# 5 Points Available

## Instructions

Please write your **Name and Student Number** at the top of this page.
**Remember:** you have to write quizzes in your **registered** tutorial.

*Make sure to show as many steps of your work as possible, justify as much and annotate any interesting steps or features of your work.* ***Do not just give the final answer.***

QUESTION 1 Prove that for any nonnegative integer $n$,

$$\sum \binom{n}{a_1, a_2, a_3} 2^{a_1+a_3} = 5^n$$

(Where the sum is taken over all $a_1, a_2, a_3$ so that $a_1 + a_2 + a_3 = n$.)

QUESTION 2 Using a **double counting** argument, prove the following, for any nonnegative integer $n$:

$$(n+1) \cdot 2^n = \sum_{k=0}^{n} (k+1) \binom{n+1}{k+1}$$

*Your argument could be a "committee formation" argument, as in lecture, or an argument involving subsets, as in the textbook. But it shouldn't involve induction, or breaking down binomial coefficients into factorials, apply the Binomial Theorem, or give any sort of "algebraic" argument, etc.*

**Solution:** **(1)** This follows directly from the Multinomial Theorem which stipulates that

$$(x+y+z)^n = \sum_{a_1+a_2+a_3=n} \binom{n}{a_1, a_2, a_3} x^{a_1} y^{a_2} z^{a_3}.$$

Setting $x = z = 2$ and $y = 1$, gives the result.

We can also give a combinatorial argument. Consider putting objects of types A, B, C, D, and E into $n$ boxes, for each box there are 5 choices of types of objects to put into the box, so there are $5^n$ many ways to do this. For the other side of the equation, consider

$$\sum_{a_1+a_2+a_3=n} \binom{n}{a_1, a_2, a_3} 2^{a_1+a_3} = \sum_{a_1+a_2+a_3=n} \binom{n}{a_1, a_2, a_3} 2^{a_1} 1^{a_2} 2^{a_3}$$

For a fixed triple $(a_1, a_2, a_3)$, suppose that we will pick $a_1$ many boxes to put in objects of type A and B, $a_2$ many boxes to put in objects of type C, and $a_3$ many boxes to put in objects of type D and E, then there are $\binom{n}{a_1, a_2, a_3}$ many ways to partition the boxes into subsets of size $a_1$, $a_2$ and $a_3$, and for each respective subset, there are are $2^{a_1}$, $1^{a_2}$ and $2^{a_3}$ many ways to put objects of type A or B, type C, and type D and E into those boxes, respectively. Thus, we have the desired equality.

**(2)** We show that the given equality holds by giving a combinatorial argument. Consider counting the number of ways there are to construct a committee of arbitrary size out of $n+1$ people, where we impose the additional requirement that the committee must have a president.

**LHS:** Here we choose the head of the committee first - there are $\binom{n+1}{1} = n+1$ many ways to do this, then we form the rest of the committee by picking a subset of any size from the remaining people - there are $2^n$ many ways to do this. Thus, there are

$$\binom{n+1}{1}2^n = (n+1)2^n$$

many ways to form such a committee.

**RHS:** For fixed $k \geqslant 0$, suppose we first choose $k+1$ people out of the $n+1$ available people to be on the committee - there are $\binom{n+1}{k+1}$ many ways of doing this, then choose the president of the committee from the $k+1$ chosen people - there are $\binom{k+1}{1} = k+1$ many ways of doing this. To count the number of ways of forming a committee with a president of any size we sum over all $k \in \{0, 1, \ldots, n\}$. Thus, we see that there are

$$\sum_{k=0}^{n} \binom{n+1}{k+1}\binom{k+1}{1} = \sum_{k=0}^{n} (k+1)\binom{n+1}{k+1}$$

many ways to form such a committee.

Since both of the above expression count the number of ways of forming a committee of arbitrary size with a president they must be equal, as desired.