Debiasing Word Embeddings

Andrew Maurer

July 26, 2018

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Cohomology Rings and Geometry

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Plan

1. Groups and Modules for Groups
2. Cohomology
3. Realizability and Consequences

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G-modules as defined above are really just modules for a certain ring denoted kG.

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And the new sequence has the form

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Define a commutative, finitely generated ring:

$$H^c(G;k) = \begin{cases} H^\bullet(G;k) & \text{if } p = 2 \\ \bigoplus H^{2n}(G;k) & \text{if } p > 2 \end{cases}$$

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$$I_M = \left\{x \in H^{\bullet}(G; k) \mid x.\,\mathrm{Ext}^{\bullet}_{kG}(M, M) = 0\right\}.$$

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For a G-module M,

$$\mathcal{V}_G(M) = \mathcal{Z}(I_M) \subseteq \mathcal{V}_G(k)$$

is the *support variety* of M.

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Describe the representation theory of G in terms of the geometry of $\mathcal{V}_G(k)$

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But these are statements about the geometry of $\mathcal{V}_G(k)$. Can we figure out the representation theory of G by studying varieties $\mathcal{V}_G(M)$?

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Theorem

$$\mathcal{V}_G(L_\zeta) = \mathcal{Z}(\zeta)$$

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And these two processes yield the same subvariety.

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Partial converse to direct sum theorem

Theorem

If M is a G-module, and $\mathcal{V}_G(M) = V_1 \cup V_2$, with $V_1 \cap V_2 = \{0\}$, then there are modules with $V_1 = \mathcal{V}_G(M_1)$ and $V_2 = \mathcal{V}_G(M_2)$ such that $M = M_1 \oplus M_2$ and .

Thank You