12. Carrie Tune will receive $18,000 a year for the next 20 years as payment for a song she has just written. If a present 10 percent discount rate is applied,
    a. Should she be willing to sell out her future rights now for $160,000?
    b. Would she be willing to sell her future rights now for $160,000, if the payments will be made at the beginning of each year?
13. George Penny will receive $32,250 for the next 10 years as a payment for a slogan he coined. Currently a 6 percent discount rate is appropriate.
    a. Should he be willing to sell his future rights now for $240,000?
    b. Should he be willing to sell his future rights now for $240,000, if payments will be made at the beginning of the year?
14. The Epic Contest awards $10,000,000. It will be paid over the next 50 years at the rate of $250,000 per year with the first payment today. With a discount rate of 9 percent, what is the present value of this prize?
15. Joan Lucky won the $80 million lottery. She is to receive $1 million a year for the next 50 years plus an additional lump-sum payment of $30 million after 50 years. The discount rate is 12 percent. What is the current value of her winnings?
16. Larry Doby invests $50,000 in a mint condition 1952 "Rocket" Richard Topps hockey card. He expects the card to increase in value 8 percent per year for the next five years. How much will his card be worth after five years?
17. Dr. Sisters has been secretly depositing $10,500 in her savings account every December starting in 1994. Her account earns 6 percent compounded annually. How much did she have in December 2005? (Assume a deposit is made in 2005.) Make sure to carefully count the years.
18. At a growth (interest) rate of 8 percent annually, how long will it take for a sum to double? To triple? Select the year that is closest to the correct answer.
19. If you owe $30,000 at the end of seven years, how much should your creditor accept in payment immediately if she could earn 11 percent on her money?
20. Jack Hammer invests in a stock that will pay dividends of $2.00 at the end of the first year; $2.20 at the end of the second year; and $2.40 at the end of the third year. Also at the end of the third year he believes he will be able to sell the stock for $33. What is the present value of these future benefits if a discount rate of 11 percent is applied?
21. S. Ken Flint retired as president of Colour Tile Company, but he is currently on a consulting contract for $45,000 per year for the next 10 years.
    a. If Mr. Flint's opportunity cost (potential return) is 10 percent, what is the present value of his consulting contract?
    b. Assuming Mr. Flint will not retire for two more years and will not start to receive his ten payments until the end of the third year, what would be the value of his deferred annuity?
    c. Recalculate part ***a*** assuming the contract stipulates that payments are to be made at the beginning of each year.
22. Cousin Berta invested $100,000 ten years ago at 12 percent, compounded quarterly.
    a. How much has she accumulated?
    b. What is her effective annual interest rate (rate of return)?

23. Determine the amount of money in a savings account at the end of five years, given an initial deposit of $3,000 and an 8 percent annual interest rate when interest is compounded (a) annually, (b) semiannually, and (c) quarterly. Calculate the effective annual interest rate of each compounding possibility.

24. Joe Macro wishes to have accumulated $60,000 ten years from today by making an equal annual deposit into an account that pays 10 percent, compounded quarterly.
    a. What is the effective annual interest rate?
    b. How large an annual deposit is required to meet Joe's objective?
    c. How large an annual deposit is required if the deposits are made at the beginning of each year?

25. Sally Gravita has received a settlement from an insurance company that will pay her $23,500 annually for 12 years. Current interest rates are 8 percent, compounded semiannually.
    a. What is the effective annual interest rate?
    b. How much is the present worth of Sally's settlement?
    c. How much is the present worth of Sally's settlement if payments are made at the beginning of each year?

26. Your grandfather has offered you a choice of one of the three following alternatives: $5,000 now; $1,000 a year for eight years; or $12,000 at the end of eight years. Assuming you could earn 11 percent annually, which alternative would you choose? If you could earn 12 percent annually, would you still choose the same alternative?

27. You need $23,000 at the end of 7 years, and your only investment outlet is a 9 percent long-term certificate of deposit (compounded annually). With the certificate of deposit, you make an initial investment at the beginning of the first year.
    a. What single payment could be made at the beginning of the first year to achieve this objective?
    b. What amount could you invest at the end of each year annually for 7 years to achieve this same objective?

28. Amy Hirt started a paper route on January 1, 2012. Every three months, she deposits $300 in her bank account, which earns 8 percent annually but is compounded quarterly. On December 31, 2015, she used the entire balance in her bank account to invest in a contract that pays 14 percent annually. How much will she have on December 31, 2018?

29. On January 1, 2016, Charley Dow bought 1,000 shares of stock at $12 per share. On December 31, 2018, he sold the stock for $18 per share. What was his annual rate of return?

30. April Wine bought 425 shares of stock at $5.50 per share. Four years later she sold the stock for $21 per share. What was her annual growth rate (rate of return, ROR) for her capital?

31. Al Counsel purchased 357 shares of Eco-Survival Tours on July 1, 2016 for $5.00 per share. Find his annual rate of return if he sold the stock
    a. On June 30, 2017, for $6.00 per share.
    b. On December 31, 2019, for $10.92 per share.
    c. On June 30, 2022, for $8.39 per share.

32. John Foresight has just invested $8,370 for his son (age one). The money will be used for his son's education 17 years from now. He calculates that he will need $90,000 for his son's education by the time the boy goes to school. What rate of return will Mr. Foresight need to achieve this goal?

33. Chris Seals has just given an insurance company $56,521. In return, she will receive an annuity of $7,500 for 12 years.
    a. At what rate of return must the insurance company invest this $56,521 to make the annual payments?
    b. What rate of return is required if the annuity is payable at the beginning of each year?

34. Mr. G. Day has approached his bank about a loan. He expects to receive $30,000 in three years and $85,000 nine years from now. These funds will be applied against the loan as they are received. The bank suggests that interest rates will be 9 percent for the next five years and 7 percent in subsequent years. Calculate the maximum amount Mr. G. Day can borrow.

35. Ms. R. Emm has purchased land for $90,000 in cash today and another $45,000 four years from today. Interest rates over a four-year period are currently 8 percent, compounded semiannually. Calculate the cash value of the property.

36. Count Crow wishes to have a large celebration eight years from today costing $150,000. Currently, he has an investment of $625,000 in a financial institution earning 7.5 percent interest annually. Count Crow also wishes to receive an annual payment from his investment over this period at the beginning of each year starting today. Calculate how much of an annual payment the Count can expect.

37. Graham Bell has just retired after 30 years with the telephone company. His total pension funds have an accumulated value of $300,000, and his life expectancy is 16 more years. His pension fund manager assumes he can earn a 7 percent return on his assets. What will be his yearly annuity for the next 16 years?

38. River Babylon, an archaeology professor, invests $65,000 in a parcel of land that is expected to increase in value by 8 percent per year for the next five years. He will take the proceeds and provide himself with a 12-year annuity. Assuming a 9 percent interest rate, how much will this annuity be?

39. Una Day is planning to retire in 20 years, at which time she hopes to have accumulated enough money to receive an annuity of $12,000 a year for 25 years of retirement. During her pre-retirement period she expects to earn 8 percent annually, while during retirement she expects to earn 10 percent annually on her money. What annual contributions to this retirement fund are required for Una to achieve her objective and sleep well at night?

40. Louisa and Bart are twins, aged 20. Advanced for their age in some respects, they were both working and also planning their financial future. Both felt they could commit $5,000 per year for their retirement expected at age 65.

    However, Louisa planned to start right away on her commitment by investing $5,000 per year in her tax-free savings account for the next 15 years and then make no further commitment to this account until she reached retirement at age 65 (thirty years). She would leave her money in the tax-free account, accumulating interest during this 30-year period.

    Bart, on the other hand, was going to spend freely for the next 15 years and then make his commitment of $5,000 per year to the tax-free savings account for the following 30 years, until his retirement at age 65.
    a. How much would Louisa and Bart have in their tax-free retirement account if yields (interest rates) are 10 percent throughout their lifetime?
    b. How much would Louisa and Bart have in their tax-free retirement account if yields (interest rates) are 3 percent throughout their lifetime?

41. You wish to retire after 30 years, at which time you want to have accumulated enough money to receive an annuity of $55,000 a year for 18 years of retirement. During the period before retirement, you can earn 9 percent annually, while after retirement you can earn 7 percent on your money.
    a. What annual contributions to the retirement fund will allow you to receive the $55,000 annually?
    b. What annual contributions are required if the contributions are made at the beginning of each year?
42. Your retirement planning suggests a goal of $57,000 a year in today's dollars for 30 years of retirement. Retirement will begin 35 years from today, at which time you will expect your first annuity payment. Inflation between now and retirement is expected to be 4 percent annually (do not consider inflation during retirement). The anticipated yield over the pre-retirement period is 7 percent annually, and 8 percent per annum is anticipated during retirement. Calculate how much you should set aside each year between now and retirement to achieve your goal. (Ignore taxes.)
43. For your retirement you would like to receive $75,000 a year in today's dollars for a period of 25 years. A problem, of course, is that you expect inflation to average 2.5 percent a year for the next 33 years until your retirement. (Inflation will not be a concern during retirement.) Interest rates (borrowing rates equal lending rates in this perfect market without taxes) are expected to be 7 percent until retirement and 5 percent during retirement. Your first retirement annuity is to be received 33 years from today, and your first contribution to your retirement will be at the end of this year and will be made 33 times. You will also require $125,000 (do not inflate) from your retirement funds in 17 years for a sabbatical that you are planning. Calculate the equal annual (33) contributions to your retirement fund required for this all to happen.
44. Del Monty will receive the following payments at the end of the next three years: $2,000, $3,500, and $4,500. Then from the end of the fourth year through the end of the tenth year, he will receive an annuity of $5,000 per year. At a discount rate of 9 percent, what is the present value of these future benefits?
45. Bridget Jones has a contract in which she will receive the following payments for the next five years: $1,000, $2,000, $3,000, $4,000, $5,000. She will then receive an annuity of $8,500 a year from the end of the sixth year through the end of the fifteenth year. The appropriate discount rate is 14 percent. If she is offered a buyout of the contract for $30,000, should she do it?
46. Darla White has just purchased an annuity to begin payment at the end of 2019 (that is the date of the first payment). Assume it is now the beginning of 2016. The annuity is for $12,000 per year and is designed to last 8 years. If the interest rate for this problem is 11 percent, what is the most she should have paid for the annuity?
47. Emphatically Square and heirs will receive $1,000 a year forever with a long-term annual expected interest rate of 7 percent. What is the current worth of this annuity?
48. Forever College will provide a scholarship of $7,500 a year forever with a long-term annual expected interest rate of 6 percent. What is the current worth of this annuity?
49. On second thought, Emphatically Square and heirs will receive $1,000 a year forever that will grow by 3 percent annually. The long-term annual expected interest rate is 7 percent. What is the current worth of this annuity?
50. On third thought, Forever College will provide a scholarship of $7,500 a year forever, growing in value by 2 percent per year. The long-term annual expected interest rate is 6 percent. What is the current worth of this annuity?

51. On fourth thought, Emphatically Square and heirs will receive $1,000 a year for only 25 years, but it will grow by 3 percent annually. The long-term annual expected interest rate is 7 percent. What is the current worth of this annuity?

52. On fifth thought, Forever College will provide a scholarship of $7,500 a year for only 30 years, growing in value by 2 percent per year. The long-term annual expected interest rate is 6 percent. What is the current worth of this annuity?

53. For your retirement you would like to receive the equivalent of $90,000 a year in today's dollars for a period of 30 years. You expect inflation to average 3 percent a year for the next 70 years. Yields (borrowing rates equal lending rates in this perfect market without taxes) are expected to be 5 percent until retirement and 4 percent during retirement. Your first retirement annuity is to be received 40 years from today and your first contribution to your retirement will be at the end of this year and will be made 40 times. You will also require $250,000 (do not inflate) from your retirement funds in 10 years for an anniversary bash that you are planning. Calculate the equal annual (40) contributions to your retirement fund required for this all to happen.

54. If you borrow $9,725 and are required to pay back the loan in five equal annual instalments of $2,500, what is the interest rate associated with the loan?

55. Sarah Adia owes $15,000 now. A lender will carry the debt for three more years at 8 percent interest. That is, in this particular case, the amount owed will go up by 8 percent per year for three years. The lender then will require that Sarah pay off the loan over the next 5 years at 9 percent interest. What will her annual payment be?

56. If your uncle borrows $50,000 from the bank at 10 percent interest over the eight-year life of the loan, what equal annual payments must be made to discharge the loan, plus pay the bank its required rate of interest (round to the nearest dollar)? How much of his first payment will be applied to interest? To principal? How much of his second payment will be applied to each?

57. Jim Thomas borrows $70,000 at 12 percent interest toward the purchase of a home. His mortgage is for 30 years.
    a. How much will his annual payments be? (Although home payments are usually on a monthly basis, we shall do our analysis on an annual basis for ease of computation. We get a reasonably accurate answer.)
    b. How much interest will he pay over the life of the loan?
    c. How much should he be willing to pay to get out of a 12 percent mortgage and into a 10 percent mortgage with 30 years remaining on the mortgage? Assume current interest rates are 10 percent. Carefully consider the time value of money. Disregard taxes.

58. Larry Davis borrows $80,000 at 14 percent interest toward the purchase of a home. His mortgage is for 25 years.
    a. How much will his annual payments be? (Although home payments are usually on a monthly basis, we shall do our analysis on an annual basis for ease of computation. We will get a reasonably accurate answer.)
    b. How much interest will he pay over the life of the loan?
    c. How much should he be willing to pay to get out of a 14 percent mortgage and into a 10 percent mortgage with 25 years remaining on the mortgage? Assume current interest rates (yields) are 10 percent.

59 Peter Piper has applied for a mortgage of $120,000. Interest is computed at 8.5 percent compounded semiannually. The mortgage will be paid off over 20 years.
   a. Calculate Peter's monthly payment.
   b. Calculate Peter's weekly payment.
   c. Calculate Peter's biweekly (every 2nd week) payment.

60. Ocean Spray has applied for a mortgage of $200,000. Interest is computed at 4.5 percent compounded semiannually. The mortgage will be paid off over 25 years.
   a. Calculate Ocean's monthly payment.
   b. Calculate Ocean's weekly payment.
   c. Calculate Ocean's biweekly (every 2nd week) payment.

61. Bing and Monica Cherrie require a mortgage of $145,000 and can afford monthly payments of $1,150 on the mortgage. Current interest rates are 4 percent compounded semiannually. How long should the Cherries select to pay off the mortgage (the amortization period)?

62. Deidre Hall can afford monthly payments of $690 on a mortgage. Current mortgage rates are 3.5 percent, compounded semiannually. The longest period over which a mortgage can be amortized is 25 years. What size mortgage can Deidre afford?

63. Your younger sister, Barbara, will start college in five years. She has just informed your parents that she wants to go to Eastern University, which will cost $15,000 per year for four years (assumed to come at the end of each year). Anticipating Barbara's ambitions, your parents started investing $2,000 per year five years ago and will continue to do so for five more years. How much more will your parents have to invest each year for the next five years to have the necessary funds for Barbara's education? Use 10 percent as the appropriate interest rate throughout this problem (for discounting or compounding).

64. Barbara (from the previous problem) is now 18 years old (five years have passed), and she wants to get married instead of going to school. Your parents have accumulated the necessary funds for her education.

    Instead of her schooling, your parents are paying $7,000 for her upcoming wedding and plan to take a year-end vacation costing $4,000 per year for the next three years.

    How much will your parents have at the end of three years to help you with graduate school, which you will start then? You plan to work on a master's and perhaps a Ph.D. If graduate school costs $12,850 per year, approximately how long will you be able to stay in school based on these funds? Use 10 percent as the appropriate interest rate throughout this problem.

65. You are chairperson of the investment fund for Middle Hockey League. You are asked to set up a fund of quarterly payments to be compounded quarterly to accumulate a sum of $250,000 after 10 years at an 8 percent annual rate (40 payments). The first payment into the fund is to occur three months from today, and the last payment is to take place at the end of the tenth year.
   a. Determine how much the quarterly payment should be. (Round to whole numbers.) On the day after the sixteenth payment is made (the beginning of the fourth year) the interest rate goes up to a 12 percent annual rate, and you can earn a 12 percent annual rate on funds that have been accumulated as well as all future payments into the fund. Interest is to be compounded quarterly on all funds.
   b. Determine how much the revised quarterly payments should be after this rate change (there are 24 payments and compounding dates). The next payment will be in the fourth quarter of the fourth year. (Round all values to whole numbers.)

COMPREHENSIVE PROBLEM

66. Mr. Rambo, President of Assault Weapons Inc. was pleased to hear that he had three offers from major defence companies for his latest missile firing automatic ejector. He will use a discount rate of 12 percent to evaluate each offer.

Offer I $500,000 now plus $120,000 from the end of year 6 through 15. Also, if the product goes over $50 million in cumulative sales by the end of year 15, he will receive an additional $1,500,000. Rambo thought there was a 75 percent probability this would happen.

Offer II A trust fund would be set up for the next nine years. At the end of that period, Rambo would receive the proceeds (and discount them back to the present at 12 percent). The trust fund called for semiannual payments for the next nine years of $80,000 (a total of $160,000 per year). The payments would start immediately (beginning).

Offer III A trust fund would be set up for the next nine years. At the end of that period, Rambo would receive the proceeds (and discount them back to the present at 12 percent). The trust fund called for semiannual payments for the next nine years of $80,000 (a total of $160,000 per year). The payments would start immediately (beginning).

Determine the present value of each offer and select the best offer.

MINI CASE

Allison Boone, M.D.

Allison Boone had been practising medicine for seven years. Her specialty was neurology. She had received her bachelor's degree in chemistry from the University of Toronto and her M.D. from McMaster University. She did her residency at Toronto General Hospital. Allison practised neurology in a clinic with three other doctors in Toronto.

Her husband, Samuel L. Boone, held an administrative position at the Toronto Dominion Bank. Allison and Samuel had been married for five years and were the parents of young twin sons, Todd and Trey. They lived in the Beaches area in a beautiful four-room house overlooking Lake Ontario.

Allison normally left for work at 7:30 a.m. and closed her office at 5:30 p.m. to return home. On Tuesday, July 6, 20XX, at 5:15 p.m., she received an emergency call from Toronto General Hospital and immediately went to the hospital to help a patient who had suffered serious brain damage. By the time she had administered aid and helped prepare the patient for surgery it was 11:00 p.m.

On her way home along Lakeshore Boulevard, she was confronted head-on by a drunken driver going over 110 kilometres an hour. A crash was inevitable, and Allison and the other driver were killed instantly. The drunken driver was making a late delivery for Wayland Frozen Foods Inc.

Legal Considerations The families of both drivers were devastated by the news of the accident. After the funeral and explaining the situation to the children, Samuel Boone knew he must seek legal redress for his family's enormous loss. Following interviews with a number of lawyers, he decided to hire Sloan Whitaker.

Whitaker was with a Toronto law firm (Hanson, Whitaker, and Thomason) that specialized in plaintiff's lawsuits. He had been in practice for over 20 years since graduating from Osgoode Law School.

When Whitaker began his investigation on behalf of Samuel Boone and his family, he was surprised to find out the driver of the delivery vehicle had a prior record of alcohol

abuse and that Wayland Frozen Foods Inc. had knowledge of the problem when it hired him. It appears the driver was a relative of the owner, and at the time of employment he revealed what he termed "a past alcoholic problem that was now under control." In any event, he was acting as an employee for Wayland Frozen Foods in using its truck to make a business-related delivery at the time of the accident. The fact that he was speeding and intoxicated at the time of the impact only increased the legal exposure for Wayland Frozen Foods.

After much negotiating with the law firm that represented Wayland Frozen Foods (and its insurance company), Whitaker received three proposals for an out-of-court settlement to be paid to Allison Boone's family. The intent of the proposals was to replace the future earnings of Allison Boone, less any of the earnings she would have personally needed for her normal living requirements. Also, the value that she provided for her family as a wife and mother, quite aside from her earning power, had to be considered. Finally, there was the issue of punitive damages that Wayland Frozen Foods was exposed to as a result of letting an unqualified driver operate its truck. If the case went to court, there was no telling how much a jury might assign to this last factor.

The three proposals are listed below. An actuarial table indicated that Allison, age 37 at the time of the accident, had an anticipated life expectancy of 40 more years.

Proposal 1	Pay the family of Allison Boone $300,000 a year for the next 20 years, and $500,000 a year for the remaining 20 years.
Proposal 2	Pay the family a lump-sum payment of $5 million today.
Proposal 3	Pay the family of Allison Boone a relatively small amount of $50,000 a year for the next 40 years, but also guarantee them a final payment of $75 million at the end of 40 years.

In order to analyze the present value of these three proposals, Whitaker called on a financial expert to do the analysis. You will aid in the process.

a. Using a current long-term interest rate, recommend a proposal to the Boone family. Justify your choice of discount rate.

b. Now assume that a discount rate of 11 percent is used. Which of the three alternatives provides the highest present value?

c. Explain why the change in outcome takes place between part ***a*** and part ***b***.

d. If Whitaker thinks punitive damages are likely to be $4 million in a jury trial, should he be more likely to settle out of court or go before a jury?

APPENDIX 9A

Derivation of Time-Value-of-Money Formulas

Equation 9–1 (Future value)

$$FV_1 = PV + iPV = PV(1+i)$$
$$FV_2 = PV(1+i) + iPV(1+i) = PV(1+i) \times (1+i) = PV(1+i)^2$$
$$FV_3 = PV(1+i)^2 + iPV(1+i)^2 = PV(1+i)^2 \times (1+i) = PV(1+i)^3$$
$$FV_n = PV(1+i)^{n-1} + iPV(1+i)^{n-1} = PV(1+i)^{n-1} \times (1+i) = PV(1+i)^n$$
$$FV = PV(1+i)^n$$

Equation 9–3 (Present value)

$$FV = PV(1+i)^n \qquad PV = FV \times \left[\frac{1}{(1+i)^n}\right]$$

Equation 9–4a (Future value—annuity)

$$\begin{aligned}
FV_1 &= A + 0 &&= A\\
FV_2 &= A + A + Ai &&= A + A(1+i)\\
FV_3 &= A + [A + A(1+i)] + i[A + A(1+i)] &&= A + [A + A(1+i)](1+i)\\
&= A + A(1+i) + A(1+i)^2\\
FV_4 &= A + A + A(1+i) + A(1+i)^2 + i[A + A(1+i) + A(1+i)^2]\\
&= A + A(1+i) + A(1+i)^2 + A(1+i)^3\\
FV_A &= A + A(1+i) + A(1+i)^2 + A(1+i)^3 + \ldots + A(1+i)^{n-1}
\end{aligned}$$

To get the sum of this geometric series, multiply by $(1 + i)$ and deduct the original equation.

$$\begin{aligned}
FV_A(1+i) &= A(1+i) + A(1+i)^2 + A(1+i)^3 + \ldots + A(1+i)^{n-1} + A(1+i)^n\\
-FV_A &\quad -A + A(1+i) + A(1+i)^2 + A(1+i)^3 + \ldots + A(1+i)^{n-1}\\
\hline
FV_A(1+i) - FV_A &= A(1+i)^n - A\\
FV_A + iFV_A - FV_A &= A(1+i)^n - A\\
iFV_A &= A[(1+i)^n - 1]\\
FV_A &= \left[\frac{A(1+i)^n - 1}{i}\right]
\end{aligned}$$

Equation 9–4b (Future value—annuity in advance)

$$\begin{aligned}
FV_1 &= A + iA = A(1+i)\\
FV_2 &= A + iA + A(1+i) + i[A(1+i)] = A(1+i) + (1+i)[A(1+i)] = A(1+i) + A(1+i)^2\\
FV_3 &= A + iA + A(1+i) + A(1+i)^2 + i[A(1+i)] + A(1+i)^2]\\
&= A(1+i) + A(1+i)^2 + A(1+i)^3\\
FV_A &= A(1+i) + A(1+i)^2 + A(1+i)^3 + \ldots + A(1+i)^n
\end{aligned}$$

To get the sum of this geometric series, multiply by $(1 + i)$ and deduct the original equation.

$$\begin{aligned}
FV_A(1+i) &= A(1+i)^2 + A(1+i)^3 + A(1+i)^4 + \ldots + A(1+i)^n + A(1+i)^{n+1}\\
-FV_A &\quad -A(1+i) + A(1+i)^2 + A(1+i)^3 + A(1+i)^4 + \ldots + A(1+i)^n\\
\hline
FV_A(1+i) - FV_A &= A(1+i)^{n+1} - A(1+i)\\
FV_A + iFV_A - FV_A &= A(1+i)^{n+1} - A(1+i)\\
iFV_A &= A[(1+i)^{n+1} - (1+i)]\\
FV_A &= A_{BGN}\left[\frac{(1+i)^{n+1} - (1+i)}{i}\right]
\end{aligned}$$

Equation 9–5a (Present value of an annuity)

$$\begin{aligned}
PV_1 &= A(1+i)^{-1}\\
PV_2 &= A(1+i)^{-1} + A(1+i)^{-2}\\
PV_3 &= A(1+i)^{-1} + A(1+i)^{-2} + A(1+i)^{-3}\\
PV_n &= A(1+i)^{-1} + A(1+i)^{-2} + A(1+i)^{-3} + \ldots + A(1+i)^{-n}
\end{aligned}$$

To get the sum of this geometric series, multiply by $(1 + i)$ and deduct the original equation.

$$\begin{aligned}
PV_n(1+i) &= A(1+i)^0 + A(1+i)^{-1} + A(1+i)^{-2} + A(1+i)^{-3} + \ldots + A(1+i)^{-(n-1)}\\
-PV_n &\quad -A(1+i)^{-1} + A(1+i)^{-2} + A(1+i)^{-3} + \ldots + A(1+i)^{-n}\\
\hline
PV_n(1+i) - PV_n &= A - A(1+i)^{-n}\\
PV_n + iPV_n - PV_n &= A[1 - (1+i)^{-n}]\\
iPV_n &= A[1 - (1+i)^{-n}]\\
PV_A &= A\left[\frac{1 - \frac{1}{(1+i)^n}}{i}\right]
\end{aligned}$$

Equation 9–5b (Present value of an annuity in advance)

$$PV_1 = A$$
$$PV_2 = A + A(1+i)^{-1}$$
$$PV_3 = A + A(1+i)^{-1} + A(1+i)^{-2}$$
$$PV_n = A + A(1+i)^{-1} + A(1+i)^{-2} + \ldots + A(1+i)^{-(n-1)}$$

To get the sum of this geometric series, multiply by $(1+i)$ and deduct the original equation.

$$PV_n(1+i) = A(1+i) + A(1+i)^0 + A(1+i)^{-1} + A(1+i)^{-2} + \ldots + A(1+i)^{-(n-2)}$$
$$-PV_n \quad -A + A(1+i)^{-1} + A(1+i)^{-2} + \ldots + A(1+i)^{-(n-1)}$$

$$PV_n(1+i) - PV_n = A(1+i) - A(1+i)^{-(n-1)}$$
$$PV_n + iPV_n - PV_n = A[(1+i) - (1+i)^{-(n-1)}]$$
$$iPV_n = A[(1+i) - (1+i)^{-(n-1)}]$$
$$PV_A = A_{BGN}\left[\frac{(1+i) - \frac{1}{(1+i)^{n-1}}}{i}\right]$$

Equation 9–6a (Annuity equalling a future value)

$$FV_A = A\left[\frac{(1+i)^n - 1}{i}\right] \qquad A = FV_A\left[\frac{i}{(1+i)^n - 1}\right] \quad (9\text{–}4a)$$

Equation 9–6b (Annuity in advance equalling a future value)

$$FV_A = A_{BGN}\left[\frac{(1+i)^{n+1} - (1+i)}{i}\right] \qquad A_{BGN} = FV_A\left[\frac{i}{(1+i)^{n+1} - (1+i)}\right] \quad (9\text{–}4b)$$

Equation 9–7a (Annuity equalling a present value)

$$PV_A = A\left[\frac{1 - \frac{1}{(1+i)^n}}{i}\right] \qquad A = PV_A\left[\frac{i}{1 - \frac{1}{(1+i)^n}}\right] \quad (9\text{–}5a)$$

Equation 9–7b (Annuity in advance equalling a present value)

$$PV_A = A_{BGN}\left[\frac{(1+i) - \frac{1}{(1+i)^{n-1}}}{i}\right] \qquad A_{BGN} = PV_A\left[\frac{i}{1+i - \frac{1}{(1+i)^{n-1}}}\right] \quad (9\text{–}5b)$$

Equation 9–9 (Perpetual annuity)

$$PV_\infty = A(1+i)^{-1} + A(1+i)^{-2} + A(1+i)^{-3} + \ldots \; (n \to \infty)$$

To get the sum of this geometric series, multiply by $(1+i)$ and deduct the original equation.

$$PV_\infty(1+i) = A(1+i)^0 + A(1+i)^{-1} + A(1+i)^{-2} + A(1+i)^{-3} + \ldots \; (n \to \infty)$$
$$-PV_n \qquad -A(1+i)^{-1} + A(1+i)^{-2} + A(1+i)^{-3} + \ldots \; (n \to \infty)$$

$$PV_\infty(1+i) - PV_\infty = A$$
$$PV_\infty + iPV_\infty - PV_\infty = A$$
$$iPV_\infty = A$$
$$PV = \frac{A}{i}$$