Instructions

Due: October 26th, 2017, during lecture ***(drop it off either at the beginning of class, at the end, or during the mid-class 10 minute break - i.e. do not interrupt class to hand it in.)*** **Alternately, you may hand in your assignment during tutorial (Oct 25th) or lecture (Oct 24th).**

Follow the **Assignment Formatting** and **Groupwork and Plagiarism** guidelines on the course website.

Exercises

(1) Chapter 5, # 33

Solution: LHS: $B(n)$ counts the number of partitions of $[n]$ with any number of parts.

RHS: Here we are counting the number of partition of $[n]$ with any number of parts, however we split up counting the partitions into two cases.

$F(n)$ counts precisely the partitions of $[n]$ with no singleton parts (by definition).

$F(n+1)$ can be thought of as counting the partitions of $[n]$ with *at least one* singleton part, in particular we get a bijective mapping $\mathcal{A} \longrightarrow \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\mathcal{A} := \{\text{partitions of } [n] \text{ with } \geqslant 1 \text{ singleton part}\}$$
$$\mathcal{B} := \{\text{partitions of } [n+1] \text{ with no singleton parts}\}$$

By taking all the singleton parts in any given partition $\Pi \in \mathcal{A}$, and grouping them into one part along with the element $n+1 \in [n+1]$, resulting in a partition in the collection $\mathcal{B}$. (Note: this map is clearly invertible, and thus a bijection)

(2) Chapter 5, # 35

(3) Chapter 8, # 26

Solution: Multiply the recurrence relation $a_{n+2} = 8a_{n+1} - 16a_n$, by x^{n+2} and sum over all $n \geqslant 0$, resulting in

$$\sum_{n\geqslant 0} a_{n+2}x^{n+2} = 8x \sum_{n\geqslant 0} a_{n+1}x^{n+1} - 16x^2 \sum_{n\geqslant 0} a_n x^n$$

Reindexing yields

$$\sum_{n\geqslant 2} a_n x^n = 8x\sum_{n\geqslant 1} a_n x^n - 16x^2\sum_{n\geqslant 0} a_n x^n$$

If $F(x) := \sum_{n\geqslant 0} a_n x^n$ is the generating function for the sequence $(a_n)_{n\geqslant 0}$, then

$$F(x) - 4x - 1 = 8x(F(x) - 1) - 16x^2F(x)$$

Solving for $F(x)$ we find

$$\begin{aligned} F(x) &= \frac{-4x+1}{16x^2-8x+1} \\ &= \frac{1}{1-(4x)} \\ &= \sum_{n\geqslant 0}(4x)^n \end{aligned}$$

Two formal power series are equal if and only if all of the coefficients are equal, thus we have that $a_n = 4^n$.

(4) Chapter 8, # 27

Solution: Let a_n denote the number of insects at the end of the nth year. a_n then satisfies the recurrence relation

$$a_{n+1} = 2a_n + 1000, \qquad a_0 = 50, \qquad \forall n \geqslant 0$$

As in the above question, multiply the leading term of the recurrence relation by the corresponding power of x and sum over all $n \geqslant 0$.

$$\sum_{n\geqslant 0} a_{n+1}x^{n+1} = 2x\sum_{n\geqslant 0} a_n x^n + 1000x\sum_{n\geqslant 0} x^n$$

Letting $F(x) := \sum_{n\geqslant 0} a_n x^n$, be the generating function for the recurrence relation, and reindexing results in

$$F(x) - 50 = 2xF(x) + 1000x\frac{1}{1-x}$$

Solving for $F(x)$ yields

$$\begin{aligned} F(x) &= \frac{50(19x+1)}{(x-1)(2x-1)} \\ &= \frac{1000}{x-1} - \frac{1050}{2x-1} \\ &= -1000\left(\frac{1}{1-x}\right) + 1050\left(\frac{1}{1-(2x)}\right) \\ &= \sum_{n\geqslant 0}(1050\cdot 2^n - 1000)x^n \end{aligned}$$

As above, this implies that $a_n = 1050 \cdot 2^n - 1000$ for all $n \geqslant 0$.