CS1231 Part 8 - Functions

Based on lectures by Terence Sim and Aaron Tan
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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

1 Functions

Let f be a relation such that $f \subseteq S \times T$. Then f is a **function** from S to T, denoted $f : S \to T$ iff

$$\forall x \in S, \exists y \in T \ (x \ f \ y \wedge (\forall z \in T \ (x \ f \ z \to y = z)))$$

Essentially, for a relation to be a function, each input to the function must only have one output.

Let $f : S \to T$ be a function. We can also write $f(x) = y$ (or $x \mapsto y$) iff $(x, y) \in f$. We say that x *maps* to y.

We will note a few more definitions:
Let $f : S \to T$ be a function.

- Let $x \in S$. Let $y \in T$ such that $f(x) = y$. Then x is called a **pre-image** of y.
- Let $y \in T$. The **inverse image** of y is the set of all its pre-images: $\{x \in S \mid f(x) = y\}$
- Let $U \subseteq T$. The **inverse image** of U is the set that contains all the pre-images of all elements of U: $\{x \in S \mid \exists y \in U, \ f(x) = y\}$
- Let $V \subseteq S$. The **restriction** of f to U is the set: $\{(x, y) \in V \times T \mid f(x) = y\}$

2 Properties

2.1 Injective

Let $f : S \to T$ be a function. f is **injective** iff

$$\forall y \in T, \forall x_1, x_2 \in S \ ((f(x_1) = y \wedge f(x_2) = y) \to x_1 = x_2)$$

This implies that every value in T has *at most* one value in S that maps to it.

2.2 Surjective

Let $f : S \to T$ be a function. f is **surjective** iff

$$\forall y \in T, \ \exists x \in S \ (f(x) = y)$$

We also say that f is a **surjection** or that f is **onto**.
This implies that every value in T has *at least* one value in S that maps to it.

2.3 Bijective

Let $f : S \to T$ be a function. f is **bijective** iff f is injective and f is surjective. We also say that f is a **bijection**.

Combining the two, this essentially states that every value in T has *exactly* one value in S that maps to it.

2.4 Inverse

Let $f : S \to T$ be a function and let f^{-1} be the inverse relation of f from T to S. Then f is a bijective iff f^{-1} is a function.

3 Composition

Let $f : S \to T$ be a function. Let $g : T \to U$ be a function. The composition of f and g, denoted as $g \circ f$, is a function from S to U.

We also denote $(g \circ f)(x)$ to mean $g(f(x))$.

3.1 Identity

Given a set A, we can define a function $\mathcal{I}_A$ from A to A by:

$$\forall x \in A \ (\mathcal{I}_A(x) = x)$$

This is the **identity function** on A.

Let $f : A \to A$ be an injective function on A. Then $f^{-1} \circ f = \mathcal{I}_A$.

3.2 Generalization

We've explored functions that take in a single argument. We can generalize functions to accept two or more arguments simply by making the domain a Cartesian product.

For example, let $f : A \times B \to C$ be a function. Then the argument to f is an ordered pair (a, b) where $a \in A$ and $b \in B$. Hence we can write $f((a, b))$ which we simplify to $f(a, b)$.

Likewise, let $g : A \times B \times C \to D$ be a function. Then we will write $g(a, b, c)$. In this way, we can allow functions to accept any finite number of arguments.

Finally, we may also allow functions to return "multiple values" by making the co0domain a Cartesian product.