ST2334 Part 1 - Basic Concepts on Probability

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AY18/19 Semester 1

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

1 Sample space and sample points

1.1 Sample space

An **observation** refers to any numerical or categorical recording of information.
A **statistical experiment** is a procedure that generates a set of data or observations.
The **sample space** of a statistical experiment is the set of all possible outcomes, and is represented by the symbol S.

1.2 Sample points

Every outcome in a sample space is thus called an element of the sample space or simply a **sample point**.

2 Events

An **event** is a subset of a sample space.

2.1 Simple and compound events

A **Simple event** is an event with exactly one outcome (i.e. one sample point).
A **compound event** is an event that consists of more than one outcomes (or sample points).

Note that the sample space is itself an event, and it is called a **sure event**. Furthermore, a subset of S that contains no elements at all is the empty set, denoted by $\emptyset$, and is called a **null event**.

2.2 Operations with events

2.2.1 Union and intersection events

Let S denote a sample space, A and B are any two events of S.
The **union** of two events A and B, denoted $A \cup B$, is the event containing all the elements that belong to A or B or to both. That is,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Furthermore, the union of n events $A_1, A_2, ..., A_n$, denoted by $A_1 \cup A_2 \cup ... \cup A_n$, is the event containing all the elements that belong to one or more of the events A_1, or A_2, or ..., or A_n. That is,

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup ... \cup A_n = \{x : x \in A_1 \text{ or ... or } x \in A_n\}$$

The **intersection** of two events A and B, denoted $A \cap B$ or AB, is the event containing elements that are common to both A and B. That is,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Furthermore, the intersection of n events $A_1, A_2, ..., A_n$, denoted by $A_1 \cap A_2 \cap ... \cap A_n$, is the event containing all the elements that are common to all the events A_1, and A_2, and ..., and A_n. That is,

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap ... \cap A_n = \{x : x \in A_1 \text{ and ... and } x \in A_n\}$$

2.2.2 Complement event

The **complement** of an event A with respect to S, denoted by A' or $A^{\complement}$, is the set of all elements of S that are not in A. That is,

$$A' = \{x : x \in S \text{ and } x \notin A\}$$

2.3 Mutually exclusive events

Two events A and B are said to be **mutually exclusive** or **mutually disjoint** if $A \cap B = \emptyset$. That is, if A and B have no elements in common.

2.4 De Morgan's Law

For any n events $A_1, A_2, ..., A_n$,

1. $(A_1 \cup A_2 \cup ... \cup A_n)' = A_1' \cap A_2' \cap ... \cap A_n'$
2. $(A_1 \cap A_2 \cap ... \cap A_n)' = A_1' \cup A_2' \cup ... \cup A_n'$

2.5 Contained events

If all of the elements in an event A are also in an event B, then event A is **contained** in event B, denoted by

$$A \subset B$$

If $A \subset B$ and $B \subset A$, then $A = B$

3 Counting Methods

3.1 Multiplication principle

If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, and for each of the first two ways, a third operation can be performed in n_3 ways, and so forth, then the number of ways the sequence of k operations can be performed is

$$n_1 n_2 \dots n_k$$

3.2 Addition principle

Suppose that a procedure, designated by 1 can be performed in n_1 ways, and there was another procedure, designated by 2, can be performed in n_2 ways, and so forth, and suppose furthermore that no two procedures may be performed together, then the number of ways in which we can perform 1 or 2 or ... or k is given by

$$n_1 + n_2 + \cdots + n_k$$

3.3 Permutations

A **permutation** is an ordered arrangement of r objects from a set of n objects, where $r \leq n$.

The number of permutations of n distinct objects taken r at a time is denoted by

$$^nP_k = n(n-1)(n-2)\dots(n-(r-1)) = \frac{n!}{(n-r)!}$$

Hence, where $r = n$, the number of permutations of n distinct objects taken all together is $n!$.

3.3.1 Circular permutations

The number of permutations of n distinct objects arranged in a circle is $(n-1)!$

3.3.2 Permutaions with non-distinct objects

Suppose we have n objects such that there are n_1 of one kind, n_2 of a second kind, $\dots$, n_k of a k^{th} kind, where

$$n_1 + n_2 + \dots + n_k = n$$

Then the number of distinct permutations of these n objects taken all together is given by

$$^nP_{n_1,n_2,\dots,n_k} = \frac{n!}{n_1!n_2!\dots n_k!}$$

3.4 Combination

A **combination** is an unordered arrangement of r objects from a set of n objects, where $r \leq n$. A combination creates a partition with 2 groups, one group containing the r objects selected and the other group containing the $n-r$ objects that are left.

The number of such combinations is denoted by

$$\binom{n}{r} \text{ or } {}^nC_r$$

and the number of combinations of n distinct objects taken r at a time is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

This can be calculated by observing that there are $r!$ permutations of any r objects we select from a set of n distinct objects, hence

$$^nC_r r! = {}^nP_r$$

3.5 Binomial coefficient

The quantity $\binom{n}{r}$ is called a **binomial coefficient** because it is the coefficient of the term $a^r b^{n-r}$ in the binomial expansion of $(a+b)^n$.

4 Relative frequency and definition of probability

The **relative frequency** of an event is the fraction of the number of occurences of the event over the total number of events. Let n_A be the number of times that the event A occured among the n repetitions respectively. Then the relative frequency can be denoted as

$$f_A = \frac{n_A}{n}$$

If A and B are two mutually exclusive events and if $f_{A\cup B}$ is the relative frequency associated with the event $A \cup B$, then $f_{A\cup B} = f_A + f_B$
Furthermore, f_A approaches some definite numerical value as the total number of events increase.

4.1 Axioms of probability

Consider an experiment whose sample space is S. The objective of probability is to assign to each event A_i, a number $Pr(A_i)$, called the probabiity of the event A_i, which gives a precise measure of the chance that A_i will occur.
Consider the collection of all events and denote it P. For each event A of the sample space S we assume that a number $Pr(A_i)$, which si called the probability of the event A, is defined and satisfies the following three axioms:

Axiom 1: $0 \leq Pr(A_i) \leq 1$

Axiom 2: $Pr(S) = 1$

Axiom 3: If $A_1, A_2, \ldots$ are mutually exclusive (disjoint) events, then

$$Pr(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} Pr(A_i)$$

in particular, if A and B are two mutually exclusive events, then $Pr(A \cup B) = Pr(A) + Pr(B)$

5 Conditional probability

Let A and B be two events associated with an experiment E. The **conditional probability** of the event B, given that event A has occured is defined as

$$Pr(B|A) = \frac{Pr(A \cap B)}{Pr(A)}, \qquad \text{if } Pr(A) \neq 0$$

For a fixed A, The conditional probability $Pr(B|A)$ still satisfies the various axioms of probability:

Axiom 1: $0 \leq Pr(B|A) \leq 1$

Axiom 2: $Pr(S|A) = 1$

Axiom 3: If $B_1, B_2, \ldots$ are mutually exclusive (disjoint) events, then

$$Pr(\bigcup_{i=1}^{\infty} B|A_i) = \sum_{i=1}^{\infty} Pr(B_i|A)$$

in particular, if B_1 and B_2 are two mutually exclusive events, then
$Pr(B_1 \cup B_2|A) = Pr(B_1|A) + Pr(B_2|A)$

5.1 Multiplication rule of probability

The probability of both events A and B occuring is the product of the probability of one event occuring and the conditional probability that the other event occurs given that the first event has occured.

$$Pr(A \cap B) = Pr(A)Pr(B|A)$$
$$Pr(A \cap B) = Pr(B)Pr(A|B)$$

In general,

$$Pr(A_1 \cap \cdots \cap A_n) = Pr(A_1)Pr(A_2|A_1)Pr(A_3|A_1 \cap A_2) \ldots Pr(A_n|A_1 \cap \cdots \cap A_{n-1})$$

provided that $Pr(A_n|A_1 \cap \cdots \cap A_{n-1}) > 0$.

5.2 The law of total probability

Let $A_1, A_2, \ldots, A_n$ be a **partition** of the sample space S. That is, $A_1, A_2, \ldots, A_n$ are mutually exlusive and exhaustive events such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{n} A_i = S$
Then for any event B,

$$Pr(B) = \sum_{i=1}^{n} Pr(B \cap A_i) = \sum_{i=1}^{n} Pr(A_i)Pr(B|A)$$

5.3 Bayes' Theorem

Let $A_1, A_2, \ldots, A_n$ be a partition of the sample space S. Then,

$$Pr(A_k|B) = \frac{Pr(A_k)Pr(B|A_k)}{\sum_{i=1}^{n} Pr(A_i)Pr(B|A)} = \frac{Pr(A_k)Pr(B|A_k)}{Pr(B)}$$

for $k = 1, 2, \ldots, n$.

6 Independent events

Two events A and B are said to be **independent** if and only if

$$Pr(A \cap B) = Pr(A)Pr(B)$$

Two events A and B that are not independent are said to be **dependent**.

6.1 Properties of independent events

Suppose $Pr(A) > 0$, $Pr(B) > 0$.

1. If A and B are independent, then

$$Pr(B|A) = Pr(B) \text{ and } Pr(A|B) = Pr(A)$$

2. If A and B are independent events, then events A and B cannot be mutually exclusive.
3. If A and B are mutually excusive, then A and B cannot be independent.
4. The sample space S as well as the empty set $\emptyset$ are independent of any event.
5. If $A \subset B$, then A and B are dependent unless $B = S$.
6. If A and B are independent, then so are A and B', A' and B, A' and B'.

6.2 n Independent Events

6.2.1 Pairwise independent events

A set of events A_1, A_2, $\ldots$, A_n are said to be pairwise independent if and only if

$$Pr(A_i \cap A_j) = Pr(A_i)Pr(A_j)$$

for $i \neq j$ and i, $j = 1, \ldots, n$

6.2.2 n Mutually independent events

The events A_1, A_2, $\ldots$, A_n are called **mutually independent** if and only if for any subset A_{i_1}, A_{i_2}, $\ldots$, A_{i_k} of A_1, A_2, $\ldots$, A_n,

$$Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = Pr(A_{i_1})Pr(A_{i_2}) \ldots Pr(A_{i_k})$$

A Basic properties of operations of events

1. $A \cap A' = \emptyset$
2. $A \cap \emptyset = \emptyset$
3. $A \cup A' = S$
4. $(A')' = A$
5. $(A \cap B)' = A' \cup B'$
6. $(A \cup B)' = A' \cap B'$
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
9. $A \cup B = A \cup (B \cap A')$
10. $A = (A \cap B) \cup (A \cap B')$

B Basic properties of probability

1. $Pr(\emptyset) = 0$
2. If $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then

$$Pr(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} Pr(A_i)$$

3. For any event A, $Pr(A') = 1 - Pr(A)$
4. For any two events A and B,

$$Pr(A) = Pr(A \cap B) + Pr(A \cap B')$$

5. For any two events A and B,

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$$

6. **The Inclusion-Exclusion Principle** - For any n events A_1, A_2, $\ldots$, A_n,

$$Pr(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} Pr(A_i) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr(A_i \cap A_j) +$$
$$\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} Pr(A_i \cap A_j \cap A_k) - \ldots \cdots + (-1)^{n+1} Pr(A_1 \cap A_2 \cap \cdots \cap A_n)$$

7. If $A \subset B$, then $Pr(A) \leq Pr(B)$.