Avoiding Blue Edges in 3-uniform Hyperpaths

Justin Clifton and Tommy Meek

Western Carolina University

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Acknowledgements

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A Question

"How many ways are there to flip a coin n times without seeing 2 heads in a row?"

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$F(n) =$ The number of ways to flip a coin n times without any consecutive heads.

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- $F(4) = 8.$
- $F(n) = FIB(n + 2).$[2]

where $FIB(n)$ is the nth Fibonacci number.

Another Question

"How many ways are there to flip a coin n times and get exactly j heads without seeing 2 heads in a row?"

Another Question

$F(n, j) =$ The number of ways to flip a coin n times and see exactly j heads without any consecutive heads.

Base cases

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- $F(n, 1) = n$.
- $F(n, j) = 0$ if $j > \lceil \frac{n}{2} \rceil$.

Finding the Recurrence

Assume we have flipped n coins and seen j heads but no 2 heads in a row.

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Take out the last coin flip.

Finding the Recurrence

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1. The next to last coin is tails and we have flipped $j-1$ heads in the first $n-1$ flips.

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2. The next to last coin is tails and we have flipped j heads in the first $n-1$ flips.

Finding the Recurrence

There are 4 cases.

3. The next to last coin is heads and we have flipped $j-1$ heads in the first $n-1$ flips.

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4. The next to last coin is heads and we have flipped j heads in the first $n-1$ flips

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Note that $F(n, j) = F_T(n, j) + F_H(n, j)$ for all n, j

Finding the Recurrence

Observing our 3 relevant cases from above, we now have

$$F(n,j) = F_T(n-1, j-1) + F_T(n-1, j) + F_H(n-1, j)$$

Finding the Recurrence

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which becomes

$$F(n,j) = F_T(n-1,j-1) + F(n-1,j).$$

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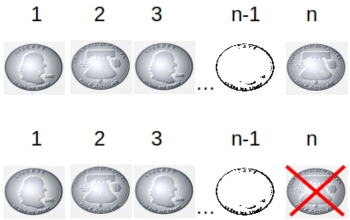

Finding the Recurrence

Given this, our recurrence that was

$$F(n,j) = F_T(n-1, j-1) + F(n-1, j)$$

now becomes

$$F(n,j) = F(n-2, j-1) + F(n-1, j).$$

Finding the Recurrence

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now becomes

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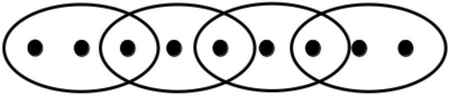
It can be shown that the closed form solution of this recurrence is:

$$F(n, j) = \binom{n-1+j}{j}$$

Prepare for the Jump to Hyperspace

Let $V = \{v_1, v_2, \ldots, v_n\}$ be a finite, non-empty set and let $E = \{e_1, e_2, \ldots, e_m\}$ be a set of subsets of V. The pair $H = (V, E)$ is called a *hypergraph* of size m with vertex set V and edge set E.



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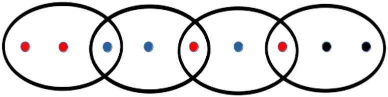
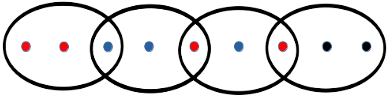


"How many ways are there to color the vertices of a size-n, 3-uniform hypergraph so that there are exactly j blue vertices, but no hyperedge contains all blue vertices?"

Prepare for the Jump to Hyperspace



Let $F^3(n, j)$ denote the number of ways to color the vertices of a size-n, 3-uniform hypergraph so that there are exactly j blue vertices, but no hyperedge contains all blue vertices.

Base cases

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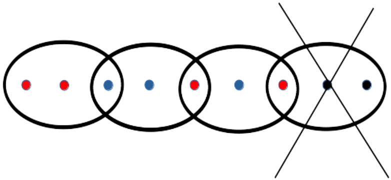
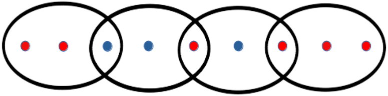
- $F^3(n, 2) = \binom{2n+1}{2}$.
- $F^3(n, j) = 0$ if $j > 2n + 1 - \lceil \frac{n}{2} \rceil$.

Finding the Recurrence



Begin by removing the last hyperedge.

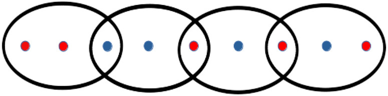
Finding the Recurrence



Case 1: No blue vertices were lost in the edge removal.

$$\boldsymbol{F^3(n,j) = F^3(n-1,j)} + \dots$$

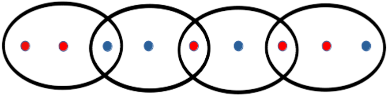
Finding the Recurrence



Case 2: One blue vertex was removed.

$$F^3(n,j) = F^3(n-1,j) \boldsymbol{+F^3(n-1,j-1)} + \dots$$

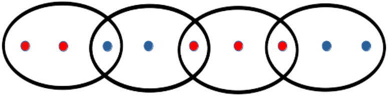
Finding the Recurrence



Case 2: One blue vertex was removed.

$$F^3(n, j) = F^3(n-1, j) + \mathbf{2}F^3(n-1, j-1) + \ldots$$

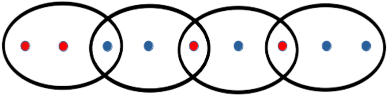
Finding the Recurrence



Case 3: Remove two blue vertices and the middle vertex of the $(n-1)$th edge is red.

$$F^3(n,j) = F^3(n-1,j) + 2F^3(n-1,j-1) + \boldsymbol{F^3(n-2,j-2)} + \dots$$

Finding the Recurrence



Case 4: Remove two blue vertices and the middle vertex of the $(n-1)$th edge is blue.

$$F^3(n,j) = F^3(n-1,j) + 2F^3(n-1,j-1) + F^3(n-2,j-2) \boldsymbol{+ F^3(n-2,j-3).}$$

Finding the Recurrence

"How many ways are there to color the vertices of a size-n, 3-uniform hypergraph so that there are exactly j blue vertices, but no hyperedge contains all blue vertices?"

$$F^3(n,j) = F^3(n-1,j) + 2F^3(n-1,j-1) + F^3(n-2,j-2) + F^3(n-2,j-3).$$

Conclusions and Future Work

- In our research, we have generalized this idea to avoiding sub-hyperpaths of arbitrary length and uniformity. [1]

Conclusions and Future Work

- In our research, we have generalized this idea to avoiding sub-hyperpaths of arbitrary length and uniformity. [1]
- We are currently using multivariate generating functions to find closed form solutions to all of our results.

- Billings et al., *Avoiding monochromatic sub-paths in uniform hypergraph paths and cycles*, (in preparation).
- William Feller, *"An Introduction to Probability Theory and Its Applications"*, Wiley, 1968.