Andrew Milas
Grade: Freshman

What Are Number Bases

The number system that we are all used to revolves around the number 10. For example, the number 1234 can be rewritten as

$$1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10 + 4$$

and this is why we say that the digit 4 is in the ones place, 3 is in the tens place, 2 is in the hundreds place, and 1 is in the thousands place. In base 10 (decimal) we are choosing to represent numbers based on how many bundles of each power of 10 they contain.

However, we could choose to represent this number in different bases as well (we have simply grown used to using base 10 for centuries which is why it is the most common, likely because humans have 10 fingers). In base 9, we would represent this number based on how many bundles of $9^3 = 729$, $9^2 = 81$, $9^1 = 9$, and $9^0 = 1$ it has. Therefore, when trying to convert 1234 into base 9, we first realize that only 1 bundle of 729 will fit and then $1234 - 729 = 505$ will be leftover. Next, 6 bundles of 81 will now fit and $505 - 6 \cdot 81 = 505 - 486 = 19$ will be left. And finally, 2 more bundles of 9 must be used and 1 bundle of 1, so 1234 in base 10 equals 1621 in base 9. This equality can also be written as

$$1234_{10} = 1621_9$$

where we write the base we are using at the lower-right corner of the number we are representing as a subscript (no subscript can be assumed to mean base 10). One thing to note is that the number seems bigger in base 9. This makes sense since clearly we will be able to fit more bundles of the powers of 9 into the number than bundles of the powers of 10 since $9 < 10$. So one question that you may ask is why would we ever choose to use base 9 instead of base 10 since we can always represent numbers "more efficiently/compactly" in base 10 rather than in base 9. The upside is that in base 9 we use one less digit; we never use the digit 9. For example $9_{10} = 10_9$ since there is 1 bundle of 9 and 0 bundles of 1 needed. This is the same reason as to why we don't have any special symbol to represent 10 in base 10 and we must use 2 symbols.

If we take this to the extreme we get base 2 (binary), which is the most used base other than 10 because it only has two possible digits 0 and 1 (this is usually used by computers because the symbols 0 and 1 can easily be represented by electricity, 0 is no electricity, and for 1 there is electricity). For example $1234_{10} = 10011010010_2$. This means that there is 1 bundle of $2^1 = 2$, $2^4 = 16$, $2^6 = 64$, $2^7 = 128$, and $2^{10} = 1024$. Indeed $2 + 16 + 64 + 128 + 1024 = 1234$, but we had to use 11 digits to represent this same number instead of 4 that's needed in base 10.

On the other end of the spectrum we have base 16 (hexadecimal/hex). However, for base 16 we need more digits in order to represent $10, 11, 12, 13, 14$, and 15. Though we could truly use whatever symbols we want, we usually use the capital letters A, B, C, D, E, and F. For example, let's convert 1234 into base 16. We will need 4 bundles of $16^2 = 256$, and there will be $1234 - 4 \cdot 256 = 1234 - 1024 = 210$ left. Now we will need 13 bundles of 16 with $210 - 13 \cdot 16 = 210 - 208 = 2$ left. Therefore, $1234 = 4D2_{16}$ and we only needed 3 digits. Because of how compact base 16 is, computers usually use it to store information such as about images and colors using less space.

Conversion

Converting to base 10 is easy. We simply expand everything out. For example,

$$\begin{aligned} 1231_5 &= 1 \cdot 5^3 + 2 \cdot 5^2 + 3 \cdot 5 + 1 \cdot 5^0 \\ &= 1 \cdot 125 + 2 \cdot 25 + 3 \cdot 5 + 1 \\ &= 125 + 50 + 15 + 1 \\ &= 191. \end{aligned}$$

On the other hand, converting from base 10 to other bases can be more tricky. However, for most numbers, I would just use the guess and check approach we did on the previous page. That is, I would list out the powers of the base less than the number, and just manually find how many bundles of each power will fit into the number. For example, if converting 50 to base 6, I would just see that there is 1 bundle of 36, 2 bundles of 6 and 2 bundles of 1 so $50 = 122_6$. Nevertheless, there do exist more systematic ways.

To show this lets again convert 1234 into base 9. All you have to do is keep dividing the number by the base, and the answer will be the remainders written backwards. The first step is to divide 1234 by 9 to get 137 with a remainder of 1. Now we do this again, dividing 137 by 9 gives 15 with a remainder of 2. Next, 15 divided by 9 is 1 remainder 6. At this point we stop since $1 < 9$ (if you want we can do $1/9$ one more time to get 0 remainder 1). The remainders were 1, 2, and 6, and 1, so we find that $1234_{10} = 1621_9$ as we got on the previous page using the guess and check approach.

As another example, lets convert 50 to base 6.

$$\begin{aligned} 50/6 &= 8 \quad R\ \boxed{2} \\ 8/6 &= \boxed{1}\ R\ \boxed{2} \end{aligned}$$

so $50 = 122_6$ as we found earlier.

Also, to convert from one non-ten base (say base x) number to another non-ten base (base y) number, usually it is easier to convert from base x to base 10 first and then from base 10 to base y.

Arithmetic

Arithmetic in other bases is done essentially the same as in base 10 except using slightly altered logic. For example, when doing addition in base 8, we must carry when the sum is bigger than 8 (which is 10 in base 8) not 10 as we normally do. However, often it is just easier to convert everything to base 10, do the arithmetic, and then convert back because we are all much more used to base 10.

First, lets go over an example of addition by adding 123_4 with 222_4:

$$\begin{array}{cccc} & 1 & 2 & 3 \\ + & 2 & 2 & 2 \\ \hline \end{array}$$

When we add the units digits (3 and 2) we get $5 = 11_4$ so we must carry the 1.

$$\begin{array}{cccc} & 1 & 2^1 & 3 \\ + & 2 & 2 & 2 \\ \hline & & & 1 \end{array}$$

Adding the 4s digits and the carry again gives 11_4 so we must carry

$$\begin{array}{cccc} & 1^1 & 2^1 & 3 \\ + & 2 & 2 & 2 \\ \hline & & 1 & 1 \end{array}$$

and adding the 16s digits and the carry gives 10_4

$$\begin{array}{cccc} & 1^1 & 2^1 & 3 \\ + & 2 & 2 & 2 \\ \hline 1 & 0 & 1 & 1 \end{array}$$

so the answer is 1011_4. If we convert to base 10 first, we find that $123_4 = 16+8+3 = 27$ and $222_4 = 32+8+2 = 42$, so their sum is 69 which will have 1 bundle of 64, 4, and 1 and is, indeed, 1011_4.

We'll also do one example of subtraction (multiplication and division aren't that different and are not as likely to appear on the NSML anyway) by subtracting 2235_7 from 4216_7.

$$\begin{array}{ccccc} & 4 & 2 & 1 & 6 \\ - & 2 & 2 & 3 & 5 \\ \hline & & & & 1 \end{array}$$

Subtracting the units digit is pretty easy ($6-5=1$), but now we must borrow to subtract the 7s digits. We'll take 1 from the 49s place (turning the 2 into a 1) and turn the 1 in the 7s place into a $11_7 = 8$. Then $11_7 - 3_7 = 5$.

$$\begin{array}{ccccc} & 4 & 1 & 11 & 6 \\ - & 2 & 2 & 3 & 5 \\ \hline & & & 5 & 1 \end{array}$$

But now we need to borrow again, so we'll turn the 4 into a 3 and the 1 in the 49s place into an 11_7. Then, $11_7 - 2_7 = 6$.

$$\begin{array}{ccccc} & 3 & 11 & 11 & 6 \\ - & 2 & 2 & 3 & 5 \\ \hline & & 6 & 5 & 1 \end{array}$$

And finally subtracting in the 343s place gives a final answer of 1651_7.

$$\begin{array}{ccccc} & 3 & 11 & 11 & 6 \\ - & 2 & 2 & 3 & 5 \\ \hline & 1 & 6 & 5 & 1 \end{array}$$

If we converted to base 10 first, then $4216_7 = 4 \cdot 343 + 2 \cdot 49 + 7 + 6 = 1372 + 98 + 13 = 1483$ and $2235_7 = 2 \cdot 343 + 2 \cdot 49 + 21 + 5 = 686 + 98 + 26 = 810$, and $1483 - 810 = 673$. Converting this to base 7 as shown below also gives 1651_7.

$$\begin{aligned} 673/7 &= 96 \quad R\ \boxed{1} \\ 96/7 &= 13 \quad R\ \boxed{5} \\ 13/7 &= \boxed{1} \quad R\ \boxed{6} \end{aligned}$$

The Radix Point

In base 10, not all numbers are integers. Numbers can have digits after the decimal point such as in the number

$$1.23_{10} = 1 + \frac{2}{10} + \frac{3}{100} = 1 \cdot 10^0 + 2 \cdot 10^{-1} + 3 \cdot 10^{-2}.$$

This is true for other bases as well, except that the point isn't called the decimal point; it is called the radix point. For example,

$$9A.B_{16} = 9 \cdot 16 + 10 + \frac{11}{16} = 9 \cdot 16^1 + 10 \cdot 16^0 + 11 \cdot 16^{-1}$$

The only difference is that the digits after the radix point are now multiplied by negative powers of the new base ($\frac{1}{16}$ instead of $\frac{1}{10}$ in the above example). If we want to convert these numbers from a non-ten base into base 10, we would simply expand everything out as before (in our example we have $9A.B_{16} = 154\frac{11}{16} = 154.6875_{10}$). To convert from decimal into a non-ten base, we would have to express the decimal as a fraction and use long division in the new base to find what comes after the radix point.

Variables

Sometimes questions will have unknown variables as digits or as the base that you will need to find. We usually treat this by expanding out the number. For example lets solve the equation $134_x = \overline{3x}_{13}$ for x. The left side has base x and is equal to $x^2 + 3x + 4$, and the right side as x as one of its digits and is equal to $3 \cdot 13 + x$. Solving this we find

$$\begin{aligned} & x^2 + 3x + 4 = 39 + x \\ \implies & x^2 + 2x - 35 = 0 \\ \implies & (x-5)(x+7) = 0. \end{aligned}$$

Since x can't be -7, we must have $x = 5$. Also, its important to note that this works since $1, 3$, and 4 are possible digits base 5 and since 5 is a possible digit base 13.