# 5 Points Available

## Instructions

Please write your **Name and Student Number** at the top of this page.
**Remember:** you have to write quizzes in your **registered** tutorial.

*Make sure to show as many steps of your work as possible, justify as much and annotate any interesting steps or features of your work.* ***Do not just give the final answer.***

QUESTION 1 Let $a_n$ be the number of partitions of $n \geqslant 1$ using only the numbers $2, 3$ and $11$. Which of the following are correct expression(s) of the generating function $f(x) = \sum_{n \geqslant 1} a_n x^n$?

*(Select all and only those that apply! You don't need to justify or prove that your choices are correct.)*

(a) $\boxed{\dfrac{1}{(1-x^2)(1-x^3)(1-x^{11})}}$

(b) $(1+x^2)(1+x^3)(1+x^{11})$

(c) $\boxed{\left(\sum_{i \geqslant 0} x^{2i}\right) \cdot \left(\sum_{j \geqslant 0} x^{3j}\right) \cdot \left(\sum_{k \geqslant 0} x^{11k}\right)}$

(d) $(2x^2 + 3x^3 + 11x^{11})^n$

**Solution:** The expressions in (a) and (c) are equivalent, therefore an argument for why one is valid constitutes an argument for why the other is. Consider the expression in (c),

$$(1 + x^2 + x^4 + \ldots)(1 + x^3 + x^6 + \ldots)(1 + x^{11} + x^{22} + \ldots)$$

Any choice of monomial terms from each of the 3 products in which the sum of the exponents equals $n$ results from something of the form

$$x^{a_1 \cdot 2 + a_2 \cdot 3 + a_3 \cdot 11} = x^n, \qquad a_1, a_2, a_3 \in \mathbb{Z}_{\geqslant 0}$$

Namely, we have $a_1 \cdot 2 + a_2 \cdot 3 + a_3 \cdot 11 = n$, or rewriting

$$\underbrace{2 + 2 + \cdots + 2}_{a_1-\text{times}} + \underbrace{3 + 3 + \cdots + 3}_{a_2-\text{times}} + \underbrace{11 + 11 + \cdots + 11}_{a_3-\text{times}} = n$$

Thus every such choice of monomials results in a partition of $n$ using only the numbers 2, 3 and 11, and in turn this means that the function in (a) and (c) correspond to $f(x)$.

The expression in (b) is invalid because then the power series is finite (which isn't a problem in itself), this is problematic because a quick thought experiment tells us that every integer $n > 1$ can be written as a sum of 2's and 3's, thus every $n$ has *at least* one partition of the specified form. If $f(x)$ we equal to $(1+x^2)(1+x^3)(1+x^{11})$ then this would imply that every $n > 16$ would admit no such partitions which is clearly false.

Finally the expression in (d) (once we wrap our heads around the confusing appearance of the $n$ in the exponent) cannot be the generating function $f(x)$ for any choice of $n$, for essentially the same reason that rules (b) out.

QUESTION 2 Suppose that $A(x)$ is the generating function for a sequence $a_n$ with $a_0 = 1$. Then what is the generating function $B(x)$ for the sequence $b_n$ given by $b_n = \displaystyle\sum_{i=1}^{n} a_i$ for $n \geqslant 1$ and $b_0 = 0$?

**Solution:** In the case of $c_n = \displaystyle\sum_{i=0}^{n} a_i$ (that is, the sum starts at $i = 0$ and not $i = 1$ as in the question). Then the solution would be given by

$$\begin{aligned} C(x) &:= c_0 + c_1 x + c_2 x^2 + \cdots \\ &= \left(\sum_{i=0}^{0} a_i\right) + \left(\sum_{i=0}^{1} a_i\right) x + \left(\sum_{i=0}^{2} a_i\right) x^2 + \cdots + \left(\sum_{i=0}^{n} a_i\right) x^n + \cdots \\ &= (a_0 + a_1 x + a_2 x^2 + \cdots) \cdot (1 + x + x^2 + \cdots) \\ &= A(x) \cdot \frac{1}{1-x} \end{aligned}$$

However, for the $b_n$'s each sum actually starts at $i = 1$, $b_n = \sum_{i=1}^{n} a_i$, since $a_0 = 1$, we are losing 1 from each of the terms. We can think of $b_n$ as

$$b_n = \sum_{i=1}^{n} a_i = \left(\sum_{i=0}^{n} a_i\right) - 1$$

which means that $B(x)$ is given by

$$B(x) = A(x) \cdot \frac{1}{1-x} - \frac{1}{1-x}$$