Monochromatic Subhypergraphs in Stochastic Processes on Hypergraphs

Zane Billings, Justin Clifton, Tommy Meek and Wesley Rogers. Faculty Advisor: Andrew Penland

Western Carolina University

What did we study?

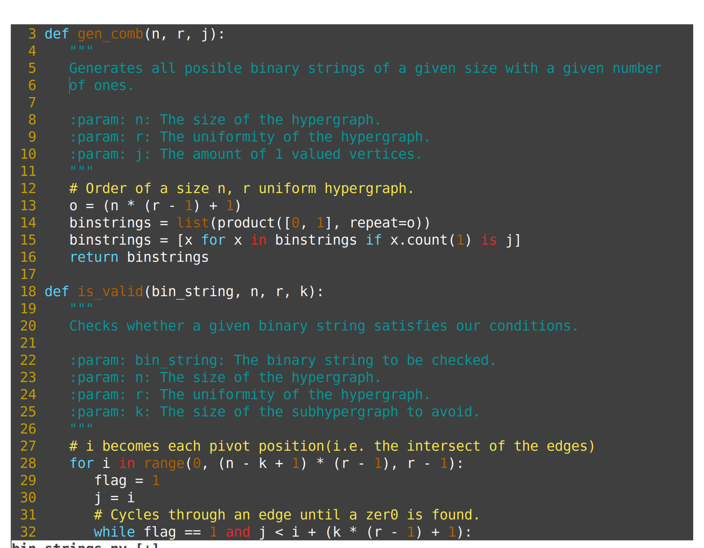
We are examining a stochastic process defined on hypergraphs where each vertex begins colored red. Each vertex will switch to blue at a time given by some probability distribution. Our aim is to determine when all vertices in a given subhypergraph have switched to blue. Specifically, we are examining 3-uniform loose hyperpaths and are concerned with when all of the vertices in a single hyperedge switch to blue.

Why should you care?

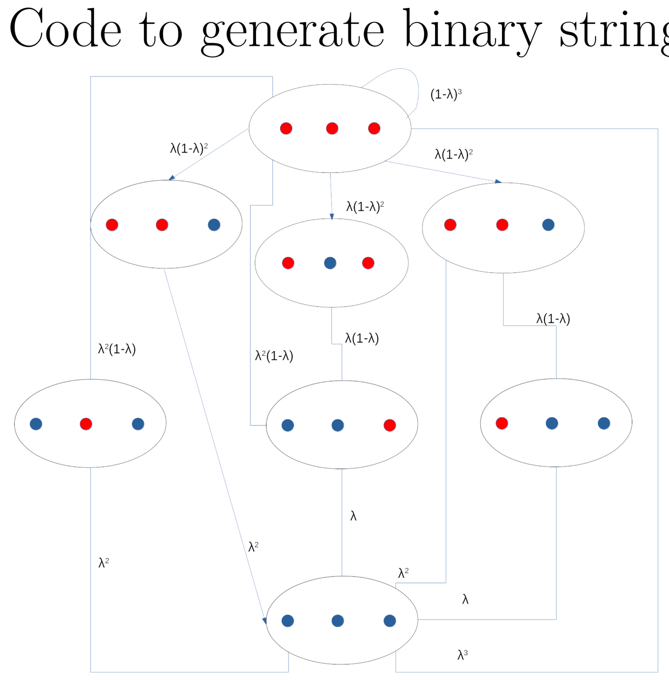
There are many applications that can be considered from this research:

- Coin flipping probability questions. [1]
- *Consecutive k-out-of-n:F systems* in reliability engineering. [2]
- The Armitage-Doll model of carcinogenesis. [3]

How did we explore this problem?



Code to generate binary strings.



An example of an abbreviated Markov chain.

What are our results?

To illustrate our general results, we will focus on this specific question:

How many ways can we color the vertices of a size n 3-uniform hyperpath with exactly j blue vertices where no hyperedge is all blue?

What is the distribution?

If $p(t)$ is the probability a vertex remains red at time t, then probability that hypergraph G has exactly j blue vertices is

$$F_1^{(r)}(n,j)\cdot p(t)^{|V(G)|-j}\cdot(1-p(t))^j.$$

and the overall hypergraph survival probability is:

$$\sum_{j=0}^{|V(G)|} F_1^{(r)}(n,j)\cdot p(t)^{|V(G)|-j}\cdot(1-p(t))^j.$$

General case for an r-uniform hyperpath

Let $r \geq 3$ and let $F_1^{(r)}(n,j)$ count size-n red/blue-colored r-uniform hyperpaths that (1) have exactly j blue vertices and (2) have no blue hyperedge. For $n > 1$, $F_1^{(r)}(n,j)$ satisfies the recurrence

$$F_1^{(r)}(n,j) = \sum_{i=0}^{r-2}\binom{r-1}{i}F_1^{(r)}(n-1,j-i) + \sum_{i=0}^{r-2}\binom{r-2}{i}F_1^{(r)}(n-2,j-(r-1)-i)$$

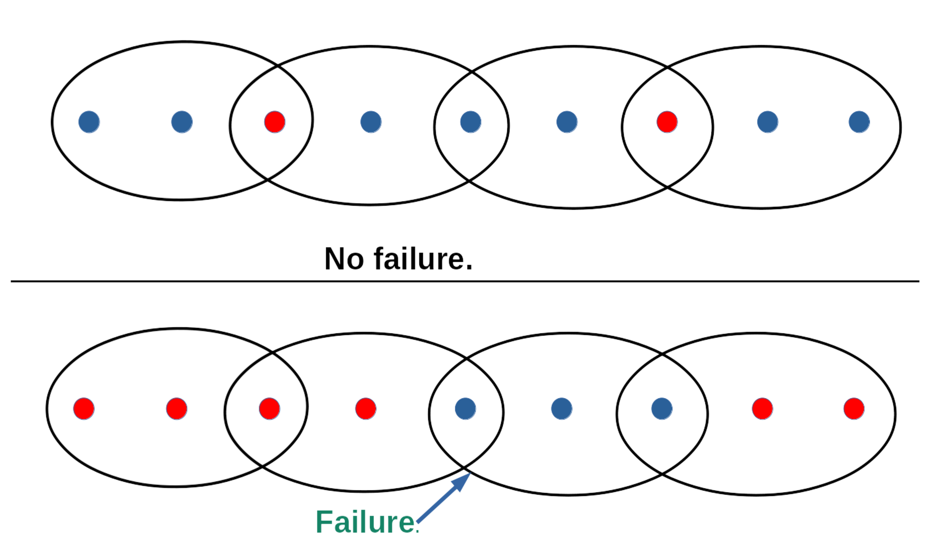
with base cases $F_1^{(r)}(n,j) = \binom{(r-1)n+1}{j}$ for $0 \leq j < r$ and $F_1^{(r)}(n,j) = 0$ for $j > n(r-1)+1-\left\lceil\frac{n}{2}\right\rceil$.

Specific case for a 3-uniform hyperpath

$$F_1^{(3)}(n,j) = F_1^{(3)}(n-1,j) + 2\cdot F_1^{(3)}(n-1,j-1) + F_1^{(3)}(n-2,j-2) + F_1^{(3)}(n-2,j-3)$$

with base cases $F_1^{(3)}(n,0) = 1$, $F_1^{(3)}(n,1) = 2n+1$, $F_1^{(3)}(n,2) = 2n^2+n$,
and $F_1^{(3)}(n,j) = 0$ for $j > 2n+1-\left\lceil\frac{n}{2}\right\rceil$.

What does a hypergraph look like?



What are some specific results?

Here are some values of $F_1^3(n,j)$,

	j = 0	j = 1	j = 2	j = 3	j = 4	j = 5	j = 6	j = 7
n = 0	1	1	0	0	0	0	0	0
n = 1	1	3	3	0	0	0	0	0
n = 2	1	5	10	8	1	0	0	0
n = 3	1	7	21	32	23	5	0	0
n = 4	1	9	36	80	102	69	19	1

Where do we go from here?

- Expand our research to include other types of hypergraphs.
- Examine hypergraphs of varying tightness.

Where did we start?

[1] W.Feller.
An introduction to probability theory and its applications.
Wiley, 1968.

[2] Pooja Mohan Manju Agarwal and Kanwar Sen.
Reliability of consecutive-k-out-of-n:f system.
IEEE Transactions on Reliability, 1:87–89, 1981.

[3] Peter Armitage and Richard Doll.
The age distribution of cancer and a multi-stage theory of carcinogenesis.
British journal of cancer, 8(1):1, 1954.

Who helped us?

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Contact Information

- Email: thmeek1@catamount.wcu.edu
- Email: jdclifton2@catamount.wcu.edu