Cohomology and Support Varieties

for Classical Lie Superalgebras

by

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(Under the direction of Daniel Nakano)

Abstract

The author establishes finite-generation of the cohomology ring of a classical Lie superalgebra relative to an even subsuperalgebra. A spectral sequence is constructed to provide conditions for when this relative cohomology ring is Cohen-Macaulay. With finite generation established, support varieties for modules are defined via the relative cohomology, which generalize those of [BKN10].

Index words: Cohomology, Lie superalgebra, Geometric invariant theory, Representation theory, Support Variety, Algebraic geometry

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Chapter 1

Introduction

1.1 Motivation

Establishing finite generation of cohomology rings is a powerful result in representation theory which links cohomology theory with commutative algebra and algebraic geometry. For example, Evens [Eve61] and Venkov [Ven59] each independently proved that the cohomology ring of a finite group is finitely generated. This result was used by Quillen [Qui71], Carlson [Car83], Chouinard [Cho76], and Alperin-Evens [AE81] to study the cohomology variety of the finite group. This allowed those listed, among others, to use techniques from classical algebraic geometry in the study of representation theory of finite groups. Similar work has been carried out in other contexts; by Friedlander-Parshall [FP87, FP86] for restricted Lie algebras, and by Friedlander-Suslin [FS97] for finite-dimensional cocommutative Hopf algebras.

Relative cohomology, as defined by Hochschild [Hoc56] is less understood than ordinary cohomology. For instance, the cohomology ring of a finite group relative to a subgroup need not be finitely generated. Indeed, Brown [Bro94] provided an example of a finite group whose relative cohomology is infinitely generated. Surprisingly, in the case $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a (finite-dimensional) classical Lie superalgebra, the cohomology ring of $\mathfrak{g}$ relative to $\mathfrak{g}_{\bar{0}}$ is always finitely generated. Specifically, Boe-Kujawa-Nakano [BKN10] realized this relative cohomology ring as the invariants of a polynomial ring under the action of a reductive group. In fact, in the case of Lie superalgebras, ordinary cohomology is often times finite-dimensional as a vector space, as proved by Fuks-Leites [?] . This implies relative cohomology rings carry more representation theoretic information than their ordinary counterparts. Furthermore, Boe-Kujawa-Nakano [BKN10] demonstrated the atypicality of a supermodule – a combinatorial invariant defined by Kac-Wakimoto [KW94] – is realized as the dimension of the support variety of that module. The geometrization of combinatorial ideas makes support variety theory useful and powerful.

One of the main results of this paper asserts that for a classical Lie superalgebra, cohomology rings relative to even subalgebras are finitely-generated over $\mathbb{C}$, and the relative cohomology of a finite-dimensional module is a Noetherian module for this ring. In proving the main theorem, a spectral sequence is constructed which relates relative Lie algebra cohomology to odd degree elements of the Lie superalgebra in an interesting way. The main theorem paves the way to define and investigate support varieties for supermodules relative to a broader class of subalgebras. The importance of this result is apparent in that cohomology relative to an even subalgebra provides a middle ground between the case of absolute cohomology of Fuks-Leites and cohomology relative to $\mathfrak{g}_{\bar{0}}$ of [BKN10].

1.2 Overview of Dissertation

Revamp: This was written for my paper. Once dissertation is completed I will rework this for the contents of the dissertation.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra, $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ a subalgebra, and M a finite-dimensional $\mathfrak{g}$-module. The theory of relative cohomology [Hoc56] can be used to define relative cohomology groups $\mathrm{H}^n(\mathfrak{g}, \mathfrak{a}; M)$, which may be viewed as relative derived functors of $\mathrm{Hom}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}, -)$. In [BKN10, Theorem 2.5.2] it was shown that when $\mathfrak{g}$ is a classical Lie superalgebra and $\mathfrak{a} = \mathfrak{g}_{\bar{0}}$, the cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ is the subring $S(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}$ of invariants under a reductive group action, and is thus finitely generated over $\mathbb{C}$. This paper's main result extends this work to arbitrary subalgebras $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$.

Main Theorem. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra, and $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ an (even) subalgebra, and M a $\mathfrak{g}$-module.*

(a) *There is a spectral sequence* $\{E_r^{p,q}\}$ *which computes cohomology and satisfies*

$$E_2^{p,q}(M) \cong \mathrm{H}^p(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M) \otimes \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathbb{C}) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{g}, \mathfrak{a}; M)$$

For $1 \leq r \leq \infty$, $E_r^{\bullet,\bullet}(M)$ *is a module for* $E_2^{\bullet,\bullet}(\mathbb{C})$. *When* M *is finite-dimensional,* $E_2^{\bullet,\bullet}(M)$ *is a Noetherian* $E_2^{\bullet,\bullet}(\mathbb{C})$*-module.*

(b) *Moreover, the cohomology ring* $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ *is a finitely-generated* $\mathbb{C}$*-algebra.*

The paper is outlined as follows. In Section **??**, Lie superalgebras, modules for Lie superalgebras, and cohomology of Lie superalgebras are defined. The pace is brisk and the interested reader will find a more thorough overview in [BKN10, Kac77]. In Section **??** the author establishes finite generation of the relative cohomology ring. To do so, a first-quadrant spectral sequence as described above is constructed, similar to that of Hochschild and Serre [HS53], pages are identified, and a standard argument is used. Additionally, the edge homomorphism of the $E_2^{\bullet,0} \to E_\infty^{\bullet,0}$ is identified as restriction, making $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ an integral extension of a homomorphic image of $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$. Equipped with the spectral sequence of the previous section, we devote Section 5.6 to investigating the structure of these relative cohomology rings. For the relative cohomology ring to be Cohen-Macaulay, it is shown to be sufficient that the spectral sequence of Section **??** collapse at the E_2 page. This is used to compute a broad class of examples. Finally, we are in a position to systematically study support varieties for Lie superalgebras, which we do in Section **??**. In this section, support varieties are defined and several basic properties are stated before addressing the more difficult questions of realizability and connectedness. Our realizability theorem demonstrates a naturality between support varieties for $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$ and those for $(\mathfrak{g}, \mathfrak{a})$.

The author would like to thank his advisor, Dr Daniel Nakano for much motivation and support, and Dr William Graham for helpful conversations involving spectral sequences and Lie algebra cohomology. This work was completed as a part of the author's dissertation at University of Georgia, and was partially supported by the Research and Training Group in Algebraic Geometry, Algebra and Number Theory grant DMS-1344994 funded by the National Science Foundation.

Chapter 2

Algebraic Geometry

2.1 Motivation

Algebraic geometry is a remarkably powerful discipline which allows one to view the hidden geometry arising in commutative ring theory. It turns out, many of these techniques that arose from commutative ring theory hold equally well for graded-commutative rings, i.e., graded rings satisfying $a \cdot b = (-1)^{\bar{a} \cdot \bar{b}} b \cdot a$. This is handy for representation theorists because cohomology rings are very often graded-commutative.

Algebraic geometry behaves nicely when the rings in question are finitely generated over a ground field, in which case the associated geometric object is a variety.

2.2 The Spectrum of a Ring

Definition 2.2.1. Let $\mathrm{H}^\bullet$ be a graded-commutative ring, which is finitely generated as a $\mathbb{C}$-algebra. The *spectrum* of $\mathrm{H}^\bullet$ is the set:

$$\mathrm{Spec}(\mathrm{H}^\bullet) = \{[\mathfrak{p}] \mid \mathfrak{p} \leq \mathrm{H}^\bullet \text{ is a prime ideal}\}$$

The set $\mathrm{Spec}(\mathrm{H}^\bullet)$ is equipped with a topology whose closed sets are the vanishing sets of ideals

$$\mathrm{Z}(I) = \{[\mathfrak{p}] \in \mathrm{Spec}(\mathrm{H}^\bullet) \mid I \subseteq \mathfrak{p}\}$$

Note 2.2.2. *Despite the fact that* $\mathrm{H}^\bullet$ *is a graded ring, we choose to use* Spec *and not* Proj*. As a result, the varieties considered are conical affine varieties and not projective varieties. This is especially useful because*

$$\mathrm{Hom}_{ring}(\mathrm{H}_1^\bullet, \mathrm{H}_2^\bullet) \longleftrightarrow \mathrm{Mor}_{var}(\mathrm{Spec}(\mathrm{H}_2^\bullet), \mathrm{Spec}(\mathrm{H}_1^\bullet))$$

In what follows, we will primarily be concerned with the topological space $\mathrm{Spec}(\mathrm{H}^\bullet)$, not with the sheaf of rings associated to it. The interested reader may consult Chapter II of Hartshorne[Har77].

Proposition 2.2.3. *Let* $\mathrm{H}_1^\bullet$ *and* $\mathrm{H}_2^\bullet$ *be commutative or graded-commutative rings. Let* $I_1 \leq \mathrm{H}_1^\bullet$ *be an ideal.*

(a) A homomorphism of rings $\varphi : \mathrm{H}_1^\bullet \to \mathrm{H}_2^\bullet$ *corresponds to a morphism of varieties* $\varphi^* : \mathrm{Spec}(\mathrm{H}_2^\bullet) \to \mathrm{Spec}(\mathrm{H}_1^\bullet)$. *This map is determined on points by* $\varphi^*([\mathfrak{p}_2]) = \varphi^{-1}(\mathfrak{p}_2) \leq \mathrm{H}_1^\bullet$.

(b) The spectrum of a quotient $\mathrm{Spec}(\mathrm{H}_1^\bullet/I_1)$ *is isomorphic to* $\mathrm{Z}(I_1)$. *Furthermore, the quotient morphism* $\pi : \mathrm{H}_1^\bullet \twoheadrightarrow \mathrm{H}_1^\bullet/I_1$ *corresponds to the inclusion morphism* $\pi^* : \mathrm{Z}(I) \hookrightarrow \mathrm{Spec}\,\mathrm{H}_1^\bullet$.

2.3 Modules and Subvarieties

As above, consider a graded-commutative ring $\mathrm{H}^\bullet$. Let M be a finitely-generated $\mathrm{H}^\bullet$-module. This module has an *annihilator*

$$\mathrm{Ann}_{\mathrm{H}^\bullet}(\mathrm{M}) = \{x \in \mathrm{H}^\bullet \mid x.M = 0\}$$

Definition 2.3.1. The *support* of M is the vanishing set of the annihilator of M:

$$\mathrm{Supp}_{\mathrm{H}^\bullet}(\mathrm{M}) = \mathrm{Z}\left(\mathrm{Ann}_{\mathrm{H}^\bullet}(\mathrm{M})\right) \subseteq \mathrm{Spec}(\mathrm{H}^\bullet)$$

Proposition 2.3.2. *Let* $\mathrm{H}^\bullet$ *be a commutative or graded-commutative ring, and let* M_i *be* $\mathrm{H}^\bullet$*-modules.*

(a) If $\mathrm{H}^\bullet$ *is graded-commutative, then* $\mathrm{Supp}_{\mathrm{H}^\bullet}(\mathrm{M})$ *is closed and conical.*

(b) $\mathrm{Supp}_{\mathrm{H}^\bullet}(\mathrm{M}_1 \oplus \mathrm{M}_2) = \mathrm{Supp}_{\mathrm{H}^\bullet}(\mathrm{M}_1) \cup \mathrm{Supp}_{\mathrm{H}^\bullet}(\mathrm{M}_2)$

(c) If $0 \to \mathrm{M}_1 \to \mathrm{M}_2 \to \mathrm{M}_3 \to 0$ *is a short exact sequence, and* $\sigma \in S_3$ *is a permutation of* $\{1, 2, 3\}$, *then*

$$\mathrm{Supp}_{\mathrm{H}^\bullet}(\mathrm{M}_{\sigma(1)}) \subseteq \mathrm{Supp}_{\mathrm{H}^\bullet}(\mathrm{M}_{\sigma(2)}) \cup \mathrm{Supp}_{\mathrm{H}^\bullet}(\mathrm{M}_{\sigma(3)})$$

Proof. (a) This is immediate. Vanishing sets are closed by definition. Because $\mathrm{Supp}_{\mathrm{H}^\bullet}(\mathrm{M})$ is the vanishing of a homogeneous ideal, it must be conical.

(b)

(c)

$\square$

Chapter 3

Algebraic Groups

3.1 Overview

3.2 Algebraic Groups ABCs

Definition 3.2.1. An *algebraic group* is a complex affine algebraic variety G equipped with an identity element $e : \mathrm{mSpec}(\mathbb{C}) \to G$, multiplication morphism $m : G \times G \to G$ and an inverse morphism $i : G \to G$. These morphisms satisfy the typical group-theoretic axioms, expressed as commutative diagrams.

Chapter 4

Lie Superalgebras

4.1 Motivation

A *Lie superalgebra* is a $\mathbb{Z}_2$-graded analogue of a Lie algebra. Lie superalgebras originated in the physical theory of *supersymmetry* and play a similar role as Lie algebras, in that they arise as tangent spaces to *Lie supergroups* at the identity element.

A thorough overview of Lie superalgebra theory is provided by Victor Kac [Kac77]. A main result of this paper is the classification of simple classical Lie superalgebras.

Theorem 4.1.1 (Kac, [Kac77])**.** *A simple classical Lie superalgebra is isomorphic to either to one of the simple Lie algebras A_n, B_n, $\ldots$, E_8 or to one of $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1;\alpha)$, $F(4)$, $G(3)$, $P(n)$, or $Q(n)$.*

4.2 Definition and Examples

Definition 4.2.1. A *superspace* is a $\mathbb{Z}_2$-graded complex vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. An element of V_i is called *homogenous of degree i*. The *superdimension* of V is the ordered pair $\dim_s V = (\dim V_{\bar{0}}, \dim V_{\bar{1}})$.

Example 4.2.2. Let V and W be superspaces. The vector space of linear homomorphisms $\mathrm{Hom}_{\mathbb{C}}(V,W)$ is naturally a superspace:

$$\mathrm{Hom}_{\mathbb{C}}(V,W)_{\bar{0}} = \{\varphi \mid \varphi(V_i) \subseteq \varphi(W_i)\}$$

$$\mathrm{Hom}_{\mathbb{C}}(V,W)_{\bar{1}} = \{\varphi \mid \varphi(V_i) \subseteq \varphi(W_{i+\bar{1}})\}$$

In this way,

$$\mathrm{Hom}_{\mathbb{C}}(V,W) = \mathrm{Hom}_{\mathbb{C}}(V,W)_{\bar{0}} \oplus \mathrm{Hom}_{\mathbb{C}}(V,W)_{\bar{1}}$$

Definition 4.2.3. A *Lie superalgebra* is a superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, equipped with a bilinear bracket $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying the following two properties:

(S1) For x, y, z homogeneous elements of $\mathfrak{g}$,

$$[x,[y,z]] = [[x,y],z] + (-1)^{\bar{x}\cdot\bar{y}}[y,[x,z]]$$

(S2) For x, y homogeneous elements of $\mathfrak{g}$,

$$[x,y] + (-1)^{\bar{x}\cdot\bar{y}}[y,x] = 0$$

It is worth noting that the even subsuperalgebra $\mathfrak{g}_{\bar{0}}$ is, in fact, a Lie algebra. Furthermore, the subset of odd elements $\mathfrak{g}_{\bar{1}}$ is a module for $\mathfrak{g}_{\bar{0}}$.

Example 4.2.4.

1. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a superspace of superdimension (m, n). The *general linear Lie superalgebra* $\mathfrak{gl}(V)$ or $\mathfrak{gl}(m|n)$ is the superspace $\mathrm{Hom}_{\mathbb{C}}(V, V)$, with grading of Example 3.2.2, visualized as

$$\mathfrak{gl}(m|n)_{\bar{0}} = \left(\begin{array}{c|c} A_{m\times m} & 0 \\ \hline 0 & A_{n\times n} \end{array}\right) \text{ and } \mathfrak{gl}(m|n)_{\bar{1}} = \left(\begin{array}{c|c} 0 & A_{m\times n} \\ \hline A_{n\times m} & 0 \end{array}\right)$$

   The bracket operation on $\mathfrak{gl}(m|n)$ is defined for homogeneous elements via

$$[A,B] = A\cdot B - (-1)^{\bar{A}\cdot\bar{B}}B\cdot A$$

2. Consider the matrix $A \in \mathfrak{gl}(m|n)$, decomposed as

$$A = \left(\begin{array}{c|c} A_{m\times m} & A_{m\times n} \\ \hline A_{n\times m} & A_{n\times n} \end{array}\right)$$

   The *supertrace of* A is $\mathrm{sTr}(A) = \mathrm{Tr}(A_{m\times m}) - \mathrm{Tr}(A_{n\times n})$. The *special linear Lie superalgebra* is $\mathfrak{sl}(V)$ or $\mathfrak{sl}(m|n)$ and consists of all matrices in $\mathfrak{gl}(V)$ with supertrace 0, i.e.,

$$\mathfrak{sl}(m|n) = \{A \in \mathfrak{gl}(m|n) \mid \mathrm{sTr}(A) = 0\}$$

Example 4.2.5. Let $\mathfrak{g}$ be a Lie superalgebra. We will classify all subsuperalgebras generated by a single homogeneous element $x \in \mathfrak{g}$.

1. If $x \in \mathfrak{g}_{\bar{0}}$, then $[x, x] = 0$. As such $\langle x \rangle$ is a one-dimensional simple Lie algebra $\mathfrak{g} \cong \mathbb{C} \oplus \{0\}$

2. If $x \in \mathfrak{g}_{\bar{1}}$, and $[x, x] = 0$ then there are no even elements and thus $\mathfrak{g} \cong 0 \oplus \mathbb{C}$.

3. If $x \in \mathfrak{g}_{\bar{1}}$ and $[x, x] = y \neq 0$, then the super Jacobi axiom says $[x, y] = [x, [x, x]] = [[x, x], x] - [x, [x, x]]$. Applying super anticommutativity yields $[x, y] = 0$. The multiplication table for this Lie superalgebra is presented in Figure 3. Lie superalgebras isomorphic to this one are referred to as *of type* $\mathbf{q}(1)$.

$[\cdot,\cdot]$	y	x
y	x	0
x	0	0

Figure 4.1: Multiplication table for $\mathbf{q}(1)$

4.3 Classical Lie Superalgebras

This section introduces a broad class of Lie superalgebras whose structure is governed by the theory of reductive algebraic groups. Later, we will see that the cohomology theory of these Lie superalgebras is also determined by the invariant theory, which behaves particularly nicely.

Definition 4.3.1. A *classical Lie superalgebra* is a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that there exists a reductive algebraic group $G_{\bar{0}}$ which acts on $\mathfrak{g}_{\bar{1}}$ which satisfies

1. $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G_{\bar{0}})$

2. The action of $G_{\bar{0}} \curvearrowright \mathfrak{g}_{\bar{1}}$ differentiates to yield the adjoint action $\mathfrak{g}_{\bar{0}} \curvearrowright \mathfrak{g}_{\bar{1}}$.

Example 4.3.2 (Lie superalgebra of type $\mathfrak{q}(n)$, as in [BKN10, §8.3]). We define a Lie superalgebra called $\mathfrak{q}(n)$ as a Lie subsuperalgebra of the special linear Lie superalgebra $\mathfrak{q}(n) \leq \mathfrak{sl}(n+1 \mid n+1)$.

$$\mathfrak{q}(n) = \left\{ \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right) \middle| A, B \in \mathrm{M}_{n\times n}(\mathbb{C}) \right\}$$

A quick computation shows that $\dim_s \mathfrak{q}(n) = (n^2 \mid n^2)$ (and therefore $\dim_{\mathbb{C}} \mathfrak{q}(n) = 2n^2$), $\mathfrak{q}(n)_{\bar{0}} \cong \mathfrak{gl}(n)$, $\mathfrak{q}(n)_{\bar{1}} \cong \mathfrak{gl}(n)$, and $\mathfrak{g}_{\bar{1}}$ is the adjoint representation of $\mathfrak{g}_{\bar{0}}$. In this way, $\mathfrak{q}(n)$ is a classical Lie superalgebra with $G_{\bar{0}} = \mathrm{GL}(n)$, and $G_{\bar{0}} \curvearrowright \mathfrak{g}_{\bar{1}}$ via conjugation, yielding the adjoint action of $\mathfrak{g}_{\bar{0}} \curvearrowright \mathfrak{g}_{\bar{1}}$.

Additionally, we may verify the Lie superalgebra of Example 3.2.5 Part 3 is indeed the classical Lie superalgebra described above. This follows by taking a basis of the form

$$x = \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array} \right) \text{ and } y = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array} \right)$$

and verifying that the multiplication table of Figure 3 is valid.

Convention / Inconsistency: Kac has a different (inequivalent) construction of Q(n)

4.4 Modules for Lie Superalgebras

As with any object in abstract algebra, we care not simply about Lie superalgebras on their own, but also about their actions on vector spaces. Because of the grading on $\mathcal{U}(\mathfrak{g})$ (introduced in Section 3.5), we require $\mathfrak{g}$-modules to be graded $M = M_{\bar{0}} \oplus M_{\bar{1}}$. With this requirement, the category of $\mathfrak{g}$-modules is no longer Abelian. In order to make use of the tools of homological algebra, we make use of the subcategory whose objects are $\mathfrak{g}$-modules and whose morphisms are *even* homomorphisms of $\mathfrak{g}$-modules. This subcategory is useful when the *parity change functor* Π is used, in which case all data contained in the catogory of $\mathfrak{g}$-modules may be recovered.

Definition 4.4.1. A module may be defined in the following three equivalent ways, each of which is useful in certain cases.

1. $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a graded module for the universal enveloping superalgebra $\mathcal{U}(\mathfrak{g})$ (to be defined in 3.5).

2. $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a graded complex vector space and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is an even homomorphism of vector spaces. The action is $g.v = \rho(g)(v)$.

3. $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a graded complex vector space and $\mathfrak{g}$ acts on V in a linear fashion such that the following conditions hold:

   (i) $g_1.(g_2.x) - (-1)^{\bar{g}_1\bar{g}_2} g_2.(g_1.x) = [g_1, g_2].x$

Definition 4.4.2. A *homomorphism* of $\mathfrak{g}$-modules $f : M \to N$ is a homogeneous linear map (i.e., $f \in \mathrm{Hom}(M, N)_{\bar{0}} \cup \mathrm{Hom}(M, N)_{\bar{1}}$) satisfying the following property:

$$f(g.m) = (-1)^{\bar{f}\cdot\bar{g}} g.f(m)$$

for $g \in \mathfrak{g}$, $m \in M$.

definition: This definition seems limited. I'll check it's actually the right one.

Unfortunately, the category of $\mathfrak{g}$-modules is not an Abelian category. We remedy this situation by considering the even subcategory, whose objects are $\mathfrak{g}$-modules and whose morphisms are even homomorphisms $\mathrm{Hom}_{\mathfrak{g}}(M, N)_{\bar{0}}$.

Proposition 4.4.3. *The category* $\mathrm{Mod}(\mathfrak{g})_{\bar{0}}$ *is an Abelian category.*

Definition 4.4.4. The *parity change functor* is a functor $\Pi : \mathrm{Mod}(\mathfrak{g}) \to \mathrm{Mod}(\mathfrak{g})$ which switches the grading of modules.

Proposition 4.4.5. $\mathfrak{gl}(M) = \mathfrak{gl}(\Pi(M))$.

The above proposition allows us to glean all information from $\mathrm{Mod}(\mathfrak{g})$ from $\mathrm{Mod}(\mathfrak{g})_{\bar{0}}$ in the following way.

Corollary. *If M is a $\mathfrak{g}$-module, $\Pi(M)$ is a $\mathfrak{g}$-module. Further, an odd homomorphism $M \to N$ is simply an even homomorphism $M \to \Pi(N)$.*

4.5 Universal Enveloping Superalgebras

When studying representations of an algebraic object G, it is useful to find a ring R whose modules correspond precisely to G-representations. This section is devoted to constructing the universal enveloping superalgebra $\mathcal{U}_s(\mathfrak{g})$ associated to a Lie superalgebra, such that $\mathfrak{g}$-modules

Definition 4.5.1. The *universal enveloping superalgebra* of a Lie superalgebra $\mathfrak{g}$ is an associative superalgebra $\mathcal{U}_s(\mathfrak{g})$ equipped with a morphism $i : \mathfrak{g} \to \mathcal{U}_s(\mathfrak{g})$ such that given any other Lie superalgebra $\mathcal{V}$ with a morphism $j : \mathfrak{g} \to \mathcal{V}$ there exists a unique homomorphism $\theta : \mathcal{U}_s(\mathfrak{g}) \to \mathcal{V}$ such that $j = \theta \circ i$.

Explicitly, a universal enveloping superalgebra may be obtained as a quotient of the tensor superalgebra[1] by the ideal generated by elements of the form $[x, y] - x \otimes y(-1)^{\bar{x}\bar{y}} y \otimes x$.

Proposition 4.5.2. *The following categories are equivalent:*

1. *The category of graded* $\mathcal{U}_s(\mathfrak{g})$*-modules in the sense of ring theory.*

2. *The category of* $\mathfrak{g}$*-modules in the sense of Section 3.4.*

[1]Simply the tensor algebra, with grading remembered.

Chapter 5

Relative Cohomology of Lie Superalgebras

5.1 Overview

Relative cohomology of Lie superalgebras generalizes the cohomology theory of Lie algebras in two ways. When both generalizations are utilized simultaneously, geometrically meaningful cohomology rings arise. This is in stark contrast to ordinary Lie algebra cohomology rings, which have Krull dimension zero and are indeed finite-dimensional vector spaces.

The first generalization is to consider Lie superalgebras rather than Lie algebras. The Koszul complex used to compute Lie superalgebra cohomology is nonzero in infinitely many degrees, potentially leading to cohomology rings of positive Krull dimension. Unfortunately, it was proved by Fuks-Leites that this is rarely the case [FL84].

The second generalization is to consider cohomology *relative* to a subsuperalgebra. Remarkably, in Lie superalgebra theory, relative cohomology often times yields cohomology groups that are larger than their absolute counterparts. Relative cohomology of Lie algebras was first considered by Fuks [Fuk86], and fits into the relative cohomology theory of Hochschild [Hoc56], considered in Appendix B.

5.2 Koszul Complex

Let $\mathfrak{g}$ be a Lie superalgebra, $\mathfrak{a} \leq \mathfrak{g}$ a subsuperalgebra, and M a $\mathfrak{g}$-supermodule. The p^{th} *cochain of* $(\mathfrak{g}, \mathfrak{a})$ *with coefficients in* M is the $\mathfrak{a}$-module

$$\mathrm{C}^p(\mathfrak{g}, \mathfrak{a}; M) = \mathrm{Hom}_{\mathfrak{a}}\left(\bigwedge\nolimits_s^p(\mathfrak{g}/\mathfrak{a}), M\right)$$

The *coboundary map* $d : \mathrm{C}^p(\mathfrak{g}, \mathfrak{a}; M) \to \mathrm{C}^{p+1}(\mathfrak{g}, \mathfrak{a}; M)$ is defined by

$$df(\omega_0 \wedge \ldots \wedge \omega_n) = \sum_{i=0}^{n} (-1)^{\tau_i(\bar{\omega}_0, \ldots, \bar{\omega}_n, \bar{f})} \omega_i . f(\omega_0 \wedge \ldots \hat{\omega}_i \ldots \wedge \omega_n)$$
$$+ \sum_{i<j} (-1)^{\sigma_{i,j}(\bar{\omega}_0, \ldots, \bar{\omega}_n)} f([\omega_i, \omega_j] \wedge \omega_0 \wedge \ldots \hat{\omega}_i \ldots \hat{\omega}_j \ldots \wedge \omega_n)$$

where parities τ_i and $\sigma_{i,j}$ follow the formulae

$$\tau_i(\alpha_0, \ldots, \alpha_n, \beta) = i + \alpha_i(\alpha_0 + \ldots + \alpha_{i-1} + \beta)$$
$$\sigma_{i,j}(\alpha_0, \ldots, \alpha_n) = i + j + \alpha_i \alpha_j + \alpha_i(\alpha_0 + \ldots + \alpha_{i-1}) + \alpha_j(\alpha_0 + \ldots + \alpha_{j-1})$$

Composing these maps yields a diagram:

$$\ldots \xrightarrow{d} \mathrm{C}^{p-1}(\mathfrak{g}, \mathfrak{a}; M) \xrightarrow{d} \mathrm{C}^p(\mathfrak{g}, \mathfrak{a}; M) \xrightarrow{d} \mathrm{C}^{p+1}(\mathfrak{g}, \mathfrak{a}; M) \xrightarrow{d} \ldots \tag{5.2.1}$$

Proposition 5.2.1. *Let $\mathfrak{g}$ be a Lie superalgebra, $\mathfrak{a} \leq \mathfrak{g}$ a submodule, and M a $\mathfrak{g}$-module. The morphism*

$$d \circ d : \mathrm{H}^{p-1}(\mathfrak{g}, \mathfrak{a}; M) \to \mathrm{H}^{p+1}(\mathfrak{g}, \mathfrak{a}; M)$$

is equal to zero. In other words, Diagram 4.2.1 is a complex.

Definition 5.2.2. Let $\mathfrak{g}$ be a Lie superalgebra, $\mathfrak{a} \leq \mathfrak{g}$ a subsuperalgebra, and M a $\mathfrak{g}$-supermodule. The p^{th} *cohomology group of* $(\mathfrak{g}, \mathfrak{a})$ *with coefficients in* M is the $\mathfrak{a}$-module

$$\mathrm{H}^p(\mathfrak{g}, \mathfrak{a}; M) = \frac{\ker\left(d : \mathrm{C}^p(\mathfrak{g}, \mathfrak{a}; M) \to \mathrm{C}^{p+1}(\mathfrak{g}, \mathfrak{a}; M)\right)}{\mathrm{im}\left(d : \mathrm{C}^{p-1}(\mathfrak{g}, \mathfrak{a}; M) \to \mathrm{C}^p(\mathfrak{g}, \mathfrak{a}; M)\right)}$$

5.3 Products on Cochains and Cohomology

Consider modules M_1, M_2, and N, with a pairing, i.e., a map of $\mathfrak{g}$-modules $m : M_1 \otimes M_2 \to N$. Cochains may be paired

$$C^p(\mathfrak{g}, \mathfrak{a}; M_1) \otimes C^q(\mathfrak{g}, \mathfrak{a}; M_2) \to C^{p+q}(\mathfrak{g}, \mathfrak{a}; N)$$

by making use of the super anaologue of ordinary Grassmann multiplication $\mu : \bigwedge_s^p(\mathfrak{g}/\mathfrak{a}) \otimes \bigwedge_s^q(\mathfrak{g}/\mathfrak{a}) \to \bigwedge_s^{p+q}(\mathfrak{g}/\mathfrak{a})$ as follows:

$$\begin{aligned} C^p(\mathfrak{g},\mathfrak{a};M_1) \otimes C^q(\mathfrak{g},\mathfrak{a};M_2) &\cong \mathrm{Hom}_{\mathfrak{a}}\left(\bigwedge\nolimits_s^p(\mathfrak{g}/\mathfrak{a}), M_1\right) \otimes \mathrm{Hom}_{\mathfrak{a}}\left(\bigwedge\nolimits_s^q(\mathfrak{g},\mathfrak{a}), M_2\right) \\ &\to \mathrm{Hom}_{\mathfrak{a}}\left(\bigwedge\nolimits_s^p(\mathfrak{g}/\mathfrak{a}) \otimes \bigwedge\nolimits_s^q(\mathfrak{g}/\mathfrak{a}), M_1 \otimes M_2\right) \\ &\to \mathrm{Hom}_{\mathfrak{a}}\left(\bigwedge\nolimits_s^{p+q}(\mathfrak{g}/\mathfrak{a}), N\right) \\ &= C^{p+q}(\mathfrak{g},\mathfrak{a};N) \end{aligned} \tag{5.3.1}$$

We will be most interest in the case $M_1 = M_2 = N = \mathbb{C}$ and $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ is ordinary multiplication. Of secondary interest is the case when $M_1 = M^*$, $M_2 = M$ and $N = M * \otimes M$ with pairing given by the natural action $\gamma \otimes x \mapsto \gamma(x)$.

This pairing of cochains descends to a well-defined pairing of cohomology groups

$$\mathrm{H}^p(\mathfrak{g},\mathfrak{a};M_1) \otimes \mathrm{H}^q(\mathfrak{g},\mathfrak{a};M_2) \to H^{p+q}(\mathfrak{g},\mathfrak{a};N) \tag{5.3.2}$$

which leads to the following definition and theorem.

Definition 5.3.1. The *cohomology ring* of $\mathfrak{g}$ relative to $\mathfrak{a}$ is the set

$$\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{a};\mathbb{C}) = \bigoplus_{p\geq 0} \mathrm{H}^p(\mathfrak{g},\mathfrak{a};\mathbb{C})$$

Theorem 5.3.2. *Let $\mathfrak{g}$ be a Lie superalgebra and $\mathfrak{a} \leq \mathfrak{g}$ a subsuperalgebra. The morphism of Equation 4.3.2*

$$\mathrm{H}^p(\mathfrak{g},\mathfrak{a};M_1) \otimes \mathrm{H}^q(\mathfrak{g},\mathfrak{a};M_2) \to \mathrm{H}^{p+q}(\mathfrak{g},\mathfrak{a};N)$$

defines a ring structure on $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{a};M)$. Furthermore, this ring is graded-commutative, meaning that for homogeneous elements $\alpha,\beta \in \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{a};\mathbb{C})$, $\alpha\cdot\beta = (-1)^{\bar{\alpha}\cdot\bar{\beta}}\beta\cdot\alpha$.

5.4 Classical Results on the Relative Cohomology of Lie Superalgebras

This section is devoted to presenting two theorems which describe the bahavior of relative cohomology at extreme values of $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$. Namely, the result of Fuks-Leites states that cohomology relative to $\mathfrak{a} = 0$ contains very little geometric information. In other words, the cohomology ring is a finite-dimensional vector space. The result of Boe-Kujawa-Nakano states that cohomology relative to $\mathfrak{a} = \mathfrak{g}_{\bar{0}}$ carries geometric information and the behavior of this cohomology ring is governed by invariant theory.

Theorem 5.4.1 (Fuks-Leites, [Fuk86, §2.6]). *There are ring isomorphisms relating Lie superalgebra cohomology to Lie algebra cohomology, from which it follows that the Lie superalgebra cohomology is finite-dimensional as a vector space.*

(a)

$$\mathrm{H}^\bullet(\mathfrak{gl}(m|n), 0; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{gl}(\max(m,n)), 0; \mathbb{C})$$

(b)

$$\mathrm{H}^\bullet(\mathfrak{osp}(m|2n), 0; \mathbb{C}) \cong \begin{cases} \mathrm{H}^\bullet(\mathfrak{o}(m), 0; \mathbb{C}) \ \textit{if } m \geq 2n \\ \mathrm{H}^\bullet(\mathfrak{sp}(2n), 0; \mathbb{C}) \ \textit{if } m < 2n \end{cases}$$

A similar statement holds for Lie superalgebras of type $G(3)$, $F(4)$, and $D(2,1;\alpha)$

Chapter 6

Finite Generation of Relative Cohomology

6.1 Motivation

In this chapter, we will prove the following theorem.

Theorem 6.1.1. *Let* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *be a classical Lie superalgebra, and* $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ *an (even) subalgebra, and* M *a* $\mathfrak{g}$*-module.*

(a) *There is a spectral sequence* $\{E_r^{p,q}\}$ *which computes cohomology and satisfies*

$$E_2^{p,q}(M) \cong \mathrm{H}^p(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M) \otimes \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathbb{C}) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{g}, \mathfrak{a}; M)$$

For $1 \leq r \leq \infty$*,* $E_r^{\bullet,\bullet}(M)$ *is a module for* $E_2^{\bullet,\bullet}(\mathbb{C})$*. When* M *is finite-dimensional,* $E_2^{\bullet,\bullet}(M)$ *is a Noetherian* $E_2^{\bullet,\bullet}(\mathbb{C})$*-module.*

(b) *Moreover, the cohomology ring* $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ *is a finitely-generated* $\mathbb{C}$*-algebra.*

6.2 Filtration on Cochains

Let $\mathfrak{g}$ be a classical Lie superalgebra and $\mathfrak{a} \leq \mathfrak{g}$ any Lie subsuperalgebra. Recall the cochains are defined by

$$C^n(\mathfrak{g}, \mathfrak{a}; M) = \mathrm{Hom}_{\mathfrak{a}}\left(\bigwedge\nolimits_s^n(\mathfrak{g}/\mathfrak{a}), M\right)$$

Because $\mathfrak{a} \leq \mathfrak{g}$ is a subsuperalgebra, the equality

$$\mathfrak{g}/\mathfrak{a} \cong \mathfrak{g}_{\bar{0}}/\mathfrak{a}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}/\mathfrak{a}_{\bar{1}}$$

holds, allowing the cochains to be decomposed (as $\mathfrak{a}$-modules) as follows.

$$\begin{aligned}
C^n(\mathfrak{g},\mathfrak{a};M) &= \mathrm{Hom}_{\mathfrak{a}}\left(\textstyle\bigwedge_s^n(\mathfrak{g}/\mathfrak{a}), M\right) \\
&= \mathrm{Hom}_{\mathfrak{a}}\left(\textstyle\bigwedge_s^n\left(\mathfrak{g}_{\bar 0}/\mathfrak{a}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}/\mathfrak{a}_{\bar 1}\right), M\right) \\
&= \mathrm{Hom}_{\mathfrak{a}}\left(\bigoplus_{i+j=n}\textstyle\bigwedge_s^i\left(\mathfrak{g}_{\bar 0}/\mathfrak{a}_{\bar 0}\right)\otimes\bigwedge_s^j\left(\mathfrak{g}_{\bar 1}/\mathfrak{a}_{\bar 1}\right), M\right) \\
&= \bigoplus_{i+j=n}\mathrm{Hom}_{\mathfrak{a}}\left(\textstyle\bigwedge^i\left(\mathfrak{g}_{\bar 0}/\mathfrak{a}_{\bar 0}\right)\otimes S^j\left(\mathfrak{g}_{\bar 1}/\mathfrak{a}_{\bar 1}\right), M\right) \\
&= \bigoplus_{i+j=n}\mathrm{Hom}_{\mathfrak{a}}\left(\textstyle\bigwedge^i\left(\mathfrak{g}_{\bar 0}/\mathfrak{a}_{\bar 0}\right), S^j\left(\mathfrak{g}_{\bar 1}/\mathfrak{a}_{\bar 1}\right)\otimes M\right) \\
&= \bigoplus_{i+j=n} C^i\left(\mathfrak{g}_{\bar 0},\mathfrak{a}_{\bar 0}; S^j(\mathfrak{g}_{\bar 1}/\mathfrak{a}_{\bar 1})\otimes M\right)
\end{aligned} \tag{6.2.1}$$

Equation 5.2.1 expresses arbitrary superalternating functions as sums of superalternating functions with i arguments coming from $\mathfrak{g}_{\bar 0}/\mathfrak{a}_{\bar 0}$, and j arguments coming from $\mathfrak{g}_{\bar 1}/\mathfrak{a}_{\bar 1}$.

Our filtration is inspired by that of [HS53], and corresponds to limiting the number of arguments that may come from $\mathfrak{g}_{\bar 0}/\mathfrak{a}_{\bar 0}$. Explicitly, define

$$C^n(\mathfrak{g},\mathfrak{a};M)_{(p)} = \bigoplus_{\substack{i+j=n\\ i\le n-p}} C^i\left(\mathfrak{g}_{\bar 0},\mathfrak{a}_{\bar 0}; S^j(\mathfrak{g}_{\bar 1}/\mathfrak{a}_{\bar 1})\otimes M\right) \tag{6.2.2}$$

This defines a descending filtration

$$\begin{aligned}
C^n(\mathfrak{g},\mathfrak{a};M) = C^n(\mathfrak{g},\mathfrak{a};M)_{(0)} \supseteq C^n(\mathfrak{g},\mathfrak{a};M)_{(1)} \supseteq \dots \\
\dots \supseteq C^n(\mathfrak{g},\mathfrak{a};M)_{(n)} \supseteq C^n(\mathfrak{g},\mathfrak{a};M)_{(n+1)} = 0
\end{aligned} \tag{6.2.3}$$

This filtration satisfies some basic desired properties.

Proposition 6.2.1. *Let $\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$ be a Lie superalgebra, $\mathfrak{a}\le\mathfrak{g}_{\bar 0}$ an even subalgebra, M a $\mathfrak{g}$-module, and $C^n(\mathfrak{g},\mathfrak{a};M)_{(p)}$ the filtration defined in Equation 5.2.2.*

(a) *This grading respects the differential, i.e., $d(C^n(\mathfrak{g},\mathfrak{a};M)_{(p)}) \subseteq C^{n+1}(\mathfrak{g},\mathfrak{a};M)_{(p)}$, and thus $C^\bullet(\mathfrak{g},\mathfrak{a};M)_{(p)}$ is a subcomplex of $C^\bullet(\mathfrak{g},\mathfrak{a};M)$ for all p.*

(b) *$C^n(\mathfrak{g},\mathfrak{a};M)_{(p)}$ is an $\mathfrak{a}$-submodule of $C^n(\mathfrak{g},\mathfrak{a};M)$, so $C^\bullet(\mathfrak{g},\mathfrak{a};M)_{(p)}$ is a subcomplex of $\mathfrak{a}$-modules.*

(c) *The filtration is exhaustive, i.e.,* $C^\bullet(\mathfrak{g},\mathfrak{a};M)_{(0)} = C^\bullet(\mathfrak{g},\mathfrak{a};M)$ *and* $\bigcap_{p\geq 0} C^\bullet(\mathfrak{g},\mathfrak{a};M)_{(p)} = 0$.

Proof. (a) Let $f \in C^n(\mathfrak{g},\mathfrak{a};M)_{(p)}$. This means f vanishes when more than $n-p$ arguments belong to $\mathfrak{g}_{\bar{0}}/\mathfrak{a}$. We wish to show that $df \in C^{n+1}(\mathfrak{g},\mathfrak{a};M)_{(p)}$, i.e., that df vanishes when more than $n-p+1$ arguments belong to $\mathfrak{g}_{\bar{0}}/\mathfrak{a}$. Let $\alpha_0,\dots,\alpha_{n-p+1} \in \mathfrak{g}_{\bar{0}}/\mathfrak{a}$, while $\beta_{n-p+2},\dots,\beta_n \in \mathfrak{g}_{\bar{1}}$. Plugging these into the coboundary formula

$$\begin{aligned}
df(\alpha_0\wedge\ldots\wedge\beta_n) = &\sum_{0\leq i\leq n-p+1} (-1)^{\tau_i(-)}\alpha_i.f(\alpha_0\wedge\ldots\hat{\alpha}_i\ldots\wedge\beta_n)\\
&+\sum_{n-p+2\leq i\leq n} (-1)^{\tau_i(-)}\beta_i.f(\alpha_0\wedge\ldots\hat{\beta}_i\ldots\wedge\beta_n)\\
&+\sum_{0\leq i<j\leq n-p+1} (-1)^{\sigma_{i,j}(-)}f([\alpha_i,\alpha_j]\wedge\alpha_0\ldots\hat{\alpha}_i\ldots\hat{\alpha}_j\ldots\wedge\beta_n)\\
&+\sum_{\substack{0\leq i\leq n-p+1\\ n-p+2\leq j\leq n}} (-1)^{\sigma_{i,j}(-)}f([\alpha_i,\beta_j]\wedge\alpha_0\ldots\hat{\alpha}_i\ldots\hat{\beta}_j\ldots\wedge\beta_n)\\
&+\sum_{n-p+2\leq i<j\leq n} (-1)^{\sigma_{i,j}(-)}f([\beta_i,\beta_j]\wedge\alpha_0\ldots\hat{\beta}_i\ldots\hat{\beta}_j\ldots\wedge\beta_n)
\end{aligned}$$

Looking at each line of the previous equation, notice that f takes in, respectively, $n-p+1$, $n-p+2$, $n-p+1$, $n-p+2$, and $n-p+3$ arguments lying in $\mathfrak{g}_{\bar{0}}/\mathfrak{a}$. Thus each term in each summation individually vanishes. Thus we conclude $df \in C^{n+1}(\mathfrak{g},\mathfrak{a};M)_{(p)}$.

(b) Let $x \in \mathfrak{a}$, $f \in C^n(\mathfrak{g},\mathfrak{a};M)_{(p)}$. Thus $f(\omega_0\wedge\ldots\omega_{n-1})$ vanishes when $n-p+1$ of the ω_i belong to $\mathfrak{a}$. writing out the definition of $(x.f)(\omega_0\wedge\ldots\wedge\omega_{n-1})$ we realize that each term vanishes when $n-p+1$ of the ω_i belong to $\mathfrak{a}$, and thus $x.f \in C^n(\mathfrak{g},\mathfrak{a};M)_{(p)}$.

(c) This follows from writing out the definitions and noting $C^n(\mathfrak{g},\mathfrak{a};M)_{(n+1)} = 0$

□

Because of the properties established in Proposition 5.2.1,

$$E_r^{p,q} \Rightarrow \mathrm{H}\left(C^\bullet(\mathfrak{g},\mathfrak{a};M)\right) = \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{a};\mathbb{C}) \tag{6.2.4}$$

which computes cohomology.

Decide: Do I need a chapter on spectral sequences?

6.3 Pages of the Spectral Sequence

This section is devoted to investigating the pages of the spectral sequence defined by Equation 5.2.4, and the necessary information is summarized in the following lemma.

Proposition 6.3.1. *The first three pages of the spectral sequence associated to the filtration of Equation 5.2.4 may be identified as follows.*

(a) $E_0^{p,q} \cong C^q\left(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathbb{C}}\left(\bigwedge_s^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M\right)\right)$,

(b) $E_1^{p,q} \cong \mathrm{H}^q\left(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathbb{C}}\left(\bigwedge_s^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M\right)\right)$,

(c) $E_2^{p,q} \cong \mathrm{H}^p(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M) \otimes \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathbb{C})$.

The proof of Proposition 5.3.1 requires the following lemma.

Lemma 6.3.2. *Let $\mathfrak{g}_{\bar{0}}$ be a reductive Lie algebra, M be a finite-dimensional semisimple $\mathfrak{g}_{\bar{0}}$-module such that $M^{\mathfrak{g}_{\bar{0}}} = 0$. Then $\mathrm{H}^n(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; M) = 0$ for all $n \geq 0$.*

Proof of Lemma 5.3.2. Suppose M is simple and that $n \geq 0$. The group $Z^n(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; M) \subseteq C^n(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; M) \subseteq C^n(\mathfrak{g}_{\bar{0}}; M)$ is semisimple. The group $d(C^{n-1}(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; M))$ is a submodule of $Z^n(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; M)$, and as such there exists a $\mathfrak{g}_{\bar{0}}$-module complement V so that $Z^n(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; M) = d(C^{n-1}(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; M)) \oplus V$. We notice that $\mathfrak{g}_{\bar{0}}.Z^n(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; M) \subseteq d(C^{n-1}(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; M))$, meaning $\mathfrak{g}_{\bar{0}}.V = 0$. Thus it suffices to show that every cocycle which is annihilated by $\mathfrak{g}_{\bar{0}}$ is a coboundary.

Since $\mathfrak{g}_{\bar{0}}$ is reductive we may write $\mathfrak{g}_{\bar{0}} = [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] \oplus \mathfrak{z}$ where $\mathfrak{z}$ denotes the center of $\mathfrak{g}_{\bar{0}}$. Since M is simple either $\mathfrak{z}.M = 0$ or no non-zero element of M is annihilated by Z. Let f be a cocycle which is annihilated by $\mathfrak{g}_{\bar{0}}$, let $z \in \mathfrak{z}$, and let $\omega_1, \ldots, \omega_n \in \mathfrak{g}_{\bar{0}}$. Then $0 = (z.f)(\omega_1 \wedge \ldots \wedge \omega_n) = z.f(\omega_1 \wedge \ldots \wedge \omega_n)$. Thus if $\mathfrak{z}.M \neq 0$, it follows that $f = 0$. Now we may suppose $\mathfrak{z}.M = 0$ and $M \neq 0$.

Let C be the annihilator in $\mathfrak{g}_{\bar{0}}$ of M, so that $C \supseteq Z$. Since the invariant submodule $M_{\bar{0}}^{\mathfrak{g}} = 0$, it must be the case $C \neq \mathfrak{g}_{\bar{0}}$. Now $C \cap [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$ is an ideal in the semisimple Lie algebra $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$, meaning there must be a complementary ideal S. Of course, S is a non-zero semisimple ideal of $\mathfrak{g}_{\bar{0}}$, which may be decomposed as $\mathfrak{g}_{\bar{0}} = S \oplus C$. Now $M{\downarrow_s}$ is simple and the representation of S is one-to-one. Thus the Casimir operator of this representation, Γ, is an automorphism of M which commutes with all $\mathfrak{g}_{\bar{0}}$-operators on M. Furthermore, since $[S, C] = 0$, it is seen that for any relative cocycle f, $\Gamma \circ f = dg$ is a coboundary. Hence $f = \Gamma^{-1} \circ dg = d(\Gamma^{-1} \circ g)$ as desired. □

With Lemma 5.3.2 established, we are now ready to prove Proposition 5.3.1.

Proof of Proposition 5.3.1. We proceed in steps, identifying the pages in sequence.

(a) By definition, $E_0^{p,q} = C^{p+q}(\mathfrak{g}, \mathfrak{a}; M)_{(p)}/C^{p+q}(\mathfrak{g}, \mathfrak{a}; M)_{(p+1)}$. Using the direct sum decomposition of Equation 5.2.2, this is exactly $C^q\left(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathbb{C}}\left(\bigwedge_s^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M\right)\right)$.

(b) Functoriality of the isomorphism of (a), i.e., $E_0^{p,\bullet} \cong C^\bullet(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathbb{C}}(S^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M))$ as complexes will imply their cohomologies are equal, i.e., $E_1^{p,q} \cong \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathbb{C}}(S^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M))$.

To deduce functoriality of the isomorphism it will suffice to chase the following diagram.

$$\begin{tikzcd}
C^{p+q}(\mathfrak{g}, \mathfrak{a}; M)_{(p)} \arrow[r, "d_{(\mathfrak{g},\mathfrak{a})}"] \arrow[d, two heads] & C^{p+q+1}(\mathfrak{g}, \mathfrak{a}; M)_{(p)} \arrow[d, two heads] \arrow[dd, bend left, "\pi"] \\
E_0^{p,q} \arrow[r, "d_0"] \arrow[d, "\cong"] & E_0^{p,q+1} \arrow[d, "\cong"] \\
C^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathbb{C}}(S^p(\mathfrak{g}_{\bar{1}}), M)) \arrow[r, "d_{(\mathfrak{g}_{\bar{0}},\mathfrak{a})}"] \arrow[uu, bend left, "i"] & C^{q+1}(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathbb{C}}(S^p(\mathfrak{g}_{\bar{1}}), M))
\end{tikzcd}$$

With section i corresponding to the direct sum decomposition given in Equation 5.2.1. The goal is to show the composition $\pi \circ d_{(\mathfrak{g},\mathfrak{a})} \circ i = d_{(\mathfrak{g}_{\bar{0}},\mathfrak{a})}$. Since d_0 is defined by $d_{(\mathfrak{g},\mathfrak{a})}$, this will show the bottom square commutes, resulting in an isomorphism of complexes.

Choose $f \in C^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathbb{C}}(S^p(\mathfrak{g}_{\bar{1}}), M))$, and notice that df is given by usual Lie algebra differential

$$df(\omega_0 \wedge \ldots \omega_q) = \sum_{i=0}^{q} (-1)^i \omega_i . f(\omega_0 \wedge \ldots \hat{\omega}_i \ldots \wedge \omega_q) + \sum_{i<j} (-1)^{i+j} f([\omega_i, \omega_j] \wedge \omega_0 \wedge \ldots \hat{\omega}_i \ldots \hat{\omega}_j \ldots \wedge \omega_q)$$

Set $\tilde{f} = i(f) \in C^{p+q}(\mathfrak{g}, \mathfrak{a}; M)$. The differential is given by the Lie superalgebra cohomology differential, and we arrive at a formula for $d_{(\mathfrak{g},\mathfrak{a})} f(\omega_0 \wedge \ldots \wedge \omega_{p+q})$. However, because we are taking a quotient π, it only matters how $d_{(\mathfrak{g},\mathfrak{a})}\tilde{f}$ behaves with $q+1$ even arguments and p odd arguments. Thus we investigate

$$\begin{aligned}
d_{(\mathfrak{g},\mathfrak{a})}f(\alpha_0\wedge\ldots\wedge\alpha_q\wedge\beta_1\wedge\ldots\wedge\beta_p) &= \sum_{i=0}^{q}(-1)^{\tau_i(-)}\alpha_i.\tilde{f}(\alpha_0\wedge\ldots\hat{\alpha}_0\ldots\wedge\alpha_q\wedge\beta_1\wedge\ldots\wedge\beta_p)\\
&+\sum_{i=q+1}^{p+q}(-1)^{\tau_i(-)}\beta_{i-q}.f(\alpha_0\wedge\ldots\wedge\alpha_q\wedge\beta_1\wedge\ldots\hat{\beta}_{i-q}\ldots\wedge\beta_p)\\
&+\sum_{0\le i<j\le q}(-1)^{\sigma_{i,j}(-)}f([\alpha_i,\alpha_j]\wedge\alpha_0\ldots\hat{\alpha}_i\ldots\hat{\alpha}_j\ldots\beta_p)\\
&+\sum_{\substack{0\le i\le q\\ q+1\le j\le p+q}}(-1)^{\sigma_{i,j}(-)}f([\alpha_i,\beta_{j-q}]\wedge\alpha_0\ldots\hat{\alpha}_i\ldots\hat{\beta}_{j-q}\ldots\beta_p)\\
&+\sum_{q+1\le i<j\le p+q}(-1)^{\sigma_{i,j}(-)}f([\beta_{i-q},\beta_{j-q}]\wedge\alpha_0\ldots\hat{\beta}_i\ldots\hat{\beta}_j\ldots\wedge\beta_p)
\end{aligned}$$

By construction, $\tilde{f}$ vanishes unless exactly q arguments are even and p arguments are odd. This only occurs in the first, third, and fourth lines of the preceding sum. Working out the relevant signs yields

$$\tau_i(\underbrace{\bar{0},\ldots,\bar{0}}_{q+1},\underbrace{\bar{1},\ldots,\bar{1}}_{p},\bar{f}) = i \text{ when } i\le q$$

$$\sigma_{i,j}(\underbrace{\bar{0},\ldots,\bar{0}}_{q+1},\underbrace{\bar{1},\ldots,\bar{1}}_{p}) = \begin{cases} i+j & \text{if } i,j\le q\\ i-q-1 & \text{if } i\le q, j\ge q+1\end{cases}$$

So the previous equation for $d_{(\mathfrak{g},\mathfrak{a})}\tilde{f}$ becomes

$$\begin{aligned}
d_{(\mathfrak{g},\mathfrak{a})}f(\alpha_0\wedge\ldots\wedge\alpha_q\wedge\beta_1\wedge\ldots\wedge\beta_p) &= \sum_{i=0}^{q}(-1)^{i}\alpha_i.\tilde{f}(\alpha_0\wedge\ldots\hat{\alpha}_0\ldots\wedge\alpha_q\wedge\beta_1\wedge\ldots\wedge\beta_p)\\
&+\sum_{0\le i<j\le q}(-1)^{i+j}f([\alpha_i,\alpha_j]\wedge\alpha_0\ldots\hat{\alpha}_i\ldots\hat{\alpha}_j\ldots\beta_p)\\
&-\sum_{\substack{0\le i\le q\\ q+1\le j\le p+q}}(-1)^{i}f(\alpha_0\ldots\hat{\alpha}_i\ldots\wedge\alpha_q\wedge[\alpha_i,\beta_{j-q}]\wedge\beta_1\ldots\hat{\beta}_{j-q}\ldots\beta_p)
\end{aligned}$$

Now if we compute $d_{(\mathfrak{g}_{\bar{0}},\mathfrak{a})}f$, accounting for the action on $\mathrm{Hom}_{\mathbb{C}}(S^p(\mathfrak{g}_{\bar{1}}),M)$, we arrive at the same formula.

(c) Notice first that by semisimplicity $\mathrm{Hom}_{\mathbb{C}}(S^n(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M) \cong \mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}(S^n(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M) \oplus V$ where V is some complement with $V^{\mathfrak{g}_{\bar{0}}} = 0$. By the lemma,

$$E_1^{p,q} \cong \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}(S^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M)) \oplus \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; V) = \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}(S^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M)).$$

Because $\mathfrak{g}_{\bar{0}}$ acts trivially on $\mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}(S^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M)$, we may conclude that $E_1^{p,q} \cong \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathbb{C}) \otimes \mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}(S^p(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M)$. This association is functorial, i.e., induces an isomorphism $E_1^{\bullet,q} \cong \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathbb{C}) \otimes \mathrm{Hom}_{\mathbb{C}}(S^\bullet(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M)$ as complexes. Therefore, we may conclude that $E_2^{p,q} \cong \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathbb{C}) \otimes \mathrm{H}^p(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M)$.

This completes the proof of Proposition 5.3.1. □

6.4 Proof of Finite Generation

Recall the statement of Theorem 5.1.1:

Numbering: How do I restate a Theorem the right way?

Theorem 6.4.1. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra, and $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ an (even) subalgebra, and M a $\mathfrak{g}$-module.*

(a) *There is a spectral sequence $\{E_r^{p,q}\}$ which computes cohomology and satisfies*

$$E_2^{p,q}(M) \cong \mathrm{H}^p(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M) \otimes \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathbb{C}) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{g}, \mathfrak{a}; M)$$

For $1 \leq r \leq \infty$, $E_r^{\bullet,\bullet}(M)$ is a module for $E_2^{\bullet,\bullet}(\mathbb{C})$. When M is finite-dimensional, $E_2^{\bullet,\bullet}(M)$ is a Noetherian $E_2^{\bullet,\bullet}(\mathbb{C})$-module.

(b) *Moreover, the cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ is a finitely-generated $\mathbb{C}$-algebra.*

Proof. In fact, the all that is left to show is that for M finite-dimensional, $E_2^{\bullet,\bullet}(M)$ is a Noetherian $E_2^{\bullet,\bullet}(\mathbb{C})$-module, and that (b) follows from (a). The $E_2^{p,q}(M)$-page identification appears in Proposition 5.3.1 of the previous section.

As such, let M be a finite-dimensional $\mathfrak{g}$-module. $E_2^{\bullet,\bullet}(M)$ is a Noetherian $S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}$-module via the map

$$S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}} \hookrightarrow E_2^{\bullet,0}(\mathbb{C}) \subseteq E_2^{\bullet,\bullet}(\mathbb{C}).$$

$E_\infty^{\bullet,\bullet}(M)$, being a section of $E_2^{\bullet,\bullet}(M)$ is a Noetherian $S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}$-module via the map

$$S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{\mathfrak{g}_{\bar{0}}} \to E_\infty^{\bullet,0}(\mathbb{C}) \subseteq E_\infty^{\bullet,\bullet}(\mathbb{C}).$$

Consequently, $E_\infty^{\bullet,\bullet}(M)$ is a Noetherian $E_\infty^{\bullet,\bullet}(\mathbb{C})$-module.

That the cohomology ring is finitely generated follows from this: Because $\mathbb{C}$ is a Noetherian $\mathfrak{g}$-module, $E_\infty^{\bullet,\bullet}(\mathbb{C})$ is a Noetherian $E_\infty^{\bullet,\bullet}(\mathbb{C})$-module, however $E_\infty^{\bullet,\bullet}(\mathbb{C}) = \mathrm{Gr}(\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{a};\mathbb{C}))$, the associated graded module of the cohomology ring. Because Noetherian associated graded modules come from Noetherian modules, we may conclude that $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{a};\mathbb{C})$ is a Noetherian $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{a};\mathbb{C})$-module, and the cohomology ring is therefore finitely generated. □

We conclude this section by identifying the edge homomorphsim of the spectral sequence as the natural restriction morphism

$$\mathrm{res} : \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_{\bar{0}};M) \to \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{a};M) \tag{6.4.1}$$

Which, in the case $M = \mathbb{C}$, makes $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{a};\mathbb{C})$ into an integral extension of a quotient of $\mathrm{H}^\bullet(\mathfrak{g},g_{\bar{0}};\mathbb{C})$.

Proposition 6.4.2. *The edge homomorphism of the spectral sequence corresponds to the natural restriction homomorphism of cohomology rings.*

Proof. The restriction map $C^n(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C}) \xrightarrow{\mathrm{res}} C^n(\mathfrak{g},\mathfrak{a};\mathbb{C})$ induces a map on cohomology $\mathrm{H}^n(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C}) \xrightarrow{\mathrm{res}^*} \mathrm{H}^n(\mathfrak{g},\mathfrak{a};\mathbb{C})$. Because restriction repects the filtration of Section 5.2, the map res^* will respect the induced filtration on cohomology, i.e., $F^p\,\mathrm{H}^n(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C}) \xrightarrow{\mathrm{res}^*} F^p\,\mathrm{H}^n(\mathfrak{g},\mathfrak{a};\mathbb{C})$. This descends to a map on the associated graded of each cohomology ring, which may be precomposed with the projection onto associated graded as follows

$$\mathrm{H}^n(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C}) \to \mathrm{Gr}\,(\mathrm{H}^n(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C})) \to \mathrm{Gr}\,(\mathrm{H}^n(\mathfrak{g},\mathfrak{a};\mathbb{C}))$$

□

Corollary. *When* $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$,

$$\dim_{Kr} \mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{a}; \mathbb{C}) \leq \dim_{Kr} \mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$$

Proof. The corollary follows from the fact that an integral extension has Krull dimension no greater than the base, and quotients can have smaller Krull dimension. $\square$

6.5 A Cohomology Ring of Intermediate Dimension

In many instances, Lie superalgebra cohomology $\mathrm{H}^{\bullet}(\mathfrak{g}; \mathbb{C}) = \mathrm{H}^{\bullet}(\mathfrak{g}, 0; \mathbb{C})$ will vanish in all but finitely many degrees (see [FL84] or [Gru97, Théorème 5.3]), leading one to conclude the ring has Krull dimension zero and thus uninteresting geometry. Here it is shown that for $\mathfrak{g} = \mathfrak{gl}(1|1)$ and $\mathfrak{a}$ generated by $\mathrm{diag}(1 \mid 1) \in \mathfrak{gl}(1|1)$, $\mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ is nonzero in infinitely many degrees. From this, we may conclude $\mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ has positive Krull dimension. This is an especially nice case; $\mathfrak{a}$ acts trivially on $\mathfrak{gl}(1|1)$ so every map $\bigwedge_s^n(\mathfrak{g}/\mathfrak{a}) \to \mathbb{C}$ is $\mathfrak{a}$-invariant.

Take the basis for $\mathfrak{gl}(1|1)/\mathfrak{a}$

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \beta_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \beta_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$\bigwedge_s^{2n}(\mathfrak{g}/\mathfrak{a})$ has basis $\{\alpha \otimes \beta_1^i \beta_2^j\}_{i+j+1=n} \cup \{\beta_1^i \beta_2^j\}_{i+j=n}$. Consider $f \in C^{2i}(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ which maps $\beta_1^n \beta_2^n$ to 1 and all other basis vectors to zero. Since $\mathbb{C}$ has the trivial action, df has the form

$$df(\omega_0 \wedge \ldots \wedge \omega_{2n}) = \sum_{i=0}^{p} (-1)^{\sigma_{i,j}(\bar{\omega}_0, \ldots, \bar{\omega}_{2n})} f([\omega_i, \omega_j] \wedge \omega_0 \wedge \ldots \hat{\omega}_i \ldots \hat{\omega}_j \ldots \wedge \omega_{2n})$$

By inspection, df will vanish on all basis vectors $\beta_1^i \beta_2^j$ and $df(\alpha \otimes \beta_1^i \beta_2^j) = (i-j) f(\beta_1^i \beta_2^j)$. This is 0 when $i, j \neq n$ by definition of f, and when $i = j = n$ this is zero because the coefficient vanishes. So f is a cocycle.

Suppose $dg = f$ for some $g \in C^{2n-1}(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$. Then we compute $dg(\beta_1^n \beta_2^n)$, which is a sum of terms of the form $(-1)^{\sigma_{i,j}(-)} g([\beta_k, \beta_l] \wedge \beta_1^{n_1} \wedge \beta_2^{n_2}$, each of which vanishes individually so that $dg(\beta_1^n \beta_2^n) = 0$.

Therefore, f is *not* a coboundary. So for every $n \geq 2$, $\mathrm{H}^{2n}(\mathfrak{g}, \mathfrak{a}; \mathbb{C}) \neq 0$. This shows that cohomology relative to an even subalgebra heuristically lies somewhere between the results of Fuks-Leites [FL84] and Boe-Kujawa-Nakano [BKN10].

6.6 Structure of Cohomology Rings

The spectral sequence of Section 5.2 allows us to investigate the properties of cohomology rings in certain cases. There are certain conditions on the spectral sequence that appear quite often and it is shown that these cohomology rings are particularly nicely behaved.

6.6.1 Cohen-Macaulay Cohomology Rings

The following theorem is motivated by [CN14, Proposition 3.1]. The reader should recall that an algebra A is *Cohen-Macaulay* if there is a polynomial subalgebra over which A is a finite and free module, see [Ben98, §5.4].

Proposition 6.6.1. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra, and $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ a subalgebra. If the spectral sequence constructed in Section* **??** *collapses at E_2 (i.e., if $E_2^{\bullet,\bullet}(\mathbb{C}) \cong E_\infty^{\bullet,\bullet}(\mathbb{C})$), then $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ is a Cohen-Macaulay ring.*

Proof. The spectral sequence $E_2^{\bullet,\bullet} = E_\infty^{\bullet,\bullet}$ is a filtered version of the cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$. As such, if $\zeta \in E_2^{i,j}$ and $\eta \in E_2^{r,s}$, then $\zeta \cdot \eta \in \sum_{\ell \geq 0} E_2^{i+r+\ell, j+s-\ell}$. Because of this, for any $m \geq 0$, the direct sum of the lowest m rows, denoted $U_m = \sum_{q \leq m} E_2^{\bullet,q}$, is a module for the bottom row $U_0 = E_0^{\bullet,0} \cong \mathrm{H}^0(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathbb{C}) \otimes S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}} \cong S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}$, which by [HR74] is a Cohen-Macaulay ring. Because the spectral sequence collapses, $E_2 = E_\infty$ and the quotients $U_m/U_{m-1} \cong \mathrm{H}^m(\mathfrak{g}_{\bar{0}}, \mathfrak{a}; \mathbb{C}) \otimes S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}$ are free $S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}$-modules. This means the quotient maps $U_m \to U_m/U_{m-1}$ split as maps of $S(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}$-modules and the proposition follows. $\square$

This example restricts to the case that cohomology of $\mathfrak{g}$ relative to $\mathfrak{g}_{\bar{0}}$ vanishes in odd degrees. While this may seem restrictive, [BKN10, Table 1] reveals that there are a great many classical Lie superalgebras whose cohomology lives in even degree.

Corollary. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra of type $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$, $\mathfrak{psl}(2n|2n)$, $\mathfrak{osp}(2m+1|2n)$, $\mathfrak{osp}(2m|2n)$, $P(4\ell-1)$, $D(2,1;\alpha)$, $G(3)$, or $F(4)$. Let $\mathfrak{l} \leq \mathfrak{g}_{\bar{0}}$ be a standard Levi subalgebra. The following hold:*

(a) $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{l}; \mathbb{C})$ *is a Cohen-Macaulay ring.*

(b) $\dim_{\mathrm{Kr}} \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{l}; \mathbb{C}) = \dim_{\mathrm{Kr}} S^\bullet(\mathfrak{g}_{\bar{1}}^*)$.

6.6.2 Krull Dimensions

In this section we present some applications in which we use the spectral sequence of Section **??** to compute Krull dimensions of cohomology rings in particularly nice cases. The reader should notice these results rely on deep results from representation theory in the relative Category $\mathcal{O}$ (cf. [Hum08, §8]).

Theorem 6.6.2. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra such that $S^\bullet(\mathfrak{g}_{\bar{1}})^{G_{\bar{0}}}$ vanishes in odd degrees, and $\mathfrak{l} \leq \mathfrak{g}_{\bar{0}}$ a standard Levi subalgebra (i.e., nonzero and generated by simple roots). The following hold.*

(a) *The spectral sequence of Section **??** collapses at the E_2 page and $E_2^{\bullet,\bullet}(\mathbb{C}) \cong E_\infty^{\bullet,\bullet}(\mathbb{C})$.*

(b) $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{l}; \mathbb{C})$ *is Cohen-Macualay,*

(c) $\dim_{\mathrm{Kr}} \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) = \dim_{\mathrm{Kr}} \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{l}; \mathbb{C})$.

Proof. We establish (a). Parts (b) and (c) follow by application of Proposition 5.6.1.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra such that $S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}$ is zero in odd degrees, and $\mathfrak{l} \leq \mathfrak{g}_{\bar{0}}$ a Levi subalgebra. According to the Kazhdan-Lusztig conjectures[1], $\mathrm{H}^\bullet(\mathfrak{g}_{\bar{0}}, \mathfrak{l}; \mathbb{C})$

[1]When $\mathfrak{h} \leq \mathfrak{g}_{\bar{0}}$ is a Cartan subalgebra, $\mathrm{Ext}^n_{\mathcal{O}}(M,N) \cong \mathrm{Ext}^n_{(\mathfrak{g}_{\bar{0}},\mathfrak{h})}(M,N)$ (see [Hum08, Theorem 6.15]). The fact that $\mathrm{Ext}^n_{(\mathfrak{g}_{\bar{0}},\mathfrak{h})}(\mathbb{C},\mathbb{C})$ vanishes in odd degrees follows from [CPS93].

is only nonzero in even degrees. Section 5.3 realizes the E_2 page of the Hochschild-Serre spectral sequence as

$$E_2^{p,q}(\mathbb{C}) \cong \mathrm{H}^q(\mathfrak{g}_{\bar{0}}, \mathfrak{l}; \mathbb{C}) \otimes S^p(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}.$$

Because the differential $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$ descends one row, either $E_2^{p,q} = 0$ or $E_2^{p+2,q-1} = 0$. In either case, $d_2 = 0$ and thus $E_3^{p,q} = E_2^{p,q}$ meaning that $E_3^{p,q}$ vanishes unless p and q are both even. By a similar argument, the differential $d_3 : E_3^{p,q} \to E_3^{p+3,q-2}$ must be zero since one of $E_3^{p,q}$ or $E_3^{p+3,q-2}$ will have odd horizontal coordinate and thus be zero. So $E_3^{p,q} \cong E_4^{p,q}$. By induction, this trend continues to arrive at the conclusion that $E_2^{p,q} \cong E_\infty^{p,q}$. This yields the following statement. $\square$

This example restricts to the case that cohomology of $\mathfrak{g}$ relative to $\mathfrak{g}_{\bar{0}}$ vanishes in odd degrees. While this may seem restrictive, [BKN10, Table 1] reveals that there are a great many classical Lie superalgebras whose cohomology lives in even degree.

Corollary. *Let* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *be a Lie superalgebra of type* $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$, $\mathfrak{psl}(2n|2n)$, $\mathfrak{osp}(2m+1|2n)$, $\mathfrak{osp}(2m|2n)$, $P(4\ell-1)$, $D(2,1;\alpha)$, $G(3)$, *or* $F(4)$. *Let* $\mathfrak{l} \leq \mathfrak{g}_{\bar{0}}$ *be a standard Levi subalgebra. The following hold:*

(a) $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{l}; \mathbb{C})$ *is a Cohen-Macaulay ring.*

(b) $\dim_{\mathrm{Kr}} \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{l}; \mathbb{C}) = \dim_{\mathrm{Kr}} S^\bullet(\mathfrak{g}_{\bar{1}}^*)$.

Chapter 7

Support Variety Theory

7.1 Motivation

The finite-generation result of Chapter 5 opens the door to use the powerful machinery of algebraic geometry when studying cohomology of classical Lie superalgebras relative to an even subsuperalgebra.

In this chapter, the cohomology variety $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})$ is defined, along with support varieties $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M) \subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})$ for each module M. Natural mappings of cohomology rings yield natural mappings of cohomology varieties.

Conjectures are presented, and of particular interest is the elusive tensor-product-theorem, i.e., $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M \otimes N) = \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M) \cap \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(N)$. The tensor product theorem has been established in several contexts in varying levels of generality. The proof always relies on concrete details, primarily through the use of explicit rank varieties.

7.2 Definition and Basic Properties

Let $\mathfrak{g}$ be a classical Lie superalgebra and $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ a subsuperalgebra. The cohomology ring $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{a};\mathbb{C})$ is a graded-commutative ring, and as such the subring

$$\mathrm{H}^{ev}(\mathfrak{g},\mathfrak{a};\mathbb{C}) = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathrm{H}^{2n}(\mathfrak{g},\mathfrak{a};\mathbb{C}) \subseteq \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{a};\mathbb{C}) \tag{7.2.1}$$

is a commutative, finitely-generated subring of $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{a};\mathbb{C})$, by Theorem [??]. This leads to the first definition of this chapter.

Definition 7.2.1. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra with $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ an even subsuperalgebra. The *cohomology variety of* $\mathfrak{g}$ *relative to* $\mathfrak{a}$ is the spectrum of the even subring of Equation 6.2.1:

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}) = \mathrm{mSpec}\left(\mathrm{H}^{ev}(\mathfrak{g},\mathfrak{a};\mathbb{C})\right)$$

For each $\mathfrak{g}$-module M, $\mathrm{Ext}_{(\mathfrak{g},\mathfrak{a})}(M,M)$ is a graded module for the cohomology ring $\mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{a};\mathbb{C}) = \mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C},\mathbb{C})$ via the tensor product or cup product, as in Section [??]. Of course, $\mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{a})}(M,M)$ is a graded module for the subring $\mathrm{H}^{ev}(\mathfrak{g},\mathfrak{a};\mathbb{C})$. This means the annihilator

$$\mathrm{Ann}_{\mathrm{H}^{ev}(\mathfrak{g},\mathfrak{a};\mathbb{C})}\left(\mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{a})}(M,M)\right) \trianglelefteq \mathrm{H}^{ev}(\mathfrak{g},\mathfrak{a};\mathbb{C})$$

is a homogeneous ideal for the even-degree subring of the cohomology ring.

Definition 7.2.2. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra with $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ an even subsuperalgebra. The *relative support variety* of M is the vanishing set of its annihilator. In other words,

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M) = \mathrm{Z}\left(\mathrm{Ann}_{\mathrm{H}^{ev}(\mathfrak{g},\mathfrak{a};\mathbb{C})}\left(\mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{a})}(M,M)\right)\right) \subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})$$

Immediately, we may rephrase common properties of modules in terms of support varieties.

Proposition 7.2.3.

1. *For any* $\mathfrak{g}$*-module* M*,* $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)$ *is a closed, conical subvariety of* $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})$.
2. *For any* $\mathfrak{g}$*-modules* M_1 *and* M_2*,* $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_1 \oplus M_2) = \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_1) \cup \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_2)$.
3. *Whenever* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is a short exact sequence of* $\mathfrak{g}$*-modules, and* $\sigma \in \mathfrak{S}_3$ *is a permutation of three letters,* $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_{\sigma(1)}) \subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_{\sigma(2)}) \cup \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_{\sigma(3)})$.

7.3 Natural Maps of Cohomology Varieties

In this section we exploit the realization of cohomology groups as n-fold extensions to see how relations between Lie superalgebras become morphisms of their associated support varieties.

Recall the realization

$$\mathrm{H}^n(\mathfrak{g},\mathfrak{a};\mathbb{C}) = \{0 \to \mathbb{C} \to E_1 \to \ldots \to E_n \to \mathbb{C} \to 0 \mid \circledast\}/\sim \tag{7.3.1}$$

where $\circledast$ is the condition that the sequence is exact as a sequence of $\mathfrak{g}$-modules and splits on restriction to $\mathfrak{a}$, and $\sim$ is an equivalence reaction obtained from the pre-equivalence relation of there existing morphisms between extensions.

Structure: Maybe this should go into the section on cohomology rings.

Definition 7.3.1. A *relative subsuperalgebra* is a quadruple $(\mathfrak{b} \leq \mathfrak{h}, \mathfrak{a} \leq \mathfrak{g})$. Here $\mathfrak{h} \leq \mathfrak{g}$ is a classical subsuperalgebras, in the sense that $\mathfrak{h}_{\bar{0}} \leq \mathfrak{g}_{\bar{0}}$ and $\mathfrak{h}_{\bar{1}} \leq \mathfrak{g}_{\bar{1}}$. Further, $\mathfrak{a}$ is a subsuperalgebra of $\mathfrak{g}$ and $\mathfrak{b}$ is a subsuperalgebra of $\mathfrak{h}$ which is also contained in $\mathfrak{a}$. See Figure 6.1 for a pictographic definition.

$$\begin{tikzcd}
\mathfrak{a} \arrow[r, hook] & \mathfrak{g} \\
\mathfrak{b} \arrow[r, hook] \arrow[u, hook] & \mathfrak{h} \arrow[u, hook]
\end{tikzcd}$$

Figure 7.1: Relative subsuperalgebra

In the case that $\mathfrak{b} \leq \mathfrak{h}$ is a relative subsuperalgebra of the pair $\mathfrak{a} \leq \mathfrak{g}$, there is a natural restriction morphism of cohomology rings:

$$\mathrm{res} : \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{a};\mathbb{C}) \to \mathrm{H}^\bullet(\mathfrak{h},\mathfrak{b};\mathbb{C}) \tag{7.3.2}$$

This yields a natural morphism of cohomology varieties

$$\mathrm{res}^* : \mathcal{V}_{(\mathfrak{h},\mathfrak{b})}(\mathbb{C}) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}). \tag{7.3.3}$$

There are several special cases in which the morphism of Equation 6.3.3 is especially useful. By Theorem [??], $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{a};\mathbb{C})$ is an integral extension of a quotient of $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C})$ via the restriction morphism (which by [??] is the edge homomorphism of the spectral sequence). This means that the morphism of varieties

$$\mathrm{res}^* : \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C})$$

is a finite-to-one map. Further, by the results of Boe-Kujawa-Nakano [BKN10] the cohomology variety $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C})$ may be realized as closed orbits

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C}) = \{G_{\bar{0}}.x \mid x \in \mathfrak{g}_{\bar{1}} \text{ and } G_{\bar{0}}.x \text{ is closed}\}\,.$$

This proves to be an invaluable morphism, allowing us to realize elements of the support variety $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})$ as closed orbits in the space $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C})$.

7.4 Rank Varieties

While many common properties of support varieties follow from the general theory of modules for rings, one result that requires explicit, context-dependent computations is the proof of the elusive *tensor product theorem*, stated below as a conjecture.

Conjecture 7.4.1 (Tensor Product Theorem). *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra and $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ an even subsuperalgebra. Let M and N be two $\mathfrak{g}$-modules, then we may identify the support variety of their tensor product as follows:*

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M \otimes N) = \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M) \cap \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(N).$$

In many cases, the path to this theorem depends on the establishment of a rank variety description of the support variety $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)$.

Definition 7.4.2. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra and $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ an even subsuperalgebra. The *rank variety* of $\mathfrak{g}$ relative to $\mathfrak{a}$ is the variety

$$\mathcal{V}^{\#}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) = \left\{G_{\bar{0}}.x \mid x \in \mathfrak{g}_{\bar{1}}, G_{\bar{0}}.x \text{ is closed, and } M{\downarrow}_{\langle x\rangle} \text{ is not projective}\right\} \cup \{0\}$$

The study of the structure of $\langle x\rangle$ was conducted in Example 3.2.5 and the cohomology rings were identified in Example [??].

7.5

Let $\mathfrak{g}$ be a classical Lie superalgebra with $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ a subalgebra. We showed in Theorem **??** that $\mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{a};\mathbb{C})$ is a finitely-generated graded-commutative $\mathbb{C}$-algebra. Therefore, the subring of even cohomology classes $\mathrm{H}^{ev}(\mathfrak{g},\mathfrak{a};\mathbb{C}) = \bigoplus \mathrm{H}^{2\bullet}(\mathfrak{g},\mathfrak{a};\mathbb{C})$ is commutative and finitely-generated over $\mathbb{C}$. The *cohomology variety* of $\mathfrak{g}$ relative to $\mathfrak{a}$ is the algebraic variety

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}) = \mathrm{mSpec}\left(\mathrm{H}^{ev}(\mathfrak{g},\mathfrak{a};\mathbb{C})\right).$$

Note that since $\mathrm{H}^{ev}(\mathfrak{g},\mathfrak{a};\mathbb{C})$ is graded we just as well could have looked at the projectivization of $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})$. When dealing with questions of connectivity it will be advantageous to use the projectivization, but in other contexts we will focus exclusively on the conical affine variety.

Recall that $\mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{a})}(M,M)$ is a module over $\mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{a};\mathbb{C})$, so its annihilator defines a subvariety called the *support variety* of M, denoted

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M) = \mathrm{Z}\left(\mathrm{Ann}_{\mathrm{H}^{ev}(\mathfrak{g},\mathfrak{a};\mathbb{C})} \mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{a})}(M,M)\right) \subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})$$

where $\mathrm{Z}(I)$ denotes the vanishing set of I.

An alternative definition of the support variety is

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M) = \left\{\ \mathfrak{m} \in \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}) \mid \mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{a})}(M,M)_{\mathfrak{m}} \neq 0\ \right\}.$$

The following are basic properties whose proof is standard and may be found in [FP87].

1. For any $\mathfrak{g}$-module M, $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)$ is a closed, conical subvariety of $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})$.
2. For any $\mathfrak{g}$-modules M_1 and M_2, $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_1 \oplus M_2) = \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_1) \cup \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_2)$.
3. Whenever $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of $\mathfrak{g}$-modules, and $\sigma \in \mathfrak{S}_3$ is a permutation of three letters, $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_{\sigma(1)}) \subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_{\sigma(2)}) \cup \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_{\sigma(3)})$.

In this section, we use the realization map $\Phi : \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C})$ induced by restriction $\mathrm{res} : \mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C}) \to \mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{a};\mathbb{C})$ to determine properties of (the image of) $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)$. This has the advantage of taking the elusive, abstract support variety and embedding it inside of something concrete – indeed, $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C})$ is simply the set of closed orbits of the action $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$.

7.6 Realizability

In this section we address the question of realizability, initially studied by Carlson [Car83]. As we are using results of Bagci-Kujawa-Nakano [BKN08], we need additional assumptions on the Lie superalgebra $\mathfrak{g}$, namely we require the superalgebra is *stable* and *polar* in addition to being classical. These assumptions originate in geometric invariant theory, and hold for $\mathfrak{gl}(m|n)$ – see [BKN10, §3.2-3.3] for a thorough description.

Definition 7.6.1. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical, stable, and polar Lie superalgebra with $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ a subalgebra. We say $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$-module M is *natural* (with respect to $\mathfrak{a}$) if $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) \cap \Phi\left(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})\right) = \Phi\left(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)\right)$. The subalgebra $\mathfrak{a}$ is *natural* if every $\mathfrak{g}$-module is natural with respect to $\mathfrak{a}$.

The paper of Bagci-Kujawa-Nakano [BKN08, Theorem 8.8.1] demonstrated that every closed conical subvariety of $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C})$ is realized as the support variety of a $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$-module.

Proposition 7.6.2. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical, stable, and polar Lie superalgebra with $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ a natural subalgebra. Let $X \subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})$ be a closed, conical subvariety. There exists a $(\mathfrak{g}, \mathfrak{a})$-module M such that $\Phi\left(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)\right) = \Phi(X)$.*

Proof. The realization theorem holds for $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$-modules, so choose M such that $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) = \Phi(X)$. By naturality, $\Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)) = \Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})) \cap \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) = \Phi(X)$. $\square$

7.7 Tensor products

A tensor product theorem gives us the ability to geometrically control the support theory of tensor products of modules. Historically, this has been a very elusive property of support varieties, often times requiring support varieties recognized in some other way. For example, in the case of finite groups, the tensor product theorem was not shown until support varieties were determined to be isomorphic to the very concrete rank varieties [AE81].

In this section, we circumvent this issue by considering only superalgebras which satisfy the tensor product theorem relative to $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$, and using the realization map to intersect supports of $(\mathfrak{g}, \mathfrak{a})$-modules inside $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C})$.

Definition 7.7.1. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra with subalgebra $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$. The pair $(\mathfrak{g}, \mathfrak{a})$ is said to satisfy the *tensor product theorem* if $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M \otimes N) = \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M) \cap \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(N)$ for all modules M, N.

Lehrer-Nakano-Zhang proved the tensor product theorem hold for the pair $(\mathfrak{gl}(m|n), \mathfrak{gl}(m|n)_{\bar{0}})$, [LNZ11, Theorem 5.2.1]

Proposition 7.7.2. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra which satisfies the tensor product theorem relative to $\mathfrak{g}_{\bar{0}}$, and $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ a natural subalgebra of $\mathfrak{g}$. Then $\Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M \otimes N)) = \Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)) \cap \Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(N))$.*

Proof. One has:

$$\begin{aligned}
\Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M \otimes N)) &= \Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})) \cap \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M \otimes N) \\
&= \big(\Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})) \cap \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)\big) \cap \big(\Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})) \cap \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(N)\big) \\
&= \Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)) \cap \Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(N)).
\end{aligned}$$

□

Proposition 7.7.3. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra with $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ a submodule. Denote by $\Phi : \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C})$ the restriction morphism. If M is a $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$-module, then*

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) \cap \Phi\left(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})\right) = \Phi\left(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)\right).$$

Proof. First, suppose $\mathfrak{m} \in \Phi\left(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)\right)$. This means there is a maximal ideal $\widetilde{\mathfrak{m}} \trianglelefteq \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{a}; \mathbb{C})$ such that $\mathrm{res}^{-1}(\widetilde{\mathfrak{m}}) = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} \supseteq \mathrm{Ann}_{\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{a};\mathbb{C})} \mathrm{Ext}^\bullet_{(\mathfrak{g},\mathfrak{a})}(M, M)$. Of course, $\mathfrak{m} \in \Phi\left(\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C})}(\mathbb{C})\right)$, so it suffices to show $\mathfrak{m} \in \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$, i.e., that every element $\zeta \in \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ which annihilates $\mathrm{Ext}^\bullet_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, M)$ is an element of $\mathfrak{m}$. Consider such a ζ and an extension $0 \to M \to$

$\ldots \to M \to 0$. We wish to show $\operatorname{res}\zeta$ annihilates $\operatorname{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{a})}(M,M)$ – it is only clear that $\operatorname{res}\zeta$ annihilates $\operatorname{res}\left(\operatorname{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M,M)\right)$, i.e., it may not annihilate some sequences which split on restriction to $\mathfrak{a}$ but not on restriction to $\mathfrak{g}_{\bar{0}}$.

Coversely, consider an element $\mathfrak{m} \in \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) \cap \Phi\left(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})\right)$. Choose $\widetilde{\mathfrak{m}} \trianglelefteq \mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{a};\mathbb{C})$ such that $\mathfrak{m} = \operatorname{res}^{-1}(\widetilde{\mathfrak{m}})$. We wish to show $\widetilde{\mathfrak{m}} \supseteq \operatorname{Ann}_{\mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{a};\mathbb{C})} \operatorname{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{a})}(M,M)$. Choose $\zeta \in \operatorname{Ann}_{\mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{a};\mathbb{C})} \operatorname{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{a})}(M,M)$. Because the restriction homomorphism is surjective (on the level of rings), there exists an element $\hat{\zeta} \in \mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C})$ which restricts to ζ. Such a $\hat{\zeta}$ annihilates all elements of $\operatorname{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M,M)$, since the action is defined by tensor product (over $\mathbb{C}$) of exact sequences. Thus $\hat{\zeta} \in \operatorname{Ann}_{\mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C})} \operatorname{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M,M) \subseteq \mathfrak{m}$, and by construction $\operatorname{res}\hat{\zeta} = \zeta \in \widetilde{\mathfrak{m}}$. □

Corollary. *Suppose $\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C})$ is a closed embedding. If a variety $X \subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C})$ is realized by a $(\mathfrak{g},\mathfrak{g}_{\bar{0}})$-module, then $X \cap \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})$ is realized by a $(\mathfrak{g},\mathfrak{a})$-module.*

7.8 Connectedness of support varieties

This section investigates connectedness of support varieties, motivated by Benson's presentation [Ben98].

Proposition 7.8.1. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical, stable, and polar Lie superalgebra with $\mathfrak{a} \leq \mathfrak{g}_{\bar{0}}$ a natural subalgebra. Suppose $\Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)) = X \cup Y$ with $X \cap Y = \{0\}$. Then there exist modules M_1 and M_2 such that $M = M_1 \oplus M_2$, $X = \Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_1))$, $Y = \Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_2))$, and*

$$\Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)) = \Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_1)) \cup \Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_2)).$$

Proof. By realizability for $(\mathfrak{g},\mathfrak{g}_{\bar{0}})$, because $\Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M))$ is a closed conical subvariety of $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$, there exist M_1 and M_2 such that $\Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)) = \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M_1) \cup \mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_2)$. Using this fact, we may compute:

$$\begin{aligned}\Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M)) &= \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M_1) \cup \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M_2) \\ &= \big(\Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})) \cap \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M_1)\big) \cup \big(\Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C})) \cap \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M_2)\big) \\ &= \Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_1)) \cup \Phi(\mathcal{V}_{(\mathfrak{g},\mathfrak{a})}(M_2)).\end{aligned}$$

□

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Appendix A

Supersymmetry Theory

Here we provide a brief sketch of supersymmetry theory. More details may be found in the book "Supersymmetry for Mathematician" [Var]

Appendix B

Relative Cohomology of Hochschild