Phys 512- Problem Set 8

Andrew Lewis

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1. Leapfrog Energy Conservation

Our expression for the leapfrog method is as follows:

$$\frac{f(t+dt,x)-f(t-dt,x)}{2dt} = -v\frac{f(t,x+dx)-f(t,x-dx)}{2dx}$$

First things first, we will multiply by 2dt on both sides

$$f(t+dt,x)-f(t-dt,x) = -v\frac{dt}{dx}\left(f(t,x+dx)-f(t,x-dx)\right)$$

from class we can set

$$v\frac{dt}{dx} = a$$

and our expression becomes:

$$f(t+dt,x)-f(t-dt,x) = -a\left(f(t,x+dx)-f(t,x-dx)\right)$$

Now, let's consider how our problem works, while we don't exactly know our length scale, (it could be general), we know that for our discrete system each dt will be a time step. Therefore we can write

$$t+dt = t+1, t-dt = t-1$$

Next, we can plug in our expression for our sine wave to determine stability:

$$f(t,x) = \xi^t exp(ikx)$$

Our expression then becomes:

$$\xi^{t+1}e^{ikx} - \xi^{t-1}e^{ikx} = -a\left(\xi^t e^{ik(x+dx)} - \xi^t e^{ik(x-dx)}\right)$$

Here we can see that we can divide by ξ^t and e^{ikx} as we know both quantities are non zero.

$$\xi^1 - \xi^{-1} = -a\left(e^{ikdx} - e^{-ikdx}\right)$$

Recall that we can use the identity

$$2i\sin(x) = e^{ix} - e^{-ix}$$

plugging it into the right hand side yields:

$$\xi^1 - \xi^{-1} = -a2i\sin(kdx)$$

multiplying by ξ:

$$\xi^2 - 1 = -2ai\xi\sin(kdx)$$

Rearranging:

$$\xi^2 + 2ai\xi\sin(kdx) - 1 =$$

Which looks like a quadratic equation in ξ, using our quadratic formula yields:

$$\xi = \frac{-2ai\sin(kx) \pm \sqrt{-4a^2\sin^2(kdx) + 4}}{2}$$

Simplifying:

$$\xi = -ai\sin(kx) \pm \sqrt{-a^2\sin^2(kdx) + 1}$$

Taking the magnitude of ξ

$$|\xi|^2 = \xi \cdot \xi^*$$

$$|\xi|^2 = -ai\sin(kx) \pm \sqrt{-a^2\sin^2(kdx) + 1} * \left(ai\sin(kx) \pm \sqrt{-a^2\sin^2(kdx) + 1}\right)$$

Which will simplify to

$$|\xi|^2 = a^2\sin^2(kx) - a^2\sin^2(kx) + 1$$

IMPORTANT:
This is true if and only if the CFL condition is satisfied, i.e. $a \leq 1$. If $a > 1$ then the values under the square roots will be negative and the magnitude will be different, but we'll just continue with the CFL condition being satisfied shall we?
Simplifying:

$$|\xi|^2 = 1$$

if a $\leq$ 1. Therefore our quantity (wave) $\xi^t e^{ikx}$ will not decay, nor explode and we will have conservation of energy! yay.

2. Conjugate Gradient Things

Part a

We can solve for a potential in a region with zero charge with the following relation (finite difference of laplace's equation)

$$V_{i,j} = \frac{1}{4} * (V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1})$$

If we consider the spot at V[1,0]

$$V_{1,0} = \frac{1}{4} * (V_{2,0} + V_{0,0} + V_{1,1} + V_{1,-1})$$

Which we can then use to solve the potential at (0,0)

$$V_{0,0} = 4 * V_{1,0} - V_{2,0} - V_{1,1} - V_{1,-1}$$

Initially we can guess for the potential away from the singularity with the following relation:

$$V = ln(r)/(2*\pi)v$$

However, at 0 we would expect a singularity, so in order to get around that, we just set it to 1. However, this would end up rescaling all the potentials around it. By setting rho = 1 and then using the relaxation algorithm, we were able to get the potentials of our grid.

This gave us the following potentials at the point of interest, (to five decimal points).

$$V[0,0] = 1.0$$

$$V[1,0] = 0.0$$

$$V[2,0] = -0.45354$$

$$V[5,0] = -1.05314$$

The final value of $V[5,0]$ is very close to our hint (-1.05), so I am pretty confident in our values. All of the code used in this section can be found in partA.py

Part b

We can use convolution to determine the potential in our region. However, as is mentioned, we can actually use conjugate gradient to solve this.

the boundary conditions were that the box was held at a potential of 1, and were that the potential goes to zero at the edges.

You can find my conjugate gradient code in partBC.py, I directly ripped it from Jon's conjugate gradient solver in class.

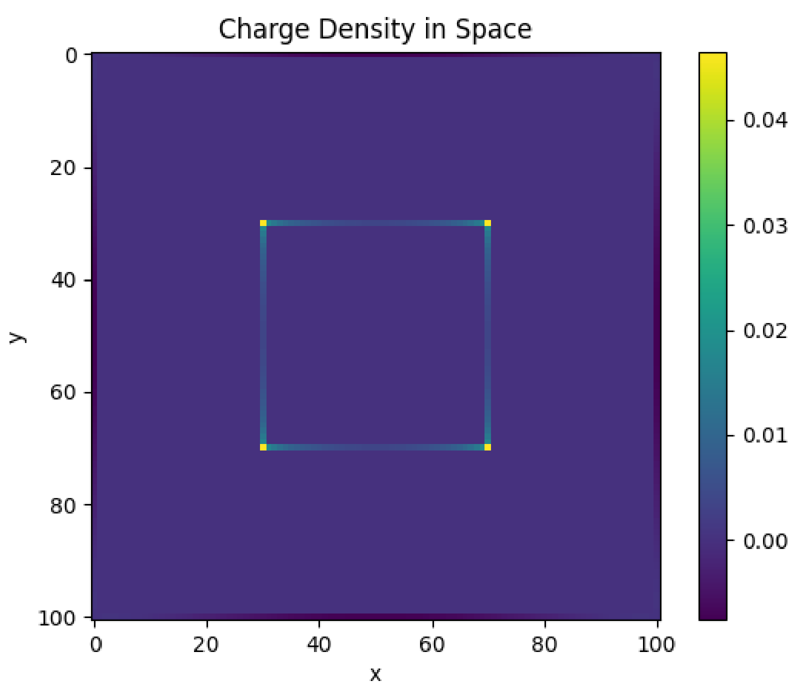

Figure 1: 2D Imshow plot of the charge density of a box held at a potential of 1. As can be seen the charge tends to gather at the corners of the box.

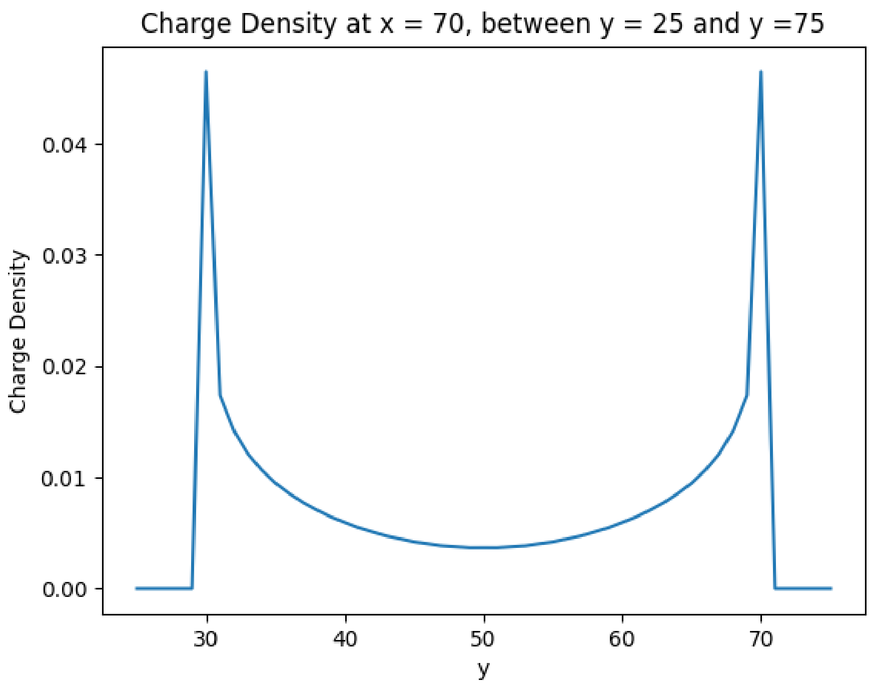

Figure 2: Plot of the charge density for the side of the box. As can be seen just outside of the box, the charge density goes to zero whereas in the box the charge density gathers near the corners.

Part c

Show the potential everywhere in space

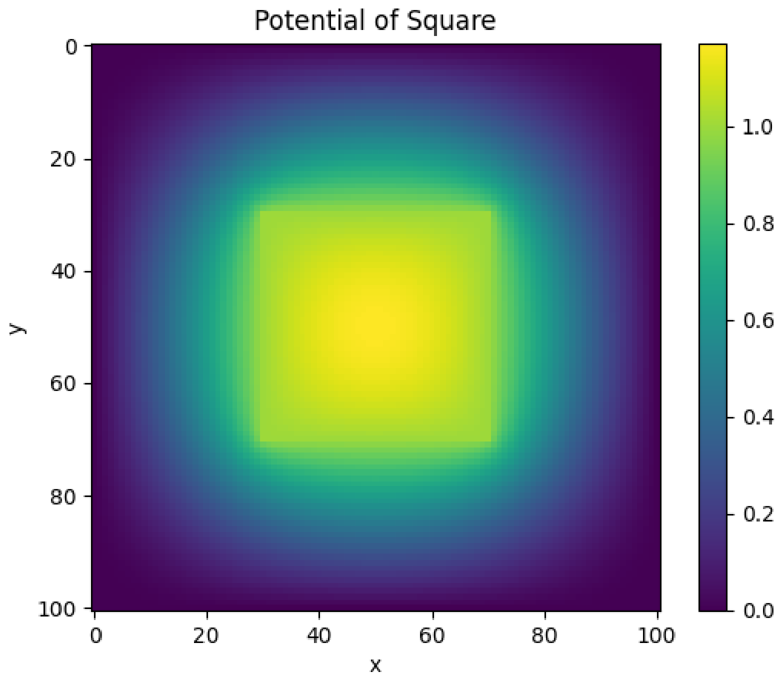

Figure 3: Plot of potential in space. As can be seen the potential within the box is almost constant while the potential outside of the box decays. The reason why the potential within the box is not exactly constant may be due to numerical errors arising from a discrete grid

How close to constant is the potential in the box?

The potential within the box is pretty close to constant. There are some fluctuations within the box however it is very telling that the fluctuations are symmetric. Perhaps there are errors arising from the charge distribution being convolved with. Additionally the grid may be too small to properly be able to make the potential constant within the box as it is only 100x100 and the box is only composed of 196 points total. That being said it is quite clear that the potential outside of the box rapidly decays which was the behavior that we expected.

Plot the electric field

The electric field is simply the gradient of the potential. It was calculated using np.gradient() and plotted with the common matplotlib package quiver (great for vector diagrams). We got the following:

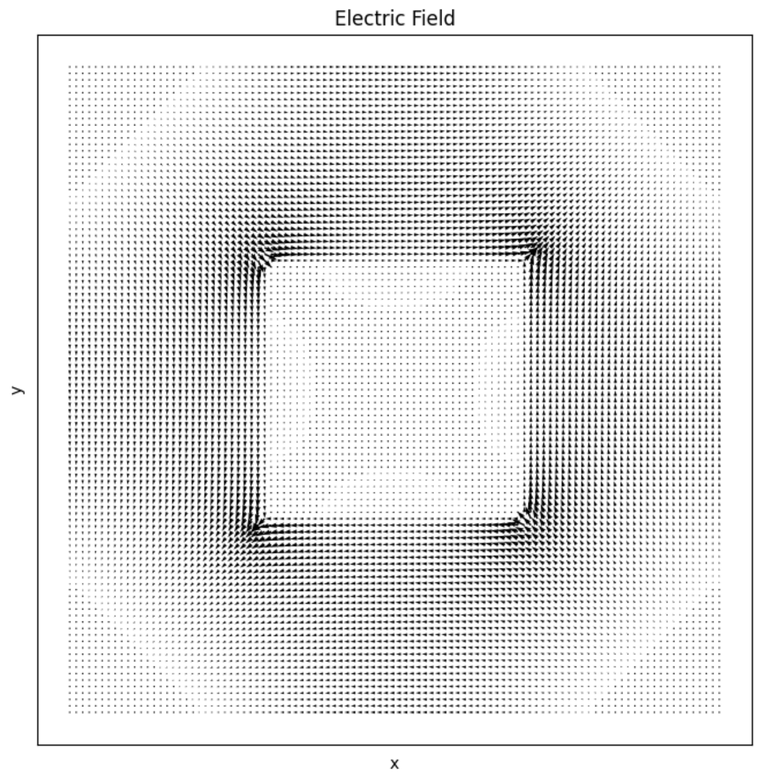

Figure 4: Electric field surrounding the box. As can see the electric field in parallel to the box along the sides and increases in intensity near the corners of the box, or points.

As can be seen the field is perpendicular to equipotential box, held at V=1, and at the corners the field is much stronger. This agrees with the standard lore and as such is what we'd expected.