5 Points Available

Instructions

Please write your **Name and Student Number** at the top of this page.
Remember: you have to write quizzes in your **registered** tutorial.

Make sure to show as many steps of your work as possible, justify as much and annotate any interesting steps or features of your work. ***Do not just give the final answer.***

Definition 1

A simple graph G is called **critically k-chromatic** if G is k-colourable, but not $k-1$-colourable (i.e. $\chi(G) = k$) and for every vertex v in G, the graph $G - \{v\}$ is $k-1$-colourable.

(The idea is that G requires k colours, but take away any vertex and the remaining graph can be $k-1$-coloured.)

Question 1

Show that for every positive integer k, every **critically k-chromatic** simple graph G has no vertex with degree strictly less than $k-1$.

Solution: For contradiction, let G be critically k-chromatic, and suppose that a vertex $v \in G$ has $\deg(v) \leqslant k-2$. By definition $H := G - \{v\}$ is $(k-1)$-colourable; consider one of these $(k-1)$-colourings of H. Use this colouring of H to obtain a colouring of all the vertices of G, *except* for the vertex v. (We will try to colour this vertex in a moment.)

Since $\deg(v) \leqslant k-2$, this implies that v has at most $k-2$ neighbours, this further implies that the neighbours of v use up *at most* $k-2$ out of the $k-1$ available colours, thus among the neighbours of v there is at least one unused colour; colour v with one of these unused colours and we have a $(k-1)$-colouring of G, but G is not $(k-1)$-colourable - a contradiction.