Assignment 4

Instructions

Due: November 10th, 2017, by the end of Prof. Rennet's office hours (i.e 12:30pm at the latest) *(Drop it off in the "MAT344" box outside the door, please. Feel free though to first come in and ask last minute questions!).*)

Alternately, you may hand in your assignment during lecture on Thursday Nov 9th, or during tutorial on Wednesday, Nov 8th.

Follow the **Assignment Formatting** and **Groupwork and Plagiarism** guidelines on the course website.

Exercises

Exercise 1

Find a closed-form of the generating function $f(x)$ for the sequence $\{a_n\}$, where a_n the equal to the number of partitions of n using only even numbers between 6 and 16, inclusively, and using exactly one or two odd numbers.

For example, if $n = 24$, some allowable partitions are $16+6+1+1$, $8+6+6+3+1$ and $23+1$. Some non-allowed partitions are $18+2+1+1$, and $16+8$.

*The **"closed-form"** means that you should express f as a sum/product of functions of x with no infinite sums involved. For example, using pieces like $1/(1-x)$.*

Solution: Here we have that the desired generating function is given by

$$\frac{1}{(1-x^6)}\frac{1}{(1-x^8)}\frac{1}{(1-x^{10})}\frac{1}{(1-x^{12})}\frac{1}{(1-x^{14})}\frac{1}{(1-x^{16})}\left(x+x^3+x^5+\cdots\right)\left(1+x+x^3+x^5+\cdots\right)$$

However, the infinite sums at the end are not in closed forms. This can be remedied by noting that we have

$$(x+x^3+x^5+\cdots) = x\frac{1}{1-x^2}, \qquad \text{and} \qquad (1+x+x^3+x^5+\cdots) = x\frac{1}{1-x^2}+1$$

Effectively we have:

(a) $\frac{1}{(1-x^i)}$, $i \in \{6, 8, 10, 12, 14, 16\}$: accounts for any number of parts of size i.

(b) $(x+x^3+x^5+\cdots)$: accounts for the (at least *one*) odd part that we must have.

(c) $(1+x+x^3+x^5+\cdots)$: accounts for the optional second odd part.

EXERCISE 2

Let w_n be the number of strings of length n using A and B which do not contain consecutive B's.

(a) Explain why w_n satisfies the recurrence relation

$$w_n = w_{n-1} + w_{n-2}, \text{ with } w_0 = 1, w_1 = 2.$$

Solution: We can count the number of these types of strings by counting two distinct subsets of these types of strings: those starting with A's and those starting with B's. If the string starts with an A then there are $n-1$ spots remaining, thus there are w_{n-1} many strings of the given kind which start with an A. If the string starts with a B then the second character must be an A so as not to have two B's in a row, thus there are $n-2$ many remaining spots and therefore there are w_{n-2} many strings of the given kind which start with B. Hence,

$$w_n = w_{n-1} + w_{n-2}.$$

(b) *In what follows, we will obtain the generating function $W(x)$ for w_n in a different way than our textbook (i.e. rather than attacking the recurrence relation directly.)*

Notice that a string of our type can be made by stringing together substrings of the form A and BA, followed by either a B or nothing at the end.

Explain how this observation allows us to conclude that $W(x) = \left(\frac{1}{1-(x+x^2)}\right) \cdot (1+x)$

Solution: Here the $x + x^2$ represents the choice of A or BA (since the first uses 1 from the length n; the second uses 2 from the length n.) The (1+x) represents our choice of a B at the end or nothing.

In the context of (*the proof of*) Theorem 8.13, suppose we have decide that we will use k-many A's and BA's (we don't know how many of each - just that there will be k "parts") and not have a B in the last position, then the number of ways of doing this is encoded in the coefficient of x^n in

$$(\underbrace{(x+x^2)(x+x^2)\cdots(x+x^2)}_{k-\text{times}} = (x+x^2)^k$$

summing over all possible k results in

$$\sum_{k\geqslant 0}(x+x^2)^k = \frac{1}{1-(x+x^2)}$$

Similarly, if we want k-many A's and BA's with the last part being a B, then this is encoded in the coefficient of x^n in

$$(\underbrace{(x+x^2)(x+x^2)\cdots(x+x^2)}_{k-\text{times}}x = (x+x^2)^k x$$

Summing over all possible k results in

$$x\sum_{k\geqslant 0}(x+x^2)^k = \frac{x}{1-(x+x^2)}$$

adding up these two expressions and factoring yields the given expression, as desired.

(c) Noticing that $(x+x^2)^n$ can be rewritten using an application of the Binomial Theorem, rewrite the above representation of $W(x)$ in a way that allows us to derive an explicit formula for w_n (i.e. a formula just in terms of n, and in particular, not involving w_{n-1}, w_{n-2}, ... etc.).

Solution: Consider the expression

$$\frac{1}{1-(x+x^2)}$$

From this we can rewrite and get

$$\begin{aligned}
\frac{1}{1-(x+x^2)} &= \sum_{k\geqslant 0}(x+x^2)^k \\
&= \sum_{k\geqslant 0} x^k(1+x)^k \\
&= \sum_{k\geqslant 0} x^k\left(\sum_{i=0}^{k}\binom{k}{i}x^i\right) \\
&= \sum_{k\geqslant 0}\left(\sum_{i=0}^{k}\binom{k}{i}x^{k+i}\right)
\end{aligned}$$

Thus the coefficient of x^n is given by

$$x^n : \qquad \sum_{i=0}^{n}\binom{n-i}{i}$$

By a completely analogous calculation, we have that the coefficient of $x\left(\frac{1}{1-(x+x^2)}\right)$ is given by

$$x^n : \qquad \sum_{i=0}^{n-1}\binom{n-1-i}{i}$$

That is, every coefficient is simply "bumped down" one position. This gives us an explicit formula for w_n since

$$W(x) = \sum_{k\geqslant 0} w_n x^n = \frac{1+x}{1-(x+x^2)} = \frac{1}{1-(x+x^2)} + x\frac{1}{1-(x+x^2)}$$

And we now know the series expansions for the two right most expressions.

(d) Relate our closed-form representation of $W(x)$ in part (b) to relate $W(x)$ to the generating function $F(x)$ for the sequence f_n of Fibonnaci numbers (i.e. the sequence of numbers satisfying $f_n = f_{n-1} + f_{n-2}$, with $f_0 = 1 = f_1$. Use this (simple) relationship between W and F to write a closed-form for $F(x)$.

Solution: Notice that we have

$$w_0 = 1 \quad w_1 = 2 \quad w_2 = 3 \quad w_3 = 5 \quad w_4 = 8 \quad w_5 = 13 \quad w_6 = 21 \ \cdots$$

$$f_0 = 1 \quad f_1 = 1 \quad f_2 = 2 \quad f_3 = 3 \quad f_4 = 5 \quad f_5 = 8 \quad f_6 = 13 \quad f_7 = 21 \ \cdots$$

Thus we see that the sequences are just shifted by 1, namely, $w_n = f_{n+1}$ (if one really felt like it, strong induction gives rigorous justification of this statement), this means that $xW(x) + 1 = F(x)$, that is

$$F(x) = \frac{x}{1-(x+x^2)}$$

and consequently, $f_n = \sum_{i=1}^{n-1} \binom{n-1-i}{i}$.

EXERCISE 3

Let $a_{n,k}$ denote the number of compositions (not *weak* compositions!) of n into k parts. And let $A(x, y)$ be the *multi-variable* generating function for $a_{n,k}$. This just means that $A(x, y) = \sum_n a_{n,k} x^n y^k$.

We will confirm that $a_{n,k} = \binom{n-1}{k-1}$ by finding a closed-form for $A(x, y)$ as follows:

(a) Notice that $A(x, y)$ will consist of a sum of terms of the form $x^n y^k$, where a term like $x^8 y$ represents the choice to put a single instance of the number "8" into our composition of n.[1]

Explain why this means that $A(x, y) = \dfrac{1}{1 - B(x, y)}$, where

$$B(x, y) = \frac{xy}{1-x} = (xy + x^2y + x^3y + ...)$$

(See Theorem 8.13!)

Solution: This is essentially exactly the proof of Theorem 8.13. Suppose that we would like to count the number of compositions of n into k parts, then the coefficient of $x^n y^k$ in

$$\underbrace{(xy + x^2y + x^3y + \cdots) \ \cdots \ (xy + x^2y + x^3y + \cdots)}_{k-\text{times}} = B(x, y)^k$$

counts the number of ways to do this, namely, that coefficient is $a_{n,k}$. Summing over all such k, gives the desired generating function

$$A(x, y) = \sum_{k \geqslant 0} B(x, y)^k = \frac{1}{1 - B(x, y)}$$

[1] An instance of the number "8", so x^8. A *single* instance of it, so y^1, giving $x^8 y$.

(b) Simplify the expression for $A(x,y)$ to show that $A(x,y)$ can be written as $1+xy\left(\frac{1}{1-x(1+y)}\right)$.

Solution:

$$\begin{aligned}\frac{1}{1-\left(\frac{xy}{1-x}\right)} &= \frac{1-x}{1-x-xy}\\ &= \frac{1-x+(-xy+xy)}{1-x-xy}\\ &= 1+\frac{xy}{1-x(1+y)}\end{aligned}$$

(c) Show how we can rewrite this as

$$1+xy\left(\sum_j x^j\left[\sum_i \binom{j}{i}y^i\right]\right)$$

Solution:

$$\begin{aligned}1+\frac{xy}{1-x(1+y)} &= 1+xy\sum_j [x(1+y)]^j\\ &= 1+xy\left(\sum_j x^j(1+y)^j\right)\\ &= 1+xy\left(\sum_j x^j\left(\sum_{i=0}^{j}\binom{j}{i}y^i\right)\right)\end{aligned}$$

(d) Show that the coefficient on $x^n y^k$ here is $\binom{n-1}{k-1}$ (as expected).

Solution: Consider the last expression in part (c), above. Expanded out fully this looks something like

$$\begin{aligned}&1+xy\\ &\quad+x^2\left[\binom{1}{0}y+\binom{1}{1}y^2\right]\\ &\quad+x^3\left[\binom{2}{0}y+\binom{2}{1}y^2+\binom{2}{2}y^3\right]\\ &\quad+x^4\left[\binom{3}{0}y+\binom{3}{1}y^2+\binom{3}{2}y^3+\binom{3}{3}y^4\right]\\ &\quad+\cdots\end{aligned}$$

So to find the coefficient of $x^n y^k$, we go to the n^{th} row as depicted above (corresponding to x^n), and then once in the n^{th} row, we find the coefficient of y^k, which we can see will be $\binom{n-1}{k-1}$, as desired.

EXERCISE 4

For a fixed k, let $q_{n,k}$ be equal to the number of partitions of n into k distinct parts, and let $Q_k(x)$ be this sequence's generating function.

Recall that (in lecture) we counted partitions of a similar type by taking a partition of a smaller number (which one?) into k parts, then adding a "staircase" to the left of it to make the resulting partition a partition of n with k *distinct* parts.

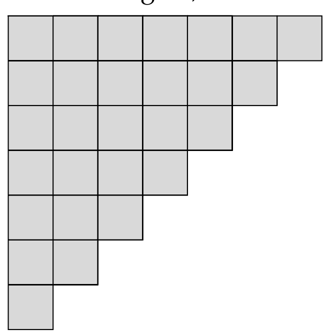
(a) Use this observation to determine a closed-form for the generating function $Q_k(x)$.

Solution: Notice that to count the number of partitions of n into k distinct parts, we can count the number of Ferrers diagrams in which we first fill out a "staircase" to ensure that each of the k parts has a distinct size (see Figure 1), and then "stack" *any* partition of the remaining blocks onto this staircase.

There are $k(k+1)/2 = \binom{k+1}{2}$ many blocks used in creating the staircase, thus there are $n - \binom{k+1}{2}$ many blocks remaining, which we can partition into at most k parts, however we like. That is,

$$q_{n,k} = p_{\leqslant k}\left(n - \binom{k+1}{2}\right)$$

Figure 1: The staircase which we add, when $k = 7$, to maker sure that each partition made from the remaining blocks has parts with distinct sizes. Again, here $k = 7$ and $n \geqslant \binom{8}{2} = 28$.



Thus we see that

$$Q_k(x) = \sum_{n\geqslant 0} q_{n,k}x^n = \sum_{n\geqslant 0} p_{\leqslant k}\left(n - \binom{k+1}{2}\right)x^n = x^{\binom{k+1}{2}}\left[\sum_{n\geqslant 0} p_{\leqslant k}(n)x^n\right] = \frac{x^{\binom{k+1}{2}}}{(1-x)(1-x^2)\cdots(1-x^k)}$$

(b) Now let

$$Q(x,y) = \sum_{k,n\geqslant 0} q_{n,k}x^n y^k = \sum_{k\geqslant 0}\left(\sum_{n\geqslant 0} q_{n,k}x^n\right)y^k = \sum_{k\geqslant 0} Q_k(x)y^k$$

and use (a) to rewrite this as an infinite product of finite sums of the form $(1+x^i y)$. Then explain how to interpret this infinite product so that it makes sense as the generating function for the "two-variable" sequence $q_{n,k}$ (i.e. where n and k are allowed to vary).

Note: interestingly, you will have written an infinite sum of (increasingly large) finite products as an infinite product of finite sums (of fixed length)! "Algebraic identities" like these are often discovered in other areas of math before a combinatorial explanation like the one above are found. (Crazy!)

Solution: We give a combinatorial interpretation of the infinite product

$$\prod_{n \geqslant 1}(1+x^n y)$$

Consider the terms in the expansion of the expression above involving $x^n y^k$, these are terms which all look like $x^{a_1+a_2+\cdots+a_k}y^k$, where $a_1+a_2+\cdots+a_k = n$ and $a_1 > a_2 > \cdots > a_k \geqslant 1$ (notice that the product above starts at $n=1$), thus the coefficient of the $x^n y^k$ term is precisely $q_{n,k}$. In particular, this means that

$$\sum_{k \geqslant 0} Q_k(x) y^k = Q(x,y) = \prod_{n \geqslant 1}(1+x^n y)$$

as desired.