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Grade: Freshman
Number Bases

Base Powers

Though conversion between different bases can be a fairly long process, there do exist tricks in order to quickly convert between powers of 2. To showcase this, we'll go through an example converting 1101001001011_2 into base 4. The idea is convert digits two at a time starting from the right as shown below

$$\boxed{1}\ \boxed{10}\ \boxed{10}\ \boxed{01}\ \boxed{00}\ \boxed{10}\ \boxed{11}$$
$$\boxed{1}\ \boxed{2}\ \boxed{2}\ \boxed{1}\ \boxed{0}\ \boxed{2}\ \boxed{3}$$

to get 1221023_4. This same trick works in reverse, such as when converting 32123_4 into base 2

$$\boxed{3}\ \boxed{2}\ \boxed{1}\ \boxed{2}\ \boxed{3}$$
$$\boxed{11}\ \boxed{10}\ \boxed{01}\ \boxed{10}\ \boxed{11}$$

giving us 1110011011_2. This works because $4 = 2^2$. Similarly, we could use this same trick to covert between bases 3 and 9 as well as bases 4 and 16. Also, we could apply the same idea to convert between bases 2 and 8, except taking digits three at a time, as shown below when converting 11010101100110111_2 to base 8

$$\boxed{11}\ \boxed{010}\ \boxed{101}\ \boxed{100}\ \boxed{110}\ \boxed{111}$$
$$\boxed{3}\ \boxed{2}\ \boxed{5}\ \boxed{4}\ \boxed{6}\ \boxed{7}$$

in order to get 325467_8. If you need to convert between bases 4 and 8, however, you need to go to base 2 first. Either way, its still much faster than converting to base 10 first.

Repeating Fractions

This section revolves a round a neat little trick that happens in base 10 whenever we divide a number by 99. If we divide 23 by 99 we get $0.\overline{23}$ where the 23 repeats forever. And this works for any number: $\frac{07}{99} = 0.\overline{07}$, $\frac{98}{99} = 0.\overline{98}$ and so on. The reason this works is because we can say that $1 = 0.\overline{99}$ where the 9s repeat forever. Of course this doesn't only work with 99, it works with $10^k - 1$ for any integer k. For example, $\frac{4}{9} = 0.\overline{4}$ and $\frac{1234}{999} = 0.\overline{1234}$.

Furthermore, it doesn't only work with base 10. If we were working in base 6 for example, we would have $1_6 = 0.\overline{55}_6$, so we can divide by 5_6, 55_6, 555_6, $\cdots$ very easily. In general, in base b, the numerator will continuously repeat whenever we divide by a number of the form $b^k - 1$ where k is any positive integer. To see how this can be helpful, lets solve the following question from the 2019 AMC 12A.

For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is $0.\overline{23}_k = 0.232323..._k$. What is k?

Solution:

Seeing the repetition after the radix point, it is time to use our trick. Since the 2 and 3 keep repeating in base k, we know that $.\overline{23}_k = \frac{23_k}{k^2-1} = \frac{2k+3}{k^2-1}$. All that's left is to set this equal to $\frac{7}{51}$ and solve:

$$\begin{aligned} \frac{7}{51} &= \frac{2k+3}{k^2-1} \\ 7k^2 - 7 &= 102k + 153 \\ 7k^2 - 102k - 160 \\ (k-16)(7k-10) &= 0 \end{aligned}$$

so $\boxed{\text{k=16}}$. Alternatively, we could have guessed this by looking at non-simplified versions of $\frac{7}{51}$ and seeing which one has a denominator one less than a perfect square. The first few denominators will be $51, 102, 153, 204, 255$, and we can quickly see that $255 = 16^2 - 1$ so k is likely 16.

Divisibility Tricks

We've all probably heard of the common divisibility tricks in base 10, such as those for 2, 3, 4, 5, 9, 10, and 11. It turns out that many of these tricks apply for other numbers in other bases as well. These divisibility tricks come in 3 main categories and we'll go through each of them separately.

First, there are divisibility tricks for each factor of the base and their powers. In base 10 this means divisibility tricks for 2, 5, 10 and their respective powers. The trick is that a number will be divisible by 2, 5, or 10 if the last digit of the number is divisible by 2, 5, or 10 respectively (which is why all multiples of 5 end in 0 or 5 and all multiples of 10 end in 0). Similarly, a number will be divisible by 4, 25, or 100 if the last two digits are divisible by 4, 25, or 100, and, in general, a number will be divisible by 2^k, 5^k, or 10^k if the last k digits of the number is divisible by 2^k, 5^k, or 10^k respectively. This works because a number $\overline{a_n a_{n-1} \cdots a_2 a_1 a_0}$ with digits $a_n, \cdots, a_1, a_0$ can be written as

$$a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \cdots a_2 \cdot 10^2 + a_1 \cdot 10 + a_0$$

and each of these terms except for the last k terms will already by divisible by 2^k, 5^k, or 10^k because of the power of 10 that it is multiplied with. This trick works for all bases for the same reason. For example, in base 6 a number is divisible by $3^2 = 9 = 13_6$ only if the last two digits are 00_6, 13_6, 30_6, or 43_6, and a number is divisible by 7 in base 7 only if it ends in a 0. (Also, another useful fact is that multiplying or dividing by the base works exactly like multiplying/dividing by 10 in base 10 in that it will just shift the radix point right or left, adding or removing 0s if necessary).

Next comes the divisibility tricks for the factors of the base minus one. In base 10, these are divisibility tricks for 3 and 9. It turns out that a number is divisible by 3 or 9 in base 10 if the sum of the digits is divisible by 3 or 9 respectively, and this works exactly the same in all bases. In base 8, a number will be divisible by 7 if the sum of the digits is divisible by 7. For people who know about mods, this works because when we take the number $\overline{a_n a_{n-1} \cdots a_1 a_0}_b \pmod{b-1}$, or mod any factor of $b-1$, we get

$$\begin{aligned}\overline{a_n a_{n-1} \cdots a_1 a_0}_b &\equiv a_n \cdot b^n + a_{n-1} \cdot b^{n-1} + \cdots + a_1 \cdot b + a_0 \pmod{b-1}\\ &\equiv a_n \cdot 1^n + a_{n-1} \cdot 1^{n-1} + \cdots + a_1 \cdot 1 + a_0 \pmod{b-1}\\ &\equiv a_n + a_{n-1} + \cdots a_1 + a_0 \pmod{b-1}.\end{aligned}$$

Also, as a result, not only do we get the divisibility trick, we also have the fact that the remainder of a number base b when divided by $b-1$ (or any of its divisors) will be equivalent to the sum of the number's digits.

Finally, we have the very similar divisibility trick when dividing by factors of the base plus one. The only difference is that we must take the alternating sum of the digits. For example, for 11 in base 10, to check if 3619 is divisible by 11, we check if $9-1+6-3 = 11$ is divisible by 11 (which it is). Also, note that we must always start from the right. Again, this works in all bases, so, for example, we could use this alternating digit trick to see if a number is divisible by 3, 5, or 15 in base 14. Similarly, this works because

$$\begin{aligned}\overline{a_n a_{n-1} \cdots a_1 a_0}_b &\equiv a_n \cdot b^n + a_{n-1} \cdot b^{n-1} + \cdots + a_1 \cdot b + a_0 \pmod{b+1}\\ &\equiv a_n \cdot (-1)^n + a_{n-1} \cdot (-1)^{n-1} + \cdots + a_1 \cdot (-1) + a_0 \pmod{b+1}.\end{aligned}$$

While it is very rare for divisibility tricks to be useful in other bases, it is helpful to understand why these tricks work, and they do appear occasionally such as in the following question:

The base-14 number $\overline{82X3YZ}_{14}$ (with unknown digits X, Y, and Z) is divisible by 455_{10}. Find the value of $X+Y$ in base 10.

Solution:

First of all, we see that $455 = 5 \cdot 7 \cdot 13$ so the number must be divisible by all three of these factors. Also, because we are working base 14 and the number must be divisible by 7, using the first divisibility trick, we know that the last digit, Z, equals 0 or 7.

Now we'll deal with the fact that the number is divisible by 13. Using the second divisibility trick, we know that $8+2+X+3+Y+Z = 13+X+Y+Z$ must be divisible by 13 so $X+Y+Z$ must be divisible by 13. If $Z=0$, this means that $X+Y$ equals or 0, 13 or 26 (it can't be bigger because X and Y are digits base 14). If $Z=7$, then $X+Y$ equals 6 or 19. To differentiate between these cases, we'll have to use the fact that the number is divisible by 5.

Z	X+Y	Z-3-(X+Y)	Divisible by 5
0	0	-3	No
0	13	-16	No
0	26	-29	No
7	6	-2	No
7	19	-15	Yes

Since 5 divides $15 = 14 + 1$, we can use our third divisibility trick to see that $Z - Y + 3 - X + 2 - 8 = Z - 3 - (X + Y)$ must be divisible by 5. If we plug in our values for Z and $X + Y$ as shown above we see that only $Z = 7$ and $X + Y = 19$ works, so $\boxed{\text{X+Y=19}}$ is the answer.