1 ~~C~~ Counting path colorings

Let's consider the question

"How many ways are there to color the vertices of a path of size ~~N~~ N so that there are j blue vertices, but no edge contains all blue vertices."

Let's ~~de~~ denote the number of ways to color ~~the~~ a path of size N under these conditions by:

$\boxed{F(N, J).}$

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Let's consider some base cases;

* $F(N,0) = 1$; there is only one coloring with 0 blue vertices.

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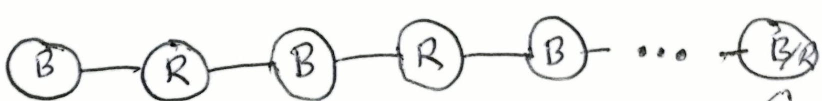
* $F(N,J) = 0$ if $J > \lfloor \frac{N}{2} \rfloor + N$ (MOD

why? the best-case scenario is to alternate red & blue, starting with blue.



last be B or R depending on

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* $F(N,1) = N+1$; a size-N path has $(N+1)$ blue vertices; you just decide which one of these should be blue.

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Now let's look at our recurrence.

Assume we have a path of size N whose vertices we want to color to have exactly J blue vertices, but no all-blue edge.

For the moment, let's "chop off" the last vertex, leaving a path of size (N-1).

There are 4 cases:

1. the next-to-last vertex is red, and we have (J-1) blue vertices from 1 to N.
2. the next-to-last vertex is red, and we used all J blue vertices from 1 to N.

(see next page)

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(4 Cases, cont'd)

3. the next-to-last vertex is blue, and we have $(J-1)$ blue vertices from 1 to N; this case actually can't happen, as it would require ending with two blue vertices

4. the next-to-last vertex is blue, and we have J blue vertices from to N.

To help us think about this additional information of what a single vertex is colored, let's introduce two new, related functions:

$F_R(N,J)$ = # of ways to color size-N paths, avoiding blue edges, with J blue vertices and last vertex red.

$F_B(N,J)$ = same as above, but with last vertex blue.

Notice that $F(M,K) = F_R(M,K) + F_B(M,K)$ for all M,K

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With this notation in hand, our cases tell us that

$$\boxed{F(N,J) = \underbrace{F_R(N-1,J-1)}_{(\text{case } 1)} + \underbrace{F_R(N-1,J)}_{(\text{case } 2)} + \underbrace{F_B(N-1,J)}_{(\text{case } 4)}}$$

which becomes

$$\boxed{F(N,J) = F_R(N-1,J-1) + F(N-1,J).}$$

We'd still like to clean up the $F_R(N-1,J-1)$

Now we notice that

$$F_R(M,K) = F(M-1,K).$$ (Exercise left ~~to~~ reader.)

Hence our count satisfies the reccurence

$$\boxed{F(N,J) = F(N-2,J-1) + F(N-1,J).}$$