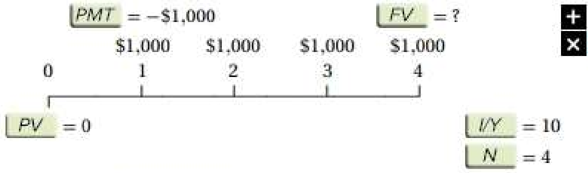
We shall let **A** = Series of equal payments. This will be PMT on most calculators. The formula for the future value of an annuity is as follows:

$$FV_A = A\left[\frac{(1+i)^n - 1}{i}\right] \quad \text{(9–4a)}$$

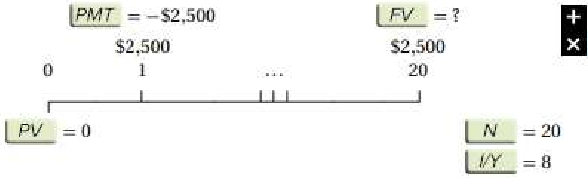


CPT FV = **$4,641.00.**

Tables (optional) Special tables are also available for annuity computations. We shall refer to Appendix C, The Future Value of an Annuity of $1. Let us define *A* as the annuity value and use formula 9–3 for the future value of an annuity. Note that the *A* part of the subscript on both the left and right sides of the formula indicates that we are dealing with tables for an annuity rather than a single amount. Using Appendix C

$$FV_A = A \times FV_{IFA} \quad (n = 4, i = 10\%)$$
$$FV_A = \$1{,}000 \times 4.641 = \$4{,}641$$

Suppose a wealthy relative offered to set aside $2,500 a year for you for the next 20 years; how much would you have to your credit after 20 years if the funds grew at 8 percent?



CPT FV = **$114,404.91.**

A rather tidy sum, considering that only a total of $50,000 ($2,500 per year) has been invested over the 20 years.

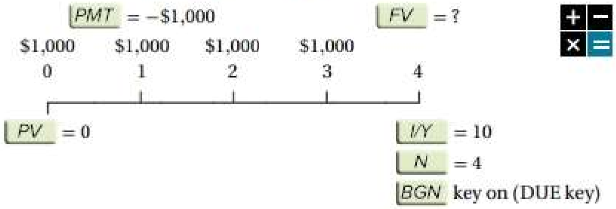
FUTURE VALUE—ANNUITY IN ADVANCE (ANNUITY DUE)

There may be an occasion when the annuity payments occur at the beginning of the time period instead of the end, as we have assumed to this point. These earlier payments increase the future value because the payments have a longer time to earn interest. An annuity in advance places payments at the beginning of each period. This is also referred to as an annuity due. Some older calculators use this term rather than a "Begin" key. Annuity in advance tables are available, but a financial calculator handles the problem easily.

We shall let A_{BGN} = Series of equal payments at the beginning of each period.

The formula for the future value of an annuity (in advance) is as follows:

$$\textbf{(In advance) } FV_A = A_{BGN}\left[\frac{(1+i)^{n+1}-(1+i)}{i}\right] \quad (9\text{–}4b)$$



CPT FV = **$5,105.10.**

Also note that (in advance) $FV_A = FV_A \times (1 + i)$.

Comparing to the calculation for formula 9–4 **a** we obtain

$$\mathbf{\$5,105.10 = \$4,641 \times [1 + 0.10]}$$

Tables (optional) We note that this result could be obtained with the tables (Appendix C) with $n = 5$. This gives the factor 6.105, which is reduced by 1.000 to take account of the payment that does not occur at $t = 4$.

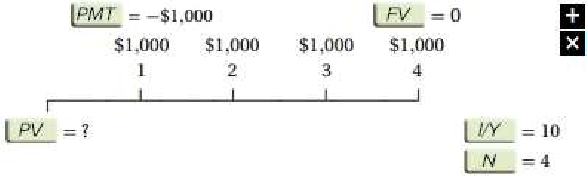
PRESENT VALUE (CUMULATIVE PRESENT VALUE)—ANNUITY

The present value of an annuity is today's worth of a series of consecutive payments, at a given interest rate over a time period. Each individual payment is discounted back to the present and then all of the discounted payments are added up, determining the present value of an annuity.

The formula for the present value of an annuity is as follows:

$$PV_A = A\left[\frac{1-\frac{1}{(1+i)^n}}{i}\right] = A\left[\frac{1-(1+i)^{-n}}{i}\right] \quad (9\text{–}5a)$$

An investment pays $1,000 a year for four years at a discount, or interest, rate of 10 percent. The present value of this annuity would be

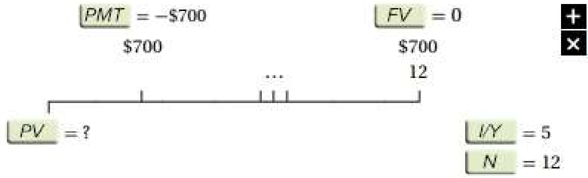


CPT PV = **$3,169.87.**

Tables (optional) Appendix D allows us to eliminate extensive calculations and to find our answer directly. In formula 9–5a the term PV_A refers to the present value of the annuity. Once again, assume $A = \$1{,}000$, $n = 4$, and $i = 10$ percent—only now we want to know the present value of the annuity. Using Appendix D

$$PV_A = A \times PV_{IFA}(n = 4, i = 10\%)$$
$$PV_A = \$1{,}000 \times 3.170 = \$3{,}170$$

A debt requires payment of $700 a year for 12 years at a discount, or interest, rate of 5 percent. The present value of this annuity would be



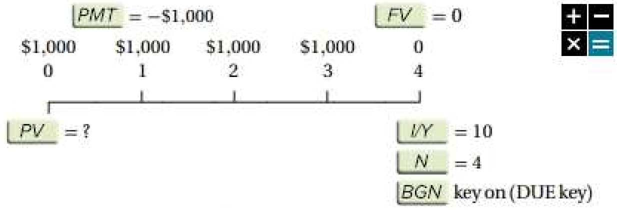
CPT PV = **$6,204.28.**

PRESENT VALUE—ANNUITY IN ADVANCE

We may want to determine the value of an annuity when the first contribution is made immediately. Calculations follow as compared to the previous annuity, with the contributions at the end of each time period.

The formula for the present value of an annuity (in advance) or annuity due is as follows:

$$\text{(In advance) } PV_A = A_{BGN}\left[\frac{(1+i) - \dfrac{1}{(1+i)^{n-1}}}{i}\right] = A_{BGN}\left[\frac{(1+i) - (1+i)^{-n+1}}{i}\right] \quad \text{(9–5b)}$$



CPT PV = **$3,486.85.**

Also, note that (in advance) $PV_A = PV_A \times (1 + i)$.

Comparing to the calculation for formula 9–5a we obtain

$$\mathbf{\$3{,}486.85 = \$3{,}169.87 \times [1 + .10]}$$

Tables (optional) We note that this result could be obtained with the tables (Appendix D) with $n = 3$. This gives the factor 2.487, to which we add 1.000 to take account of the payment that occurs at $t = 0$ and is already stated at present value.

DETERMINING THE ANNUITY VALUE

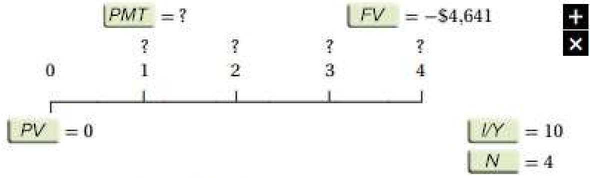
In our prior discussion of annuities, we assumed the unknown variable was the future value or the present value—with specific information available on the annuity value **(A),** the interest rate, and the number of periods or years. In certain cases, our emphasis may shift to solving for one of these other values (on the assumption that future value or present value is given). For now we will concentrate on determining an unknown annuity rate.

Annuity Equalling a Future Value (Sinking-Fund Value)

Assuming we wish to accumulate $4,641 after four years at a 10 percent interest rate, how much must be set aside at the end of each of the four periods?

The formula for an annuity equal to a future value is as follows:

$$A = \mathrm{FV}_A\left[\frac{i}{(1+i)^n - 1}\right] \quad \text{(9–6a)}$$



CPT PMT **= $1,000.**

The solution is the exact reverse of that previously presented under the discussion of the future value of an annuity.

Tables (optional) Or we could take the previously developed statement for the future value of an annuity and solve for A.

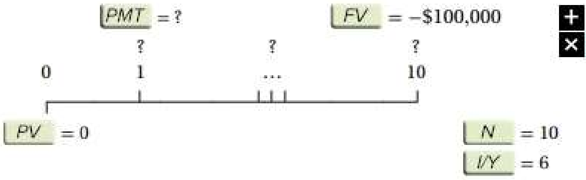
$$\mathrm{FV}_A = A \times \mathrm{FV}_{\mathrm{IFA}}$$

$$A = \frac{\mathrm{FV}_A}{\mathrm{FV}_{\mathrm{IFA}}}$$

The future value of an annuity is given as $4,641, and $\mathrm{FV}_{\mathrm{IFA}}$ may be determined from Appendix C. Whenever you are working with an annuity problem relating to future value, you employ Appendix C, regardless of the variable that is unknown. For $n = 4$, and $i = 10$ percent, $\mathrm{FV}_{\mathrm{IFA}}$ is 4.641. Thus, A equals $1,000.

$$A = \frac{\mathrm{FV}_A}{\mathrm{FV}_{\mathrm{IFA}}} = \frac{\$4{,}641}{4.641} = \$1.000$$

As a second example, assume the director of the Women's Tennis Association must set aside an equal amount for each of the next 10 years to accumulate $100,000 in retirement funds, and that the return on deposited funds is 6 percent.



CPT PMT = **$7,586.80.**

The formula for an annuity in advance equalling a future value is as follows:

$$A_{BGN} = FV_A\left[\frac{i}{(1+i)^{n+1}-(1+i)}\right] \quad (9\text{–}6b)$$

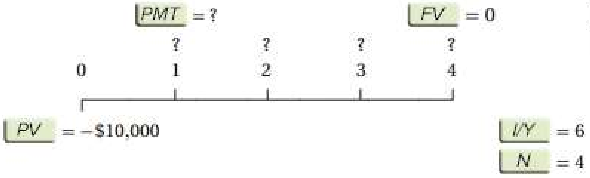
For the same example as above, the required payment or annuity would be $7,157.35. With the annuity in advance a smaller payment is required annually.

Annuity Equalling a Present Value (Capital Recovery Value)

In this instance, we assume that you know the present value and wish to determine what size annuity can be equated to that amount. Suppose your wealthy uncle presents you with $10,000 now to help you get through the next four years of college or university. If you are able to earn 6 percent on deposited funds, how much can you withdraw at the end of each year for four years? We need to know the value of an annuity equal to a given present value.

The formula for an annuity equal to a present value is as follows:

$$A = PV_A\left[\frac{i}{1-\frac{1}{(1+i)^n}}\right] = PV_A\left[\frac{i}{1-(1+i)^{-n}}\right] \quad (9\text{–}7a)$$



CPT PMT = **$2,885.91.**

Tables (optional) We can take the previously developed statement for the present value of an annuity and reverse it to solve for *A*.

$$PV_A = A \times PV_{IFA}$$

$$A = \frac{PV_A}{PV_{IFA}}$$

The appropriate table is Appendix D (present value of an annuity). We determine an answer of $2,886.

$$A = \frac{PV_A}{PV_{IFA}}(n = 4, i = 6\%)$$

$$A = \frac{\$10{,}000}{3.465} = \$2{,}886$$

The flow of funds would follow the pattern in Table 9–1. Annual interest is based on the beginning balance for each year.

Year	Beginning Balance	Annual Interest (6 percent)	Annual Withdrawal	Ending Balance
1	$10,000.00	$600.00	$2,885.91	$7,714.09
2	7,714.09	462.85	2,885.91	5,291.03
3	5,291.03	317.46	2,885.91	2,722.58
4	2,722.58	163.35	2,885.91	0

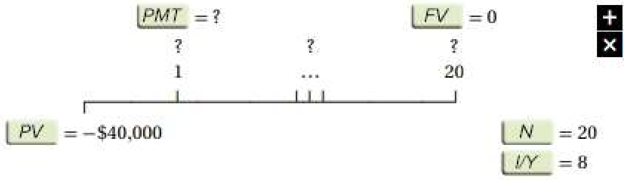
Table 9–1 Relationship of Present Value to Annuity (rounding differences)

The formula for an annuity in advance equalling a present value is as follows:

$$A_{BGN} = PV_A\left[\frac{i}{(1+i) - \frac{1}{(1+i)^{n-1}}}\right] = PV_A\left[\frac{i}{(1+i) - (1+i)^{-n+1}}\right] \quad \text{(9–7b)}$$

For the same example as above (formula 9–7*a*), the available payment or annuity would be $2,722.56, suggesting a lower annual payment, although received sooner.

The same process can be used to indicate necessary repayments on a loan. Suppose a homeowner signs a $40,000 mortgage to be repaid over 20 years at 8 percent interest. How much must he or she pay annually to eventually liquidate the loan? In other words, what annuity paid over 20 years is the equivalent of a $40,000 present value with an 8 percent interest rate? This assumes payments in arrears (9–7**a**).



CPT PMT = **$4,074.09.**

Part of the payment to the mortgage company will go toward the payment of interest, with the remainder applied to debt reduction, as indicated in Table 9–2.

	1	2	3	4	(1 − 4)
Period	Beginning Balance	Annual Payment	Annual Interest (8 percent)	Repayment on Principal	Ending Balance
1	$40,000	$4,074	$3,200	$ 874	$39,126
2	39,126	4,074	3,130	944	38,182
3	38,182	4,074	3,055	1,019	37,163

Table 9–2 Payoff table for loan (amortization table)

If this same process is followed over 20 years, the balance will be reduced to zero. The student might note that the homeowner will pay over $41,000 of ***interest*** during the term of the loan, as indicated below.

Total payments ($4,074 for 20 years)	$ 81,480
Repayment of principal	–40,000
Payments applied to interest	$ 41,480

FORMULA SUMMARY

In our discussion thus far, we have considered the following time-value-of-money problems with our calculator, by formula, or with tables. In each case we knew three or four variables and solved for an unknown.

	Formula	Appendix
Future value—single amount (9–1)	$FV = PV(1 + i)^n$	A
Present value—single amount (9–3)	$PV = FV\left[\frac{1}{(1 + i)^n}\right]$	B
Future value—annuity (9–4*a*)	$FV_A = A\left[\frac{(1 + i)^n - 1}{i}\right]$	C
Future value—annuity in advance (9–4*b*)	$FV_A = A_{BGN}\left[\frac{(1 + i)^{n+1} - (1 + i)}{i}\right]$	—
Present value—annuity (9–5*a*)	$PV_A = A_{BGN}\left[\frac{1 - \frac{1}{(1 + i)^n}}{i}\right]$	D
Present value—annuity in advance (9–5*b*)	$PV_A = A\left[\frac{(1 + i) - \frac{1}{(1 + i)^{n-1}}}{i}\right]$	—
Annuity equalling a future value (9–6*a*)	$A = FV_A\left[\frac{i}{(1 + i)^n - 1}\right]$	C
Annuity in advance equalling a future value (9–6*b*)	$A_{BGN} = FV_A\left[\frac{i}{(1 + i)^{n+1} - (1 + i)}\right]$	—
Annuity equalling a present value (9–7*a*)	$A = PV_A\left[\frac{i}{1 - \frac{1}{(1 + i)^n}}\right]$	D
Annuity in advance equalling a present value (9–7*b*)	$A_{BGN} = PV_A\left[\frac{i}{(1 + i) - \frac{1}{(1 + i)^{n-1}}}\right]$	—

DETERMINING THE YIELD ON AN INVESTMENT

LO3 We will follow the prior procedure once again, but now the unknown variable will be i, the interest rate, yield, or return on the investment. Yield is a measure equating values across different time periods.

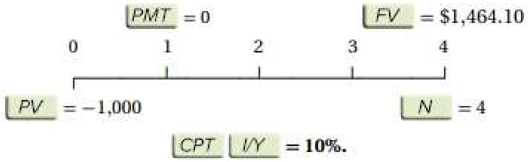
Yield—Present Value of a Single Amount

An investment producing $1,464 after four years has a present value of $1,000. What is the interest rate, or yield, on the investment?

The ***I/Y*** can also be used to determine the growth rate of an investment or pattern of payments over time.

The formula is as follows:

$$i = \left(\frac{FV}{PV}\right)^{\frac{1}{n}} - 1 \quad (9\text{–}8)$$



Tables (optional) We can also use the basic formula for the present value of a single amount and rearrange the terms.

$$PV = FV \times PV_{IF}$$

$$PV_{IF} = \frac{PV}{FV} = \frac{\$1,000}{\$1,464} = 0.683$$

The determination of PV_{IF} does not give us the final answer, but it scales down the problem so that we may ascertain the answer from Appendix B. A portion of Appendix B is reproduced below.

Periods	1%	2%	3%	4%	5%	6%	8%	10%
2	0.980	0.961	0.943	0.925	0.907	0.890	0.857	0.826
3	0.971	0.942	0.915	0.889	0.864	0.840	0.794	0.751
4	0.961	0.924	0.888	0.855	0.823	0.792	0.735	0.683

Read down the left-hand column of the table until you have located the number of periods in question (in this case $n = 4$), and read across the table for $n = 4$ until you have located the computed value of PV_{IF} from above. We see that for $n = 4$ and PV_{IF} equal to 0.683, the interest rate, or yield, is 10 percent. This is the rate that will equate $1,464 received in four years to $1,000 today.

If a PV_{IF} value does not fall under a given interest rate, an approximation is possible. For example, with $n = 3$ and $PV_{IF} = 0.861$, 5 percent may be suggested as an approximate answer.

Interpolation may also be used to find a more precise answer. In the above example, we write out the two PV_{IF} values between which the designated PV_{IF} (0.861) falls and take the difference between the two.

PV_{IF} at 5%	0.864
PV_{IF} at 6%	0.840
	0.024

We then find the difference between the PV_{IF} value at the lowest interest rate and the designated PV_{IF} value.

PV_{IF} at 5%	0.864
PV_{IF} designated.	0.861
	0.003

We next express this value (0.003) as a fraction of the preceding value (0.024) and multiply by the difference between the two interest rates (6 percent minus 5 percent). The value is added to the lower interest rate (5 percent) to get a more exact answer of 5.125 percent rather than the estimated 5 percent.

$$5\% + \frac{0.003}{0.024}(1\%) =$$

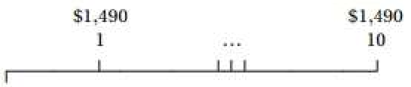
$$5\% + 0.125(1\%) =$$

$$5\% + 0.125\% = 5.125\%$$

Yield—Present Value of an Annuity

Assuming a $10,000 investment will produce $1,490 a year for the next 10 years, what is the yield on the investment?

FV = 0



PV = −$10,000

N = 10

PMT = $1,490

CPT I/Y = **7.996%.**

Tables (optional) Let's look at the present value of an annuity. Take the basic formula for the present value of an annuity, and rearrange the terms.

$$PV_A = A \times PV_{IFA}$$

$$PV_{IFA} = \frac{PV_A}{A}$$

The appropriate table is Appendix D (the present value of an annuity of $1).

$$PV_{IFA} = \frac{PV_A}{A} = \frac{\$10,000}{\$1,490} = 6.711$$

If the student will flip to Appendix D and read across the columns for $n = 10$ periods, he or she will see that the yield is 8 percent.

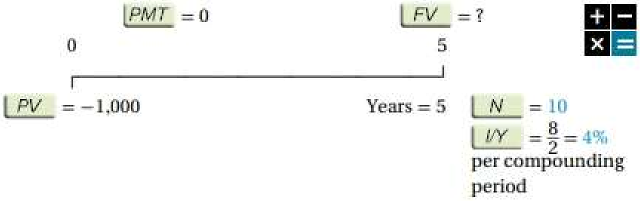
The same type of approximated or interpolated yield that applied to a single amount can also be applied to an annuity when necessary.

SPECIAL CONSIDERATIONS IN TIME VALUE ANALYSIS

We have assumed that interest was compounded or discounted on an annual basis. This assumption will now be relaxed. Contractual arrangements, such as an instalment purchase agreement or a corporate bond contract, may call for semiannual, quarterly, or monthly compounding periods. The adjustment to the normal formula is simple.

To determine ***n***, multiply the number of years by the number of compounding periods during the year. The factor for ***i*** is then determined by dividing the quoted annual interest rate by the number of compounding periods.

Case 1: Find the future value of a $1,000 investment after five years at 8 percent annual interest, compounded semiannually.

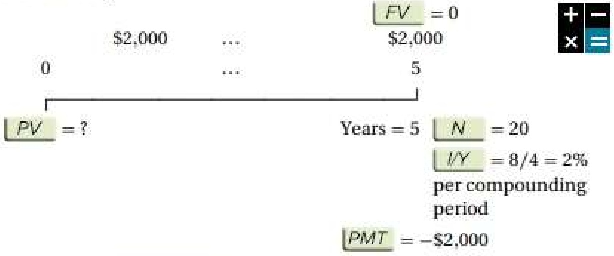


CPT FV = **$1,480.24.**

Tables (optional) Since the problem calls for the future value of a single amount, the formula is FV = PV × FV_{IF}. Using Appendix A for $n = 10$ and $i = 4$ percent, the answer is $1,480.

$$FV = PV \times FV_{IF}$$
$$FV = \$1{,}000 \times 1.480 = \$1{,}480$$

Case 2: Find the present value of 20 quarterly payments of $2,000 each to be received over the next five years. The stated interest rate is 8 percent per annum. The problem calls for the present value of an annuity.



CPT PV = **$32,702.87.**

Tables (optional) We again follow the same procedure as in Case 1 in regard to n and i.

$$PV_A = A \times PV_{IFA}\ (n = 20, i = 2\%)$$
(from Appendix D)
$$PV_A = \$2{,}000 \times 16.351 = \$32{,}702$$

Patterns of Payment

Time-value-of-money problems may evolve around a number of different payment or receipt patterns. Not every situation will involve a single amount or an annuity.