

1 Finitely generated nilpotent spaces

In these notes, we will present a proof of the following theorem, which is Theorem 4.5.2 of [1].

Theorem 1.0.1: *Let X be a nilpotent space. Then the following statements are equivalent:*

- i) X is weakly equivalent to a CW complex with finite skeleta,*
- ii) X is f -nilpotent,*
- iii) $\pi_i(X)$ is finitely generated for each $i \geq 1$,*
- iv) $\pi_1(X)$ and, for $i \geq 2$, $H_i(\tilde{X})$ are finitely generated,*
- v) $H_i(X)$ is finitely generated for each $i \geq 1$.*

The strategy of proof is as follows. We first develop some basic algebra concerning nilpotent groups which will immediately imply $ii) \iff iii)$. We then present a result from Wall's classical paper "Finiteness Conditions on CW complexes" which, along with the algebraic theory developed earlier, will allow us prove the implication $iv) \iff i)$. We also use our previous algebraic work to show that $K(A, n)$ can be modelled by a CW complex with finite skeleta whenever A is a finitely generated abelian group. We will use this information to prove Serre's classical result showing that, for simply connected spaces, $H_i(X)$ is finitely generated for all i iff $\pi_i(X)$ is. This allows us to prove $iii) \iff iv)$. We finish with a proof that $v) \implies iv)$, again making use of our previous work. Observe that the implication $i) \implies v)$ is trivial.

1.1 Algebraic Results

Let G be a group. We record here the definition of a nilpotent group and some of their basic properties. The proofs of the results of this page can be found in Chapter 2 of [3]

Definition: A group G is called nilpotent if there exists a finite series of subgroups of the form:

$$1 = G_0 \subset G_1 \subset \dots \subset G_k = G,$$

such that each G_i is normal in G and, for every i , $\frac{G_{i+1}}{G_i} \subset Z(\frac{G}{G_i})$.

Definition: The lower central series of G is the normal series defined by:

$$\begin{aligned}\gamma_0(G) &= G \\ \gamma_{i+1}(G) &= [G, \gamma_i(G)].\end{aligned}$$

Definition: The upper central series of G is the normal series defined by:

$$\begin{aligned}\zeta_0(G) &= 1 \\ \zeta_{i+1}(G) &= \pi^{-1}(Z(\frac{G}{\zeta_i(G)})),\end{aligned}$$

Almost by definition, if either series terminates then the obtained series expresses G as a nilpotent group. Moreover, the upper central series ascends faster than any other series expressing G as a nilpotent group in the sense that if $1 = G_0 \subset G_1 \subset \dots \subset G_k = G$ expresses G as a nilpotent group, then $G_i \subset \zeta_i(G)$. Similarly, the lower central series descends faster than any other series expressing G as a nilpotent group. It follows that G is nilpotent iff the upper central series terminates iff the lower central series terminates.

There is also an epimorphism:

$$\phi : \frac{\gamma_i(G)}{\gamma_{i+1}(G)} \otimes Ab(G) \rightarrow \frac{\gamma_{i+1}(G)}{\gamma_{i+2}(G)}$$

defined by:

$$\phi(\{a\}, \{g\}) = \{[a, g]\}.$$

The results of the previous page allow us to prove the bulk of the following theorem. The remainder of this subsection will be devoted to proving the final sentence.

Theorem 1.1.1: *Let G be a nilpotent group. Then the following are equivalent:*

- i) G is finitely generated,*
- ii) $\text{Ab}(G)$ is finitely generated,*
- iii) G is f -nilpotent,*
- iv) Every subgroup of G is finitely generated.*

Moreover, if these conditions are satisfied, then G is finitely presentable and $\mathbb{Z}[G]$ is a Noetherian ring.

Proof: The implications $iv) \implies i) \implies ii)$ are trivial. The implication $ii) \implies iii)$ follows, inductively, from the epimorphism described on the previous page. It remains to prove $iii) \implies iv)$. For this we start with the fact that all subgroups of a finitely generated abelian group are finitely generated. Suppose that the series:

$$1 = G_0 \rightarrow G_1 \rightarrow \dots \rightarrow G_m = G$$

expresses G as an f -nilpotent group. Assume that every subgroup of G_k is finitely generated and consider the short exact sequence:

$$1 \rightarrow G_k \rightarrow G_{k+1} \rightarrow \frac{G_{k+1}}{G_k} \rightarrow 1$$

in which both G_k and the finitely generated abelian group $\frac{G_{k+1}}{G_k}$ satisfy Max, that is every subgroup of them is finitely generated. Now if H is a subgroup of G_{k+1} , then the image of H in $\frac{G_{k+1}}{G_k}$ is finitely generated, as well as $H \cap G_k$. It follows that H is finitely generated, and so G_{k+1} satisfies Max. It follows, inductively, that G satisfies Max. □

For the remainder of this subsection, assume that G is a finitely generated nilpotent group. The fact that $\mathbb{Z}[G]$ is Noetherian holds more generally for polycyclic groups, and we begin by showing that G is polycyclic.

Definition: A group G is called polycyclic if it has a subnormal series of the form:

$$1 = G_0 \rightarrow G_1 \rightarrow \dots \rightarrow G_m = G$$

in which each quotient is a cyclic group.

By the structure theorem for finitely generated abelian groups, we know that finitely generated abelian groups are polycyclic. The fact that G is polycyclic follows from this fact and an induction up a series expressing G as an f -nilpotent group, using the next lemma.

Lemma 1.1.2: *If K and H are polycyclic in the short exact sequence of groups:*

$$1 \rightarrow K \rightarrow G \rightarrow H \rightarrow 1,$$

then G is polycyclic.

Proof: Let the subnormal series $\{K_i\}$ and $\{H_i\}$ express K and H as polycyclic groups. Then we define a subnormal series on G by:

$$1 = K_0 \rightarrow K_1 \rightarrow \dots \rightarrow K_m = K = f^{-1}(H_0) \rightarrow f^{-1}(H_1) \rightarrow \dots \rightarrow f^{-1}(H_n) = G$$

The fact that this series expresses G as a polycyclic group follows from the third isomorphism theorem for groups. \square

The following lemma, and corollary, is due to P.Hall ([4], Theorem 1).

Lemma 1.1.3: *Suppose that H is a normal subgroup of G such that $\frac{G}{H}$ is either finite or infinite cyclic, and that N is an H -submodule of the right G -module M such that $M = NG$. Then if N satisfies Max- H , M satisfies Max- G (that is all G -submodules of M are finitely generated as G -modules).*

Proof: If $\frac{G}{H}$ is finite, then let g_0, \dots, g_n be elements of G representing each element of $\frac{G}{H}$. Then Ng_i is an H -submodule of M for each i and we have an epimorphism $\oplus_i Ng_i \rightarrow M$, since $M = NG$. If A is an H -submodule of Ng_i , then $A' = \{n \in N | ng_i \in A\}$ is an H -submodule of N and so is finitely generated. It follows that Ng_i satisfies Max- H and, therefore, so do $\oplus_i Ng_i$ and M . It follows that M satisfies Max- G .

If $\frac{G}{H} \cong \mathbb{Z}$, let g be an element of G representing $1 \in \frac{G}{H}$. Then, since $M = NG$, every element $m \in M$ is of the form:

$$m = \sum_{k \in \mathbb{Z}} n_k g^k$$

where $n_k \in N$ for all k , and all but finitely many of the n_k are 0. Let A be a G -submodule of M . If p, q are integers such that $p < q$, define the H -submodule $N_{p,q}$ of N by:

$$N_{p,q} = \{n \in N | \text{there exists } \sum_k n_k g^k \in A \text{ such that } n_i = 0 \text{ if } i < p \text{ or } i > q \text{ and } n_p = n\}$$

Multiplication by g and its inverse shows that $N_{p,q}$ depends only on the value of $q - p$, so define $N_i = N_{0,i}$ for $i \in \mathbb{N}$. Then we have an ascending chain of H -submodules of N :

$$N_1 \subset N_2 \subset \dots$$

Since N satisfies Max- H , this sequence must terminate after finitely many steps, say at N_j , $j \in \mathbb{N}$.

For each $i \leq j$, let $\{m_{i,0}, m_{i,1}, \dots, m_{i,t_i}\}$ be a set of elements of A representing a generating set for N_i . We claim that this is a generating set for A as a G -module. Suppose that:

$$m = \sum_{k \in \mathbb{Z}} n_k g^k$$

is an element of A . By subtracting elements of the form $m_{j,t} g^k h$ and multiplying by some g^s , we may assume that $n_k = 0$ for $k < 0$ and $k > j - 1$. Then, by subtracting elements of the form $m_{i,t} h$, where $i < j$, we can reduce all the way to 0. Hence, A is a finitely generated G -module, as desired. \square

Corollary: If G is a polycyclic group, then $\mathbb{Z}[G]$ is a Noetherian ring.

Proof: This follows from the previous lemma, the definition of a polycyclic group and the observation that if $H \leq G$, then $\mathbb{Z}[H]$ is an H -submodule of $\mathbb{Z}[G]$, and $\mathbb{Z}[G] = \mathbb{Z}[H]G$. Also, $\mathbb{Z}[1] = \mathbb{Z}$ is Noetherian. \square

This shows that f -nilpotent groups have Noetherian group rings. The final task of Theorem 1.1.1 is to show that they are also finitely presented. We know that finitely generated abelian groups are finitely presented, and so the result will follow by inducting up a series expressing G as an f -nilpotent group, using the following lemma:

Lemma 1.1.4: *If K and H are finitely presented in a short exact sequence of groups:*

$$1 \rightarrow K \rightarrow G \rightarrow H \rightarrow 1$$

then G is also finitely presented.

Proof: Let $H = \langle S \mid R \rangle$ be a presentation of H , where S is a finite set and $R \subset W(S, S^{-1})$ is a finite subset of the words in S and their inverses. Similarly, let $K = \langle P, Q \rangle$ be a finite presentation of K . Then we have an epimorphism $\phi : \langle S \cup P \rangle \rightarrow G$ defined by sending elements of P to their images in G and elements of S to a chosen preimage in G .

We will define three finite subsets of the kernel of ϕ . Firstly, we have

$$Q \subset W(P, P^{-1}) \subset W((S \cup P), (S \cup P)^{-1}).$$

For each $r \in R$, we have $\phi(r) \in K$. Let w_r be a word in P and P^{-1} representing $\phi(r)^{-1}$. Then we define:

$$R' = \{rw_r \mid r \in R\} \subset W(S \cup P, (S \cup P)^{-1})$$

Now let (s, p) be a pair consisting of an element $s \in S \cup S^{-1}$ and an element $p \in P \cup P^{-1}$. The image of the conjugate p^s is in K and so let $w_{(s,p)}$ be a word in P and P^{-1} representing $\phi(p^s)^{-1}$. Define:

$$T = \{p^s w_{(s,p)} \mid s \in S \cup S^{-1}, p \in P \cup P^{-1}\}.$$

We claim that $G = \langle S \cup P \mid Q \cup R' \cup T \rangle$.

Suppose that:

$$w_1 v_1 w_2 v_2 \dots w_k v_k$$

is a product of words in P and P^{-1} (the w_i) and words in S and S^{-1} (the v_i), which is in the kernel of ϕ . Then $v_1 \dots v_k$ is a product of conjugates of elements of R . Therefore, by multiplying by conjugates of elements of R' , we may assume that $v_1 \dots v_k = 1$. In this case our word is of the form:

$$w_1^{v_1} \dots w_k^{v_k}$$

where the w_i are words in P and the v_i are words in S . In fact we may assume that each v_i is an element of $S \cup S^{-1}$, and each w_i is an element of $P \cup P^{-1}$. Then this word is of the form:

$$t_{(s_1, p_1)} w_{(s_1, p_1)}^{-1} \dots t_{(s_k, p_k)} w_{(s_k, p_k)}^{-1}$$

where $t_{(s_i, p_i)}$ is the element of T corresponding to (s_i, p_i) . Hence, by multiplying by conjugates of elements of T , we may assume our original word is of the form w_1 , a word in P and P^{-1} . This case is then dealt with by multiplying by conjugates of elements of Q to reduce to 1. \square

Corollary: If G is an f -nilpotent group, then G is finitely presentable. \square

We finish this algebraic section by providing the proof of the implication $ii) \iff iii)$ of Theorem 2.0.1 which is purely algebraic. Firstly, we recall what we mean by a nilpotent space. Let X be a space and let $\pi = \pi_1(X)$. We call a (left) module over the group ring $\mathbb{Z}[\pi]$ a π -module. So, in particular, $\pi_i(X)$ is a π -module for $i \geq 2$. By a π -group, we mean a group with a left action of π by homomorphisms. So all π -modules are π -groups.

Definition: Let G be a π -group. Then G is a nilpotent π -group if there is a sequence of π -subgroups of G of the form:

$$1 = G_0 \subset G_1 \subset \dots \subset G_k = G$$

such that, for every i , G_i is normal in G , $\frac{G_{i+1}}{G_i} \subset Z(\frac{G}{G_i})$ and π acts trivially on $\frac{G_{i+1}}{G_i}$.

We let π act on itself via conjugation. We then have the following definition:

Definition: A nilpotent space X is a space for which $\pi_i(X)$ is a nilpotent π -group for $i \geq 1$.

Observe that when $i = 1$, the definitions reduce to saying that $\pi_1(X)$ is a nilpotent group. We say that a space is f -nilpotent if the quotients $\frac{G_{i+1}}{G_i}$ can be taken to be finitely generated as abelian groups, that is $\pi_i(X)$ is an f -nilpotent π -group for all i .

With our algebraic work in hand we can now prove:

Corollary 1.1.5: *Let X be a nilpotent space. Then X is an f -nilpotent space iff $\pi_i(X)$ is finitely generated for $i \geq 1$.*

Proof: If X is f -nilpotent, then $\pi_i(X)$ is an f -nilpotent group for all $i \geq 1$. It follows that $\pi_i(X)$ is finitely generated for $i \geq 1$.

If $\pi_i(X)$ is finitely generated for all $i \geq 1$, then we have two cases. If $i \geq 2$ then the group $\pi_i(X)$ is abelian and so all subgroups are finitely generated. In particular, any series expressing $\pi_i(X)$ as a nilpotent π -group will have finitely generated quotients.

For the case $i = 1$, we have that $\pi_1(X)$ is a finitely generated nilpotent group and, hence, an f -nilpotent group. A normal subgroup of π is (equivalent to) a π -subgroup, since π is acting on itself by conjugation. Moreover, π acts trivially on the quotient groups of any series expressing $\pi_1(X)$ as an f -nilpotent group. It follows that $\pi_1(X)$ is an f -nilpotent π -group, as desired. Taking both cases together, we have that X is an f -nilpotent space. \square

1.2 A Theorem of Wall

In this subsection we will prove a theorem of Wall concerning when a space is weakly equivalent to a CW complex with finite skeleta ([2], Theorem A). This will allow us to immediately prove the implication $iv) \iff i)$ of Theorem 2.0.1, and the theorem will continue to be useful for the remaining implications.

Theorem 1.2.1: *A space X is weakly equivalent to a CW complex with finite skeleta iff each of the following conditions is satisfied:*

- i) $\pi = \pi_1(X)$ is finitely presented,*
- ii) for every map $\sigma : K \rightarrow X$ from a connected finite CW complex K which induces an isomorphism on fundamental groups, $\pi_1(F\sigma)$ is finitely generated as a π -module,*
- iii) for every n -connected ($n \geq 2$) map $\sigma : K \rightarrow X$ from a finite CW complex K , $\pi_n(F\sigma)$ is finitely generated as a π -module.*

Proof: (\Leftarrow) We'll first show that if each of the conditions is satisfied, then X is weakly equivalent to a CW complex with finite skeleta. Since π is finitely presented, we can construct, using the van Kampen theorem, a finite CW complex K , with cells of dimension ≤ 2 , equipped with a map $\sigma : K \rightarrow X$ inducing an isomorphism on fundamental groups. We have the exact sequence of π -modules:

$$\dots \rightarrow \pi_2(F\sigma) \rightarrow \pi_2(K) \rightarrow \pi_2(X) \rightarrow \pi_1(F\sigma) \rightarrow 0$$

where $\pi_1(F\sigma)$ is abelian since it is a quotient of $\pi_2(X)$, and the action of π is induced by $\pi = \pi_1(K)$ in the underlying fiber sequence. By assumption, $\pi_1(F\sigma)$ is finitely generated as a π -module, so choose finitely many elements of $\pi_2(X)$ which correspond to generators of $\pi_1(F\sigma)$ under the given surjection. Form a finite CW complex \hat{K} , and a map $\hat{\sigma} : \hat{K} \rightarrow X$, by wedging a copy of S^2 to K for each chosen element of $\pi_2(X)$, and define the map $\hat{\sigma}$ by sending each copy of S^2 to a representative of the corresponding element of $\pi_2(X)$. Given the exact sequence above, it is straightforward to see that the map $\hat{\sigma} : \hat{K} \rightarrow X$ is 2-connected.

Now suppose that we have constructed an n -connected map $\sigma : K \rightarrow X$ from a finite CW complex K , where $n \geq 2$. We will show that by adding finitely many $(n+1)$ -cells to K , we can extend σ to an $(n+1)$ -connected map. In light of the previous paragraph, this will complete the first half of the proof. We have the exact sequence:

$$\dots \rightarrow \pi_{n+1}(K) \rightarrow \pi_{n+1}(X) \rightarrow \pi_n(F\sigma) \rightarrow \pi_n(K) \rightarrow \pi_n(X) \rightarrow 0$$

By assumption, $\pi_n(F\sigma)$ is finitely generated as a π -module. Therefore, the kernel of $\sigma_* : \pi_n(K) \rightarrow \pi_n(X)$ is finitely generated as a π -module. Form a finite CW complex \hat{K} , and a map $\hat{\sigma} : \hat{K} \rightarrow X$, by attaching $(n+1)$ -cells along representatives of a finite generating set for the kernel of σ_* , with the map $\hat{\sigma}$ defined using chosen nullhomotopies in X . Then we have a surjection $\pi_n(K) \rightarrow \pi_n(\hat{K})$, since $\pi_n(\hat{K}, K) = 0$, with kernel equal to the kernel of σ_* . It follows that $\hat{\sigma}_* : \pi_n(\hat{K}) \rightarrow \pi_n(X)$ is an isomorphism. We then have the exact sequence:

$$\dots \rightarrow \pi_{n+1}(\hat{K}) \rightarrow \pi_{n+1}(X) \rightarrow \pi_n(F\hat{\sigma}) \rightarrow 0$$

induced by $\hat{\sigma}$. By assumption, $\pi_n(F\hat{\sigma})$ is a finitely generated π -module. Therefore, we can wedge finitely many copies of S^{n+1} to \hat{K} , in the exact same way as in the $n = 1$ case, to form a finite CW complex L and an $(n + 1)$ -connected map $L \rightarrow X$, as desired.

(\implies) For the converse, we may assume that X is a CW complex with finite skeleta, for example by repeatedly using the fill-in lemma ([1], Lemma 1.2.3). By the van Kampen theorem, it is clear that π is finitely presented. We will first show that if (Y, B) is an n -connected CW-pair ($n \geq 1$) such that Y has finite skeleta, then there exists a weak equivalence of CW pairs $(Y, B) \rightarrow (\hat{Y}, \hat{B})$ such that \hat{Y} has finite $(n + 1)$ -skeleton and \hat{B} contains the n -skeleton of \hat{Y} . By enlarging the subcomplex B by attaching 1-cells between a vertex of B and a vertex of $Y \setminus B$, we may as well assume that B contains every vertex of Y .

For each 1-cell e_α of Y which is not in B , let $f_\alpha : \partial\Delta^1 \rightarrow B$ be its attaching map. Since (Y, K) is connected, f_α is nullhomotopic. This nullhomotopy can be expressed as a map $g_\alpha : \Lambda_0^2 \rightarrow B$, which restricts to f_α on the boundary of the 0th face. We can extend g_α to a map $h_\alpha : \partial\Delta^2 \rightarrow Y$ by letting the restriction of h_α to the 0th face be the inclusion of the cell e_α . Since (Y, B) is 1-connected, we can modify g_α so that h_α is nullhomotopic. Define \hat{B}_1 to be the pushout:

$$\begin{array}{ccc} \sqcup_\alpha \Lambda_0^2 & \xrightarrow{\sqcup_\alpha g_\alpha} & B \\ \downarrow & & \downarrow \\ \sqcup_\alpha \Delta^2 & \longrightarrow & \hat{B}_1 \end{array}$$

Then we have an inclusion $j : Y_{(1)} \rightarrow \hat{B}_1$ and we can use the fact that h_α is nullhomotopic for all α to define a map $r : \hat{B}_1 \rightarrow Y$ such that rj is the inclusion of the 1-skeleton of Y . It is also clear that the inclusion $B \rightarrow \hat{B}_1$ is a weak equivalence.

Now we repeat this process to construct a CW pair (\hat{B}_n, B) equipped with an inclusion $j : Y_{(n)} \rightarrow \hat{B}_n$, and a map $r : \hat{B}_n \rightarrow Y$ such that rj is the inclusion of the n -skeleton and the inclusion $B \rightarrow \hat{B}_n$ is a weak equivalence. We define $\hat{B} = \hat{B}_n$. Let Z be the CW complex obtained by attaching the remaining cells of Y (of dimension greater than n) onto the subcomplex $Y_{(n)}$ of \hat{B} . Then Z contains Y as a subcomplex and we can extend the maps r and j in the obvious way and we have $rj = 1$. It is clear that $r_\star : \pi_\star(Z) \rightarrow \pi_\star(Y)$ is a surjection for all \star and is an isomorphism for $\star \leq n - 1$. We claim that r_n is also an isomorphism. Suppose that $\phi : S^n \rightarrow Z$ becomes nullhomotopic after applying r . By the cellular approximation theorem, ϕ is homotopic to a map which factors through $Z_{(n)} \subset \hat{B}$. Since the inclusion $B \rightarrow \hat{B}$ is a weak equivalence, we may even assume ϕ factors through B . Now the restriction of r to B is the inclusion of B into Y , and so ϕ is nullhomotopic in Y . This means that ϕ is nullhomotopic in Z , since Z contains Y as a subcomplex. It follows that r_n is injective, and, hence, an isomorphism as desired. Finally, we construct \hat{Y} by attaching cells of dimension $\geq n + 2$ to Z to extend r to a weak equivalence. Then the inclusion $(Y, B) \rightarrow (\hat{Y}, \hat{B})$ is a

weak equivalence where \hat{Y} has finite $(n+1)$ -skeleton and $\hat{Y}_{(n)} \subset \hat{B}$ as desired.

To complete the proof, let $\sigma : K \rightarrow X$ be a map from a finite CW complex K , as in condition ii) or iii). Then we may as well assume that σ is a cellular map and then replace σ by the inclusion $i : K \rightarrow M\sigma$, where $M\sigma$ is a CW complex with finite skeleta and K is a subcomplex. By assumption, i is n -connected with $n \geq 1$ and so we may consider a CW pair (Y, B) in which i corresponds to the inclusion $B \rightarrow Y$, B contains $Y_{(n)}$ and Y has finite $(n+1)$ -skeleton. We have:

$$\pi_n(F\sigma) = \pi_{n+1}(Y, B) = \pi_{n+1}(\tilde{Y}, \tilde{B}) = H_{n+1}(\tilde{Y}, \tilde{B}),$$

where the final equality follows from the relative Hurewicz theorem and the middle equality follows, when $n = 1$, from the fact that σ induces an isomorphism on fundamental groups. Considering the cellular chain complex of (\tilde{Y}, \tilde{B}) , we see that it is zero at degree n and is a finitely generated π -module at degree $n+1$. It follows that the quotient group $H_{n+1}(\tilde{Y}, \tilde{B})$ is a finitely generated π -module as desired. \square

The following corollary is useful for applications:

Corollary 1.2.2: *If X is a space such that $\pi_1(X)$ is finitely presented, $\mathbb{Z}[\pi]$ is a Noetherian ring and $H_i(\tilde{X})$ is a finitely generated π -module for all i , then X is weakly equivalent to a CW complex with finite skeleta.*

Proof: Suppose that $\sigma : K \rightarrow X$ is an n -connected map ($n \geq 1$) from a finite CW complex K which induces an isomorphism on fundamental groups. Then we have $\pi_i(F\sigma) \cong H_{i+1}(\tilde{X}, \tilde{K})$. Consider the exact sequence:

$$\dots \rightarrow H_{i+1}(\tilde{K}) \rightarrow H_{i+1}(\tilde{X}) \rightarrow H_{i+1}(\tilde{X}, \tilde{K}) \rightarrow H_i(\tilde{K}) \rightarrow H_i(\tilde{X}) \rightarrow 0$$

$H_{i+1}(\tilde{X})$ is a finitely generated π -module by assumption and $H_i(\tilde{K})$ is too since K is a finite cell complex. Since $\mathbb{Z}[\pi]$ is Noetherian, any submodule of $H_i(\tilde{K})$ is also a finitely generated π -module. It follows that $H_{i+1}(\tilde{X}, \tilde{K})$ is a finitely generated π -module, which means that X satisfies the conditions i) - iii) of the theorem. \square

Corollary 1.2.3: *If A is a finitely generated abelian group, then $K(A, 1)$ is weakly equivalent to a CW complex with finite skeleta.*

\square

Finally, for this subsection, we prove the implication $iv) \iff i)$ of Theorem 2.0.1. For the proof, we will need to use the fact that, if X is a nilpotent space, then $H_i(\tilde{X})$ is a nilpotent π -module for all i .

Corollary 1.2.4: *If X is a nilpotent space, then X is weakly equivalent to a CW complex with finite skeleta iff $\pi_1(X)$ and $H_i(\tilde{X})$ are finitely generated for all i .*

Proof: If X is a nilpotent space which is weakly equivalent to a finite CW complex, then π is finitely presented, and, hence, a finitely generated nilpotent group. Moreover, $H_i(\tilde{X})$ is finitely generated as a π -module. Since $\mathbb{Z}[\pi]$ is Noetherian, it follows that all quotients of a series expressing $H_i(\tilde{X})$ as a nilpotent π -module are finitely generated π -modules on which π acts trivially. This means they are finitely generated abelian groups. Inducting up the series, we see that this implies that $H_i(\tilde{X})$ is also a finitely generated abelian group, as desired.

Now assume that $\pi_1(X)$ and $H_i(\tilde{X})$ are finitely generated. Then π is f -nilpotent and, therefore, π is finitely presented and $\mathbb{Z}[\pi]$ is Noetherian. Since $H_i(\tilde{X})$ is also finitely generated viewed as a π -module, Corollary 2.2.2 implies that X is weakly equivalent to a CW complex with finite skeleta. \square

1.3 A Theorem of Serre

In this subsection we will prove a classical result of Serre - that for simply connected spaces X , all homotopy groups of X are finitely generated iff all homology groups of X are. Following [1], the strategy of proof will be to replace the simply connected space X by a Postnikov tower and induct up it using the Serre spectral sequence. In the previous subsection, we showed that if A is a finitely generated abelian group, then $K(A, 1)$ is weakly equivalent to a CW complex with finite skeleta, and so, in particular, has finitely generated homology groups. The next lemma allows us to extend this result to $K(A, n)$ for $n > 1$:

Lemma 1.3.1: *If X is simply connected, then $H_i(X)$ is finitely generated for all i iff $H_i(\Omega X)$ is finitely generated for all i .*

Proof: Consider the Serre spectral sequence for the fibration:

$$\Omega X \rightarrow PX \rightarrow X$$

where the local coefficient is trivial since X is simply connected. Since PX is contractible, the only non-zero term of the E^∞ -page is $E_{0,0}^\infty \cong \mathbb{Z}$. We have exact sequences:

$$E_{r,q-r+1}^r \rightarrow E_{0,q}^r \rightarrow E_{0,q}^{r+1} \rightarrow 0$$

Suppose that $H_i(X)$ is finitely generated for all i , and that $H_i(\Omega X)$ is finitely generated for $i < q$. Then, for $r \geq 2$:

$$E_{r,q-r+1}^2 = H_r(X, H_{q-r+1}(\Omega X))$$

is finitely generated, since if A, B are finitely generated abelian groups, then $\text{Tor}_1(A, B)$ is finitely generated since it can be expressed as a homology group of a chain complex of finitely generated abelian groups. It follows that $E_{r,q-r+1}^r$ is finitely generated, and by induction using the exact sequence above, that $E_{0,q}^2 = H_q(\Omega X)$ is finitely generated. Therefore, by induction, if $H_i(X)$ is finitely generated for all i , then $H_i(\Omega X)$ is finitely generated for all i . The proof of the reverse implication is entirely analogous. \square

Corollary 1.3.2: *If A is a finitely generated abelian group and $n > 1$, then $K(A, n)$ is weakly equivalent to a CW complex with finite skeleta.*

Proof: Inductively, Lemma 2.3.1 implies that $H_i(K(A, n))$ is finitely generated for all i , so the result follows from Corollary 2.2.2. \square

Theorem 1.3.3: *If X is simply connected, then $\pi_i(X)$ is finitely generated for all i iff $H_i(X)$ is finitely generated for all i .*

Proof: Since X is simple, we can assume that X is the limit of a Postnikov tower:

$$\dots \rightarrow X_{n+1} \rightarrow X_n \rightarrow \dots \rightarrow X_1 = \star$$

Therefore, we have fibrations:

$$X_{n+1} \rightarrow X_n \rightarrow K(\pi_{n+1}(X), n+2)$$

If $H_i(X_n)$ is finitely generated for all i , then inspection of the argument given in Lemma 2.3.1 shows that we only used the fact that the E^∞ page was finitely generated. Therefore, the argument can be generalised to show that $H_i(X_{n+1})$ is finitely generated for all i iff $H_i(K(\pi_{n+1}(X), n+2))$ is finitely generated for all i . Therefore, if $\pi_i(X)$ is finitely generated for all i , an inductive argument using the fact that $H_i(K(\pi_{n+1}(X), n+2))$ is finitely generated for all i and n , shows that $H_i(X_n)$ is finitely generated for all i and n . It follows that $H_i(X)$ is finitely generated for all i , since the groups $H_i(X_n)$ eventually stabilise at $H_i(X)$ for large n .

Now suppose that $H_i(X)$ is finitely generated for all i . The map $X \rightarrow X_n$ is an $(n+1)$ -equivalence, since $\pi_{n+1}(X_n) = 0$, and so $H_i(X) \rightarrow H_i(X_n)$ is an isomorphism for $i \leq n$ and a surjection when $i = n+1$. It follows that $H_i(X_n)$ is finitely generated whenever $i \leq n+1$. Suppose that we have proved that $\pi_i(X)$ is finitely generated for $i \leq n$. Then inductively, similarly to our previous work, we can show that $H_j(X_i)$ is finitely generated for $i \leq n$ and all j . Consider the Serre spectral sequence for the fibration:

$$X_{n+1} \rightarrow X_n \rightarrow K(\pi_{n+1}(X), n+2)$$

Then $E_{p,q}^\infty$ is finitely generated for all p and q , and $E_{0,q}^r$ is finitely generated whenever $q \leq n+2$. We also have $E_{n+2,0}^2 = \pi_{n+1}(X)$. We have exact sequences:

$$0 \rightarrow E_{n+2,0}^{r+1} \rightarrow E_{n+2,0}^r \rightarrow E_{n-r+2,r-1}^r$$

When $r = n+2$, $E_{n+2,0}^{r+1} = E_{n+2,0}^\infty$ and $E_{n-r+2,r-1}^r = E_{0,n+1}^{n+2}$ and so both of these are finitely generated. It follows that $E_{n+2,0}^{n+2}$ is finitely generated. For $2 \leq r < n+2$, $E_{n-r+2,r-1}^r = 0$ and so, inductively, it follows that $E_{n+2,0}^2 = \pi_{n+1}(X)$ is finitely generated, as desired. It follows, again inductively, that $\pi_i(X)$ is finitely generated for all i . \square

We can now prove another implication of Theorem 2.0.1:

Corollary 1.3.4: *If X is a nilpotent space, then $\pi_i(X)$ finitely generated for each $i \geq 1$ iff $\pi_1(X)$ and, for $i \geq 2$, $H_i(\tilde{X})$ are finitely generated.*

Proof: If $\pi_i(X)$ is finitely generated for $i \geq 1$, then so is $\pi_i(\tilde{X})$ and so Theorem 2.3.3 applies to show that $H_i(\tilde{X})$ is finitely generated for all i . On the other hand, if $\pi_1(X)$ and, for $i \geq 2$, $H_i(\tilde{X})$ are finitely generated, then Theorem 2.3.3 tells us that $\pi_i(\tilde{X})$ ($= \pi_i(X)$) is finitely generated for $i \geq 2$ and so all homotopy groups of X are finitely generated. \square

1.4 The final implication

We have already shown that the first four conditions of Theorem 2.0.1 are equivalent and that $i) \implies v)$. Therefore, to complete the proof of Theorem 2.0.1 we just need to show that $v)$ implies any one of $i) - iv)$. We will show that $v) \implies iv)$. For the proof, we will want to assume that the nilpotent space X is, in fact, the limit of a Postnikov \mathcal{A} -tower and to define the 'universal cover' of X to be the fiber of the fibration:

$$\tilde{X} \rightarrow X \rightarrow K(\pi_1(X), 1)$$

induced by the tower. As justification for doing this, consider a fibrant cofibrant approximation \hat{X} of X in the Quillen model structure on topological spaces, so that there is a weak equivalence between the fiber of the composite fibration $\hat{X} \rightarrow K(\pi_1(X), 1)$ and \tilde{X} . Finally, for the justification, note there is a weak equivalence from the universal cover of \hat{X} to the fiber of this composite fibration. Recall also that, if X is nilpotent, then $\pi_1(X)$ acts nilpotently on $H_i(\tilde{X})$, since this fact plays a key role in the following proof:

Theorem 1.4.1: *Let X be a nilpotent space. If $H_i(X)$ is finitely generated for each $i \geq 1$, then $\pi_1(X)$ and, for $i \geq 2$, $H_i(\tilde{X})$ are finitely generated.*

Proof: By the Hurewicz theorem, the abelianisation of the nilpotent group $\pi = \pi_1(X)$ is finitely generated, and so it follows that π itself is finitely generated by our algebraic result, Theorem 2.1.1. Consider the Serre spectral sequence of the fibration:

$$\tilde{X} \rightarrow X \rightarrow K(\pi, 1)$$

Since π is finitely presented and $\mathbb{Z}[\pi]$ is Noetherian, it follows that $K(\pi, 1)$ is weakly equivalent to a CW complex with finite skeleta. The E^∞ page is finitely generated by assumption, and we have that $E_{p,0}^2 = H_p(K(\pi, 1); \mathcal{H}_0(\tilde{X})) = H_p(K(\pi, 1))$ is finitely generated. Suppose that we have shown $H_i(\tilde{X})$ is finitely generated for $i \leq n$. We have exact sequences:

$$E_{r,n-r+2}^r \rightarrow E_{0,n+1}^r \rightarrow E_{0,n+1}^{r+1} \rightarrow 0$$

Now, $E_{r,n-r+2}^2 = H_r(K(\pi, 1); \mathcal{H}_{n-r+2}(\tilde{X}))$ and, if $r \geq 2$, this is finitely generated since it is the homology of a complex of finitely generated abelian groups, namely the cellular chain complex, with finitely generated local coefficients, of a CW complex with finite skeleta. It follows, inductively, that $E_{0,n+1}^2$ is finitely generated. We have:

$$E_{0,n+1}^2 = H_0(K(\pi, 1); \mathcal{H}_{n+1}(\tilde{X})) = H_{n+1}(\tilde{X})/\pi$$

Let:

$$1 = G_0 \rightarrow \dots \rightarrow G_k = H_{n+1}(\tilde{X})$$

be a sequence of π -submodules of $H_{n+1}(\tilde{X})$ expressing $H_{n+1}(\tilde{X})$ as a nilpotent π -group.

Consider the exact sequence of homology groups with local coefficients:

$$\dots \rightarrow H_1(K(\pi, 1); \frac{G_{i+1}}{G_i}) \rightarrow H_0(K(\pi, 1); \mathcal{G}_i) \rightarrow H_0(K(\pi, 1); \mathcal{G}_{i+1}) \rightarrow H_0(K(\pi, 1); \frac{G_{i+1}}{G_i}) \rightarrow 0$$

When $i = k - 1$, the end surjection implies that $H_0(K(\pi, 1); \frac{G_k}{G_{k-1}}) = \frac{G_k}{G_{k-1}}$ is finitely generated. This implies, along with the fact that $K(\pi, 1)$ is weakly equivalent to a CW complex with finite skeleta, that $H_1(K(\pi, 1); \frac{G_k}{G_{k-1}})$ is finitely generated. It follows that $H_0(K(\pi, 1); \mathcal{G}_{k-1})$ is finitely generated. Continuing in this fashion down the series express G_k as a nilpotent π -group, we find that $H_0(K(\pi, 1); \frac{G_{i+1}}{G_i}) = \frac{G_{i+1}}{G_i}$ is finitely generated for all i . It follows that $H_{n+1}(\tilde{X})$ is f -nilpotent and, hence, finitely generated. Inductively, it now follows that $H_i(\tilde{X})$ is finitely generated for all i , as desired. \square

1.5 Bibliography

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