Math 1060: Lecture Notes

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Abstract

Lecture notes for the section of *Math 1060: Mathematics of Decision Making* I taught in Summer 2017.

1 Management Science and Graph Theory

Mathematics is a tool which allows us to take complicated, noisy, real-world problems and isolate the interesting or difficult bits. In this unit, we will be concerned with the notion of a *graph*, which abstracts the notion of *binary connections*, i.e., two objects of interest are either connected or not connected. In this unit, we will see how to turn appropriate real-world problems into problems about graphs. We will also investigate some algorithms to solve efficiency problems in this context.

1.1 Urban services

Lesson 1

Goal: Identify situations where graphs will be helpful. Learn basic definitions involving graphs.
Objectives: Do examples of real world graphs. Identify connected graphs and disconnected graphs, coming from pictures and from applications.

- Talk about syllabus, course expectations etc. Office hours are a good use of time. Drop/Add ends Friday, Drop deadline is June 29.
- In this course we'll distill real-world problems into mathematical problems, and learn how to solve the mathematical problems.
- Motivating example: Meter reader starts at town hall and wants to check all meters (on foot) as efficiently as possible.
  1. Start and end at same place
  2. Walks on every side of the street with meters
  3. Repeats streets as little as possible.
- A **graph** is a mathematical object which consists of two pieces of information.
  1. Vertex information: A finite set of **vertices**.
  2. Edge information: A finite set of **edges** which connect pairs of vertices.

  In the future we may deal with **directed graphs** (**digraphs** for short) in which edges point from their start vertex to their end vertex.
- Examples of graphs and their visual representations. Note that *one graph may be drawn in many, many different ways – these are still the same graph.*
- Definitions:
  1. A **vertex** x **is connected** to a vertex y if there is a sequence of vertices, each connected to the next, which starts at x and ends with y.
  2. A **graph is connected** if every vertex is connected to every other vertex.
- Examples of real-world graphs:
  1. Facebook / Real-World friends. Vertices are people. Edges connect those who are friends.

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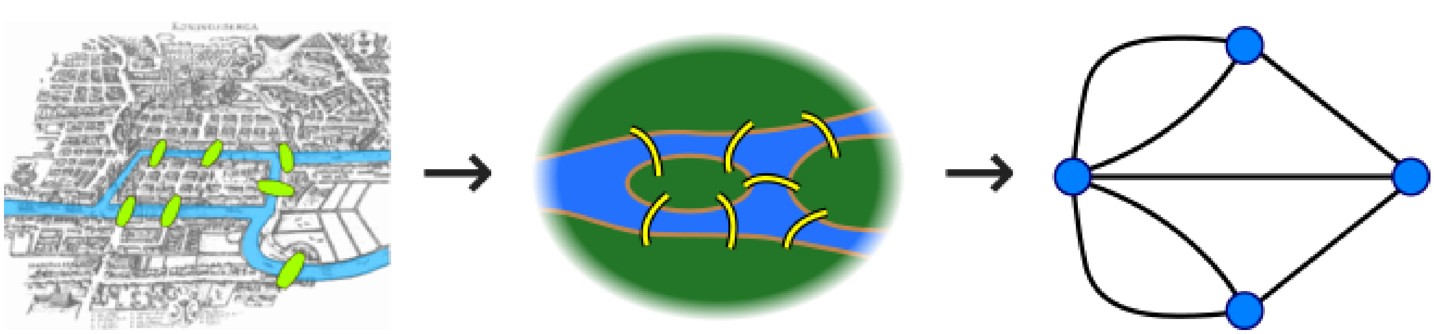
2. Airports and Direct flights. Vertices are airports. Edges correspond to direct flights. This is a digraph.

- Reality check:
  1. Should the airport graph be connected?
  2. If you built a friendship graph out of your graduating class from high school, would it be connected? Why or why not?
- Definitions:
  1. A **path** is a sequence of vertices, each connected to the next. We already saw this implicitly in the definition of *connected*.
  2. A **circuit** is a path that starts and ends at the same vertex.

  You'd want to find a circuit if you're planning a pilot's daily route. You'd want to find a path if you're looking for someone to introduce you to someone who knows Elon Musk.
- Back to the meter checker. Build this into a graph. Translate the problem into this:

  We are looking for a *circuit* which *passes through each edge* at least once, but no more than necessary.
- Definition: An **Euler circuit** (pronounced Oy-ler) is a circuit which passes through every edge exactly once.
- Example: the seven bridges of Königsberg. Real world problem → math problem



- Think about:
  1. Does the graph from class have an Euler circuit?
  2. Why would the graph of Königsberg not have an Euler circuit?

Lesson 2

Goals: Use Euler's theorem to identify when graphs have an Euler circuit. Use Euler's theorem to Eulerize graphs. Apply theorems correctly by thinking of them as an ingredient list.
Objectives:

- Announcement: Updated website to have yesterday's notes. This class is maleable – if there's something that interests you and you want to hear more about, let me know and I'll do my best to work it in.
- Last time: distilling problems, graphs, paths, circuits, Euler circuits, examples (abstract and real-world).
- Euler circuits in city block graph – did anyone find one? How?
- If a graph has an Euler circuit, what can we say about it?
  1. Connected – for any vertices x and y, just start at x and follow the Euler circuit until you end up at y.
  2. Every vertex has even valence – Any time the Euler circuit enters a vertex, it must also exit the same vertex.
- Back to seven bridges: why does it not have an Euler circuit?
- Theorems are mathematical statements that are always true. A theorem always has a proof, which is a logical justification for why it is true. Corollaries are theorems that are easily deduced from other theorems.

- **Theorem**: The sum of valences is twice the number of edges, i.e. $\sum$ valence = # Edges.
  Corollary: The sum of valences is always an even number.
  Corollary: There is always an even number of vertices with odd valence.

- **Theorem** (Euler's Theorem): A graph has an Euler circuit *if and only if* it is connected and each vertex has even valence.

  (The term if and only if means that this is really two statements combined into one. This is the same as saying "Any graph with an Euler circuit is connected and every vertex has even valence. Also, Any graph that is connected with every vertex having even valence must have an Euler circuit.")

  Conditional theorems – those containing the word – should be thought of as a recipe. *If I have flour, yeast, and water, then I can make bread.* If you're missing one of the ingredients, you cannot make bread.

- Practice finding Euler circuits.

- Back to the meter reader problem – if there isn't an Euler circuit, can we still solve the problem?

  Add the fewest number of *repeat* edges to "Eulerize" a connected graph. You can always do this because of the parity theorem.

- The problem of finding the smallest number of edges we must repeat to get a graph with an Euler circuit is called the *Simplified Chinese Postman Problem.*

- In the SCPP we are assuming all "streets" (a.k.a., edges) have the same length. For a regular *Chinese Postman Problem* edges can have different lengths. We may get to it in this chapter, but these will definitely come up in the next chapter.

Lecture 3

- Previously: Euler's theorem, Parity theorem. Telling whether a graph has an Euler circuit.
  Today: Finding Euler circuits and solving the Chinese Postman Problem

- **Complete graph on n vertices** (where n is a whole number). This is a graph which has n vertices, and where every vertex is connected to every other vertex. These will be important in chapter 2.

- Recall that a graph has an Euler circuit if and only if every vertex has valence an even number and the graph is connected.

- The Simplified Chinese Postman Problem asks to add the fewest number of *repeated edges* to get a graph with an Euler circuit. Phrasing this in real-world terms, "How many extra streets must our meter-reader walk down to check all the meters in the most efficient way possible?"

  The process of adding additional edges to a graph is called **Eulerizing the graph**.

- Eulerize a graph on the board.

- Euler's theorem tells us exactly where the problem lies – we must add edges to the vertices which have odd valence.

- Worksheet on Eulerizing graphs and finding Euler circuits. Collect this and use it to count participation.

- In real life, streets have lengths. We can modify our definition of graph by adding a number to each edge. Now the length of a circuit is the total length of all edges in the circuit.

- The Chinese Postman Problem is the problem of Eulerizing the graph in such a way that the resulting Euler circuit has minimal possible length.

1.2 Chapter 2: Business efficiency

1.3 Chapter 3: Planning and scheduling

2 Voting and Social Choice

2.1 Chapter 9: Social choice: thie impossible dream

2.2 Chapter 10: The manipulability of voting systems

2.3 Chapter 11: Weighted voting systems

3 Fairness and Apportionment

3.1 Chapter 13: Fair division

3.2 Chapter 14: Apportionment

4 Further Topics

4.1 Identification Numbers

4.2 Information Science