Galois Theory

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1 Polynomial Rings

1.1 Adjuctions and their Properties

Definition 1. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Then an adjunction from $\mathscr{C}$ to $\mathscr{D}$ is a 4-tuple $(\eta, \epsilon) : F \vdash G$, where $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ are functors and $\eta : 1_{\mathscr{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow 1_{\mathscr{D}}$ are natural transformations and the following hold:*

1. *For every object $X \in Ob\mathscr{C}$, $\epsilon_{FX} \circ F\eta_X = 1_{FX}$*
2. *For every object $Y \in Ob\mathscr{D}$, $G\epsilon_Y \circ \eta_{GX} = 1_{GX}$*

1.2 Polynomial Ring Adjoints

Definition 2. *Let $CRing$ be the category of commutative rings and ring homomorphisms and $CRing^*$ be the category of pointed commutative rings and pointed ring homomorphisms. Then the functor*

$$U : CRing^* \to CRing : (\phi : (R, a) \to (S, b)) \mapsto (\phi : R \to S)$$

is called the forgetful functor. We call U the forgetful functor.

Definition 3. *A polynomial ring adjoint is a triple (Π, η, ϵ) such that $(\eta, \epsilon) : \Pi \vdash U$ is an adjunction. We denote the composite functor $U\Pi$ as π.*

Definition 4. *Let (Π, η, ϵ) be a polynomial ring adjoint and (R, a) be a pointed commutative ring. Then we call the component of the counit at (R, a), $\epsilon_{(R,a)} : \Pi R \to (R, a)$, the evaluation homomorphism at a.*

Theorem 1. *Let (Π, η, ϵ) be a polynomial ring adjoint. Then for any pointed commutative ring (R, a), the ring homomorphism $\eta_R : R \to \pi R$ is injective and the pointed ring homomorphism $\epsilon_{(R,a)} : \Pi R \to (R, a)$ is surjective.*

Proof. Let (R, a) be a pointed commutative ring. By definition of adjunctions,

$$U\epsilon_{(R,a)} \circ \eta_{U(R,a)} = 1_{U(R,a)}$$

It follows that $\epsilon_{(R,a)}$ and η_R are right and left invertible functions, respectively. This implies that they are surjective and injective functions, respectively. □

Definition 5. *Let (Π, η, ϵ) be a polynomial ring adjoint and R be a commutative ring. Then $\overline{R} \subseteq \pi R$ denotes the image of R under η_R. The elements of $\overline{R}$ are called constant polynomials.*

Lemma 1. *For any commutative ring R, $R \simeq \overline{R}$.*

Proof. This is a simple consequence of the fact that η_R is injective. □

Definition 6. *Let (Π, η, ϵ) be a polynomial ring adjoint and R be a commutative ring. Then, if $\Pi R = (\pi R, x)$, we call $x \in \pi R$ the indeterminate element of πR.*

Theorem 2. *Let (Π, η, ϵ) be a polynomial ring adjoint and R be a commutative ring. Suppose $x \in \pi R$ is the indeterminate element. Then the following are equivalent:*

1. *R is a zero ring.*
2. *πR is a zero ring.*
3. *x is a unit in πR.*

Theorem 3. *Let (Π, η, ϵ) be a polynomial ring adjoint, R be a commutative ring. Then the indeterminate element $x \in \pi R$ is not a zero divisor in πR.*