# Game Theory: An introduction

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# What is game theory

*The study of mathematical models of conflict and cooperation between intelligent rational decision-makers.*

- Roger B. Myerson (1991). Game Theory: Analysis of Conflict

# Rationality

*Suppose that you are perfectly rational. So am I. You know that I am rational. I know you are rational too. You know that I know you are rational. And so on... In this perfectly logical world, how do agents take decisions?*

# Start with an example...

- You and 2 other friends go to Chipotle for lunch.
- All of you have 3 options to choose from: $\{\text{burrito}(B), \text{tacos}(T), \text{salad bowl}(S)\}$.
- Suppose you can measure your satisfaction in dollars! Imagine that eating a food item gives you the same amount of pleasure as getting a particular amount of money. We call this the *utility* of the food item.
- All of you value the food items equally: $U(B) = \$9, U(T) = \$8, U(S) = \$5$.
- Costs at Chipotle are as follows: $C(B) = \$7, C(T) = \$5, C(S) = \$4$.

# Example

- What would you choose if you just pay for yourself?
- What would you choose if you split the bill evenly?

# Example

<u>To each his own:</u>

- $U(B) = \$9, U(T) = \$8, U(S) = \$5.$
- $C(B) = \$7, C(T) = \$5, C(S) = \$4.$
- Net benefit (call it *payoff*, denoted as $\pi$) = Utility - Cost.
- $\pi(B) = \$2, \pi(T) = \$3, \pi(S) = \$1.$
- Oh! Those delicious tacos! Remember you pay \$5.

# Example

Let's split the bill:

- $U(B) = \$9, U(T) = \$8, U(S) = \$5.$
- $C(B) = \$7, C(T) = \$5, C(S) = \$4.$
- Suppose your friends choose items that cost $c_2$ and $c_3$ respectively.
- Calculate your own payoffs for each item.

$$\begin{aligned}\pi(B) &= U(B) - \left(C(B) + c_2 + c_3\right)/3 \\ &= \$(9 - 7/3) - (c_2 + c_3)/3 \\ &= \$(20/3) - (c_2 + c_3)/3.\end{aligned}$$

- Similarly we get,

$$\begin{aligned}\pi(T) &= \$(19/3) - (c_2 + c_3)/3. \\ \pi(S) &= \$(11/3) - (c_2 + c_3)/3.\end{aligned}$$

# Example

Let's split the bill contd:

- $U(B) = \$9, U(T) = \$8, U(S) = \$5.$
- $C(B) = \$7, C(T) = \$5, C(S) = \$4.$
- Payoffs are

$$\pi(B) = \$(20/3) - (c_2 + c_3)/3.$$
$$\pi(T) = \$(19/3) - (c_2 + c_3)/3.$$
$$\pi(S) = \$(11/3) - (c_2 + c_3)/3.$$

- Aren't those burritos amazing? How much do you pay?
- Wait! Everyone gets a burrito! (Why?) And $7 it is.
- *Rationality hurts!*

# Common Definition

Let us define a common framework to study the effect of rationality!

## Definition

Definition of a Game A game consists of:

- A set of players. Call it $P = \{p_1, p_2, \ldots, p_n\}$.
- A set of actions for each player.
  Player $i$'s action set $A_i = \{a_i^1, a_i^2, \ldots, a_i^{k_i}\}$.
- Payoffs for each player as a function of the actions taken by all players.
  Payoff for player $i$ is given as $\pi_i(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$.

# A Digression

Clarification of terms: *Utility, Cost, Payoff*.

Texts use them in various contexts. Broadly, we have:

- Utility: A quantitative measure of your satisfaction in consumption.
- Cost: How much you pay for it.
- Payoff: The net benefit measured quantitatively.
- Payoff $=$ Utility - Cost.

# Back to the example

- Q: Who are the players?
  A: You and your 2 friends.
- Q: What are their action sets?
  A: Choices of food each player has. Here it is $\{$burrito, tacos, salad bowl$\}$ for all.
- Q: What are the payoffs?
  A: My payoff for a particular set of choices is:
    - *Each his own:*
      $\pi_{\text{me}}(\text{set of choices}) = (\text{utility from my food}) - (\text{cost of my food})$.
    - *Split the bill:* $\pi_{\text{me}}(\text{set of choices}) =$
      $(\text{utility from my food}) - (\text{average cost of food for all})$.

# The Classic Example: Prisoners' Dilemma

Consider two prisoners. They are held in different cells and asked independently if they are going to cooperate (C) with each other and stay silent or defect (D) and rat each other out.

| | C | D |
|---|---|---|
| C | -0.2, -0.2 | -2, 0 |
| D | 0, -2 | -1, -1 |

Table: The Prisoner's Dilemma

- The *payoff matrix* for this game is given above. -2 is 2 years in jail.
- Any 2-player one-shot game with finite action sets can be represented as a payoff matrix. This representation is called *normal form* or *strategic form*.
- $(a, b)$ in the payoff matrix denotes payoffs to players 1 and 2 respectively.

# The Classic Example: Prisoners' Dilemma

**Examples**

Prisoner's dilemma

| | C | D |
|---|---|---|
| C | -0.2, -0.2 | -2, 0 |
| D | 0, -2 | -1, -1 |

Consider how Prisoner 1 (row) thinks.

- Q: If Prisoner 2 (column) plays Cooperate, what should I play?
- A: I should play Defect. (Why?)
- Q: If Prisoner 2 plays Defect, what should I play?
- A: I should play Defect. (Why?)
- I should always play Defect!

# The Classic Example: Prisoners' Dilemma

## Examples

Prisoner's dilemma

| | C | D |
|---|---|---|
| C | -0.2, -0.2 | -2, 0 |
| D | 0, -2 | -1, -1 |

Similarly (by symmetry), Prisoner 2 should always Defect!

# The Classic Example: Prisoners' Dilemma

## Examples

Prisoner's dilemma

| | C | D |
|---|---|---|
| C | -0.2, -0.2 | -2, 0 |
| D | 0, -2 | -1, -1 |

Thus we end up with this outcome for the game.
*Rationality hurts again!*

# Looking ahead...

Formalizing notions already seen through examples:

- Dominant/ dominated strategy.
- Nash Equilibrium

# Looking back...

Remember these lines from the previous slides?

- Q: If Prisoner 2 (column) plays Cooperate, what should I play?
- A: I should play Defect. (Why?)
- Q: If Prisoner 2 plays Defect, what should I play?
- A: I should play Defect. (Why?)
- I should always play Defect!

Here Prisoner 1 finds what her *best response* to Prisoner 2's actions are.

# Dominant Strategy

Define $a_{-i}$ as actions of other players.

**Definition**

A strategy or action $a_i^*$ for a player $i$ is a **Dominant Strategy** if and only if

$$\pi_i(a_i^*, a_{-i}) \geq \pi_i(a_i, a_{-i}) \quad \forall a_i \in A_i \quad \forall a_{-i} \in A_{-i}$$

# Dominant Strategy

Consider the Prisoners Dilemma example again.

## Examples

Prisoner's dilemma

| | C | D |
|---|---|---|
| C | -0.2, -0.2 | -2, 0 |
| D | 0, -2 | -1, -1 |

- Whatever Prisoner 2 plays, it's best for Prisoner 1 to play Defect.
- Thus, Defect is a **dominant strategy** for Prisoner 1. A strategy is *dominant* for a player if she is better off playing it regardless of what the other player chooses.
- Cooperate is a **dominated** strategy. since Prisoner 1 is always better off not playing Cooperate.
- Thus, Prisoner 1 removes Cooperate from her set of strategies.

# Dominant Strategy

- When dominant strategies exist, they make the game analysis relatively easy.
- However, such strategies do not always exist.
- Need more tools to help us...

# Player $i$'s motive

- Player $i$ seeks action $a_i$ that maximizes $\pi_i(a_i, a_{-i})$.
- Leads us to a natural definition of *equilibrium*, where everyone is playing their best response to others.

# Nash Equilibrium

Define $a_{-i}$ as actions of other players.
Define $a_i^* \in A_i$ as the action which solves $a_i^*(a_{-i}) = \max_{a_i \in A_i} \pi_i(a_i, a_{-i})$.

### Definition

A strategy or action profile $(a_1^*, a_2^*, \ldots, a_n^*)$ is a **Nash Equilirbium** if and only if

$$\pi_i(a_i^*, a_{-i}^*) \geq \pi_i(a_i, a_{-i}^*) \quad \forall a_i \in A_i \quad \forall i$$

Comparing with the definition of Dominant Strategy:

### Definition

A strategy or action $a_i^*$ for a player $i$ is a **Dominant Strategy** if and only if

$$\pi_i(a_i^*, a_{-i}) \geq \pi_i(a_i, a_{-i}) \quad \forall a_i \in A_i \quad \forall a_{-i} \in A_{-i}$$

# Back to Prisoners Dilemma Example

**Examples**

Prisoner's dilemma

| | C | D |
|---|---|---|
| C | -0.2, -0.2 | -2, 0 |
| D | 0, -2 | -1, -1 |

- Note that (Defect, Defect) is a Nash Equilibrium.
- Why? Ask yourself the following questions:
- When Prisoner 2 is playing Defect, what should Prisoner 1 play?
- When Prisoner 1 is playing Defect, what should Prisoner 2 play?
- Check with the definition of NE!

# Limits of Nash Equilibra

- Q: Do NE's always exist?
- A: No! Check this example.

## Examples

| | | $P_2$ | |
|---|---|---|---|
| | | X | Y |
| $P_1$ | X | 2, 1 | 1, 2 |
| | Y | 1, 2 | 2, 1 |

- Can you make a story for this pay-off matrix?
- This is an example of a *matching pennies* game!

# Limits of Nash Equilibria

- Q: Are NE's unique if they exist?
- A: No! Check this example.

## Examples

| | | $P_2$ X | $P_2$ Y |
|---|---|---|---|
| $P_1$ | X | 2, 2 | 0, 0 |
| $P_1$ | Y | 0, 0 | 1, 1 |

- Can you make a story for this pay-off matrix?
- This is an example of a *coordination* game!

# Recap...

What have we studied so far?

- Definition of a game.
- How rationality works.
- Nash Equilibrium.
- Dominant strategies.

# Moving beyond

Consider the game we've already seen before:

## Examples

Matching Pennies

| | | $P_2$ X | $P_2$ Y |
|---|---|---|---|
| $P_1$ | X | 2, 1 | 1, 2 |
| $P_1$ | Y | 1, 2 | 2, 1 |

- NE does not exist here.
- Q: Can we do better when we play this game a million times?
- A: Yes! Play probabilistically. Goal is to maximize *expected* payoffs.
- If other player knows for sure what you play, you will suffer.

# Strategy and action

Here we distinguish between an action and a strategy.

- *Action*: It still remains the same.
- *Strategy*: The rule that a player follows every time he has to take a decision in the game.
- In a probabilistic setting, a strategy for a player is a probability distribution over his action set.
- This is called a **mixed strategy**.
- Choosing individual actions is a special case. It's called a **pure strategy**.

# Mixed Strategy Equilibrium

Define a Nash Equilibrium in this probabilistic setting.

## Definition

Suppose $\mathbf{p_1^*}, \mathbf{p_2^*}, \ldots, \mathbf{p_N^*}$ be probability distributions over the action sets $A_1, A_2, \ldots, A_N$ respectively. This set of probability distributions constitutes a **Mixed Strategy Nash Equilibrium** iff

$$\begin{aligned} & \mathbb{E}[\pi_i(\mathbf{p_1^*}, \ldots, \mathbf{p_i^*}, \ldots, \mathbf{p_N^*})] \\ \geq \quad & \mathbb{E}[\pi_i(\mathbf{p_1^*}, \ldots, \mathbf{p_i}, \ldots, \mathbf{p_N^*})] \end{aligned}$$

for all distributions $p_i$ over $A_i$, for all $i$.

# Mixed Strategy Equilibrium

Salient points in the definition:

- We maximize **expected payoff** here.
- The maximization is over all possible **distributions** over the action sets.
- Each player chooses a best response **distribution**.

Explore this definition through an example...

# Example

Consider the game we saw before.

## Examples

| | | $P_2$ X | $P_2$ Y |
|---|---|---|---|
| $P_1$ | X | 2, 1 | 1, 2 |
| $P_1$ | Y | 1, 2 | 2, 1 |

Let's see how player 1 thinks...

# Example

- $P_1$ knows $P_2$ will randomize between $X$ and $Y$ with probabilities $q$ and $1 - q$.
- $P_1$ will randomize between $X$ and $Y$ with probability $p$ and $1 - p$.
- $P_1$ tries to find the best response to $P_2$'s distribution, i.e., what should $P_1$ choose for $p$, as a function of $q$?
- Let's find that.

# $p$ as a function of $q$.

- Let us calculate the expected payoff.

## Examples

| | | $P_2$ q X | 1-q Y |
|---|---|---|---|
| $P_1$ | p X | 2, 1 | 1, 2 |
| | 1-p Y | 1, 2 | 2, 1 |

- Expected payoff for $P_1$:

$$\begin{aligned}\mathbb{E}\pi_1(p,q) &= 2pq + 1p(1-q) + \\ &\quad 1(1-p)q + 2(1-p)(1-q) \\ &= \underbrace{p(2q-1)}_{P_1 \text{ chooses } p} + \underbrace{(2-q)}_{\text{indep. of } p}.\end{aligned}$$

# $p$ as a function of $q$

Expected payoff for $P_1$:

$$\mathbb{E}\pi_1(p,q) = \underbrace{p(2q-1)}_{P_1 \text{ chooses } p} + \underbrace{(2-q)}_{\text{indep. of } p}.$$

Goal: Find $p$ that maximizes $\mathbb{E}\pi_1(p,q)$:

- If $q > 1/2$, $p = 1$ maximizes $P_1$'s profit.
- If $q < 1/2$, $p = 0$ maximizes $P_1$'s profit.
- If $q = 1/2$, $P_1$ is indifferent. Thus $p \in [0,1]$.
- We have computed: $P_1$'s strategy of $p = f(q)$.
- Similarly find $P_2$'s best response as $q = g(p)$.

# Simplify the Calculation

To calculate the equilibrium, it is enough to find a strategy for $P_1$ that makes $P_2$ indifferent, and a strategy for $P_2$ that makes $P_1$ indifferent.

$P_1$'s expected payoff from playing $X$ is $2q + (1 - q)$.
And the payoff from playing $Y$ is $q + 2(1 - q)$.
So that indifference requires that:

$$2q + (1 - q) = q + 2(1 - q),$$

which implies that $q = \frac{1}{2}$.

# Exercise

**Rock, Paper, Scissors**

(Denote a win by 1, a loss by -1 and a tie by 0)

# Exercise

**Cournot Duopoly**

# Common Definition

Let us define a common framework to study the effect of rationality!
A game consists of:

- A set of players. Call it $P = \{p_1, p_2, \ldots, p_n\}$.
- A set of actions for each player.
  Player $i$'s action set $A_i = \{a_i^1, a_i^2, \ldots, a_i^{k_i}\}$.
- Payoffs for each player as a function of the actions taken by all players.
  Payoff for player $i$ is given as $\pi_i(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$.

# Mixed Strategy NE

- We already saw that Pure Strategy NE might not always exist.
- What about Mixed Strategy NE?
- For a game with finite number of players and finite action sets, it **always exists**.
- The last statement is a loose version of Nash's Theorem.

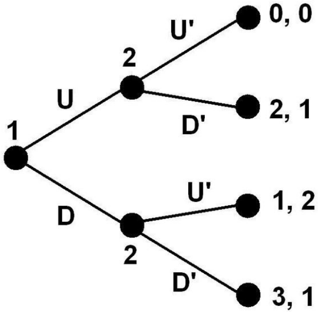
# Timing

- So far we've seen one-shot or *simultaneous move* games.
- How to represent games with a timing aspect, with sequential moves?
- Example: Board games, tic-tac-toe, etc.

- Extensive form games!
- Use a game tree to capture sequence/timing.

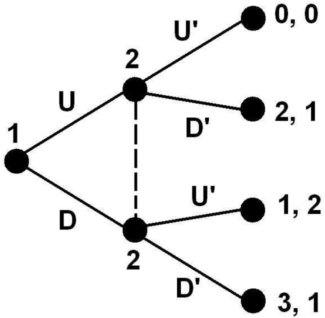
# Games in Extensive Form

- Example (perfect information):



# Games in Extensive Form

- Example (imperfect information):



# Recap

- How rationality works.
- Dominant strategies.
- Nash Equilibria: Pure and Mixed Strategies.
- Extensive form games.

# Later...

- Continuum of players?
  - Routing games!

- Continuous action sets?
  - Auctions!

Thank you.