Representation Theory via Geometry

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Overview

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G-modules as defined above are really just modules for a certain ring denoted kG.

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Addition in the cohomology group corresponds to Baer sum of extensions.

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And the new sequence has the form

$$0 \to k \to \underbrace{E_1 \to \dots \to E_n \to F_1 \to \dots \to F_m}_{m+n} \to k \to 0$$

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Define a commutative, finitely generated ring:

$$H^c(G; k) = \begin{cases} H^\bullet(G; k) & \text{if } p = 2 \\ \bigoplus H^{2n}(G; k) & \text{if } p > 2 \end{cases}$$

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So $\mathrm{Ext}^\bullet_{kG}(M, M)$ has an annihilator $I_M \trianglelefteq H^\bullet(G; k)$ which is a homogeneous ideal.

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Modules L_ζ so that $V_G(L_\zeta) = Z(\zeta)$, where $\zeta \in H^\bullet(G; k)$.

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If $V_G(M) = V_1 \cup V_2$ with $V_1 \cap V_2 = \{0\}$, then there exist modules M_1 and M_2 with $V_G(M_1) = V_1$ and $V_G(M_2) = V_2$.