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CMB Foreground Removal

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Abstract

The cosmic microwave background allows for astrophysicists to observe electromagnetic radiation from the early universe. Efforts to map CMB anisotropies are complicated by the presence of foreground radiation. By finding a linear combination of full-sky radio intensity maps at different frequencies which minimizes the variance in the intensity, we have produced a reasonably accurate map of the cosmic microwave background (CMB). This minimization problem is first approached using the method of Lagrange multipliers. The intensity maps used come from the Planck public data release. We also implement two other methods of approaching this minimization one problem; one wherein each frequency map is partitioned and another wherein the angular scales of the frequency maps are taken into account using spherical harmonics.

1 Introduction

1.1 The Cosmic Microwave Background

The Cosmic Microwave Background (CMB) is a faint electromagnetic radiation which fills the entire universe, and is a relic of the early universe. The very early universe was hot, dense, and opaque, and as it expanded it cooled. Once the universe cooled enough for atoms to form, photons were thermally decoupled from matter and were free to propagate through empty space (i.e. the universe became transparent). This moment in time created the surface of last scattering, which is the set of points in space at a distance from us such that we are now receiving photons originally emitted from those points at the moment of photon-matter decoupling. The radiation emanating from all points of the sky on the surface of last scattering is the cosmic microwave background. It is extremely an extremely uniform signal—the largest temperature fluctuations in the CMB are on the order of one part in 10^5. The CMB is overwhelmingly comprised of the thermal spectrum of the early universe, and almost perfectly fits a blackbody spectrum curve [2].

1.2 Planck

Planck was a spacecraft operated by the European Space Agency between 2009 and 2013 which mapped the CMB at microwave and infra-red frequencies. The full-sky maps created by Planck at various frequencies are used in our demonstration. These maps have already removed the CMB dipole, so all that remains to be removed is the Milky Way itself, which is the focus of this project.

1.3 Power Spectra

The structure of the CMB is relatively uniform, however there are anisotropies that exist. The origin of these anisotropies are gravitation-pressure fluctuations in the photo-baryonic fluid. These give rise to a unique acoustic scale in the CMB. These oscillations can be analysed by breaking up the CMB into spherical harmonics, akin to how a wave may be broken up into a sum of sines or cosines. After achieving this, the absolute magnitude of the decomposition at various scales of l can be taken, forming a power spectrum. In the CMB, the most prevalent of these scales occurs at l = 200, and therefore, if the methods outlined in this paper work, a power spectra speak should be visible at l = 200.

2 Mathematical Methods

In this project, we combine seven sky maps, each at a different frequency, to produce a single sky map which captures the CMB without the inclusion of the Milky Way foreground signal. We may express this as

$$T_{\text{clean}}(r) = \Sigma_i^{n_{\text{freq}}} \omega_i T(r, \nu_i), \tag{1}$$

where $T_{\text{clean}}(r)$ is the 'cleaned' sky temperature map, n_{freq} is the number of frequencies included in the sum, ω_i is the i$^{\text{th}}$ weight in the weighted sum, and $T(r, \nu_i)$ is the temperature map of the i$^{\text{th}}$ frequency.

The idea is to find the combination of weights which cancel out the Milky Way signal entirely, leaving only the CMB signal in the combined image. A good proxy for the presence of foregrounds in an image is the variance in the image. The Milky Way galaxy is much brighter than the rest of the map, and greatly contributes to the map's variance. Therefore, if the weights are assigned such that the variance of the combined map is minimized, the Milky Way's presence should be mostly eliminated in such a map.

The variance of a random variable X is the expected value of the square deviation from the mean of X ($\mu = E[X]$). In equation form, we may express this as $\sigma^2 = E[(X - \mu)^2]$, where σ is the standard deviation from the mean, X is the random variable, and μ is the mean. For our purposes, this means

$$\sigma^2 \propto \Sigma_k T^2_{\text{clean}}(r_k) = \Sigma_{ijk} \omega_i \omega_j T(r_k, \nu_i) T(r_k, \nu_j). \tag{2}$$

We may express this more succinctly by defining the matrix $H_{ij} = \Sigma_k T(r_k, \nu_i) T(r_k, \nu_j)$ and writing now $\sigma^2 \propto \Sigma_{ij} \omega_i \omega_j H_{ij}$. This gives us an easily calculable quantity to minimize. Clearly, however, a trivial solution exists wherein all weights are set equal to zero. In order to avoid this, we also require that the sum of all weights must equal one ($\Sigma_i \omega_i = 1$).

As we are minimizing a function with respect to a constant, we may proceed using the method of Lagrange multipliers. Let us write

$$f = \Sigma_{ijk} \omega_i \omega_j H_{ij} - \lambda \Sigma_i \omega_i. \tag{3}$$

Now we extremize by setting the first derivative with respect to ω_k equal to zero:

$$\frac{\partial f}{\partial \omega_k} = 2\Sigma_{ij} \delta_{ik} \omega_j H_{ij} - \lambda = 2\Sigma_j H_{kj} \omega_j - \lambda = 0. \tag{4}$$

$2\Sigma_j H_{kj} \omega_j - \lambda = 0$ may be rewritten in matrix form by defining $\mathbf{e} = (1, 1, ..., 1)$ and thereby writing

$$2H\boldsymbol{\omega} = \lambda \boldsymbol{e}, \tag{5}$$

which may be rearranged to obtain our final equation

$$\boldsymbol{\omega} = \frac{H^{-1}\boldsymbol{e}}{\boldsymbol{e}^T H^{-1} \boldsymbol{e}}. \tag{6}$$

Using the above equation, we may solve for the weight vector $\boldsymbol{\omega}$ which minimizes the variance in the CMB.

2.1 Partitioning

The foreground signals present in the map of the CMB is highly non-isotropic, meaning that using a single set of weights for the entire map will result in some imperfections. One way to get around this is by partitioning each map into N regions, and calculating a separate optimal set of weights for each region, which is the method used by C.L. Bennet et al. in their 2003 report on the foregrounds detected in the WMAP mission [1]. The weights in each region are calculated exactly as outlined above, but are only applied to the region over which they are calculated. Therefore, the final CMB map using partitioning is calculated as

$$T_{TOT} = \Sigma_{n=0}^{N} \omega_n T_n, \tag{7}$$

where T_{TOT} is the final CMB map, ω_n is the vector of weights for the n$^{\text{th}}$ region and T_n is the vector of CMB temperature maps for the n$^{\text{th}}$ region.

2.2 Power Spectra

The power spectra were created by using the *healpy* package's *anafast* function. *anafast* uses spherical transforms to compute a power spectrum with respect to l, the degree of the transform. In order to maintain a consistent scale, all maps were normalized with the smallest value being 0 and the largest being 1. This was done with the following two line of code,

```
opt_map = opt_map + abs(min(opt_map))
opt_map = opt_map * 1/max(opt-map)
```

Then, when the maps were passed to *anafast* function, the resulting power spectra had the same scale, and were thus comparable.

2.3 Spherical Harmonics

The third method of foreground removal we explored uses spherical harmonic decomposition. Our approach to this method is based on the approach used by Tegmark et al. in their 2003 paper on cleaning CMB maps from WMAP data [3]. This method is similar to partitioning, in that several sets of weights are calculated, but rather than these weight vectors corresponding to regions of the map, they correspond to particular spherical harmonic modes of the map. This requires transforming between spherical harmonic coefficients, defined as

$$a_{lm}^{i} = \int Y_{lm}(r) x_i(r) d\Omega, \tag{8}$$

and the real space maps,

$$x_i(r) = \Sigma a^i_{lm} Y_{lm}(r). \tag{9}$$

First, the list of real-space maps is transformed into a list of a_{lm} coefficients using the *healpy* package in python ($\{x\} \to \{a_{lm}\}$). Then we define the matrix C_l as the matrix-valued cross-spectrum between each a_{lm} map, i.e. $C^{ij}_l = \langle a^i_{lm}{}^* a^j_{lm} \rangle$. Now a separate weight vector may be calculated for each l via $\mathbf{w}_l = \frac{C_l^{-1}\mathbf{e}}{\mathbf{e}^T C_l^{-1}\mathbf{e}}$. With the weights for each l determined, the transformed coefficients may simply be determined by taking the dot product of each a_{lm} vector and the weight vector corresponding to that l state. This product produces a one-dimensional array of spherical harmonic coefficients, which may then be transformed back into a real-space map.

A notable exclusion from our calculations from the paper is the that we did not consider the beam function. The beam function is a function that determines the sensitivity of the data, and, since we have more recent data than from the paper, our data has a higher resolution and hence the beam function is flat; therefore we are not obliged to consider it.

3 Results

Using the basic linear algebra approach, a weighted sum of the CMB temperature maps was produced which minimized the map's variance. This shown in Figure 2, and may be compared to an equal-weighted sum map shown in Figure 1. The optimized sum still shows some level of foreground contamination near the equator.

3.1 Partitioning

The following maps present the partitioning method with 12, 24 and 48 slices, as well as a map showing how the partitioning method treats different parts of the sky with different weights.

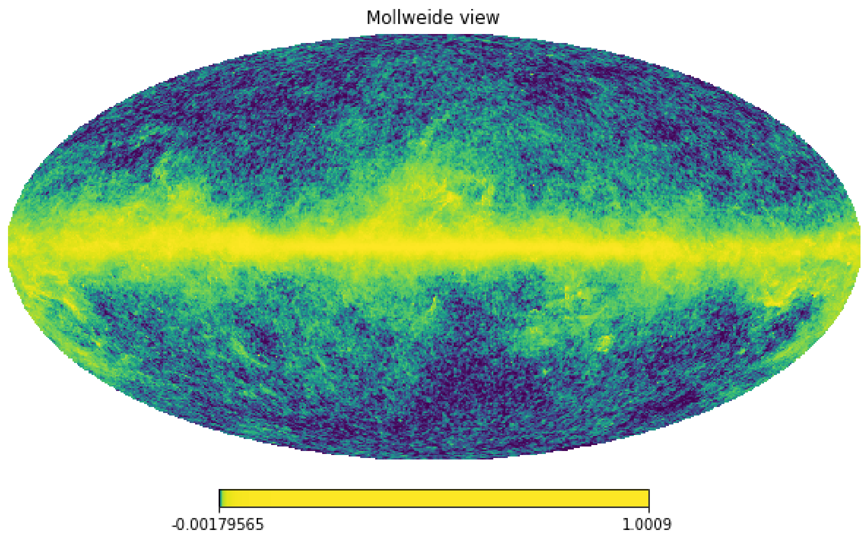


Figure 1: An equal-weight sum of the CMB temperature maps visualized in galactic coordinates on a Mollweide projection map. The Milky Way galaxy is clearly visible as the yellow band across the center of the map.

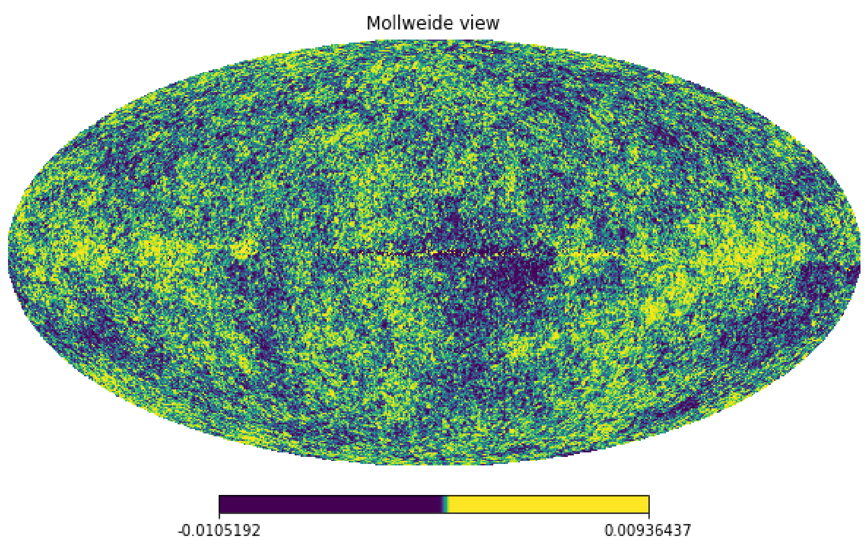
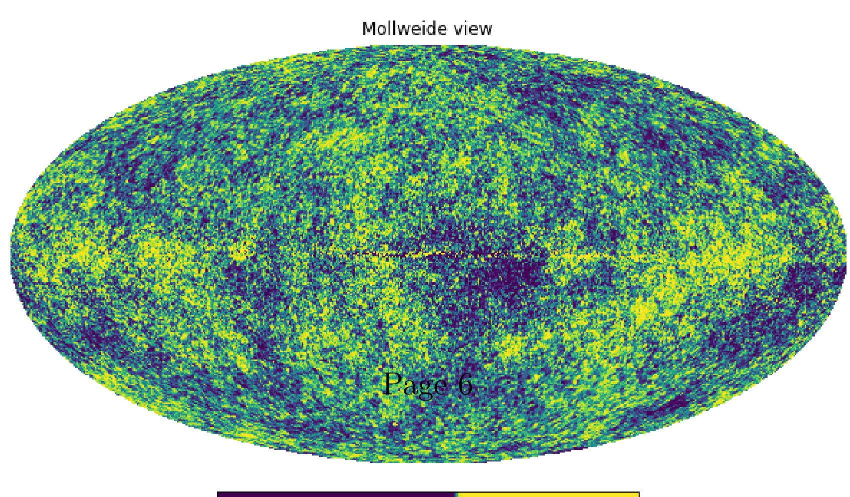


Figure 2: An optimized sum generated using the basic linear algebra method outlined in the introduction. Although some foreground artefacts are visible near the center of the map, the Milky Way foreground has mostly been extirpated from this map.



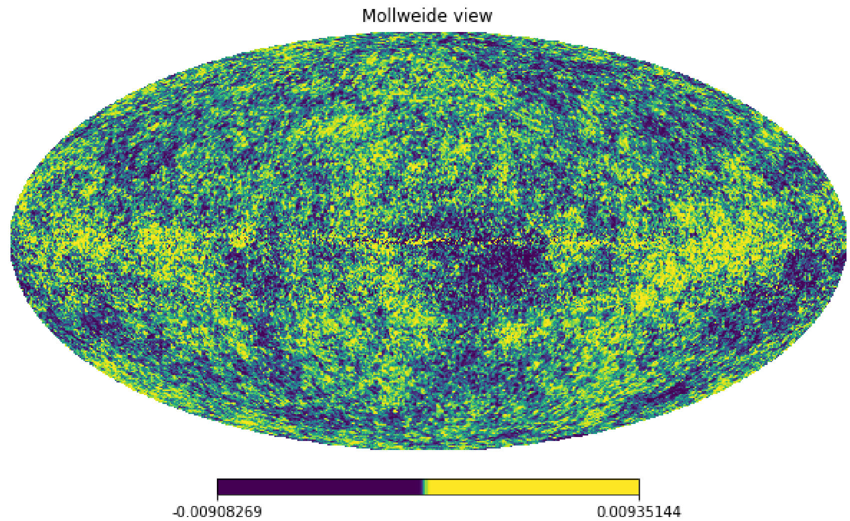


Figure 4: A normalized, optimized sum generated by applying the basic linear algebra method over 24 slices of the CMB. The smaller area of the 24 slices shows minimal improvement over 12 slices, reinforcing the idea that most of the foreground noise is on the equator.

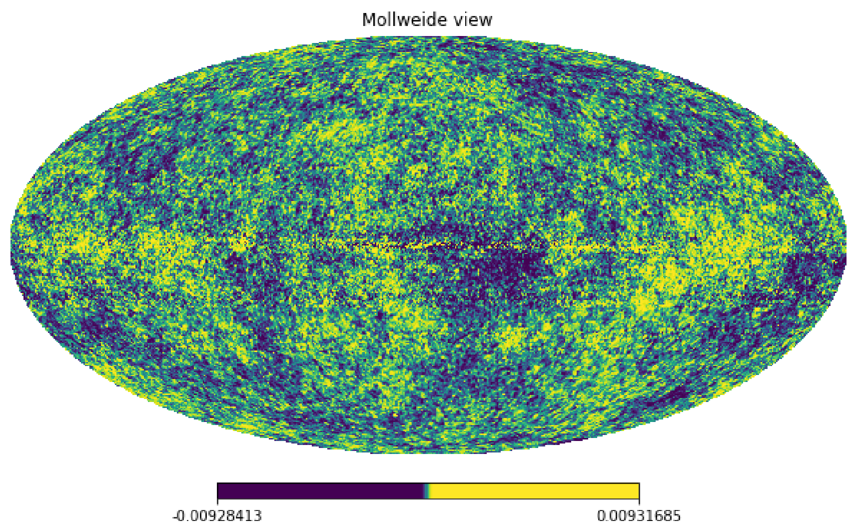


Figure 5: A normalized, optimized sum generated by applying he basic] linear algebra method over 48 slices of the CMB.

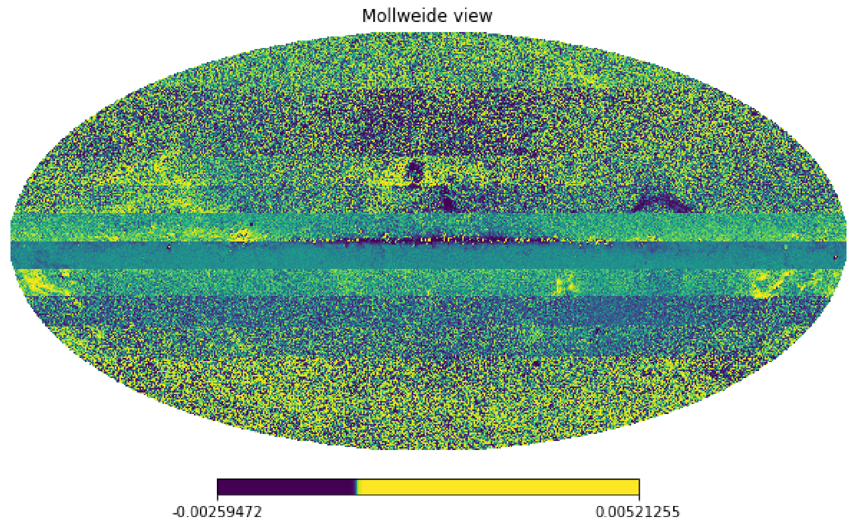


Figure 6: A map of the CMB showing how the partitioned method treated different areas of the sky different compared with the basic linear algebra method. This map was produced by subtracting the CMB map with 12 slices from the basic optimized map. The turquoise color is 0, suggesting that the linear and N sliced method had the same weight.

3.2 Power Spectra

The following figures were created to visualize the difference between the methods being implemented and a naive equal weighting of the maps.

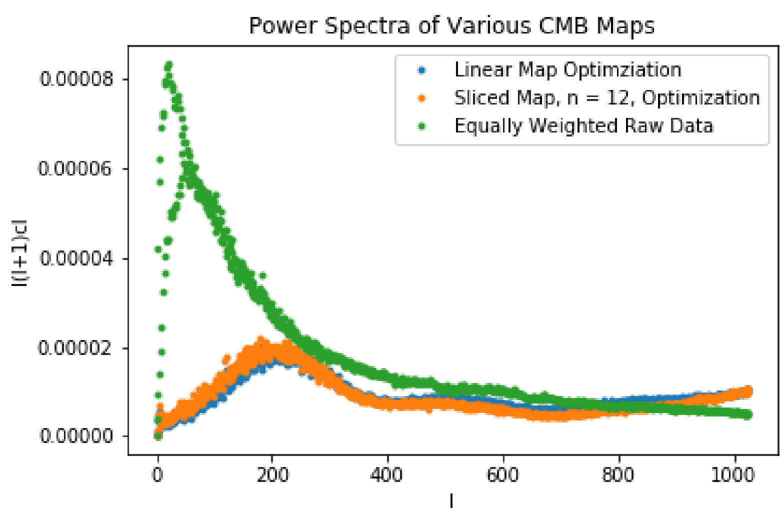


Figure 7: Power Spectrum of the CMB taken from three different maps. In the two maps that were cleared of foreground, blue and orange, it is clear that there is an acoustic peak around l = 200, while in the naively, uncleared green map there is not such peak.

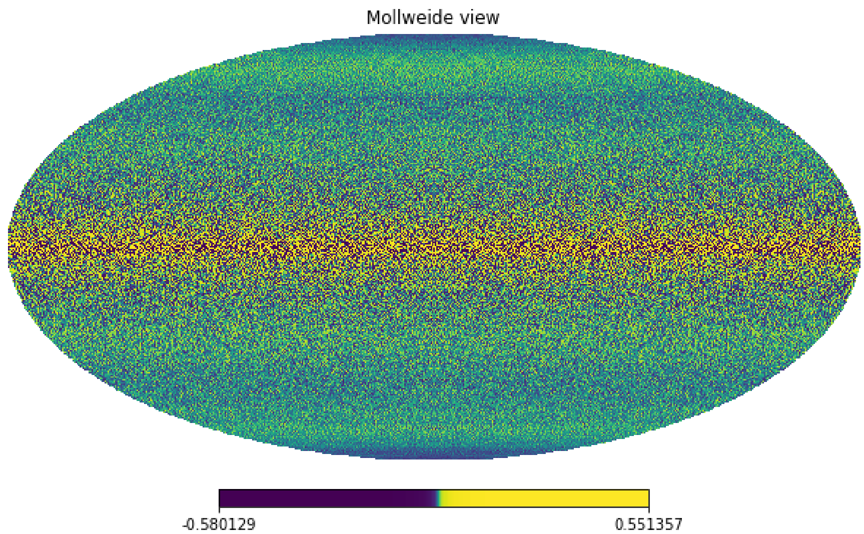


Figure 9: The map produced by taking an optimized linear combination of spherical harmonic modes within each frequency map. This combination did not accurately reproduce the CMB.

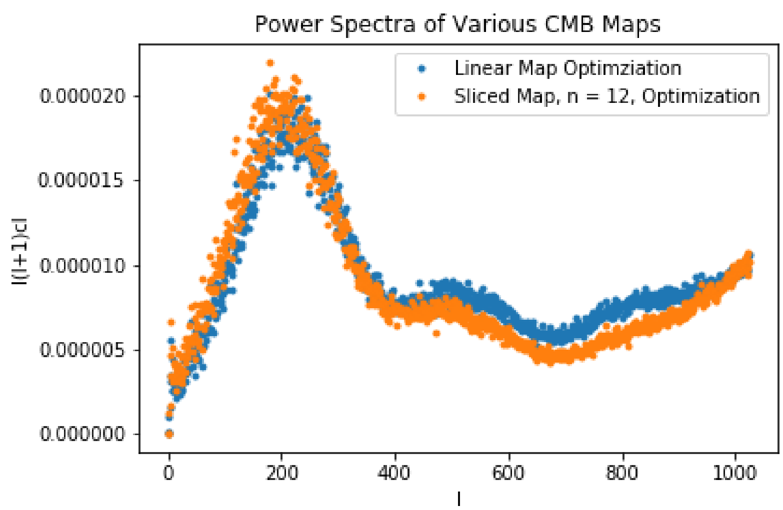


Figure 8: Power Spectrum of the two optimized maps. The blue spectrum was optimized using the method from 6. The orange spectrum was optimized by applying the same equations to smaller portions of the sky. these two maps have similar acoustic peaks at similar l values

3.3 Spherical Harmonics

This method failed to reproduce the CMB, and significant foreground contamination remains, as shown in Figure 9.

4 Discussion

4.1 Partitioning

By partitioning, it was possible to minimize the foreground artifacts near the equator. While there was still a part that remained, it was more minimal. Treating the CMB like a black body is a good approximation, as it results in a background that is mostly cleared of Milky Way foreground artifacts. While this method has the benefit of treating each section of sky as an idealised black body, therefore allowing for better refinement, it cannot filter out all the foreground noise. Additionally, if too many slices are taken (i.e. ≈ 393216 slices or more), then the resolution of these slices will be too low and then the data is lost in the noise of the optimizer. Therefore, this method is only effective for relatively small N (i.e. $N \leq 200$).

The improvement of this technique over a pure linear combination is visualized in Figure 6. This figure was produced by subtracting the the 12 sliced CMB map from the map created using the linearly optimized maps. By subtracting one from the other, it was clear to see that the linear optimizer without partitioning focuses on the bright equator, where the milky way is, while neglecting the poles. The partitioning method is treating different parts of the sky differently, allowing for superior noise reduction at the equator while preserving the information at the poles.. By treating discrete regions differently, we can still adhere to the black body assumption, while allowing for different levels of foreground in different parts of sky. Unfortunately there is still foreground visible in this method. This is to be expected as it is difficult to completely remove the foreground.

4.2 Power Spectra

By investigating the power spectra of the optimized maps, and comparing them with the power spectrum of equally weighted maps, it is possible to see how effective the various techniques are. With respect to Figure 7, it is clear to see that the methods have a substantial effect on cutting down on foreground noise. This produces a power spectrum that is more akin to what is expected. The green map in Figure 7 is the power spectrum that would be created by naively summing the maps together. This results in a curve that is not consistent with the CMB data previously seen. The two other curves in Figure 7 are more consistent, with a pronounced peak around l = 200 and smaller peaks with increasing l.

Investigating the two power spectrum curves in Figure 8 reveals that they have similar behavior. However, the sliced map has a more pronounced peak around l = 200, and lower values for the power spectrum everywhere else. This is to be expected as it is a superior method to linearly combining the maps. However, the presence in both methods of a increasing tail as l increases suggests that foreground noise is still a source of error in the analysis.

4.3 Spherical Harmonics

As seen in Figure 9, the map produced using the spherical harmonic method did not accurately reconstruct the CMB. The method used by Tegmark et al. considered both angular scales and distance to the galactic plane in the calculation of the weights, while our method only considered the first of these components [3]. Neglecting the latter may explain why our results do not match those of Tegmark et al.. Notably, Figure 9 shows significant latitudinal variation, indicating that the distance to the galactic plane should be taken into account in the weight calculation.

5 Conclusions

Of the three methods of foreground removal that we investigated, we were most successful using the method of partitioning, although this yielded only slightly better results than those garnered through the simple linear combination approach. In all cases, some foreground contamination remained, particularly near the galactic plane. One reason for this could be that we neglected smoothing in both our partitioning and spherical harmonics approaches. When the different weights used in each map partition cause discontinuities, Gaussian smoothing is necessary in order to remove the artifacts of these discontinuities from the map. Such artifacts could be the source of the contamination at the galactic plane seen in Figures 2, 3, 4, and 5. The failure of the spherical harmonics method to accurately reproduce the CMB may be attributed to its ignorance of the distance of each part of the map to the galactic plane, which constitutes a major break between our method and that of Tegmark et al. [3].

6 Appendix

6.1 Author Contribution Statements

Andrew Lewis (260786909) Contributed to writing the code and tests for the partitioning method. Additionally, he contributed to the discussion regarding this method as well as the power spectra.
Samuel Gagnon-Hartman (260762240) wrote the code for the basic linear algebra method, the mathematical methods sections about linear algebra and spherical harmonics, the introduction, the conclusion, and the abstract.
Louis Croquette (260723478) contributed to writing code and tests for the spherical harmonic method, as well as some discussion about the procedure for this method.

References

[1] C. L. Bennett, R. S. Hill, G. Hinshaw, M. R. Nolta, N. Odegard, L. Page, D. N. Spergel, J. L. Weiland, E. L. Wright, M. Halpern, N. Jarosik, A. Kogut, M. Limon, S. S. Meyer, G. S. Tucker, and E. Wollack. First-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Foreground Emission. *The Astrophysical Journal*, 148(1):97–117, September 2003. 4

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