5 POINTS AVAILABLE

INSTRUCTIONS

Please write your **Name and Student Number** at the top of this page.
Remember: you have to write quizzes in your **registered** tutorial.

Make sure to show as many steps of your work as possible, justify as much and annotate any interesting steps or features of your work. **Do not just give the final answer.**

QUESTION 1 Let $C(x)$ be the generating function for the sequence c_n, where c_n is the number of ways to take $[n]$, split it into some unspecified number of disjoint non-empty intervals and then pick exactly one number from each part.

*Note that **intervals** are sets of the form $\{i, i+1, ..., i+j\} \subseteq [n]$. So for example, c_{10} counts the number of ways to take $[10]$, split it into sets like $\{1, 2, 3\}, \{4, 5\}, \{6\}, \{7, 8, 9, 10\}$ and then pick one number from each of these subsets.*

Determine a simple, closed form for the generating function $C(x)$.

Solution: Let $B(x) := \sum_{k \geq 0} \binom{k}{1} x^n = \sum_{k \geq 0} k x^k$, this is the generating function for the sequence $b_k = \binom{k}{1}$, namely b_k counts the number of ways to choose 1 object from k objects. (We consider the sequence, b_k, because given an interval of size k we are choosing 1 number from those k numbers.) Notice

$$B(x) = \frac{x}{(1-x)^2}$$

Since we are partitioning $[n]$ into an unspecified number of disjoint non-empty intervals (and then choosing one number from each interval) this is the composition of generating functions (cf. Theorem 8.13), that is,

$$C(x) = \frac{1}{1 - B(x)} = \frac{1}{1 - \frac{x}{(1-x)^2}} = \frac{x^2 - 2x + 1}{x^2 - 3x + 1}$$