Avoiding Strings of 1's

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1 How Many 1's Can we Pack Into A String Before Two Have to Sit Next to Each Other?

1.1 Getting Started

Let's try to answer the question posed in the section title. As you might expect at this point, we're going to set up a recurrence.

Let $M(n)$ represent the largest number of 1's that can occur in a string of length n without having two consecutive 1's. As a first observation, we see $M(0) = 0$ and $M(1) = 1$. (Stop to consider this for a moment.)

As practice, it might be worth making a table of some brute force/best guess observations for $M(n)$.

n	M(n)
0	0
1	1
2	
3	
4	
5	
6	
7	
8	

1.2 Setting Up A Recurrence

At this point, we have an idea:

Conjecture: The sequence $M(n)$ satisfies the recurrence

$$M(n) = M(n-2) + 1, n \geq 2; M(0) = 0, M(1) = 1.$$

Let's prove the conjecture now.

Proof: We will use induction on n. First, for a string of length 0, we can't put in any 1's, so $M(0) = 0$. A string of length 1 can contain at most a single 1 (no worries about two consecutive), so $M(1) = 1$, Thus, the two base cases hold. Now for the (strong) induction step: suppose there exists $k \geq 0$ such that whenever $k \geq n \geq 0$, the equality is valid. Consider the string of length $k+1$ that contains the most 1's, without two consecutive. Let's call this string S. Notice that S can not end in "11", so we can write $S = T01$, $S = T01$, or $S = T00$, where T consists of the first $k-1$ letters of S. Now, since T has $k-1$ letters, then by the induction hypothesis, T contains at most $M(k-1)$ 1's, so S contains at most $M(k-1)+1 = M(k+1-2)+1$ 1's. Thus $M(k+1) \leq M(k-1)+1$. On the other hand, let us take a word V of length $k-1$ that has $M(k-1)$ 1's. Then we can always throw on "01" to V to get a word of length $k+1$ that has $M(k-1)+1$ 1's. Thus, we have $M(k+1) \geq M(k-1)+1$. It follows that $M(k+1) = M(k-1)+1$, and thus by induction that $M(n) = M(n-2)+1$ for all $n \geq 0$.

1.3 Solving the Recurrence

Using generating functions to solve this particular recurrence is overkill. Instead, we go back to our table:

n	M(n)
0	0
1	1
2	1
3	2
4	2
5	3
6	3
7	4
8	4

Based on this, it seems that $M(n) = \left\lceil \frac{n+1}{2} \right\rceil$. Since *solutions to one variable recurrences with a given initial condition are unique*, all we have to do is prove that the formula we've proposed satisfies the given recurrence.

There are two cases. **Case 1:** If n is even, then $n = 2k$ for some integer k. So

$$M(n) = \left\lceil \frac{2k+1}{2} \right\rceil$$
$$= \left\lceil k + \frac{1}{2} \right\rceil$$
$$= k + 1.$$

On the other hand, in this case,

$$\begin{aligned} M(n-2)+1 &= \left\lceil \frac{2k-2+1}{2} \right\rceil + 1 \\ &= \left\lceil \frac{2k-1}{2} \right\rceil + 1 \\ &= \left\lceil k - \frac{1}{2} \right\rceil + 1 \\ &= k+1 \end{aligned}$$

Thus, when n is even, $M(n) = M(n-2) + 1$.

Case 2: The case when n is odd is left as an exercise for the reader.

2 Counting Binary Strings That Avoid Two Consecutive 1's

2.1 Definitions

Let $F_2(n, k)$ represent the number of binary strings that satisfy these properties:

- the total length of the string is n
- there are exactly k 1's in the string (we call this the *weight* of the string).
- there are two consecutive 1's anywhere in the string

We will call strings that meet these three conditions *legal strings*.

Q: What is $F_2(4,2)$?

A: There are six **total** binary strings of length four that have exactly two 1's.

$$0011 \quad 0101 \quad 0110 \quad 1001 \quad 1010 \quad 1100$$

but only three of these are legal:

$$\cancel{0011} \quad \boxed{0101} \quad \cancel{0110} \quad \boxed{1001} \quad \boxed{1010} \quad \cancel{1100}$$

Our Goal: Find a formula for $F_2(n,k)$. That is, given positive integers n and k, find a tool to determine how many legal strings have length n and weight k.

2.2 Finding a Recurrence

To help us find a formula for $F(n,k)$, we set up a recurrence that we know it should satisfy.

Suppose we have a legal string of length n and weight k. Let's look at the last character of the string.

Case 1: The last character is 0. If we delete this last 0, we have a legal string of length $n-1$ and weight k. Thus, there are $F_2(n-1,k)$ strings that fall into this case.

Case 2: The last character is 1. Since this word is legal, we know it does not contain two consecutive 1's, so the next-to-last character must be 0. If we delete this "01", we have two fewer characters, and one fewer 1. Thus the words in this case correspond to legal words of length $n-2$ and weight $k-1$, so there are $F(n-2,k-1)$ of them.

So we have the recurrence

$$F_2(n,k) = F_2(n-1,k) + F_2(n-2,k-1). \textit{ (subject to some conditions........).}$$

Now let's talk about those nitpicky conditions. We need to be a bit careful: *we assumed that we could delete at least two characters and still have a legal word.* We were implicitly assuming that $n \geq 2$.

We also need to create some "reality checks" – what if someone asks us to have a legal string of length -3 and weight -1? **Impossible!** - so we need to be sure we guard against it.

2.3 Finding the conditions

It's not hard to pick out the first condition that we should have. It doesn't make sense to talk about binary strings of length or weight less than 0:

- If $n < 0$ or $k < 0$, then $F_2(n, k) = 0$.

  What if the length or the weight is exactly 0?

- If $n = 0$, $k = 0$, then $F_2(n, k) = 1$.

  When counting binary strings, we usually assume that there's one *empty string* of length 0.[1]

Another thing to consider is how many 1's we can pack into a string of length n without putting two of them next to each other. This was solved in Section 1, so we know that if $k > \left\lceil \frac{n+1}{2} \right\rceil$, it must be true that $F_2(n, k) = 0$. **This is important.** Notice that without including this, our recurrence would give $F_2(1, 2) = F_2(1, 1) + F_2(0, 2) = 1 + 0$, and would tell us that we could have two 1's in a string of length one. Impossible.

One way to visualize this is as a triangle. We have boundaries for both sides of the triangle, as pictured below (**todo:** picture).

2.4 Setting up the recurrence using generating functions

We are going to set up a generating function for the sequence $F_2(n, k)$. We define

$$\alpha_2(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} F_2(n, k) x^n y^k.$$

Beginning with the recurrence above, we have

$$F_2(n, k) = F_2(n-1, k) x^n y^k + F_2(n-2, k-1) \text{ for } n \geq 2 \text{ and } k \geq 1.$$

Multiplying by through by $x^n y^k$ and summing over the appropriate indices, we have

$$\sum_{n=2}^{\infty} \sum_{k=1}^{\infty} F_2(n, k) = \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} F_2(n-1, k) x^n y^k + \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} F_2(n-2, k-1) x^n y^k$$

[1] This is partially because binary strings often correspond to types of subsets, and we always have one empty set.

This equation contains three fairly involved terms, so let's consider them one at a time.

1. **Dealing with:** $\sum_{n=2}^{\infty}\sum_{k=1}^{\infty} F_2(n,k)$.

   This looks a lot like $\alpha(x,y)$, except that the indices on the summation are different. In particular, this sum misses any terms that appeared in $\alpha(x,y)$ but had $n < 2$ or $k < 1$. In particular, we are missing $\boxed{\text{n = 0, k = 0}}$, $\boxed{\text{n = 0, k = 1}}$, $\boxed{\text{n=1, k = 0}}$, $\boxed{\text{n = 1 , k = 1}}$, $\boxed{\text{n = 2, k =0}}$. So we have

$$\begin{aligned}\sum_{n=2}^{\infty}\sum_{k=1}^{\infty} F_2(n,k) &= \alpha(x,y) - F_2(2,0)x^2 - F_2(1,1)xy - F_2(0,1)y - F_2(1,0)x - F_2(0,0)\\ &= \alpha(x,y) - x^2 - xy - x - 1\end{aligned}$$

2. **Dealing with:** $\sum_{n=2}^{\infty}\sum_{k=1}^{\infty} F_2(n-1,k)x^n y^k$

   First, we can factor out an x, so we have

$$x\sum_{n=2}^{\infty}\sum_{k=1}^{\infty} F_2(n-1,k)x^{n-1}y^k$$

   Then, we can do an index substitution $m = n-1$, so when $n = 2, m = 1$, giving us

$$x\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} F_2(m,k)x^m y^k$$

   Now notice this is similar in appearance to $x\alpha(x,y)$, except that the indices are a bit "off". We are missing $F_2(1,0)x$, $F_2(0,1)y$ and $F_2(0,0)$, so we can rewrite this as

$$x\left(\alpha(x,y) - x - 1\right) = x\alpha(x,y) - x^2 - x.$$

3. **Dealing with** $\sum_{n=2}^{\infty}\sum_{k=1}^{\infty} F_2(n-2,k-1)x^n y^k$

   We can factor out an x^2y to obtain

$$x^2y\sum_{n=2}^{\infty}\sum_{k=1}^{\infty} F_2(n-2,k-1)x^{n-2}y^{k-1}$$

   Now we do the substitution $m = n-2, j = k-1$, to obtain $x^2y\alpha(x,y)$.

2.5 Our new equation

Taking the results from the previous section, we know that $\alpha(x,y)$ satisfies the equation:

$$\alpha(x,y) - x^2 - xy - x - 1 = x\alpha(x,y) - x^2 - x + x^2y\alpha(x,y)$$

and we do algebra to solve that

$$\alpha(x,y) = \frac{1+xy}{1-x-x^2y}.$$

We rewrite this as

$$\alpha(x,y) = (1+xy)\frac{1}{1-(x+x^2y)}$$

Let's expand $\frac{1}{1-(x+x^2y)}$. Once again, geometric series and Binomial Theorem show up.

$$\begin{aligned}
\frac{1}{1-(x+x^2y)} &= \sum_{r=0}^{\infty}\left(x+x^2y\right)^r \\
&= \sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}x^{r-s}(x^2y)^s \\
&= \sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}x^{r-s}x^{2s}y^s \\
&= \sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}x^{r+s}y^s.
\end{aligned}$$

So

$$
\begin{aligned}
(1+xy)\frac{1}{1-(x+x^2y)} &= (1+xy)\sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}x^{r+s}y^s \\
&= \sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}(1+xy)x^{r+s}y^s \\
&= \sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}\left(x^{r+s}y^s + xyx^{r+s}y^s\right) \\
&= \sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}x^{r+s}y^s + \binom{r}{s}x^{r+s+1}y^{s+1}
\end{aligned}
$$

We want to find a formula for the coefficient on x^ny^k, which means we need to relate n and k to r and s. There are two possibilites.

$\boxed{\textbf{Case 1:}}$ We have $x^{r+s}y^s$, so $s=k$ and $n=r+s$. Then $r=n-s=n-k$. In this case

$$\binom{r}{s}x^{r+s}y^s = \binom{n-k}{k}x^ny^k$$

$\boxed{\textbf{Case 2:}}$ We have $x^{r+s+1}y^{s+1}$, so $k=s+1$, and $s=k-1$, while $r+k=n$, which means $r=n-k$.

In this case $\binom{r}{s}x^{r+s}y^s = \binom{n-k}{k-1}x^ny^k$.

Now remember that our whole goal in this work is to find the coefficient on x^ny^k, since that coefficient gives the value $F_2(n,k)$. We have found that

$$\alpha(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\binom{n-k}{k} + \binom{n-k}{k-1}\right)x^ny^k$$

We could call this a final answer, but we know an identity for binomial coefficients that let us write this as

$$\alpha(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n-k+1}{k}x^ny^k$$

Thus $F_2(n,k) = \binom{n-k+1}{k}$.

Conclusion: For $n \geq 0$ and $0 \leq k \leq \left\lceil \frac{n+1}{2} \right\rceil$, the number of ways to have a string of length n that has k total 1's and does not have two consecutive 1's is $\binom{n-k+1}{k}$.