5 Points Available

Instructions

Please write your **Name and Student Number** at the top of this page.
Remember: you have to write quizzes in your **registered** tutorial.

Make sure to show as many steps of your work as possible, justify as much and annotate any interesting steps or features of your work. ***Do not just give the final answer.***

Question 1 Prove that there is a positive integer n so that $117^n - 1$ is divisible by 11. (Note: $117 = 3^2 \cdot 13$.)

Hint: consider the remainders (mod 11) of the numbers $117 - 1, 117^2 - 1, 117^3 - 1, ..., 117^{11} - 1$.

Solution: If we have $117^i - 1 \mod 11 \equiv 0$ for some $1 \leqslant i \leqslant 11$ then we are done.

Suppose for contradiction we have $117^i - 1 \mod 11 \in \{1, 2, \dots, 10\}$, for all $1 \leqslant i \leqslant 11$. In this case, letting the residues of the numbers $117 - 1, 117^2 - 1, ..., 117^{11} - 1$ be the pigeons, and letting the numbers $1, 2, \dots, 10$ be the holes, we have 11 pigeons and 10 holes, therefore by the PHP one of the holes must contain at least two pigeons, that is, we must have that

$$117^i - 1 \equiv 117^j - 1 \mod 11 \qquad \text{for some } 1 \leqslant i < j \leqslant 11$$

Taking the difference gives

$$(117^j - 1) - (117^i - 1) = 117^i(117^{j-i} - 1) \equiv 0 \mod 11$$

This implies $11 \mid 117^i(117^{j-i} - 1)$. Since $117 = 3^2 \cdot 13$, we have that $11 \nmid 117$, thus it must be the case that $11 \mid 117^{j-i} - 1$, contradicting the initial assumption that $117^i - 1 \mod 11 \in \{1, 2, \dots, 10\}$, for all $1 \leqslant i \leqslant 11$.