$\boxed{1}$

$$A(x,y) = \frac{1+xy+3xy^2}{1-xy-x-xy-x^2y^2-x^2y^3}$$

$$= \frac{1+xy+3xy^2}{1-x(y+1+y+xy^2+xy^3}$$

$$= \frac{1+xy+3xy^2}{1-x(y+(1+y)\cdot 1+xy^2(1+y))}$$

$$= \frac{1+xy+3xy^2}{1-x(y+(1+xy^2)(1+y))}$$

$$= (1+xy+3xy^2)\sum_{n=0}^{\infty}\left[x(y+(1+xy^2)(1+y))\right]^{M}$$

$\boxed{2}$

$$= \left(1+xY+3xY^2\right) \sum_{M=0}^{\infty} X^M \cdot \left[Y + (1+xY^2)(1+Y)\right]^M$$

$$= \left(1+xY+3xY^2\right) \sum_{M=0}^{\infty} X^M \cdot \left[\sum_{R=0}^{M} \binom{M}{R} Y^{M-R} \cdot (1+xY^2)^R (1+Y)^R\right]$$

$$= \left(1+xY+3xY^2\right) \sum_{M=0}^{\infty} X^M \left[\sum_{R=0}^{M} \binom{M}{R} Y^{M-R} \left[\sum_{S=0}^{R} \binom{R}{S} (xY^2)^S \sum_{T=0}^{R} \binom{R}{T} Y^T\right]\right]$$

$$= \left(1+xY+3xY^2\right) \sum_{M=0}^{\infty} \sum_{R=0}^{M} \sum_{S=0}^{R} \sum_{T=0}^{R} \binom{M}{R} \binom{R}{S} \binom{R}{T} X^{M+S} Y^{M-R+2S+T}$$

$$= \sum_{M=0}^{\infty} \sum_{R=0}^{M} \sum_{S=0}^{R} \sum_{T=0}^{R} \binom{M}{R}\binom{R}{S}\binom{R}{T} \Big(x^{M+S} y^{M-R+2S+T} + x^{M+S+1} y^{M-R+2S+T+1} + 3x^{M+S+1} y^{M-R+2S+T+2} \Big)$$

We are interested in the coefficient of

$x^N y^J$ for a fixed N, J.

There are three cases:

$\boxed{A}$ $x^{M+S} y^{M-R+2S+T} = x^N y^J$

$\boxed{B}$ $x^{M+S+1} y^{M-R+2S+T+1} = x^N y^J$

$\boxed{C}$ $x^{M+S+1} y^{M-R+2S+T+2} = x^N y^J$

CASE A

$$M+S=N \implies M=N-S$$

$$M-R+2S+T=J \implies N-S-R+2S+T=J$$

$$\implies R=N+S+T-J$$

So the coefficient is:

$$\sum_{R=0}^{N-S}\sum_{S=0}^{R}\sum_{T=0}^{R}\binom{N-S}{N+S+T-J}\binom{N+S+T-J}{S}\binom{N+S+T-J}{T}$$

CASE B

$$M+S+1=N \implies M=N-S-1$$

$$M-R+2S+J+1=J \implies R=M-J+2S+T+1$$

$$\implies R=N-S-1-J+2S+T+1$$

$$\implies R=N+S+T-J$$

$$\sum_{R=0}^{N-S}\sum_{S=0}^{R}\sum_{T=0}^{R}\binom{N-S-1}{N+S+T-J}\binom{N+S+T-J}{S}\binom{N+S+T-J}{T}$$

$\boxed{\text{CASE C}}$

$$M+S+1=N \to M=N-S-1$$

$$M-R+2S+T+2=J \to R = M-J+2S+T+2$$

$$R=(N-S-1)-J+2S+T+2$$

$$=N+S+1+T-J$$

$$\cancel{\sum_{R=0}^{N-S}} 3\cdot \binom{N-S-1}{N+S+T-J+1}\binom{N+S+1+T-J}{S}\binom{N+S+1+T-J}{T}$$

So the coefficient on $X^N Y^J$

is

$$\sum_{R=0}^{N-S}\sum_{S=0}^{R}\sum_{T=0}^{R}\Bigg[\binom{N-S}{N+S+T-J}\binom{N+S+T-J}{S}\binom{N+S+T-J}{T}$$

$$+\binom{N-S-1}{N+S+T-J}\binom{N+S+T-J}{S}\binom{N+S+T-J}{T}$$

$$+\,3\cdot\binom{N-S-1}{N+S+T-J+1}\binom{N+S+1+T-J}{S}$$