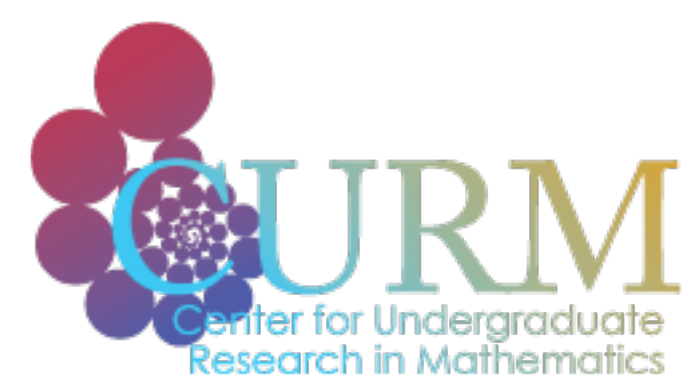


Monochromatic Subhypergraphs in Stochastic Processes on Hypergraphs



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What did we study?

We are examining a stochastic process defined on hypergraphs where each vertex begins colored red. Each vertex will switch to blue at a time given by some probability distribution. Our aim is to determine when all vertices in a given subhypergraph have switched to blue. Specifically, we are examining 3-uniform loose hyperpaths and are concerned with when all of the vertices in a single hyperedge switch to blue.

Why should you care?

There are many applications that can be considered from this research:

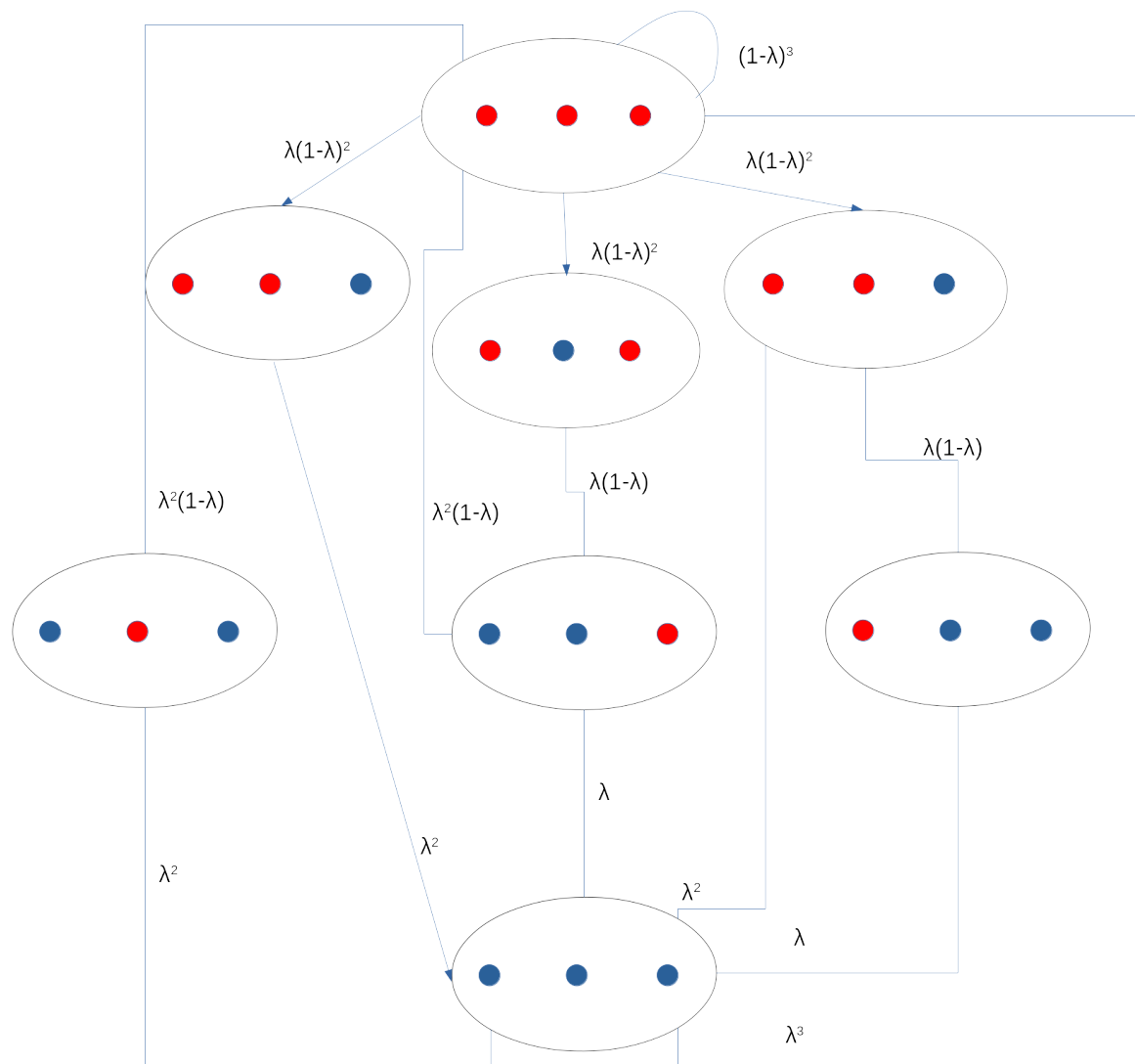
- Coin flipping probability questions. [1]
- *Consecutive k-out-of-n:F systems* in reliability engineering. [2]
- The Armitage-Doll model of carcinogenesis. [3]

How did we explore this problem?

```
def gen_bin(n, r, j):
    """
    Generates all possible binary strings of a given size with a given number
    of ones.
    """
    # Parameters
    # (param: n) The size of the hypergraph.
    # (param: r) The uniformity of the hypergraph.
    # (param: j) The amount of 1 valued vertices.
    # Order of a size n, r uniform hypergraph.
    o = (n * (r - 1) * j)
    binstrings = []
    for x in range(0, o):
        binstrings.append(''.join('1' if x >= i else '0' for i in range(n)))
    return binstrings

def is_valid(bin_string, n, r, k):
    """
    Checks whether a given binary string satisfies our conditions.
    """
    # Parameters
    # (param: bin_string) The binary string to be checked.
    # (param: n) The size of the hypergraph.
    # (param: r) The uniformity of the hypergraph.
    # (param: k) The size of the subhypergraph to avoid.
    # i becomes each pivot position (i.e. the intersect of the edges)
    for i in range(0, (n - k + 1) * (r - 1), r - 1):
        flag = 1
        j = 1
        # cycles through an edge until a zero is found.
        while flag == 1 and i + (k * (r - 1) + 1) <= n:
            bin_strings.py (x)
```

Code to generate binary strings.



An example of an abbreviated Markov chain.

What are our results?

To illustrate our general results, we will focus on this specific question:

How many ways can we color the vertices of a size n 3-uniform hyperpath with exactly j blue vertices where no hyperedge is all blue?

What is the distribution?

If $p(t)$ is the probability a vertex remains red at time t , then probability that hypergraph G has exactly j blue vertices is

$$F_1^{(r)}(n, j) \cdot p(t)^{|V(G)|-j} \cdot (1 - p(t))^j.$$

and the overall hypergraph survival probability is:

$$\sum_{j=0}^{|V(G)|} F_1^{(r)}(n, j) \cdot p(t)^{|V(G)|-j} \cdot (1 - p(t))^j.$$

General case for an r-uniform hyperpath

Let $r \geq 3$ and let $F_1^{(r)}(n, j)$ count size- n red/blue-colored r -uniform hyperpaths that (1) have exactly j blue vertices and (2) have no blue hyperedge. For $n > 1$, $F_1^{(r)}(n, j)$ satisfies the recurrence

$$F_1^{(r)}(n, j) = \sum_{i=0}^{r-2} \binom{r-1}{i} F_1^{(r)}(n-1, j-i) + \sum_{i=0}^{r-2} \binom{r-2}{i} F_1^{(r)}(n-2, j-(r-1)-i)$$

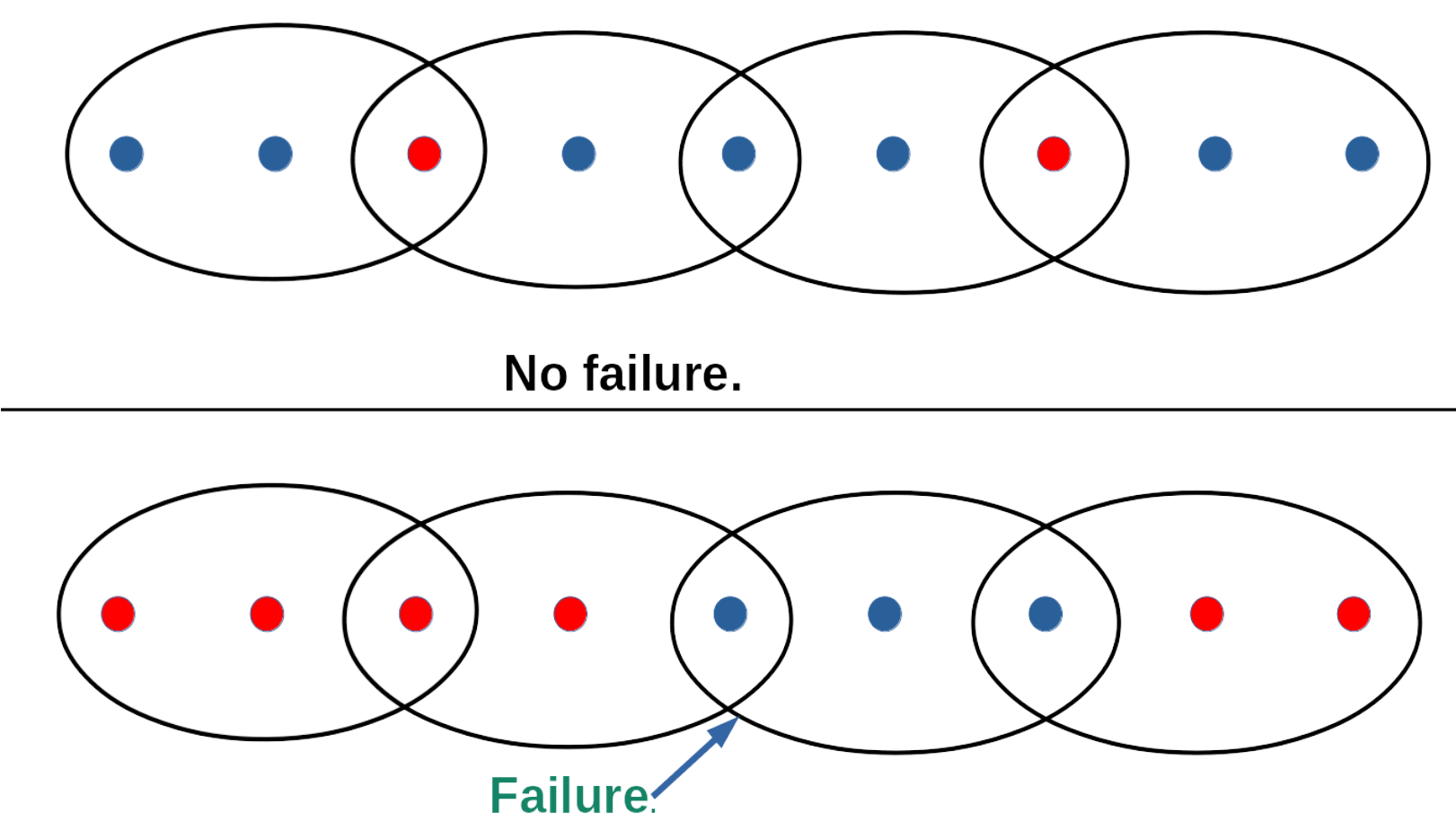
with base cases $F_1^{(r)}(n, j) = \binom{(r-1)n+1}{j}$ for $0 \leq j < r$ and $F_1^{(r)}(n, j) = 0$ for $j > n(r-1) + 1 - \lfloor \frac{n}{2} \rfloor$.

Specific case for a 3-uniform hyperpath

$$F_1^{(3)}(n, j) = F_1^{(3)}(n-1, j) + 2 \cdot F_1^{(3)}(n-1, j-1) + F_1^{(3)}(n-2, j-2) + F_1^{(3)}(n-2, j-3)$$

with base cases $F_1^{(3)}(n, 0) = 1$, $F_1^{(3)}(n, 1) = 2n + 1$, $F_1^{(3)}(n, 2) = 2n^2 + n$, and $F_1^{(3)}(n, j) = 0$ for $j > 2n + 1 - \lfloor \frac{n}{2} \rfloor$.

What does a hypergraph look like?



What are some specific results?

Here are some values of $F_1^3(n, j)$,

	j = 0	j = 1	j = 2	j = 3	j = 4	j = 5	j = 6	j = 7
n = 0	1	1	0	0	0	0	0	0
n = 1	1	3	3	0	0	0	0	0
n = 2	1	5	10	8	1	0	0	0
n = 3	1	7	21	32	23	5	0	0
n = 4	1	9	36	80	102	69	19	1

Where do we go from here?

- Expand our research to include other types of hypergraphs.
- Examine hypergraphs of varying tightness.

Where did we start?

[1] W.Feller.
An introduction to probability theory and its applications.
Wiley, 1968.

[2] Pooja Mohan Manju Agarwal and Kanwar Sen.
Reliability of consecutive-k-out-of-n:f system.
IEEE Transactions on Reliability, 1:87–89, 1981.

[3] Peter Armitage and Richard Doll.
The age distribution of cancer and a multi-stage theory of carcinogenesis.
British journal of cancer, 8(1):1, 1954.

Who helped us?

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