5 POINTS AVAILABLE

INSTRUCTIONS

Please write your **Name and Student Number** at the top of this page.
Remember: you have to write quizzes in your **registered** tutorial.

Make sure to show as many steps of your work as possible, justify as much and annotate any interesting steps or features of your work. ***Do not just give the final answer.***

QUESTION 1 Prove that for any nonnegative integer n,

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$$

QUESTION 2 Using a **double counting** argument, prove the following identity, for any *fixed* nonnegative integers $m \leqslant n$:

$$n\binom{n-1}{m-1} = m\binom{n}{m}$$

Your argument could be a "committee formation" argument, as in lecture, or an argument involving subsets, as in the textbook. But it shouldn't involve induction, or breaking down binomial coefficients into factorials, apply the Binomial Theorem, or give any sort of "algebraic" argument, etc.

Solution: **(1)** This is a direct application of the Binomial Theorem (or the Multinomial Theorem if you're really going for the overkill ...), that is,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Setting $x = 2$, $y = 1$ gives us the result.

An alternative way of doing this is using a combinatorial argument. First we rewrite the equation a little,

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k = \sum_{k=0}^{n} \binom{n}{n-k} 2^k$$

Now we first interpret the right-most expression in the string of equalities above. Consider filling n boxes with 3 different kinds of objects (objects of type 1, 2, and 3). For fixed k, suppose we will fill $n-k$ of the boxes with objects of type 1, then there are $\binom{n}{n-k}$ many ways of choosing the $n-k$ boxes of which will will put the objects of type 1. For the remaining k boxes, there are 2^k many subsets of boxes - the number of subsets of the remaining k boxes corresponds to the number of ways of choosing boxes to fill with objects of type 2, and the boxes in the complement of those subsets will be filled with objects of type 3. Sum over all $k \in \{0, 1, \ldots, n\}$ and this represents the total number of ways to fill n boxes with objects of 3 difference types.

clearly 3^n also denotes the number of ways to distribute objects of 3 different types into n boxes (by the combinatorial product rule.

(2) We argue that this equality holds using a combinatorial argument. Consider counting the number of ways to pick a team of m players from n people, where one of the m players is the team captain.

LHS: We first pick the team captain, there are $\binom{n}{1} = n$ many ways to do this, we then choose the rest of the $m-1$ players from the remaining $n-1$ people; there are $\binom{n-1}{m-1}$ many ways of doing this. Thus, there are

$$\binom{n}{1}\binom{n-1}{m-1} = n\binom{n-1}{m-1}$$

many ways to choose a team of m player from n people, where one of the m players is the team captain.

RHS: First we pick the m players from the n people, there are $\binom{n}{m}$ many ways of doing this, then we choose the team captain from the m chosen players, there are $\binom{m}{1} = m$ many ways of doing this. Thus there are

$$\binom{n}{m}\binom{m}{1} = m\binom{n}{m}$$

many ways to choose a team of m player from n people, where one of the m players is the team captain.

Since both expressions count the number of ways of forming a team (plus choosing a captain) they expressions must be equal, as desired.