Research Part 1: Generation Function of $A(x, y)$

Andrew Velasquez-Berroteran

September 2020

Introduction

Let $F(n, j)$ be the number of ways to color a 3-uniform hypergraph of n hyperedges with j blue vertices without getting a blue hyperedge.
Some Base Cases and useful equations:
$F(0, 0) = 1$
$F(0, 1) = 0$
$F(0, 2) = 0$
$F(0, 3) = 0$
$F(1, 0) = 1$
$F(1, 1) = 3$
$F(1, 2) = 3$
$F(1, 3) = 0$
$F(2, 0) = 1$
$F(2, 1) = 5$
$F(2, 2) = 10$
$F(1, j) = 0$ when $j \geq 3$
$F(n, 0) = 1$ for every $n \geq 0$

Generating $A(x, y)$

Let us define $A(x, y)$ as

$$A(x, y) = \sum_{n,j \geq 0} F(n, j) x^n y^j, \tag{1}$$

where we have the recursive function $F(n, j) = F(n-1, j) + 2F(n-1, j-1) + F(n-2, j-2) + F(n-2, j-3)$, where we assume $n \geq 2$ and $j \geq 3$. By the recursive relationship establish above for $F(n, j)$, we have that

$$\sum_{(n,j) \geq (2,3)} F(n, j) x^n y^j = \sum_{(n,j) \geq (2,3)} [F(n-1, j) + 2F(n-1, j-1) + F(n-2, j-2) + F(n-2, j-3)] x^n y^j \tag{2}$$

Let us attempt to fin $A(x,y)$ by first writing all 5 individual sums in terms of $A(x,y)$ so we can then have an equation where both sides contain $A(x,y)$. Our goal is then accomplished through normal algebraic methods. We have

$$\sum_{(n,j)\geq(2,3)} F(n,j)x^n y^j = A(x,y) - F(2,2)x^2y^2 - F(2,1)x^2y - F(2,0)x^2$$
$$- F(1,2)xy^2 - F(1,1)xy - F(1,0)x - F(0,0)$$

Thus, we have that

$$\sum_{(n,j)\geq(2,3)} F(n,j)x^n y^j = A(x,y) - 10x^2y^2 - 5x^2y - x^2 - 3xy^2 - 3xy - x - 1 \quad (3)$$

Let us look at the sum with $F(n-1,j)$:

$$\begin{aligned}\sum_{(n,j)\geq(2,3)} F(n-1,j)x^n y^j &= \sum_{(m,j)\geq(1,3)} F(m,j)x^{m+1}y^j \quad [m = n-1]\\ &= x(\sum_{(m,j)\geq(1,3)} F(m,j)x^m y^j)\\ &= x(A(x,y) - F(1,2)xy^2 - F(1,1)xy - F(1,0)x - F(0,0))\\ &= x(A(x,y) - 3xy^2 - 3xy - x - 1)\end{aligned}$$

Thus, we have that

$$\sum_{(n,j)\geq(2,3)} F(n-1,j)x^n y^j = xA(x,y) - 3x^2y^2 - 3x^2y - x^2 - x \quad (4)$$

Let us look at the sum with $2F(n-1,j-1)$:

$$\begin{aligned}\sum_{(n,j)\geq(2,3)} 2F(n-1,j-1)x^n y^j &= 2\sum_{(m,p)\geq(1,2)} F(m,p)x^{m+1}y^{p+1} \quad [m = n-1,\ p = j-1]\\ &= 2xy(\sum_{(m,j)\geq(1,2)} F(m,p)x^m y^p)\\ &= 2xy(A(x,y) - F(2,1)x^2y - F(2,0)x^2\\ &\quad - F(1,1)xy - F(1,0)x - F(0,0))\\ &= 2xy(A(x,y) - 5x^2y - x^2 - 3xy - x - 1)\end{aligned}$$

Thus, we have that

$$\sum_{(n,j)\geq(2,3)} 2F(n-1,j-1)x^n y^j = 2xyA(x,y) - 10x^3y^2 - 2x^3y - 6x^2y^2 - 2x^2y - 2xy \quad (5)$$

Let us now look at the sum with $F(n-2, j-2)$:

$$\begin{aligned}
\sum_{(n,j)\geq(2,3)} F(n-2, j-2)x^n y^j &= \sum_{(m,p)\geq(0,1)} F(m,p)x^{m+2}y^{p+2} \quad [m = n-2,\ p = j-2] \\
&= x^2y^2\Big(\sum_{(m,p)\geq(0,1)} F(m,p)x^m y^p\Big) \\
&= x^2y^2(A(x,y) - F(0,0)) \\
&= x^2y^2(A(x,y) - 1)
\end{aligned}$$

Thus, we have that

$$\sum_{(n,j)\geq(2,3)} F(n-2, j-2)x^n y^j = x^2y^2A(x,y) - x^2y^2 \tag{6}$$

Finally, let us look at the sum with $F(n-2, j-3)$:

$$\begin{aligned}
\sum_{(n,j)\geq(2,3)} F(n-2, j-3)x^n y^j &= \sum_{(m,p)\geq(0,0)} F(m,p)x^{m+2}y^{p+3} \quad [m = n-2,\ p = j-3] \\
&= x^2y^3\Big(\sum_{(m,p)\geq(0,0)} F(m,p)x^m y^p\Big) \\
&= x^2y^3A(x,y)
\end{aligned}$$

Thus, we have that

$$\sum_{(n,j)\geq(2,3)} F(n-2, j-3)x^n y^j = x^2y^3A(x,y) \tag{7}$$

Putting Everything Together

By (2), we have that (3) is simply the sum of (4), (5), (6) and (7). So,

$(x + 2xy + x^2y^2 + x^2y^3)A(x,y) - 10x^3y^2 - 2x^3y - 10x^2y^2 - 5x^2y - x^2 - 2xy - x$

is equal to

$$A(x,y) - 10x^2y^2 - 5x^2y - x^2 - 3xy^2 - 3xy - x - 1$$

So,

$A(x,y) - 3xy^2 - 3xy - 1 = (x + 2xy + x^2y^2 + x^2y^3)A(x,y) - 10x^3y^2 - 2x^3y - 2xy$

With this, we obtain

$$A(x,y) = \frac{3xy^2 + xy + 1 - 10x^3y^2 - 2x^3y}{1 - (x + 2xy + x^2y^2 + x^2y^3)}$$