Instructions

Due: October 6th, 2017, during Prof. Rennet's 11:30-12:30 office hours (please drop it off in the box for MAT344 assignments). Alternately, you may hand in your assignment during tutorial (Oct 4th) or lecture (Oct 5th).

Follow the **Assignment Formatting** and **Groupwork and Plagiarism** guidelines on the course website.

Exercises

EXERCISE 1

For positive integers $k \leqslant n$, how many ways are there to split the set $[n]$ into a_1-many subsets of size 1, a_2-many subsets of size 2, ..., a_k-many subsets of size k, if we assume that the numbers $a_1, ..., a_k$ satisfy $n = \sum_{1 \leqslant i \leqslant k} i \cdot a_i$?

Solution: tl;dr: $\binom{n}{a_1,...,a_k} \cdot \frac{1}{(1!)^{a_1} \cdot ... \cdot (k!)^{a_k}}$

First suppose each of above subsets (boxes) is distinct, then we have $a_1 + a_2 + \cdots + a_k$ many (*distinct*) subsets. Place each integer into one of these subsets; there are

$$S := \binom{n}{\underbrace{1, 1, \ldots, 1}_{a_1-\text{times}}, \underbrace{2, 2, \ldots, 2}_{a_2-\text{times}}, \ldots, \underbrace{k, k, \ldots, k}_{a_k-\text{times}}} = \frac{n!}{(1!)^{a_1} \cdot ... \cdot (k!)^{a_k}}$$

many ways to do this.

If we return back to the situation given in the question then each of the a_1-many subsets of size 1 are really indistinguishable form one another, thus we have created $a_1!$-many duplicate arrangements, so we must correct our count by dividing S by $a_1!$. Similarly, each of the a_2-many subsets of size 2 are indistinguishable from one another, thus we have also over counted by a factor of $a_2!$ and therefore must divide S by $a_2!$, and so on. Therefore the total number of ways to splitting $[n]$ as described above is given by

$$S \cdot \frac{1}{a_1! a_2! \cdots a_k!} = \binom{n}{a_1, ..., a_k} \cdot \frac{1}{(1!)^{a_1} \cdot ... \cdot (k!)^{a_k}}$$

EXERCISE 2 *(Chapter 3, #42)*

A host invites n couples to a party. She wants to ask a subset of the $2n$ guests to give a speech, but she does not want to ask *both* members of any couple to give speeches. In how many ways can she proceed?

Note: we don't just want the final answer - you need to justify your answer.

Solution: Let's assume for a second that *couple* means a pairing of a man an a woman (this is really just for ease of notation). We can then liken this to counting the number of n-bit strings formed from a 3 letter alphabet $\{n, w, m\}$. Each position in the string corresponds to one of the n couples, each character from the alphabet determines whether no one (n), the woman (w) or the man (m) speaks from that couple. Thus, there are 3^n-many ways for her guests to give speeches.

EXERCISE 3 **Only hand in your solution for (b), and make sure you solve it with a combinatorial counting argument (not induction, etc.)**

(a) (Chapter 5, #28) Find a closed formula for $S(n, n-2)$, for all $n \geqslant 2$.

Solution: Since we are counting ways of putting n (distinguishable) balls into $n-2$ non-empty (indistinguishable) boxes we must first put at least one ball into each box (so as to make each box non-empty). Since there are $n-2$ boxes, this leaves two remaining balls to be distributed.

From this we se that any two configurations differ only by the balls which are grouped together. There are two disjoint scenarios for the groupings: One box with 3 balls in it with the remaining boxes being singletons - there are

$$\binom{n}{3}$$

many ways of grouping (selecting/choosing) 3 balls. The other scenario is two boxes, each with two balls two balls. If the boxes we distinct, there would be $\binom{n}{2}\binom{n-2}{2}$ many ways of putting two balls into box 1 and two balls into box 2, however since the boxes are indistinguishable, we have over counted by 2!, thus there are, in fact,

$$\binom{n}{2}\binom{n-2}{2}\Big/2!$$

Since we have counted all possible configurations from each of the two scenarios, we have that

$$S(n,n-2) = \binom{n}{3} + \binom{n}{2}\binom{n-2}{2}\Big/2!$$

(b) (Chapter 5, #29) Find a closed formula for $S(n,n-3)$, for all $n \geqslant 3$.

Solution: We proceed in a similar manner as in part (a). Fill each of the boxes with a ball, since there are $n-3$ boxes this leaves 3 balls to distribute. Analogously to part (a) we are left with 3 disjoint sets of configurations: One box with 4 balls - $\binom{n}{4}$, one box with 3 balls and another box with 2 balls $\binom{n}{3}\binom{n-3}{2}$, and 3 boxes with 2 balls $\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}\Big/3!$. Thus, we have

$$S(n,n-3) = \binom{n}{4} + \binom{n}{3}\binom{n-3}{2} + \binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}\Big/3!$$

EXERCISE 4 **Only hand in your solution for (b), and make sure you solve it with a combinatorial counting argument (not induction, etc.)**

(a) (Chapter 5, #36) Let a_n be the number of compositions of n into parts that are larger than 1. Express a_n using a_{n-1} and a_{n-2}. *(Justify your answer, of course).*

Solution: Notice that each *part* must have size at least 2. Thus we divide the set of all possible configurations into two disjoint subsets which we subsequently count. The two subsets are characterized by the size of the first part. Either the first *part* has size equal to 2 or first *part* has size greater than 2.

If the first *part* has size equal to 2 then we get a bijection between each of these compositions of n and the compositions of $n-2$ via the map that deletes the first *part*. There are a_{n-2}-many such compositions. If the first *part* has size greater than 2 then we get a bijection between each of these compositions and the compositions of $n-1$ via the map that removes one element from the first *part*. There are a_{n-1}-many such compositions. Thus, we have

$$a_n = a_{n-1} + a_{n-2}$$

(b) (Chapter 5, #37) Let b_n be the number of compositions of n into parts that are larger than 2. Find a recurrence relation satisfied by b_n similar to the one you found in part (a) for a_n. *(Justify your answer.)*

Solution: This argument is structurally identical to the one above. Each *part* must have size at least 3. Thus we divide the set of all possible configurations into two disjoint subsets which we subsequently count. The two subsets are characterized by the size of the first part. Either the first *part* has size equal to 3 or the first *part* has size greater than 3.

If the first *part* has size equal to 3 then we get a bijection between each of these compositions of n and the compositions of $n-3$ via the map that deletes the first *part*. There are b_{n-3}-many such compositions. If the first *part* has size greater than 3 then we get a bijection between each of these compositions and the compositions of $n-1$ via the map that removes one element from the first *part*. There are b_{n-1}-many such compositions. Thus, we have

$$b_n = b_{n-1} + b_{n-3}$$