On the Finite Generation of Relative Cohomology for Lie Superalgebras

Andrew Maurer

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ON THE FINITE GENERATION OF RELATIVE COHOMOLOGY FOR LIE SUPERALGEBRAS

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ABSTRACT. In this paper, the author establishes finite-generation of the cohomology ring of a classical Lie superalgebra relative to an even subsuperalgebra. A spectral sequence is constructed to provide conditions for when this relative cohomology ring is Cohen-Macaulay. With finite generation established, support varieties for modules are defined via the relative cohomology, which generalize those of [5].

1. INTRODUCTION

1.1. Establishing finite generation of cohomology rings is a powerful result in representation theory which links cohomology theory with commutative algebra and algebraic geometry. For example, Evens [11] and Venkov [26] each independently proved that the cohomology ring of a finite group is finitely generated. This result was used by Quillen [25], Carlson [7], Chouinard [9], and Alperin-Evens [1] to study the cohomology variety of the finite group. This allowed those listed, among others, to use techniques from classical algebraic

Overview

1. Introduction & Motivation
2. Spectral Sequence
3. Algebra
4. Geometry

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These grow polynomially, as opposed to $\bigwedge^p(\mathfrak{g}_{\bar{0}}/\mathfrak{a})$ which are finite. So Lie superalgebras can have infinite cohomology.

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Theorem (Fuks-Leites)

There are isomorphisms:

$$\mathrm{H}^\bullet(\mathfrak{osp}(m|2n), 0; \mathbb{C}) \cong \begin{cases} \mathrm{H}^\bullet(\mathfrak{o}(m), 0; \mathbb{C}) & \text{if } m \geq 2n \\ \mathrm{H}^\bullet(\mathfrak{sp}(2n), 0; \mathbb{C}) & \text{if } m < 2n \end{cases}$$

$$\mathrm{H}^\bullet(\mathfrak{gl}(m|n), 0; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{gl}(\max(m,n)), 0; \mathbb{C})$$

A similar statement holds for Lie superalgebras of type $G(3)$, $F(4)$, and $D(2,1;\alpha)$

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Theorem (Restated)

Regular Lie superalgebra cohomology usually doesn't have interesting geometry.

A result of Boe-Kujawa-Nakano

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Theorem (Boe, Kujawa, Nakano 2006)

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *be a classical Lie superalgebra. The relative cohomology ring* $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ *is isomorphic to* $S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}$*, and is thus a finitely generated* $\mathbb{C}$*-algebra.*

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Theorem (Restated)

Cohomology of a classical Lie superalgebra relative to its even subsuperalgebra has interesting geometry determined by invariant theory.

Spectral sequence theorem

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Theorem (M–)

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra, $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ a subalgebra, and M a $\mathfrak{g}$-module. There exists a spectral sequence $\{E_r^{p,q}(M)\}$ which computes cohomology and satisfies

$$E_2^{p,q}(M) \cong \mathrm{H}^p(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M) \otimes \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathbb{C}) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{g}, \mathfrak{a}; M).$$

Moreover, when M is finite-dimensional, $E_2^{\bullet,\bullet}(M)$ is a Noetherian $E_2^{\bullet,\bullet}(\mathbb{C})$-module.

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Moreover, when M is finite-dimensional, $E_2^{\bullet,\bullet}(M)$ is a Noetherian $E_2^{\bullet,\bullet}(\mathbb{C})$-module.

Corollary

The relative cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ is a finitely-generated $\mathbb{C}$-algebra.

Filtration

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1. Decompose:

$$C^n(\mathfrak{g}, \mathfrak{a}; M) = \bigoplus_{i+j=n} C^i\left(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathbb{C}}\left(\bigwedge\nolimits_s^j (\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M\right)\right)$$

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2. Filter:

$$C^n(\mathfrak{g}, \mathfrak{a}; M)_{(p)} = \bigoplus_{\substack{i+j=n \\ i \leq n-p}} C^i\left(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathbb{C}}\left(\bigwedge\nolimits_s^j (\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M\right)\right)$$

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We are guaranteed a spectral sequence of $\mathfrak{a}$-modules:

$$E_r^{p,q}(M) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{g}, \mathfrak{a}; M).$$

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3. $E_2^{p,q}(M) \cong \mathrm{H}^p(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M) \otimes \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathbb{C})$: This uses the fact that $\mathfrak{g}$ is classical and a vanishing theorem for cohomology.
$\mathrm{Hom}_{\mathbb{C}}(\bigwedge_s^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M) = \mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}(\bigwedge_s^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M) \oplus V = C^p(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M) \oplus V$.

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The edge homomorphism $E_2^{\bullet,0}(M) \to E_\infty^{\bullet,0}(M)$ is induced by the restriction map

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Corollary

Under this map $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ is an integral extension of a homomorphic image of $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$, and thus a finitely generated $\mathbb{C}$-algebra.

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If the spectral sequence collapses at E_2, then $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ is a Cohen-Macaulay ring.

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Theorem

Let $\mathfrak{g}$ *be classical such that* $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ *vanishes in odd degrees,* $\mathfrak{l} \leq \mathfrak{g}_{\bar{0}}$ *be a standard Levi.*

$$\dim_{\mathrm{Kr}} \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) = \dim_{\mathrm{Kr}} \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{l}; \mathbb{C}).$$

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Example

This result applies to many $\mathfrak{g}$. E.g., $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$, $\mathfrak{psl}(2n|2n)$, $\mathfrak{osp}(2m+1|2n)$, $\mathfrak{osp}(2m, 2n)$, $P(4\ell-1)$, $D(2,1;\alpha)$, $G(3)$, and $F(4)$.

Cohomology varieties

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Definition

The *relative cohomology variety* of $\mathfrak{g}$ relative to $\mathfrak{a}$ is

$$V_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}) = \mathrm{mSpec}\left(\mathrm{H}^{ev}(\mathfrak{g}, \mathfrak{a}; \mathbb{C})\right).$$

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And the *relative support variety* of a $\mathfrak{g}$-module M is

$$V_{(\mathfrak{g},\mathfrak{a})}(M) = \mathrm{Z}\left(\mathrm{Ann}_{\mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{a};\mathbb{C})} \mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{a})}(M, M)\right) \subseteq V_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})$$

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Definition

We say a $\mathfrak{g}$-module is *natural* relative to $\mathfrak{a}$ if

$$\Phi(V_{(\mathfrak{g},\mathfrak{a})}(M)) = \Phi(V_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})) \cap V_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M).$$

The subalgebra $\mathfrak{a}$ is *natural in* $\mathfrak{g}$ if every $\mathfrak{g}$-module is natural relative to $\mathfrak{a}$.

Tensor products

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for all modules M and N.

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For finite group cohomology, this corresponds to Carlson's conjecture, proved by [Alperin-Evens].

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Theorem

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *be classical, stable, and polar, with* $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ *natural. Suppose* $\Phi(V_{(\mathfrak{g},\mathfrak{a})}(M)) = X \cup Y$ *with* $X \cap Y = \{0\}$*. Then there exist modules* M_1 *and* M_2

$$X = \Phi(V_{(\mathfrak{g},\mathfrak{a})}(M_1)) \text{ and } Y = \Phi(V_{(\mathfrak{g},\mathfrak{a})}(M_1)).$$

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The result in the finite groups case is due to [Carlson].

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Theorem

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be classical, stable, and polar, with $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ natural. Let X be a closed, conical, subvariety of $V_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})$. There is a $\mathfrak{g}$-module M with

$$\Phi(X) = \Phi(V_{(\mathfrak{g},\mathfrak{a})}(M)).$$

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3. When is $\mathfrak{a}$ natural?
4. The dimension $\dim V_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}(M) = \operatorname{atyp} M$ for simple modules when $\mathfrak{g} = \mathfrak{gl}(m|n)$ [Boe-Kujawa-Nakano]. Is there a combinatorial interpretation for $\dim V_{(\mathfrak{g}, \mathfrak{a})} M$ for simple M?

Thanks!

Thank you!