, there's a map

$$\varphi : G \to \mathrm{GL}(V), g \mapsto \varphi_g$$

Examples:

- *Trivial Representation*: k is a G -module via the action $\varphi_g(v) = v$.

Modules

Groups are modeled on symmetries. A G-**module** is just a vector space where G acts as symmetries, i.e., linear automorphisms.

Namely, there's a map

$$\varphi : G \to \mathrm{GL}(V), g \mapsto \varphi_g$$

Examples:

- *Trivial Representation*: k is a G -module via the action $\varphi_g(v) = v$.
- *Natural Representation*: V is a $\mathrm{GL}(V)$ module.

Modules

Groups are modeled on symmetries. A G-**module** is just a vector space where G acts as symmetries, i.e., linear automorphisms.

Namely, there's a map

$$\varphi : G \to \mathrm{GL}(V), g \mapsto \varphi_g$$

Examples:

- *Trivial Representation*: k is a G -module via the action $\varphi_g(v) = v$.
- *Natural Representation*: V is a $\mathrm{GL}(V)$ module.
- *Permutation Representation*: The vector space with basis $\{e_1, \ldots, e_n\}$ is a S_n-module by $\varphi_\sigma(e_i) = e_{\sigma(i)}$, and extend by linearity.

Modules

Groups are modeled on symmetries. A G-**module** is just a vector space where G acts as symmetries, i.e., linear automorphisms.

Namely, there's a map

$$\varphi : G \to \mathrm{GL}(V), g \mapsto \varphi_g$$

Examples:

- *Trivial Representation*: k is a G -module via the action $\varphi_g(v) = v$.
- *Natural Representation*: V is a $\mathrm{GL}(V)$ module.
- *Permutation Representation*: The vector space with basis $\{e_1, \dots, e_n\}$ is a S_n-module by $\varphi_\sigma(e_i) = e_{\sigma(i)}$, and extend by linearity.

G-modules as defined above are really just modules for a certain ring denoted kG.

Cohomology groups are Ext groups

Cohomology groups are Ext groups

The cohomology group $H^n(G; k)$ can be defined in many ways.

Cohomology groups are Ext groups

The cohomology group $H^n(G;k)$ can be defined in many ways.

Most concrete is as $H^n(G;k) \cong \mathrm{Ext}^n_{kG}(k, M)$.

$$\mathrm{Ext}^n_{kG}(k, M) = \left\{0 \to M \to E_1 \to \ldots \to E_n \to k \to 0\right\} / \sim$$

i.e., as all n-fold extensions of k by M, modulo a certain equivalence relation.

Cohomology groups are Ext groups

The cohomology group $H^n(G; k)$ can be defined in many ways.

Most concrete is as $H^n(G; k) \cong \mathrm{Ext}^n_{kG}(k, M)$.

$$\mathrm{Ext}^n_{kG}(k, M) = \left\{0 \to M \to E_1 \to \ldots \to E_n \to k \to 0\right\} / \sim$$

i.e., as all n-fold extensions of k by M, modulo a certain equivalence relation.

Addition in the cohomology group corresponds to Baer sum of extensions.

Cohomology groups are Ext groups

The cohomology group $H^n(G; k)$ can be defined in many ways.

Most concrete is as $H^n(G; k) \cong \mathrm{Ext}^n_{kG}(k, M)$.

$$\mathrm{Ext}^n_{kG}(k, M) = \{0 \to M \to E_1 \to \ldots \to E_n \to k \to 0\} / \sim$$

i.e., as all n-fold extensions of k by M, modulo a certain equivalence relation.

Addition in the cohomology group corresponds to Baer sum of extensions.This is complicated.

Yoneda splicing

Yoneda splicing

Realizing cohomology groups $H^n(G; k)$ in this way means we can multiply two sequences together.

$$\mathrm{Ext}^n_{kG}(k,k) \times \mathrm{Ext}^m_{kG}(k,k) \to \mathrm{Ext}^{m+n}_{kG}(k,k)$$

Yoneda splicing

Realizing cohomology groups $H^n(G;k)$ in this way means we can multiply two sequences together.

$$\mathrm{Ext}^n_{kG}(k,k) \times \mathrm{Ext}^m_{kG}(k,k) \to \mathrm{Ext}^{m+n}_{kG}(k,k)$$

$$\begin{tikzcd}
\dots \arrow[r] & E_n \arrow[r] \arrow[drr, dashed] & k \arrow[r] \arrow[d, equal] & 0 & \\
 & 0 \arrow[r] & k \arrow[r] & F_1 \arrow[r] & \dots
\end{tikzcd}$$

Yoneda splicing

Realizing cohomology groups $H^n(G;k)$ in this way means we can multiply two sequences together.

$$\mathrm{Ext}^n_{kG}(k,k) \times \mathrm{Ext}^m_{kG}(k,k) \to \mathrm{Ext}^{m+n}_{kG}(k,k)$$

$$\begin{tikzcd}
\dots \arrow[r] & E_n \arrow[r] \arrow[dr, dashed] & k \arrow[r] \arrow[d, equal] & 0 & \\
& 0 \arrow[r] & k \arrow[r] & F_1 \arrow[r] & \dots
\end{tikzcd}$$

And the new sequence has the form

$$0 \to k \to \underbrace{E_1 \to \dots \to E_n \to F_1 \to \dots \to F_m}_{m+n} \to k \to 0$$

$H^\bullet(G;k) = \mathrm{Ext}^\bullet_{kG}(k,k)$ is a ring

$H^\bullet(G;k) = \mathrm{Ext}^\bullet_{kG}(k,k)$ is a ring

Under this $\mathrm{Ext}^n_{kG}(k,k) \times \mathrm{Ext}^m_{kG}(k,k) \to \mathrm{Ext}^{n+m}_{kG}(k,k)$ association,

$$\mathrm{Ext}^\bullet_{kG}(k,k) = \bigoplus \mathrm{Ext}^n_{kG}(k,k)$$

becomes a graded ring.

$H^\bullet(G;k) = \mathrm{Ext}^\bullet_{kG}(k,k)$ is a ring

Under this $\mathrm{Ext}^n_{kG}(k,k) \times \mathrm{Ext}^m_{kG}(k,k) \to \mathrm{Ext}^{n+m}_{kG}(k,k)$ association,

$$\mathrm{Ext}^\bullet_{kG}(k,k) = \bigoplus \mathrm{Ext}^n_{kG}(k,k)$$

becomes a graded ring. What can we say about it?

$H^\bullet(G; k) = \mathrm{Ext}^\bullet_{kG}(k, k)$ is a ring

Under this $\mathrm{Ext}^n_{kG}(k, k) \times \mathrm{Ext}^m_{kG}(k, k) \to \mathrm{Ext}^{n+m}_{kG}(k, k)$ association,

$$\mathrm{Ext}^\bullet_{kG}(k, k) = \bigoplus \mathrm{Ext}^n_{kG}(k, k)$$

becomes a graded ring. What can we say about it?

- It is *graded-commutative*, i.e., $\alpha \cdot \beta = (-1)^{\bar{\alpha} \cdot \bar{\beta}} \beta \cdot \alpha$.

$H^\bullet(G;k) = \mathrm{Ext}^\bullet_{kG}(k,k)$ is a ring

Under this $\mathrm{Ext}^n_{kG}(k,k) \times \mathrm{Ext}^m_{kG}(k,k) \to \mathrm{Ext}^{n+m}_{kG}(k,k)$ association,

$$\mathrm{Ext}^\bullet_{kG}(k,k) = \bigoplus \mathrm{Ext}^n_{kG}(k,k)$$

becomes a graded ring. What can we say about it?

- It is *graded-commutative*, i.e., $\alpha \cdot \beta = (-1)^{\bar{\alpha}\cdot\bar{\beta}} \beta \cdot \alpha$.
- (Evens 1961) It is finitely-generated over $\mathrm{Ext}^0_{kG}(k,k) \cong k$.

$H^\bullet(G;k) = \mathrm{Ext}^\bullet_{kG}(k,k)$ is a ring

Under this $\mathrm{Ext}^n_{kG}(k,k) \times \mathrm{Ext}^m_{kG}(k,k) \to \mathrm{Ext}^{n+m}_{kG}(k,k)$ association,

$$\mathrm{Ext}^\bullet_{kG}(k,k) = \bigoplus \mathrm{Ext}^n_{kG}(k,k)$$

becomes a graded ring. What can we say about it?

- It is *graded-commutative*, i.e., $\alpha \cdot \beta = (-1)^{\bar{\alpha}\cdot\bar{\beta}} \beta \cdot \alpha$.
- (Evens 1961) It is finitely-generated over $\mathrm{Ext}^0_{kG}(k,k) \cong k$.

Define a commutative, finitely generated ring:

$$H^c(G;k) = \begin{cases} H^\bullet(G;k) & \text{if } p = 2 \\ \bigoplus H^{2n}(G;k) & \text{if } p > 2 \end{cases}$$

One more thing about rings

One more thing about rings

$\mathrm{Ext}^{\bullet}_{kG}(M, M)$ is a ring in the exact same way. It is far less nice, and usually not finitely generated.

One more thing about rings

$\mathrm{Ext}^{\bullet}_{kG}(M, M)$ is a ring in the exact same way. It is far less nice, and usually not finitely generated.

$\mathrm{Ext}^{\bullet}_{kG}(M, M)$ is a graded module for $H^{\bullet}(G; k)$ (or $H^{c}(G; k)$) using the tensor product.

One more thing about rings

$\mathrm{Ext}^{\bullet}_{kG}(M, M)$ is a ring in the exact same way. It is far less nice, and usually not finitely generated.

$\mathrm{Ext}^{\bullet}_{kG}(M, M)$ is a graded module for $H^{\bullet}(G; k)$ (or $H^{c}(G; k)$) using the tensor product.

So $\mathrm{Ext}^{\bullet}_{kG}(M, M)$ has an annihilator $I_M \trianglelefteq H^{\bullet}(G; k)$ which is a homogeneous ideal.

One more thing about rings

$\operatorname{Ext}^{\bullet}_{kG}(M, M)$ is a ring in the exact same way. It is far less nice, and usually not finitely generated.

$\operatorname{Ext}^{\bullet}_{kG}(M, M)$ is a graded module for $H^{\bullet}(G; k)$ (or $H^{c}(G; k)$) using the tensor product.

So $\operatorname{Ext}^{\bullet}_{kG}(M, M)$ has an annihilator $I_M \trianglelefteq H^{\bullet}(G; k)$ which is a homogeneous ideal.

$$I_M = \left\{x \in H^{\bullet}(G; k) \mid x.\operatorname{Ext}^{\bullet}_{kG}(M, M) = 0\right\}.$$

Cohomology varieties and support varieties

Cohomology varieties and support varieties

To every finitely generated ring R, there is a geometric space $\operatorname{Spec} R$ whose functions are R.

Cohomology varieties and support varieties

To every finitely generated ring R, there is a geometric space $\operatorname{Spec} R$ whose functions are R.

The main example is if $R = k[x_1, \ldots, x_n]$, then $\operatorname{Spec} R \cong k^n$, with some "special" points added.

Cohomology varieties and support varieties

To every finitely generated ring R, there is a geometric space $\operatorname{Spec} R$ whose functions are R.

The main example is if $R = k[x_1, \ldots, x_n]$, then $\operatorname{Spec} R \cong k^n$, with some "special" points added.

Define $\mathcal{V}_G(k) = \operatorname{Spec}(H^c(G; k))$. This is the *cohomology variety* of G/k.

Cohomology varieties and support varieties

To every finitely generated ring R, there is a geometric space $\operatorname{Spec} R$ whose functions are R.

The main example is if $R = k[x_1, \ldots, x_n]$, then $\operatorname{Spec} R \cong k^n$, with some "special" points added.

Define $\mathcal{V}_G(k) = \operatorname{Spec}(H^c(G; k))$. This is the *cohomology variety* of G/k.

For a G-module M,

$$\mathcal{V}_G(M) = \mathcal{Z}(I_M) \subseteq \mathcal{V}_G(k)$$

is the *support variety* of M.

Geometry via Representation Theory

Geometry via Representation Theory

Goal

Describe the representation theory of G in terms of the geometry of $\mathcal{V}_G(k)$

Geometry via Representation Theory

Goal

Describe the representation theory of G in terms of the geometry of $\mathcal{V}_G(k)$

Theorem

Let M_1 and M_2 be G-modules.

Geometry via Representation Theory

Goal

Describe the representation theory of G in terms of the geometry of $\mathcal{V}_G(k)$

Theorem

Let M_1 and M_2 be G-modules.

- $\mathcal{V}_G(M_1) = \{0\}$ *if* M_1 *is projective.*

Geometry via Representation Theory

Goal

Describe the representation theory of G in terms of the geometry of $\mathcal{V}_G(k)$

Theorem

Let M_1 and M_2 be G-modules.

- $\mathcal{V}_G(M_1) = \{0\}$ *if M_1 is projective.*
- $\mathcal{V}_G(M_1 \oplus M_2) = \mathcal{V}_G(M_1) \cup \mathcal{V}_G(M_2)$.

Geometry via Representation Theory

Goal

Describe the representation theory of G in terms of the geometry of $\mathcal{V}_G(k)$

Theorem

Let M_1 and M_2 be G-modules.

- $\mathcal{V}_G(M_1) = \{0\}$ *if M_1 is projective.*
- $\mathcal{V}_G(M_1 \oplus M_2) = \mathcal{V}_G(M_1) \cup \mathcal{V}_G(M_2)$.
- $\mathcal{V}_G(M_1 \otimes M_2) = \mathcal{V}_G(M_1) \cap \mathcal{V}_G(M_2)$.

Geometry via Representation Theory

Goal

Describe the representation theory of G in terms of the geometry of $\mathcal{V}_G(k)$

Theorem

Let M_1 and M_2 be G-modules.

- $\mathcal{V}_G(M_1) = \{0\}$ *if M_1 is projective.*
- $\mathcal{V}_G(M_1 \oplus M_2) = \mathcal{V}_G(M_1) \cup \mathcal{V}_G(M_2)$.
- $\mathcal{V}_G(M_1 \otimes M_2) = \mathcal{V}_G(M_1) \cap \mathcal{V}_G(M_2)$.

But these are statements about the geometry of $\mathcal{V}_G(k)$. Can we figure out the representation theory of G by studying varieties $\mathcal{V}_G(M)$?

Carlson Modules

Carlson Modules

Each nonzero element $\zeta \in H^\bullet(G; k)$ of the cohomology ring determines a surjective homomorphism $\Omega^n(k) \to k$

Carlson Modules

Each nonzero element $\zeta \in H^\bullet(G;k)$ of the cohomology ring determines a surjective homomorphism $\Omega^n(k) \to k$, which can be fit into a short exact sequence

$$0 \to L_\zeta \to \Omega^n(k) \to k \to 0.$$

Carlson Modules

Each nonzero element $\zeta \in H^\bullet(G; k)$ of the cohomology ring determines a surjective homomorphism $\Omega^n(k) \to k$, which can be fit into a short exact sequence

$$0 \to L_\zeta \to \Omega^n(k) \to k \to 0.$$

Theorem

$$\mathcal{V}_G(L_\zeta) = \mathcal{Z}(\zeta)$$

Rehash previous slide

Rehash previous slide

On the one hand:

Rehash previous slide

On the one hand:

> *Each element* $\zeta \in H^\bullet(G;k)$ *determines a subvariety of* $\operatorname{Spec} H^\bullet(G;k) = \mathcal{V}_G(k)$.

Rehash previous slide

On the one hand:

> *Each element* $\zeta \in H^\bullet(G;k)$ *determines a subvariety of* $\operatorname{Spec} H^\bullet(G;k) = \mathcal{V}_G(k)$*.*

On the other hand:

Rehash previous slide

On the one hand:

> *Each element* $\zeta \in H^\bullet(G; k)$ *determines a subvariety of* $\operatorname{Spec} H^\bullet(G; k) = \mathcal{V}_G(k)$.

On the other hand:

> *Each element* $\zeta \in H^\bullet(G; k)$ *defines a morphism* $\Omega^n(k) \to k$*, which has a kernel* L_ζ*, which determines a subvariety* $\mathcal{V}_G(L_\zeta) \subseteq \mathcal{V}_G(k)$.

Rehash previous slide

On the one hand:

> *Each element* $\zeta \in H^\bullet(G;k)$ *determines a subvariety of* $\operatorname{Spec} H^\bullet(G;k) = \mathcal{V}_G(k)$.

On the other hand:

> *Each element* $\zeta \in H^\bullet(G;k)$ *defines a morphism* $\Omega^n(k) \to k$, *which has a kernel* L_ζ, *which determines a subvariety* $\mathcal{V}_G(L_\zeta) \subseteq \mathcal{V}_G(k)$.

And these two processes yield the same subvariety.

Realization

Realization

Theorem

Every variety that could arise as the support variety of a module, does arise as the support variety of a module.

Realization

Theorem

Every variety that could arise as the support variety of a module, does arise as the support variety of a module.

- Closed

Realization

Theorem

Every variety that could arise as the support variety of a module, does arise as the support variety of a module.

- Closed
- Conical

Realization

Theorem

Every variety that could arise as the support variety of a module, does arise as the support variety of a module.

- Closed
- Conical

Take $X \subseteq \mathcal{V}_G(k)$ to be a closed conical subvariety:

Realization

Theorem

Every variety that could arise as the support variety of a module, does arise as the support variety of a module.

- Closed
- Conical

Take $X \subseteq \mathcal{V}_G(k)$ to be a closed conical subvariety:

X

Realization

Theorem

Every variety that could arise as the support variety of a module, does arise as the support variety of a module.

- Closed
- Conical

Take $X \subseteq \mathcal{V}_G(k)$ to be a closed conical subvariety:

$$X = \bigcap_{\text{i}=1}^{n} \mathcal{Z}(\zeta_i)$$

Realization

Theorem

Every variety that could arise as the support variety of a module, does arise as the support variety of a module.

- Closed
- Conical

Take $X \subseteq \mathcal{V}_G(k)$ to be a closed conical subvariety:

$$X = \bigcap_{\mathrm{i}=1}^{n} \mathcal{Z}(\zeta_i) = \bigcap_{i=1}^{n} \mathcal{V}_G(L_\zeta)$$

Realization

Theorem

Every variety that could arise as the support variety of a module, does arise as the support variety of a module.

- Closed
- Conical

Take $X \subseteq \mathcal{V}_G(k)$ to be a closed conical subvariety:

$$X = \bigcap_{\text{i}=1}^{n} \mathcal{Z}(\zeta_i) = \bigcap_{i=1}^{n} \mathcal{V}_G(L_\zeta) = \mathcal{V}_G\left(\bigotimes_{i=1}^{n} L_\zeta\right).$$

Partial converse to direct sum theorem

Theorem

If M is a G-module, and $\mathcal{V}_G(M) = V_1 \cup V_2$, with $V_1 \cap V_2 = \{0\}$, then there are modules with $V_1 = \mathcal{V}_G(M_1)$ and $V_2 = \mathcal{V}_G(M_2)$ such that $M = M_1 \oplus M_2$ and .

Thank You