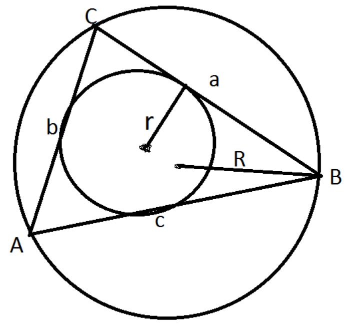
Andrew Milas
Grade: Sophomore

Advanced Geometrical Concepts

Triangles



Triangle Areas:

$$\text{Area} = \frac{b \cdot h}{2} = \frac{a \cdot b \cdot \sin(C)}{2} = sr = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R}$$

where s is the semi-perimeter, r is the inradius, and R is the circumradius.

Law of Sines:

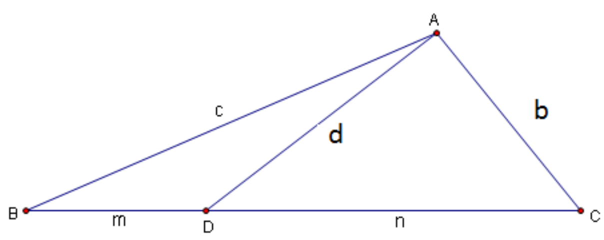
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(B)} = 2R$$

Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab \cdot \cos(C)$$

These top three sections aren't that common on NSML but appear very often in contest math in general.

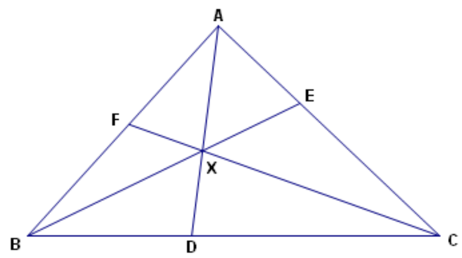
Stewart's Theorem:



$$man + dad = bmb + cnc$$

Useful for finding medians and angle bisectors. Easily proved by law of cosines on Point D. Remember a man and his dad put a bomb in their sink (b and m go on opposite sides of AD).

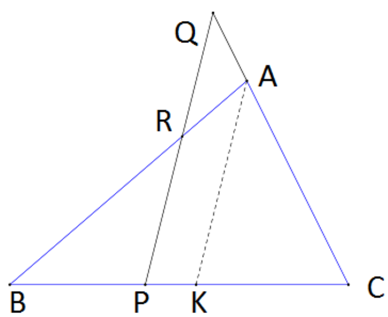
Ceva's Theorem:



Lines AD, BE, and CF are concurrent if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Menelaus's Theorem:



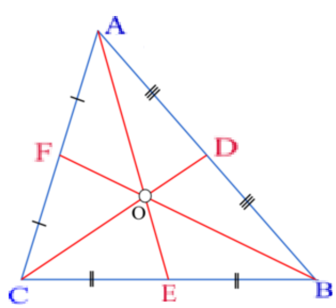
Points P, R, and Q on lines (BC, AB, and AC) are collinear if and only if

$$\frac{AR}{RB} \cdot \frac{QC}{QA} \cdot \frac{PB}{PC} = -1.$$

Easily proven by drawing AK parallel to PR and noting $\triangle RBP \sim \triangle ABK$ and $\triangle QCP \sim \triangle ACK$.

The negative 1 means that the lengths are directed (have positive or negative signs). Specifically, $\frac{AR}{RB}$ is negative if and only if R is outside AB so AR and RB travel in opposite directions. For the NSML contest, directed lengths can be ignored and positive 1 can be used instead. The reason the -1 is included is because P, Q, and R clearly can't be collinear if they all lie on a triangle's side or if exactly 2 lie outside the triangle's sides.

Medians:

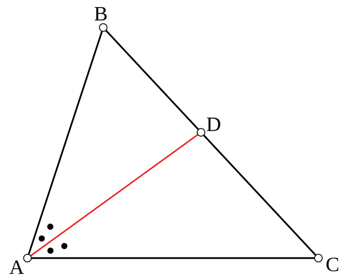


- Point O (place where medians meet) is the triangle's center.
- All six smaller triangles have the same area.
- $AO = 2OE$, $CO = 2OD$, $BO = 2OF$

Andrew Milas
Grade: Sophomore

Advanced Geometrical Concepts

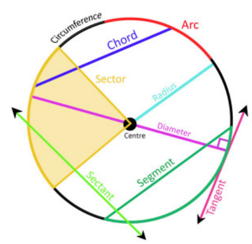
Angle Bisector Theorem:



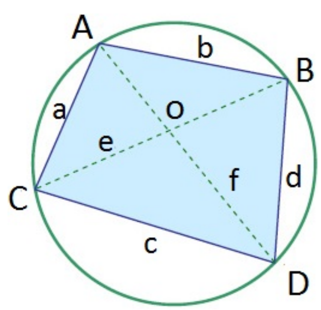
$$\frac{AB}{BD} = \frac{AC}{DC}$$

Easily proved by law of sines. Almost always used when angle bisectors are present.

Circles and Cyclic Quadrilaterals

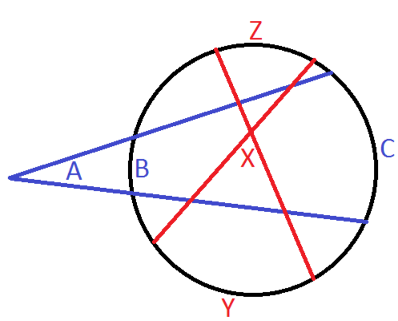


Cyclic Quadrilaterals:



- Opposite angles are supplementary
- Angles which subtend the same side (such as $\angle ACO$ and $\angle BDO$) are congruent
- **Ptolemy's Theorem**: $ef = ad + bc$. Very common
- **Brahmagupta's Formula**: Area $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$ where s is the semiperimeter. Rarely used other than on NSML.

Angles and Arcs:

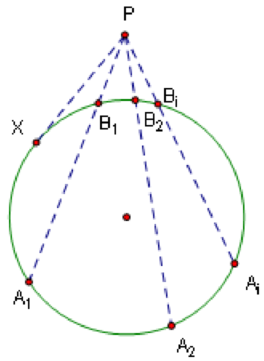


$$m\angle A = \frac{m\widehat{C} - m\widehat{B}}{2}$$

$$m\angle X = \frac{m\widehat{Y} + m\widehat{Z}}{2}$$

Check to see that this makes sense with central and inscribed angles.

Power of a Point Theorem:



$$PX^2 = PA_1 \cdot PB_1 = PA_2 \cdot PB_2 = \cdots = PA_i \cdot PB_i$$

Basically, for any line through P the product of the line segments between P and the first point where the line intersects the circle and P and the second point where the line intersects the circle is constant for all lines. This also holds if P is inside the circle.