3.36pt

# Cohomology and Support Varieties for Classical Lie Superalgebras

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March 4, 2019

# Overview

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# History

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**Theorem (Evens, '61)**

- *The cohomology ring* $\mathrm{H}^\bullet(G; k) = \mathrm{Ext}^\bullet_{kG}(k, k)$ *is finitely generated.*
- *If* $M$ *is a finite-dimensional* $kG$*-module,* $\mathrm{Ext}^\bullet_{kG}(M, M)$ *is a Noetherian graded module for* $\mathrm{H}^\bullet(G; k)$.

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*Idea for proof:* Reduce to Sylow subgroups using spectral sequence.

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## Example

If $G = \mathbb{Z}_p^r$, $p \neq 2$, then

$$\mathrm{H}^\bullet(\mathbb{Z}_p^r;k) \cong k[x_1,\ldots,x_r] \otimes \Lambda(y_1,\ldots,y_r)$$

with $\deg x_i = 2$, $\deg y_i = 1$.
This implies $\mathrm{H}^{ev}(\mathbb{Z}_p^r;k) \cong k[x_1,\ldots,x_r]$ and thus $\mathcal{V}_{\mathbb{Z}_p^r} = \mathbb{A}^r$.

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## Idea

Build a dictionary:

$$\{\text{Algebraic Data About } M\} \leftrightarrow \{\text{Geometric Data About } \mathcal{V}_G(M)\}$$

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- *If $\mathcal{V}_G(M) = X \cup Y$ with $X \cap Y = \{0\}$, then $M = M_X \oplus M_Y$ with $\mathcal{V}_G(M_X) = X$ and $\mathcal{V}_G(M_Y) = Y$.*

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Here $c_G(M)$ is the *complexity*, meaning the *rate of growth* of a minimal $kG$-projective resolution

$$\ldots \to P_2 \to P_1 \to P_0 \to M \to 0$$

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## Definition

Let $E = \mathbb{Z}_p^r = \langle g_1, \ldots, g_r \rangle$ be elementary Abelian.

$$\mathcal{V}_E^{\#} = \langle g_1 - 1, \ldots, g_r - 1 \rangle \subseteq kE$$

And the *rank variety* of a module $M$ is the subset of $\mathcal{V}_E^{\#}$

$$\mathcal{V}_E^{\#}(M) = \left\{ v \in \mathcal{V}_E^{\#} \middle| M{\downarrow}_{\langle 1+v \rangle} \text{ is not free } \right\} \cup \{0\}$$

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**Corollary**

$$\mathcal{V}_G(M \otimes N) = \mathcal{V}_G(M) \cap \mathcal{V}_G(N)$$

# New Situation

Classical Results

# Lie Superalgebras

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A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a $\mathbb{Z}_2$-graded (complex) vector space with a bilinear bracket $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying:

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## Example

The degree-zero elements $\mathfrak{g}_{\bar{0}}$ form a Lie algebra.

# Examples of Lie superalgebras

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- $\mathfrak{gl}(m|n) =$ all $(m+n) \times (m+n)$ matrices.

$$\mathfrak{gl}(m|n)_{\bar{0}} = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right) \text{ and } \mathfrak{gl}(m|n)_{\bar{1}} = \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right)$$

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The simple, complex, finite-dimensional Lie superalgebras were classified [Kac].

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A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ is *classical* if there is a reductive algebraic group $G_{\bar{0}}$ such that

- $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G_{\bar{0}})$, and
- An action $G_{\bar{0}} \curvearrowright \mathfrak{g}_{\bar{1}}$ differentiates to the adjoint action.

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**Example**

The following are classical Lie superalgebras: $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(m|n)$.

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**Definition**

A *module* for $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded vector space $M = M_{\bar{0}} \oplus M_{\bar{1}}$ with a homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(M)$. In this way $\mathfrak{g}$ acts on $M$.

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A $\mathfrak{g}$-module is really a graded module for $U_s(\mathfrak{g})$, the universal enveloping superalgebra:

$$U_s(\mathfrak{g}) = \mathcal{T}(\mathfrak{g}) \Big/ \left\langle \lambda \otimes \mu - (-1)^{\bar{\mu}\cdot\bar{\lambda}} \mu \otimes \lambda - [\lambda, \mu] \right\rangle .$$

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This is naturally a $\mathfrak{g}$-module.

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$$C^p(\mathfrak{g}, \mathfrak{a}; M) = \mathrm{Hom}_{\mathfrak{a}}\left(\bigwedge\nolimits_s^p (\mathfrak{g}/\mathfrak{a}), M\right)$$

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$$C^p(\mathfrak{g}, \mathfrak{a}; M) = \mathrm{Hom}_{\mathfrak{a}} \left( \bigwedge\nolimits_s^p (\mathfrak{g}/\mathfrak{a}), M \right)$$

Equip this with differentials:

$$d : C^p(\mathfrak{g}, \mathfrak{a}; M) \to C^{p+1}(\mathfrak{g}, \mathfrak{a}; M)$$

$$\begin{aligned} df(\omega_0 \wedge \ldots \wedge \omega_p) = &\sum_i (-1)^{\tau_i(-)} \omega_i . f(\omega_0 \wedge \ldots \hat{\omega}_i \ldots \wedge \omega_p) \\ &+ \sum_{i<j} (-1)^{\sigma_{i,j}(-)} f([\omega_i, \omega_j] \wedge \omega_0 \ldots \hat{\omega}_i \ldots \hat{\omega}_j \ldots \wedge \omega_p) \end{aligned}$$

Of course,

$$d \circ d = 0$$

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We can give the following direct sum of cohomology groups the structure of a graded $\mathbb{C}$-algebra by Yoneda splice or tensor product:

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$$\mathrm{Ext}^{\bullet}_{(\mathfrak{g}, \mathfrak{a})}(M, M) = \bigoplus_{p \in \mathbb{Z}} \mathrm{Ext}^{p}_{(\mathfrak{g}, \mathfrak{a})}(M, M).$$

# A result of Fuks-Leites

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**Theorem (Fuks-Leites)**

*There are isomorphisms:*

$$\mathrm{H}^\bullet(\mathfrak{osp}(m|2n), 0; \mathbb{C}) \cong \begin{cases} \mathrm{H}^\bullet(\mathfrak{o}(m), 0; \mathbb{C}) \text{ if } m \geq 2n \\ \mathrm{H}^\bullet(\mathfrak{sp}(2n), 0; \mathbb{C}) \text{ if } m < 2n \end{cases}$$

$$\mathrm{H}^\bullet(\mathfrak{gl}(m|n), 0; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{gl}(\max(m,n)), 0; \mathbb{C})$$

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*There's a similar statement for Lie superalgebras of type $G(3)$, $F(4)$, and $D(2, 1; \alpha)$*

**Theorem (Restated)**

*In many cases, ordinary Lie superalgebra cohomology carries no geometric information.*

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**Theorem (Boe, Kujawa, Nakano 2006)**

*Let* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *be a classical Lie superalgebra. The relative cohomology ring may be related to invariant theory*

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \cong S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}$$

*Therefore,* $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ *is a finitely generated* $\mathbb{C}$*-algebra.*

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**Theorem (Boe, Kujawa, Nakano 2006)**

*Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra. The relative cohomology ring may be related to invariant theory*

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \cong S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}$$

*Therefore, $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ is a finitely generated $\mathbb{C}$-algebra.*

**Theorem (Restated)**

*Cohomology relative to the even subsuperalgebra carries geometric information, which is determined by geometric invariant theory.*

# Detecting Subalgebras

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**Theorem (Boe-Kujawa-Nakano)**

*If $\mathfrak{g}$ is a classical, simple Lie superalgebra with stable and polar action $G_{\bar{0}} \curvearrowright \mathfrak{g}_{\bar{1}}$, then there exists a Lie subsuperalgebra $\mathfrak{e} \leq \mathfrak{g}$ such that*

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^W$$

*where $W$ is a finite pseudoreflection group. Such a subsuperalgebra $\mathfrak{e}$ is called a detecting subalgebra.*

# Spectral Sequence

# Spectral sequence theorem

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**Theorem**

*Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra, $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ a subalgebra, and $M$ a $\mathfrak{g}$-module. There exists a spectral sequence $\{E_r^{p,q}(M)\}$ which computes cohomology and satisfies*

$$E_2^{p,q}(M) \cong \mathrm{H}^p(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M) \otimes \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathbb{C}) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{g}, \mathfrak{a}; \mathbb{C}).$$

*Moreover, when $M$ is finite-dimensional, $E_2^{\bullet,\bullet}(M)$ is a Noetherian $E_2^{\bullet,\bullet}(\mathbb{C})$-module.*

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*Moreover, when $M$ is finite-dimensional, $E_2^{\bullet,\bullet}(M)$ is a Noetherian $E_2^{\bullet,\bullet}(\mathbb{C})$-module.*

**Corollary**

*The relative cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ is a finitely-generated $\mathbb{C}$-algebra.*

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- Decompose:

$$C^n(\mathfrak{g}, \mathfrak{a}; M) = \bigoplus_{i+j=n} C^i \left( \mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathbb{C}} \left( \textstyle\bigwedge_s^j \left( \mathfrak{g}/\mathfrak{g}_{\bar{0}} \right), M \right) \right)$$

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We are guaranteed a spectral sequence of $\mathfrak{a}$-modules:

$$E_r^{p,q}(M) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{g}, \mathfrak{a}; M).$$

# ID Pages

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1. $E_0^{p,q}(M) \cong C^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathbb{C}}(\bigwedge_s^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M))$: Just a quotient.

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3. $E_2^{p,q}(M) \cong \mathrm{H}^p(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M) \otimes \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathbb{C})$: This uses the fact that $\mathfrak{g}$ is classical and a vanishing theorem for cohomology:

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}}\left(\bigwedge\nolimits_s^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M\right) &= \mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}\left(\bigwedge\nolimits_s^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M\right) \oplus V \\ &= C^p(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M) \oplus V \end{aligned}$$

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**Theorem**

*The edge homomorphism $E_2^{\bullet,0}(M) \to E_\infty^{\bullet,0}(M)$ is induced by the restriction map*

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$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ *is Noetherian* $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$*-module, and is therefore finitely-generated as a* $\mathbb{C}$*-algebra.*

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Recall that a $\mathbb{C}$-algebra $A$ is *Cohen-Macaulay* if there is a polynomial subring $R$ such that $A$ is a finite and free $R$-module.

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*If the spectral sequence collapses at $E_2$, then $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ is a Cohen-Macaulay ring.*

This uses a rather serious result of Hochster-Roberts regarding invariants under a reductive group action being Cohen-Macaulay.

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*Let* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *be a classical Lie superalgebra such that* $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ *vanishes in odd degrees. Let* $\mathfrak{l} \leq \mathfrak{g}_{\bar{0}}$ *be a standard Levi subalgebra (i.e., nonzero and generated by simple roots). The following hold:*

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## Example

This result applies to many $\mathfrak{g}$. E.g., $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$, $\mathfrak{psl}(2n|2n)$, $\mathfrak{osp}(2m+1|2n)$, $\mathfrak{osp}(2m,2n)$, $P(4\ell-1)$, $D(2,1;\alpha)$, $G(3)$, and $F(4)$.

# Geometry

# Cohomology varieties

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To do geometry, we need a finitely-generated commutative subring. As in Friedlander-Parshall, we use the even cohomology groups:

$$\mathrm{H}^{ev}(\mathfrak{g}, \mathfrak{a}; \mathbb{C}) = \bigoplus_{n \geq 0} \mathrm{H}^{2n}(\mathfrak{g}, \mathfrak{a}; \mathbb{C}).$$

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## Definition

The *relative cohomology variety* of $\mathfrak{g}$ relative to $\mathfrak{a}$ is

$$V_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}) = \operatorname{mSpec}\left(\mathrm{H}^{ev}(\mathfrak{g}, \mathfrak{a}; \mathbb{C})\right).$$

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Call this the *relative support variety* $V_{(\mathfrak{g},\mathfrak{a})}(M) \subseteq V_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})$.

# Rank Varieties

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**Definition**

For $\mathfrak{g}$ classical, $M$ a $\mathfrak{g}$-module, define the *rank variety* of $M$ to be

$$\mathcal{V}^{\#}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) = \left\{ G_{\bar{0}}.x \,\middle|\, G_{\bar{0}}.x \text{ is closed}, x \in \mathfrak{g}_{\bar{1}}, \text{ and } M{\downarrow_{\langle x \rangle}} \text{ is not projective} \right\} \cup \{0\}$$

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For $\mathfrak{g}$ simple, classical, stable, and polar, and $M$ a $\mathfrak{g}$-module, define the *BKN rank variety* to be

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## Conjecture

*When both varieties exist,* $\mathcal{V}^{rank}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C}) \cong \mathcal{V}^{\#}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C})$, *and under this association* $\mathcal{V}^{rank}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) = \mathcal{V}^{\#}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$.

# Tensor products

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We say a Lie superalgebra $\mathfrak{g}$ *satisfies the tensor product theorem* relative to $\mathfrak{a}$ if

$$V_{(\mathfrak{g},\mathfrak{a})}(M \otimes N) = V_{(\mathfrak{g},\mathfrak{a})}(M) \cap V_{(\mathfrak{g},\mathfrak{a})}(N)$$

for all modules $M$ and $N$.

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for all modules $M$ and $N$.

**Theorem (Grantcharov-Grantcharov-Nakano-Wu)**

*Let $\mathfrak{g}$ be simple, classical, polar, and stable. Then $\mathfrak{g}$ satisfies the tensor product theorem relative to $\mathfrak{g}_{\bar{0}}$.*

[GGNW] Idea: Define rank varieties for detecting subalgebra, and prove

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) \cong \mathcal{V}^{rank}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)/W$$

**Example**

The following Lie superalgebras satisfy the tensor product theorem: $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$, $\mathfrak{psl}(m|n)$, $\mathfrak{q}(n)$, $\mathfrak{osp}(m|n)$, $D(2,1;\alpha)$, $G(3)$, and $F(4)$.

# Tensor products (cont)

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**Conjecture (Kujawa-Nakano-Talian, 2014)**

*The tensor product theorem holds for all classical Lie superalgebras.*

# Natural Subalgebras I

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The restriction morphism $\mathrm{res} : \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \to \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ yields a finite-to-one realization morphism:

$$\Phi : \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C})$$

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## Definition

For a classical Lie superalgebra $\mathfrak{g}$, a subalgebra $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ is *natural* if

$$\Phi\left(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)\right) = \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) \cap \Phi\left(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})\right)$$

# Natural Subalgebras II

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- *Let $\mathfrak{g}$ be stable and polar, and $\Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)) = X_1 \cup X_2$ with $X_i$ closed conical and $X_1 \cap X_2 = \{0\}$. Then there exist modules $M_1$ and $M_2$ with $M = M_1 \oplus M_2$ and $\Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_i)) = X_i$.*

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- *Let $X$ be a closed conical subset of $\Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}))$. There exists a $\mathfrak{g}$-module $M$ with*

$$\Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)) = X$$

Thank you!