We've seen that auctions are a natural choice for advertising on the web....
now let's take a step away from the web and go over the basics of auction theory.

Auctioning 1 item.

ex:

value: v_i v_j
bid: b_i b_j

Consider the classical setting of Auctioning 1 good to a set of interested bidders. Each bidder has an intrinsic valuation v_i for the item.

There are 4 main types of auctions for this setting:

1) Ascending-bid (English) auctions:
   Seller gradually raises price until there is only 1 bidder left.

2) Descending-bid (Dutch) auctions:
   Seller gradually lowers price until someone accepts the price.

3) First-price sealed bid auctions:
   Bidders submit a "sealed bid" that others don't know. Highest bid wins and winner pays there bid.

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4) <u>Second-price (Vickery) sealed bid auctions:</u>
Bidders submit sealed-bid and highest bid wins but pays the second highest price.

Let's start by comparing these formats...

<u>Q:</u> Are any of these "equivalent"?
i.e. would any choose the same winner and have her pay the same price?

<u>A1:</u> Descending-bid $\approx$ 1st Price

In descending bid, bidders do not learn anything about the other bidder valuations (except that noone has accepted current price).
So, there will be some first price where a bidder will break their silence, and they will pay that price. This price is chosen with the same information as in the first-price auction.

<u>A2:</u> Ascending auction $\approx$ 2nd price auction

Suppose you're in an ascending auction.
Does it ever make sense to stay in beyond your valuation?
No $\rightarrow$ you will pay more than v_i

Does it ever make sense to drop out before you valuation? ...
No $\rightarrow$ you will pay less than v_i

... with the highest valuation

So, the person with the highest valuation will win and pay the 2nd highest players valuation.

This is the same as 2nd price
if bidding your valuation is the dominant strategy
→ which will show that it is

Q: Which would you rather be a bidder in?

Q: Which would you rather run?

→ Unclear, it seems that you could make more money running a 1st price auction, but this depends on bidding strategy. Similarly it seems that you would pay less in 2nd price as a bidder, but maybe you can exploit behavior in 1st price.

... need to study these as games to get a better understanding.

Auctions as a game

Players → Bidders

Payoffs → if win $v_i - p_i$ ← price assigned

if lose 0

(if tie choose a player according to a predefined ordering) ← we'll ignore this and assume $b_i \neq b_j \; \forall i,j$

Difference compared to what we've studied
player i doesn't know v_j's for $j \neq i$

$\Rightarrow$ we want to design auctions with dominant strategies so that this isn't an issue.
(we don't want actions to depend on beliefs about others)

Thm: In a 2^{nd} price auction, it is a dominant strategy to bid $b_i = v_i$

i.e. Truthfulness is the dominant strategy.

pf: We need to show that if $b_i = v_i$, then no deviation of b_i can improve payoff.

Case 1: Deviate by raising bid

i.e.: $b_i' > b_i$

Affects payoff only if b_i' wins but b_i loses.

$\Rightarrow$ highest other bid $b_j \in [b_i, b_i']$

So, payoff is $v_i - b_j \leq 0$.

Case 2: Deviate by decreasing bid

i.e. $b_i' < b_i$

Affects payoff only if b_i'' loses & b_i wins.

So, $\exists\, b_k \in [b_i'', b_i]$.

& payoff is 0 with b_i''

& $v_i - b_k \geq 0$ with b_i

$\square$

Key: Your bid determines whether you win or lose, but has no impact on the price you pay if you win.

(parallel to ascending bid... point you choose to stay in until doesn't impact price paid, only whether you win)

Note: Truthfullness is still best strategy even if others are colluding, overbidding, etc.

We'll now study behavior in 1st price auctions and see that the story is more complicated...

Q: Is $b_i = v_i$ a dominant strategy?

A: No... this ensures a payoff of 0.

So, you bid less than your valuation... but how much less?

low bid → high payoff, but less likely to win
high bid → low payoff, but more likely to win.

⇓

your best bid depends on distribution of valuations

We'll do the analysis for a special case

ex: $v_i \sim \text{Uniform}(0,1)$, 2 bidders

Let's consider strategies of the form

$S(v_i)$ where $S()$ is increasing, differentiable & $S(v_i) \le v_i$

☆ all players use the same $S()$ in symmetric equil.

Q: What is the expected payoff of a player with v_i in a sym. equil.?

A: $$\underbrace{g(v_i)}_{\text{payoff}} = \underbrace{v_i}_{\substack{Pr(win) \\ = Pr(v_i > v_j) = v_i}} \underbrace{(v_i - S(v_i))}_{\text{payoff if win}}$$

Now, we need to find an equilibrium strategy $S()$.

Key idea: Deviations in $S()$ ⟺ Deviations in valuation revealed.

i.e. instead of searching for different $s()$ we can fix $s()$ and allow players to evaluat $s()$ at v_i' instead of v_i

Simple version of the "Revalation Principle"

$\Downarrow$

Deviations in strategy $\equiv$ Deviations in revealed value.

Q: So, what is the equilibrium condition?

A: Equillibrium condition becomes

$$v_i(v_i - s(v_i)) \geq v(v_i - s(v)) \quad \forall v.$$

i.e. if player j is using $s(v_j)$ but i uses valuation v instead of v_i, i should be no better off.

Q: What $s()$ satisfies this?

i.e. what $s()$ is an equilibrium strategy?

A: $s(v) = v/2$!

To see this we must find $s()$ st

$g(v) = v(v_i - s(v))$ is maximized at v_i.

$$g'(v) = v_i - s(v) - v s'(v)$$

$$g'(v_i) = 0 \iff v_i s'(v_i) = -s(v_i) + v_i$$

$$s'(v_i) = 1 - \frac{s(v_i)}{v_i}$$

solved by $s(v) = v/2$

We can verify this by checking

$$v_i(v_i - \frac{v_i}{2}) \geq ? \; v(v_i - \frac{v}{2})$$

$$v_i^2/2 \geq ? \; v v_i - \frac{v^2}{2}$$

$$\frac{v_i^2}{2} \geq ? \; v v_i - \frac{v^2}{2}$$

$$\frac{1}{2}\left(v_i^2 - 2 v v_i + v^2\right) \geq ? \; 0$$

$$\frac{1}{2}\left(v_i - v\right)^2 \geq ? \; 0$$

✓

So, Shading your bid by 1/2 is an equilibrium strategy. If your opponent is doing it, it is optimal for you too.

Q: What do you think happens if we move to n bidders?

A: You shade your bid by less, since there is more likely to be another bidder with a similar valuation

... $\left(\frac{n-1}{n}\right) v_i$ is the equilibrium

[The only change in the analysis is that $g(v_i) = v_i^{n-1}\left(v_i - s(v_i)\right)$ instead of $1-1=1$.]

☆ This can be easily worked out.

HW will probably ask for case of n bidders (& general distribution).

Now that we know the equilibrium bidding strategy in both 1st & 2nd price auctions, we can return to our initial questions about which you'd want to use as a bidder / seller?

1st though we need to know:

Q: Given n samples from a uniform distribution, what is the expected value of the ith largest?

A: $\frac{i}{n+1}$ ... you should be able to prove this!

Bidder perspective:

When truthfulness is a dominant thing are certainly easier on bidders, but in terms of expected payoff, it's not clear.

2nd price winner: $\frac{n}{n+1} - \frac{n-1}{n+1} = \frac{1}{n+1}$ payoff

1st price winner: $\frac{n}{n+1} - \left(\frac{n}{n+1} \cdot \frac{n-1}{n}\right) = \frac{1}{n+1}$ payoff

Seller perspective:

2nd price seems on the surface to generate less revenue, but 1st price bids are lower than true valuation...

2nd price payement: $\frac{n-1}{n+1}$

1st price payment: $\frac{n}{n+1} \cdot \frac{n-1}{n} = \frac{n-1}{n+1}$

So 1st price & 2nd price are the "same" if bidders use their equilibria strategies.

☆ This is a peak into a deep result called "revenue equivalence" which shows that this holds in a very broad class of auctions, bidder distributions, etc.

One last interesting observation.

Consider the following change to the situation. Suppose everyone is hoping to resell the item.

then $\exists$ a common value V
and each bidder has a valuation that estimates V w/ some $V_c = V + \varepsilon$
error term

Q: Is 2nd price still truthful?

A: No!

... the winner is likely the person with the worst over-estimate

and has to pay the second worst over estimate.

→ "Winner's curse"
winner has negative payoff

So... every bidder should under bid their estimated valuation!

That's it for the
basics of auctions...
Next time we'll do
Ad auctions.