FALL 2017 | # MAT344 MIDTERM | October 19th

*Fill in your **Name** (as it appears on Portal) and **Student ID**, **sign** below, and select your tutorial.*

NAME (Last, First): ______________________________

STUDENT ID: ______________________________

SIGNATURE: ______________________________

TUTORIALS - Indicate Your Registered Tutorial (✓)
WEDNESDAY
☐ TUT101 - 3pm
☐ TUT102 - 4pm
☐ TUT103 - 5pm

- There are **seven questions** on this test, some with multiple parts.
- There is a total of **30 available points.**
- **No aids are allowed.** (i.e. no calculators, cheat sheets, devices etc.)
- This test has **11 pages** including this page, and a **page of scrap**.
- **Nothing on the scrap page will be marked. You may remove them.**

MARKING - Leave This Blank							
/5	/4	/5	/3	/3	/4	/6	/30

QUESTION 1 **(5 points)**

Seventy-three points are given inside a hexagon with side lengths all 1. Prove that there are three of these points that span a triangle of area at most 1/8.

Note: the area of a triangle is $\frac{1}{2}bh$.

For future reference notice that the area of an equilateral triangle with side length, s, is given by

$$\frac{1}{2}bh = \frac{1}{2} \cdot s \cdot \frac{s\sqrt{3}}{2} = \frac{\sqrt{3}}{4}s^2$$

We also operate under the assumption that the hexagon is a regular hexagon, that is, one with all interior angles being $2\pi/3$ (this seems like a fair assumption).

Solution: To prove the above assertion, subdivide the hexagon into 6 equilateral triangles, each with side length 1, then further subdivide each of these triangles into 4 equilateral triangles each with side lengths equal to $\frac{1}{2}$.

The GPHP then tells us that one of these 24 equilateral triangles with sides of length $\frac{1}{2}$ (and area $\frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{16} < \frac{1}{8}$) must contain 4 points (although we only need 3). Thus, any triangle spanned using 3 of these 4 points must have an area less than that of the containing triangle (that is, one of the 24 equilateral triangles with areas equal to $\frac{\sqrt{3}}{16}$).

Question 2 **(4 points)**

Using a **double counting** argument, prove the following identity, for any *fixed* nonnegative integers n, and $r \leqslant n$:

$$\sum_{k=0}^{n-r+1} \binom{r+k}{r} = \binom{n+2}{r+1}$$

Your argument could be a "committee formation" argument, as in lecture, or an argument involving subsets, as in the textbook. But it shouldn't involve induction, or breaking down binomial coefficients into factorials, apply the Binomial Theorem, or give any sort of "algebraic" argument, etc.

Solution: RHS: The RHS counts the number of ways to choose $r+1$ people (to form a committee, say) from $n+2$ people.

LHS: **tl;dr**: Assume person m is on the committee, choose the remaining committee members from the first $1, \ldots, m-1$ people, and choose none of the $m+1, \ldots, n+2$ people. Do this for all $m \in \{r+1, r+2, \ldots, n+1, n+2\}$.

Long version: For the LHS, first write the sum in "long form", that is

$$\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} + \binom{n+1}{r}$$

Then we split the process of choosing $r+1$ people from $n+2$ into cases.

First suppose that the $(r+1)$st person is on the committee and we choose the remaining people from the first r people, there are $\binom{r}{r}$-many ways of doing this.

Next suppose that the $(r+2)$nd person is on the committee and we choose the remaining people from the first $r+1$ people, there are $\binom{r+1}{r}$-many ways of doing this.

Continuing on like this we reach the last case where we assume that the $(n+2)$nd person is on the committee and we choose the remaining people from the first $n+1$ people, there are $\binom{n+1}{r}$-many ways of doing this.

We then sum up the counts of all of these committees of size $r+1$ from $n+2$ people. Thus, we see that the LHS and RHS count the same thing.

QUESTION 3 **(5 points)**

Let $q_m(n)$ be the number of partitions of n with exactly m parts and with first part of size exactly m, and let $r_m(n)$ be the number of partitions of n with no part of size greater than m and no more than m parts.

Prove that for any positive integers n, m with $2m+1 \leqslant n$, we have

$$q_m(n) = r_m(n-2m+1)$$

Solution: *NOTE:* The definition of $r_m(n)$ was altered during the test to read: "let $r_m(n)$ be the number of partitions of n with no part of size greater than $m-1$ and no more than $m-1$ parts."

The proof of the above statement is probably best coupled with an explicit example of how the identification of q_m and r_m can be done. Suppose $m = 8$, and let $n = 32$. Figure 1 shows a possible partition of n - there isn't any difference between the black and grey squares; the black squares are coloured this way to drive home the point that the partitions which q_m counts will always have the black boxes filled.

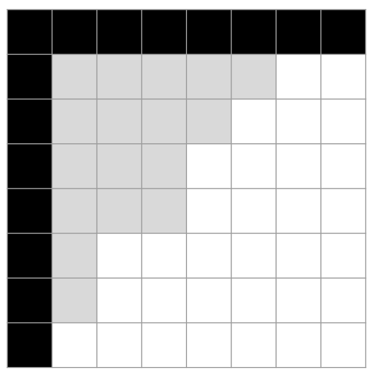


Figure 1: One possible partition of n counted by $q_m(n)$, where $m = 8$ and $n = 32$.

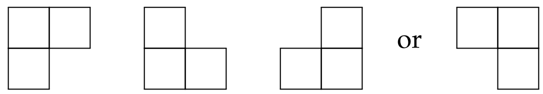
Thus, we can say that $q_m(n) = q_8(32)$ is counting the number of Ferrers diagrams with $n = 32$ boxes (or $n = 32$ *filled* boxes - if your using a diagram like the one above) which fit in an $m \times m = 8 \times 8$ grid and always have the black boxes filled.

What about $r_m(n-2m+1)$? If we look at the above example, the number $n-2m+1 = 32-2(8)+1 = 17$ is precisely the number of grey boxes, that is, $n-2m+1$ is exactly the result of deleting the black boxes from n, and in particular we can then interpret $r_m(n-2m+1) = r_8(17)$ as counting the number of Ferrers diagrams with $n-2m+1 = 17$ boxes which fit into an $(m-1) \times (m-1) = 7 \times 7$ grid.

Returning to $q_m(n)$ quickly, we see that since the black boxes are always filled $q_m(n)$ is effectively counting the number of Ferrers diagrams with $n-2m+1$ boxes which fit into a $(m-1) \times (m-1)$ grid. Thus, $q_m(n)$ and $r_m(n-2m+1)$ count the same thing, so the equality follows.

QUESTION 4 **(3 points)**

Prove that for any positive integer n, it is possible to tile any $2^n \times 2^n$ grid with exactly one square removed, using only "L"-shaped tiles with three squares, as in:



Hint: induction on n.

Solution: We take the advice given in the hint and proceed by induction on n. Although, let's alter the proposition to a slightly stronger proposition:

Prove that for any positive integer n, it is possible to tile any $2^n \times 2^n$ grid with exactly one *corner* square removed, using only "L"-shaped tiles with three squares

[Base Case]: For our base case we have $n = 1$. in this case we are trying to cover a 2×2 gird leaving only a corner square uncovered. This is done by using any one of the 4 pieces shown above.

[Inductive Step]: Assuming that a grid of size $2^n \times 2^n$ can be covered leaving only a corner square uncovered consider a grid of size $2^{n+1} \times 2^{n+1}$, This grid is in fact made up of 4 grids of size $2^n \times 2^n$, tile each of these subgrids using the Inductive hypothesis (for each of them) and then orient the tiled subgrids as shown to the left in Figure 2 (here we're really just rotating the individual tilings on each subgrid). There are 4 uncovered squares forming a 2×2 grid. Cover 3 of theses squares using the first of the 4 "L"-shaped pieces above and then rotate the bottom right subgrid as shown to the right in Figure 2. We have covered all the squares in the $2^{n+1} \times 2^{n+1}$ grid leaving only a corner square uncovered; completing the induction.

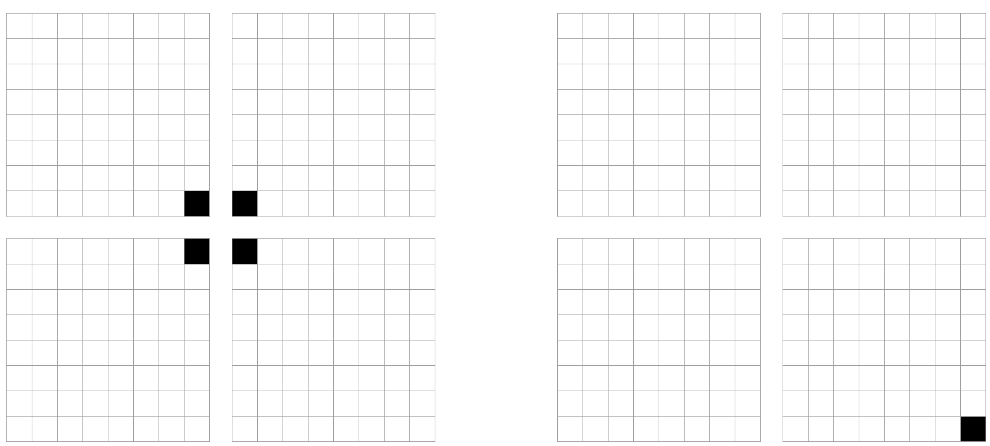


Figure 2: *Left:* An illustration of how to orient the 4 tilings of the subgrids of size $2^n \times 2^n$ given by applying the inductive hypothesis to each of the subgrids. Here the black squares denote the uncovered corner tiles of the $2^n \times 2^n$ grid. *Right:* Completion of the inductive step by covering 3 of the 4 uncovered tiles and rotating the bottom right tiled subgrid.

QUESTION 5 **(3 points)**

Prove the following identity for any integer $n \geqslant 2$:

$$n(n-1)4^n = \sum ac2^{b+4}\binom{n}{a,b,c}$$

(Where the sum is taken over all a, b, c so that $a+b+c=n$.)

Solution: Consider the expansion of the following product

$$(x+y+z)^n = \sum x^a y^b z^c \binom{n}{a,b,c}$$

Taking partials with respect to x and z results in

$$n(n-1)(x+y+z)^{n-2} = \frac{\partial^2}{\partial x \partial z}(x+y+z)^n = \frac{\partial^2}{\partial x \partial z}\left(\sum x^a y^b z^c \binom{n}{a,b,c}\right) = \sum (ax^{a-1})y^b(cz^{c-1})\binom{n}{a,b,c}$$

Setting $x=1$, $y=2$, $z=1$ gives

$$n(n-1)4^{n-2} = \sum a2^b c\binom{n}{a,b,c}$$

Multiplying both sides by $4^2 = 2^4$ yields the desired equality.

Question 6 **(4 points)**

Recall that $S(n,k)$, *the Sterling numbers of the second kind*, stand for the number of partitions of $[n]$ into k non-empty subsets.

Prove, using a **combinatorial argument**, that for all positive integers $k \leqslant n$,

$$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$$

Solution: $S(n,k)$ counts the number of ways of putting n *distinguishable* items into k *indistinguishable* boxes. These distributions of balls into boxes can be split into two disjoint sets defined by whether or not the "first" object is in a box by itself or not.

If the first object is in a box by itself, then there are $S(n-1,k-1)$-many ways to distribute the remaining $n-1$ objects into the remaining $k-1$ boxes.

If the first object is not in a box by itself, then there are $S(n-1,k)$-many ways of distributing the 2nd through nth objects into k boxes, but for each configuration there are k choices for which box we put the first object in; thus, there are $k \cdot S(n-1,k)$-many distributions.

Summing these two quantities yields the desired identity.

QUESTION 7 **(2 points per part - 6 points)**

You are on the sidewalk handing out 500 identical flyers (lucky you!). You hand them out to people who pass you in the street, *trying* to get one to each person. But **sometimes you miss people**, and **sometimes you accidentally hand out multiple copies at once**.

(a) Suppose that by the end, 200 people walked by you on the street, and that you managed to hand out all of the flyers. Furthermore, suppose that the first ten people all got exactly one flyer each, and that the last ten people each got at least two. Then how many ways are there for you to have handed out the flyers?

Solution: This is the weak compositions of $470 = 500 - 10 - 20$ into 190 parts (removing 10 flyers which the first 10 people received, and removing 20 flyers which the is the minimum number of flyers that the last ten people walked away with - at least 2 a piece). As such, this is $\binom{470+190-1}{190-1}$.

(b) Suppose that instead, you don't know how many people walked by you, but you do know that everyone who walked by got at least one flyer, and you know that the first ten people got exactly one flyer each again. How many ways are there for you to have handed out the flyers in this case?

Solution: This time we're talking about all compositions (not weak compositions) of 500, with exactly 1 ball in each of the first ten boxes (that is, 1 flyer per person for the first 10 people). Then this is the total number of compositions of 490 into an unknown number of boxes, i.e. 2^{489}.

(c) Finally, suppose that 250 people walk by, and that you've handed out all of the flyers. But unbeknownst to you, this time *each flyer had a (single) QR-code number* on it, and that there were exactly 25 different QR-codes, each appearing the same number of times amongst the flyers. This means, of course, that the flyers weren't actually identical after all. Now how many ways are there for you to have handed out the flyers?

Solution: This is still weak compositions, as in (a), but first, we put the flyers into some ordering. Since there are 25 different QR-codes and each QR-code appears on the same number of flyers, each distinct QR-code appears on $500/25 = 20$ flyers. This means that we want to order 500 objects where there are 25 different types of object and there are 20 of each type, this number of orderings is given by the multinomial coefficient

$$\binom{500}{\underbrace{20, 20, \ldots., 20}_{25 \text{ times}}}$$

This is the number of all possible orderings, and for each such ordering there are $\binom{500+250-1}{250-1}$-many ways (i.e., the number of weak compositions of 500 into 250 boxes) to hand out this ordering of flyers to the passersby. Thus, the answer is

$$\binom{500}{20, 20, \ldots., 20} \cdot \binom{500+250-1}{250-1}.$$

The End

Scrap Paper

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