5 Points Available

Instructions

Please write your **Name, Student Number, and Tutorial Number** at the top of this page.
Remember: unless you are on the waitlist for the course, you have to write this in your **registered** tutorial.

Make sure to show as many steps of your work as possible, justify as much and annotate any interesting steps or features of your work. ***Do not just give the final answer.***

QUESTION 1 Fix some $n \geqslant 2$. We select $n+1$ different elements of the set $[2n] = \{1, 2, ..., 2n-1, 2n\}$. Prove that (no matter our choice of n) there will always be two numbers from our selection which are relatively prime (i.e. so that their greatest common divisor is 1.)

Hint: what can we say about consecutive integers, like k & $k+1$?

Solution: Notice that there are precisely n, pairs of consecutive integers when we partition the set $[2n]$ into the sets $\{1,2\}, \{3,4\}, \ldots, \{2n-1, 2n\}$. Let the number of distinct integers we pick $(n+1)$ be the pigeons, and let the number of subsets of pairs (n) be the holes, by the PHP we have that at least one of the holes must have two pigeons, that is we must have picked two consecutive integers, i and $i+1$, and since these two numbers are co-prime, we are done.

Aside: Suppose that $a \mid i$ and $a \mid i+1$, that is, suppose that i and $i+1$ are not co-prime. We can write $i = ab$ and $i+1 = ac$, taking the difference $1 = (i+1) - i = a(c-b)$, since $a, b, c \in \mathbb{Z}$, we have that $a = 1$. Thus any common divisor of i and $i+1$ is 1.