1 EXAMPLE

Problem: Let $F(N,J)$ be the number of words over the alphabet $\{0,1,2\}$ that have

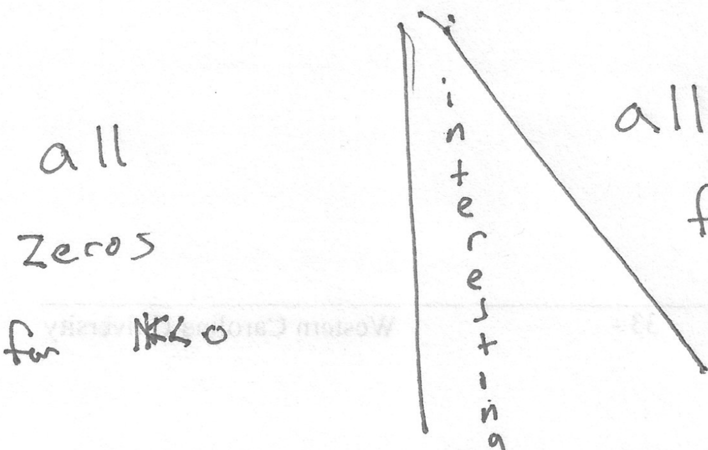
- length N
- exactly J 1's.

Come up with a generating function for $F(N,J)$.

Notice that

- $F(N,J)=0$ if $J>N$
- $F(N,J)=0$ if $N<0$
- $F(N,J)=0$ if $J<0$.

So $F(N,J)$ has a classic triangle shape often seen in counting problems



$\boxed{2}$ We set up a recurrence for $F(N,J)$.

Notice that for a word of length N with J 2's, the prefix of length $(N-1)$ has either

- $(J-1)$ 2's, in which case there is only one way to extend it.
- J 2's, in which case there are two ways to extend it.

Thus we have the recurrence

$$F(N,J) = F(N-1,J-1) + 2F(N-1,J); \quad N \geq 1, \; J \geq 1$$

with initial conditions

$$\boxed{F(1,1)=1; \quad F(1,0)=2, \quad F(0,1)=0, \quad F(0,0)=1}$$

Let us declare the generating function

$$A(x,y) = \sum_{N=0}^{\infty} \sum_{J=0}^{\infty} F(N,J)\, x^N y^J .$$

or more compactly

$$\sum_{N,J \geq 0} F(N,J)\, x^N y^J .$$

We "do the trick"

- multiply each term in recurrence by $x^N y^J$
- add up over $N, J \geq 1$
- simplify

4 This gives

$$\sum_{N,J\geq 1} F(N,J)x^N y^J = \sum_{N,J\geq 1} F(N-1,J-1)x^N y^J + \sum_{N,J\geq 1} 2F(N-1,J)x^N y^J$$

$$(*) \qquad\qquad (**) \qquad\qquad (***)$$

We will analyze the three terms $(*)$, $(**)$, and $(***)$ separately.

$(*)$ this is almost $A(x,y)$, but the indices are a bit off: we are missing $\boxed{N=1, J=0}$, $\boxed{N=0, J=1}$ and $\boxed{N=0, J=0}$. So this term is equal to

$$A(x,y) - F(1,0)x - F(0,1)y - F(0,0) = A(x,y) - 2x - 1.$$

(**) For

$$\sum_{N,J\geq 1} F(N-1,J-1)\,X^N Y^J,$$ we

introduce substitutions: $M = N-1$
$K = J-1$.

When $N=1$, $M=0$; when $J=1$, $K=0$. This becomes

$$\sum_{M,K\geq 0} F(M,K)\,X^{M+1} Y^{K+1}.$$

Factor out a XY to obtain

$$XY \sum_{M,K\geq 0} F(M,K)\,X^M Y^K = XY \cdot A(X,Y).$$

$\boxed{6}$ ***

For $\displaystyle\sum_{N,J\geq 1} 2f(N-1,J)\,x^N y^J$, we

use $M=N-1$ to obtain

$$\sum_{\substack{M\geq 0\\ J\geq 1}} 2f(M,J)\,x^{M+1}y^J = 2x\sum_{\substack{M\geq 0\\ J\geq 1}} f(M,J)\,x^M y^J$$

$$= 2x\Big(A(x,y)-1\Big)$$

So now our equation is

$$A(x,y) - 2x - 1 = xy\cdot A(x,y) + 2x\big(A(x,y)-1\big),$$

To find the generating function, just solve algebraically.

7

Dear Student,

Please finish solving algebraically before moving on. What follows is my solution for you to double-check.

8

doing the algebra

$$A(x,y) - \cancel{2x} - 1 = xy \cdot A(x,y) - 2x \cdot A(x,y) - \cancel{2x}$$

$$A(x,y) - 1 = xy \cdot A(x,y) - 2x \cdot A(x,y)$$

$$A(x,y) - xy\,(A(x,y) - 2x \cdot A(x,y) = 1$$

$$(1 - xy - 2x)\,A(x,y) = 1$$

$$\boxed{A(x,y) = \frac{1}{1 - xy - 2x}}$$

or

$$\boxed{A(x,y) = \frac{1}{1 - x(y+2)}}$$

either of these is perfectly acceptable as generating functions.

☺ ! it is possible to use this to get a nice expression for $F(N,J)$. if you're interested, read on.

$\boxed{\text{CLEANING UP TO OBTAIN A NICE EXPRESSION}}$

$\boxed{9}$

$$\sum_{\substack{N\ge 0\\ J\ge 0}} F(N,J)\, x^N Y^J = A(x,Y)$$

$$= \frac{1}{1-x(Y+2)}$$

$$= \sum_{N=0}^{\infty} \left(x(Y+2)\right)^N$$

$$= \sum_{N=0}^{\infty} (Y+2)^N x^N$$

$\boxed{\text{Now apply Binomial Theorem to obtain.}}$

$$= \sum_{N=0}^{\infty} \left(\sum_{J=0}^{N} \binom{N}{J} 2^{N-J} Y^J \right) x^N$$

$$= \sum_{N=0}^{\infty} \sum_{J=0}^{\infty} \binom{N}{J} 2^{N-J} x^N Y^J$$

Hence $F(N,J) = \binom{N}{J} 2^{N-J}$, since coefficients on generating functions are unique.