Manuscript Title: Avoiding monochromatic subpaths in uniform hyptergraph paths and cycles.

Authors: Hiller et. al.

In this paper, the authors generalize the well-known result that the number of binary strings of length n with no two consecutive 1s is given by the Fibonacci numbers by generalizing the problem to r-uniform hypergraphs where each vertex is assigned one of two colors. In this context, the binary string result corresponds to having a path graph with no monochromatic blue edge.

On the pro side: the problem is nicely motivated by reliability engineering. The authors cite known results in the context of graphs and generalize them to the context of hypergraphs. The paper is both well-organized and well-motivated.

There are a number of small items that must be edited for consistency/correctness, which are detailed below But more deeply: the primary proof technique is an analysis in cases, which results in increasingly unwieldy recurrences involving (nested) summations. The original Fibonacci recurrence that is used for motivation in the introduction corresponds to a nice generating function. The paper would be stronger if the authors could push the mathematics further to find equivalent non-recursive formulas, or at least comment on the predicted difficulty of doing so. Similarly, if we hold all but one of the variables constant in $F_k(n, r, j)$ or $F_k(m, r, n, j)$, we obtain families of integer sequences. Do any of these sequences appear in `oeis.org` for other reasons that could be made clear?

In summary, I enjoyed the motivation and thoughtful generalization the authors considered in this paper. While the current proofs in cases are a correct and solid way to approach the problem, the paper would be stronger if the authors pushed the mathematics further than the recursive results, either by exploring equivalent non-recursive formulas, approximations (suggested question (3) on p. 12) or connect the sequences to other enumeration questions. I cannot recommend the current version for publication in *Involve*, but with further work in one or more of these directions, I believe it could be made suitable.

More particular comments follow:

(1) p. 2, line 11: "generalizers" should be "generalizes".
(2) p. 2, line 27: n should be in math mode.
(3) p. 2, line 32–33: You say "cases when H and K are certain types of graphs". Can you add half a sentence that lists these types of graphs?
(4) p. 2–3, bulleted remarks: If G is a subhypergraph of H, I would expect H to contain at least as many copies of K as G since H is larger, (you've reversed the roles of G and H). You also say "a failed copy of K must exist before a failed copy of K can." One of these K's is a typo. Should the first K be G?
(5) p. 3, line 10: x should be in math mode.
(6) p. 3, line 11: $_aR_b$ is nonstandard notation for remainders. Why not use a mod b?
(7) p. 3, line 16: r-uniform should have the r in math mode.
(8) p. 3, line 17: you say $|E_i \cap E_j| = 0$ otherwise. But really, you mean when $|i - j| \geq 2$. For $i = j$, this quantity won't be 0.

(9) p. 3: At the end of Definition 2.3, E_{i+1} has already been defined for the situation when $i = 0$. You mean $0 < i \le n - 1$.

(10) p. 3: Remark 2.4 is false. By your definition, if we use $r = 3$ and $m = 1$, we get $E_1 = \{0, 1, 2\}$, $E_2 = \{1, 2, 3\}$, $E_3 = \{2, 3, 4\}$, etc., and two consecutive edges shared two vertices rather than 1. Perhaps you want to change the definition of m-tight so the remark is true? Or do you want to change the tightness number in the remark?

(11) p.3: In Definition 2.5, using x as a variable for colors is awkward when x was used with floor and ceiling earlier in this page. Make sure to use variables consistently. Also, there are three times x appears in text rather than in math mode.

(12) p. 3 and throughout: In the spirit of using variables consistently, on p. 3, vertices are give by numbers such as 0, 1, 2, etc. However, later in the paper vertices are called V_i. Pick one notation and use it consistently throughout. If you go with using v, since $V(H)$ is already the vertex set of H, I might expect $V(H) = \{v_1, v_2, v_3, \dots\}$, using capital letters for sets and lower case for the actual vertices.

(13) p. 3, line 35: n should be in math mode.

(14) p. 4, Figure: Did you use a Figure environment with the caption command? I don't see a Figure number to reference. Notice that if a reader prints your paper in grayscale or is colorblind, red and blue vertices are indistinguishable in the figure. Consider something like making red vertices circles and blue vertices squares to further distinguish them. Consider using the loose/tight nomenclature as appropriate here too since it's your one visual illustration of a hyperpath.

(15) p. 4, lines 4–5: "the following two easy proposition" should be " the following easy proposition" (delete "two"). Also, why do you state it without proof? Even if it's a simple proof, talking through the details can help your readers cement their understanding of the role of each variable in your definitions. You can give intuition for the inequalities in each if-then statement. For example, $k(r-1)+1$ is the number of vertices in a loose subpath of the required length, so in this case, j is too small for a monochromatic blue subpath, so we may pick any j vertices in the hypergraph without restriction. Similar (brief) arguments could be given for (2) and (3) to help build intuition. "If" should be capitalized in line (3) of the Proposition too.

(16) p. 4, line 16: "touple" should be "tuple" (twice).

(17) p. 4, line 18: What is h here? Should it be r?

(18) p. 4, line 22: Is there a more formal word choice than " we have no worries"?

(19) p. 5, line 2: While "leaf" is defined in the context of a tree graph as a vertex of degree 1, this is the one place where you use the word "leaf" in this paper, and it's in the context of a hypergraph. Why not just say "in the edge E_1"?

(20) p. 5, line 5: "theremaining" should be "the remaining".

(21) p. 5, In the proof of Lemma 3.6, you used b as the summation variable of the Lemma statement as well as a variable in the proof. Could you pick a consistent variable for the proof of Lemma 3.7 too? (instead of k in the argument and i in the problem statement). In fact, k already has a job as the subscript in $F_k(r, n, j)$, so I'd avoid using it in the proof.

(22) p. 5, line 15: "Applying Lemma 3.6 follows that" is awkward grammar. Are you missing a word?

(23) p. 5, line 20: Should "2-Path" really be "2-path"?

(24) p. 6, lines 2–3: Are you using the same hyphens/dashes for r-uniform throughout the paper? These look like dashes rather than hyphens.

(25) p. 6, line 6: The superscript i is never defined. It should be either defined before this, or, since it's only used in this proof the cases should be phrased so that the superscript is unnecessary.

(26) p. 6, line 19: Should $E_l \cap E_{i+1}$ be $E_i \cap E_{i+1}$? Also, the intersection of two edges should be a set but you have no braces around your vertices.

(27) p. 7, line 4: M and N are defined in Theorem 3.10, Proposition 3.11, Theorem 4.10, and Theorem 4.11 but never actually used.

(28) p. 7, line 6–7: You have two consecutive lines that begin with "Note that". Can you vary the grammar?

(29) p. 7, line 13: This line shouldn't be indented since it's not the start of a new paragraph. Also x has too many jobs in this paper. Consider using a different variable.

(30) p. 7, line 16: There's a missing comma after V_3.

(31) p. 8, line 17: "re-state" should be "restate".

(32) p. 10–11: Theorem 4.10 and Theorem 4.11 both involve long multi-line equations. Consider using an align* environment rather than centering them for better formatting.

(33) p. 11, line 8: You defined M in the theorem statement, but now it seems to have a different job altogether.

(34) p. 12: In multiple places you talk about m-tight and r-uniform without putting m or r in math mode.

(35) p. 12: Make sure your references are formatted consistently. e.g. are there missing periods after initials in reference 2? How are you deciding when to use an initial vs. a name? Is there an extra p in the first author's name in reference 9? Is 41–48 really the correct page range for two references in a row or was one copied over? I was also surprised to see the same team of authors (references 7 and 8) with papers nearly 3 decades apart, but had trouble locating them with the given citations. Is all information in both citations correct? How are you deciding when to abbreviate journal titles and when to spell them out?