Techniques in Spacetime Visualization

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1 Introduction

Gravity as a concept is often one introduced to students at the beginning of their scientific careers. This concept is characterized by its nature of drawing objects in toward a center of mass, like the way a pencil falls to the floor when rolled off of a desk. The Newtonian view of gravity allows for the field to be represented with vectors pointing towards the center of mass while the size of the vectors, representing the strength of the field, decreases with the square of the distance from the center of mass. When using Einstein's relativity to describe gravity, the field is instead represented by tensors showing curvature throughout a manifold. This field of tensors cannot easily be depicted like the vectors in Newtonian gravity, so instead we look for different methods of visualization. Attempts at visualization are made using embedding diagrams to offer intuition as to the inner workings of this phenomenon.[1] These diagrams have a long history in physics, dating as far back as the early 20th century.[2] They work by embedding the three dimensional curvature into a 2D diagram using different techniques to better represent some of the effects of a specific spacetime manifold.

Phenomenon such as orbital motion can also be visualized through embedding diagrams by simply outlining a path in the diagram which represents a geodesic. Unfortunately, for almost any system other than one that is single-bodied and spherically symmetric (i.e Kerr black holes, binary systems) these diagrams fail to capture the effects and no longer offer the complexity needed to describe the system.

The last two semesters, I have been working towards a new visualization technique that offers a much more comprehensive and easy to understand way of fully describing more complex gravitational systems with accuracy while still offering the level of intuition that one might gain by looking at a vector field for an electromagnetic system.

2 Developing the Theory

To progress in efficiently designing a model for visualizing spacetime, one must first have the necessary mathematics to accurately base this visualization on. In order to simplify the calculations so that they may better be adapted into code, different methods of arranging Einstein's equations have been tested by many scientists in search of similar results to our own goals. For our research, we have used the linearized equations of general relativity, adapted to resemble Maxwell's equations so that the gravitational field may be broken into two components, a gravito-electric and gravito-magnetic component. This will propel the understanding of gravity by allowing scientists to relate the fields to something they are far more familiar with in terms of intuition, and that is electromagnetism. Professor David Nichols and others have already written about the development of this arranging of the fields into two sub-fields[3] so I will only briefly paraphrase the process and then get into the actual technique used for visualization.

2.1 Electromagnetic Fields

The electric field $\vec{E}$ and magnetic field $\vec{B}$ are often defined through the expression of a particles acceleration $\vec{a}$ due to the total electromagnetic force, known as the Lorentz force equation (where m, q, and $\vec{v}$ are a particles mass, charge, and velocity, respectively). Using the Cartesian basis, the particle's acceleration is given by

$$a^j = \frac{q}{m}\left(E^j + \sum_{k,p} \epsilon_{jkp} v^k B^p\right) \tag{1}$$

The fields then obey Maxwell's equations,

$$\sum_j \frac{\partial E_j}{\partial x^j} = 0 \qquad \sum_{j,k} \epsilon_{ijk} \frac{\partial E_k}{\partial x^j} + \frac{\partial B_i}{\partial t} = 0 \tag{2}$$

$$\sum_{j,k} \epsilon_{ijk} \frac{\partial B_k}{\partial x^j} - \frac{1}{c^2}\frac{\partial E_i}{\partial t} = 0 \qquad \sum_j \frac{\partial B_j}{\partial x^j} = 0. \tag{3}$$

Here the equations are written such that there are no sources or material properties so that we may take the charge and current density to be zero and assume vacuum values for the dielectric constant and magnetic permeability.

2.2 Gravitational Fields

Similarly to how we began reviewing electromagnetic fields in the absence of sources, etc., we will now take gravitation in its simplest form, gravitostatics. For this special case of static gravitational fields which are weak, i.e. not accelerating particles to speeds near c, then Newtonian ideas can be used to modify and simplify relativistic gravitation.

In the relativistic viewpoint, we consider the way that naturally free-falling motions vary from place to place and time to time. Clearly, in a static configuration the information does not change from time to time and is thus contained purely in the description of changes in space, defined by $\nabla\Phi_g$, the gradient of the Newtonian potential. Referred to as the gravitoelectric field, this tensor describes the raising of tides on astrophysical objects in a Newtonian setting, and hence is also called the tidal tensor. In a Cartesian basis, we can write this as the set of tensor components

$$\mathcal{E}_{jk} = \frac{\partial^2 \Phi_g}{\partial x^j \partial x^k}, \tag{4}$$

where the trace is written as

$$\sum_k \mathcal{E}_{kk} = \sum_k \frac{\partial^2 \Phi_g}{\partial x^k \partial x^k} = \nabla^2 \Phi_g. \tag{5}$$

Just as the electric field is divergenceless outside sources, the gravitoelectric field is traceless outside sources. However, equation (5) is still valid in the presence of sources. This equation allows us to describe the field in terms of our sources, however, only for a static field. In order to deal with events such as gravitational waves (or radiation in general, per se) we need more general definitions similar to what we had for the electromagnetic field. In this case, we will not be able to describe the field based on the acceleration of a particle, since there is no concept of "force" in relativistic gravitation, but rather we will describe the field based on point particles separated by a small displacement **s** with a relative velocity **v**. The relative acceleration of the two particles, given by

$$\frac{d^2 s^j}{dt^2} = -\sum_k \mathcal{E}_{jk} s^k - 2 \sum_{k,p,m} \epsilon_{jkp} \mathcal{B}_{pm} v^k s^m, \tag{6}$$

defines the tensors $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$. This equation has striking similarity to the Lorentz acceleration in Eq. (1). A key difference, however, is that in Eq. (1) the particles mass and charge are used to prescribe uniqueness in the system and dependence on the object moving through the field, whereas there is no reference to any characteristic of the particles in Eq. (6) in accordance with the equivalence principle.[4]

The gravito-electric and gravito-magnetic fields which are encompassed in Eq. (6) are symmetric ($\mathcal{E}_{jk} = \mathcal{E}_{kj}, \mathcal{B}_{jk} = \mathcal{B}_{kj}$). Outside of sources, these fields are traceless and obey the Maxwell-like relations that we have been waiting to get to.[3]

$$\sum_j \frac{\partial \mathcal{E}_{jk}}{\partial x^j} = 0 \qquad \frac{1}{2}\left(\sum_{jk} \epsilon_{pjk} \frac{\partial \mathcal{E}_{qk}}{\partial x^j} + \sum_{jk} \epsilon_{qjk} \frac{\partial \mathcal{E}_{pk}}{\partial x^j}\right) + \frac{\partial \mathcal{B}_{pq}}{\partial t} = 0 \tag{7}$$

$$\frac{1}{2}\left(\sum_{jk} \epsilon_{pjk} \frac{\partial \mathcal{B}_{qk}}{\partial x^j} + \sum_{jk} \epsilon_{qjk} \frac{\partial \mathcal{B}_{pk}}{\partial x^j}\right) - \frac{1}{c^2}\frac{\partial \mathcal{E}_{pq}}{\partial t} = 0 \qquad \sum_j \frac{\partial \mathcal{B}_{jk}}{\partial x^j} = 0 \tag{8}$$

which can be written elegantly as

$$\nabla \cdot \mathcal{E} = 0 \qquad \nabla \times \mathcal{E} + \frac{\partial \mathcal{B}}{\partial t} = 0 \tag{9}$$

$$\nabla \times \mathcal{B} - \frac{1}{c^2}\frac{\partial \mathcal{E}}{\partial t} = 0 \qquad \nabla \cdot \mathcal{B} = 0, \tag{10}$$

with the appropriate interpretation of the divergence and curl, noting that the divergence can be taken on either index since the tensors are symmetric and the curl is symmetrized in these equations.

2.3 Radiation

To provide the actual information we are looking for, the gravitational radiation, we take the equations listed and adapt them such that they offer field information for an oscillating gravitational point quadrupole, the simplest configuration which emits radiation. To develop this quadrupole moment, we require a configuration shown below in Figure 1. Here, M represents the mass and d is the separation distance between the objects.

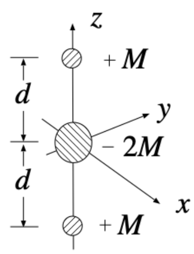

Figure 1: A simple model of a gravitational quadrupole. Here, M represents the mass and d is the separation distance between the objects. This was originally published in a paper worked on by David Nichols.[3]

We can replace q with M and $1/4\pi\epsilon_0$ with G in our electric potential and then take the limit as $d \to 0$ and $Q = Md^2$ to be fixed. Using these substitutions and limits from the electric quadrupole moment in the Cartesian basis and plugging them into the potential for the gravitational system we find the Newtonian potential. It is easier to then convert this into the spherical orthonormal basis, rather than using spherical from the start. The Newtonian potential in spherical coordinates is then

$$\Phi_g = -\frac{GQ}{r^3}(3\cos^2\theta - 1), \tag{11}$$

where G represents the gravitational constant and $Q = Md^2$. Extracting the gravitoelectric components of the potential is straightforward. Analogous to the electric case, there are no gravitomagnetic fields. We obtain the equations

$$\begin{aligned}
\mathcal{E}_{rr} &= -\frac{12GQ}{r^5}(3\cos^2\theta - 1) \\
\mathcal{E}_{r\theta} &= -\frac{24GQ}{r^5}\sin\theta\cos\theta \\
\mathcal{E}_{\phi\phi} - \mathcal{E}_{\theta\theta} &= \frac{6GQ}{r^5}\sin^2\theta.
\end{aligned} \tag{12}$$

The components $\mathcal{E}_{r\phi}$ and $\mathcal{E}_{\theta\phi}$ are zero because of the axisymmetry, and using the traceless condition, $\mathcal{E}_{\phi\phi} + \mathcal{E}_{\theta\theta} = -\mathcal{E}_{rr}$ we can obtain the remaining components. For gravitational radiation, however, a static field is not enough. To produce radiation, we oscillate the point quadrupole at frequency ω. We then take the distance d to have the form $d_0 + \Delta d\cos\omega t$ and take Q to be $2Md_0\Delta d$. The equations describing the oscillating field are listed below.[3]

$$\begin{aligned}
\mathcal{E}_{rr} &= \frac{-4GQk^2}{r^3}\left[\left(-1+\frac{3}{k^2r^2}\right)\cos(kr-\omega t) + \frac{3\sin(kr-\omega t)}{kr}\right](3\cos^2\theta - 1) \\
\mathcal{E}_{r\theta} &= \frac{-4GQk^2}{r^3}\left[\left(\frac{6}{(kr)^2}-3\right)\cos(kr-\omega t) - \left(kr - \frac{6}{kr}\right)\sin(kr-\omega t)\right]\sin\theta\cos\theta \\
\mathcal{B}_{r\phi} &= \frac{-4GQk^2}{cr^3}\left[-3\cos(kr-\omega t) - \left(kr-\frac{3}{kr}\right)\sin(kr-\omega t)\right]\sin\theta\cos\theta \\
\mathcal{E}_{\phi\phi} - \mathcal{E}_{\theta\theta} &= \frac{-2GQk^2}{r^3}\left[\left(-\frac{3}{k^2r^2}+3-(kr)^2\right)\cos(kr-\omega t) - \left(\frac{3}{kr}-2kr\right)\sin(kr-\omega t)\right]\sin^2\theta \\
\mathcal{B}_{\theta\phi} &= \frac{-GQk^2}{cr^3}\left[(-3+k^2r^2)\cos(kr-\omega t) - \left(-\frac{3}{kr}+2kr\right)\sin(kr-\omega t)\right]\sin^2\theta,
\end{aligned} \tag{13}$$

where $k \equiv \frac{\omega}{c}$. These equations are then used to find the principal axes corresponding to the three eigenvectors of the gravito-magnetic and gravito-electric fields. With this information, we are able to find out information about the strength and direction of the fields at certain times and locations based on their eigenvector and eigenvalue information. This will be discussed further in the next section.

3 Computation and Visualization

So far with our research we have limited ourselves to only exploring the static case of the gravito-electric component. Doing this allows us to focus on correcting the method of visualization on a more simple case, so that it may then be easily adapted to time evolving fields and fields involving the gravitomagnetic component, as well.

Taking only the gravito-electric components out of (13) and putting them into matrix form, we begin computing eigenvectors and eigenvalues using a routine built into Numpy.[5] This routine then ejects values for each of the eigenvalues and eigenvectors corresponding to different locations on a graph. We then take the values given for the eigenvalues at each point and plot them on two separate plots, one referencing the positive eigenvalues and the other negative eigenvalues. The plot shown below gives insight into the field strength, where color corresponds to the size of the eigenvalue as shown on the color-bar (log scale) to the side of the graph, and to the directions of propagation demonstrated by the overlaying vectors. We took the vectors that came from our eigenvector calculation and scaled them such that the eigenvector corresponding the the larger eigenvalue would be equal to 1, and the eigenvector corresponding to the smaller eigenvalue would be scaled down from one by whatever the ratio of the eigenvalues was. This allows for us to see which directions of the field are more dominant in certain regions.

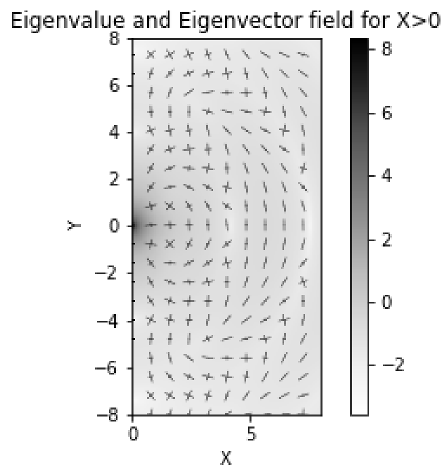

Figure 2: Vectors representing the direction of propagation of gravitational radiation overlaid on a log plot of the corresponding eigenvalues.

4 Conclusions and Future Directions

We began by drawing comparisons between electromagnetic and gravitational fields and moved into expanding those into oscillating quadrupole moments. We found that using the oscillating quadrupole moment in its Maxwell-like form, we could use a routine built into python to solve for the eigenvectors and eigenvalues of the gravitoelectric tensor. Using this information, we were able to plot the eigenvectors of the tensor in a clear way that represents the direction of propagation of the tensor with the same intuition that one might have when looking at a more basic electric field plot. This step was an important part of the process of deciding how best to represent the gravitational field so that all of its information is captured while still being easily interpreted.

We then began experimenting with different ways of encompassing more information with the same simple plots. Our first step was to scale the eigenvectors such that they would represent, based on the ratio of the eigenvalues, which direction of propagation was stronger at a certain point. After this, I was tasked with assigning a certain color, blue or red, to each vector depending on what component of the gravitational field the vector represented (stretching or squeezing). Unfortunately, I have not finished that yet, but I plan on getting that fixed before starting any new projects this summer.

Other things to look into as we move forward would be learning how to evolve our system with time. As previously stated, we are currently only retrieving information for the gravito-electric component of the field and at time $t = 0$. As we continue, we hope to evolve our systems with time so that we can see the effect that the gravitational system has as it evolves. Further, we will eventually begin to incorporate the gravito-magnetic component such that we would have a full description of the gravitational field at each location, and at every time for as long as we evolve the system.

5 Refrences

[1] Stanislav Hledík, Zdeněk Stuchlík, Alois Cipko arxiv.org/pdf/astro-ph/0701237

[2] Ludwig Flamm, doi.org/10.1007/s10714-015-1908-2

[3] Richard H. Price, John W. Belcher, David A. Nichols, arxiv.org/abs/1212.4730

[4] www.npl.washington.edu/eotwash/equivalence-principle

[5] numpy.linalg.eig