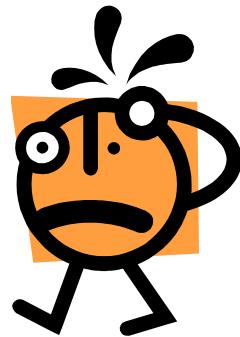


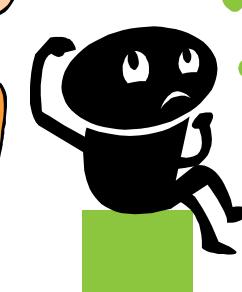
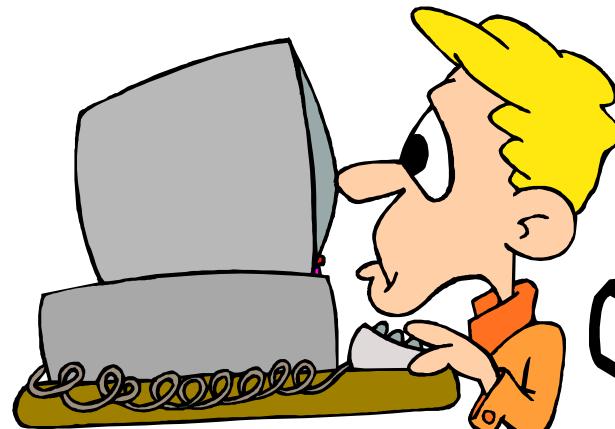
CS I

Introduction to Computer Programming



Lecture 24: December 2, 2015

Advanced topics, part 1



Last time

- C++, part 2



Today

- Advanced topics, lecture 1
 - recursion
 - first-class functions
 - **lambda** expressions
 - higher-order functions
 - **map**, **filter**, **reduce**



Admin notes

- This is the last week of classes!
- Assignment 7 is due Saturday at 2 AM
 - last assignment!
- Make sure you get caught up on all old assignments/redos!
- Last ombuds lunch at Athenaeum tomorrow!



Recursion

- A function is known as *recursive* when it calls itself
- Python allows you to define recursive functions
- Recursion has a reputation for difficulty, but it's really no big deal
- Let's look at a simple example



Sums

- How would you compute the sum of all the numbers from 0 to N (where $N \geq 0$)?
- In Python, could just use the built-in `sum` function or a `for` or `while` loop
- Mathematically, we can define a solution like this:
 - `sum_to_N(0) = 0`
 - `sum_to_N(n) = n + sum_to_N(n-1)`



Sums

- **sum_to_N** definition:
 - $\text{sum_to_N}(0) = 0$
 - $\text{sum_to_N}(n) = n + \text{sum_to_N}(n-1)$
- We state that $\text{sum_to_N}(0)$ is 0 by definition
- This is called a *base case*
 - it can be solved *immediately* (no recursion needed)
- To compute $\text{sum_to_N}(n)$ for $n > 0$:
 - first compute $\text{sum_to_N}(n-1)$
 - then add n to it
 - This *must* be the correct answer!



Sums

- **sum_to_N** definition:
 - $\text{sum_to_N}(0) = 0$
 - $\text{sum_to_N}(n) = n + \text{sum_to_N}(n-1)$
- Another way to look at this:
- If $\text{sum_to_N}(n-1)$ gives the correct sum from 0 to $n-1$
- then $\text{sum_to_N}(n)$ will be the correct sum from 0 to n , because we've just added n to $\text{sum_to_N}(n-1)$



Sums

- **sum_to_N** definition:
 - $\text{sum_to_N}(0) = 0$
 - $\text{sum_to_N}(n) = n + \text{sum_to_N}(n-1)$
- The hard part:
- We assume that **sum_to_N** works correctly for all numbers $< n$ when we define **sum_to_N** for the argument **n**
- How can we do this?



Sums

- **sum_to_N** definition:
 - $\text{sum_to_N}(0) = 0$
 - $\text{sum_to_N}(n) = n + \text{sum_to_N}(n-1)$
- This is an example of *mathematical induction*:
 - if **sum_to_N** works for some *base case* (**0**)
 - and we know that *if sum_to_N works for some value n-1, then it will also work for the value n*
 - then we conclude that **sum_to_N** works for all **n** from **0** to infinity!
 - works for 0, 1, 2, 3, ... as far up as we want to go



sum_to_N in Python

```
def sum_to_N(n):
    if n == 0:
        return 0      # base case
    else:
        return (n + sum_to_N(n - 1))
```

recursive call to `sum_to_N`



sum_to_N in Python

- Question 1: How is it possible that you can call a function while defining it?
- Answer: In fact, you *don't* call the function while defining it
 - You *never* call any functions while defining a function!
 - Instead, you define it with an internal call to what the function *will be* once it has been completely defined
 - The language implementation can handle the bookkeeping for you



sum_to_N in Python

- Question 2: What happens to the computation in progress in **sum_to_N** when a new recursive call to the same function starts?
- Answer: Every call to **sum_to_N** gets its own frame on the runtime stack, so when a recursive call to **sum_to_N** starts, a new frame is put on the stack, and the original call to **sum_to_N** still has all its own data in its own frame
 - recursion wouldn't work without a runtime stack!



A better example: sorting

- Recursion may just seem like a cute trick with no practical value
- But recursion can sometimes allow us to solve problems that would be difficult to solve without it
- Case study: sorting a list of numbers
 - Could use the Python `sort` method, but we want to do it ourselves
 - Recursion will be an essential component of our solution



Quicksort

- We will write a function which is a variation on a sorting algorithm called "quicksort"
- The idea:
 - Take the first number of a list of numbers, called the *pivot*
 - Divide the rest of the numbers into (a) a list of numbers *smaller than* the pivot, (b) a list of numbers *greater than or equal to* the pivot
 - Sort these sublists (recursively) using quicksort again!
 - Combine the sorted lists to get the final sorted list



Quicksort

```
def quicksort(lst):
    if lst == []:
        # base case (no items in list)
        return []
        # nothing to sort, return as is
    pivot = lst[0]
    less = []
    greater = []
    for item in lst[1:]:
        # examine rest of list
        if item < pivot:
            less.append(item)
        else:
            greater.append(item)
    all = quicksort(less) + [pivot] + quicksort(greater)
    return all
```



Quicksort

- While writing the **quicksort** function, we assume that the **quicksort** function itself will work on all smaller lists than the list given as an argument
- We make sure that all the recursive calls to **quicksort** are on lists that are strictly smaller than the list that was given as the argument
- We make sure that we return the correct value for the **base case** (empty list, no recursion needed)
- And then our function will work!



Recursion

- Recursion is also very useful when using tree-like data structures
 - e.g. made out of nodes which have a value, a left subtree, and a right subtree
- In this case, functions that act on the tree-like data must first deal with smaller trees
- Whenever a function has to deal with data which is a "smaller version of the original data" (lists → sublists, trees → subtrees), then recursion is useful



Recursion

- You'll learn more about recursion if you take CS 2
 - and *lots* more about recursion if you take CS 4!
- We have to move on to other topics



First-class functions

- We have seen that Python functions can be treated as data
 - e.g. callback functions for event handlers in Tkinter
- This is a general phenomenon
- Python functions are also Python objects, so they can be treated like any other Python object
 - assigned to a different name
 - stored in data structures
 - returned from functions
 - etc.



First-class functions

```
def double(x):  
    return 2 * x  
  
->>> foo = double  
  
->>> foo(10)  
20  
  
->>> lst = [foo, foo, foo]  
->>> lst[0](42)  
84
```



First-class functions

- You can return functions from functions:

```
def make_adder(n):  
    def add(x):  
        return x + n  
    return add  
  
">>>> add5 = make_adder(5)  
>>> add5(10)
```

15

```
>>> make_adder(15)(45)
```

60



First-class functions

```
def make_adder(n):
    def add(x):
        return x + n
    return add
```

- The **n** in **add** is the **n** which was the argument to **make_adder**
- We say that the **n** from **make_adder** is in scope inside the **add** function
- Even when **add** is returned, the **n** that was the argument to **make_adder** is retained



First-class functions

```
def make_adder(n):
    def add(x):
        return x + n
    return add
```

- Notice that the **add** function is a very trivial function that is only used in a single place
- The name **add** is not really important
- It would be nice if there was a more compact way to create a "function object" without having to give it a name



lambda

- In Python, an "anonymous function" can be created using a **lambda** expression
- The syntax:

lambda x: <expression involving x>

- is the same as writing

```
def func(x) :  
    return <expression involving x>
```

- except that the **lambda** expression has no name



lambda

- Examples:

```
lambda x : x * 2
```

- is the same as:

```
def double(x) :  
    return (x * 2)
```



lambda

- Examples:

```
lambda x, y: x * x + y * y
```

- is the same as:

```
def sum_squares(x, y):  
    return (x * x + y * y)
```



lambda

- `make_adder` again:

```
def make_adder(n):  
    return (lambda x: x + n)
```

- and that's it!
- We use `lambda` expressions primarily when we are defining a function that will only be used in one place
- `lambda` really only works for one-line functions
 - Python's indentation-based syntax gets in the way for longer functions



lambda

- As we will see shortly, **lambda** is often very handy when used with "higher-order" functions that take functions as arguments
- But first...



Interlude

- Another look at recursion!



Higher-order functions

- Python functions can be used as data by just referring to them by name (no arguments) or by creating a **lambda** expression
- What's the use of this?
- In Python, can define functions which take other functions as their arguments
- These are known as "higher-order" functions
- Several useful ones are built-in



map

- The **map** function takes two arguments:
 - a function requiring one argument
 - a list
- and applies the function argument to every element of the list, collecting all the results into a new list
- Let's see some examples



map

```
def double(x):  
    return x * 2  
  
>>> map(double, [1, 3, 5, 7, 9])  
[2, 6, 10, 14, 18]  
  
>>> map(lambda x: x * 2, [1, 2, 3, 4, 5])  
[2, 4, 6, 8, 10]
```

- Notice: we can use a **lambda** expression anywhere we can use a function



map

- **map** allows us to convert a list into another (related) list of the same size where the elements of the second list are functions of the elements of the first list
 - *n.b.* **map** doesn't change the original list
- **map** takes a function as its first argument, so it's a higher-order function
 - sounds scary, but it's no big deal



filter

- Sometimes, we want to pick out certain elements of a list satisfying particular properties
- The properties can be represented by a *predicate* (a function returning a boolean (**True/False**) value)



filter

- For instance, if we wanted to pick out all positive elements of a list, a predicate could be:

```
def positive(x):  
    '''Return True if x is positive.'''  
    return (x > 0)
```

- This is so simple, we can just write it as:

```
lambda x: x > 0
```



filter

- There is a higher-order function called **filter** that takes two arguments:
 - a function (a predicate *i.e.* returning a boolean)
 - a list
- and returns a new list consisting of all of the elements of the original list that satisfied the predicate (for which the predicate returned **True**)
- The original list is "filtered" to give the new list



filter

- Examples:

```
>>> filter(positive, [-3, 1, -4, 1, -5, 9, -2, 6])
[1, 1, 9, 6]
>>> filter(lambda x: x > 0, [5, -3, -8, 9, 7, -9])
[5, 9, 7]
>>> filter(lambda x: x != 0, [1, 0, 0, 2, 0, 0, 0])
[1, 2]
>>> filter(lambda x: x > 5, [4, 1, -2, 0, 3])
[]
>>> filter(lambda x: x > 10, [])
[]
```



reduce

- Another common thing to want to do is to take a list and collapse it into a single value which is a function of all the elements of the list
- For instance:
 - sum all the elements of a list together
 - multiply all the elements of a list together
 - find the largest/smallest elements of a list
- In all cases, we are "reducing" a list into a single value



reduce

- Example: sum the elements of a list
- If list elements are $[i, j, k, l \dots]$ we want to compute
 - $(i + j)$
 - $((i + j) + k)$
 - $((((i + j) + k) + l)$
- etc. until all the elements of the list are added together
- In each case, we're adding the previous sum to the next element



reduce

- Python has a built-in function called **reduce** that does this:

```
>>> reduce(lambda x, y: x + y, [1, 2, 3, 4, 5])  
15  
>>> reduce(lambda x, y: x * y, [1, 2, 3, 4, 5])  
120  
>>> reduce(lambda x, y: max(x, y), [3, 1, 4, 1, 5])  
5
```



reduce

- The first argument to **reduce** must be a function of two arguments
- Let's call it **f(x, y)**
- Then **reduce(f, [1, 2, 3, 4, 5])** is:
 - **f(f(f(f(1, 2), 3), 4), 5)**
- Notice the nested **f**s
 - second argument to **f** is the next element of the list
 - first argument is the result of reducing all previous elements of the list



reduce

- If we have:

```
def f(x, y):  
    return x + y
```

- Then `reduce(f, [1, 2, 3, 4, 5])` becomes
 $((((1 + 2) + 3) + 4) + 5)$
- which is just the sum of the list
- `reduce` makes it easy to do repetitive computations like this without having to write explicit loops



reduce

- If we use reduce on an empty list:

```
reduce(lambda x, y: x + y, [])
```

- we get:

TypeError: reduce() of empty sequence with no initial value

- **reduce** takes an extra (optional) argument that (if present) gives a value to use for the empty list

```
>>> reduce(lambda x, y: x + y, [], 0)
```

```
0
```



Functional programming

- There is a style of programming which uses higher-order functions and **lambda** expressions a lot
- It's called *functional programming*
 - in contrast to e.g. object-oriented programming
- Python supports functional programming, but not as well as some other languages
 - e.g. Scheme, Ocaml, Haskell
- CS 4 will cover functional programming in much greater depth (using Ocaml)



Next time

- Last lecture in the class!
- Advanced topics, lecture 2
 - command-line arguments
 - list comprehensions
 - iterators
 - generators
- Where to go from here



Code summary

```
def sum_to_N(n):
    if n == 0:
        return 0      # base case
    else:
        return (n + sum_to_N(n - 1))
```



Code summary

```
def quicksort(lst):
    if lst == []:
        # base case (no items in list)
        return []
        # nothing to sort, return as is
    pivot = lst[0]
    less = []
    greater = []
    for item in lst[1:]:
        # examine rest of list
        if item < pivot:
            less.append(item)
        else:
            greater.append(item)
    all = quicksort(less) + [pivot] + quicksort(greater)
    return all
```



Code summary

```
def make_adder(n):  
    def add(x):  
        return x + n  
    return add
```

```
# Shorter:  
def make_adder(n):  
    return (lambda x: x + n)
```

