On the Finite Generation of Relative Cohomology for Lie Superalgebras

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Overview

1. Context
2. Lie superalgebras and their cohomology
3. Spectral Sequence
4. Finite Generation
5. Geometry

History

or

an extended motivating example

The Main Players

- $k = \bar{k}$ field of characteristic $p > 0$,
- G = **finite group** with $p \mid |G|$, p-Lie algebra, pointed Hopf algebra, etc.
- $\mathrm{H}^n(G; M)$ = cohomology groups. Can be multiplied using Yoneda product or cup product.

Theorem (Evens, '61)

- *The cohomology ring* $\mathrm{H}^\bullet(G; k) = \mathrm{Ext}^\bullet_{kG}(k, k)$ *is finitely generated.*
- *If* M *is a finite-dimensional* kG*-module,* $\mathrm{Ext}^\bullet_{kG}(M, M)$ *is a Noetherian graded module for* $\mathrm{H}^\bullet(G; k)$*.*

Idea for proof: Reduce to Sylow subgroups using spectral sequence.

Varieties

Definition

- *Cohomology Variety:* $\mathcal{V}_G = \operatorname{Spec}\left(\mathrm{H}^\bullet(G;k)\right)$
- *Support Variety:*

$$\mathcal{V}_G(M) = \mathrm{Z}\left(\operatorname{Ann}_{\mathrm{H}^\bullet(G;k)}\left(\operatorname{Ext}^\bullet_{kG}(M,M)\right)\right) \subseteq \mathcal{V}_G$$

Notice that $\mathcal{V}_G(M)$ is a conical subvariety.

Example

If $G = \mathbb{Z}_p^r$.

Algebra $\leftrightarrow$ Geometry

Idea

Build a dictionary:

$$\{\text{Algebraic Data About } M\} \leftrightarrow \{\text{Geometric Data About } \mathcal{V}_G(M)\}$$

Theorem

- *M is injective if and only if $\mathcal{V}_G(M) = \{0\}$.*
- $c_G(M) = \dim \mathcal{V}_G(M)$
- *If $\mathcal{V}_G(M) = X \cup Y$ with $X \cap Y = \{0\}$, then $M = M_X \oplus M_Y$ with $\mathcal{V}_G(M_X) = X$ and $\mathcal{V}_G(M_Y) = Y$.*

Here $c_G(M)$ is the *complexity*, meaning the *rate of growth* of a minimal kG-projective resolution

$$\ldots \to P_2 \to P_1 \to P_0 \to M \to 0$$

Tensor Products

Theorem (Tensor Product Theorem)

Let M and N be kG-modules. The following varieties are equal:

$$\mathcal{V}_G(M \otimes N) = \mathcal{V}_G(M) \cap \mathcal{V}_G(N)$$

Tensor product theorems as above do not follow from general nonsense, and require explicit constructions.

Definition

Let $E = \mathbb{Z}_p^r = \langle g_1, \ldots, g_r \rangle$ be elementary Abelian.

$$\mathcal{V}_E^{\#} = \langle g_1 - 1, \ldots, g_r - 1 \rangle \subseteq kE$$

And the *rank variety* of a module M is the subset of $\mathcal{V}_E^{\#}$

$$\mathcal{V}_E^{\#}(M) = \left\{ v \in \mathcal{V}_E^{\#} \middle| M{\downarrow}_{\langle 1+v \rangle} \text{ is not free } \right\} \cup \{0\}$$

Tensor Products (cont)

Theorem (Avrunin-Scott)

There is an isomorphism $\mathcal{V}_E \cong \mathcal{V}_E^{\#}$, and under this isomorphism:

$$\mathcal{V}_E(M) \cong \mathcal{V}_E^{\#}(M)$$

Theorem (Quillen)

The variety $\mathcal{V}_G$ may be stratified by varieties for elementary Abelian subgroups.

$$\mathcal{V}_G = \bigcup_{\substack{E \leq G \\ el.ab.}} \operatorname{res}^*_{G,E} \mathcal{V}_E$$

Corollary

$$\mathcal{V}_G(M \otimes N) = \mathcal{V}_G(M) \cap \mathcal{V}_G(N)$$

Lie Superalgebras

Definition

A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a $\mathbb{Z}_2$-graded (complex) vector space with a bilinear bracket $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying:

- $[x, [y, z]] = [[x, y], z] + (-1)^{\bar{x}\bar{y}}[y, [x, z]]$
- $[x, y] + (-1)^{\bar{x}\bar{y}}[y, x] = 0$

for homogeneous $x, y, z \in \mathfrak{g}$.

Example

The degree-zero elements $\mathfrak{g}_{\bar{0}}$ form a Lie algebra.

Examples of Lie superalgebras

- $\mathfrak{gl}(m|n) =$ all $(m+n) \times (m+n)$ matrices.

$$\mathfrak{gl}(m|n)_{\bar{0}} = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right) \text{ and } \mathfrak{gl}(m|n)_{\bar{1}} = \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right)$$

  Bracket is given by $[X, Y] = X \circ Y - (-1)^{\bar{X}\bar{Y}} Y \circ X$.
- $\mathfrak{sl}(m|n)$ is the subsuperalgebra of $\mathfrak{gl}(m|n)$ given by all matrices of *supertrace* 0, i.e., those such that $\operatorname{Tr} A - \operatorname{Tr} D = 0$.
- Those defined similarly to Lie algebras: linear transformations preserving certain forms on $\mathbb{Z}_2$-graded vector spaces.

The simple, complex, finite-dimensional Lie superalgebras were classified [Kac].

Classical Lie superalgebras

Definition

A Lie superalgebra $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ is *classical* if there is a reductive algebraic group $G_{\bar{0}}$ such that

- $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G_{\bar{0}})$, and
- An action $G_{\bar{0}} \curvearrowright \mathfrak{g}_{\bar{1}}$ differentiates to the adjoint action.

Example

$\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(m|n)$.

Modules for Lie superalgebras

Definition

A *module* for $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded vector space $M = M_{\bar{0}} \oplus M_{\bar{1}}$ with a homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(M)$. In this way $\mathfrak{g}$ acts on M.

Example

- $\mathbb{C}$ is the trivial $\mathfrak{g}$-module.
- $\mathfrak{g}$ is a $\mathfrak{g}$-module via the adjoint action.

A $\mathfrak{g}$-module is really a graded module for $U_s(\mathfrak{g})$, the universal enveloping superalgebra:

$$U_s(\mathfrak{g}) = \mathcal{T}(\mathfrak{g}) \Big/ \left\langle \lambda \otimes \mu - (-1)^{\bar{\mu}\cdot\bar{\lambda}} \mu \otimes \lambda - [\lambda, \mu] \right\rangle .$$

Wedges

Definition

The *exterior product* of a $\mathfrak{g}$-module $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is

$$\bigwedge\nolimits_s^p (M) = \mathcal{T}(M) \,/ \left\langle \lambda \otimes \mu + (-1)^{\bar{\mu}\cdot\bar{\lambda}} \mu \otimes \lambda \right\rangle$$

or

$$\bigwedge\nolimits_s^p (M) = \bigoplus_{i+j=p} \bigwedge\nolimits^i (M_{\bar{0}}) \otimes S^j (M_{\bar{1}})$$

This is naturally a $\mathfrak{g}$-module.

Relative cohomology I

Relative cohomology fits into Hochschild's relative cohomology theory. We use an explicit Koszul complex. Let $\mathfrak{a} \leq \mathfrak{g}$ be a subsuperalgebra:

$$C^p(\mathfrak{g}, \mathfrak{a}; M) = \mathrm{Hom}_{\mathfrak{a}}\left(\bigwedge\nolimits_s^p (\mathfrak{g}/\mathfrak{a}), M\right)$$

Equip this with differentials:

$$d : C^p(\mathfrak{g}, \mathfrak{a}; M) \to C^{p+1}(\mathfrak{g}, \mathfrak{a}; M)$$

$$\begin{aligned} df(\omega_0 \wedge \ldots \wedge \omega_p) = &\sum_i (-1)^{\tau_i(-)} \omega_i . f(\omega_0 \wedge \ldots \hat{\omega}_i \ldots \wedge \omega_p) \\ &+ \sum_{i<j} (-1)^{\sigma_{i,j}(-)} f([\omega_i, \omega_j] \wedge \omega_0 \ldots \hat{\omega}_i \ldots \hat{\omega}_j \ldots \wedge \omega_p) \end{aligned}$$

Relative cohomology II

Definition

$$\mathrm{H}^p(\mathfrak{g}, \mathfrak{a}; M) = \frac{\ker\left(d : C^p(\mathfrak{g}, \mathfrak{a}; M) \to C^{p+1}(\mathfrak{g}, \mathfrak{a}; M)\right)}{\operatorname{im}\left(d : C^{p-1}(\mathfrak{g}, \mathfrak{a}; M) \to C^p(\mathfrak{g}, \mathfrak{a}; M)\right)}$$

Or

$$\mathrm{H}^p(\mathfrak{g}, \mathfrak{a}; M) = \mathrm{Ext}^p_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}, M)$$

$$\mathrm{H}^p(\mathfrak{g}, \mathfrak{a}; M) = \frac{\{0 \to M \to E_1 \to \ldots \to E_p \to \mathbb{C} \to 0 \text{ which are } \mathfrak{a}\text{-split}\}}{\sim}$$

Relative cohomology III

We can give the following direct sum of cohomology groups the structure of a graded $\mathbb{C}$-algebra by Yoneda splice or tensor product:

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C}) = \bigoplus_{p \in \mathbb{Z}} \mathrm{H}^p(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$$

Under tensor product, the direct sum of cohomology groups for a module becomes a module for $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$:

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; M) = \bigoplus_{p \in \mathbb{Z}} \mathrm{H}^p(\mathfrak{g}, \mathfrak{a}; M).$$

As does

$$\mathrm{Ext}^\bullet_{(\mathfrak{g}, \mathfrak{a})}(M, M) = \bigoplus_{p \in \mathbb{Z}} \mathrm{Ext}^p(\mathfrak{g}, \mathfrak{a}; M).$$

A result of Fuks-Leites

Theorem (Fuks-Leites)

There are isomorphisms:

$$\mathrm{H}^\bullet(\mathfrak{osp}(m|2n), 0; \mathbb{C}) \cong \begin{cases} \mathrm{H}^\bullet(\mathfrak{o}(m), 0; \mathbb{C}) \text{ if } m \geq 2n \\ \mathrm{H}^\bullet(\mathfrak{sp}(2n), 0; \mathbb{C}) \text{ if } m < 2n \end{cases}$$

$$\mathrm{H}^\bullet(\mathfrak{gl}(m|n), 0; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{gl}(\max(m, n)), 0; \mathbb{C})$$

There's a similar statement for Lie superalgebras of type $G(3)$, $F(4)$, and $D(2, 1; \alpha)$

A result of Boe-Kujawa-Nakano

Theorem (Boe, Kujawa, Nakano 2006)

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *be a classical Lie superalgebra. The relative cohomology ring* $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ *is isomorphic to* $S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}$*, and is thus a finitely generated* $\mathbb{C}$*-algebra.*

Spectral sequence theorem

Theorem

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *be a classical Lie superalgebra,* $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ *a subalgebra, and* M *a* $\mathfrak{g}$*-module. There exists a spectral sequence* $\{E_r^{p,q}(M)\}$ *which computes cohomology and satisfies*

$$E_2^{p,q}(M) \cong \mathrm{H}^p(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M) \otimes \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathbb{C}) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{g}, \mathfrak{a}; \mathbb{C}).$$

Moreover, when M *is finite-dimensional,* $E_2^{\bullet,\bullet}(M)$ *is a Noetherian* $E_2^{\bullet,\bullet}(\mathbb{C})$*-module.*

Corollary

The relative cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ *is a finitely-generated* $\mathbb{C}$*-algebra.*

Filtration

We filter the cochains in a way inspired by Hochschild and Serre:

1. Decompose:

$$C^n(\mathfrak{g}, \mathfrak{a}; M) = \bigoplus_{i+j=n} C^i\left(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathbb{C}}\left(\textstyle\bigwedge_s^j(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M\right)\right)$$

2. Filter:

$$C^n(\mathfrak{g}, \mathfrak{a}; M)_{(p)} = \bigoplus_{\substack{i+j=n \\ i \le n-p}} C^i\left(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathbb{C}}\left(\textstyle\bigwedge_s^j(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M\right)\right)$$

We are guaranteed a spectral sequence of $\mathfrak{a}$-modules:

$$E_r^{p,q}(M) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{g}, \mathfrak{a}; M).$$

ID Pages

1. $E_0^{p,q}(M) \cong C^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathbb{C}}(\bigwedge_s^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M))$: Just a quotient.

2. $E_1^{p,q}(M) \cong \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathbb{C}}(\bigwedge_s^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M))$: Requires some algebraic gymnastics and a diagram chase.

3. $E_2^{p,q}(M) \cong \mathrm{H}^p(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M) \otimes \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathbb{C})$: This uses the fact that $\mathfrak{g}$ is classical and a vanishing theorem for cohomology.
$\mathrm{Hom}_{\mathbb{C}}(\bigwedge_s^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M) = \mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}(\bigwedge_s^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M) \oplus V = C^p(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M) \oplus V$.

ID Edge

We have an edge homomorphism:

$$E_2^{\bullet,0}(M) \to E_\infty^{\bullet,0}(M)$$

But recall $E_2^{\bullet,0}(M) \cong \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M)$.

Theorem

The edge homomorphism $E_2^{\bullet,0}(M) \to E_\infty^{\bullet,0}(M)$ is induced by the restriction map

$$\mathrm{res} : \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M) \to \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; M).$$

Finite generation

Theorem

When M is finite-dimensional, $E_r^{\bullet,\bullet}(M)$ is a Noetherian $E_r^{\bullet,\bullet}(\mathbb{C})$-module for $2 \leq r \leq \infty$.

Corollary

$E_\infty^{\bullet,\bullet}(\mathbb{C})$ is a Noetherian $E_2^{\bullet,0}(\mathbb{C})$-module, and is thus finitely-generated.

Corollary

$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ *is Noetherian* $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$*-module, and is therefore finitely-generated as a $\mathbb{C}$-algebra.*

Collapsing at E_2

Definition

Recall that a $\mathbb{C}$-algebra A is *Cohen-Macaulay* if there is a polynomial subring R such that A is a finite and free R-module.

Theorem

If the spectral sequence collapses at E_2, then $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ *is a Cohen-Macaulay ring.*

This uses a rather serious result of Hochster-Roberts regarding invariants under a reductive group action being Cohen-Macaulay.

Application

Theorem

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *be a classical Lie superalgebra such that* $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ *vanishes in odd degrees. Let* $\mathfrak{l} \leq \mathfrak{g}_{\bar{0}}$ *be a standard Levi subalgebra (i.e., nonzero and generated by simple roots). The following hold:*

- *Spectral sequence collapses at* E_2*.*
- $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{l}; \mathbb{C})$ *is Cohen-Macaulay.*
- $\dim_{\mathrm{Kr}} \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) = \dim_{\mathrm{Kr}} \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{l}; \mathbb{C})$.

This statement uses deep results from Kazhdan-Lusztig theory and Category $\mathcal{O}$ cohomology.

Example

This result applies to many $\mathfrak{g}$. E.g., $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$, $\mathfrak{psl}(2n|2n)$, $\mathfrak{osp}(2m+1|2n)$, $\mathfrak{osp}(2m, 2n)$, $P(4\ell - 1)$, $D(2, 1; \alpha)$, $G(3)$, and $F(4)$.

Cohomology varieties

To do geometry, we need a finitely-generated commutative subring. As in Friedlander-Parshall, we use the even cohomology groups:

$$\mathrm{H}^{ev}(\mathfrak{g}, \mathfrak{a}; \mathbb{C}) = \bigoplus_{n \geq 0} \mathrm{H}^{2n}(\mathfrak{g}, \mathfrak{a}; \mathbb{C}).$$

Definition

The *relative cohomology variety* of $\mathfrak{g}$ relative to $\mathfrak{a}$ is

$$V_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}) = \mathrm{mSpec}\left(\mathrm{H}^{ev}(\mathfrak{g}, \mathfrak{a}; \mathbb{C})\right).$$

Support varieties

A $\mathfrak{g}$-module M determines a $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; M)$-module $\mathrm{Ext}^\bullet_{(\mathfrak{g},\mathfrak{a})}(M, M)$. This determines a variety

$$\mathrm{Z}\left(\mathrm{Ann}_{\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{a};\mathbb{C})} \mathrm{Ext}^\bullet_{(\mathfrak{g},\mathfrak{a})}(M, M)\right) \subseteq V_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}).$$

Call this the *support variety* $V_{(\mathfrak{g},\mathfrak{a})}(M) \subseteq V_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})$.

Realization morphism and Natural Modules

We have the restriction map

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \xrightarrow{\text{res}} \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$$

And a corresponding *realization morphism*

$$V_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C}) \xleftarrow{\Phi} V_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})$$

Definition

We say a $\mathfrak{g}$-module is *natural* relative to $\mathfrak{a}$ if

$$\Phi(V_{(\mathfrak{g},\mathfrak{a})}(M)) = \Phi(V_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})) \cap V_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M).$$

The subalgebra $\mathfrak{a}$ is *natural in* $\mathfrak{g}$ if every $\mathfrak{g}$-module is natural relative to $\mathfrak{a}$.

Tensor products

We say a Lie superalgebra $\mathfrak{g}$ *satisfies the tensor product theorem* relative to $\mathfrak{a}$ if

$$V_{(\mathfrak{g},\mathfrak{a})}(M \otimes N) = V_{(\mathfrak{g},\mathfrak{a})}(M) \cap V_{(\mathfrak{g},\mathfrak{a})}(N)$$

for all modules M and N.

Theorem

Let $\mathfrak{g}$ be a Lie superalgebra which satisfies the tensor product theorem relative to $\mathfrak{g}_{\bar{0}}$. Then

$$\Phi(V_{(\mathfrak{g},\mathfrak{a})}(M \otimes N)) = \Phi(V_{(\mathfrak{g},\mathfrak{a})}(M)) \cap \Phi(V_{(\mathfrak{g},\mathfrak{a})}(N)).$$

Connectedness

We need to assume further $\mathfrak{g}$ is *stable* and *polar*, some GIT conditions appearing in [Bagci-Kujawa-Nakano].

Theorem

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be classical, stable, and polar, with $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ natural. Suppose $\Phi(V_{(\mathfrak{g},\mathfrak{a})}(M)) = X \cup Y$ with $X \cap Y = \{0\}$. Then there exist modules M_1 and M_2 with

$$X = \Phi(V_{(\mathfrak{g},\mathfrak{a})}(M_1)) \text{ and } Y = \Phi(V_{(\mathfrak{g},\mathfrak{a})})(M_1).$$

Realizability

Theorem

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *be classical, stable, and polar, with* $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ *natural. Let* X *be a closed, conical, subvariety of* $V_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})$*.*

What's next?

1. What happens when $\mathfrak{a} \not\leq \mathfrak{g}_{\bar{0}}$? Are there cases when $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ is still finitely-generated?
2. We saw that the spectral sequence collapses at E_2 implies $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ is CM. What about the converse? (It does for p-nilpotent Lie algebras, [Carlson-Nakano].)
3. When is $\mathfrak{a}$ natural?
4. The dimension $\dim V_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) = \operatorname{atyp} M$ for simple modules when $\mathfrak{g} = \mathfrak{gl}(m|n)$ [Boe-Kujawa-Nakano]. Is there a combinatorial interpretation for $\dim V_{(\mathfrak{g},\mathfrak{a})} M$ for simple M?

Thanks!

Thank you!