Instructions

Due: September 20th, 2017, at the start of your registered tutorial.

Make sure to follow the **Assignment Formatting** and **Groupwork and Plagiarism** guidelines on the course website.

Exercises

EXERCISE 1 ***Only hand in your answer to (b).***

Suppose $a_1, ..., a_n$ is a sequence of real numbers.

(a) Let μ be the *mean* of our sequence:

$$\mu = \frac{1}{n}\sum_{i=1}^{n} a_i.$$

Prove that there are positive integers $i, j \leqslant n$ so that $a_i \leqslant \mu \leqslant a_j$.

(b) Let H be the *harmonic mean* of our sequence:

$$H = n\left(\sum_{i=1}^{n}\frac{1}{a_i}\right)^{-1}$$

Prove that there are positive integers $i, j \leqslant n$ so that $a_i \leqslant H \leqslant a_j$.

Solution:

(a) Suppose that we have $a_i < \mu$ for all $i \in \{1, 2, \ldots, n\}$. This implies

$$\sum_{i=1}^{n} a_i < \underbrace{\mu + \cdots + \mu}_{n-\text{times}} = n\mu = \sum_{i=1}^{n} a_i.$$

A contradiction. This means that there must exist some $i \in \{1, 2, \ldots, n\}$ such that $a_i \geqslant \mu$. A structurally identical argument shows that there must also exist $j \in \{1, 2, \ldots, n\}$ such that $a_j \leqslant \mu$.

(b) *NOTE:* We need to impose the additional requirement that either $a_i > 0$ or $a_i < 0$ for all $i \in \{1, 2, \ldots, n\}$. If we allow real numbers of mixed sign then the proposition is false, for example $H(-1, 1, 1) = 3$; this violates the proposition.

We use part (a) to prove (b). Rewrite the harmonic mean as follows

$$\frac{1}{H} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}$$

Let $\mu = \frac{1}{H}$, and $b_i = \frac{1}{a_i}$. The equation defining the harmonic series then becomes

$$\mu = \frac{1}{n}\sum_{i=1}^{n} b_i$$

Part (a) tells us there are $i, j \in \{1, 2, \dots, n\}$ such that $b_i \leqslant \mu \leqslant b_j$, or equivalently

$$\frac{1}{a_i} \leqslant \frac{1}{H} \leqslant \frac{1}{a_j}$$

Suppose $0 < \frac{1}{a_i} \leqslant \frac{1}{H} \leqslant \frac{1}{a_j}$, then we have $0 < a_j \leqslant H \leqslant a_i$, similarly if $\frac{1}{a_i} \leqslant \frac{1}{H} \leqslant \frac{1}{a_j} < 0$, we have $a_j \leqslant H \leqslant a_i < 0$, as desired.

EXERCISE 2

Suppose that the integers from 1 to n are arranged around a circle (for some fixed n). Let $k \leqslant n$ be a positive integer. Show that there exists a sequence of k adjacent numbers in the arrangement whose sum is at least

$$\left\lceil \frac{k(n+1)}{2} \right\rceil$$

Solution:

Given a distribution of integers arranged in a circle, choose a starting position on the circle and in a clockwise direction label each position in ascending order. Let S_i denote the sum starting in the i^{th} position and ending at the $(i+k-1)^{th}$ position, that is, in a clockwise manner we sum k elements starting at the i^{th} position. We the have that

$$\frac{1}{n}\sum_{i=1}^{n} S_i \overset{(\star)}{=} \frac{1}{n}\sum_{i=1}^{n} k \cdot i = k\frac{1}{n}\frac{n(n+1)}{2} = \frac{k(n+1)}{2}$$

By Exercise 1 (a) this means that there exists S_i such that $S_i \geqslant \frac{k(n+1)}{2}$, since we are dealing with integers this translates to $S_i \geqslant \lceil \frac{k(n+1)}{2} \rceil$, as desired.

NOTE: The equality $(\star)$ comes from the fact that every integer arranged in the circle appears in exactly k of the sums, S_i. This argument becomes pretty intuitive if you draw yourself a picture of which elements are summed over for each S_i, take $n = 6$ and $k = 3$, for example.

EXERCISE 3 *(Chapter 1, Exercise #31)* A teacher receives a paycheck every two weeks, always the same day of the week. Is it true that in any six consecutive calendar months she receives exactly 13 paychecks?

Solution:

This is False. Notice that the smallest number of days in a 6 month span is given by $28+31+30+31+30+31 = 181$ (Feb - July; or Jan - June), and notice that the most number of days it can take to collect 13 pay checks is given by $13 * 14 = 182$ (this happens when you collect a pay check the day before the 6mo window starts). In this case you would collect 12 pay checks between Feb. 1st and July 31st, with the 13th pay check coming a day late, on Aug. 1st.

Alternate Solution:

Let me preface this by saying that I think what one should do is just take a few minutes and think of a counter example, however, if one really wants to explicitly turn this into a PHP-type solution, then one possibility is to proceed as follows.

Step 1: [Suppose we get 26 pay checks this year.]

This is a reasonable assumption since there are 365 days in a (non-leap) year, with pay checks coming every 14 days, the most number of pay checks one can collect in a given year is 27 (note that the minimum number of days it takes to collect 27 pay checks is: $26 * 14 + 1 = 365$), however this happens only when the first pay check of the year comes on Jan. 1st. Thus, we can consider a case where we receive our first pay check on Jan. 2nd or later (i.e. we are only going to receive 26 pay checks this year.)

Step 2 : [Let there be 365 balls, i.e. $n = 365$]

We are going to try and use the generalized PHP to show the existence of a 6 consecutive calendar months where only 12 pay checks are collected. Thus, we will take the days of the year to be the balls, which we will later place into boxes.

Step 3: [Let there be 2 boxes, i.e. $m = 2$]

Consider splitting the calendar months up into two sets, the first set contains Jan. - June, the second set contains the set July - Dec.

Step 4: [Let $r = 182$]

Step 5: [One of the boxes contains 183 balls.]

Here we use the Generalize PHP. We are putting 365 balls into 2 boxes, since $365 = n > rm = 182 \cdot 2 = 364$, one of the boxes (6 month intervals) must contain at least 183 balls (days).

What does this buy us? We now know that one of Jan. - June or July - Dec. must contain 183 days. The minimum number of days it takes to collect 14 pay checks is $14 * 13 + 1 = 183$ days. Therefore there is a scenario in which we collect 14 pay checks in one of Jan. - June or July - Dec. and in turn we collect $12 = 26 - 14$ pay checks in the other 6 month interval.

One thing we should check is that the specific case where we collect 14 pay checks in one 6 month period doesn't coincide with the case that we collect our first pay check of the year on Jan. 1st (because then this argument doesn't go through) ... however a quick check of how many days are in the Jan. - June interval tells us that there are 182 days (not enough to ever collect 14 pay checks) meaning we must have collected the 14 pay checks in the other interval, to collect 14 pay checks in July - Dec. means we would have had to have collected a pay check on July 1st, meaning that our first January pay check came on Jan. 14 $\neq$ Jan 1st.

In short, come up with an explicit counter example.

EXERCISE 4

This one is similar to Chapter 2, Exercise #35, but I didn't like its wording.

Let m be a positive integer with n digits, $d_0, ..., d_{n-1}$ written as:

$$m = d_{n-1}\ d_{n-2}\ ...\ d_1\ d_0.$$

For example, the number $m = 1982137$ would have $d_0 = 7, d_1 = 3, ..., d_6 = 1$.

Prove, using induction on n, that m is divisible by 11 if and only if the following sum is divisible by 11:

$$\sum_{i=0}^{n-1} (-1)^i d_i$$

Hint: How do we write an integer (any integer) in terms of powers of 10 and its digits?

Solution:

We first show that $10^n \equiv (-1)^n \mod 11$ for $n \geqslant 0$. Consider $n = 0$, we have $10^0 \equiv (-1)^0 \mod 11$, for the inductive step ($n = i+1$) we have $10^{i+1} = 10 \cdot 10^i \overset{\text{IH}}{\equiv} (-1) \cdot (-1)^i \equiv (-1)^i \mod 11$. With this:

$$m = \sum_{i=0}^{n-1} d_i 10^i \equiv \sum_{i=0}^{n-1} d_i (-1)^i \mod 11$$

Thus $m \equiv 0 \mod 11 \iff \sum_{i=0}^{n-1} d_i(-1)^i \equiv 0 \mod 11$, as desired.