5 POINTS AVAILABLE

INSTRUCTIONS

Please write your **Name and Student Number** at the top of this page.
Remember: you have to write quizzes in your **registered** tutorial.

Make sure to show as many steps of your work as possible, justify as much and annotate any interesting steps or features of your work. ***Do not just give the final answer.***

QUESTION 1 Using a **double counting** argument, prove the following, for any nonnegative integer n:

$$n\binom{2n-1}{n-1} = \sum_{k=1}^{n} k\binom{n}{k}^2$$

Your argument could be a "committee formation" argument, as in lecture, or an argument involving subsets, as in the textbook. But it shouldn't involve induction, or breaking down binomial coefficients into factorials, apply the Binomial Theorem, or give any sort of "algebraic" argument, etc.

Hint: you may use the identity $\frac{n}{k}\binom{n-1}{k-1} = \binom{n}{k}$ without proof. Also, note that $n = \binom{n}{1}$.

Solution: We show that this equality holds using a combinatorial (or equivalently, a "double counting") argument. Suppose we have n men and n women candidates for a space program. We would like to count the number of ways in which we can select n astronauts from the $2n$ candidates, where we impose the additional requirement that the mission commander is one of the n women candidates.

LHS: From the n women candidates we first pick the mission commander, there are $n = \binom{n}{1}$ many ways of doing this, and we subsequently pick the rest of the crew ($n-1$ astronauts) from the remaining $(2n-1)$ candidates, there are $\binom{2n-1}{n-1}$ ways of doing this. Thus, we see that there are

$$n\binom{2n-1}{n-1} \tag{1}$$

many ways of picking a team of astronauts, as outlined above.

RHS: First we consider the case of fixed $k \geqslant 1$. Suppose we have decided to pick k women and $n-k$ men from the pool of candidates, in this case there are $\binom{n}{n-k}$ many ways of picking the men astronauts from the n men candidates, $\binom{n}{k}$ many ways of picking the k woman astronauts from the n women candidates, and subsequently $\binom{k}{1} = k$ many ways of picking the mission commander from the k women astronauts. Thus, for fixed k, there are

$$\binom{n}{n-k}\binom{n}{k}\binom{k}{1} = k\binom{n}{k}^2$$

many ways of picking n astronauts with k women astronauts, 1 of which is the mission commander

Therefore, to get the total number of ways of picking n astronauts, with the mission commander being a woman, we sum over all $k \in \{1, 2, \ldots, n\}$, that is, there are

$$\sum_{k=1}^{n} k \binom{n}{k}^2 \tag{2}$$

many ways of picking a team of astronauts, as outlined above.

Since (1) and (2) both count the same thing they must be equal, as desired.