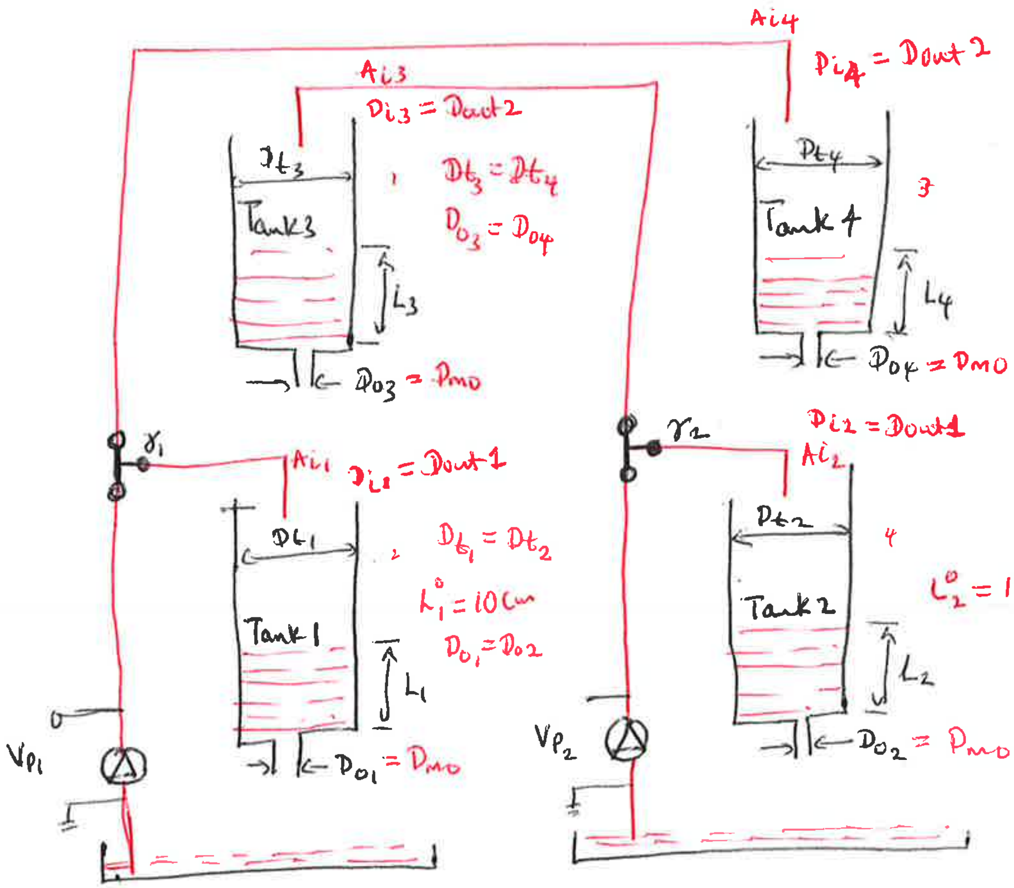
Quadruple Tank System



A_{t_i} – Cross section of Tank i

a_{o_i} – Cross-section of the outlet pipe for tank i

L_i – water level

A_i1 – Tank 1 inlet area

Equation of motion

$$\underbrace{A_{t_i}\frac{dL_i}{dt}}_{\text{Rate of change of volume in tank } i} = \underbrace{F_{in_i}}_{\text{Volumetric flow rate (in)}} - \overbrace{f_{out\,i}}^{\text{volumetric flow rate (out)}}$$

Flow velocity × cross-sectional area = Volumetric flow rate

$$v_i = \sqrt{2gL_i} \qquad A_{t_i} = \frac{\pi D_{t_i}^2}{4} \qquad v_i A_{t_i}$$

$$A_{t_1}\frac{dL_1}{dt} = a_{o3}\sqrt{2gL_3} + \gamma_1 k_{p_1} V_{p_1} - a_{o1}\sqrt{2gL_1}$$

$$A_{t_2}\frac{dL_2}{dt} = a_{o4}\sqrt{2gL_4} + \gamma_2 k_{p_2} V_{p_2} - a_{o2}\sqrt{2gL_2}$$

$$A_{t_3}\frac{dL_3}{dt} = (1-\gamma_2) k_{p_2} V_{p_2} - a_{o3}\sqrt{2gL_3}$$

$$A_{t_4}\frac{dL_4}{dt} = (1-\gamma_1) k_{p_1} V_{p_1} - a_{o4}\sqrt{2gL_4}$$

The control target is to Control the levels in the lower two tanks with the two pumps.

k_p — Pump flow constant

k_L — Tank water level sensor sensitivity

In short form:

$$\frac{dL_1}{dt} = -\frac{a_1}{A_1}\sqrt{2gL_1} + \frac{a_3}{A_1}\sqrt{2gL_3} + \frac{\gamma_1 K_{P_1} V_{P_1}}{A_1}$$

$$\frac{dL_2}{dt} = -\frac{a_2}{A_2}\sqrt{2gL_2} + \frac{a_4}{A_2}\sqrt{2gL_4} + \frac{\gamma_2 K_{P_2} V_{P_2}}{A_2}$$

$$\frac{dL_3}{dt} = -\frac{a_3}{A_3}\sqrt{2gL_3} + \frac{(1-\gamma_2) K_{P_2} V_{P_2}}{A_3}$$

$$\frac{dL_4}{dt} = -\frac{a_4}{A_4}\sqrt{2gL_4} + \frac{(1-\gamma_1) K_{P_1} V_{P_1}}{A_4}$$

Linearization

Define $x_i = L_i - l_i^0$ and $u_i = V_{P_i} - V_i^0$

Using truncated Taylor's Series expansion

$$\sqrt{2gl_i} \approx \sqrt{2gl_i^0} + \left.\frac{g}{\sqrt{2gl_i}}\right|_{i_i = i_i^0} (l_i - l_i^0) = \sqrt{2gl_i^0} + \frac{g}{\sqrt{2gl_i^0}} x_i$$

At equilibrium, we have of tank 1 and tank 2

$$-\frac{a_i}{A_i}\sqrt{2gl_i^0} + \frac{a_{i+2}}{A_i}\sqrt{2gl_{i+2}^0} + \frac{\gamma_i K_{P_i} V_{P_i}^0}{A_i} = 0$$

and for tank 3 and 4, we have,

$$-\frac{a_i}{A_i}\sqrt{2gl_i^0} + \frac{(1-\gamma_j) K_{P_j} V_{P_j}^0}{A_i} = 0 \quad \text{where } j=1 \text{ for tank 4 and } j=2 \text{ for tank 3}$$

we have

$$\dot{x}_1 = -\frac{a_1}{A_1}\sqrt{\frac{g}{2l_1^0}}\, x_1 + \frac{a_3}{A_1}\sqrt{\frac{g}{2l_3^0}}\, x_3 + \frac{\gamma_1 K_1 u_1}{A_1}$$

$$\dot{x}_2 = -\frac{a_2}{A_2}\sqrt{\frac{g}{2l_2^0}}\, x_2 + \frac{a_4}{A_2}\sqrt{\frac{g}{2l_4^0}}\, x_4 + \frac{\gamma_2 K_2 u_2}{A_2}$$

$$\dot{x}_3 = -\frac{a_3}{A_3}\sqrt{\frac{g}{2l_3^0}}\, x_3 + \frac{(1-\gamma_2) K_2 u_2}{A_3}$$

$$\dot{x}_4 = -\frac{a_4}{A_4}\sqrt{\frac{g}{2l_4^0}}\, x_4 + \frac{(1-\gamma_1) K_1 u_1}{A_4}$$

Equilibrium Conditions

$$\frac{dL_3}{dt} = \frac{-a_3}{A_3}\sqrt{2gL_3} + \frac{(1-\gamma_2)K_p V_{p_1}}{A_3}$$

at equilibrium

$$0 = \frac{-a_3}{A_3}\sqrt{2gL_3^0} + \frac{(1-\gamma_2)K_p V_{p_2}^0}{A_3}$$

$$a_3\sqrt{2gL_3^0} = (1-\gamma_2)K_p V_{p_2}^0 \Rightarrow V_{p_2}^0 = \frac{a_3\sqrt{2gL_3^0}}{(1-\gamma_2)K_p} \text{ or } L_3^0 = \left[\frac{(1-\gamma_2)K_p V_{p_2}^0}{a_3}\right]^2 \Big/ 2g \quad \text{—— (1)}$$

$$\Rightarrow a_3\sqrt{2gL_3^0} = (1-\gamma_2)K_p V_{p_2}^0$$

Also,

$$\frac{dl_4}{dt} = \frac{-a_4}{A_4}\sqrt{2gL_4} + \frac{(1-\gamma_1)K_p V_{p_1}}{A_4}$$

at equilibrium.

$$0 = -\frac{a_4}{A_4}\sqrt{2gL_4^0} + \frac{(1-\gamma_1)K_p V_{p_1}^0}{A_4}$$

$$V_{p_1}^0 = \frac{a_4\sqrt{2gL_4^0}}{(1-\gamma_1)K_p} \text{ or } \sqrt{2gL_4^0} = \frac{(1-\gamma_1)K_p V_{p_1}^0}{a_4} \Rightarrow L_4^0 = \left[\frac{(1-\gamma_1)K_p V_{p_1}^0}{a_4}\right]^2 \Big/ 2g \quad \text{—— (2)}$$

$$\Rightarrow a_4\sqrt{2gL_4^0} = (1-\gamma_1)K_p V_{p_1}^0$$

$$\frac{dL_1}{dt} = -\frac{a_1}{A_1}\sqrt{2gL_1} + \frac{a_3}{A_1}\sqrt{2gL_3} + \frac{\gamma_1 K_p V_{p_1}}{A_1}$$

At equilibrium

$$0 = -\frac{a_1}{A_1}\sqrt{2gL_1^0} + \frac{a_3}{A_1}\sqrt{2gL_3^0} + \frac{\gamma_1 K_p V_{p_1}^0}{A_1} \quad \text{—— (3)}$$

finally

$$\frac{dL_2}{dt} = \frac{-a_2}{A_2}\sqrt{2gL_2} + \frac{a_4}{A_2}\sqrt{2gL_4} + \frac{\gamma_2 K_{p_2} V_{p_2}}{A_2}$$

At equilibrium.

$$0 = -\frac{a_2}{A_2}\sqrt{2gL_2^0} + \frac{a_4}{A_2}\sqrt{2gL_4^0} + \frac{\gamma_2 K_p V_{p_2}^0}{A_2} \quad \text{—— (4)}$$

from ③ and ①, we have

$$0 = -\frac{a_1}{A_1}\sqrt{2gL_1^0} + \frac{(1-\gamma_2)K_p V_{p_2}^0}{A_1} + \frac{\gamma_1 K_p V_{p_1}^0}{A_1}$$

from ④ and ②, we have

$$0 = -\frac{a_2}{A_2}\sqrt{2gL_2^0} + \frac{(1-\gamma_1)K_p V_{p_1}^0}{A_2} + \frac{\gamma_2 K_p V_{p_2}^0}{A_2}$$

Two equations, two unknowns

$$\gamma_1 K_p V_{p_1}^0 + (1-\gamma_2)K_p V_{p_2}^0 = a_1\sqrt{2gL_1^0}$$

$$(1-\gamma_1)K_p V_{p_1}^0 + \gamma_2 K_p V_{p_2}^0 = a_2\sqrt{2gL_2^0}$$

$$V_{p_1}^0 + \frac{1-\gamma_2}{\gamma_1}V_{p_2}^0 = \frac{a_1}{\gamma_1 K_p}\sqrt{2gL_1^0}$$

$$V_{p_1}^0 + \frac{\gamma_2}{1-\gamma_1}V_{p_2}^0 = \frac{a_2}{(1-\gamma_1)K_p}\sqrt{2gL_2^0}$$

$$\left[\frac{1-\gamma_2}{\gamma_1} - \frac{\gamma_2}{1-\gamma_1}\right]V_{p_2}^0 = \frac{a_1}{\gamma_1 K_p}\sqrt{2gL_1^0} - \frac{a_2}{(1-\gamma_1)K_p}\sqrt{2gL_2^0}$$

$$\left[\frac{(1-\gamma_2)(1-\gamma_1) - \gamma_1\gamma_2}{\gamma_1(1-\gamma_1)}\right]V_{p_2}^0 = \frac{a_1}{\gamma_1 K_p}\sqrt{2gL^0} - \frac{a_2}{(1-\gamma_1)K_p}\sqrt{2gL_2^0}$$

or

$$\boxed{V_{p_2}^0 = \frac{a_1(1-\gamma_1)\sqrt{2gL_1^0}/K_p}{(1-\gamma_1)(1-\gamma_2)-\gamma_1\gamma_2} - \frac{a_2\gamma_1\sqrt{2gL_2^0}/K_p}{(1-\gamma_1)(1-\gamma_2)-\gamma_1\gamma_2}}$$

or

$$\boxed{V_{p_1}^0 = \frac{a_1}{\gamma_1 K_p}\sqrt{2gL_1^0} - \frac{1-\gamma_2}{\gamma_1}V_{p_2}^0}$$

$K_p = 3.3\ cm^3/s/V$

$K_L = 25/4.15\ cm/V$

$g = 981\ cm/s^2$

$$\alpha_1 = \frac{0.3167}{0.3167+0.1781} = 0.6401$$

$$\alpha_2 = \frac{0.3167}{0.3167+0.1781} = 0.6401$$

$$a_1 = \frac{\pi D_{o_1}^2}{4} = \frac{\pi D_{mo}^2}{4} = \frac{\pi(0.47625)^2}{4} = 0.1781\ cm^2$$

$$a_2 = \frac{\pi D_{o2}^2}{4} = \frac{\pi D_{mo}^2}{4} = 0.1781\ cm^2$$

$$a_3 = \frac{\pi D_{o3}^2}{4}$$

$$a_4 = \frac{\pi D_{o4}^2}{4}$$

Then L_3^0 and L_4^0 are recovered from (1) and (2) as:

$$L_3^0 = \left[\frac{(1-\gamma_2)K_p V_{p_2}^0}{a_3}\right]^2 \Big/ 2g \quad \text{and} \quad L_4^0 = \left[\frac{(1-\gamma_1)K_p V_{p_1}^0}{a_4}\right]^2 \Big/ 2g$$

$$A_{i3} = \frac{\pi D_{i3}^2}{4} = \frac{\pi D_{out_2}^2}{4} = 0.1781\ cm^2$$

$$A_{i4} = \frac{\pi D_{i4}^2}{4} = \frac{\pi D_{out_2}^2}{4} = 0.1781\ cm^2$$

$$\gamma_1 = \frac{A_{i1}}{A_{i1} + A_{i4}}$$

$$\gamma_2 = \frac{A_{i2}}{A_{i2} + A_{i3}}$$

A_{i1} – Tank 1 inlet Area (cm^2) $= \frac{\pi D_{i1}^2}{4} = \frac{\pi D_{out1}^2}{4} = 0.3167\ cm^2$

A_{i2} – Tank 2 inlet Area (cm^2) $= \frac{\pi D_{i2}^2}{4} = \frac{\pi D_{out1}^2}{4} = 0.3167\ cm^2$

D_{out1} – Out 1" Orifice Diameter (cm) $= 0.635$ cm

D_{out2} – Out 2" Orifice Diameter (cm) $= 0.47625$ cm

In ~~Compact~~ form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_1} & 0 & \frac{A_3}{A_1 T_3} & 0 \\ 0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2 T_4} \\ 0 & 0 & -\frac{1}{T_3} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} \frac{\gamma_1 K_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 K_2}{A_2} \\ 0 & \frac{(1-\gamma_2)K_2}{A_3} \\ \frac{(1-\gamma_1)K_1}{A_4} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} k_{L_1} & 0 & 0 & 0 \\ 0 & k_{L_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

where

k_{L_i} - Tank i water level sensor sensitivity.

$$T_i = \frac{A_i}{a_i}\sqrt{\frac{2 i_i^0}{g}} \; ; \; i = 1, \dots 4.$$

In transfer function

$$sX_1 = -\frac{1}{T_1}X_1 + \frac{A_3}{A_1 T_3}X_3 + \frac{\gamma_1 K_1}{A_1}U_1 \Rightarrow X_1 = \frac{A_3/A_1 T_3}{s+\frac{1}{T_1}}X_3 + \frac{\gamma_1 K_1/A_1}{s+\frac{1}{T_1}}U_1$$

$$sX_2 = -\frac{1}{T_2}X_2 + \frac{A_4}{A_2 T_4}X_4 + \frac{\gamma_2 k_2}{A_2}U_2 \qquad X_2 = \frac{A_4/A_2 T_4}{s+\frac{1}{T_2}}X_4 + \frac{\gamma_2 K_2/A_2}{s+\frac{1}{T_2}}U_2$$

$$sX_3 = -\frac{1}{T_3}X_3 + \frac{(1-\gamma_2)K_2}{A_3}U_2 \Rightarrow X_3 = \frac{(1-\gamma_2)K_2/A_3}{s+\frac{1}{T_3}}U_2$$

$$sX_4 = -\frac{1}{T_4}X_4 + \frac{(1-\gamma_1)K_1 U_1}{A_4} \Rightarrow X_4 = \frac{(1-\gamma_1)K_1/A_4}{s+\frac{1}{T_4}}U_1$$

$$\Downarrow$$

$$X_1 = \frac{(1-\gamma_2)K_2/A_1 T_3}{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_3}\right)}U_2 + \frac{\gamma_1 K_1/A_1}{s+\frac{1}{T_1}}U_1$$

$$X_2 = \frac{(1-\gamma_1)K_1/A_2 T_4}{\left(s+\frac{1}{T_2}\right)\left(s+\frac{1}{T_4}\right)}U_1 + \frac{\gamma_2 K_2/A_2}{s+\frac{1}{T_2}}U_2$$

This yields

$$Y_1(s) = \frac{\gamma_1 K_1 K_{L_1} T_1/A_1}{T_1 s + 1} U_1(s) + \frac{(1-\gamma_2) K_2 K_{L_1} T_1/A_1}{(T_1 s+1)(T_3 s+1)} U_2(s)$$

$$Y_2(s) = \frac{(1-\gamma_1) K_1 K_{L_2} T_2/A_2}{(T_2 s+1)(T_4 s+1)} U_1(s) + \frac{\gamma_2 K_2 K_{L_2} T_2/A_2}{T_2 s + 1} U_2(s)$$

or

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{\gamma_1 c_1}{T_1 s+1} & \frac{(1-\gamma_2) c_2}{(T_1 s+1)(T_3 s+1)} \\ \frac{(1-\gamma_1) c_3}{(T_2 s+1)(T_4 s+1)} & \frac{\gamma_2 c_4}{T_2 s+1} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

where $c_1 = K_1 K_{L_1} T_1/A_1$, $c_2 = K_2 K_{L_1} T_1/A_1$
$c_3 = K_1 K_{L_2} T_2/A_2$, $c_4 = K_2 K_{L_2} T_2/A_2$

Note that if $K_1 = K_2$ and $K_{L_1} = K_{L_2}$, then $c_1 = c_2$ and $c_3 = c_4$.

Time Constants

$$T_1 = \frac{A_1}{a_1}\sqrt{\frac{2L_1^0}{g}}$$ where $A_1 = \frac{\pi D_{t_1}^2}{4} = \frac{\pi \times (4.445)^2}{4} cm^2 = 15.518 cm^2$

$L_1^0 = 10 cm$

$a_1 = \frac{\pi D_{MO}^2}{4} = \frac{\pi \times (0.47625)^2}{4} cm^2 = 0.178 cm^2$

$$= \frac{15.518}{0.178}\sqrt{\frac{2\times 10}{981}} = \text{[illegible]} \; 12.448 \text{ sec.}$$

$L_1^0 = 10$ cm

$g = 981 \; cm/s^2$

$$T_2 = \frac{A_2}{a_2}\sqrt{\frac{2L_2^0}{g}}$$ where $A_2 = \frac{\pi D_{t_2}^2}{4} = \frac{\pi \times (4.445)^2}{4} cm^2 = 15.518 cm^2$

$L_2^0 = 14 cm$

$a_2 = \frac{\pi D_{MO}^2}{4} = \frac{\pi \times (0.47625)^2}{4} cm^2 = 0.178 cm^2$

$$= \frac{15.518}{0.178}\sqrt{\frac{2\times 15}{981}} = \text{[illegible]} \; 14.729 \text{ sec.}$$

$L_2^0 = 14$ cm

Here

$$L_3^0 = \left[\frac{(1-\gamma_2)\, a_1(1-\gamma_1)\sqrt{2gL_1^0}}{a_3[(1-\gamma_1)(1-\gamma_2)-\gamma_1\gamma_2]} - \frac{(1-\gamma_2)\, a_2\gamma_1\sqrt{2gL_2^0}}{a_3[(1-\gamma_1)(1-\gamma_2)-\gamma_1\gamma_2]}\right]^2 \Big/ 2g$$

$$1-\gamma_1-\gamma_2+\cancel{\gamma_1\gamma_2}-\cancel{\gamma_1\gamma_2}$$

$$L_3^2 = \left[\frac{a_1(1-\gamma_1-\gamma_2+\gamma_1\gamma_2)\sqrt{2gL_1^0} - a_2(\gamma_1-\gamma_1\gamma_2)\sqrt{2gL_2^0}}{a_3[1-\gamma_1-\gamma_2]}\right]^2 \Big/ 2g$$

Setting $L_{10} = 10$ cm and $L_{20} = 14$ cm, we computed L_{30} and L_{40} as

$$L_{30} = 2.6088 \text{ cm}$$

$$L_{40} = ~~[illegible]~~\ 0.7573 \text{ cm}$$

Also

$$V_{p10} = ~~[illegible]~~\ 5.7801 \text{ V}$$

$$V_{p20} = 10.7278$$

The open-loop transfer function is given by

$$G(s) = \begin{bmatrix} \dfrac{\gamma_1 c_1}{T_1 s+1} & \dfrac{(1-\gamma_2)c_2}{(T_1 s+1)(T_3 s+1)} \\[2ex] \dfrac{(1-\gamma_1)c_3}{(T_2 s+1)(T_4 s+1)} & \dfrac{\gamma_2 c_4}{T_2 s+1} \end{bmatrix}$$

where

$$\gamma_1 = \frac{A_{i1}}{A_{i1}+A_{i4}}$$

$$\gamma_2 = \frac{A_{i2}}{A_{i2}+A_{i3}}$$

$$c_1 = K_1 K_{L_1} T_1/A_1, \quad c_2 = K_2 K_{L_1} T_1/A_1 \quad c_3 = K_1 K_{L_2} T_2/A_2$$

$$c_4 = K_2 K_{L_2} T_2/A_2$$

$$T_1 = \frac{A_1}{a_1}\sqrt{\frac{2L_1^0}{g}} \quad T_2 = \frac{A_2}{a_2}\sqrt{\frac{2L_2^0}{g}} \quad T_2 = \frac{A_3}{a_3}\sqrt{\frac{2L_3^0}{g}} \text{ and } T_4 = \frac{A_4}{a_4}\sqrt{\frac{2L_4^0}{g}}$$

with $T_1 = 12.4381$ sec $\quad c_1 = 15.9341 \quad \gamma_1 = 0.64$

$T_2 = 14.7170$ sec $\quad c_2 = 15.9341 \quad \gamma_2 = 0.64$

$T_3 = 6.3529$ sec $\quad c_3 = 18.8534$

$T_4 = 3.4224$ sec $\quad c_4 = 18.8534$

which gives

$$G(s) = \begin{bmatrix} \dfrac{10.2}{12.44s+1} & \dfrac{5.736}{(12.44s+1)(6.35s+1)} \\ \dfrac{6.787}{(14.72s+1)(3.042s+1)} & \dfrac{12.07}{14.72s+1} \end{bmatrix}$$

Internal model Control

$$G(s)^{-1} = \begin{bmatrix} G(2,2) & -G(1,2) \\ -G(2,1) & G(1,1) \end{bmatrix} \Big/ Det(G)$$

$$Det(G) = \frac{10.2 \times 12.07}{(12.44s+1)(14.72s+1)} - \frac{6.787^2}{(12.44s+1)(6.35s+1)(14.72s+1)(12.76s+1)}$$

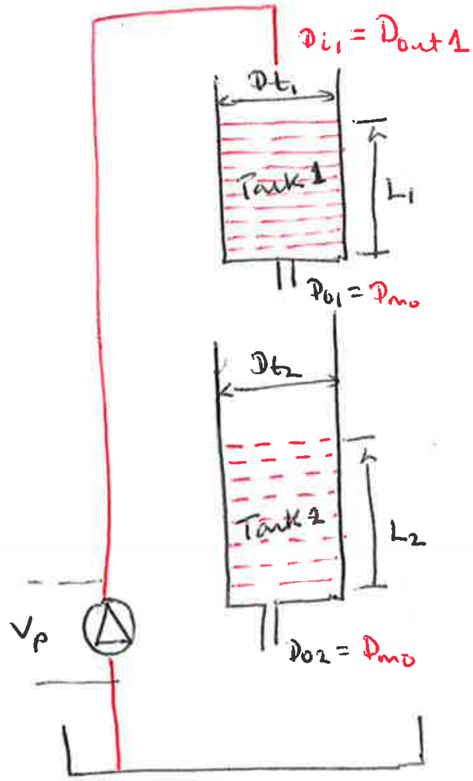
$$= \frac{123.114(6.35s+1)(12.76s+1) - 46.0634}{(12.44s+1)(14.72s+1)(6.35s+1)(12.76s+1)}$$

$$G(s)^{-1} = \begin{bmatrix} \dfrac{12.07}{14.72s+1} & \dfrac{-6.787}{(12.44s+1)(6.35s+1)} \\ \dfrac{-6.787}{(14.72s+1)(12.76s+1)} & \dfrac{10.2}{12.44s+1} \end{bmatrix} \Big/ \frac{123.114(6.35s+1)(12.76s+1) - 46.0634}{(12.44s+1)(14.72s+1)(6.35s+1)(12.76s+1)}$$

$$G(s)^{-1} = \frac{\begin{bmatrix} (12.44s+1)(6.35s+1)(12.76s+1)(12.07) & -6.787(14.72s+1)(12.76s+1) \\ -6.787(12.44s+1)(6.35s+1) & 10.2(14.72s+1)(6.35s+1)(12.76s+1) \end{bmatrix}}{123.114(6.35s+1)(12.76s+1) - 46.0634}$$

$$9975s^2 + 2353s + 77.05$$

Coupled Tank System
(Configuration #2)



A controller is designed to regulate or track the level in Tank 2.

The pump feeds into Tank 1 which in turn feeds into Tank 2.

Equation of motion

$$A_{t_2} \frac{dL_2}{dt} = a_{01}\sqrt{2gL_1} - a_{02}\sqrt{2gL_2}$$

Linearized

Define $x_1 = L_1 - L_1^0$ and $x_2 = L_2 - L_2^0$

and using truncated Taylor's series expansion

$$\sqrt{2gL_i} \approx \sqrt{2gL_i^0} + \frac{g}{\sqrt{2gL_i}}\Bigg|_{L_i = L_i^0} (L_i - L_i^0) = \sqrt{2gL_i^0} + \frac{g}{\sqrt{2gL_i^0}} x_i$$

Hence

$$A_2 \frac{d}{dt}(x_2 + L_2^0) = a_1\sqrt{2gL_1^0} + \frac{a_1 g}{\sqrt{2gL_1^0}} x_1 - a_2\sqrt{2gL_2^0} - \frac{a_2 g}{\sqrt{2gL_2^0}} x_2$$

or

$$A_2 \frac{dx_2}{dt} = a_1\sqrt{\frac{g}{2L_1^0}}\, x_1 - a_2\sqrt{\frac{g}{2L_2^0}}\, x_2 \Rightarrow \frac{dx_2}{dt} = \frac{a_1}{A_2}\sqrt{\frac{g}{2L_1^0}}\, x_1 - \frac{a_2}{A_2}\sqrt{\frac{g}{2L_2^0}}\, x_2$$

Let $T_2 = \frac{A_2}{a_2}\sqrt{\frac{2L_2^0}{g}}$ and $T_1 = \frac{A_1}{a_1}\sqrt{\frac{2L_1^0}{g}}$

Hence we have

$$\dot{x}_2 = -\frac{1}{T_2}x_2 + \frac{A_1}{A_2 T_1}x_1$$

Taking Laplace transforms yields

$$sX_2(s) = -\frac{1}{T_2}X_2(s) + \frac{A_1}{A_2 T_1}X_1(s)$$

$$\left(s + \frac{1}{T_2}\right)X_2(s) = \frac{A_1}{A_2 T_1}X_1(s)$$

$$\frac{X_2(s)}{X_1(s)} = \frac{\frac{A_1}{A_2 T_1}}{s + \frac{1}{T_2}} \text{ or } \frac{\frac{A_1}{A_2} \times \frac{T_2}{T_1}}{T_2 s + 1} = \frac{K_{dc-2}}{T_2 s + 1}$$

$$K_{dc-2} = \frac{\cancel{A_1}}{\cancel{A_2}} \times \left(\frac{\cancel{A_2}}{a_2}\sqrt{\frac{2L_2^0}{g}}\right) \times \left(\frac{a_1}{\cancel{A_1}}\sqrt{\frac{g}{2L_1^0}}\right) = \frac{a_1}{a_2}\sqrt{\frac{L_2^0}{L_1^0}}$$

Note: At equilibrium

$$~~0 = \frac{a_1}{A_2}\sqrt{\frac{g}{2L_1^0}}~~ \qquad A_2\frac{dL_2^0}{dt} = a_1\sqrt{2gL_1^0} - a_2\sqrt{2gL_2^0}$$

or $$a_2\sqrt{2gL_2^0} = a_1\sqrt{2gL_1^0}$$

and $$L_1^0 = \left(\frac{a_2}{a_1}\right)^2 L_2^0 = \left(\frac{0.1781\ cm^2}{0.1481\ cm^2}\right)^2 \times 15\ cm = 15\ cm$$

$$K_{dc-2} = \left(\frac{a_1}{a_2}\right)\sqrt{\frac{L_2^0}{L_1^0}} = \left(\frac{0.1781\ cm^2}{0.1781\ cm^2}\right) \times \sqrt{\frac{15\ cm}{15\ cm}} = 1$$

$$T_2 = \frac{A_2}{a_2}\sqrt{\frac{2L_2^0}{g}} = \left(\frac{15.5179\ cm^2}{0.1781\ cm^2}\right)\sqrt{\frac{2 \times 15\ cm}{981\ cm/s^2}} = 15.2335\ s$$

Hence

$$\frac{X_2(s)}{X_1(s)} = \frac{1}{15.2\,s+1}$$

Now we derive the dynamics of tank 1 with respect to the tank level and the input voltage.

Equation of motion

$$A_{t_1}\frac{dL_1}{dt} = K_p V_p - a_{01}\sqrt{2gL_1}$$

Linearized

$$A_1 \frac{d(x_1+L_1^0)}{dt} = K_p(u+V_p^0) - a_1\sqrt{2gL_1^0} - \frac{a_1 g}{\sqrt{2gL_1^0}}x_1$$

or

$$A_1\frac{dx_1}{dt} = K_p u + K_p V_p^0 - a_1\sqrt{2gL_1^0} - a_1\sqrt{\frac{g}{2L_1^0}}\,x_1$$

$$\Rightarrow \quad A_1\frac{dx_1}{dt} = K_p u - a_1\sqrt{\frac{g}{2L_1^0}}\,x_1$$

or

$$\frac{dx_1}{dt} = -\frac{a_1}{A_1}\sqrt{\frac{g}{2L_1^0}}\,x_1 + \frac{K_p}{A_1}u$$

or

$$\dot{x}_1 = -\frac{1}{T_1}x_1 + \frac{K_p}{A_1}u$$

$$sX_1(s) = -\frac{1}{T_1}X_1(s) + \frac{K_p}{A_1}U(s)$$

or $$\frac{X_1(s)}{U(s)} = \frac{K_p/A_1}{s+\frac{1}{T_1}} = \frac{K_pT_1/A_1}{T_1s+1} =$$

or

$$\frac{X_1(s)}{U(s)} = \frac{K_{dc-1}}{T_1 s + 1}$$

$$T_1 = \frac{A_1}{a_1}\sqrt{\frac{2L_1^0}{g}} = \left(\frac{15.5179\ cm^2}{0.1781\ cm^2}\right) \times \sqrt{\frac{2 \times 15\ cm}{981\ cm/s^2}} = 15.2335\ s.$$

$$K_{dc-1} = \frac{K_p \times T_1}{A_1} = \frac{3.3\ cm^3/s/V \times 15.2335\ s}{15.5179\ cm^2} = 3.2395\ cm/V$$

Put together

Tank 2 dynamics   Tank 1 dynamics

$$\frac{X_2(s)}{U(s)} = \boxed{\frac{X_2(s)}{X_1(s)}} \times \boxed{\frac{X_1(s)}{U(s)}} = \frac{K_{dc-2}}{T_2 s + 1} \times \frac{K_{dc-1}}{T_1 s + 1} = \frac{K_{dc}}{(T_1 s + 1)(T_2 s + 1)}$$

$$= \frac{3.2395}{(15.23 s + 1)(15.23 s + 1)}$$

We can develop a feedback controller directly for the above transfer-function. However, in the quanser workbook, a controller is first developed for ~~the~~ Tank 1 such that ~~at~~ steady-state the ~~height of~~ level in tank 1 is about the equilibrium level of 15cm. ~~and then a second~~ This is the inner loop controller. Then a Second controller is designed to regulate the level of tank 2. In this case the inner loop controller must be such that ~~the~~ tank 1 dynamics is much faster than that of tank 2.

Following the above Strategy, I develop an IMC-based Control as follows.

For tank 1.

we have $G_1(s) = \frac{K_{dc-1}}{T_1 s + 1} = \frac{X_1(s)}{U(s)}$

The internal model controller is obtained by

Inverting the plant.

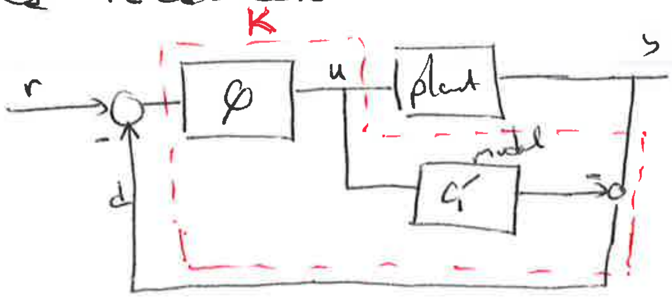
Hence $\phi_0 = G^{-1} = \dfrac{T_1 s + 1}{K_{dc-1}} = \dfrac{15.2 s + 1}{3.24}$ ← This is improper

Augmenting with a first order filter $f(s) = \dfrac{1}{\lambda s + 1}$

we have

$$\phi = \phi_0 f = \frac{15.2 s + 1}{3.24} \times \frac{1}{\lambda s + 1} = \frac{15.2 s + 1}{3.24(\lambda s + 1)}$$

Now using the connection between internal model controller and feedback controller:



$u = \phi(r - d)$ where $d = y - Gu$

$= \phi(r + Gu - y)$

$= \phi r + \phi G u - \phi y$

$= \phi(r - y) + \phi G u$

or $u - \phi G u = \phi(r - y)$

$(1 - \phi G) u = \phi(r - y)$

or $u = \dfrac{\phi}{1 - \phi G}(r - y)$ ⟺ This exactly like your feedback controller where $u = K(r - y)$ ↑ error.

Hence

$$K(s) = \frac{\phi(s)}{1 - \phi G(s)}$$

But $\phi G = \dfrac{\cancel{15.2s+1}}{\cancel{3.24}(\lambda s + 1)} \times \dfrac{\cancel{3.24}}{\cancel{15.2s+1}} = \dfrac{1}{\lambda s + 1}$

$1 - \phi G = 1 - \dfrac{1}{\lambda s + 1} = \dfrac{\lambda s}{\lambda s + 1}$

Hence

$$K(s) = \frac{\phi(s)}{1 - \phi G(s)} = \frac{15.2 s + 1}{3.24(\cancel{\lambda s + 1})} \times \frac{\cancel{\lambda s + 1}}{\lambda s} = \frac{15.2 s + 1}{3.24 \lambda s}$$

$= \dfrac{15.2}{3.24\lambda} + \dfrac{1}{3.24\lambda} \dfrac{1}{s}$ ← This is your PI controller where $K_p = \dfrac{15.2}{3.24\lambda}$ and $K_I = \dfrac{1}{3.24\lambda}$

Here λ becomes your tuning parameter (in fact the only tuning parameter). Just choose λ, and your have your PI gains.

Following similar procedure, the controller for tank 2 is developed as follows:

$$G_2(s) = \frac{K_{dc-2}}{T_2 s + 1} = \frac{1}{15.2s+1} = \frac{X_2(s)}{X_1(s)}$$

and $$Q_o(s) = G_2^{-1}(s) = \frac{15.2s+1}{1}$$ — This is improper

$$Q(s) = Q_o(s)\, f(s) = \frac{15.2s+1}{\lambda_2 s + 1}$$

and then

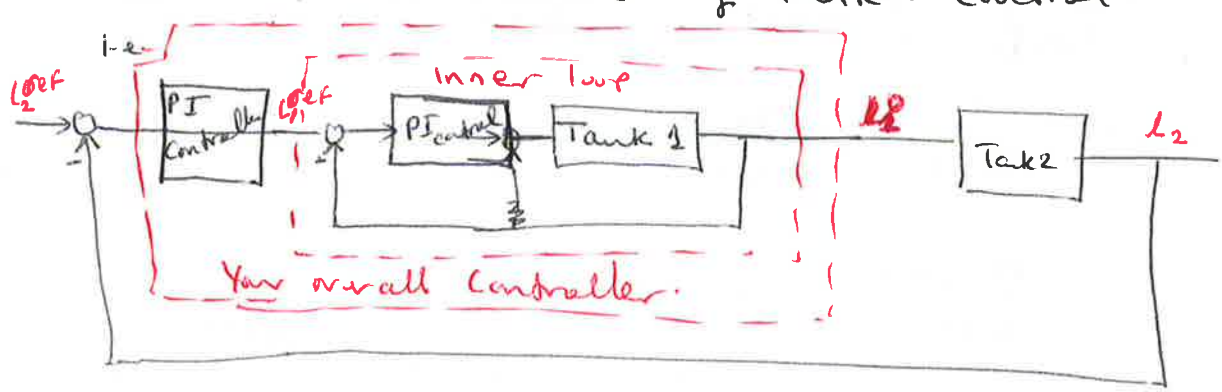
$$K(s) = \frac{Q(s)}{1 - G(s)\,Q(s)} = \frac{15.2s+1}{\lambda_2 s+1} * \frac{\lambda_2 s+1}{\lambda_2 s} = \frac{15.2s+1}{\lambda_2 s}$$

$$= \frac{15.2}{\lambda_2} + \frac{1}{\lambda_2 s} \qquad \text{Here } K_{P_2} = \frac{15.2}{\lambda_2} \text{ and } K_{I_2} = \frac{1}{\lambda_2}.$$

So again, choose any ~~the~~ λ_2, and you have your PI gains.

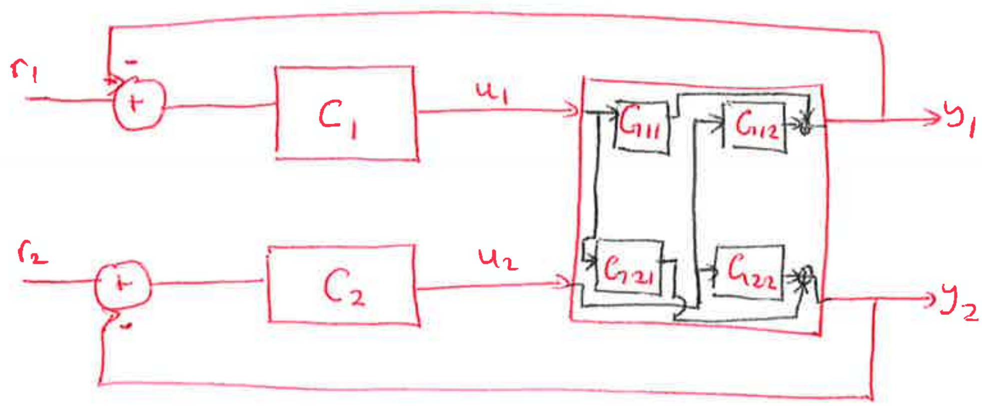
NOTE!: To implement the controller for tank 2, you must embed the controller for tank 1 into your overall control implementation. Note that the set-point for tank 2 control is determined by tank 1 control.

i.e.



Decentralized Control Design for Coupled-Tank Sys

$$G = \begin{bmatrix} \dfrac{10.2}{12.44s+1} & \dfrac{5.736}{(12.44s+1)(6.35s+1)} \\[2ex] \dfrac{6.787}{(14.72s+1)(3.42s+1)} & \dfrac{12.07}{14.72s+1} \end{bmatrix}$$



where $C_i = K_i\left(1 + \frac{1}{T_{ci}s}\right)$ $i=1,2$ or $C_i = K_{pi} + \frac{K_{Ii}}{s}$

We design C_1 based on $G_{11}(s)$ using the internal model ~~Principle~~ control techniques.

$$\varnothing_{01} = G_{11}^{-1}(s) = \frac{12.44s+1}{10.2}$$

Augumenting with a first order filter, we have

$$\varnothing_1 = G_{11}^{-1}(s)\,f(s) = \frac{12.44s+1}{10.2} \times \frac{1}{\lambda_1 s+1} = \frac{12.44s+1}{10.2\lambda_1 s + 10.2}$$

But $C_1 = \dfrac{\varnothing_1}{1-\varnothing_1 G_{11}}$ where $\varnothing_1 G_{11} = \dfrac{1}{\lambda_1 s+1}$ and $1-\varnothing G = \dfrac{\lambda_1 s}{\lambda_1 s+1}$

Hence

$$C_1 = \frac{12.44s+1}{10.2\,\cancel{(\lambda_1 s+1)}} \times \frac{\cancel{\lambda_1 s+1}}{\lambda_1 s} = \frac{12.44s+1}{10.2\lambda_1 s} = \frac{1.2197}{\lambda_1} + \frac{0.0981}{\lambda_1 s}$$

Following similar techniques, we develop C_2 as follows:

$$\phi_{02} = G_{22}^{-1}(s) = \frac{14.72s+1}{12.07}$$

Augumenting with a first-order filter, we have

$$\phi_2 = \phi_{02} f_2 = \frac{14.72s+1}{12.07} \times \frac{1}{\lambda_2 s+1} = \frac{14.72s+1}{12.07(\lambda_2 s+1)}$$

and

$$C_2 = \frac{\phi_2}{1-\phi_2 G_{22}} = \frac{14.72s+1}{12.07\cancel{(\lambda_2 s+1)}} \times \frac{\cancel{\lambda_2 s+1}}{\lambda_2 s} = \frac{14.72s+1}{12.07\lambda_2 s} = \frac{1.2197}{\lambda_2} + \frac{0.0829}{\lambda_2 s}$$

Decoupling Controller Based on Internal Model Control

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \begin{bmatrix} \dfrac{10.2}{12.44s+1} & \dfrac{5.736}{(12.44s+1)(6.35s+1)} \\ \dfrac{6.787}{(14.72s+1)(3.42s+1)} & \dfrac{12.07}{14.72s+1} \end{bmatrix}$$

$$G^{-1}(s) = \begin{bmatrix} G_{22}(s) & -G(1,2) \\ -G(2,1) & G(1,1) \end{bmatrix} \Big/ \det(G(s))$$

$$\det G(s) = \left(\frac{10.2}{12.44s+1}\right)\left(\frac{12.07}{14.72s+1}\right) - \left(\frac{6.787}{(14.72s+1)(3.42s+1)}\right)\left(\frac{5.736}{(12.44s+1)(6.35s+1)}\right)$$

$$= \frac{(10.2)(12.07)(3.42s+1)(6.35s+1) - (6.787)(5.736)}{(12.44s+1)(14.72s+1)(3.42s+1)(6.35s+1)}$$

$$= \frac{2674s^2 + 1203s + 84.18}{(12.44s+1)(14.72s+1)(3.42s+1)(6.35s+1)}$$

$$= \frac{84.2028(2.7533s+1)(11.534s+1)}{(12.44s+1)(14.72s+1)(3.42s+1)(6.35s+1)}$$

$$G^{-1}(s) = \begin{bmatrix} \dfrac{12.07}{14.72s+1} & -\dfrac{5.736}{(12.44s+1)(6.35s+1)} \\ -\dfrac{6.787}{(14.72s+1)(3.42s+1)} & \dfrac{10.2}{12.44s+1} \end{bmatrix} * \frac{(12.44s+1)(14.72s+1)(3.42s+1)(6.35s+1)}{84.18(2.7533s+1)(11.534s+1)}$$

$$= \frac{0.0119}{(2.7533s+1)(11.534s+1)} \begin{bmatrix} 12.07(12.44s+1)(3.42s+1)(6.35s+1) & -5.736(14.72s+1)(3.42s+1) \\ -6.787(12.44s+1)(6.35s+1) & 10.2(14.72s+1)(3.42s+1)(6.35s+1) \end{bmatrix}$$

$$G^{-1}(s) = \begin{bmatrix} \dfrac{0.1436(12.44s+1)(3.42s+1)(6.35s+1)}{(2.7533s+1)(11.534s+1)} & -\dfrac{0.0683(14.72s+1)(3.42s+1)}{(2.7533s+1)(11.534s+1)} \\[2em] -\dfrac{0.0808(12.44s+1)(6.35s+1)}{(2.7533s+1)(11.534s+1)} & \dfrac{0.1214(14.72s+1)(3.42s+1)(6.35s+1)}{(2.7533s+1)(11.534s+1)} \end{bmatrix}$$

To make φ bi-proper, we choose φ as

$$\varphi = G^{-1}(s)\, f(s) \quad \text{where} \quad f(s) = \begin{bmatrix} \dfrac{1}{\lambda_1 s+1} & 0 \\ 0 & \dfrac{1}{\lambda_2 s+1} \end{bmatrix}$$

$$\varphi = \begin{bmatrix} \dfrac{0.1436(12.44s+1)(3.42s+1)(6.35s+1)}{(\lambda_1 s+1)(2.7533s+1)(11.534s+1)} & -\dfrac{0.0683(14.72s+1)(3.42s+1)}{(\lambda_2 s+1)(2.7533s+1)(11.534s+1)} \\[2em] -\dfrac{0.0808(12.44s+1)(6.35s+1)}{(\lambda_1 s+1)(2.7533s+1)(11.534s+1)} & \dfrac{0.1214(14.72s+1)(3.42s+1)(6.35s+1)}{(\lambda_2 s+1)(2.7533s+1)(11.534s+1)} \end{bmatrix}$$

Using the connection between internal model controller φ and ~~the~~ feedback controller $K(s)$ which is

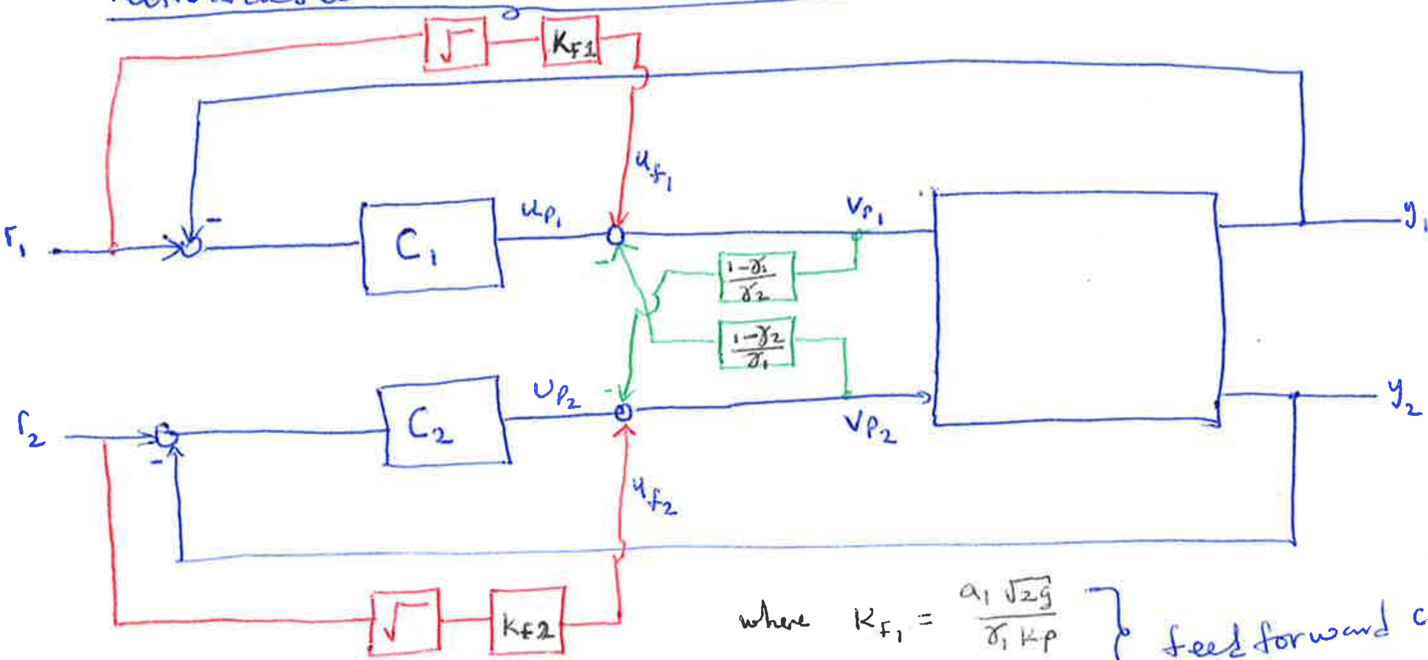
$$K(s) = \varphi(s)(1-G\varphi)^{-1} = (1-\varphi G)^{-1}\varphi$$

with
$$1-G\varphi = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} \dfrac{1}{\lambda_1 s+1} & 0 \\ 0 & \dfrac{1}{\lambda_2 s+1} \end{bmatrix} = \begin{bmatrix} \dfrac{\lambda_1 s}{\lambda_1 s+1} & 0 \\ 0 & \dfrac{\lambda_2 s}{\lambda_2 s+1} \end{bmatrix}$$

and
$$(1-G\varphi)^{-1} = \begin{bmatrix} \dfrac{\lambda_1 s+1}{\lambda_1 s} & 0 \\ 0 & \dfrac{\lambda_2 s+1}{\lambda_2 s} \end{bmatrix}$$

Hence
$$K(s) = \varphi(s)(1-G\varphi)^{-1} = \begin{bmatrix} \dfrac{0.1436(12.44s+1)(3.42s+1)(6.35s+1)}{\lambda_1 s(2.7533s+1)(11.534s+1)} & -\dfrac{0.0683(14.72s+1)(3.42s+1)}{\lambda_2 s(2.7533s+1)(11.534s+1)} \\[2em] -\dfrac{0.0808(12.44s+1)(6.35s+1)}{\lambda_1 s(2.7533s+1)(11.534s+1)} & \dfrac{0.1214(14.72s+1)(3.42s+1)(6.35s+1)}{\lambda_2 s(2.7533s+1)(11.534s+1)} \end{bmatrix}$$

Multivarable PI Control with Feedback control.



where $K_{f_1} = \frac{a_1\sqrt{2g}}{\gamma_1 K_p}$, $K_{f_2} = \frac{a_2\sqrt{2g}}{\gamma_2 K_p}$ } feedforward control gains

where $C_1 = K_1\left(1+\frac{1}{\tau_{c_1}s}\right)$, $C_2 = K_2\left(1+\frac{1}{\tau_{c_2}s}\right)$ } feedback PI Controllers

Recall from the equilibrium analysis of the open-loop nonlinear plant that

$$\boxed{v_{p_1}^0 = \frac{a_1}{\gamma_1 K_p}\sqrt{2gL_1^0} - \left(\frac{1-\gamma_2}{\gamma_1}\right)v_{p_2}^0} \quad \text{and} \quad v_{p_2}^0 = \frac{a_3}{(1-\gamma_2)K_p}\sqrt{2gL_3^0}$$

$$\boxed{v_{p_2}^0 = \frac{a_2}{\gamma_2 K_p}\sqrt{2gL_2^0} - \left(\frac{1-\gamma_1}{\gamma_2}\right)v_{p_1}^0} \quad\quad v_{p_1}^0 = \frac{a_4}{(1-\gamma_1)K_p}\sqrt{2gL_4^0}$$

From the control configuration above, equilibrium condition implies

$$r_1^0 - y_1^0 = 0 \Rightarrow u_{p_1}^0 = 0 \Rightarrow v_{p_1}^0 = u_{f_1}^0 - \frac{1-\gamma_2}{\gamma_1}v_{p_2}^0$$

$$r_2^0 - y_2^0 = 0 \Rightarrow u_{p_2}^0 = 0 \Rightarrow v_{p_2}^0 = u_{f_2}^0 - \frac{1-\gamma_1}{\gamma_2}v_{p_1}^0$$

Substituting into the equilibrium condition of the open-loop plant gives

$$\boxed{v_{p_1}^0 = K_{f_1}\sqrt{L_1^0} - \frac{1-\gamma_2}{\gamma_1}v_{p_2}^0} \quad \text{and} \quad \boxed{v_{p_2}^0 = K_{f_2}\sqrt{L_2^0} - \frac{1-\gamma_1}{\gamma_2}v_{p_1}^0}$$

Comparing the two equilibria conditions gives

$$K_{f_1} = \frac{a_1}{\gamma_1 K_p}\sqrt{2g} \quad \text{and}$$

$$K_{f_2} = \frac{a_2}{\gamma_2 K_p}\sqrt{2g}$$

Decentralised PI Control Design

We already designed C_1 and C_2 based on the internal model control design technique. We used the linearized plant and ignored the couplings.

Centralised Steady-State decoupler

To design a centralised controller which decouples the linearized plant at steady state, we ~~adopt~~ follow the ~~following~~ design technique below:

$$C = \begin{bmatrix} C_1 & \\ & C_2 \end{bmatrix} K_s^{-1}$$

where K_s is the steady-state gain of the linearized plant.

i.e. $$K_s = \lim_{s \to 0} G(s) = -CA^{-1}B$$

Model Predictive Control formulation for Quadruple Tank System

①

1. Linearized plant model (Discrete)

$$x(t+1) = \overset{n\times n}{A}\, x(t) + \overset{n\times m}{B}\, u(t)$$
$$y(t) = \overset{m\times n}{C}\, x(t)$$

2. The observer — To capture the mismatch between the actual plant and the linearized plant in steady state.

first augment the linear plant with internal model of the disturbance — constant disturbance

$$\overset{n\times 1}{x}(t+1) = \overset{n\times n}{A}\, x(t) + \overset{n\times m}{B}\, u(t)$$

$$\overset{m\times 1}{d}(t+1) = \overset{m\times 1}{d}(t)$$ — integrator — Adding integral action.

$$\overset{m\times 1}{y}(t) = \overset{m\times n}{C}\, \overset{n\times 1}{x}(t) + \overset{m\times 1}{d}(t)$$ — Disturbance assumed to be at the plant output.

The augmented system becomes

$$\begin{bmatrix} x(t+1) \\ d(t+1) \end{bmatrix} = \begin{bmatrix} \overset{n\times n}{A} & \overset{n\times m}{0} \\ \overset{m\times n}{0} & \overset{m\times m}{I} \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} + \overset{n\times m}{\underset{m\times m}{\begin{bmatrix} B \\ 0 \end{bmatrix}}} u(t)$$

$$y(t) = \begin{bmatrix} \overset{m\times n}{C} & \overset{m\times m}{I} \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}$$

We consider the state and disturbance estimator

Actual plant output (measured)

$$\begin{bmatrix} \hat{x}(t+1) \\ \hat{d}(t+1) \end{bmatrix} = \begin{bmatrix} \overset{n\times n}{A} & \overset{n\times m}{0} \\ \overset{m\times n}{0} & \overset{m\times m}{I} \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{d}(t) \end{bmatrix} + \overset{n\times m}{\underset{m\times m}{\begin{bmatrix} B \\ 0 \end{bmatrix}}} u(t) + \begin{bmatrix} L_x \\ L_d \end{bmatrix} \left(-y_m(t) + \underbrace{C\hat{x}(t) + \hat{d}(t)}_{\text{output of the estimator/observer}} \right)$$

Choose L_x and L_d such that the estimator is "stable".

i.e

$$\begin{bmatrix} \hat{x}(t+1) \\ \hat{d}(t+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A + L_x C & L_x \\ L_d C & I + L_d \end{bmatrix}}_{\text{No poles at } (1,0)} \begin{bmatrix} \hat{x}(t) \\ \hat{d}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) - \begin{bmatrix} L_x \\ L_d \end{bmatrix} y_m(t) .$$

3. Steady State Analysis:

At steady state, we have

$$\hat{x}_\infty = A\hat{x}_\infty + B u_\infty$$
$$y_{m,\infty} = C\hat{x}_\infty + \hat{d}_\infty$$

or

$$\begin{array}{cc} n\times n & n\times m \\ & \\ m\times n & m\times m \end{array} \quad \begin{bmatrix} A-I & B \\ C & 0 \end{bmatrix}\begin{bmatrix} \hat{x}_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} 0 \\ y_{m,\infty} - \hat{d}_\infty \end{bmatrix}$$

or

$$\begin{bmatrix} A-I & B \\ C & 0 \end{bmatrix}\begin{bmatrix} \hat{x}_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} 0 \\ r_\infty - \hat{d}_\infty \end{bmatrix}$$

reference.

The solution of $\begin{bmatrix} \hat{x}_\infty \\ u_\infty \end{bmatrix}$ can be parameterized by $r(\infty) - \hat{d}_\infty$. For solution to exist, the above matrix $\begin{bmatrix} A-I & B \\ C & 0 \end{bmatrix}$ must be full row-ranked, i.e. the rows of C (measured output) must be less or equal to the columns of B (no of control input), i.e.

This is satisfied for our case, since we have two measured outputs and two control inputs.

4. The MPC design

The MPC design is as follows:

$$\min_{u_0, \dots u_{N-1}} \|x_N - \bar{x}_t\|_P^2 + \sum_{k=0}^{N-1} \|x_k - \bar{x}_t\|_Q^2 + \|u_k - \bar{u}_t\|_R^2$$

Subject to

$\underline{x} \le x_k \le \bar{x}$ $\quad k=0, \dots N$ — Constraints on states

$\underline{u} \le u_k \le \bar{u}$ $\quad k=0, \dots N$ — Constraints on inputs

$x_{k+1} = Ax_k + Bu_k + d_k, \quad k=0, \dots N$

$d_{k+1} = d_k, \quad k=0, \dots N$

prediction model for state and disturbance.

$x_0 = \hat{x}(t)$ — initial state

$d_0 = \hat{d}(t)$ — initial disturbance

with $\bar{u}_t$ and $\bar{x}_t$ given by

$$\begin{bmatrix} A-I & B \\ C & 0 \end{bmatrix}\begin{bmatrix} \bar{x}_t \\ \bar{u}_t \end{bmatrix} = \begin{bmatrix} 0 \\ r(t) - \hat{d}(t) \end{bmatrix}$$

Steady state target

where $\|x\|_M^2 \triangleq x^T M x$, $Q \geq 0$, $R > 0$ and P satisfies the Riccati equation

$$P = A^T P A - (A^T P B)(B^T P B + R)^{-1}(B^T P A) + Q$$

Suppose

$$\begin{bmatrix} A-I & B \\ C & 0\end{bmatrix}^{-1} = \begin{bmatrix} * & \overset{n\times m}{M} \\ * & \overset{m\times m}{N}\end{bmatrix} \quad \text{so} \quad \begin{bmatrix} A-I & B\\ C & 0\end{bmatrix}\begin{bmatrix}\bar{x}_t\\ \bar{u}_t\end{bmatrix} = \begin{bmatrix} \overset{n\times 1}{0} \\ \overset{m\times 1}{r(t)-\hat{d}(t)}\end{bmatrix} \quad \text{can be}$$

expressed as

$$\underset{m\times 1}{\overset{n\times 1}{\begin{bmatrix}\bar x_t\\ \bar u_t\end{bmatrix}}} = \overset{n\times m}{\begin{bmatrix} M\\ \underset{m\times m}{N}\end{bmatrix}}(r(t)-\hat d(t))$$

The objective function.

So the objective function can be re-written as:

$$V_N = (x_N - Mw)^T P (x_N - Mw) + (x_0-Mw)^T \varphi (x_0-Mw) + (x_1-Mw)^T\varphi(x_1-Mw) + \dots + (x_{N-1}-Mw)^T\varphi(x_{N-1}-Mw) + (u_0-\hat N w)^T R(u_0-\hat N w) + \dots + (u_{N-1}-\hat N w)^T R (u_{N-1}-\hat N w)$$

$$\begin{bmatrix} u_0^T & x_1^T & u_1^T & \cdots & u_{N-1}^T & x_N^T\end{bmatrix} \overbrace{\begin{bmatrix} \overset{m\times m}{R} & & & & & \\ & \overset{n\times n}{\varphi} & & & & \\ & & \overset{m\times m}{R} & & & \\ & & & \ddots & & \\ & & & & \overset{m\times m}{R} & \\ & & & & & \overset{n\times n}{P}\end{bmatrix}}^{(n+m)N \text{ — prediction horizon.}} \begin{bmatrix} u_0 \\ x_1 \\ u_1 \\ \vdots \\ u_{N-1} \\ x_N\end{bmatrix}$$

$$-2\begin{bmatrix} u_0^T & x_1^T & u_1^T & \cdots & u_{N-1}^T & x_N^T\end{bmatrix}\begin{bmatrix} R\hat N w \\ \varphi M w \\ R\hat N w \\ \vdots \\ R\hat N w \\ PMw\end{bmatrix} + \underbrace{(x_0-Mw)^T\varphi(x_0-Mw) + (N-3)w^T M^T\varphi M w + w^T M^T P M w + N w^T \hat N^T R \hat N w}_{\text{Constant terms}}$$

Define

$$z = \overset{(n+m)N\times 1}{\begin{bmatrix} u_0 \\ x_1 \\ u_1 \\ \vdots \\ u_{N-1} \\ x_N \end{bmatrix}} \qquad H = \overset{(n+m)N\times(n+m)N}{\begin{bmatrix} R & & & & & \\ & \varphi & & & & \\ & & R & & & \\ & & & \ddots & & \\ & & & & R & \\ & & & & & P\end{bmatrix}} \quad \text{and} \quad \overset{(n+m)N\times m}{F = H} \overset{F_0}{\underset{(n+m)N\times m}{\begin{bmatrix} \hat N \\ M \\ \hat N \\ \vdots \\ \hat N \\ M\end{bmatrix}}}$$

we have the objective as

$$z^T H z - 2 z^T F w + \text{constant terms.}$$

where $F = H F_0$ and $w = r(t) - \hat{d}(t)$

$r(t)$: reference input

$\hat{d}(t)$: estimated disturbance. (from observer)

The Prediction.

$$\left.\begin{aligned} x_{k+1} &= A x_k + B u_k + d_k \\ d_{k+1} &= d_k \end{aligned}\right\} \quad k = 0, \ldots N$$

$k=0$:

$$\begin{aligned} x_1 &= A x_0 + B u_0 + d_0 \\ d_1 &= d_0 \end{aligned} \Rightarrow x_1 - B u_0 - d_0 = A x_0$$

or

$$x_1 - B u_0 = A x_0 + d_0$$

$k=1$:

$$\begin{aligned} x_2 &= A x_1 + B u_1 + d_1 \\ d_2 &= d_1 \end{aligned} \Rightarrow x_2 - A x_1 - B u_1 = d_0$$

$k=3$:

$$\begin{aligned} x_3 &= A x_2 + B u_2 + d_2 \\ d_3 &= d_2 \end{aligned} \Rightarrow x_3 - A x_2 - B u_2 = d_0$$

$$x_N - A x_{N-1} - B u_{N-1} = d_0$$

$k=N$

$$x_{N+1} = A x_N + B u_N + d_N, \quad d_{N+1} = d_N$$

$$nN \left\{ \underbrace{\begin{bmatrix} -B & I & & & & \\ 0 & -A & -B & I & & \\ 0 & 0 & 0 & -A & -B & I \\ & & \vdots & & & \\ & & & 0 & -A & -B \quad I \end{bmatrix}}_{\substack{(n+m)N \\ E}} \right. \underbrace{\begin{bmatrix} u_0 \\ x_1 \\ u_1 \\ \vdots \\ u_{N-1} \\ x_N \end{bmatrix}}_{(n+m)N \times 1} = \underbrace{\overbrace{\begin{bmatrix} \overset{n \times n}{A} & \overset{n \times n}{I} \\ 0 & I \\ 0 & I \\ \vdots & \vdots \\ 0 & I \end{bmatrix}}^{nN \times (n+m)}}_{e} \begin{bmatrix} x_0 \\ d_0 \end{bmatrix}$$

In compact form, we have

$$E z = e \quad \text{where} \quad e = \overset{nN \times n}{F_1} \hat{x}(t) + \overset{nN \times m}{F_2} \hat{d}(t)$$

$\hat{x}(t)$: estimated state

$\hat{d}(t)$: estimated disturbance

from the observer.

$$F_1 = \begin{bmatrix} A \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad F_2 = \begin{bmatrix} I \\ I \\ I \\ \vdots \\ I \end{bmatrix}$$

Constraints

$$\underline{x} \le x_k \le \bar{x} \qquad k = 0, \dots N$$
$$\underline{u} \le u_k \le \bar{u}$$

$$\underbrace{\begin{bmatrix} \underline{u} \\ \underline{x} \\ \underline{u} \\ \vdots \\ \underline{u} \\ \underline{x} \end{bmatrix}}_{(n+m)N\times 1} \le \underbrace{\begin{bmatrix} u_0 \\ x_1 \\ u_1 \\ \vdots \\ u_{N-1} \\ x_N \end{bmatrix}}_{(n+m)N\times 1} \le \underbrace{\begin{bmatrix} \bar{u} \\ \bar{x} \\ \bar{u} \\ \vdots \\ \bar{u} \\ \bar{x} \end{bmatrix}}_{(n+m)N\times 1}$$

In Compact form, we have

$$\underbrace{\underline{z}}_{\text{Lower bound}} \le z \le \underbrace{\bar{z}}_{\text{upper bound.}}$$

Putting all together, we ~~theave~~ the following quadratic program:

$$\min_z \ \frac{1}{2} z^T H z - z^T q(t) \qquad \text{where } q(t) = F(r(t) - \hat{d}(t)) = HF_0\,(r(t) - \hat{d}(t))$$

$$\text{Subject to } \quad Ez = e(t) \qquad \text{and } e(t) = F_1 \hat{x}(t) + F_2 \hat{d}(t)$$

$$\underline{z} \le z \le \bar{z}$$

NOTES:

1. We only apply the first element of the decision ~~sequence~~ variable z i.e u_0
2. Recall that we linearized out plant about operating points (x_0^0, u_0^0), so

$$\underbrace{x}_{\text{Actual state}} = \underbrace{x_0}_{\text{estimated state by observer}} + \underbrace{x^0}_{\text{operating point}} \quad \text{and} \quad u = \underbrace{u_0}_{\text{computed control effort by MPC}} + \underbrace{u^0}_{\text{operating point}}$$

We must apply u to the actual plant

So r must be adjusted by r^0 before passing into the MPC i.e

$$\underbrace{r_0}_{\text{What is given to MPC}} = \underbrace{r}_{\text{Actual reference}} - \underbrace{r^0}_{\text{Corresponding operating point}}$$

3. The reformulated QP can be solved either in digital form (Haley & Jeff) or in continuous time (Terrence).
4. The observer/~~estimator~~ has to be implemented

observer implementation

This has to be computed using "kalman" or "place" in Matlab.

In Discrete:

$$\begin{bmatrix}\hat{x}(t+1)\\ \hat{d}(t+1)\end{bmatrix} = \begin{bmatrix}A & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\hat{x}(t)\\ \hat{d}(t)\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}u(t) + \begin{bmatrix}L_x\\ L_d\end{bmatrix}\left(C\hat{x}(t) + \hat{d}(t) - y_m(t)\right)$$

same control applied to the actual plant
As before, this is the u_0 computed by the MPC.

Actual plant output (measured height).
As before ducted by the operating point.

In Continuous time

Linearized plant

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Augument the plant with a disturbance model as

$$\dot{x} = Ax + Bu$$
$$\dot{d} = d$$ —— integrator - Adding integral action.
$$y = Cx + d$$

Such that we have:

$$\begin{bmatrix}\dot{x}\\ \dot{d}\end{bmatrix} = \begin{bmatrix}A & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}x\\ d\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}u$$

$$y = \begin{bmatrix}C & I\end{bmatrix}\begin{bmatrix}x\\ d\end{bmatrix}$$

The observer or state estimator becomes

observer gains to be computed such that the observer is stable.

$$\begin{bmatrix}\dot{\hat{x}}\\ \dot{\hat{d}}\end{bmatrix} = \begin{bmatrix}A & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{d}\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}u + \begin{bmatrix}L_x\\ L_d\end{bmatrix}[\hat{y} - y]$$

where $\hat{y} = C\hat{x} + \hat{d}$

Output of the estimator.

output of the actual plant adjusted for the operating condition.

Note: we have abuse the notation for the system parameters A, B, C.
Here they are the actual system parameters of the plant.
For the discrete version, (A, B, C) has be discretized using appropriate sampling time to obtain new parameters

MPC Design for the Case without State constraints:

Suppose there are no state constraints so that the MPC problem becomes:

$$\min_{u_0, \dots u_{N-1}} \|x_N - \bar{x}_t\|_P^2 + \sum_{k=0}^{N-1} \|x_k - \bar{x}_t\|_Q^2 + \|u_k - \bar{u}_t\|_R^2$$

Subject to:

$$\underline{u} \le u_k \le \bar{u}, \quad k = 0 \dots N$$

$$x_{k+1} = Ax_k + Bu_k + d_k, \quad k = 0 \dots N$$

$$d_{k+1} = d_k, \quad k = 0, \dots N$$

$$x_0 = \hat{x}(t)$$

$$d_0 = \hat{d}(t)$$

with $\bar{u}_t$ and $\bar{x}_t$ given by

$$\begin{bmatrix} A - I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_t \\ \bar{u}_t \end{bmatrix} = \begin{bmatrix} 0 \\ r(t) - \hat{d}(t) \end{bmatrix}$$

As before, we parameterize the target solution in terms of the input $r(t) - \hat{d}(t)$ as

$$\begin{bmatrix} \bar{x}_t \\ \bar{u}_t \end{bmatrix} = \begin{bmatrix} M \\ \hat{N} \end{bmatrix} (r(t) - \hat{d}(t))$$

The prediction.

$k=0$ $\quad x_1 = Ax_0 + Bu_0 + d_0, \quad d_1 = d_0$

$k=1$ $\quad x_2 = Ax_1 + Bu_1 + d_1, \quad d_2 = d_1$

$$= A^2 x_0 + ABu_0 + Ad_0 + Bu_1 + d_0$$

$k=2$ $\quad x_3 = Ax_2 + Bu_2 + d_2, \quad d_3 = d_2$

$$= A^3 x_0 + A^2 Bu_0 + A^2 d_0 + ABu_1 + Ad_0 + Bu_2 + d_0$$

$k=3$ $\quad x_4 = Ax_3 + Bu_3 + d_3, \quad d_4 = d_3$

$$x_4 = A^4 x_0 + A^3 B u_0 + A^3 d_0 + A^2 B u_1 + A^2 d_0 + A B u_2 + A d_0 + B u_3 + d_0$$

$\vdots$

$$x_N = A^N x_0 + (A^{N-1} + A^{N-2} + \dots + A + I) d_0 + A^{N-1} B u_0 + A^{N-2} B u_1 + \dots + A B u_{N-2} + B u_{N-1}$$

So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} A \\ A^2 \\ A^3 \\ \vdots \\ A^N \end{bmatrix} x_0 + \begin{bmatrix} I \\ A+I \\ A^2+A+I \\ \vdots \\ A^{N-1}+\dots+A+I \end{bmatrix} d_0 + \begin{bmatrix} B & & & & \\ AB & B & & & \\ A^2B & AB & B & & \\ \vdots & & & \ddots & \\ A^{N-1}B & A^{N-2}B & \dots & AB & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

Define:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix}, \quad U = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix}, \quad \emptyset = \begin{bmatrix} A \\ A^2 \\ A^3 \\ \vdots \\ A^N \end{bmatrix}, \quad \Psi = \begin{bmatrix} I \\ A+I \\ A^2+A+I \\ \vdots \\ A^{N-1}+\dots+A+I \end{bmatrix}, \quad \Lambda = \begin{bmatrix} B & & & \\ AB & B & & \\ A^2B & AB & B & \\ \vdots & & & \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix}$$

So in compact form

$$\boxed{X = \emptyset x_0 + \Psi d_0 + \Lambda U}$$

<u>The constraints</u>

$$\underline{u} \le u_k \le \bar{u} \qquad k=0, \dots, N-1$$

$$\begin{bmatrix} \underline{u} \\ \underline{u} \\ \underline{u} \\ \vdots \\ \underline{u} \end{bmatrix} \le \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix} \le \begin{bmatrix} \bar{u} \\ \bar{u} \\ \bar{u} \\ \vdots \\ \bar{u} \end{bmatrix}$$

This can be rewritten compactly as.

$$\underline{U} \le U \le \bar{U}$$

Objective function

$$\sum_{k=0}^{M-1} \|x_k - \bar{x}_t\|^2_{\phi} + \|x_M - \bar{x}_t\|^2_P$$

$$= (x_0 - Mw)^T \phi (x_0 - Mw) + (x_1 - Mw)^T \phi (x_1 - Mw) + \cdots + (x_{M-1} - Mw)^T \phi (x_{M-1} - Mw)$$

$$+ (x_M - Mw)^T P (x_M - Mw)$$

$$= \begin{bmatrix} x_1^T & x_2^T & \cdots & x_{M-1}^T, & x_M^T \end{bmatrix} \begin{bmatrix} \phi & & & & \\ & \phi & & & \\ & & \ddots & & \\ & & & \phi & \\ & & & & P \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{M-1} \\ x_M \end{bmatrix}$$

$$-2 \begin{bmatrix} x_1^T & x_2^T & \cdots & x_{M-1}^T, & x_M^T \end{bmatrix} \begin{bmatrix} \phi & & & & \\ & \phi & & & \\ & & \ddots & & \\ & & & \phi & \\ & & & & P \end{bmatrix} \begin{bmatrix} M \\ M \\ \vdots \\ M \\ M \end{bmatrix} w$$

$$\underbrace{+ x_0^T \phi x_0 + (M-1) w^T M^T \phi M w + w^T M^T P M w}_{\text{constants}}$$

Compactly

$$= X^T \tilde{\phi} X - 2 X^T \tilde{\phi} \tilde{M} w + \text{constant terms.}$$

Also

$$\sum_{k=0}^{M-1} \|u_k - \bar{u}_t\|^2_R = (u_0 - \hat{N} w)^T R (u_0 - \hat{N} w) + \cdots + (u_{M-1} - \hat{N} w)^T R (u_{M-1} - \hat{N} w)$$

$$= \begin{bmatrix} u_0^T & u_1^T & \cdots & u_{M-1}^T \end{bmatrix} \begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{M-1} \end{bmatrix} - 2 \begin{bmatrix} u_0^T & u_1^T & \cdots & u_{M-1}^T \end{bmatrix} \begin{bmatrix} R\hat{N} & & & \\ & R\hat{N} & & \\ & & \ddots & \\ & & & R\hat{N} \end{bmatrix} w$$

$$+ N w^T \hat{N}^T R \hat{N} w$$

$$= U^T \tilde{R} U - 2 U^T \tilde{R} \tilde{N} w + \text{constant term.}$$

Putting all together and substituting for

$$X = \Phi x_0 + \Psi d_0 + \Lambda U$$

The objective becomes

$$X^T \tilde{\Phi} X - 2X^T \tilde{\Phi} \tilde{M} w + U^T \tilde{R} U - 2U^T \tilde{R} \tilde{N} w + \text{Constants}$$

$$= (\Phi x_0 + \Psi d_0 + \Lambda U)^T \tilde{\Phi} (\Phi x_0 + \Psi d_0 + \Lambda U) - 2 (\Phi x_0 + \Psi d_0 + \Lambda U)^T \tilde{\Phi} \tilde{M} w$$

$$+ U^T \tilde{R} U - 2 U^T \tilde{R} \tilde{N} w + \text{constants}$$

$$= x_0^T \Phi^T \tilde{\Phi} \Phi x_0 + x_0^T \Phi^T \tilde{\Phi} \Psi d_0 + x_0^T \Phi^T \tilde{\Phi} \Lambda U$$

$$+ d_0^T \Psi^T \tilde{\Phi} \Phi x_0 + d_0^T \Psi^T \tilde{\Phi} \Psi d_0 + d_0^T \Psi^T \tilde{\Phi} \Lambda U$$

$$+ U^T \Lambda^T \tilde{\Phi} \Phi x_0 + U^T \Lambda^T \tilde{\Phi} \Psi d_0 + U^T \Lambda^T \tilde{\Phi} \Lambda U$$

$$- 2U^T \tilde{R} \tilde{N} w - 2 x_0^T \Phi^T \tilde{\Phi} \tilde{M} w - 2 d_0^T \Psi^T \tilde{\Phi} \tilde{M} w - 2 U^T \Lambda^T \tilde{\Phi} \tilde{M} w$$

$$+ U^T \tilde{R} U + \text{constant}$$

$$= U^T (\Lambda^T \tilde{\Phi} \Lambda + \tilde{R}) U + 2 U^T \Lambda^T \tilde{\Phi} \Phi x_0 + 2 U^T \Lambda^T \tilde{\Phi} \Psi d_0$$

$$- 2U^T \Lambda^T \tilde{\Phi} \tilde{M} w - 2U^T \tilde{R} \tilde{N} w + \text{Constant terms.}$$

$$= U^T (\Lambda^T \tilde{\Phi} \Lambda + \tilde{R}) U + 2 U^T \Lambda^T \tilde{\Phi} \Phi x_0 + 2U^T (\Lambda^T \tilde{\Phi} \Psi + \tilde{\Phi} \tilde{M} + \tilde{R} \tilde{N}) d_0$$

$$-2 U^T (\Lambda^T \tilde{\Phi} \tilde{M} + \tilde{R} \tilde{N}) r(t) + \text{Constant terms.}$$

The MPC problem can the be compactly described as

$$\min_{u_0, u_1, \ldots u_{N-1}} \quad U^T H U + 2 U^T (F_1 x_0 + F_2 d_0 + F_3 r) + \text{Constant terms}$$

$$\text{subject to} \quad \underline{U} \le U \le \overline{U}$$

Consider the problem

$$\min_{z} \ \frac{1}{2} z^T H z - z^T q(t)$$

Subject to $Ez = e(t)$

$$\underline{z} \le z \le \overline{z}$$

where

$$H = \overset{(n+m)N \times (n+m)N}{\begin{bmatrix} R & & & & \\ & R & & & \\ & & \ddots & & \\ & & & R & \\ & & & & P \end{bmatrix}} \qquad E = \overset{nN \times (n+m)N}{\begin{bmatrix} -B & I & & & & & & \\ 0 & -A & -B & I & & & & \\ & & & 0 & -A & -B & I & \\ & & & & & & & \ddots \\ & & & & & 0 & -A-B & I \end{bmatrix}}$$

$$q(t) = \overset{(n+m)N \times m}{F}\,(r(t) - \hat{d}(t)) \quad \text{with} \quad F = H f_0, \quad f_0 = \begin{bmatrix} \hat{N} \\ M \\ \vdots \\ \hat{N} \\ M \end{bmatrix}$$

and

$$e(t) = \overset{nN \times n}{F_1}\,\hat{x}(t) + \overset{nN \times m}{F_2}\,\hat{d}(t) \quad \text{with} \quad F_1 = \begin{bmatrix} A \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad F_2 = \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix}$$

For problem with a single prediction Horizon $(N=1)$

$$H = \overset{(n+m) \times (n \times m)}{\begin{bmatrix} R & 0 \\ 0 & P \end{bmatrix}}, \quad E = \underset{2 \times 4}{\overset{n \times (n+m)}{\begin{bmatrix} -B & I \end{bmatrix}}}, \quad F = \underset{4 \times 2}{\overset{(n+m) \times m}{\begin{bmatrix} R\hat{N} \\ PM \end{bmatrix}}}, \quad F_1 = \underset{2 \times 2}{\overset{n \times n}{A}}, \quad F_2 = \underset{2 \times 2}{\overset{n \times m}{I}}$$

$$q(t) = \begin{bmatrix} R\hat{N} \\ PM \end{bmatrix} [r(t) - \hat{d}(t)] \quad 4 \times 4 \qquad e(t) = A\hat{x}(t) + \hat{d}(t)$$

using the fixed point iteration algorithm of ~~Pranderiele~~ Adegbege and Skibik

$$z^{k+1} = \underset{\text{Projection}}{\phi}\left(z^k - \alpha D^{-1}\left(Hz^k - E^T \lambda^k - q(t)\right)\right)$$

$$\lambda^{k+1} = \lambda^k + w \phi^{-1}\left(e(t) - E z^{k+1}\right)$$

After Euler approximation:

$$\tau \dot{z} = \phi\left(z - D^{-1}\left(Hz - E^T\lambda - q(t)\right)\right) - z$$

$$\gamma \dot{\lambda} = \phi^{-1}\left(e(t) - E z\right)$$

We can take advantage of the conditioning matrices D and ϕ to scale our signals

Suppose H is positive definite then we choose D as the diagonal elements of H. ϕ can be chosen as EE^T or $EH^{-1}E^T$ or I $\quad EE^T = [B \ \ I]\begin{bmatrix} -B^T \\ I \end{bmatrix} = BB^T + I$