Problem 2

a. We write $\tilde{r}$ as follows:

$$\tilde{r} = rG \qquad G = \alpha P + (1-\alpha)\begin{bmatrix} \frac{1}{n+1} & \frac{1}{n+1} & \cdots \\ \frac{1}{n+1} & \ddots & \frac{1}{n+1} \\ \vdots & \cdots & \frac{1}{n+1} \end{bmatrix}$$

Expanding this out, we have:

$$(\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_n, \tilde{r}_{n+1}) = (r_1, \dots, r_{n+1})\left(\begin{bmatrix} \alpha P_{1,1} & \alpha P_{1,2} & \cdots & \alpha P_{1,n+1} \\ \vdots & \ddots & & \vdots \\ \alpha P_{n+1,1} & \cdots & \cdots & \alpha P_{n+1,n+1} \end{bmatrix} + \begin{bmatrix} \frac{1-\alpha}{n+1} & \cdots & \frac{1-\alpha}{n+1} \\ \vdots & \ddots & \\ \frac{1-\alpha}{n+1} & \cdots & \frac{1-\alpha}{n+1} \end{bmatrix}\right)$$

Looking at each page rank:

$$\tilde{r}_1 = r_1\left(\alpha P_{1,1} + \frac{1-\alpha}{n+1}\right) + r_2\left(\alpha P_{2,1} + \frac{1-\alpha}{n+1}\right) + \dots + r_{n+1}\left(\alpha P_{n+1,1} + \frac{1-\alpha}{n+1}\right) = \frac{1-\alpha}{n+1}\sum_{i=1}^{n+1} r_i + \alpha \sum_{i=1}^{n+1} r_i P_{i,1}$$

$$\vdots$$

$$\tilde{r}_{n+1} = r_1\left(\alpha P_{1,n+1} + \frac{1-\alpha}{n+1}\right) + r_2\left(\alpha P_{2,n+1} + \frac{1-\alpha}{n+1}\right) + \dots + r_{n+1}\left(\alpha P_{n+1,n+1} + \frac{1-\alpha}{n+1}\right) = \frac{1-\alpha}{n+1}\sum_{i=1}^{n+1} r_i + \alpha \sum_{i=1}^{n+1} r_i P_{i,n+1}$$

$\tilde{r}_{n+1}$ is precisely the page we add which is X so we can now solve for the page rank of X.

$$X = \tilde{r}_{n+1} = \frac{1-\alpha}{n+1}\sum_{i=1}^{n+1} r_i + \alpha \sum_{i=1}^{n+1} r_i P_{i,n+1} = \frac{1-\alpha}{n+1} + \alpha X$$

because X has no out-degree, so therefore it points to itself.

Now, solving for X:

$$X = \frac{1-\alpha}{n+1} + \alpha X \Rightarrow X - \alpha X = \frac{1-\alpha}{n+1} \Rightarrow X(1-\alpha) = \frac{1-\alpha}{n+1} \Rightarrow \boxed{X = \frac{1}{n+1}}$$

B/c X has no out or in degree, it cannot give or take rank to / from the other nodes. Thus we simply have to rescale and normalize the page ranks of the other n pages. Hence:

$$\boxed{\tilde{r} = r \cdot \frac{n}{n+1}}$$

$$\boxed{X = \frac{1}{n+1}}$$

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b. We first calculate the Page Rank of Y. Similar to how we calculated X in part a, we get that

$$Y = \frac{1-\alpha}{n+2} \sum_{i=1}^{n+2} r_i + \alpha \sum_{i=1}^{n+2} r_i P_{i,n+2}$$

B/c Y has an out-degree of 1 and nothing else points to Y, we know that $\alpha \sum_{i=1}^{n+2} r_i P_{i,n+2} = 0$ and so:

$$Y = \frac{1-\alpha}{n+2} \sum_{i=1}^{n+2} r_i = \boxed{\frac{1-\alpha}{n+2}}$$

Now, for X, we not only have a link to itself (b/c it has no out-degree), we also get rank from Y.

$$X = \frac{1-\alpha}{n+2} + \alpha X + \alpha Y \Rightarrow (1-\alpha) X = \frac{1-\alpha}{n+2} + \frac{\alpha(1-\alpha)}{n+2}$$ (substit. for Y)

$$\boxed{X = \frac{1+\alpha}{n+2}}$$

This shows that the pagerank of X $\boxed{\text{does improve}}$ b/c before the addition of Y, $X = \frac{1}{n+1}$, and for large n:

$$\frac{1+\alpha}{n+2} > \frac{1}{n+1}, \quad n \to \infty.$$

Now, X and Y are still isolated from the rest of the graph of n nodes, so for the original n nodes, we simply rescale/normalize to account for these two new nodes.

$$\boxed{F = r \cdot \frac{n}{n+2}}$$

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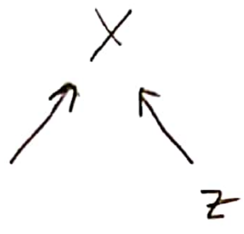
c. The setup would be as follows:



We know that b/c these three nodes are isolated from the rest of the graph, the only way for x to gain rank is for other pages to point to it, as we saw in part b. Hence, we should have both Y and Z contribute their rank to X. Likewise to maximize X's rank, we do not want X to share its rank by pointing anywhere, so we keep it so that X has the max. in-degree and min. out-degree (0).

Mathematically, b/c X, Y, and Z are isolated from the rest of the n nodes, $X+Y+Z = 3/(n+3)$. Thus, to maximize X, we minimize Y and Z by setting $y = z = \frac{1-d}{n+3}$ and pointing them both to X to have a rank of $\frac{1+2d}{n+3}$.

↑max.

d. No, adding links from page X to other pages will not improve the PageRank of X, but actually decrease it. Increasing the out-degree of X will only mean that X will potentially be giving rank to other pages. Similarly, adding links from Y or Z to older, popular pages will mean that Y and Z will not be able to donate their full ranks to X, but can only donate a portion of it b/c they are connected to additional pages now. mathematically, based off of part c, we see that $x+y+z \le \frac{3}{(n+3)}$, given that now X, Y, and Z may have links to other pages. To maximize X, we should abide by the setup in c.

e. From the previous parts, we see that adding additional pages and making them point to X will boost the rank of X. However, doing such will only boost the page rank of X marginally, especially for larger and larger n. To increase the rank of X more significantly, we should try to get popular pages that already have high rank to point to X. In this case, a portion of the popular pages' ranks will be transferred to X which can boost the rank of X.