Quantiles, Networks, Time

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Genesis

Central banks have been around quite a while. This book has deveoped from an initiative taken by the Bank of England in the early 1990s. It was decided that the Bank should create a forum where the central bankers of the world could gather to commune, discuss, and above all, learn together. Central banking had long been associated with grey men in grey suits, pondering deeply the impenetrable decisions of high finance, fuelled by cigar smoke and mystique. In truth, this Capraesque view was substantially out of date. Central bankers were no longer monochrome, or exclusively male, and needed skills their forebears could have barely imagined – and had most decidedly different priorities.

The timing, of course, was not incidental, and the initiative turned out to be a prescient one. This was at a major historical turning point, one that signalled a burgeoning new world order, as the Iron Curtain crumbled, the European experiment gathered momentum, and industrial might continued an inexorable shift eastwards. Economic policy had shifted too. Monetary policy was beginning a new and – as it turned out – lasting phase. There was indeed much to learn, and so the Centre for Central Banking Studies was instituted. More than thirty years later it remains a key forum for learning, discussion and networking, just as intended. Activities have evolved to include many more of the disparate areas of responsibility that now involve the central banking community, and with a truly global reach.

is about economics and central banking Various histories of the CCBS exist, e.g. Hammond (2006).

Placeholders abound.

Figure 1: Puppet, Yogyakarta

Figure 2: Abu Dhabi, United Arab Emirates

Figure 3: Villa Sterne, Pretoria

Figure 4: Insadong, Seoul

Figure 5: Old Town, Montevideo

Part I

Methods matter

Chapter 1

Introduction

This is a book created using Quarto and includes all examples as executable code.

See Andrew P. Blake and Mumtaz (2017).

1.1 Reading is good for you

For me, the best (although slightly dated) text is Hastie, Tibshirani, and Friedman (2009) The Elements of Statistical Learning and the best source for the mathematics, with an easy-reading version by some of the same authors James et al. (2021) Introduction to Statistical Learning.

I also rather like Boehmke and Greenwell (2019) Hands-On Machine Learning with R which is something of a cookbook rather than a technical manual but with wide scope. Taddy (2019) is more elementary.

On text, just read Silge and Robinson (2017) Text Mining with R: A Tidy Approach and then Hvitfeldt and Silge (2021) Supervised Machine Learning for Text Analysis in R. That's it.

Two books I would solidly recommend to make us all into better statisticians and not just econometricians are Gelman, Hill, and Vehtari (2019) Regression and Other Stories, and McElreath (2020) Statistical Rethinking.

Chapter 2

Non-econometric methods for econometricians

2.1 Selected ML and Dataviz techniques in R

Econometricians are used to handling data, performing analysis and reporting results. But somewhere along the line data became big and unstructured, analysis was now machines learning about something and outputs became visualisations.

This online seminar takes some big(ish) datasets, sets the machines on them and draws some great graphs. If you ever wondered what use a tree was for forecasting or why everything is a network (including neural ones), or wanted to draw a map with your house in it, or to understand a document without the bother of reading it (or a few other things besides) then you might find something to interest you. All done in R.

Specifically, this seminar is designed to introduce some of the key methods used outside of econometrics that econometricians will find very useful in their work in a central bank. This includes some important machine learning techniques as a gateway to others, particularly tree-based methods and neural networks, as well as text processing and map making. All the way through there is an emphasis on the network properties of many of these techniques. We make extensive use of the `tidyverse`, including `ggplot2` and `tidytext`, and a number of statistics, machine learning, geographical data and other packages.

The framework for each day is the following:

- Each day is divided into two two-hour sessions starting at 10.30 am and 2.00 pm GMT.

- The first hour of each will be an online presentation covering a particular topic (or topics) with a look at both techniques and code.
- After a quick break the second hour will be largely devoted to the code itself or resources to understand how to code the material.

We may run polls during the event to prioritize the topics covered in the webinars as it is not expected that everyone will be able to try out everything.

2.1.1 The code

All code and some of the data will be made available through the Juno portal. For each presentation the .Rmd (R markdown) file is supplied that creates the presentation, an HTML file of the presentation for you to step through which can be re-created from the .Rmd file, and a further .R file of the code that we use. Some additional code and data is included, including links to a number of videos that cover some additional aspects both in this file and in the presentations.

Some data will need to be downloaded from original other sites if all the examples are to be followed. All code is additionally available at https://github.com/andrewpeterblake/R2020 or https://github.com/andrewpeterblake/R2021 or through the QR codes below.

Figure 2.1: 2020 GitHub repositories for historic CCBS courses

Figure 2.2: 2021

2.2 HOW TO ENSURE RSTUDIO FINDS THE CODE

To use the code, in particular so that R Studio finds the data files etc, create a directory for each topic, (e.g. Trees, ANN etc) and copy the contents from the zip file or GitHub. Then create a new project in R Studio that uses that directory as its home directory, using "File/New Project" in the drop down menu. Opening files within a project sets the home directory to that directory, so everything (including the sub-directories) can be found.

2.3 Typical program structure

2.3.1 Day 1: Trees and maps

2.3.1.1 Trees

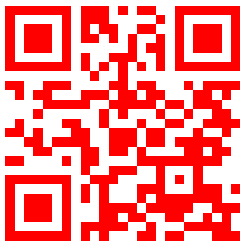
- Classification and regression trees
- Econometrics strikes back: Bootstrap/bagging and Boosting/Model selection
- Random forests
- Visualising decision trees
- Use example: House prices

The presentations for this are `Trees.html` and `LondonHP.html`; The two programs `TreeCancer.R` and `TreeNW.R` are the use examples.

2.3.1.2 Maps

- How to draw a map in R
- A guide to some resources
- Choropleths
- Use examples: Climate change, regional data, postcode wrangling

The presentation for this is `MapAER.html` (see also `Weatherpretty.html`); The program `MapAERcode.R` is the main map drawing code, I've included `ZAF.R` as as short simple way and source for two countries, and the directory `Trendz` contains the program (`app.R`) and data for the weather example.



I've included an additional video (red QR code) for more about Shiny. This uses unemployment data from the Survey of Professional Forecasters. The code we look at is for climate change data World Bank data.

A comprehensive treatment of maps is Lovelace, Nowosad, and Muenchow (2019) *Geocomputation in R*, but it is quite a lot to assimilate all at once.

2.3.2 Day 2: Networks

2.3.2.1 Neural networks

- What is an ANN? Deep learning?
- Function approximation via a network
- Data: fit, validate, test
- Network architecture
- Use examples: House prices revisited

The presentation for this is `IntroANN.html`; The program `ANN.R` replicates the ANN estimation. The data used is the same as for Day 1.

2.3.2.2 Networks (real ones)

- DAGs and ANNs as network graphs
- Incidence matrices
- Measuring connectivity: Degree and betweenness
- Plotting with `igraph`
- Use examples: Industry inter-relationships

Figure 2.3: Coding club Network examples

Figure 2.4: R-Bloggers article

The presentation used for the first part of this is `DAG.html` and the program `Draw_DAG_ANN.R` draws the ANN examples from Day 2 Session 1 as well as some of the DAG examples. The example is modified from Cunningham (2021) *Causal Inference: The Mixtape*, which is a great read with R code. The pdf `HandShake3.pdf` is the source of the director network graphs, and `Graph101a.R` is a subset of the analytical work on the corruption data set as described in the post *Graph Theory 101* (purple QR code), which is the work of Marina Medina (blue QR code link to presentation site).

2.3.3 Day 3: Text

Text modelling, a 'tidytext' approach.

2.3.3.1 Session 1

- Data cleaning
- Sentiment
- Topic modelling

2.3.3.2 Session 2

- Parts-of-speech tagging
- Text regression
- Use examples: Central bank minutes, reports

Chapter 3

Kalman filtering

Estimating unobserved components

3.1 Literature

Alternatives to what follows can be found in Harvey (1989), Hamilton (1994), Kim and Nelson (1999), Durbin and Koopman (2001) or Triantafyllopoulos (2021). There are many books devoted to the Kalman filter as a casual Amazon search mostly from an engineering perspective. However econometricians since Harvey and Pierse (1984) have used it in a way somewhat different from standard engineering applications. We will cover the filter and then look at a simple example if filtering, then develop a maximum likelihood estimation approach.

3.2 Reminder of a state-space model

Consider the trend-cycle model

$$y_t = \chi_t + \tau_t + \varepsilon_t \tag{3.1}$$

where the cycle equation is

$$\chi_t = c + \rho_1 \chi_{t-1} + \rho_2 \chi_{t-2} + v_{1t} \tag{3.2}$$

and the trend equation is

$$\tau_t = \tau_{t-1} + v_{2t} \tag{3.3}$$

In state space this can be written

$$y_t = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \chi_t \\ \chi_{t-1} \\ \tau_t \end{bmatrix} + [1]\varepsilon_t \tag{3.4}$$

$$\begin{bmatrix} \chi_t \\ \chi_{t-1} \\ \tau_t \end{bmatrix} = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \rho_1 & \rho_2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \chi_{t-1} \\ \chi_{t-2} \\ \tau_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{1t} \\ v_{2t} \end{bmatrix} \tag{3.5}$$

3.2.1 A useful class of models

We need a framework that nests this type of model (and many more). State-space models are one such framework, amenable to classical (and Bayesian) estimation. Quite a lot of apparatus required before we can apply maximum likelihood.

Key points:

- Many quantities routinely used to build models and analyse policy are *unobservable*.
- Econometricians face a major problem estimating such models: everything unobserved needs estimating simultaneously (states and parameters).
- Fortunately there is a method we can use: the ***Kalman filter*** (Kalman (1960)).
- Has the useful spin-off that we can also use it to calculate the value of the likelihood function.

3.3 What is a filter?

Imagine we had a time-varying parameter model where we estimate β_t; this becomes a time series so we have a lot of parameters to estimate: we outline an estimation method here, which we will then characterise as the *Kalman Filter*.

Simple observation and transition equations

$$y_t = H_t\beta_t + e_t, \qquad var(e_t) = R \tag{3.6}$$

$$\beta_t = \mu + F\beta_{t-1} + v_t, \qquad var(v_t) = Q \tag{3.7}$$

where β_t is a vector of stochastic variables, y_t a vector of measurements and the data forms an information set such that $\psi_T = \{y_T, y_{T-1}, ..., y_1\}$.

Jazwinski (1970) defines three type of estimation problem

- **Smoothing** is the problem of estimating β_k for any $k < T$

- **Filtering** is the problem of estimating β_k for $k = T$
- **Prediction** is the problem of estimating β_k for any $k > T$

> "The object of filtering is to update our knowledge of the system each time a new observation y_t is brought in." (Durbin and Koopman (2001))

Filtering is specifically this: perhaps we have some estimates already of β_t for $t = 1...t-1$, then given the new period-t observation of y_t how should we estimate a new value of β_t? This immediately implies a recursive structure to estimation problems, consistent with on-line (or real-time) estimation. The Kalman filter uses the period t news available from observed y_t to update our estimates of β_t using the regression lemma.

3.3.1 Forecasting

Consider the simple first-order VAR model

$$\beta_t = F\beta_{t-1} + v_t, \quad v_t \sim N(0, Q) \tag{3.8}$$

We can use to make the *conditional* forecast

$$\beta_{t|t-1} = F\beta_{t-1} \tag{3.9}$$

where $\beta_{t|t-1} = E[\beta_t|\psi_{t-1}]$ and ψ_{t-1} is the information set available at time $t-1$.

If ψ_{t-1} includes β_{t-1} we can straightforwardly forecast next period (and the next-but-one period etc) using the model. This is a standard forecasting exercise given any estimated (or even calibrated) economic model.

3.3.2 Uncertainty

How can we assess the associated forecast uncertainty? The forecast covariance $P_{t|t-1} = var(\beta_t|\psi_{t-1})$ is given by

$$\begin{aligned} P_{t|t-1} &= E\left[(\beta_t - \beta_{t|t-1})(\beta_t - \beta_{t|t-1})'\right] && (3.10)\\ &= E\left[(F\beta_{t-1} + v_t - F\beta_{t-1|t-1})\left(\beta'_{t-1}F' + v'_t - \beta'_{t-1|t-1}F'\right)\right] && (3.11)\\ &= E\left[F\left(\beta_{t-1} - \beta_{t-1|t-1}\right)\left(\beta_{t-1} - \beta_{t-1|t-1}\mu_z\right)' F'\right] + E[v_t v'_t] \\ &= FP_{t-1|t-1}F' + Q && (3.12) \end{aligned}$$

where $P_{t-1|t-1} = var(\beta_{t-1}|\psi_{t-1})$. The forecast error variance depends on the previous error variance; that value depends on the information set.

If β_{t-1} forms part of the information set ψ_{t-1} then

$$P_{t-1|t-1} = var\left(\beta_{t-1}|\psi_{t-1}\right) = 0 \tag{3.13}$$

and there is no uncertainty other than from the disturbance terms and $P_t = Q$.

If β_{t-1} does not form part of the information set ψ_{t-1} but β_{t-2} does then $P_{t-1|t-1} = Q$ and $P_{t|t-1} = FQF' + Q$. This can be continued backwards; the unconditional (steady-state) covariance of β_t is the limit $P = FPF' + Q$. We can easily calculate error bands for β_t using the appropriate information set.

3.3.3 Prediction error

We can turn this around, as it must be the *prediction errors* are given by

$$\begin{aligned}
\eta_{t|t-1} &= y_t - E[y_t|\psi_{t-1}] && (3.14)\\
&= y_t - E[H_t\beta_t + e_t|\psi_{t-1}] && (3.15)\\
&= y_t - H_t\beta_{t|t-1} && (3.16)
\end{aligned}$$

where $\eta_{t|t-1}$ is uncorrelated with ψ_{t-1}. So the 'news' over that contained in y_t above ψ_{t-1} is captured by $\eta_{t|t-1}$. It will be that $\eta_{t|t-1} \sim N(0, \Sigma_{\eta\eta})$; we need to find an expression for the covariance.

3.3.4 Current-data predictions

Now we find $E[\beta_t|\psi_t]$ – the best prediction of the unknown coefficient vector given *current* information. Using the regression lemma we know that

$$\begin{aligned}
E[\beta_t|\psi_t] &= E[\beta_t|\psi_{t-1}, \eta_{t|t-1}] && (3.17)\\
&= E[\beta_t|\psi_{t-1}] + \Sigma_{\beta\eta}\Sigma_{\eta\eta}^{-1}\eta_{t|t-1} && (3.18)\\
&= \beta_{t|t-1} + \Sigma_{\beta\eta}\Sigma_{\eta\eta}^{-1}\eta_{t|t-1} && (3.19)
\end{aligned}$$

because ψ_{t-1} and $\eta_{t|t-1}$ are uncorrelated and $\eta_{t|t-1}$ is mean zero.

Similarly, we can find $P_{t|t}$ as the best prediction of the variance of β_t given ψ_t. Using the regression lemma we know that

$$\begin{aligned}
P_{t|t} &= E[(\beta_t - \beta_{t|t})(\beta_t - \beta_{t|t})'|\psi_{t-1}, \eta_{t|t-1}] && (3.20)\\
&= E[(\beta_t - \beta_{t|t})(\beta_t - \beta_{t|t})'|\psi_{t-1}] - \Sigma_{\beta\eta}\Sigma_{\eta\eta}^{-1}\Sigma_{\eta\beta} && (3.21)\\
&= P_{t|t-1} - \Sigma_{\beta\eta}\Sigma_{\eta\eta}^{-1}\Sigma_{\eta\beta} && (3.22)
\end{aligned}$$

3.3.5 Estimated model covariances

All we need do is plug the relevant expressions into the regression lemma. So, what is $\Sigma_{\beta\eta}$?

$$\begin{aligned}
\Sigma_{\beta\eta} &= E[(\beta_t - \beta_{t|t-1})\eta'_{t|t-1}] && (3.23)\\
&= E\left[(\beta_t - \beta_{t|t-1})(y_t - H_t\beta_{t|t-1})'\right] && (3.24)\\
&= E\left[(\beta_t - \beta_{t|t-1})(H_t\beta_t + e_t - H_t\beta_{t|t-1})'\right] && (3.25)\\
&= E[(\beta_t - \beta_{t|t-1})(\beta_t - \beta_{t|t-1})'H'_t] + E[(\beta_t - \beta_{t|t-1})e'_t] && (3.26)\\
&= P_{t|t-1}H'_t && (\Sigma_{\beta\eta})
\end{aligned}$$

as $E[(\beta_t - \beta_{t|t-1})e'_t] = 0$.

What is $\Sigma_{\eta\eta}$?

$$\begin{aligned}
\Sigma_{\eta\eta} &= E[(y_t - H_t\beta_{t|t-1})(y_t - H_t\beta_{t|t-1})'] && (3.27)\\
&= E[(H_t\beta_t + e_t - H_t\beta_{t|t-1})(H_t\beta_t + e_t - H_t\beta_{t|t-1})'] && (3.28)\\
&= E[(H_t\beta_t - H_t\beta_{t|t-1})(H_t\beta_t - H_t\beta_{t|t-1})'] + E[e_te'_t] && (3.29)\\
&= E[H_t(\beta_t - \beta_{t|t-1})(\beta_t - \beta_{t|t-1})'H'_t] + R && (3.30)\\
&= H_tP_{t|t-1}H'_t + R && (3.31)\\
&= f_{t|t-1} && (\Sigma_{\eta\eta})
\end{aligned}$$

where we define $f_{t|t-1} = E[\eta_{t|t-1}\eta'_{t|t-1}]$. Now we're ready.

3.4 The Kalman filter

The equations of the filter are

- the conditional expectation depending on ψ_{t-1};
- an update that uses $\eta_{t|t-1}$ to obtain the best t-period prediction now based on ψ_t.

These must be of the form

$$\begin{aligned}
E[\beta_t|\psi_{t-1}] &= \mu + FE[\beta_t|\psi_{t-1}] && (3.32)\\
E[P_t|\psi_{t-1}] &= FE[P_{t-1}|\psi_{t-1}]F' + Q && (3.33)\\
E[\beta_t|\psi_t] &= E[\beta_t|\psi_{t-1}] + \Sigma_{\beta\eta}\Sigma^{-1}_{\eta\eta}\eta_{t|t-1} && (3.34)\\
E[P_t|\psi_t] &= E[P_t|\psi_{t-1}] - \Sigma_{\beta\eta}\Sigma^{-1}_{\eta\eta}\Sigma_{\eta\beta} && (3.35)
\end{aligned}$$

These are specifically

$$\begin{aligned}
\beta_{t|t-1} &= \mu + F\beta_{t-1|t-1} && \text{(Predicted } \beta\text{)} \\
P_{t|t-1} &= FP_{t-1|t-1}F' + Q && \text{(Predicted } P\text{)} \\
\eta_{t|t-1} &= y_t - H_t\beta_{t|t-1} && \text{(Prediction error)} \\
f_{t|t-1} &= H_tP_{t|t-1}H_t' + R && \text{(Pred. err. variance)} \\
\beta_{t|t} &= \beta_{t|t-1} + P_{t|t-1}H_t'f_{t|t-1}^{-1}\eta_{t|t-1} && \text{(Updated } \beta\text{)} \\
P_{t|t} &= P_{t|t-1} - P_{t|t-1}H_t'f_{t|t-1}^{-1}H_tP_{t|t-1} && \text{(Updated } P\text{)}
\end{aligned}$$

The filter evaluates these recursively, beginning from β_0, P_0.

Treatment of these initial condition reflects knowledge/model

- Stationary models can use the steady-state
- Non-stationary models use something which is often (confusingly) called a *diffuse prior* (zero mean, large variance)

3.4.1 Kalman filter trick

For ***known*** initial conditions – say $\beta_0 \sim N(\mu_0, P_0)$ – the likelihood of a state-space model with T observations of m variables is

$$\log L(\theta|y) = \sum_{t=1}^{T} \log\left(p\left(\theta|\psi_{t-1}, \theta\right)\right) \tag{3.36}$$

$$= -\Phi - \frac{1}{2}\sum_{t=1}^{T}\left(\log(\det(f_{t|t-1})) + \eta_{t|t-1}'f_{t|t-1}^{-1}\eta_{t|t-1}|\theta\right) \tag{3.37}$$

where $\Phi = \frac{Tm}{2}\log(2\pi)$ and θ are all the non-state parameters to be estimated. We can use the Kalman filter to obtain $\eta_{t|t-1}$ and $f_{t|t-1}$ as they are the *prediction error* and its *variance*. This is the ***prediction error decomposition*** of the log-likelihood.

A maximum likelihood estimate maximizes $\log L(y|\theta)$ by choice of θ.

Part II

Quantiles

Chapter 4

Quantile regression

This shows how to manipulate data from the SPF in R.

```
library(readxl)
library(xts)
library(lubridate)
library(tidyverse)
library(quantmod)
library(quantreg)
```

4.1 Getting the data

We download the data and save it locally.

```
h <- "https://www.philadelphiafed.org/-/media/frbp/assets/surveys-and-data/survey-of-professio
f <- "meanlevel.xlsx"

download.file(paste0(h, f), destfile=f, mode="wb")
```

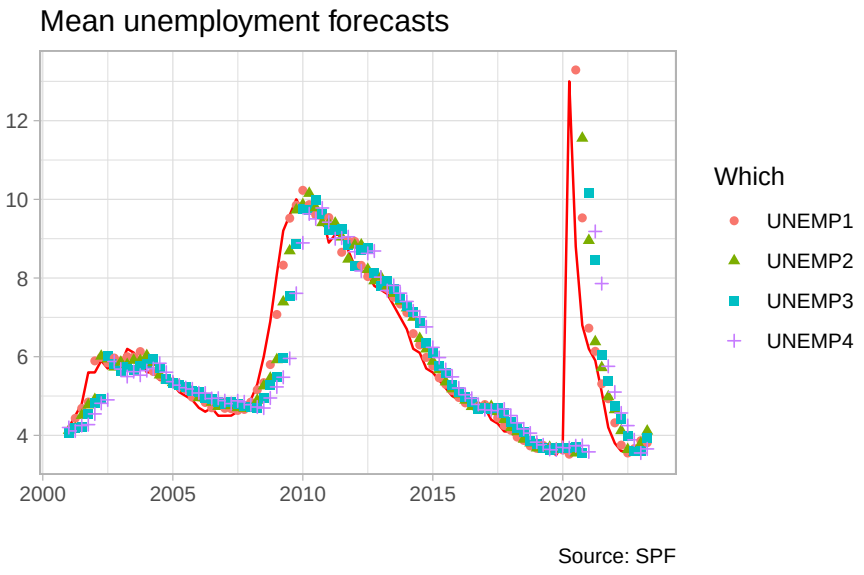
Retrieve the unemployment data for the average unemployment forecast.

```
UNEMP <- f %>%
  read_excel(na="#N/A", sheet="UNEMP") %>%
  mutate(Date=as.Date(as.yearqtr(paste(YEAR, QUARTER), format="%Y %q")))

Usel <- UNEMP %>%
  select(Date, UNEMP1, UNEMP3, UNEMP4, UNEMP5, UNEMP6) %>%
```

```
  mutate(UNRATE = lead(UNEMP1,1)) %>%
  select(Date, UNRATE,
         UNEMP1=UNEMP3, UNEMP2=UNEMP4, UNEMP3=UNEMP5, UNEMP4=UNEMP6) %>%
  mutate(UNEMP1 = lag(UNEMP1,1),
         UNEMP2 = lag(UNEMP2,2),
         UNEMP3 = lag(UNEMP3,3),
         UNEMP4 = lag(UNEMP4,4)) %>%
  pivot_longer(cols = -c(Date, UNRATE), names_to="Which", values_to="Val") %>%
  filter(year(Date) > 2000)
```

4.2 Plots

```
Usel %>%
  ggplot(aes(x=Date)) +
  geom_line(aes(y=UNRATE), colour="red") +
  geom_point(aes(y=Val, colour=Which, shape=Which)) +
  theme_light() +
  labs(title="Mean unemployment forecasts", x="", y="", caption="Source: SPF")
```



Chapter 5

Carter-Kohn

5.1 Using the Kalman Filter

Establish the usefulness of the Kalman Filter (and not just for state estimation). Refresh idea of maximum likelihood estimation in the context of state space models.

- Introduce *smoothing*
- Develop Gibbs sampling by the Carter-Kohn method
- All of these use the Kalman Filter to develop conceptually different tools

Follow Kim and Nelson (1999); also see Harvey (1989), Hamilton (1994), Durbin and Koopman (2001)

5.2 Maximum likelihood

5.2.1 Classical Maximum Likelihood Estimation

The principle of maximum likelihood is that the parameters should be chosen so that the probability of observing a given sample is maximized.

For time series models the joint density of $\psi_T = \{y_T, y_{T-1}, \dots, y_1\}$ and parameters θ in conditional form is

$$p(\theta|\psi_T) = \prod_{t=1}^{T} p(y_t|\psi_{t-1}, \theta) \tag{5.1}$$

emphasizing the serial dependence of observations.

Interpret this as the likelihood for a particular sample. Assuming (conditional) normality, the likelihood of any particular n-vector of observations is

$$p(y_t) = (2\pi)^{-\frac{n}{2}} |var(y_t)|^{-\frac{1}{2}} e^{\{-\frac{1}{2}(y_t-\mu)' var(y_t)^{-1}(y_t-\mu)\}} \tag{5.2}$$

Notice this depends on the observed data *and* the values of the parameters. It can be multivariate and for any underlying density. A maximum likelihood (ML) estimate of θ maximizes the likelihood of the parameter given an observed sample.

5.2.2 Poisson example

Th Poisson distribution is a nice one to consider as the maximum likelihood estimate can be calculated easily – essentially in your head.[1]

The Poisson distribution is

$$p(y_i,\ \theta) = \frac{e^{-\theta}\theta^{y_i}}{y_i!} \tag{5.4}$$

for $y > 0$, zero otherwise with the property $E[Y] = var(Y) = \theta$.

Count variables often modelled as a random Poisson process: numbers of road traffic accidents, sales, telephone calls, electron emissions. Greene's example is to find the most likely value of θ given observations.

$$5,\ 0,\ 1,\ 1,\ 0,\ 3,\ 2,\ 3,\ 4,\ 1 \tag{5.5}$$

For independent observations the joint density is

$$p(y,\ \theta) = \prod_{i=1}^{10} p(y_i,\ \theta) = \frac{e^{-10\theta}\theta^{\sum_i y_i}}{\prod_i (y_i!)} = \frac{e^{-10\theta}\theta^{20}}{207,360} \tag{5.6}$$

We can plot this function to see if it has a maximum

We can also find this by calculus. As ever, because the log function is monotonic it is convenient to take logs

$$\ln L(\theta) = -10\theta + 20\ln\theta - \ln(207,360) \tag{5.7}$$

First order conditions are

$$\frac{\partial \ln L(\theta)}{\partial \theta} = -10 + \frac{20}{\theta} \Rightarrow \theta = \frac{20}{10} = 2 \tag{5.8}$$

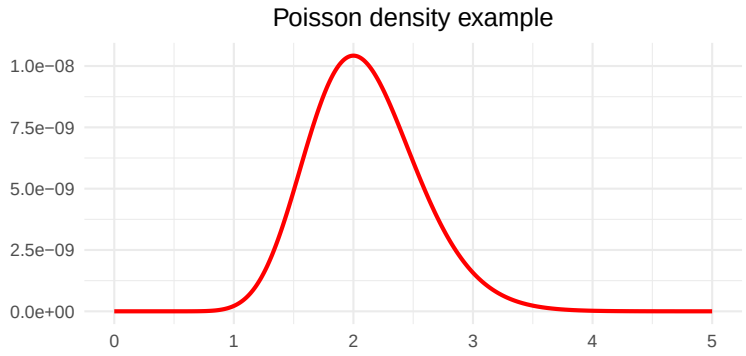
Check for maximum

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = -\frac{20}{\theta^2} < 0 \tag{5.9}$$

[1] For example Greene (1997) constructs a Poisson distribution example where the chosen observations yield an exact estimate of the underlying parameter of the distribution. The example is to find the most likely value of θ given observations

$$5,\ 0,\ 1,\ 1,\ 0,\ 3,\ 2,\ 3,\ 4,\ 1 \tag{5.3}$$

ten observations which sum to 20.



5.3 Poisson example

The Poisson density for each observation is

$$p(y_i, \theta) = \frac{e^{-\theta}\theta^{y_i}}{y_i!} \tag{5.10}$$

for $y > 0$, zero otherwise, with $E[Y] = var(Y) = \theta$. For n independent observations, joint density is

$$P(y, \theta) = \prod_{i=1}^{n} p(y_i, \theta) = \frac{e^{-n\theta}\theta^{\sum_i y_i}}{\prod_i (y_i!)} \tag{5.11}$$

Interpret this as a likelihood function, i.e. a probability measure for θ given some observed y

$$L(\theta|y) = P(y, \theta) \tag{5.12}$$

As log function is monotonic

$$\ln L(\theta|y) = -n\theta + \textstyle\sum_i y_i \ln\theta - \ln\left(\prod_i y_i!\right) \tag{5.13}$$

First order conditions are

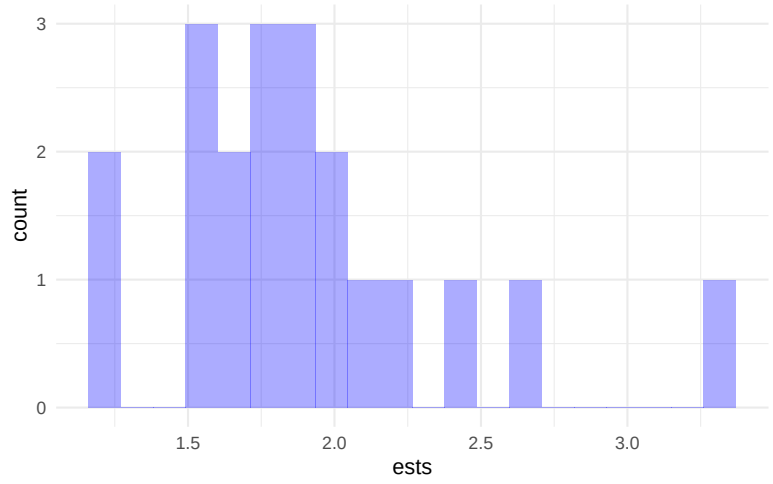
$$\frac{\partial \ln L(\theta|y)}{\partial\theta} = -n + \frac{\sum_i y_i}{\theta} \Rightarrow \theta = \frac{\sum_i y_i}{n} \tag{5.14}$$

Finally, check for maximum

$$\frac{\partial^2 \ln L(\theta)}{(\partial\theta)^2} = -\frac{\sum_i y_i}{\theta^2} < 0 \tag{5.15}$$

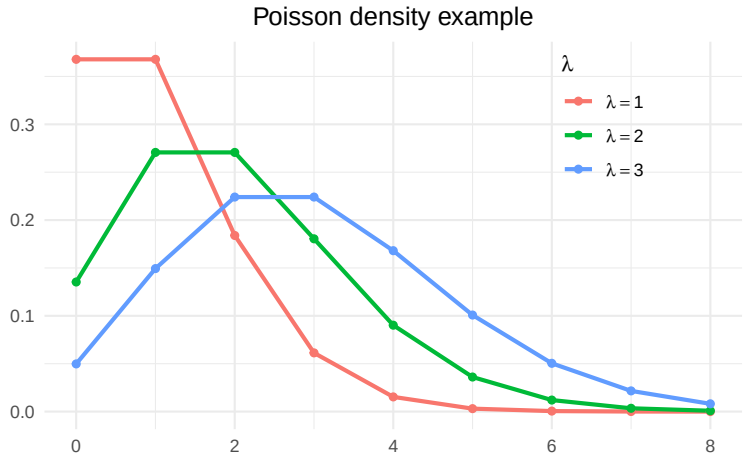
```
library(tidyverse)
library(scales)

reps   <- 20
lambda <- 2
n      <- 10

a <- tibble(x=seq(0, 3*lambda),
            p=(exp(-lambda*x)*(lambda^x))/prod(factorial(lambda)))

d <- matrix(rpois(n*reps,lambda), n, reps)

t <- as_tibble(factorial(d), .name_repair = ~paste0("Rep",1:reps)) %>%
  pivot_longer(cols = everything()) %>%
  group_by(name) %>%
  summarise(dd = prod(value)) %>%
  mutate(sum  = colSums(d),
         ests = sum/n)
```

```
ggplot(t) +
  geom_histogram(aes(x=ests), fill="blue", alpha=.33, color=NA, bins=20) +
  theme_minimal()
```



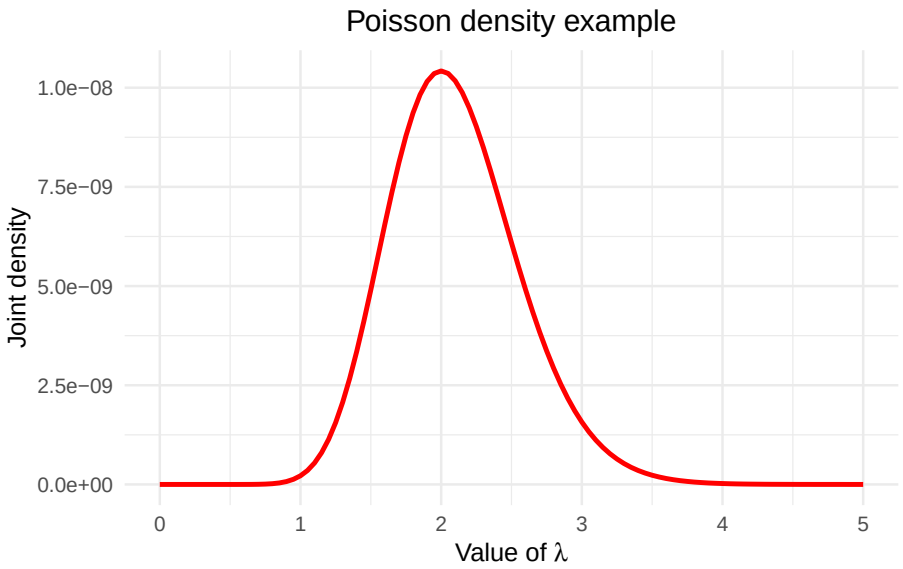

```
theta <- seq(0, 8, 1)
y     <- (exp(-lambda)*lambda^(theta))/(factorial(theta))
df    <- tibble(theta = theta) %>%
```

```
  mutate(`lambda == 1` = (exp(-1)*1^(theta))/(factorial(theta)),
         `lambda == 2` = (exp(-2)*2^(theta))/(factorial(theta)),
         `lambda == 3` = (exp(-3)*3^(theta))/(factorial(theta)))

df %>%
  pivot_longer(cols = -theta, names_to = "lambda") %>%
  ggplot() +
  geom_line(aes(x=theta, y=value, color=lambda), size=1) +
  geom_point(aes(x=theta, y=value, color=lambda)) +
  theme_minimal() +
  labs(title="Poisson density example", x="", y="",
       color=expression(lambda)) +
  theme(plot.title = element_text(hjust = 0.5),
        plot.subtitle = element_text(hjust = 0.5),
        legend.position = c(.8,.8)) +
  scale_colour_discrete(labels = parse_format())
```




```
e <- c(5,0,1,1,0,3,2,3,4,1)
l <- seq(0,5,0.05)
y <- (exp(-n*l)*l^(sum(e)))/(prod(factorial(e)))

tibble(l=l, y=y) %>%
  ggplot() +
  geom_line(aes(x=l, y=y), color="red", linewidth=1) +
  theme_minimal() +
  labs(title="Poisson density example",
```

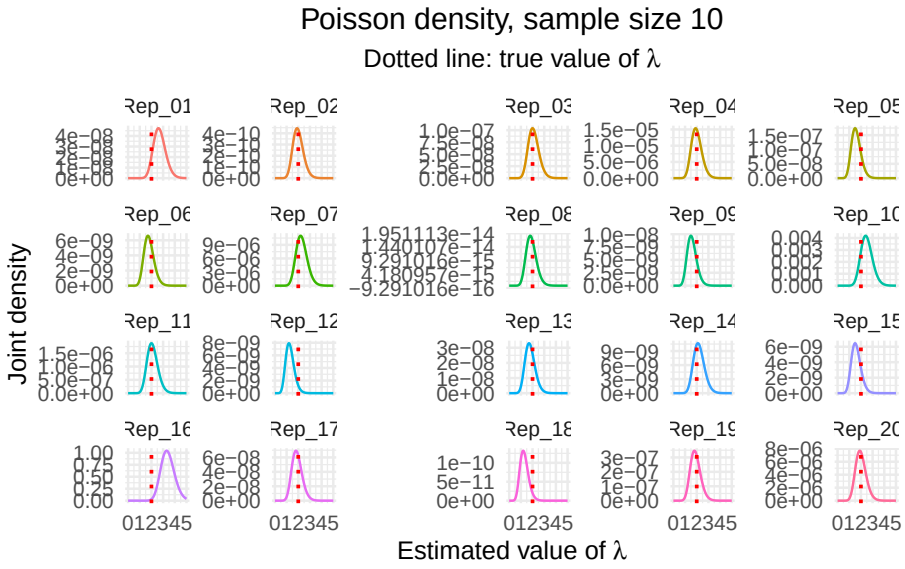
```
       x=expression(paste("Value of ", lambda)),
       y="Joint density") +
  theme(plot.title = element_text(hjust = 0.5))
```



Maintain the value of λ of 2. Now generate $reps = 20$ replications of $n = 10$ observations and plot the empirical density of each and the maximum likelihood estimate for each replication.

```
t2 <- t %>%
  group_by(name) %>%
  mutate(r = list((exp(-n*l)*l^sum)/dd))

nms <- paste0("Rep_", str_pad(1:reps, 2, pad="0"))

t2$r %>%
  bind_cols(.name_repair = ~ nms) %>%
  mutate(l = l) %>%
  pivot_longer(cols = -l) %>%
  ggplot() +
  geom_line(aes(x=l, y=value, color=name), show.legend = FALSE) +
  geom_vline(aes(xintercept=lambda), color="red", linetype=3, linewidth=0.75) +
  facet_wrap(~ name, scales = "free_y") +
  theme_minimal() +
  labs(title    = paste("Poisson density, sample size", n),
```

```
      subtitle = expression(paste("Dotted line: true value of ", lambda)),
      x        = expression(paste("Estimated value of ", lambda)),
      y        = "Joint density") +
theme(plot.title = element_text(hjust = 0.5),
      plot.subtitle = element_text(hjust = 0.5))
```



5.3.1 ML and regression

Linear regression problem is

$$L(\beta|y,X) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2}(y-X\beta)'(y-X\beta)\right] \tag{5.16}$$

The log-likelihood is

$$\ln L = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}(y-X\beta)'(y-X\beta) \tag{5.17}$$

As before, find the extremum by calculus; yields *likelihood equations*

$$\frac{\partial \ln L}{\partial \beta} = -\frac{2}{2\sigma^2}(X'y - X'X\beta) = 0 \tag{5.18}$$

$$\Rightarrow \hat{\beta}_{ml} = (X'X)^{-1}X'y \tag{5.19}$$

and

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}(y - X\beta)'(y - X\beta) = 0 \tag{5.20}$$

$$\Rightarrow -n + \sigma^{-2}(\epsilon'\epsilon) = 0 \tag{5.21}$$

$$\Rightarrow \hat{\sigma}^2_{ml} = \frac{\hat{\epsilon}'\hat{\epsilon}}{n} \tag{5.22}$$

ML estimate of σ^2 divided by n (not $n-k$) so biased in small samples but not asymptotically

5.4 Kalman filter tricks

For some initial condition – say $\beta_0 \sim N(\mu_0, P_0)$) – the conditional log-likelihood for sample 1 to T

$$\log L(\psi_t|\theta) = \sum_{t=1}^{T} \log p(y_t|\psi_{t-1}, \theta) \tag{5.23}$$

$$\propto -\sum_{t=1}^{T} \left(\log \left|f_{t|t-1}\right| + \eta'_{t|t-1} f^{-1}_{t|t-1} \eta_{t|t-1}| \ \theta\right) \tag{5.24}$$

Note we could obtain $\eta_{t|t-1}$ and $f_{t|t-1}$ from the Kalman filter, i.e.

$$f_{t|t-1} = (H_t P_{t|t-1} H'_t + Q) = \Sigma_{\eta\eta} \tag{5.25}$$

This is the *prediction error decomposition* of the log-likelihood. For a classical approach we estimate θ by numerically maximizing $\log L(\psi_T|\theta)$.This gives a point estimate for the value of θ and we typically apply classical inference using the estimated standard errors. Note to do this we need to evaluate the best estimate of the state as well as maximize the likelihood: the Kalman Filter is a key ingredient in both.

> **Maximisation**
>
> A suitable numerical maximization routine will (in principle) maximize the likelihood straightforwardly. Often use Chris Sims' **csminwel** in Matlab or R as well-suited to this type of problem.

Can show (via Cramer-Rao) that

$$\hat{\theta} \sim N\left(\theta, -\frac{\partial^2 \log L(\psi_T|\theta)}{\partial\theta\partial\theta'}\right) \tag{5.26}$$

5.5 Full sample estimates of β_t

5.5.1 The regression lemma again

You may recall that for any

$$\begin{bmatrix} z \\ y \\ \varepsilon \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_z \\ \mu_y \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_{zz} & \Sigma_{zy} & \Sigma_{z\varepsilon} \\ \Sigma_{yz} & \Sigma_{yy} & 0 \\ \Sigma_{\varepsilon z} & 0 & \Sigma_{\varepsilon\varepsilon} \end{bmatrix} \right) \tag{5.27}$$

then it must be that

$$E[z|y,\varepsilon] = \mu_z + \Sigma_{zy}\Sigma_{yy}^{-1}(y-\mu_y) + \Sigma_{z\varepsilon}\Sigma_{\varepsilon\varepsilon}^{-1}\varepsilon \tag{5.28}$$

$$= E[z|y] + \Sigma_{z\varepsilon}\Sigma_{\varepsilon\varepsilon}^{-1}\varepsilon \tag{5.29}$$

We use this to derive a recursive update to *smooth* our estimates. We will also derive an appropriate conditional expectation which we can use in Gibbs sampling.

5.6 Smoothing

The Kalman filter estimates β_t recursively: it only uses information available up until time t. This means that the estimate of $\beta_{T|T}$ uses all available information, but any previous estimate doesn't. Indeed there must be some values of $\eta_{i|t-1}$

$$\beta_{t|T} = E(\beta_t|\psi_{t-1}, \eta_{t|t-1}, \eta_{t+1|t-1}, ..., \eta_{T|t-1}) \tag{5.30}$$

where the 'news' is relative to period t.

We could update $\beta_{t|t-1}$ using the (uncorrelated) future innovations

$$\beta_{t|T} = \beta_{t|t} + \sum_{j=t}^{T} \Sigma_{\beta_t\eta_j}\Sigma_{\eta_j\eta_j}^{-1}\eta_{j|t-1} \tag{5.31}$$

and recalling $\beta_{t|t} = E(\beta_t|\psi_{t-1}, \eta_{t|t-1})$.

This is a *fixed interval smoother*; often used for full sample estimates of β_t. Remember we already have an estimate of $\beta_{T|T}$ from the Kalman filter so *smoothers work backwards*; we sketch a derivation here.

In the last but one period we have a different prediction error

$$\varsigma_{T|T-1} = \beta_{T|T} - F\beta_{T-1|T-1} - \mu \tag{5.32}$$

which is the error in predicting β_T using ψ_{T-1}.

An 'update' has to be of the form

$$\beta_{T-1|T} = \beta_{T-1|T-1} + \Sigma_{\beta\varsigma}\Sigma_{\varsigma\varsigma}^{-1}\varsigma_{T|T-1} \tag{5.33}$$

where $\Sigma_{\varsigma\varsigma} = var\left[\varsigma_T|\psi_{T-1}\right]$ and $\Sigma_{\beta\varsigma} = cov\left[\beta_{T-1}, \varsigma_T|\psi_{T-1}\right]$.

These are

$$\begin{aligned}
\Sigma_{\varsigma\varsigma} &= var(\beta_T - F\beta_{T-1|T-1} - \mu) && (5.34)\\
&= var\left(F(\beta_{T-1} - \beta_{T-1|T-1}) + e_t\right) && (5.35)\\
&= FP_{T-1|T-1}F' + Q && (5.36)
\end{aligned}$$

and

$$\begin{aligned}
\Sigma_{\beta\varsigma} &= E\left[(\beta_{T-1} - \beta_{T-1|T-1})\left(\beta_T - F\beta_{T-1|T-1} - \mu\right)'\right] && (5.37)\\
&= E\left[(\beta_{T-1} - \beta_{T-1|T-1})(\beta_T - \beta_{T-1|T-1})'\right]F' && (5.38)\\
&= P_{T-1|T-1}F' && (5.39)
\end{aligned}$$

Plugging these definitions in gives us

$$\beta_{T-1|T} = \beta_{T-1|T-1} + P_{T-1|T-1}F'P_{T|T-1}^{-1}(\beta_{T|T} - F\beta_{T-1|T-1} - \mu) \tag{5.40}$$

Applying the argument backward in time gives the recursion

$$\begin{aligned}
\beta_{t|T} &= \beta_{t|t} + P_{t|t}F'P_{t+1|t}^{-1}(\beta_{t+1|T} - F\beta_{t|t} - \mu) && (5.41)\\
&= \beta_{t|t} - K_{t|T}(\beta_{t+1|T} - F\beta_{t|t} - \mu) && \text{(smooth)}
\end{aligned}$$

All these quantities are outputs of the Kalman filter so smoothing is easy to implement.

The smoothed variance of $\beta_{t|T}$ found by multiplying out (smooth). To do this use $\beta_{t+1|t} = \mu + F\beta_{t|t}$ so rearranging gives

$$\tilde{\beta}_{t|T} + K_{t|T}\beta_{t+1|T} = \tilde{\beta}_{t|t} + K_{t|T}\beta_{t+1|t} \tag{5.42}$$

where $\tilde{\beta}_{t|t} = \beta_t - \beta_{t|t}$. Now square both sides and take expectations

$$P_{t|T} + K_{t|T}E\left[\beta_{t+1|T}\beta_{t+1|T}'\right]K_{t|T}' = P_{t|t} + K_{t|T}E\left[\beta_{t+1|t}\beta_{t+1|t}'\right]K_{t|T}' \tag{5.43}$$

Adding and subtracting $E[\beta_{t+1}\beta_{t+1}']$ we can show that

$$-E\left[\beta_{t+1|T}\beta_{t+1|T}'\right] + E\left[\beta_{t+1|t}\beta_{t+1|t}'\right] = P_{t+1|T} - P_{t+1|t} \tag{5.44}$$

to obtain

$$P_{t|T} = P_{t|t} + K_{t|T}(P_{t+1|T} - P_{t+1|t})K_{t|T}'. \tag{5.45}$$

5.7 Kalman filter in econometrics

5.7.1 Classical approach

The typical procedure is some variation on:

- Formulate state-space model
- Estimate the model by maximum likelihood
- Condition on the parameters to retrieve the (usually smoothed) state estimates and standard errors
- Use Cramer-Rao to calculate the standard errors of any other parameter estimates

For this the Kalman filter is a useful tool, as it allows a great deal of flexibility in the estimation of a variety of models, as is is an appropriate tool for models with unobserved components. However, it must be used with care: it is easy to try to estimate models that are essentially unidentified.

Further useful tools

- The *Extended Kalman filter* linearises the filter at every step and can be used for nonlinear models (such as ones where you need to estimate B_T and θ simultaneously)
- Increasingly non-Gaussian non-linear models are estimated using the *particle filter*

5.7.2 Bayesian approach

Bayesian approach is to generate the entire distribution of the model parameters.

- Now no longer just look for the point estimate obtained by maximum likelihood
- Use Gibbs sampling or some other appropriate method applied to the state space model
- In particular we treat the states and the parameters as jointly determined by the data
- As the state is estimated we need a way to draw the states conditional on the other estimates to do Gibbs sampling
- Seek a conditional updating algorithm that replicates the Gibbs sampling approach we have used before

We require a procedure such that:

- **Step 1** Conditional on θ and the data, generate the sequence $B_T = (\beta_1, \beta_2, ..., \beta_T)$
- **Step 2** Conditional on B_T and the data, generate values of θ
- **Step 3** Iterate previous two steps until convergence

In this way the joint distribution of the two can be obtained from the resulting simulation.

5.8 Carter-Kohn algorithm

- **Step 2** above relatively easy but how do we generate a sequence of states?
  - The state estimates depend on the parameter value through the Kalman filter
- Appropriate algorithm designed by Carter and Kohn (1994)
- Takes the form of a modified Kalman smoother
- Known as *multimove Gibbs sampling*
- Similar to above, define

$$B_t = [\beta_1 \quad \beta_2 \quad ... \quad \beta_t] \tag{5.46}$$

so in particular

$$B_{T-1} = [\beta_1 \quad \beta_2 \quad ... \quad \beta_{T-1}] \tag{5.47}$$

consistent with out earlier definition of ψ_t.

- Multimove Gibbs sampling generates the whole vector of states (B_T) at once
- We therefore need to generate a realization of B_T given the probability distribution $p(B_T|\psi_T)$
- We want to generate an appropriate conditional probability distribution $p(\beta_t|B_{j\neq t}, \psi_T)$ to sample from for our Gibbs sampler
- Just as for the Kalman smoother we use the outputs of the Kalman filter and a separate backward recursion to obtain the conditional distribution

5.9 Joint distribution

Deriving the appropriate distributions is easy if we know what to condition on. The joint probability density function can be split into a sequence of conditional distributions: $p(B_T|\psi_T)$ can be written recursively

$$p(B_T|\psi_T) = p(\beta_T|\psi_T) \times p(B_{T-1}|\beta_T, \psi_T) \tag{5.48}$$
$$= p(\beta_T|\psi_T) \times p(\beta_{T-1}|\beta_T, \psi_T) \times p(B_{T-2}|\beta_{T-1}, \beta_T, \psi_T) \tag{5.49}$$
$$= p(\beta_T|\psi_T) \times p(\beta_{T-1}|\beta_T, \psi_T) \times p(B_{T-2}|\beta_{T-1}, \psi_T) \tag{5.50}$$

Final simplification follows as the state vector is a Markov chain so there is no information in β_T not contained in β_{T-1} and ψ_T. Further as soon as we know β_{T-1} there is no information contained in ψ_T so we can drop that, so

$$p(B_T|\psi_T) = p(\beta_T|\psi_T) \times p(\beta_{T-1}|\beta_T, \psi_T) \times p(B_{T-2}|\beta_{T-1}, \psi_T) \tag{5.51}$$
$$= p(\beta_T|\psi_T) \times p(\beta_{T-1}|\beta_T, \psi_{T-1}) \times p(B_{T-2}|\beta_{T-1}, \psi_{T-2}) \tag{5.52}$$
$$= p(\beta_T|\psi_T) \times \prod_{t=1}^{T-1} p(\beta_t|\beta_{t+1}, \psi_t) \tag{5.53}$$

5.10 The Carter-Kohn equations

The estimated β variables are distributed

$$\beta_{T|\psi_T} \sim N(\beta_{T|T}, P_{T|T}) \tag{5.54}$$
$$\beta_{t|\psi_t,\beta_{t+1}} \sim N(\beta_{t|t,\beta_{t+1}}, P_{t|t,\beta_{t+1}}) \tag{5.55}$$

where

$$\beta_{t|t,\beta_{t+1}} = E\left[\beta_t|\psi_t, \beta_{t+1}\right] = E\left[\beta_t|\beta_{t|t}, \beta_{t+1}\right] \tag{5.56}$$
$$P_{t|t,\beta_{t+1}} = cov\left[\beta_t|\psi_t, \beta_{t+1}\right] = cov\left[\beta_t|\beta_{t|t}, \beta_{t+1}\right] \tag{5.57}$$

Carter-Kohn derive appropriate recursions so that, for example, we update the state estimate conditioning on some known value of β_{t+1}

$$\beta_{t|t,\beta_{t+1}} = \beta_{t|t} - K_{t|t+1}(\beta_{t+1} - F\beta_{t|t} - \mu) \tag{5.58}$$

Define

$$\varsigma_{t+1|t} = \beta_{t+1} - F\beta_{t|t} - \mu \tag{5.59}$$

as the 'innovation' in predicted $\beta_{t+1|t}$ where we have some realized β_{t+1} drawn from its probability distribution. The Carter-Kohn smoother comprises updates

to the conditional expectations that use this news.

$$E[\beta_t|\psi_t, \beta_{t+1}] = E[\beta_t|\psi_t] + \Sigma_{\beta\varsigma}\Sigma_{\varsigma\varsigma}^{-1}\varsigma_{t+1|t} \tag{5.60}$$
$$= \beta_{t|t} + \Sigma_{\beta\varsigma}\Sigma_{\varsigma\varsigma}^{-1}\varsigma_{t+1|t} \tag{5.61}$$
$$var[\beta_t|\psi_t, \beta_{t+1}] = var[\beta_t|\psi_t] - \Sigma_{\beta\varsigma}\Sigma_{\varsigma\varsigma}^{-1}\Sigma_{\varsigma\beta} \tag{5.62}$$
$$= P_{t|t} - \Sigma_{\beta\varsigma}\Sigma_{\varsigma\varsigma}^{-1}\Sigma_{\varsigma\beta} \tag{5.63}$$
$$= P_{t|t,\beta_{t+1}} \tag{5.64}$$

Both $\beta_{t|t}$ and $P_{t|t}$ are outputs of the Kalman filter.

5.10.1 Deriving $\Sigma_{\beta\varsigma}$ and $\Sigma_{\varsigma\varsigma}$

As before we just plug in the definitions so

$$\Sigma_{\varsigma\varsigma} = var[\beta_{t+1} - F\beta_{t|t} - \mu] \tag{5.65}$$
$$= var\left[F\beta_t + \mu + v_{t+1} - F\beta_{t|t} - \mu\right] \tag{5.66}$$
$$= var\left[F(\beta_t - \beta_{t|t}) + v_{t+1}\right] \tag{5.67}$$
$$= FP_{t|t}F' + Q \tag{5.68}$$

and

$$\Sigma_{\beta\varsigma} = E\left[(\beta_t - \beta_{t|t})(\beta_{t+1} - F\beta_{t|t} - \mu)'\right] \tag{5.69}$$
$$= E\left[(\beta_t - \beta_{t|t})\left(F(\beta_t - \beta_{t|t}) + v_{t+1}\right)'\right] \tag{5.70}$$
$$= P_{t|t}F' \tag{5.71}$$

5.11 'Kalman gain' again

So using the definitions of the covariances and the regression lemma we get

$$\beta_{t|t,\beta_{t+1}} = \beta_{t|t} + \Sigma_{s\eta}\Sigma_{\eta\eta}^{-1}\varsigma_t \tag{5.72}$$
$$= \beta_{t|t} + P_{t|t}F'\left(FP_{t|t}F' + Q\right)^{-1}(\beta_{t+1} - F\beta_{t|t} - \mu) \tag{5.73}$$
$$= \beta_{t|t} - K_{t|t}(\beta_{t+1} - F\beta_{t|t} - \mu) \tag{5.74}$$

where

$$K_{t|t+1} = -P_{t|t}F'(FP_{t|t}F' + Q)^{-1} \tag{5.75}$$

Like the Kalman smoother, this uses the filter's estimate of $P_{t|t}$ and updates β_t using the error in predicting β_{t+1} not y_t.

5.11.1 Conditional mean and variance of the state

Updating equations for the state and variance obtained directly from the regression lemma

$$\beta_{t|t,\beta_{t+1}} = \beta_{t|t} - K_{t|t+1}\varsigma_{t+1|t} \tag{5.76}$$

$$P_{t|t,\beta_{t+1}} = P_{t|t} - P_{t|t}F'(FP_{t|t}F' + Q)^{-1}FP_{t|t} \tag{5.77}$$

CK equations recursively evaluate these quantities *backwards* beginning from s_T, P_T obtained from the Kalman filter.

Generate appropriate conditional samples using

$$\beta_{t|t,\beta_{t+1}} \sim N\left(\beta_{t|t,\beta_{t+1}}, P_{t|t,\beta_{t+1}}\right) \tag{5.78}$$

to give $B_T|\psi_T$. This is a conditional sample that depends on a given parameter vector to use in a Gibbs sampling scheme that draws those parameters in turn from distributions conditioned on the states.

5.12 State-space Gibbs sampling in practice

Above approach cannot be used explicitly if $\Sigma_{\varsigma\varsigma}$ is singular, for example if we have more states than shocks (which is not uncommon). A simple modification given in KN can deal with this; we treat only those states that are shocked as observed.

In general we need conditional distributions for all the other parameters to be estimated. Need to store the complete sequence of states and covariances to implement the Gibbs sampler. We will investigate the exact implementation of Gibbs sampling for state-space models in the exercises.

5.13 Comparing the filters and smoothers

Filter	Innovation	Gain and state covariance
KF	$\eta_t = y_t - H_t\beta_{t\|t-1}$	$K_{t\|t} = -P_{t\|t-1}H_t'(H_tP_{t\|t-1}H_t' + R)^{-1}$
		$P_{t\|t} = P_{t\|t-1} - P_{t\|t-1}H_t'(H_tP_{t\|t-1}H_t' + R)^{-1}H_tP_{t\|t-1}$
KS	$\varsigma_t = \beta_{t+1\|T} - F\beta_{t\|t} - \mu$	$K_{t\|T} = -P_{t\|t}F'P_{t+1\|t}^{-1}$
		$P_{t\|T} = P_{t\|t} + K_{t\|T}(P_{t+1\|T} - P_{t+1\|t})K_{t\|T}'$
CK	$\varsigma_t = \beta_{t+1} - F\beta_{t\|t} - \mu$	$K_{t\|t,\beta_{t+1}} = -P_{t\|t}F'P_{t+1\|t}^{-1}$
		$P_{t\|t,\beta_{t+1}} = P_{t\|t} - P_{t\|t}F'(FP_{t\|t}F' + Q)^{-1}FP_{t\|t}$

Part III

Networks

Chapter 6

Causal Inference

Outline how to solve the Pearl, Glymour, and Jewell (2016) exercises in R.

6.1 Study question 1.3.2

Data:

```
library(tidyverse)
ed <- tibble(Gender = c("M","M","M","M","F","F","F","F"),
             eLevel = c("U","H","C","G","U","H","C","G"),
             num    = c(112,231,595,242,136,189,763,172)) %>%
  mutate(total = sum(num))
```

which we tabulate as

```
ed %>%
  kable()
```

Gender	eLevel	num	total
M	U	112	2440
M	H	231	2440
M	C	595	2440
M	G	242	2440
F	U	136	2440
F	H	189	2440
F	C	763	2440
F	G	172	2440

6.2 Exercises and answers

6.2.1 Find $P(eLevel = H)$

```
ed %>%
  filter(eLevel == "H") %>%
  mutate(p_H = sum(num)/total) %>%
  kable()
```

Gender	eLevel	num	total	p_H
M	H	231	2440	0.1721311
F	H	189	2440	0.1721311

6.2.2 Find $P(eLevel = H \ \vee \ Gender = F)$

```
ed %>%
  filter(Gender == "F" | eLevel == "H") %>%
  mutate(p_HorF = sum(num)/total) %>%
  kable()
```

Gender	eLevel	num	total	p_HorF
M	H	231	2440	0.6110656
F	U	136	2440	0.6110656
F	H	189	2440	0.6110656
F	C	763	2440	0.6110656
F	G	172	2440	0.6110656

6.2.3 Find $P(eLevel = H \mid Gender = F)$

```
ed %>%
  filter(Gender == "F") %>%
  mutate(tcond = sum(num)) %>%
  filter(eLevel == "H") %>%
  mutate(p_HgivenF = sum(num)/tcond) %>%
  kable()
```

Gender	eLevel	num	total	tcond	p_HgivenF
F	H	189	2440	1260	0.15

6.2.4 Find $P(Gender = F \mid eLevel = H)$

```
ed %>%
  filter(eLevel == "H") %>%
  mutate(tcond = sum(num)) %>%
  filter(Gender == "F") %>%
  mutate(p_FgivenH = sum(num)/tcond) %>%
  kable()
```

Gender	eLevel	num	total	tcond	p_FgivenH
F	H	189	2440	420	0.45

Chapter 7

Mapping regional house price inflation

7.1 How heterogenous is UK house price inflation?

A simple enough question, and one that Bahaj, Foulis, and Pinter (2020) thought was best answered with a map – actually a referee asked for one. As I know how to draw a map in R they asked me if I could do it. Well yes, but there are some particular difficulties.

- The UK (actually Great Britain) is an awkward (but not too awkward) shape.
- Population in the UK is heavily concentrated in a small number of centres, such as London or Manchester.
- There are three different periods to compare.
- It has to be in grayscale.

Before all of this we need some data, with boundaries that correspond to areas that we have data for. The regional inflation data is available at the level of the Land Registry, which almost by local authority but amalgamates a number of the areas. So a map at Local Authority level would be fine as long as we can amalgamate some of the regions.

The map data used here is available from the UK's ONS geoportal, with a lot of administrative data available including local authority boundaries. The Local Authority data is specifically available from here, where I use the clipped full extent version. There are a number of possibilities, but in general high water mark, and enough but not too much detail is needed.

```
library(tidyverse)
library(readxl)
library(sf)
```

The information in the map file is comprehensive, and by Local Authority as of December 2015.

```
fle <- "LAD_Dec_2015_GCB_GB"
shape <- read_sf(dsn=".", layer=fle)
```

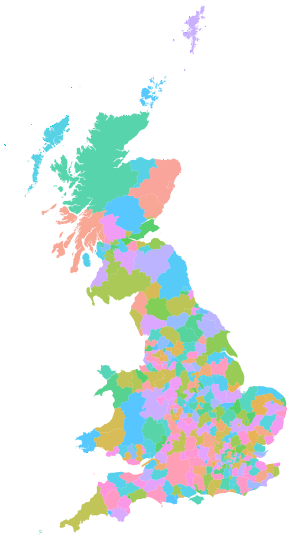
We can look at the attributes using `summary`.

```
summary(shape)
```

```
  lad15cd            lad15nm            lad15nmw           GlobalID
Length:380         Length:380         Length:380         Length:380
Class :character   Class :character   Class :character   Class :character
Mode  :character   Mode  :character   Mode  :character   Mode  :character
         geometry
MULTIPOLYGON :380
epsg:27700   :  0
+proj=tmer...:  0
```

This can be plotted straightforwardly using `ggplot`.

```
shape %>%
  ggplot() +
  geom_sf(aes(geometry=geometry, fill=lad15nm),
          color=NA, alpha=.66, show.legend=FALSE) +
  theme_void()
```



Looking at the read-out above, each of the 380 regions have some metadata associated, which are contained in each of the listed attributes. It should be obvious that `objectid` is just a sequence from 1 to 380. `lad15nm` turns out to be a list of names of the regions – I suspect `lad` for Local Authority District, `15` for 2015 and `nm` for name – and it is easy to specify this as the name to use for the region when using `tidy`.

Now this can be plotted using `ggplot`, using `geometry` for the x and y coordinates. The choice of fill colour is determined by `fill` and we can set the colour of the lines by `colour` (or `color`). The two extra arguments are for a suitable blank style and to impose an appropriate ratio of height to width.

Immediately, the awkward shape of the British Isles is apparent. (Note this is a plot of Great Britain, and there is no Northern Ireland.) The islands to the far north are somewhat unnecessary, although quite rightly the inhabitants get a bit tired of being left off maps! Nonetheless I'll do exactly the same by filtering out the polygons associated with `Orkney Islands` and `Shetland Islands`.

Fewer Scottish Islands makes the graphs a lot clearer with little loss of information, paticularly given the tiny number of transactions in the Orkneys and the Shetlands, very far to the north.

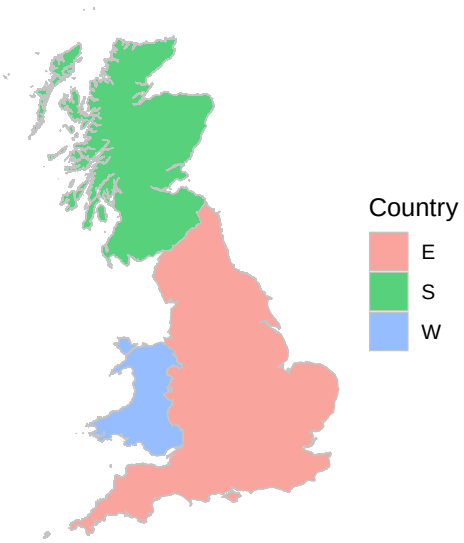
In what follows we filter out the islands using

```
shape <- read_sf(dsn=".", layer=fle) %>%
  filter(!lad15nm %in% c("Shetland Islands","Orkney Islands")) %>%
  mutate(Country=str_sub(lad15cd, 1, 1), .after=1)
```

where we also create an indicator of country using the first letter of the code string.

So the final country map is

```
shape %>%
  group_by(Country) %>%
  summarise() %>%
  ggplot() +
  geom_sf(aes(fill=Country), color="grey77", linewidth=.25, alpha=.66) +
  theme_void()
```



group and **summarise** can join geographical areas

Note the really nice feature – if we group by something, in this case country, we can summarise to amalgamate the geometries!

You may have noticed, one thing that that's missing on the LA graphs is the boundaries. They aren't, they're just invisible. That's because I set `colour = NA`, so I can fix that by choosing a colour and making the lines very thin so they don't swamp the map, as in the country one.

One further amendment, the `fill` is moved inside the `aes()` specification and made conditional. `R` now chooses unique colours for each of the regions.

Two things now need to be done to get the map colours right to illustrate re-

gional inflation rates. First we need to amalgamate some of the Local Authority boundaries to the Land Registry definitions, and second we need to assign the inflation rate to each area.

7.2 Inflation data and regions

We have a map, and we have that data in a form that is easy to understand. If we can suitably attach an inflation rate to each area then we can fill the individual areas with a colour unique to each individual inflation rates.

Recall that the Land Registry areas aren't quite what we have, and will need amalgamating. Bahaj, Foulis, and Pinter (2020) supplied me the areas that needed amalgamating (and the inflation rates) using the ONS codes. This is contained in the metadata `lad15cd` above.

The data is structured in 'wide' format with one row for each Land Registry region. The details aren't very important for us now, but what it means is I can manipulate it to get

```
# Price data by Land Registry region, converted to long format
hp_data <- read_excel("house_price_data_figure_1.xls")  %>%
  select("land_reg_region", starts_with("e_"), starts_with("av_")) %>%
  pivot_longer(names_to  = "name",
               values_to = "lad15cd",
               cols      = c(-land_reg_region, -starts_with("av_"))) %>%
  drop_na() %>%
  select(land_reg_region, lad15cd, starts_with("av_"))

codes <- hp_data %>%
  select(lad15cd, land_reg_region)
```

The important thing that the `pivot_longer` achieves is that for every `land_reg_region` I get a list of all the ONS codes that makes up the Local Authority level. So if I look at `buckinghamshire` as an example there are four ONS codes now associated with it.

```
filter(codes, land_reg_region == "buckinghamshire")
```

```
# A tibble: 4 x 2
  lad15cd   land_reg_region
  <chr>     <chr>
1 E07000004 buckinghamshire
2 E07000005 buckinghamshire
3 E07000006 buckinghamshire
```

```
4 E07000007 buckinghamshire
```

Join these together

```
# Join polygons defined by Land Registry regions
gg <- shape %>%
  select(starts_with(c("lad","C"))) %>%
  left_join(codes, by="lad15cd") %>%
  group_by(land_reg_region) %>%
  summarise()
```

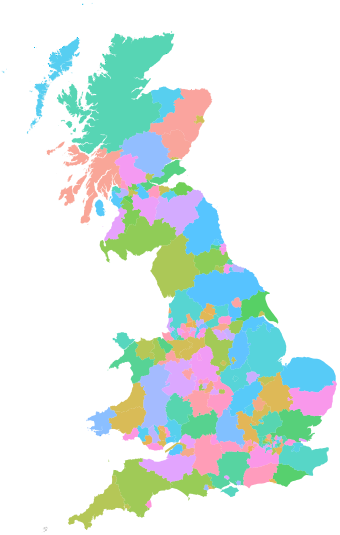
which produces a match between the Land Registry and the Local Authority areas, plus the inflation rates.

7.2.1 Inflation in grayscale

All the information required to plot the Land Registry-based regional inflation rates is now available. As you can see from the `buckinghamshire` data above, there are three average rates in three different periods, so I'll focus on one, 2002-2007 to begin with.

First, augment the geographic data with the inflation data, and call them something better.

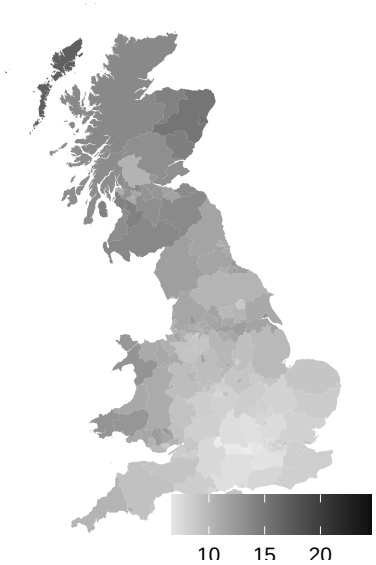
```
gg %>%
  ggplot() +
  geom_sf(aes(geometry=geometry, fill=land_reg_region),
          color=NA, alpha=.66, show.legend = FALSE) +
  theme_void()
```



Then specify gray and put the legend at the bottom.

```
nms <- gsub("av_hp_growth", "HPI", colnames(hp_data))

hp_data %>%
  rename_all( ~ nms) %>%
  select(land_reg_region, starts_with("HPI")) %>%
  distinct() %>%
  left_join(gg) %>%
  ggplot() +
  geom_sf(aes(geometry=geometry, fill=HPI_02_07),
          color=NA, alpha=.66, show.legend = TRUE) +
  theme_void() +
  scale_fill_gradient(low=grey(0.9), high=grey(0.05)) +
  theme(legend.direction = "horizontal",
        legend.position  = c(0.75,0.05),
        legend.title     = element_blank())
```

```
Joining with `by = join_by(land_reg_region)`
```


10
15
20

Part IV

Time

Chapter 8

Linear rational expectations models

8.1 Introduction

How do we solve rational expectations models? What does that even mean? Here I show how to implement versions of the Blanchard and Kahn (1980) and Klein (2000) solutions to linear rational expectations models in R. The implementation is fairly general, and copes with singular models. It is a very transparent implementation, with all the necessary code, and also shows how to calculate and plot impulse responses.

8.2 Model

We take a simple New Keynesian model

$$y_t = y^e_{t+1} - \frac{1}{\sigma}(i_t - \pi^e_{t+1}) + e^1_t \tag{8.1}$$

$$\pi_t = \beta\pi^e_{t+1} + \kappa y_t + e^2_t \tag{8.2}$$

$$i_t = \gamma i_{t-1} + (1-\gamma)\delta\pi_t + \varepsilon^3_t \tag{8.3}$$

$$e^1_t = \rho_1 e^1_{t-1} + \varepsilon^1_t \tag{8.4}$$

$$e^2_t = \rho_2 e^2_{t-1} + \varepsilon^2_t \tag{8.5}$$

The model comprises a dynamic IS curve, a Phillips Curve and a policy rule with smoothing. There are three shocks, two of which are persistent. This we need to write in the general algebraic linear state-space form:

$$E\begin{bmatrix} z_t \\ x^e_{t+1} \end{bmatrix} = A\begin{bmatrix} z_{t-1} \\ x_t \end{bmatrix} + B\varepsilon_t$$

We map our variables to their algebraic equivalent as $(z_t,\, x_t) = ((e_t^1,\, e_t^2,\, i_t),\, (y_t,\, \pi_t))$. Then the model in state-space form but including the matrix E is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -\frac{1}{\sigma} & 1 & \frac{1}{\sigma} \\ 0 & 1 & 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} e_t^1 \\ e_t^2 \\ i_t \\ y_{t+1}^e \\ \pi_{t+1}^e \end{bmatrix} = \begin{bmatrix} \rho_1 & 0 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & (1-\gamma)\delta \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\kappa & 1 \end{bmatrix} \begin{bmatrix} e_{t-1}^1 \\ e_{t-1}^2 \\ i_{t-1} \\ y_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_t^1 \\ \varepsilon_t^2 \\ \varepsilon_t^3 \end{bmatrix}$$

Anyone wanting to code up solutions should familiarize themselves with this before continuing.

8.2.1 Coding the model

Before we begin coding this in R, load the `tidyverse` libraries so we can do impulse responses with our usual tool kit and then we can forget about it.

```
library(tidyverse)
```

Set the model parameters

```
nf    <- 2
ns    <- 5
ne    <- 3
np    <- ns-nf

beta  <- 0.99   # Discount factor
sigma <- 2.0    # Elas. substitution
kappa <- 0.075  # Slope PC
delta <- 1.5    # Inflation feedback
gamma <- 0.75   # Smoothing
rho_1 <- 0.9    # AR1
rho_2 <- 0.8    # AR1
Omega <- diag(c(0.33,0.33,0.33)) # SE of 3 shocks
```

Now define the model matrices 'long hand' and some variable names, which we put in `labels`.

```
labels <- c("e^1","e^2","i","y","pi")

E <- matrix(0,ns,ns)
A <- matrix(0,ns,ns)
B <- diag(1,ns,ne)

# Now put the equations in matrix form
```

```
diag(E[1:2,1:2]) <- 1
diag(A[1:2,1:2]) <- c(rho_1, rho_2)

E[3,3]             <- 1
E[4,c(1, 3, 4, 5)] <- c(1, -1/sigma, 1, 1/sigma)
E[5,c(2, 5)]       <- c(1, beta)

A[3,c(3, 5)]       <- c(gamma, (1-gamma)*delta)
A[4,4]             <- 1
A[5,c(4,5)]        <- c(-kappa, 1)
```

where for example, E and A are

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -0.5 & 1 & 0.5 \\ 0 & 1 & 0 & 0 & 0.99 \end{bmatrix} \tag{8.6}$$

$$A = \begin{bmatrix} 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.75 & 0 & 0.375 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -0.075 & 1 \end{bmatrix} \tag{8.7}$$

Calculate the reduced form state-space model

$$\begin{bmatrix} z_t \\ x^e_{t+1} \end{bmatrix} = C \begin{bmatrix} z_{t-1} \\ x_t \end{bmatrix} + D\varepsilon_t \tag{8.8}$$

which is done in R very simply as

```
C <- solve(E,A)
D <- solve(E,B)
```

Why can't we solve this for impulse responses?

The following function simulates the impulse responses of a model in a loop within a loop[1] and returns the time series in a suitably organised data frame.

```
impulse_responses <- function(P, Q, Omega, labels, T) {
  s   <- matrix(0, ncol(Q), 1)
  z   <- matrix(0, nrow(Q), T)
  rownames(z) <- labels
  dza <- NULL
```

[1] Sometimes a loop is the right way to do something.

```
  for (j in 1:ncol(Q)) {
    s[j]  <- Omega[j,j]
    z[,1] <- Q %*% s
    for (i in 1:(T-1)) {
      z[,i+1] <- P %*% z[,i]
    }
    s[j] <- 0
    dz <- as_tibble(t(z)) %>%
      mutate(Period = 1:T, Shock = paste0("epsilon^",j))
    dza <- bind_rows(dza,dz)
  }
  return(dza)
}
```

A function to plot the impulses will be useful, so we create one.

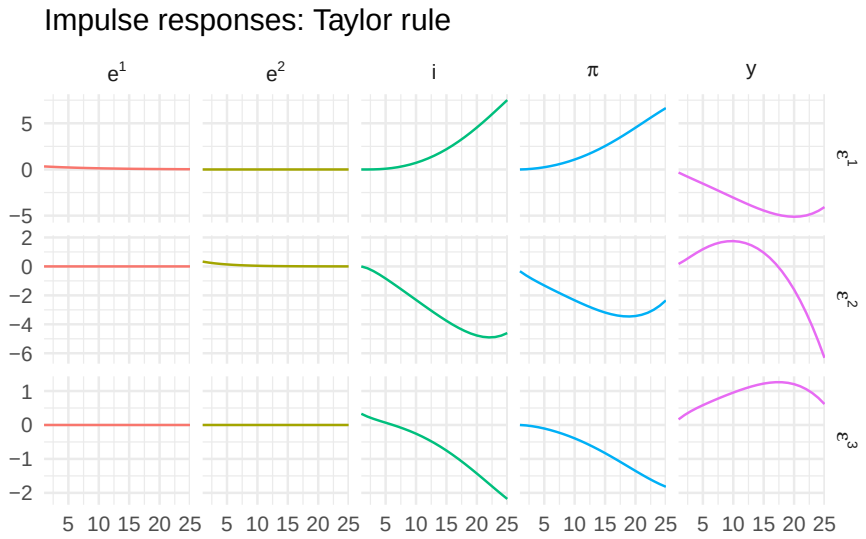
```
response_plot <- function(series, title) {
  return(pivot_longer(series, cols = -c(Period,Shock), names_to="Var", values_to = '
           ggplot() +
           geom_line(aes(x=Period, y=Val, group=Shock, colour=Var), show.legend=FALS
           facet_grid(Shock~Var, scales="free", labeller=label_parsed) +
           scale_x_continuous(expand=c(0,0)) +
           theme_minimal() +
           labs(title=title, x="",y=""))
}
```

Call the impulse response function using the model C and D.

```
T <- 25
z <- impulse_responses(C, D, Omega, labels, T)
```

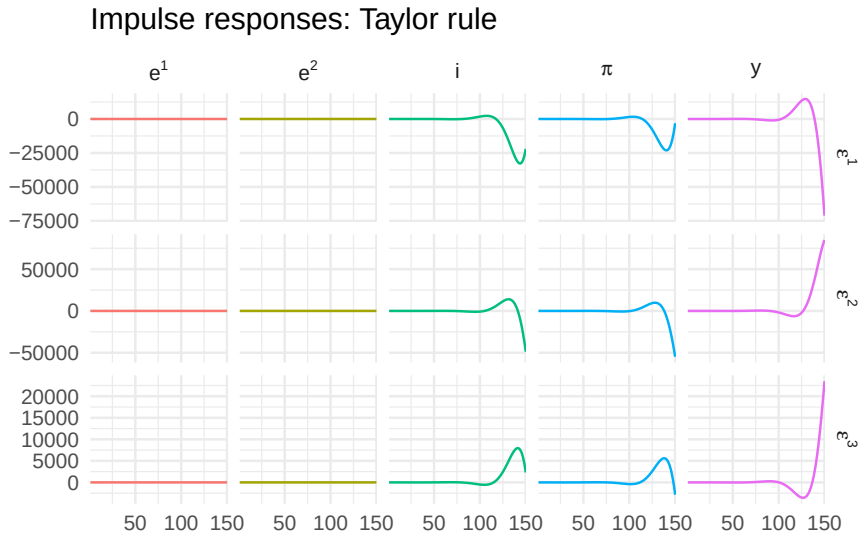
and plot

```
response_plot(z, "Impulse responses: Taylor rule")
```



Oh! That's not looking good. Let's try a few more periods.

```
T <- 150
z <- impulse_responses(C, D, Omega, labels, T)
response_plot(z, "Impulse responses: Taylor rule")
```



This is clearly exploding. But it's rational – we're solving forward so expectations are always fulfilled. This is a key the insight of the early rational expectations modellers – rational isn't enough, non-explosive is necessary too. Fortunately we know how to find this.

8.3 Blanchard and Kahn (1980)

To solve this model to give a unique *stable* rational expectations equilibrium, we appeal to the following. Consider the eigenvalue decomposition

$$MC = \Lambda M$$

where Λ is a diagonal matrix of *eigenvalues* in increasing absolute value and M is a non-singular matrix of *left eigenvectors*. Note that computer routines (including the one in R) usually calculate *right eigenvectors* such that $CV = V\Lambda$ and that $M = V^{-1}$, so be aware of this in what follows.

We can *diagonalise* C and write it as $C = M^{-1}\Lambda M$. So pre-multiplying the reduced form model by M gives

$$M \begin{bmatrix} z_t \\ x^e_{t+1} \end{bmatrix} = \Lambda M \begin{bmatrix} z_{t-1} \\ x_t \end{bmatrix} + MD\varepsilon_t$$

Blanchard and Kahn (1980) (following Vaughan (1970)) show uniqueness requires as many unstable eigenvalues as jump variables. To see this, define

$$\begin{bmatrix} \xi^s_{t-1} \\ \xi^u_t \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} z_{t-1} \\ x_t \end{bmatrix}$$

Write the normalized model as

$$\begin{bmatrix} \xi^s_t \\ \xi^u_{t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_s & 0 \\ 0 & \Lambda_u \end{bmatrix} \begin{bmatrix} \xi^s_{t-1} \\ \xi^u_t \end{bmatrix} + \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} D\varepsilon_t$$

where the eigenvalues are split into stable (Λ_s) and unstable (Λ_u). If we ignore the stochastic bit for a moment

$$\begin{bmatrix} \xi^s_t \\ \xi^u_{t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_s & 0 \\ 0 & \Lambda_u \end{bmatrix} \begin{bmatrix} \xi^s_{t-1} \\ \xi^u_t \end{bmatrix}$$

We seek a non-explosive solution, and this turns out to be easy to find using the following

- The dynamics of ξ^u_t are determined by Λ_u and nothing else;
- If they don't start at 0 they must explode;
- This implies they must start at 0 and are always 0.

Thus the definition of the canonical variables necessarily implies

$$\begin{bmatrix} \xi^s_{t-1} \\ 0 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} z_{t-1} \\ x_t \end{bmatrix}$$

From this it is clear that the jump variables themselves are only on the saddle path if

$$M_{21} z_{t-1} + M_{22} x_t = 0$$

The rational solution implies that the jump variables are linearly related to the predetermined ones through

$$x_t = -M_{22}^{-1} M_{21} z_{t-1} \tag{8.9}$$
$$= N z_{t-1} \tag{8.10}$$

We'll deal with the shocks in a moment.

How do we do this in R? First, find the eigenvalue decomposition of C using

```
m <- eigen(C, symmetric=FALSE)
```

which yields

```
eigen() decomposition
$values
[1] 1.0715518+0.092734i 1.0715518-0.092734i 0.9000000+0.000000i
[4] 0.8000000+0.000000i 0.6548762+0.000000i

$vectors
                       [,1]                  [,2]          [,3]           [,4]
[1,]  0.0000000+0.0000000i  0.0000000+0.0000000i 0.2854942+0i  0.00000000+0i
[2,]  0.0000000+0.0000000i  0.0000000+0.0000000i 0.0000000+0i  0.09783896+0i
[3,]  0.1599159-0.5089425i  0.1599159+0.5089425i 0.7830500+0i  0.49622464+0i
[4,] -0.6991064+0.0000000i -0.6991064+0.0000000i 0.4552131+0i -0.86012270+0i
[5,]  0.2629802-0.3968579i  0.2629802+0.3968579i 0.3132200+0i  0.06616328+0i
              [,5]
[1,]  0.0000000+0i
[2,]  0.0000000+0i
[3,]  0.6351203+0i
[4,] -0.7554249+0i
[5,] -0.1611069+0i
```

However this calculates *right* eigenvectors. We will need to invert it for left ones. Given the number of jump variables in the model satisfies the Blanchard-Kahn conditions of as many unstable roots (1.072+0.093i, 1.072-0.093i) as jump variables (2) we can calculate the reaction function from the eigenvectors

```
iz <- 1:np
ix <- (np+1):ns
M  <- solve(m$vectors[,ns:1])        # Invert & reverse order for increasing abs val
N  <- -Re(solve(M[ix,ix], M[ix,iz])) # Drop tiny complex bits (if any)
```

where `iz` are the indices of the first `np` variables and `ix` those of the remaining `nf` ones.

8.3.1 Stochastic part

What about the shocks? Assume the stochastic reaction function is

$$x_t = Nz_{t-1} + G\varepsilon_t$$

Following Andrew P. Blake (2004), note that $x^e_{t+1} = Nz_t$ as the expected value of $\varepsilon_{t+1} = 0$, meaning we can write

$$Nz_t = C_{21}z_{t-1} + C_{22}x_t + D_2\varepsilon_t$$

or

$$N\left(C_{11}z_{t-1} + C_{12}x_t + D_1\varepsilon_t\right) = C_{21}z_{t-1} + C_{22}x_t + D_2\varepsilon_t$$

Gathering terms we obtain

$$(C_{22} - NC_{12})x_t = (NC_{11} - C_{21})z_{t-1} + (ND_1 - D_2)\varepsilon_t$$

which implies

$$G = (C_{22} - NC_{12})^{-1}(ND_1 - D_2)$$

Notice it also implies $N = (C_{22} - NC_{12})^{-1}(NC_{11} - C_{21})$. It is this fixed point nature of the solution for N – which in turn implies the quadratic matrix equation $C_{21} = NC_{11} - C_{22}N + NC_{12}N$ – that means we need to use the Blanchard and Kahn (1980) method in the first place.

All of this means that

```
G <- solve((C[ix,ix] - N %*% C[iz,ix]), (N %*% D[iz,]- D[ix,]))
```

so for our model and parameters N and G are

$$N = \begin{bmatrix} 4.8568 & -2.7586 \\ -1.1894 & 1.7929 \\ 1.9628 & -0.2537 \end{bmatrix} \tag{8.11}$$

$$G = \begin{bmatrix} 5.3964 & -3.4483 \\ -1.5859 & 1.9921 \\ 2.4535 & -0.3382 \end{bmatrix} \tag{8.12}$$

The 'fixed point' check is that the following should be the same as N

```
solve((C[ix,ix] - N %*% C[iz,ix]), (N %*% C[iz,iz]- C[ix,iz]))
```

```
        [,1]      [,2]       [,3]
[1,] 4.85680 -2.758647 -1.1894200
[2,] 1.79286  1.962790 -0.2536635
```

which it is.

The solved model is finally

$$\begin{bmatrix} z_t \\ x_t \end{bmatrix} = \begin{bmatrix} C_{11} + C_{12}N & 0 \\ N & 0 \end{bmatrix} \begin{bmatrix} z_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} D_1 + C_{12}G \\ G \end{bmatrix} \varepsilon_t \tag{8.13}$$

$$= P \begin{bmatrix} z_{t-1} \\ x_{t-1} \end{bmatrix} + Q\varepsilon_t \tag{8.14}$$

which can be coded as

```
P  <- cbind(rbind((C[iz,iz] + C[iz,ix] %*% N), N), matrix(0,ns,nf))
Q  <- rbind(D[iz,] + C[iz,ix] %*% G, G)
```

8.3.2 Digression – right eigenvector version

It turns out that we could use the output from the standard eigenvalue/vector routine directly by exploiting the following. This time, let M be the matrix of *right eigenvectors* so

$$CM = M\Lambda \text{ or } C = M\Lambda M^{-1}$$

and

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \xi^s_{t-1} \\ \xi^u_t \end{bmatrix} = \begin{bmatrix} z_{t-1} \\ x_t \end{bmatrix}$$

Written this way around, if $\xi^u_t = 0 \ \forall \ t$ then (again ignoring stochastics)

$$M_{11}\xi^s_{t-1} = z_{t-1}, \ M_{21}\xi^s_t = x_t$$

$$\Rightarrow x_t = M_{21}M_{11}^{-1}z_t$$

so

```
M <- m$vectors[,ns:1]             # Don't invert as already right vectors, but reorder
Re(M[ix,iz] %*% solve(M[iz,iz])) # Again, drop tiny complex bits
```

```
        [,1]      [,2]       [,3]
[1,] 4.85680 -2.758647 -1.1894200
[2,] 1.79286  1.962790 -0.2536635
```

The result is identical. This method is particularly useful if there are fewer predetermined variables than jumps as the matrix we need to invert is of the same dimension as the predetermined variables this way round.

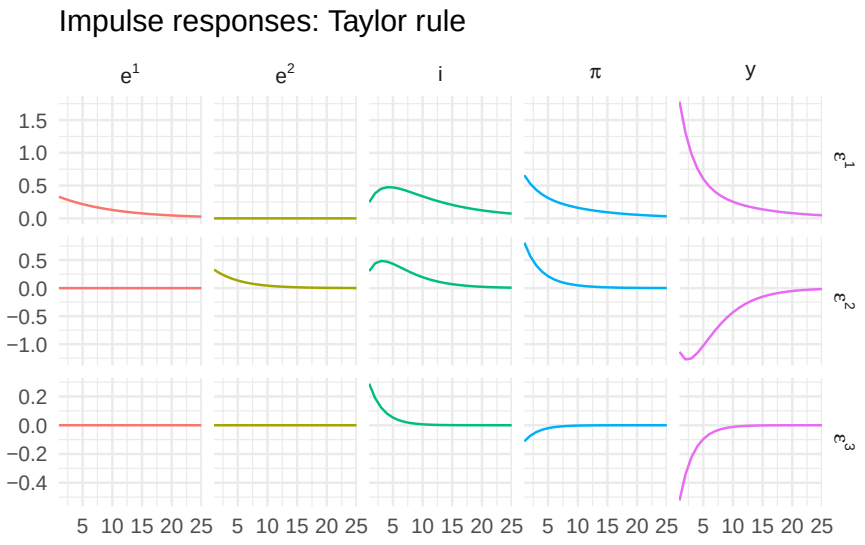
8.3.3 Impulse responses

We now call the impulse response function using the model solved for rational expectations.

```
T <- 25
z <- impulse_responses(P, Q, Omega, labels, T)
```

Now plot these responses

```
response_plot(z, "Impulse responses: Taylor rule")
```



Now, that looks better! It is no longer explosive. It also makes complete economic sense, which you can verify by going through the dynamics of the different demand, supply and monetary shocks.

8.4 Generalized solution

Sometimes for a model E is singular. A more general solution was proposed by Klein (2000), that doesn't require E to be non-singular. This uses a generalized Schur decomposition instead of an eigenvalue one and is applied to the structural model represented by the matrix pencil (A, E), and is considered much more numerically stable (see Pappas, Laub, and Sandell (1980)). The generalized Schur form of (A, E) is (QTZ', QSZ'), so we can write the model as

$$E \begin{bmatrix} z_t \\ x^e_{t+1} \end{bmatrix} \equiv QTZ' \begin{bmatrix} z_t \\ x^e_{t+1} \end{bmatrix} \equiv QT \begin{bmatrix} \xi^s_t \\ \xi^u_{t+1} \end{bmatrix}$$

and

$$A \begin{bmatrix} z_{t-1} \\ x_t \end{bmatrix} \equiv QSZ' \begin{bmatrix} z_{t-1} \\ x_t \end{bmatrix} \equiv QS \begin{bmatrix} \xi^s_{t-1} \\ \xi^u_t \end{bmatrix}$$

so the model pre-multiplied by Q' is

$$T \begin{bmatrix} \xi^s_{t+1} \\ \xi^u_{t+1} \end{bmatrix} = S \begin{bmatrix} \xi^s_t \\ \xi^u_t \end{bmatrix} + Q'B\varepsilon_t$$

We use the function `gqz` from the library `geigen` for this

```
d <- geigen::gqz(A, E, sort="S") # Option "S" puts the stable roots first
```

We can check that this is actually saddle path using `gevalues()` to get all the eigenvalues from the generalized Schur decomposition, and the unstable ones are

```
e <- geigen::gevalues(d)
e[abs(e) > 1]
```

```
[1] 1.071552+0.092734i 1.071552-0.092734i
```

The number of *stable* roots is returned in `d$sdim` which is 3.

We then modify our solution function to calculate `Ns` and `Gs` using the matrix `Z` and a generalized version of the formula for G and calculate the reduced form model `Ps` and 'Q which are

$$N_s = Z_{21}Z_{11}^{-1} \tag{8.15}$$
$$H = (E_{11} + E_{12}N_s)^{-1} \tag{8.16}$$
$$W = (E_{21} + E_{22}N_s)H \tag{8.17}$$
$$G_s = (A_{22} - WA_{12})^{-1}(WB_1 - B_2) \tag{8.18}$$
$$P_s = H(A_{11} + A_{12}N_s) \tag{8.19}$$
$$Q_s = H(B_1 + A_{12}G_s) \tag{8.20}$$

Verify this yourself with a bit of matrix algebra!

The R code for this is

```
solveGenBK <- function(E,A,B,n) {
  d  <- geigen::gqz(A, E, sort="S")
  np <- d$sdim
  ns <- nrow(E)
  print(paste("Number of unstable roots is", ns-np))
  if (n == np) {
    iz <- 1:n
    ix <- (n+1):ns
    Ns <- d$Z[ix,iz] %*% solve(d$Z[iz,iz])
    H  <- solve(E[iz,iz] + E[iz,ix] %*% Ns)
    W  <- (E[ix,iz] + E[ix,ix] %*% Ns) %*% H
    Gs <- solve((A[ix,ix] - W %*% A[iz,ix]), (W %*% B[iz,] - B[ix,]))
    As <- H %*% (A[iz,iz] + A[iz,ix] %*% Ns)
    Bs <- H %*% (B[iz,] + A[iz,ix] %*% Gs)
    return(list(P=cbind(rbind(As,Ns),matrix(0,ns,ns-n)), Q=rbind(Bs, Gs)))
    }
  else {
    return(-1)
    }
}
```

Using this on our original model gives

```
S  <- solveGenBK(E,A,B,np)
```

```
[1] "Number of unstable roots is 2"
```

```
Ps <- S$P
Qs <- S$Q
```

and comparing `Ps` and `Qs` with `P` and `Q` obtained using Blanchard-Kahn we find

```
round(max(abs(P-Ps), abs(Q-Qs)), 12)
```

```
[1] 0
```

They are, as expected, the same – at least up to 12 decimal places, which should be enough.

8.5 Singular models: optimal policy

However, this is an easy test. What we need is to use a model that can't be solved using the BK method. Under optimal policy, the interest rate instrument rule is replaced with a targeting rule, so that

$$\pi_t = -\mu \Delta y_t - \varepsilon_t^3$$

for some value of μ that reflects the optimal trade-off between output (gap) growth and inflation, and we've included a disturbance which we can loosely describe as a monetary policy shock. We modify the model above by dropping the Taylor rule in favour of the targeting rule. This requires a lagged value of y to be created. The following does the trick

```
nf <- 2
ne <- 3
ns <- 6      # One extra state
np <- ns-nf
mu <- 0.75   # Representative trade-off

labels <- c("e^1","e^2","ylag","i","y","pi") # New variable order

E <- matrix(0,ns,ns)
A <- E
B <- matrix(0,ns,ne)
B[1,1] <- 1
B[2,2] <- 1
B[4,3] <- -1

diag(E[1:3,1:3]) <- 1
diag(A[1:2,1:2]) <- c(rho_1, rho_2)
A[3,5]           <- 1

E[4,3]           <- 1
A[4,c(3, 6)]     <- c(1, -1/mu)

E[5,c(1, 4, 5, 6)] <- c(1, -1/sigma, 1, 1/sigma)
A[5,5]           <- 1

E[6,c(2, 6)]     <- c(1, beta)
A[6,c(5, 6)]     <- c(-kappa, 1)
```

The new E and A system matrices are then

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -0.5 & 1 & 0.5 \\ 0 & 1 & 0 & 0 & 0 & 0.99 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1.333 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -0.075 & 1 \end{bmatrix}$$

Now we have a singular model. The matrix E is clearly singular as rows 3 and 4 are identical. But we have a problem using the code above. To use it we need the matrices H and $(A_{22} - WA_{21})$ to be non-singular. What to do?

There are two ways out. Klein (2000) gives a solution that depends on the decomposed matrix pencil, which is what is typically implemented, but you don't actually need it although it is easiest. Instead, all you need to do is reorder the equations.

The real problem is that with a targeting rule that doesn't include the interest rate, and the interest rate is now only determined by the IS curve. But we can swap the location of any two rows of the model arbitrarily. If we swap the positions of the equations for the IS curve and the targeting rule (rows 4 and 5) using the following

```
E[4:5,] <- E[5:4,]
A[4:5,] <- A[5:4,]
B[4:5,] <- B[5:4,]
```

then the model is unchanged but now we have

$$E_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -0.5 \end{bmatrix}$$

so $E_{11} + E_{12}N$ is likely non-singular (it is). Also, note after the re-ordering A_{22} is

$$A_{22} = \begin{bmatrix} 0 & -1.33 \\ -0.07 & 1 \end{bmatrix}$$

which is guaranteed non-singular for zero W. We can now proceed as before. First, check for saddle path stability

```
e <- geigen::gevalues(geigen::gqz(A, E, sort="S"))
e[abs(e) > 1]
```

```
[1] 1.378195      Inf
```

which confirms that it has a unique saddle path stable solution. This is

```
So <- solveGenBK(E,A,B,np)
```

```
[1] "Number of unstable roots is 2"
```

```
Po <- So$P
Qo <- So$Q
```

The solved model is then

```
Po
```

```
     [,1]      [,2]          [,3] [,4] [,5] [,6]
[1,]  0.9  0.000000  0.000000e+00    0    0    0
[2,]  0.0  0.800000  6.986592e-17    0    0    0
[3,]  0.0 -1.863455  7.329156e-01    0    0    0
[4,]  1.8 -1.241330 -2.446879e-01    0    0    0
[5,]  0.0 -1.863455  7.329156e-01    0    0    0
[6,]  0.0  1.397591  2.003133e-01    0    0    0
```

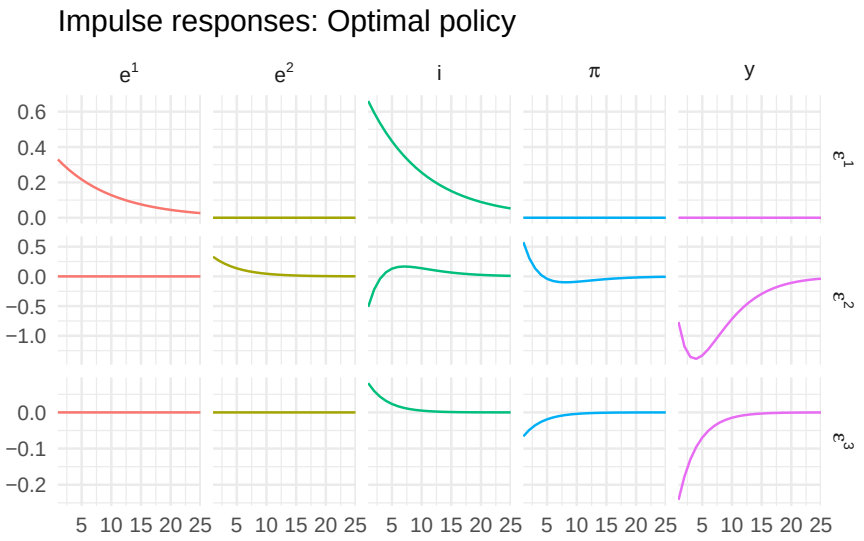
```
Qo
```

```
     [,1]      [,2]          [,3]
[1,]    1  0.000000  0.000000e+00
[2,]    0  1.000000 -6.986592e-17
[3,]    0 -2.329318 -7.329156e-01
[4,]    2 -1.551663  2.446879e-01
[5,]    0 -2.329318 -7.329156e-01
[6,]    0  1.746989 -2.003133e-01
```

8.5.1 Optimal impulse responses

We can now simulate the model under optimal policy and plot using

```
zo <- impulse_responses(Po, Qo, Omega, labels, T) %>%
  select(-ylag) # Drop duplicate series
response_plot(zo, "Impulse responses: Optimal policy")
```



8.6 Dummy jumps

But this isn't the only way to get this to work. Effectively what we just did was create an extra predetermined variable and reorder the system to give us non-singularity. What if instead of including an unused i_{t-1} on the right hand side, we instead include an unused i^e_{t+1} on the left hand side? So we swap to having one more jump variable, one less predetermined one?

Compare the following to the previous model. When we pick out the interest rate we do so on the right hand side of the matrix equation, not the left as before.

```
ns <- 6       # One extra state
nf <- 3       # And one extra jump
np <- ns-nf
labels <- c("e^1","e^2","ylag","i","y","pi") # New variable order

E <- matrix(0,ns,ns)
A <- E
```

```
B <- matrix(0,ns,ne)
B[1,1] <- 1
B[2,2] <- 1
B[4,3] <- -1

diag(E[1:3,1:3]) <- 1
diag(A[1:2,1:2]) <- c(rho_1, rho_2)
A[3,5]           <- 1

E[4,3]           <- 1
A[4,c(3, 6)]     <- c(1, -1/mu)

E[5,c(1, 5, 6)]  <- c(1, 1, 1/sigma) # One less coefficient
A[5,c(4, 5)]     <- c(1/sigma, 1)    # One more - nothing else changes

E[6,c(2, 6)]     <- c(1, beta)
A[6,c(5, 6)]     <- c(-kappa, 1)
```

This is still a singular model, as we can see from

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 1 & 0 & 0 & 0 & 0.99 \end{bmatrix}$$

with column 4 all zeros. Is *this* model saddle path stable?

```
e <- geigen::gevalues(geigen::gqz(A, E, sort="S") )
e[abs(e) > 1]
```

```
[1]      -Inf 1.378195       Inf
```

Again, it is with an extra unstable root for the extra jump variable. We could simplify the solution. As that top left 3 by 3 block, E_{11}, is the identity matrix and E_{12} is all zeros this `Ei` is always an identity matrix. However, here we simply re-use `solveGenBG`

```
So2 <- solveGenBK(E,A,B,np)
```

```
[1] "Number of unstable roots is 3"
```

```
Po2 <- So2$P
Qo2 <- So2$Q
```

Now the solved model is

```
Po2
```

```
     [,1]      [,2]       [,3] [,4] [,5] [,6]
[1,]  0.9  0.000000  0.0000000    0    0    0
[2,]  0.0  0.800000  0.0000000    0    0    0
[3,]  0.0 -1.863455  0.7329156    0    0    0
[4,]  1.8 -1.241330 -0.2446879    0    0    0
[5,]  0.0 -1.863455  0.7329156    0    0    0
[6,]  0.0  1.397591  0.2003133    0    0    0
```

```
Qo2
```

```
     [,1]      [,2]       [,3]
[1,]    1  0.000000  0.0000000
[2,]    0  1.000000  0.0000000
[3,]    0 -2.329318 -0.7329156
[4,]    2 -1.551663  0.2446879
[5,]    0 -2.329318 -0.7329156
[6,]    0  1.746989 -0.2003133
```

which is actually identical to our previous solution. This is because I have preserved the order of the solved-out variables, and shows that the swap from a predetermined to a jump variable is completely arbitrary.

8.7 Substituting out

But even this doesn't exhaust the possible re-parametrisations of the model. We can reduce the number of jump variables to 1 and find the same solution. There exist formal methods for reducing models (see King and Watson (2002)) but there is an obvious way to proceed here. From the targeting rule, it must be that

$$y^e_{t+1} = y_t - \frac{1}{\mu}\pi^e_{t+1}$$

as the expected shock is zero. This means the IS curve can be rewritten

$$y_t = y_t - \frac{1}{\mu}\pi^e_{t+1} - \frac{1}{\sigma}\left(i_t - \pi^e_{t+1}\right) + e^1_t$$

implying

$$i_t = \left(1 - \frac{\sigma}{\mu}\right)\pi^e_{t+1} + \sigma e^1_t$$

This is the required interest rate consistent with the targeting rule holding. Now the only jump variable is the inflation rate as we have eliminated the expected output gap.

$$y_t = y_{t-1} - \frac{1}{\mu}\pi_t + \frac{1}{\mu}\varepsilon^3_t \tag{8.21}$$

$$\pi_t = \beta\pi^e_{t+1} + \kappa y_t + e^2_t \tag{8.22}$$

$$i_t = \left(1 - \frac{\sigma}{\mu}\right)\pi^e_{t+1} + \sigma e^1_t \tag{8.23}$$

$$e^1_t = \rho_1 e^1_{t-1} + \varepsilon^1_t \tag{8.24}$$

$$e^2_t = \rho_2 e^2_{t-1} + \varepsilon^2_t \tag{8.25}$$

We can code this

```
ns <- 5       # Back to 5 states
nf <- 1       # Now only one jump
np <- ns-nf

labels <- c("e^1","e^2","i","y","pi") # Lose a y

E <- matrix(0,ns,ns)
A <- E
B <- matrix(0,ns,ne)
B[1,1] <- 1
B[2,2] <- 1
B[4,3] <- -1

diag(E[1:4,1:4]) <- 1
diag(A[1:2,1:2]) <- c(rho_1, rho_2)

E[3,c(1, 3, 5)]  <- c(-sigma, 1, sigma/mu-1)

A[4,c(4,5)]      <- c(1, -1/mu)

E[5,c(2, 4, 5)]  <- c(1, kappa, beta)
A[5,5]           <- 1
```

and solve it using

```
Ss <- solveGenBK(E,A,B,np)
```

```
[1] "Number of unstable roots is 1"
```

```
Ps <- Ss$P
Qs <- Ss$Q
```

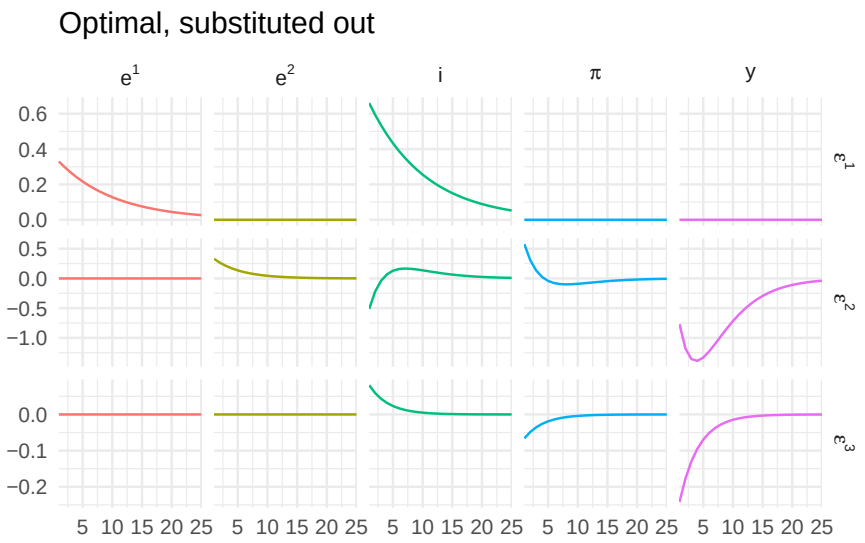
Compare the realized of `Ps`

```
Ps
```

```
     [,1]      [,2] [,3]       [,4] [,5]
[1,]  0.9  0.000000    0  0.0000000    0
[2,]  0.0  0.800000    0  0.0000000    0
[3,]  1.8 -1.241330    0 -0.2446879    0
[4,]  0.0 -1.863455    0  0.7329156    0
[5,]  0.0  1.397591    0  0.2003133    0
```

with `Po` above, say. This is the most 'efficient' way of programming the model, in that we have only five states, and indeed the repeated behavioural equations we had before have disappeared in the reduced form solution. Just to confirm this, simulating and plotting this version gives

```
response_plot(impulse_responses(Ps,Qs,Omega,labels,T), "Optimal, substituted out")
```



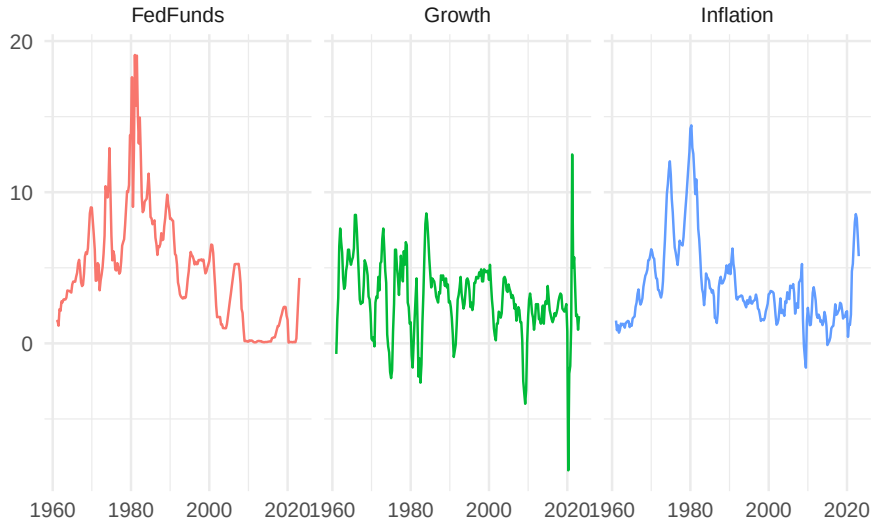
which are identical results to those above. But of course E is now invertible so we could solve this using the simplest Blanchard-Kahn variant. Try it!

Chapter 9

BVAR with dummies

9.1 Estimating BVARs using US data

We will the Fed Funds rate, annual GDP growth and annual CPI inflation data from FRED, retrieved 2023-07-01. These are:



We will build a variety and two and three variable BVARs. More details on the data are given below.

9.2 BVARs with dummy variable priors

Rather than combine a prior distribution with a likelihood and draw from the resulting joint posterior distribution there is another convenient way of parameterizing the problem. We can instead add some 'dummy variables' that have the same properties of the prior so we have a single modified likelihood that incorporates the prior information. This approach was most obviously adopted by Banbura, Giannone, and Reichlin (2010). Further discussion of this can be found in Giannone, Lenza, and Primiceri (2015). In general this is a version of the Theil and Goldberger (1961) ***mixed estimator*** given a Bayesian interpretation.

9.2.1 VAR model

Simple bi-variate two-lag VAR model:

$$\begin{bmatrix} g_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} g_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} g_{t-2} \\ \pi_{t-2} \end{bmatrix} + \begin{bmatrix} \nu_{g,t} \\ \nu_{\pi,t} \end{bmatrix} \tag{9.1}$$

$$\begin{bmatrix} \nu_{g,t} \\ \nu_{\pi,t} \end{bmatrix} \sim N(0, \Sigma) \tag{9.2}$$

9.3 BVAR hyperparameters

We will (similarly to the straightforward Minnesota prior) need some control parameters:

- τ controls the overall tightness of the prior for the AR coefficients
- d controls the prior on higher lags;
- λ controls the prior on constants;
- γ controls the prior on the sum of coefficients;
- δ controls the cointegration prior;

where

- σ_i standard deviation of error terms from individual OLS regressions;
- μ_i sample means of the data.

9.3.0.1 First lag

Now consider the following artificial data for the **first lag**. We construct some dummy observations of the dependent and explanatory variables that look like:

$$Y_{D,1} = \begin{bmatrix} \frac{1}{\tau}\sigma_1 & 0 \\ 0 & \frac{1}{\tau}\sigma_2 \end{bmatrix} \tag{9.3}$$

and

$$X_{D,1} = \begin{bmatrix} 0 & \frac{1}{\tau}\sigma_1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\tau}\sigma_2 & 0 & 0 \end{bmatrix} \tag{9.4}$$

Intuition:

$$\begin{bmatrix} \frac{\sigma_1}{\tau} & 0 \\ 0 & \frac{\sigma_2}{\tau} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sigma_1}{\tau} & 0 & 0 & 0 \\ 0 & 0 & \frac{\sigma_2}{\tau} & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ b_{11} & b_{21} \\ b_{12} & b_{22} \\ d_{11} & d_{21} \\ d_{12} & d_{22} \end{bmatrix} + \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} \tag{9.5}$$

Multiplying out we get:

$$\begin{bmatrix} \frac{\sigma_1}{\tau} & 0 \\ 0 & \frac{\sigma_2}{\tau} \end{bmatrix} = \begin{bmatrix} \frac{\sigma_1}{\tau}b_{11} & \frac{\sigma_1}{\tau}b_{21} \\ \frac{\sigma_2}{\tau}b_{12} & \frac{\sigma_2}{\tau}b_{22} \end{bmatrix} + \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} \tag{9.6}$$

Concentrating on the first row, notice:

$$\frac{\sigma_1}{\tau} = \frac{\sigma_1}{\tau}b_{11} + \xi_{11} \tag{9.7}$$

implying:

$$b_{11} = 1 - \frac{\tau}{\sigma_1}\xi_{11} \tag{9.8}$$

so we can write:

$$b_{11} \sim N\left(1, \frac{\tau^2 var(\xi_{11})}{\sigma_1^2}\right) \tag{9.9}$$

as $E[b_{11}] = 1 - \frac{\tau}{\sigma_1}E[\xi_{11}] = 1$ and the variance is easily derived. Similarly:

$$b_{12} = -\frac{\tau}{\sigma_1}\xi_{12} \tag{9.10}$$

which is clearly zero in expectation.

9.3.1 Further priors

9.3.1.1 Higher lags

Rather than derive the implications we state the rest of the dummy priors. Consider the following artificial data for the **second** lag:

$$Y_{D,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$X_{D,2} = \begin{bmatrix} 0 & 0 & 0 & \frac{\sigma_1 2^d}{\tau} & 0 \\ 0 & 0 & 0 & 0 & \frac{\sigma_2 2^d}{\tau} \end{bmatrix}$$

We can multiply these out and check the properties, in particular we can verify in the same way as for the first lag that:

$$b_{ji} \sim N\left(0, \frac{1}{4}\frac{\tau^2 var(\xi_{ji})}{2^d \sigma_j^2}\right) \tag{9.11}$$

for $j = 1, ...N$, $i = 1, ...l$.

9.3.1.2 Constant

Consider the following artificial data for the **constant**:

$$Y_{D,3} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$
$$X_{D,3} = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 \end{bmatrix}$$

so $\lambda c_1 = \varepsilon_1$ and $\lambda c_2 = \varepsilon_2$. As $\lambda \to \infty$ the prior is implemented more tightly.

9.3.1.3 Covariances

Dummy observations to implement the prior on the error covariance matrix are:

$$Y_{D,4} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$
$$X_{D,4} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with the magnitude of the diagonal elements of Σ controlled by the scale of the diagonal elements of $Y_{D,4}$, as larger diagonal elements implement the prior belief that the variance of ν_1 and ν_2 is larger.

Banbura, Giannone, and Reichlin (2010) stop here, but there are additional priors that could be added.

9.3.1.4 Sum of coefficients

We could add a prior that reflects the belief that the **sum of coefficients** on 'own' lags add up to 1. This is an additional 'unit root'-style prior. Consider:

$$Y_{D,5} = \begin{bmatrix} \gamma\mu_1 & 0 \\ 0 & \gamma\mu_2 \end{bmatrix}$$
$$X_{D,5} = \begin{bmatrix} 0 & \gamma\mu_1 & 0 & \gamma\mu_1 & 0 \\ 0 & 0 & \gamma\mu_2 & 0 & \gamma\mu_2 \end{bmatrix}$$

where μ_1 is the sample mean of y_t and μ_2 is the sample mean of x_t. Note that these dummy observations imply prior means of the form $b_{ii} + d_{ii} = 1$ where $i = 1, 2$ and γ controls the tightness of the prior. As $\gamma \to \infty$ the prior is implemented more tightly. Forecast growth rates eventually converge to their sample averages.

9.3.1.5 Trends

We can also specify **common stochastic trend** dummies:

$$Y_{D,6} = [\delta\mu_1 \quad \delta\mu_2]$$
$$X_{D,6} = [\delta \quad \delta\mu_1 \quad \delta\mu_2 \quad \delta\mu_1 \quad \delta\mu_2]$$

where this imposes that the coefficients are consistent with limiting the amount of drift between the predictions at their average values.

9.3.2 Implementation

The data and the artificial data are now stacked:

$$Y^* = \begin{bmatrix} g_3 & \pi_3 \\ \vdots & \vdots \\ g_T & \pi_T \\ \frac{1}{\tau}\sigma_1 & 0 \\ 0 & \frac{1}{\tau}\sigma_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sigma_1 & 0 \\ 0 & \sigma_2 \\ \gamma\mu_1 & 0 \\ 0 & \gamma\mu_2 \\ \delta\mu_1 & \delta\mu_2 \end{bmatrix}, \quad X^* = \begin{bmatrix} 1 & g_2 & \pi_2 & g_1 & \pi_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & g_{T-1} & \pi_{T-1} & g_{T-2} & \pi_{T-2} \\ 0 & \frac{1}{\tau}\sigma_1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\tau}\sigma_2 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sigma_1 2^d}{\tau} & 0 \\ 0 & 0 & 0 & 0 & \frac{\sigma_2 2^d}{\tau} \\ \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma\mu_1 & 0 & \gamma\mu_1 & 0 \\ 0 & 0 & \gamma\mu_2 & 0 & \gamma\mu_2 \\ \delta & \delta\mu_1 & \delta\mu_2 & \delta\mu_1 & \delta\mu_2 \end{bmatrix} \tag{9.12}$$

Estimation via Gibbs sampling now proceeds in a very straightforward way. There is no need to draw for the prior separately.

9.4 Examples

First we use quarterly US Growth (FRED series A191RO1Q156NBEA) and CPI (FRED series CPALTT01USQ661S) expressed as the annual inflation rate from 1961-01-01 to 2023-01-01 in a bi-variate BVAR. The last ten observations are:

Date	Growth	Inflation
2020-10-01	-1.5	1.224176
2021-01-01	1.2	1.905310
2021-04-01	12.5	4.776278
2021-07-01	5.0	5.264633
2021-10-01	5.7	6.765892
2022-01-01	3.7	8.023109
2022-04-01	1.8	8.556077
2022-07-01	1.9	8.284860
2022-10-01	0.9	7.110821
2023-01-01	1.8	5.769521

We specify a VAR with two lags, and use it to forecast 12 periods ahead. The BVAR are specified using the names above, with only `tau` particularly binding in this case. We set the total number of iterations in each case to 20000 and discard the first half. The parameter `nb` is used to set how much back data should appear in a fan chart.

```
#########
# Options
#########
nf <- 12 # Max forecast horizon
nb <- 21 # No. back periods plotted in graphs
l  <- 2  # Number of lags in VAR

# specify parameters of the Minnesota-type prior
tau    <- .1   # controls prior on own 1st lags (1 makes wibbly)
d      <- 1    # decay for higher lags
lambda <- 1    # prior for the constant
gamma  <- 1    # sum of coefficients unit roots
delta  <- 1    # cointegration prior

# Gibbs control
reps <- 20000 # total numbers of Gibbs iterations
burn <- 10000 # number of burn-in iterations
```

In what follows we vary `tau` and the lag length to illustrate their effects. To do this we create the augmented data and then run the Gibbs sampler, using:

```
# Create augmented data
Yplus <- augmentData(Y, l, tau, d, lambda, gamma, delta)

# Run Gibbs sampler
out   <- Gibbs_estimate(Yplus[[1]], Yplus[[2]], reps, burn, 1, nf)
```

where `Y` contains the data in a dataframe/tibble with the date in the first column as in the data example above. The code strips out the date and then uses the remaining N columns in the BVAR. See the Code Appendix for the details of the functions.

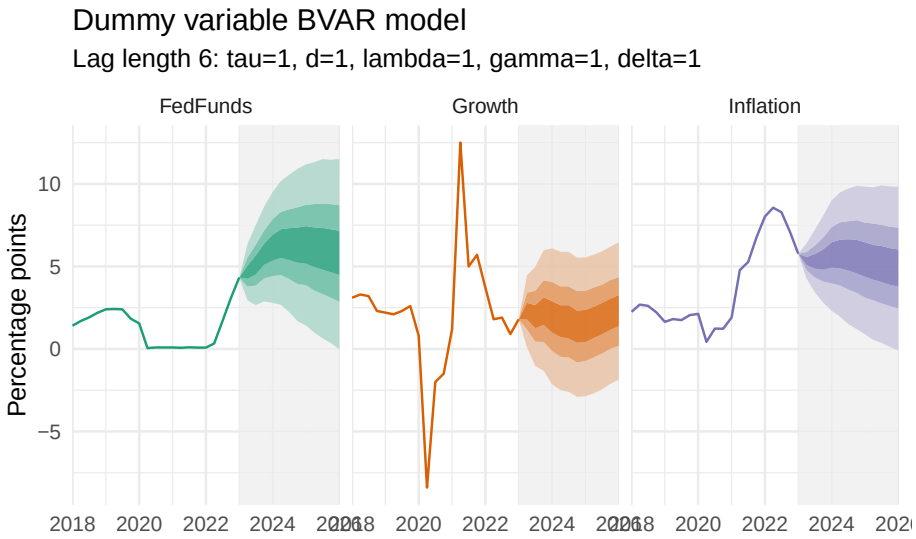
The output contains any ***forecast*** draws from the Gibbs sampler in the third list element from the `Gibbs_estimate()` function. The first two elements are coefficient draws. Two further functions plots the fan charts using the Gibbs draws:

```
# String to put in subtitle
controls <- paste0("Lag length ", l, ": tau=", tau, ", d=", d,
                   ", lambda=", lambda, ", gamma=", gamma, ", delta=", delta)

# Plots
fan_chart(Y, out[[3]], controls, nb)
p           <- coeff_plot(Y, l, out[[1]], out[[2]], 333, controls)
pnum        <- pnum+1
```

```
pce[[pnum]] <- p[[1]]
```

where the string `controls` is put in the chart subtitle and the coefficient densities. It can be anything but is a good place to remind yourself of how you specified the model. Notice we save the coefficient plots for later use.

9.4.1 Example 1: BVAR(2) with $\tau = 0.1$



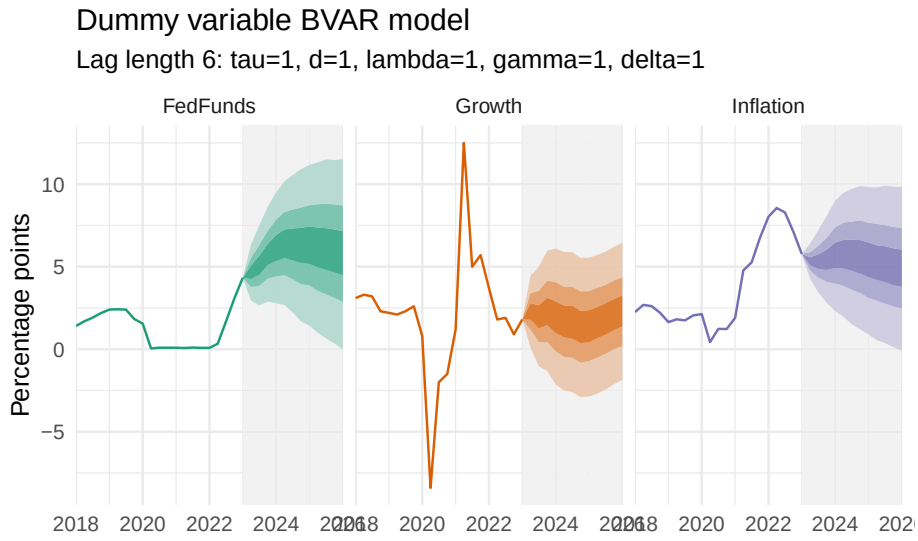
9.4.2 Example 2: BVAR(2) with $\tau = 1$

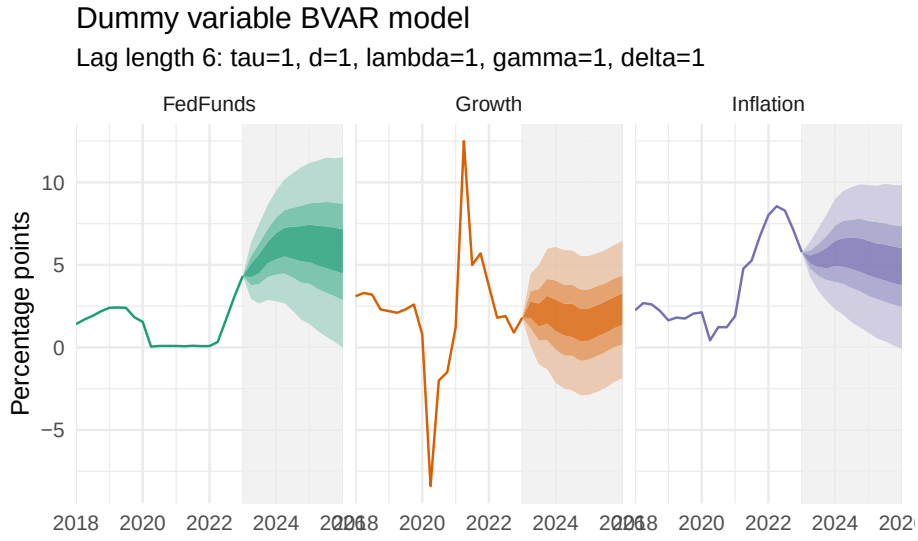
9.4.3 Example 3: BVAR(6) with $\tau = 0.1$

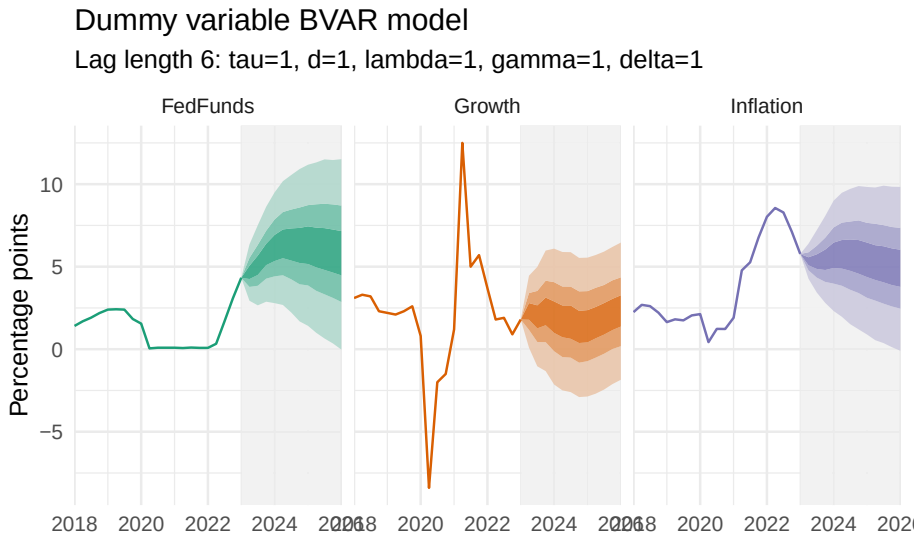
9.4.4 Example 4: BVAR(6) with $\tau = 1$

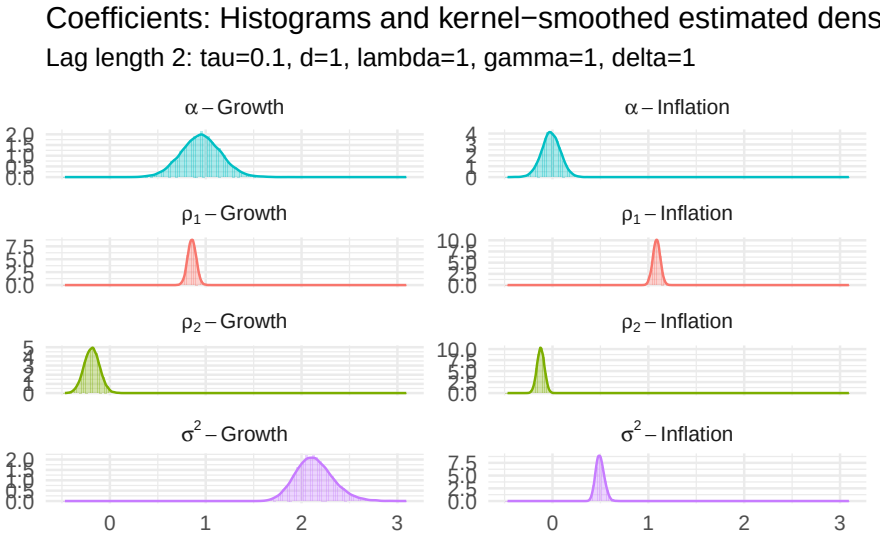
9.4.5 Coefficient estimates

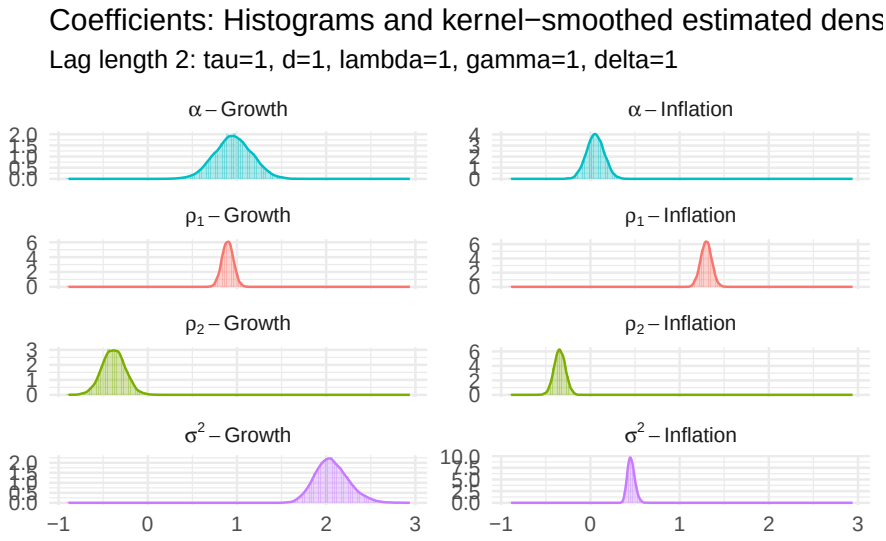
All of these have underlying parameters. Their estimated posterior densities are:

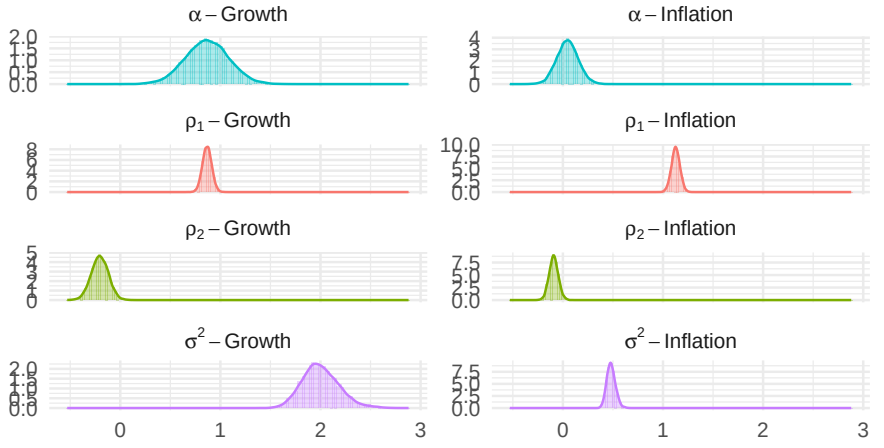

Dummy variable BVAR model
Lag length 6: tau=1, d=1, lambda=1, gamma=1, delta=1
FedFunds
Growth
Inflation
Percentage points
10
5
0
−5
2018
2020
2022
2024

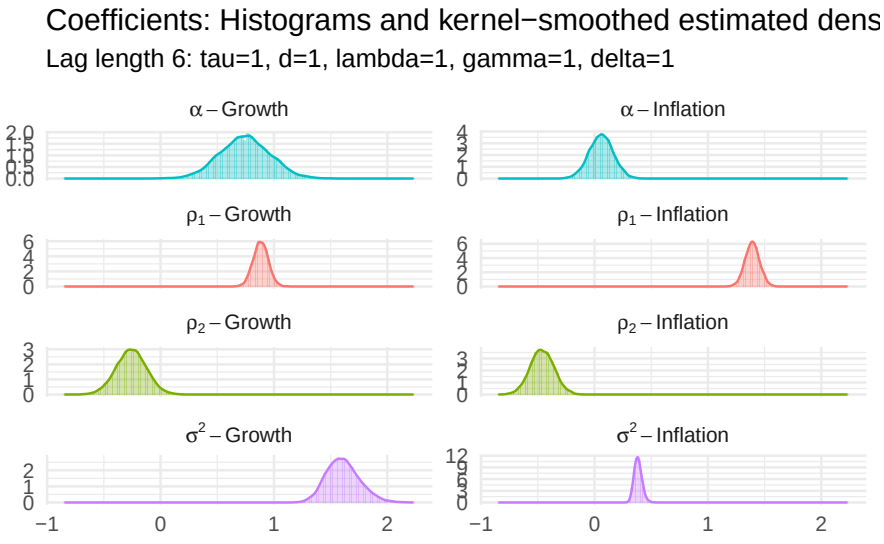

Dummy variable BVAR model
Lag length 6: tau=1, d=1, lambda=1, gamma=1, delta=1
FedFunds
Growth
Inflation
Percentage points
10
5
0
−5
2018
2020
2022
2024


Dummy variable BVAR model
Lag length 6: tau=1, d=1, lambda=1, gamma=1, delta=1
FedFunds
Growth
Inflation
Percentage points
10
5
0
−5
2018
2020
2022
2024


Coefficients: Histograms and kernel−smoothed estimated dens
Lag length 2: tau=0.1, d=1, lambda=1, gamma=1, delta=1
α – Growth
α – Inflation
ρ1 – Growth
ρ1 – Inflation
ρ2 – Growth
ρ2 – Inflation
σ2 – Growth
σ2 – Inflation
0
1
2
3


Coefficients: Histograms and kernel−smoothed estimated dens
Lag length 2: tau=1, d=1, lambda=1, gamma=1, delta=1
α – Growth
α – Inflation
ρ1 – Growth
ρ1 – Inflation
ρ2 – Growth
ρ2 – Inflation
σ2 – Growth
σ2 – Inflation


Coefficients: Histograms and kernel−smoothed estimated dens
Lag length 6: tau=0.1, d=1, lambda=1, gamma=1, delta=1
α – Growth
α – Inflation
ρ1 – Growth
ρ1 – Inflation
ρ2 – Growth
ρ2 – Inflation
σ2 – Growth
σ2 – Inflation



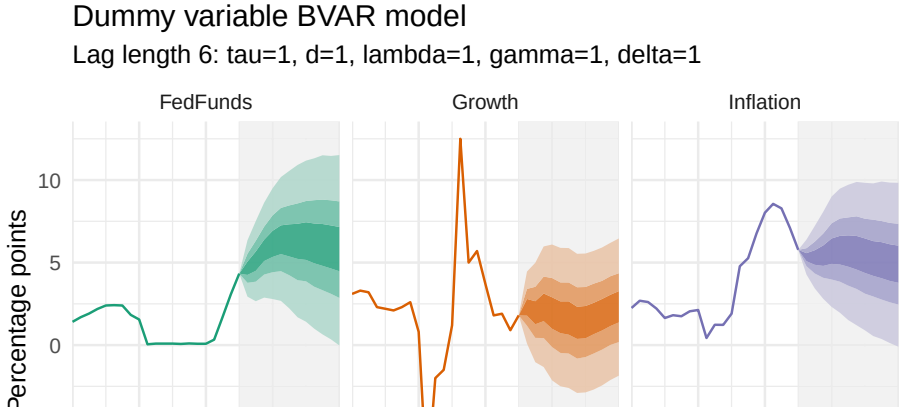
9.5 Tri-variate BVAR

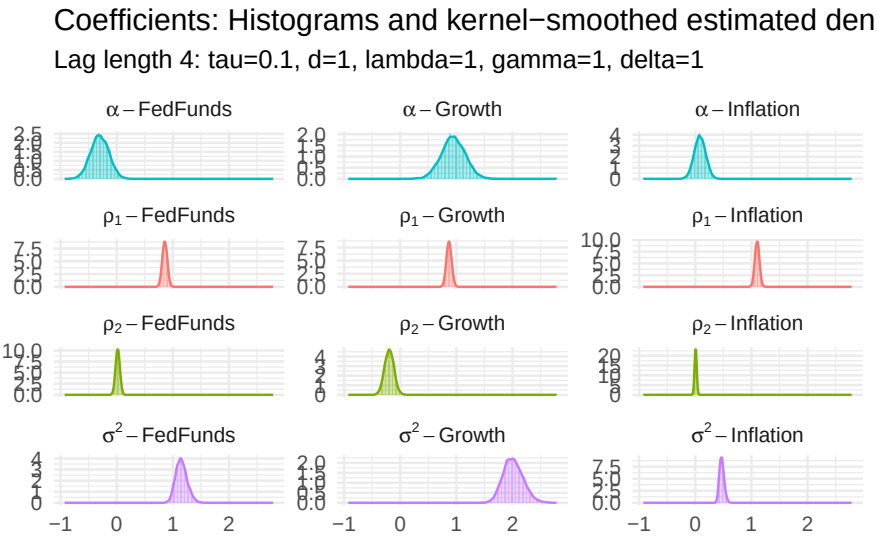
Now we add the FedFunds rate (FRED series FEDFUNDS), so the last ten periods of the data set is now:

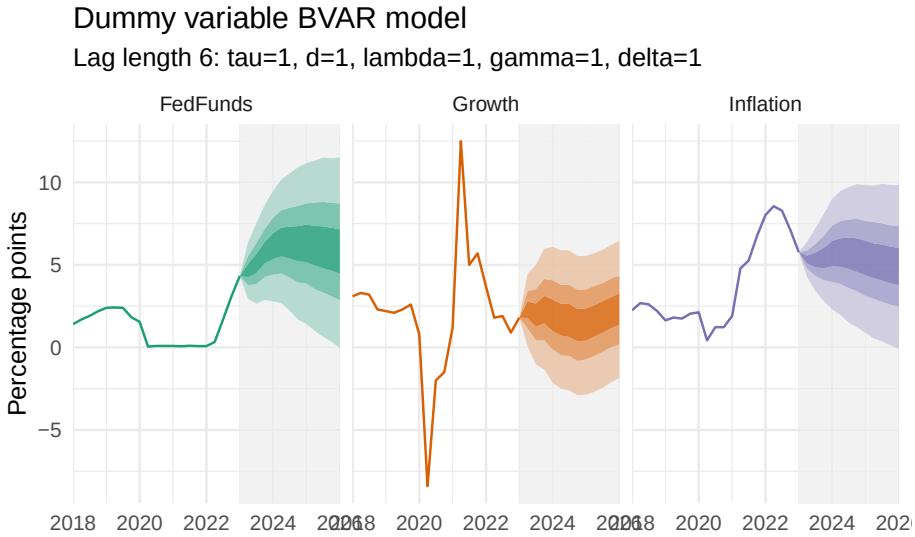
Date	Growth	Inflation	FedFunds
2020-10-01	-1.5	1.224176	0.09
2021-01-01	1.2	1.905310	0.09
2021-04-01	12.5	4.776278	0.07
2021-07-01	5.0	5.264633	0.10
2021-10-01	5.7	6.765892	0.08
2022-01-01	3.7	8.023109	0.08
2022-04-01	1.8	8.556077	0.33
2022-07-01	1.9	8.284860	1.68
2022-10-01	0.9	7.110821	3.08
2023-01-01	1.8	5.769521	4.33

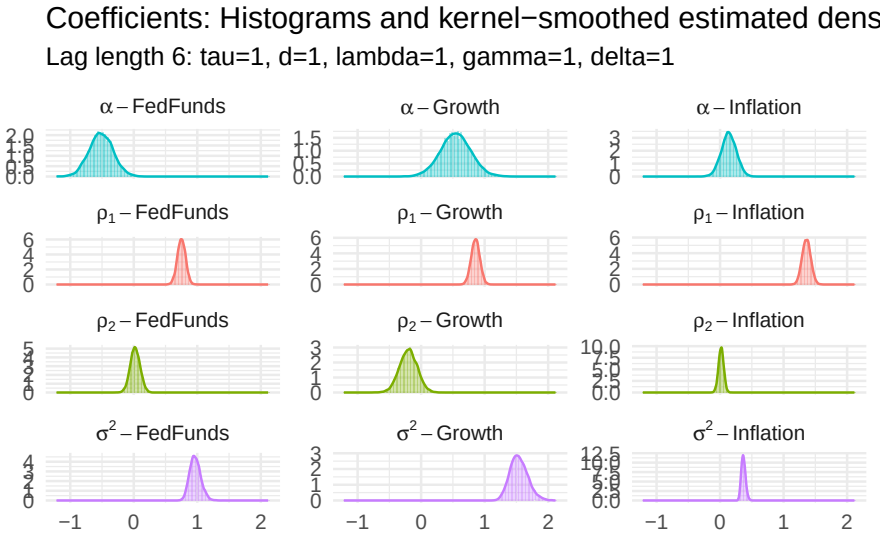
Two more examples follow.

9.5.1 Example 5: BVAR(4) with $\tau = .1$, 3 variables




Coefficients: Histograms and kernel−smoothed estimated den
Lag length 4: tau=0.1, d=1, lambda=1, gamma=1, delta=1
α – FedFunds
α – Growth
α – Inflation
ρ1 – FedFunds
ρ1 – Growth
ρ1 – Inflation
ρ2 – FedFunds
ρ2 – Growth
ρ2 – Inflation
σ2 – FedFunds
σ2 – Growth
σ2 – Inflation


Dummy variable BVAR model
Lag length 6: tau=1, d=1, lambda=1, gamma=1, delta=1
FedFunds
Growth
Inflation
Percentage points



9.5.2 Example 6: BVAR(6) with $\tau = 1$, 3 variables

9.6 Code appendix

You can download the program and functions used for the estimates above from the links below. Put them in the same directory and they should recreate exactly (within sampling error) the same graphs as above. Ensure you have all the libraries available that are loaded at the top of `BVARdum.R`.

Main program:

Functions:

Part V

End matter

Chapter 10

Summary

In summary, this book has no content whatsoever...

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Appendix A

Basic ggplot2

A.1 Plotting in the tidyverse

ggplot2 forms a key part of the tidyverse – for many the only part. It builds on the ***g**rammar of **g**raphics* proposed by the late Leland Wilkinson, Wilkinson (2013). In essence it provides rules for how graphics should be treated, simple rules that drive you mad until you get it.

The process for building a graph is something like the following.

- Initiate a plot using ggplot.
- Specify **aesthetics** which indicate *what* you want to plot from some data set.
- Call a geom (or an alternative) to say *how* you want to plot it.
- Add **modifiers** to change how it *looks*.

The order of operations is essentially always this, although quite how the ordering is apllied differs subtly, which we will show here.

A.2 Example

To illustrate, we take the wooldridge data set approval from Yong, Krosnick, and Wooldridge (2016), do a little wrangling and (eventually) produce some quite nice plots. Start with the libraries and retrieve data the data.

```
library(tidyverse)
library(wooldridge)
data("approval")
```

The first few columns and rows of this looks like:

```
  id month year   sp500   cpi cpifood approve
1 302     2 2001 1239.94 184.4   171.8   59.24
2 303     3 2001 1160.33 185.3   172.2   57.01
3 304     4 2001 1249.46 185.6   172.4   60.31
4 305     5 2001 1255.82 185.5   172.9   55.82
5 306     6 2001 1224.42 185.9   173.4   54.93
6 307     7 2001 1211.23 186.2   174.0   56.36
```

and all the available variables are

```
names(approval)
```

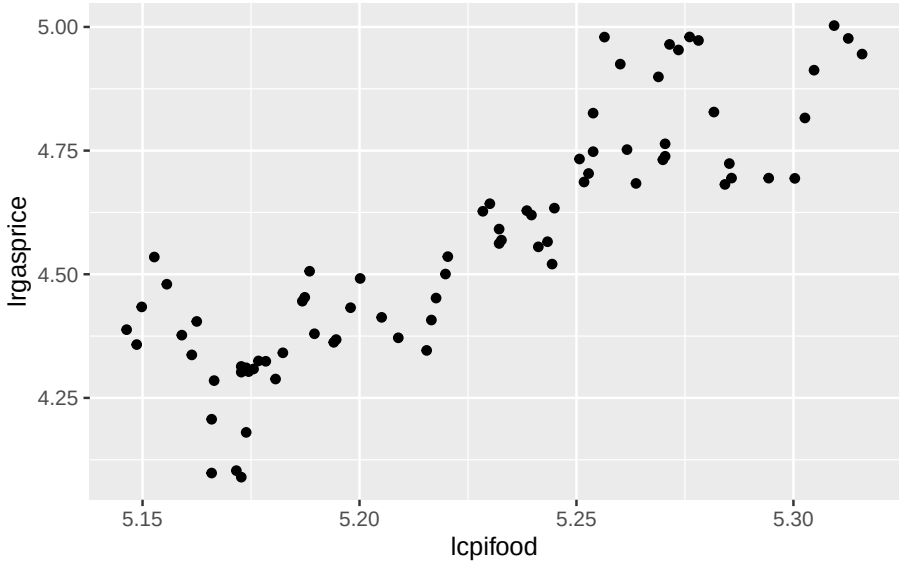
```
 [1] "id"         "month"      "year"       "sp500"      "cpi"
 [6] "cpifood"    "approve"    "gasprice"   "unemploy"   "katrina"
[11] "rgasprice"  "lrgasprice" "X11.Sep"    "iraqinvade" "lsp500"
[16] "lcpifood"
```

Typically we want to investigate trends and correlations and graphing pairs or more of series is a good way to begin.

A.2.1 Scatter plot

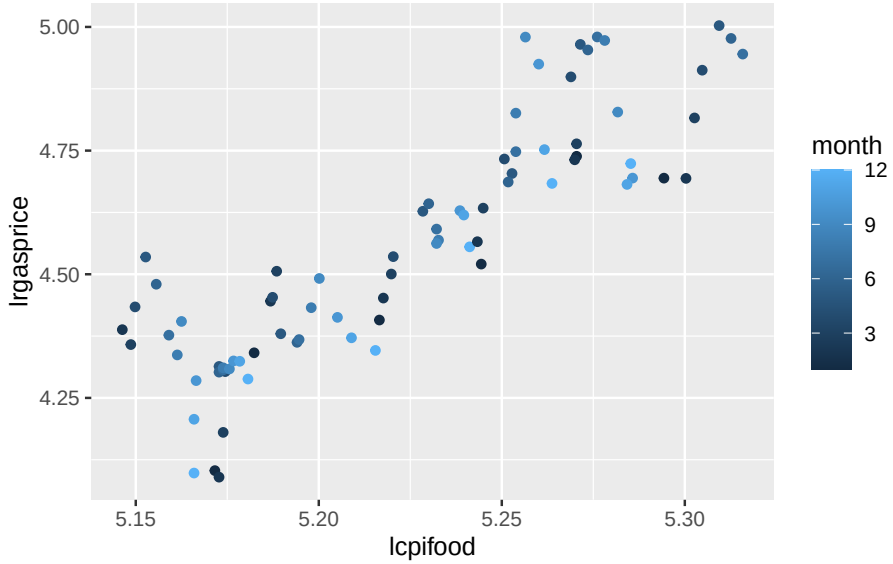
A first scatter plot, using `geom_point` of food against gas (petrol) prices

```
ggplot(approval, aes(x=lcpifood, y=lrgasprice)) +    # Initiate, set aesthetics
  geom_point()                                       # Display as points
```



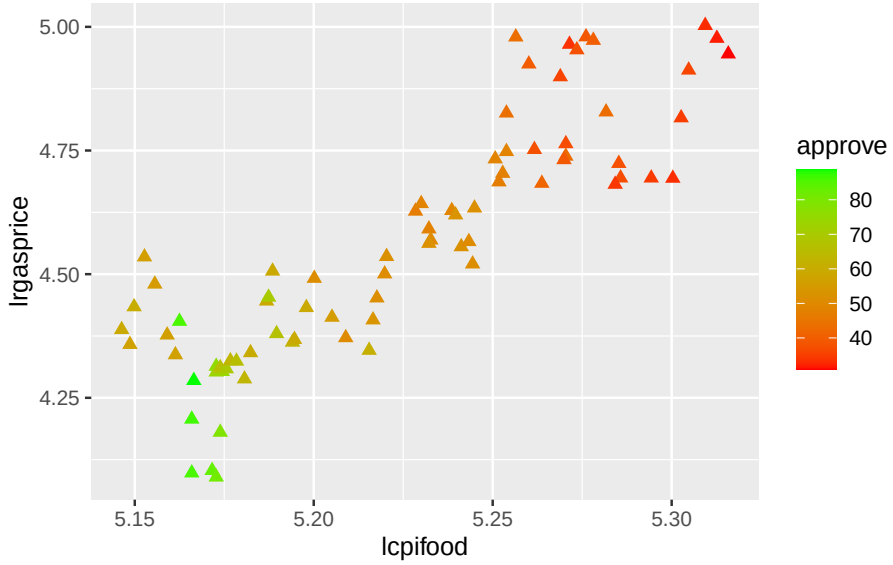
OK, I guess, but a bit dull – so add some colour. This time, `aes` is specified in the `geom` – either is fine, but there are some advantages either way which we will see shortly.

```
ggplot(approval) +
  geom_point(aes(x=lcpifood, y=lrgasprice, color=month))  # Colours by month
```



Better, but how about…

```
ggplot(approval) +
  geom_point(aes(x=lcpifood, y=lrgasprice, color=approve), size=2, shape=17) + # Col
  scale_color_gradient(low="red", high="green")
```

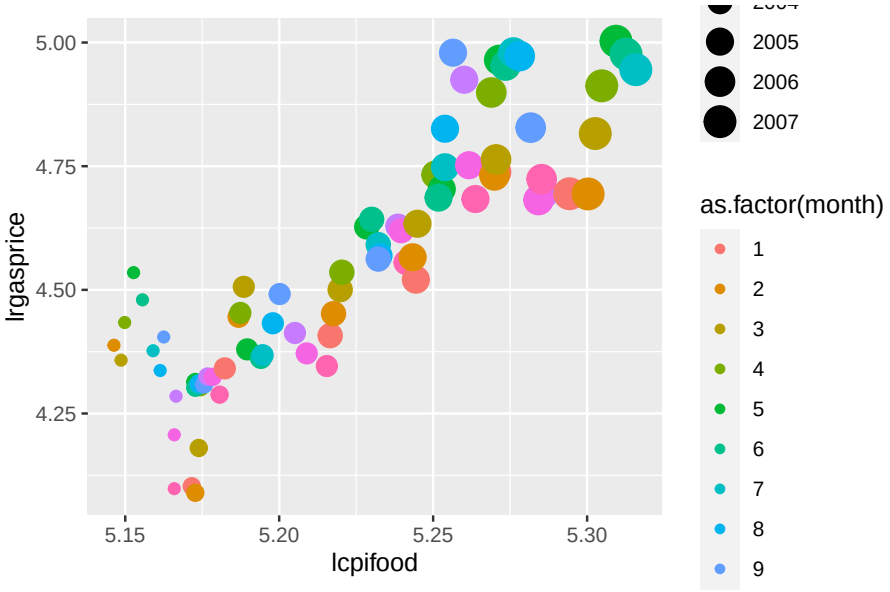


where the colours are a gradient we specify. But months can only be one of twelve categories, so a categorical variable (a *factor*) is needed to get different actual colours, otherwise for a continuous variable I get shades of one colour or a continuous change we need to specify.

Lets do this – and add a different aesthetic, size, for year.

```
ggplot(approval) +
  geom_point(aes(x=lcpifood, y=lrgasprice, color=as.factor(month), size=as.factor(year)))
```

```
Warning: Using size for a discrete variable is not advised.
```



Note there is now a lot going n, and maybe too much. `ggplot` thinks so!

A.2.2 Time series plots

Our time index is a bit odd as the data set has year and month separately. Create a proper date series using:

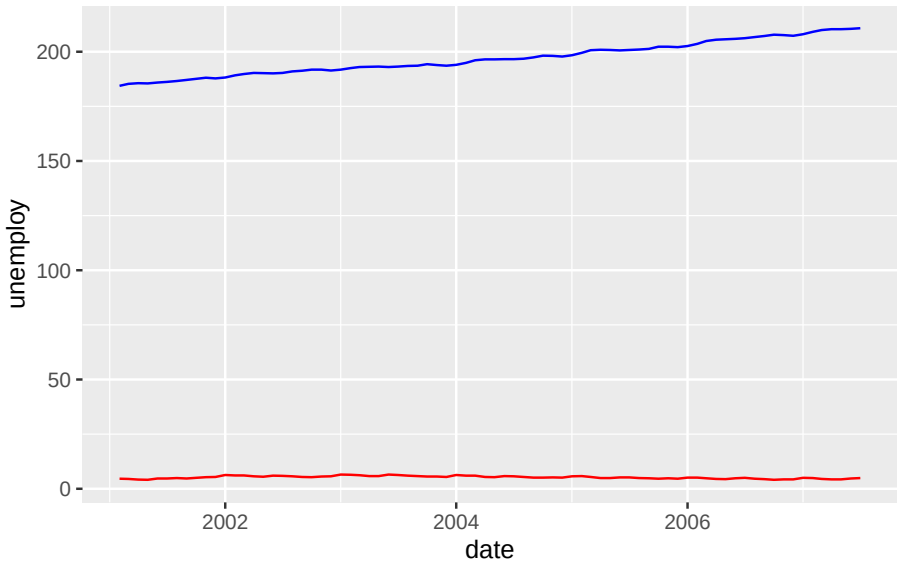
```
approval %<>%
  unite(date, year, month, sep="/") %>%
  mutate(date = as.Date(paste0(date,"/01"), "%Y/%m/%d"))
```

I've used the `%<>%` pipe operator to send and get back `approval` so this is now

```
  id       date   sp500   cpi cpifood approve gasprice unemploy katrina
1 302 2001-02-01 1239.94 184.4   171.8   59.24    148.4      4.6       0
2 303 2001-03-01 1160.33 185.3   172.2   57.01    144.7      4.5       0
3 304 2001-04-01 1249.46 185.6   172.4   60.31    156.4      4.2       0
4 305 2001-05-01 1255.82 185.5   172.9   55.82    172.9      4.1       0
5 306 2001-06-01 1224.42 185.9   173.4   54.93    164.0      4.7       0
6 307 2001-07-01 1211.23 186.2   174.0   56.36    148.2      4.7       0
  rgasprice lrgasprice X11.Sep iraqinvade   lsp500 lcpifood
1  80.47723   4.387974       0          0 7.122818 5.146331
2  78.08958   4.357857       0          0 7.056460 5.148656
3  84.26724   4.433993       0          0 7.130467 5.149817
4  93.20755   4.534829       0          0 7.135544 5.152713
5  88.21947   4.479828       0          0 7.110222 5.155601
6  79.59184   4.376912       0          0 7.099391 5.159055
```

Then I can plot a couple of series using two calls to `geom_line`

```
ggplot(approval) +
  geom_line(aes(x=date, y=unemploy), colour="red") +
  geom_line(aes(x=date, y=cpi), colour="blue")
```



But this is pretty inefficient, as I would need a call to `geom_line` for every series I wanted to plot and even then scales are unsuitable. Plus the labels are not right.

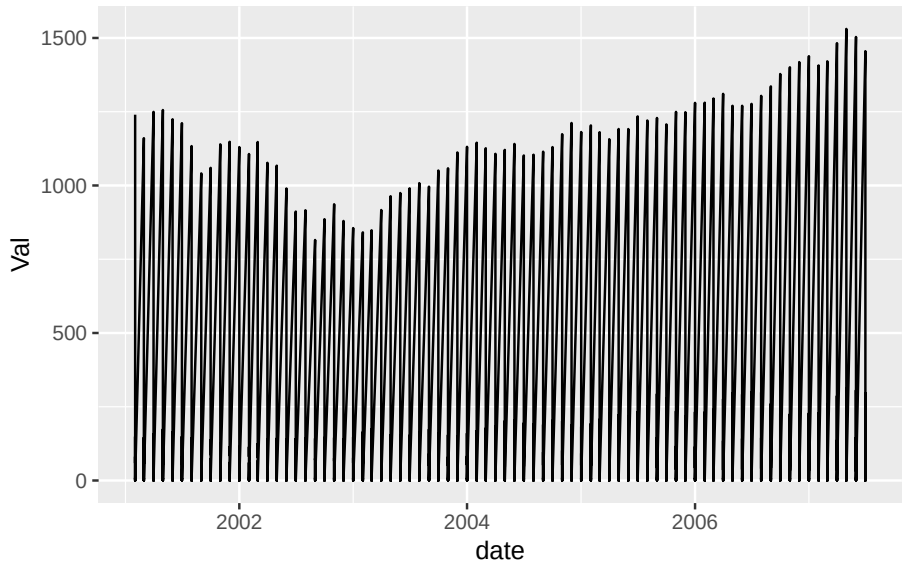
This is where things really get interesting. I `pivot_longer` all the variables into a single column.

```
df <- pivot_longer(approval, cols=-c(date, id), names_to= "Var", values_to = "Val")
head(df)
```

```
# A tibble: 6 x 4
     id date       Var          Val
  <int> <date>     <chr>      <dbl>
1   302 2001-02-01 sp500     1240.
2   302 2001-02-01 cpi        184.
3   302 2001-02-01 cpifood    172.
4   302 2001-02-01 approve     59.2
5   302 2001-02-01 gasprice   148.
6   302 2001-02-01 unemploy     4.6
```

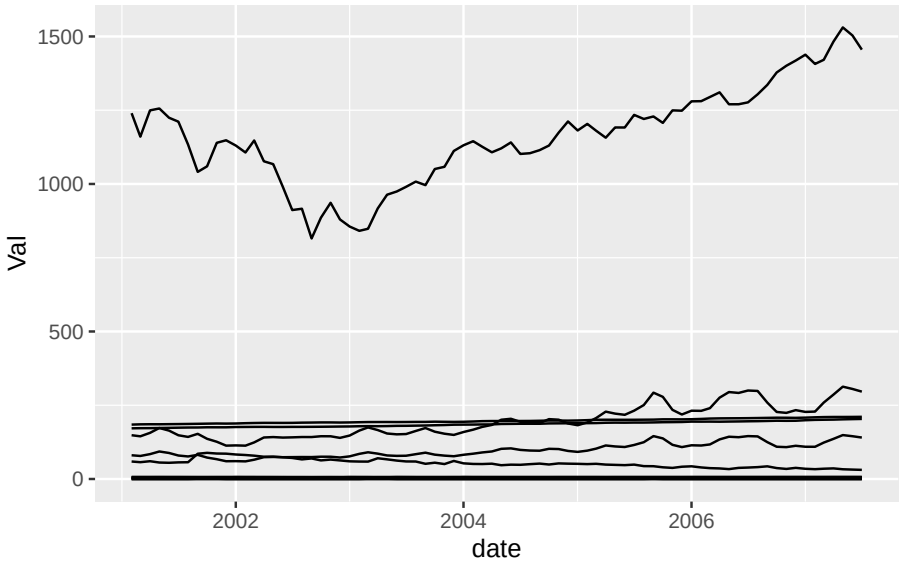
Great! Now I can plot `Val` using one call to `geom_line`. This time, put the graph object into `p` and then explicitly plot it.

```
p  <- ggplot(df) +
  geom_line(aes(x=date, y=Val))
plot(p)
```



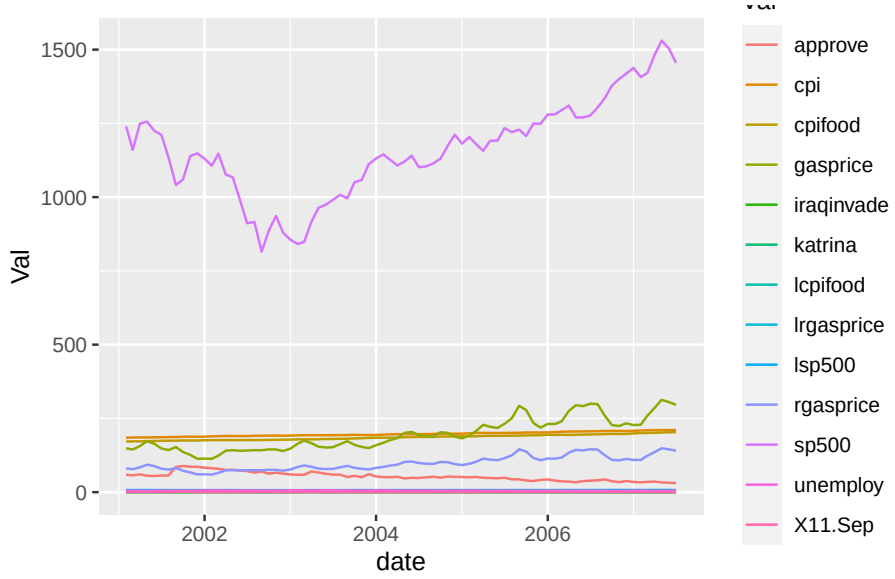
Oops! I need to tell `ggplot2` to separate out the variables which are stored in `Var`. For this, use `group`:

```
p  <- ggplot(df) +
  geom_line(aes(x=date, y=Val, group=Var))
plot(p)
```



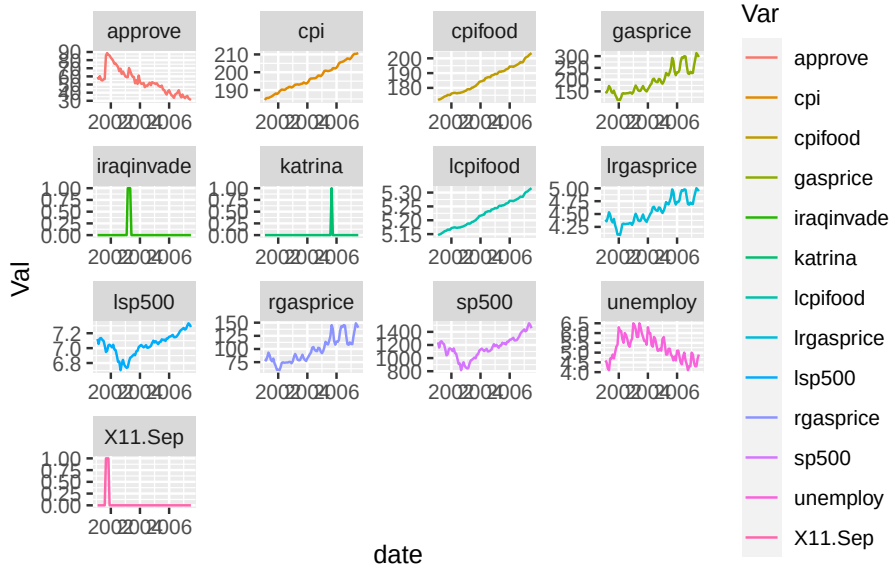
But this could better be done by using an aesthetic like colour which implies group

```
p  <- ggplot(df) +
  geom_line(aes(x=date, y=Val, colour=Var))
plot(p)
```



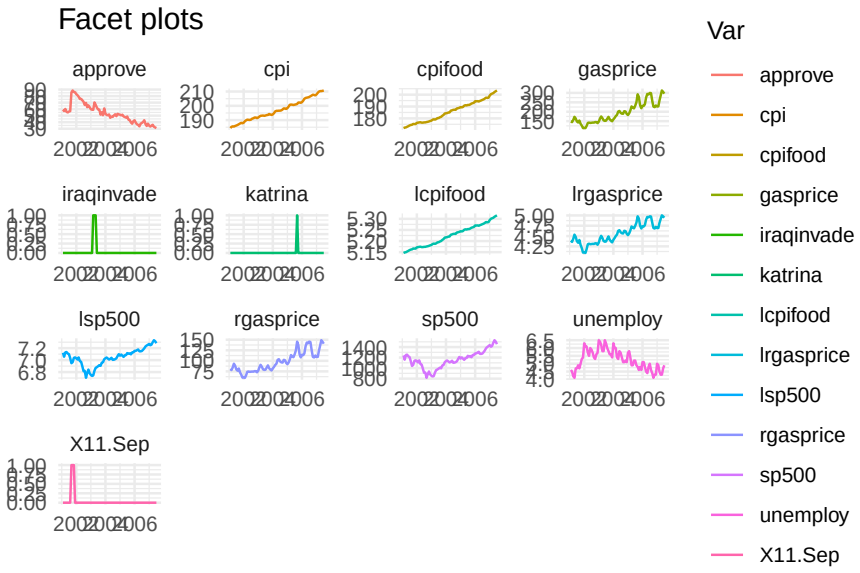
OK, but can I plot them so we can see what's going on, like in a grid? This is where `facet` comes in.

```
p  <- p +
  facet_wrap(~Var, scales = "free")
plot(p)
```



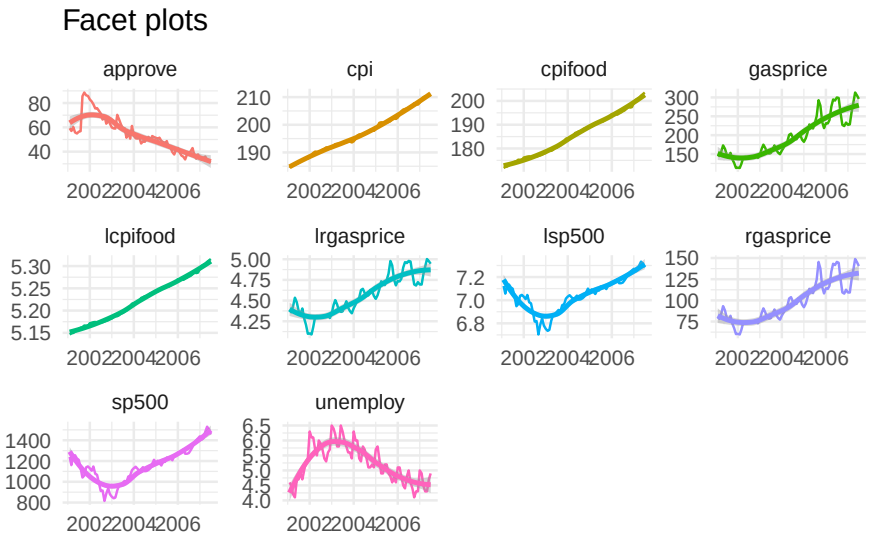
A bit more formatting...

```
p  <- p +
  theme_minimal() +
  labs(title="Facet plots", x="", y="")
plot(p)
```



Finally all in one go, dropping the dummies, don't store as an object. Also no legend, as series labelled in the facets. And I call a rather handy little function `geom_smooth` which fits (by default) a Loess smoothing line.

```
approval %>%
  select(-iraqinvade, -katrina, -X11.Sep) %>%
  pivot_longer(cols=-c(date, id), names_to="Var", values_to="Val") %>%
  ggplot(aes(x=date, y=Val, group=Var, colour=Var)) +
  geom_line() +
  geom_smooth() + # Smoother
  facet_wrap(~Var, scales = "free") +
  theme_minimal() +
  theme(legend.position = "none") +
  labs(title="Facet plots", x="", y="")
```


Facet plots
approve
cpi
cpifood
gasprice
lcpifood
lrgasprice
lsp500
rgasprice
sp500
unemploy
200220042006

Cool, huh?

Appendix B

Linear algebra

B.1 Modelling in economics is linear algebra

That's a bit extreme, but you mostly need to do linear algebra to program up many of the estimators we need, or to solve a rational expectations models.

B.2 Beginning to program

A few non-linear algebra things we will need are summarized in the following table.

Table B.1: Useful things

	R	Example	Notes
Assign a value	`<-`	`a <- 4`	Also legal is `a = 4`. But I hate it.
Create a list of values	`c(.)`	`v <- c(1, -2, 22)`	Defining 'on the fly'
Sequence	`seq(i, k, l)`	5, 7, … ,21	Create a sequence
	`i:k`	i, $i \pm 1$, … ,k	Short cut for unit in/de-crements
Loop commands	`for (var in seq) expr`	`for (i in 5:1) print(i)`	Loops. We need loops.
Draw a random number	`rnorm(k,a,b)`	`rnorm(60, 0, 5)`	Example draws 60 values $\sim N(0, 5)$
Create a matrix	`matrix(v,i,j)`	`matrix(5, 2, 2)`	Create a 2×2 matrix of 5s

B.2.1 Functions

Everything in R is a function (although it doesn't look like it). Defining a function is simple:

```
name_of_function <- function(function_arguments){
  # Body of function where stuff is done
}
```

Here's one that actually does something:

```
addaddadd <- function(x,y){
  z <- 3*(x+y)
  return(z)
}
```

and if we run it:

```
addaddadd(4,6)
```

```
[1] 30
```

B.3 Linear algebra

Assume the following: A and B are real matrices of dimension $n \times n$, b and c are real $n-$vectors, X is a real $T \times k$ matrix, and S is a symmetric real matrix.

Table B.2: Maths commands essential to linear algebra

	Maths	R	Notes
Hadamard product	$A \odot B$	`A * B`	Element-by-element, A, B same size
Matrix/vector product	$A \times B$, $A \times b$	`A %*% B`, `A %*% b`	Normal product rule
Inner product	$X'X$	`t(X) %*% X`	Also uses transpose operator, `t()`
		`crossprod(X)`	More efficient, but less mathy
	$A'B$	`t(A) %*% B`	
		`crossprod(A,B)`	
Outer product	$A \times B'$	`tcrossprod(A,B)`	

	Maths	R	Notes
Inverse	A^{-1}	`solve(A)`	Matrix inverse is a special case of...
Solve for d	$Ad = b \Rightarrow d = A^{-1}b$	`d <- solve(A, b)`	...linear solution!
Cholesky decomp	$S = R'R$	`R <- chol(S)`	S is a symmetric, positive definite matrix
Cholesky inverse	S^{-1}	`chol2inv(R)`	Fast!
Determinant	$\|A\|$	`det(A)`	
Diagonal			
of a matrix		`diag(A)`	Retrieve the elements a_{ii}, $i = 1, .., n$
in a matrix		`A <- diag(b)`	Set the diagonal of A to b, zero elsewhere
Identity matrix	I_n	`diag(n)`	
Eigenvalues/vectors		`E <- eigen(A)`	Returns a *list*: `E$values`, `E$vectors`

B.4 Starting to do linear algebra

B.4.1 Problem

Consider the following simultaneous system of equations:

$$\begin{aligned} x_1 + 2x_2 &= 6 \\ x_1 - 3x_2 + 2x_3 &= 0 \\ -2x_1 + 3x_3 &= 2 \end{aligned}$$

Find the values of x that solve this using R.

Hint – write the problem in matrix form

$$Ax = b$$

where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -3 & 2 \\ -2 & 0 & 3 \end{bmatrix}, \qquad b = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}$$

and then use `solve`.

B.4.2 Solution

R code to create these matrices is:

```
# Matrices are populated by column by default
A <- matrix(c(1,1,-2,2,-3,0,0,2,3),3,3)
b <- matrix(c(6,0,2),3,1)
```

The solution is:

```
x <- solve(A,b)
```

where `x` is:

```
     [,1]
[1,]    2
[2,]    2
[3,]    2
```

B.5 Eigenvalue decomposition

Any square matrix can be decomposed into a non-singular matrix V of eigenvectors and a diagonal matrix of eigenvalues Λ such that:

$$AV = V\Lambda \Rightarrow A = V\Lambda V^{-1} \tag{B.1}$$

Call `eigen` to decompose our previously defined matrix A

```
e <- eigen(A)
L <- e$values    # Returns a list, assign vectors/values
V <- e$vectors
```

Note A is **not** symmetric so it may have complex roots, which is does

```
L
```

```
[1] -3.682744+0.000000i  2.341372+0.873683i  2.341372-0.873683i
```

If we calculate $A = V\Lambda V^{-1}$ we get

```
s <- V %*% diag(L) %*% solve(V)
s
```

```
      [,1]              [,2]              [,3]
[1,]  1+0i  2.000000e+00+0i -2.220446e-16+0i
[2,]  1+0i -3.000000e+00+0i  2.000000e+00+0i
[3,] -2+0i -1.054712e-15+0i  3.000000e+00+0i
```

and when we drop the zero imaginary parts

```
Re(s)
```

```
     [,1]          [,2]          [,3]
[1,]    1  2.000000e+00 -2.220446e-16
[2,]    1 -3.000000e+00  2.000000e+00
[3,]   -2 -1.054712e-15  3.000000e+00
```

Round to eliminate numerical error, to get

```
round(Re(s), digits=12)
```

```
     [,1] [,2] [,3]
[1,]    1    2    0
[2,]    1   -3    2
[3,]   -2    0    3
```

B.6 Precision

Why do we round? Take a real matrix A_{mn} with $n \leq m$ and pre-multiply by its own transpose, i.e. $S = A'A$. AA is symmetric, positive semi-definite. If $rank(A) = n$, then $rank(S) = n$ and positive definite, and its inverse exists.

```
A  <- matrix(c(1, .2, 0, 1), 2, 2)
S <- t(A) %*% A
S
```

```
     [,1] [,2]
[1,] 1.04  0.2
[2,] 0.20  1.0
```

Let's invert S three different ways.

```
i1 <- solve(S)
i2 <- chol2inv(chol(S))
i3 <- qr.solve(S)
```

Pre-multiplying S by its inverse gives

$$I_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{B.2}$$

Looking at the results of doing this for each method gives

```
i1 %*% S
```

```
             [,1] [,2]
[1,] 1.000000e+00    0
[2,] 2.775558e-17    1
```

```
i2 %*% S
```

```
     [,1]         [,2]
[1,]    1 5.551115e-17
[2,]    0 1.000000e+00
```

```
i3 %*% S
```

```
     [,1]          [,2]
[1,]    1 -2.775558e-17
[2,]    0  1.000000e+00
```

which are all slightly different but by tiny – and insignificant – amounts. Don't be fooled by this, they are all numerically the same.

B.7 Programming the regression problem

Let's look at the familiar regression problem for some generated data.

$$y = XB + \epsilon \tag{B.3}$$

where $\epsilon \sim N(0, .2)$, X is a $(k+1) \times n$ matrix of regressors including a constant and B a $k+1$ vector of coefficients. Let's generate some random data of an arbitrary sized problem:

```
X <- matrix(rnorm(180, 2, 1), 60, 3)
head(X, 6) # Print first six rows
```

```
         [,1]        [,2]     [,3]
[1,] 1.906710  4.86573396 2.390330
[2,] 3.268898  4.05527385 1.705367
[3,] 1.622163  2.10549632 2.176462
[4,] 1.304953  0.40456346 3.078402
[5,] 3.106988  2.54012774 1.180277
[6,] 4.346681 -0.09252894 2.508839
```

```
X <- cbind(1, X) # Add a constant
tail(X,6) # Print last six rows
```

```
      [,1]       [,2]      [,3]      [,4]
[55,]    1 -1.0234818 1.3021403 1.4330546
[56,]    1  1.2814302 0.7535604 1.0542110
[57,]    1  2.3069226 1.4478565 0.3346868
[58,]    1 -0.2652897 3.0628273 2.0804648
[59,]    1  2.1082132 1.5461640 1.8327673
[60,]    1  2.4949328 4.4734549 2.5290737
```

Now create a dependent variable that is a linear combination of these variables plus some noise. Create the linear relationship first so we know what it is:

```
B <- matrix(c(0.5,1,-1,.2), 4, 1)
```

and then the dependent variable:

```
y <- X %*% B + 0.2*rnorm(60)
```

We could now do a regression – i.e. calculate

$$\hat{B} = (X'X)^{-1}X'y \tag{B.4}$$

which can be written:

```
Bhat <- solve(t(X)%*%X)%*%t(X)%*%y
```

which gives

```
           [,1]
[1,]  0.5013554
[2,]  1.0131585
[3,] -1.0033866
[4,]  0.1707199
```

But I wouldn't do it like this (we'll see why in a minute). A better way would be

```
Bhat2 <- chol2inv(chol(crossprod(X))) %*% crossprod(X,y)
```

or even

```
Bhat3 <- qr.solve(X,y)
```

which both evaluate to the same $\hat{B}$ values. Each is better in different circumstances.

B.7.1 Test timings

Why does it matter how you do things? It should be obvious that it might, but it turns out some fairly trivial things can make a lot of difference. We set some parameters so we can create a bigger problem.

```
n <- 400
k <- 12
```

We will use seven different methods to calculate an estimate of B. These are two variations on the three calculations below (where the brackets matter!):

$$\hat{B}_1 = ((X'X)^{-1})X'y \tag{B.5}$$

$$\hat{B}_2 = ((X'X)^{-1})(X'y) \tag{B.6}$$

$$\hat{B}_3 = ((X'X)^{-1}(X'y)) \tag{B.7}$$

where we do each of these either 'by hand' or using `crossprod`, with a final solution using `qr.solve`.

```
library(tictoc)

reps <- 10000
t    <- list()

set.seed(123)
```

```
tic()
for (i in 1:reps) {
  B <- matrix(runif(k+1), k+1, 1)
  X <- cbind(1L, matrix(rnorm(n*k, 2, 1), n, k))
  y <- X %*% B + 0.2 * rnorm(n)
  Bhat <- solve(t(X) %*% X) %*% t(X) %*% y
  }
t[[1]] <- toc(quiet = TRUE)

set.seed(123)
tic()
for (i in 1:reps) {
  B <- matrix(runif(k+1), k+1, 1)
  X <- cbind(1L, matrix(rnorm(n*k, 2, 1), n, k))
  y <- X %*% B + 0.2 * rnorm(n)
  Bhata <- solve(t(X)%*%X) %*% (t(X)%*%y)
}
t[[2]] <- toc(quiet = TRUE)

set.seed(123)
tic()
for (i in 1:reps) {
  B <- matrix(runif(k+1), k+1, 1)
  X <- cbind(1L, matrix(rnorm(n*k, 2, 1), n, k))
  y <- X %*% B + 0.2 * rnorm(n)
  Bhatb <- solve((t(X) %*% X), (t(X) %*% y))
  }
t[[3]] <- toc(quiet = TRUE)

set.seed(123)
tic()
for (i in 1:reps) {
  B <- matrix(runif(k+1), k+1, 1)
  X <- cbind(1L, matrix(rnorm(n*k, 2, 1), n, k))
  y <- X %*% B + 0.2 * rnorm(n)
  Bhat2 <- solve(crossprod(X)) %*% crossprod(X,y)
  }
t[[4]] <- toc(quiet = TRUE)

set.seed(123)
tic()
for (i in 1:reps) {
  B <- matrix(runif(k+1), k+1, 1)
  X <- cbind(1L, matrix(rnorm(n*k, 2, 1), n, k))
```

```
  y <- X %*% B + 0.2 * rnorm(n)
  Bhat2a <- solve(crossprod(X), crossprod(X,y))
  }
t[[5]] <- toc(quiet = TRUE)

set.seed(123)
tic()
for (i in 1:reps) {
  B <- matrix(runif(k+1), k+1, 1)
  X <- cbind(1L, matrix(rnorm(n*k, 2, 1), n, k))
  y <- X %*% B + 0.2 * rnorm(n)
  Bhat2b <- chol2inv(chol(crossprod(X))) %*% crossprod(X,y)
  }
t[[6]] <- toc(quiet = TRUE)

set.seed(123)
tic()
for (i in 1:reps) {
  B <- matrix(runif(k+1), k+1, 1)
  X <- cbind(1L, matrix(rnorm(n*k, 2, 1), n, k))
  y <- X %*% B + 0.2 * rnorm(n)
  Bhat3 <- qr.solve(X,y)
  }
t[[7]] <- toc(quiet = TRUE)
```

How do we display the timings we saved in `t`? We could do a simple (but dull) table, or something a bit nicer.

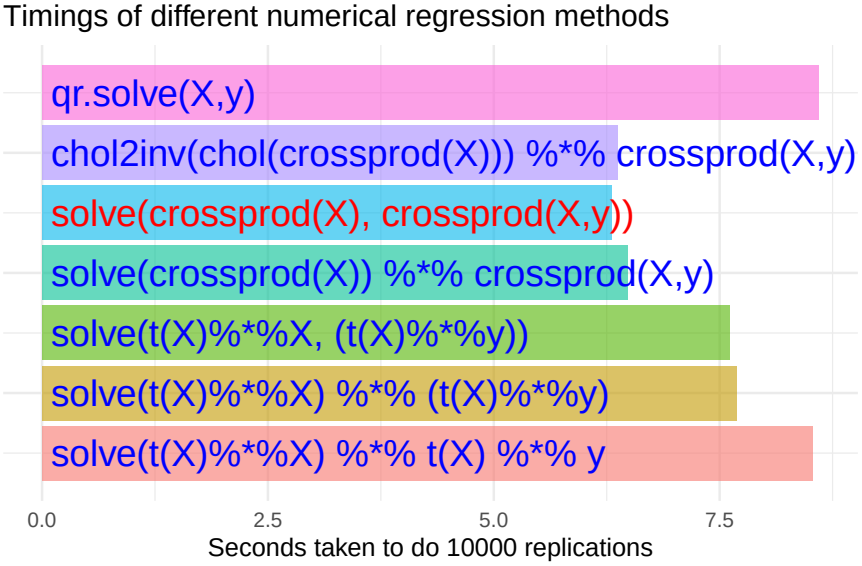
```
library(tidyverse)

FF <- c("solve(t(X)%*%X) %*% t(X) %*% y",
        "solve(t(X)%*%X) %*% (t(X)%*%y)",
        "solve(t(X)%*%X, (t(X)%*%y))",
        "solve(crossprod(X)) %*% crossprod(X,y)",
        "solve(crossprod(X), crossprod(X,y))",
        "chol2inv(chol(crossprod(X))) %*% crossprod(X,y)",
        "qr.solve(X,y)")

tms <- as.numeric(gsub(" sec elapsed", "", unlist(t)[seq(3,21,3)]))
v   <- tibble(Method  = 1:7,
              Times   = tms,
              Formula = FF) %>%
  mutate(Col = case_when(Times == min(Times) ~ "red",
                         TRUE ~ "blue"
```

```
  ))

ggplot(v) +
  geom_col(aes(x=Method, y=Times, fill=as.factor(Method)), alpha=.6) +
  geom_text(aes(x=Method, y=.1, label=Formula, color=Col), size=5.5, hjust=0) +
  theme_minimal() +
  scale_color_identity() +
  theme(legend.position = "none") +
  theme(axis.text.y = element_blank()) +
  coord_flip() +
  labs(y=paste("Seconds taken to do",reps,"replications"), x="",
       title="Timings of different numerical regression methods")
```



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