CS1231 Part 3 - The Logic of Quantified Statements

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

1 Predicates and quantified statements

A **predicate** is a sentence that contains a finite number of values and becomes a statement when specific values are substituted for the variables.
The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.
The **truth set** of a predicate $P(x)$, where x has a domain D, is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}$$

1.1 The universal quantifier

The symbol $\forall$, denoted "for all", is called the **universal quantifier**.

Let $Q(x)$ be a predicate and D the domain of x. A **universal statement** is a statement of the form

$$\forall x \in D, Q(x)$$

It is defined to be true iff $Q(x)$ is true for every x in D, and it is defined to be false iff $Q(x)$ is false for at least one x in D.
A value for x for which $Q(x)$ is false is called a **counterexample**.

The **method of exhaustion** proves that a universal statement is true by exhausting all cases or proving that the statement is true for each element in the domain.

1.2 The existential quantifier

The symbol $\exists$, denoted "there exists", is called the **existential quantifier**.

Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form

$$\exists x \in D \text{ such that } Q(x)$$

It is defined to be true iff $Q(x)$ is true for at least one x in D, and it is false iff $Q(x)$ is false for all x in D.

Furthermore, the symbol $\exists!$ is used to denote "there exists a unique" or "there is one and only one".

1.3 Universal conditional statements

The **universal conditional statement** comes in the form of:

$$\forall x, \text{ if } P(x) \text{ then } Q(x)$$

1.3.1 Equivalent forms of universal and existential statements

Given a statement $\forall x \in U$, if $P(x)$ then $Q(x)$, we can narrow the domain U to be the domain D consisting of all values of the variable x that make $P(x)$ true:

$$\forall x \in U, \text{ if } P(x) \text{ then } Q(x) \equiv \forall x \in D, \, Q(x)$$

This works similarly for existential statements.

1.4 Implicit quantification

Let $P(x)$ and $Q(x)$ be predicates and supposed the common domain of x is D.
• The notation $P(x) \implies Q(x)$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or, equivalently, $\forall x, P(x) \to Q(x)$.
• The notaion $P(x) \iff Q(x)$ means that the truth sets of $P(x)$ and $Q(x)$ are identical, or, equivalently, $\forall x, P(x) \leftrightarrow Q(x)$

1.5 Negations of quantified statements

Negation of a universal statement:

$$\sim (\forall x \text{ in } D, \, P(x)) \equiv \exists x \text{ in } D \text{ such that } \sim P(x)$$

Negation of an existential statement:

$$\sim (\exists x \text{ in } D \text{ such that } P(x)) \equiv \forall x \text{ in } D, \sim P(x)$$

Negation of universal conditional statements:

$$\sim (\forall x, \, P(x) \to Q(x)) \equiv \exists x \text{ such that } P(x) \wedge \sim Q(x)$$

1.6 Variants of universal conditional statements

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$

1. Its contrapositive is: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$
2. Its converse is: $\forall x \in D$, if $Q(x)$ then $P(x)$
3. Its inverse is: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$

2 Statements with multiple quantifiers

A statement may have multiple quantifiers, and the meaning of the statement depends on the quantifiers used, and their order within the statement.

2.1 Negation of multiply-quantified statements

We can use the equivalencies for the negation of statements with only one quantifier to deduce the negations of multiply quantified statements. For example:
As $\sim (\forall x \text{ in } D, \, P(x)) \equiv \exists x \text{ in } D \text{ such that } \sim P(x)$,
and $\sim (\exists x \text{ in } D \text{ such that } P(x)) \equiv \forall x \text{ in } D, \sim P(x)$,

$$\sim (\forall x \text{ in } D, \, \exists y \text{ in } E \text{ such that } P(x,y)) \equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x,y)$$

As a general rule, to negate a quantified statement, negate all statement variables, and switch all $\forall$'s and $\exists$'s while maintaining their order.

3 Arguments with quantified statements

The rule of **universal instantiation** states that if some property is true of *everything* in the set, then it is true of *any particular thing* in the set. This rule is the fundamental tool for deductive reasoning.

With this, we can obtain the valid form of argument, **universal modus ponens**:

$$\begin{gathered}\forall x, \text{ if } P(x) \text{ then } Q(x) \\ P(a) \text{ for a particular } a. \\ \bullet\ Q(a).\end{gathered}$$

Likewise, we can obtain the valid form of argument, **universal modus tollens**:

$$\begin{gathered}\forall x, \text{ if } P(x) \text{ then } Q(x) \\ \sim Q(a)) \text{ for a particular } a. \\ \bullet \sim P(a)\end{gathered}$$

Furthermore, we can create additional forms of arguments involving universally quantified statements simply through **universal transitivity**:

$$\begin{gathered}\forall x,\ P(x) \rightarrow Q(x) \\ \forall x,\ Q(x) \rightarrow R(x) \\ \bullet\ \forall x,\ P(x) \rightarrow R(x)\end{gathered}$$