For example, a contract may call for the payment of a different amount each year over a three-year period. To determine present value, each payment is discounted to the present and then summed. (Assume 8 percent discount rate.)

1.	\$1,000	PV = \$ 926
2.	2,000	PV = 1,715
3.	3,000	PV = 2,381
		\$5,022

A more involved problem might include a combination of single amounts and an annuity. If the annuity will be paid at some time in the future, it is referred to as a deferred annuity, and it requires special treatment. Assume the same problem as above, but with an annuity of \$1,000 that will be paid at the end of each year from the fourth through the eighth year. With a discount rate of 8 percent, what is the present value of the cash flows?

1.	\$1,000	Present value = \$5,022
2.	2,000	
3.	3,000	
4.	1,000	Five-year annuity
5.	1,000	
6.	1,000	
7.	1,000	
8.	1,000	

We know that the present value of the first three payments is \$5,022, but what about the annuity? Let's diagram the five annuity payments.

Present value				A_1 \$1,000	A_2 \$1,000	A_3 \$1,000	A_4 \$1,000	A_5 \$1,000
0	1	2	3	4	5	6	7	8

FV = 0 PMT = \$1,000
N = 5 I/Y = 8
CPT PV = **\$3,992.71.**

However, this result is discounted only to the beginning of the first stated period of an annuity—in this case the beginning of the fourth year, as diagrammed below.

				Beginning of fourth period				
Present value			**\$3,993**	**A_1 \$1,000**	**A_2 \$1,000**	**A_3 \$1,000**	**A_4 \$1,000**	**A_5 \$1,000**
0	1*	2	3	4	5	6	7	8

*Each number represents the end of the period; for example, 4 represents the end of the fourth period.

The \$3,993 must finally be discounted back to the present. Since this single amount falls at the beginning of the fourth period—in effect, the equivalent of the end of the third period—we discount back for three periods at the stated 8 percent interest rate.

FV = \$3,992.71 PMT = 0
N = 3 I/Y = 8
CPT PV = **\$3,169.54.**

Therefore, this pattern of uneven payments is worth (present worth) $3,170. The last step in the discounting process is shown below.

		End of the third period—beginning of the fourth period						
$3,170 Present value		**$3,993 (single amount)**		**A_1 $1,000**	**A_2 $1,000**	**A_3 $1,000**	**A_4 $1,000**	**A_5 $1,000**
0	1	2	3	4	5	6	7	8

Calculator To calculate the present value of uneven cash flows, calculators have special function keys requiring the net present value concept. This is discussed in Appendix E.

Perpetuities

A perpetuity is an annuity or a series of payments that has no end date and seemingly goes on forever.

Equal Payments The formula for a perpetual annuity is as follows (payments at end of period):

$$PV = \frac{A}{i} = \frac{PMT}{i} \quad (9\text{–}9)$$

Assuming the receipt of $100 payment a year forever with an annual interest rate of 5 percent, the present value is

$$PV = \frac{\$100}{0.05} = \$2{,}000$$

If we assume a very large number for the number of periods (n), say 1,000, the calculator can be used in place of the formula.

FV = 0 PMT = $100
N = 1,000 I/Y = 5
CPT PV = **$2,000.**

Growing Payments The formula for a perpetual annuity growing at a constant rate (g) is as follows (payments at end of period):

$$PV = \frac{A_1}{i - g} \quad (9\text{–}10)$$

Assuming the receipt of a first payment of $100, growing at 3 percent annually forever and with an annual interest rate of 5 percent, the present value is

$$PV = \frac{\$100}{0.05 - 0.03} = \$5{,}000$$

If we assume a very large number for the number of periods (n) the calculator can be used in place of the formula.

FV = 0 PMT = $100
N = 1,000 I/Y = 5 − 3 = 2
CPT PV = **$5,000.**

Growing Annuity (with End Date)

The formula for an annuity growing at a constant rate (***g***) for a limited period of time (***n***) is as follows (payments at end of period):

$$PV_n = A_1\left(\frac{1}{i-g}\right)\left[1-\left(\frac{1+g}{1+i}\right)^n\right] \quad (9\text{–}11)$$

Assuming the receipt of a first payment of $100, growing at 3 percent annually for 10 years and with an annual interest rate of 5 percent, the present value is

$$\begin{aligned} PV &= \$100\left(\frac{1}{0.05-0.03}\right)\left[1-\left(\frac{1+0.03}{1+0.05}\right)^{10}\right] \\ &= \$100(50)[1-0.8250481] = \$874.76 \end{aligned}$$

CANADIAN MORTGAGES

In Canada it is common to have mortgages that have interest compounded semiannually, with payments made monthly. The potential problem with blended payments of principal and interest made on a monthly basis is that the interest is being paid before it is actually due. Calculations must acknowledge the early payment of interest. We cannot just divide the semiannual interest rate by six. To adjust, we must calculate a monthly effective interest rate that, when compounded over a six-month period, is equivalent to the semiannual effective interest rate. It is with this monthly effective interest rate that we calculate the monthly payment.

Say the interest rate offered at the bank is 8 percent annually. Therefore, the rate for six months is 4 percent (8/2). We now need a rate that, when compounded six times, will equal 4 percent; by formula $(1 + i)^1 = 1.04$. Solving for ***i*** gives us 0.6558 percent.

FV = 1.04

0 1 2 3 4 5 6

PV = −1.00

N = 6

PMT = 0

$i = 0.655819692\%$

Also with EFF→ (dependent on calculator)

6 2nd EFF→ 4 = 3.9349174 (6-month equivalent)

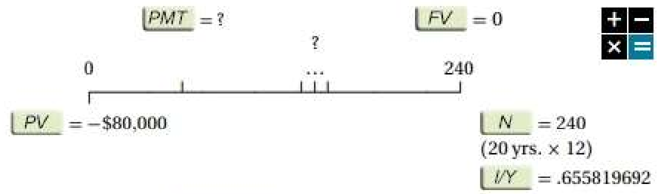
$$\left(\begin{array}{c}\textbf{number of payment periods}\\ \textbf{in compounding period}\end{array}\right)\left(\frac{\textbf{six-month}}{\textbf{interest rate}}\right)$$

Then divide by 6:

$$\frac{3.9349174}{6} = .6558196\% \text{ (monthly effective interest rate)}$$

We begin with 1, and six months later it is 1.04. We have determined the interest rate for one of the six periods, a monthly effective interest rate.

Now we can calculate the monthly payment on the mortgage. Suppose the mortgage is for $80,000, to be paid off over 20 years at our interest rate of 8 percent annually.



CPT PMT = **$662.69.**

With this calculation, the outstanding principal is the present value (PV) of the remaining payments.

FINANCE IN ACTION

Is a Weekly Mortgage a Good Idea?

The banks often promote the weekly mortgage as a great way to pay off your mortgage early. It is suggested that you can reduce the time to pay off a mortgage by perhaps four to five years, depending on circumstances. However, do these claims identify the complete picture?

We have noted that a mortgage for $80,000 paid monthly over 20 years at an 8 percent interest rate would require a monthly payment of $662.69. If you were to pay weekly, the bank would likely take that monthly payment and divide by four to represent the weeks in a month. The weekly payment will therefore be $165.67. Some banks do identify this as an accelerated payment schedule.

Principal amount	$80,000
Annual interest rate	8%
Weekly interest rate	0.15096273%*
Weekly payment	$165.67
Number of payments	865.48
Number of years	16.64

*$\left(1+\frac{0.08}{2}\right)^{1/26} - 1 \times 100\%$ For 26 weeks in a six-month period.

Sounds great until the situation is examined more closely. With weekly payments a mortgagee is actually making an extra monthly payment each year.

Weekly	52 × $165.67 = $8,614.84
Monthly	12 × $662.69 = 7,952.28
Extra payment	$662.56

Presumably, if a homeowner can afford $8,614.84 as weekly payments over one year, the homeowner could pay the same amount as monthly payments. In that case the monthly payment would be $717.90.

Principal amount	\$80,000
Annual interest rate	8%
Monthly interest rate	0.655819691%
Monthly payment	\$717.90
Number of payments	200.77
Number of years	16.73

This is very similar to the weekly plan. So what is a homeowner to do? The key is to match your cash inflows with your cash outflows. A mortgage is the major obligation (outflow) for most people, and salary the major inflow. If the homeowner is paid monthly, take out a monthly mortgage! If the homeowner is paid weekly, take out a weekly mortgage! Otherwise, cash flows to the household will be inefficiently allocated. The homeowner with a weekly mortgage, but monthly pay, would be forced to save money from each pay to meet the weekly obligation or, even worse, to borrow until the next monthly pay period. The best strategy for the homeowner is to determine the largest payment out of each pay that can be afforded and to match the amortization period and payment period to that payment.

A FINAL NOTE

The key foundation tool of financial management is the ability to understand and to calculate the time value of money. Value is determined by the ability to generate cash flows. The time value of money allows us to properly value cash flows that occur at different points in time. Therefore, it is essential that the student of finance be able to comfortably handle the problems of this chapter.

SUMMARY AND REVIEW OF FORMULAS

1. The time value of money suggests that a dollar today is worth more than a dollar tomorrow. Alternatively, a dollar invested today will grow to a larger value tomorrow. Through the discounting technique, that dollar tomorrow is equated (discounted) to a value today. Discounting values to a common time period allows for comparison.

2.,3. In working a time-value-of-money problem, the student should determine, first, whether the problem deals with future value or present value and, second, whether a single sum or an annuity is involved. The major calculations in Chapter 9 are summarized below in case a calculator is not used.

A. Future value of a single amount.

Formula: $FV = PV(1 + i)^n$ (9–1)

Appendix A

When to use: In determining the future value for a single amount.

Sample problem: You invest $1,000 for four years at 10 percent interest. What is the value at the end of the fourth year?

B. Effective interest rate.

Formula: $(1 + i)^n - 1 = \text{Effective interest rate}$ (9–2)

When to use: In determining an interest rate that captures interest compounding.

C. Present value of a single amount.

Formula: $PV = FV\left[\frac{1}{(1+i)^n}\right] = FV(1+i)^{-n}$ (9–3)

Appendix B

When to use: In determining the present value of an amount to be received in the future.

Sample problem: You will receive $1,000 after four years at a discount rate of 10 percent. How much is this worth today?

D. Future value of an annuity.

Formula: $FV_A = A\left[\frac{(1+i)^n - 1}{i}\right]$ (9–4*a*)

Appendix C

When to use: In determining the future value of a series of consecutive, equal payments (an annuity).

Sample problem: You will receive $1,000 at the end of each period for four periods.

What is the accumulated value (future worth) at the end of the fourth period if money grows at 10 percent?

When the payments are at the beginning of each period:

Formula: $FV_A = A_{BGN}\left[\frac{(1+i)^{n+1} - (1+i)}{i}\right]$ (9–4*b*)

E. Present value of an annuity.

Formula: $$\mathrm{PV}_A = A\left[\frac{1-\dfrac{1}{(1+i)^n}}{i}\right] = A\left[\frac{1-(1+i)^{-n}}{i}\right] \quad (9\text{–}5a)$$

Appendix D

When to use: In determining the present worth of an annuity.

Sample problem: You will receive $1,000 at the end of each period for four years. At a discount rate of 10 percent, what is the current worth?

When the payments are at the beginning of each period:

Formula: $$\mathrm{PV}_A = A_{\mathrm{BGN}}\left[\frac{(1+i)-\dfrac{1}{(1+i)^{n-1}}}{i}\right] = A_{\mathrm{BGN}}\left[\frac{(1+i)-(1+i)^{-n+1}}{i}\right] \quad (9\text{–}5b)$$

F. Annuity equalling a future value.

Formula: $$A = \mathrm{FV}_A\left[\frac{i}{(1+i)^n-1}\right] \quad (9\text{–}6a)$$

Appendix C

When to use: In determining the size of an annuity that will equal a future value.

Sample problem: You need $1,000 after four periods. With an interest rate of 10 percent, how much must be set aside at the end of each period to accumulate this amount?

When the payments are at the beginning of each period:

Formula: $$A_{\mathrm{BGN}} = \mathrm{FV}_A\left[\frac{i}{(1+i)^{n+1}-(1+i)}\right] \quad (9\text{–}6b)$$

G. Annuity equalling a present value.

Formula: $$A = \mathrm{PV}_A\left[\frac{i}{1-\dfrac{1}{(1+i)^n}}\right] = \mathrm{PV}_A\left[\frac{i}{1-(1+i)^{-n}}\right] \quad (9\text{–}7a)$$

Appendix D

When to use: In determining the size of an annuity equal to a given present value.

Sample problems:

a. What four-year annuity is the equivalent of $1,000 today with an interest rate of 10 percent?
b. You deposit $1,000 today and wish to withdraw funds equally over four years. How much can you withdraw at the end of each year if funds earn 10 percent?
c. You borrow $1,000 for four years at 10 percent interest. How much must be repaid at the end of each year?

When the payments are at the beginning of each period:

Formula: $$A_{\mathrm{BGN}} = \mathrm{PV}_A\left[\frac{i}{(1+i)-\dfrac{1}{(1+i)^{n-1}}}\right] = \mathrm{PV}_A\left[\frac{i}{(1+i)-(1+i)^{-n+1}}\right] \quad (9\text{–}7b)$$

H. ***Determining the yield on an investment.***

	Formulas	Tables
a.	$i = \left(\frac{FV}{PV}\right)^{\frac{1}{n}} - 1$ (9-8)	Appendix B Yield—present value of a single amount
b.	Interpolation required	Appendix D Yield—present value of an annuity

When to use: In determining the interest rate (i) that will equate an investment with future benefits.

Sample problem: You invest $1,000 now, and the funds are expected to increase to $1,360 after four periods.

What is the yield on the investment?

I. ***Less than annual compounding periods.***

Semiannual	Multiply $n \times 2$	Divide i by 2	Then use normal formula
Quarterly	Multiply $n \times 4$	Divide i by 4	
Monthly	Multiply $n \times 12$	Divide i by 12	

When to use: If the compounding period is more (or perhaps less) frequent than once a year.

Sample problem: You invest $1,000 compounded semiannually at 8 percent per annum over four years.

Determine the future value.

J. ***Patterns of payment—deferred annuity.***

Formulas	Tables
$PV_A = A\left[\frac{1 - \frac{1}{(1+i)^n}}{i}\right] = A\left[\frac{1-(1+i)^{-n}}{i}\right]$	Appendix D
$PV = FV\left[\frac{1}{(1+i)^n}\right] = FV(1+i)^{-n}$	Appendix B

When to use: If an annuity begins in the future.

Sample problem: You will receive $1,000 per period, starting at the end of the fourth period and running through the end of the eighth period. With a discount rate of 8 percent, determine the present value.

K. ***Perpetuity.***

Formula: $PV = \frac{A}{i} = \frac{PMT}{i}$ (9–9)

L. ***Perpetuity growing at a constant rate (g).***

Formula: $PV = \frac{A_1}{i - g}$ (9–10)

M. ***Growing annuity (with end date).***

Formula: $PV_n = A_1\left(\frac{1}{i-g}\right)\left[1 - \left(\frac{1+g}{1+i}\right)^n\right]$ (9–11)

4. Use a time line to set up the problem.

   Use I/Y =, N =, PV =, FV =, PMT = .

   Input the known values for the above, including a zero if necessary (this ensures memory is cleared). Calculate the unknown value.

   The student is encouraged to work on the many problems found at the end of the chapter.

DISCUSSION QUESTIONS

1. How is the future value (Appendix A) related to the present value of a single sum (Appendix B)? (LO2)
2. How is the present value of a single sum (Appendix B) related to the present value of an annuity (Appendix D)? (LO2)
3. Why does money have a time value? (LO1)
4. Does inflation have anything to do with making a dollar today worth more than a dollar tomorrow? (LO1)
5. Adjust the annual formula for a future value of a single amount at 12 percent for 10 years to a semiannual compounding formula. What are the interest factors (FV_{IF}) for the two assumptions? Why are they different? (LO2)
6. If, as an investor, you had a choice of daily, monthly, or quarterly compounding, which would you choose? Why? (LO3)
7. What is a deferred annuity? (LO2)
8. List five different financial applications of the time value of money. (LO1, LO2)
9. Discuss why the compounding of interest within a tax-sheltered plan is so effective, as opposed to paying taxes each year. (LO1)

INTERNET RESOURCES AND QUESTIONS

The Financial Consumer Agency of Canada under the Government of Canada protects rights and provides financial education for consumers. It has a website with several financial tools for time value calculations, including a mortgage calculator.
www.canada.ca/en/services/finance/tools.html

Bloomberg, under money and tools, has a mortgage calculator for U.S. mortgages:
bloomberg.com/personal-finance/calculators/mortgage

The Canadian banks have sites that have mortgage calculators. The Royal Bank calculator has a breakdown for weekly and biweekly accelerated mortgage payments:
rbcroyalbank.com/mortgages/index.html

The Bank of Montreal site, under tools and calculators, has a mortgage calculator:
bmo.com/main/personal/mortgages/calculators#

1. Problems 55 to 58 in this chapter include mortgage calculations. After you have completed these problems, use a mortgage calculator such as the one available at a site listed above to redo the calculations. Are the results the same, and, if not, why is there a difference?
2. Redo the above calculations using a mortgage calculator from a U.S. financial institution or from Bloomberg. Why is there a difference in the numbers calculated?

PROBLEMS

1. What is the present value of
   a. $8,000 in 10 years at 6 percent?
   b. $16,000 in 5 years at 12 percent?
   c. $25,000 in 15 years at 8 percent?
   d. $1,000 in 40 years at 20 percent?

2. You will receive $6,800 three years from now. The discount rate is 10 percent.
   a. What is the value of your investment two years from now?
   b. What is the value of your investment one year from now?
   c. What is the value of your investment today?
3. If you invest $12,000 today, how much will you have
   a. in 6 years at 7 percent?
   b. in 15 years at 12 percent?
   c. in 25 years at 10 percent?
   d. in 25 years at 10 percent (compounded semiannually)?
4. You invest $3,000 for three years at 12 percent.
   a. What is the value of your investment after one year?
   b. What is the value of your investment after two years?
   c. What is the value of your investment after three years?
   d. What is the future value of $3,000 in 3 years at 12 percent interest?
5. How much would you have to invest today to receive
   a. $12,000 in 6 years at 12 percent?
   b. $15,000 in 15 years at 8 percent?
   c. $5,000 each year for 10 years at 8 percent?
   d. $5,000 each year, at the beginning, for 10 years at 8 percent?
   e. $50,000 each year for 50 years at 7 percent?
   f. $50,000 each year for 50 years, at the beginning, at 7 percent?
6. If you invest $8,000 per period for the following number of periods, how much would you have?
   a. 10 years at 5 percent
   b. 20 years at 9 percent
   c. 35 periods at 11 percent
7. Rework the previous problem, assuming that the $8,000 per period is received at the beginning of each year. (Annuity in advance)
8. You invest a single amount of $20,000 for 6 years at 7 percent. At the end of 6 years you take the proceeds and invest them for 8 years at 10 percent. How much will you have after 14 years?
9. Delia has a choice between $30,000 in 50 years or $650 today. If long-term rates are 8 percent, what should be her choice?
10. "Red" Herring will receive $11,000 a year for the next 18 years as a result of his patent. At present, 9 percent is an appropriate discount rate.
    a. Should he be willing to sell out his future rights now for $100,000?
    b. Would he be willing to sell his future rights now for $100,000, if the payments will be made at the beginning of each year?
11. Phil Goode will receive $175,000 in 50 years. Sounds great! However if current interest rates suggested for discounting are 14 percent, what is the present worth of his future "pot of gold"?