A Appendix

A.1 Proposition 3 of Menon and Ong (2016)

Proposition 5. Let P be the class conditional p(C=1|s,a) and Q be the class conditional p(C=0|s,a) with marginal class probability $\frac{1}{2}$. Let $\mathcal{D}(P,Q,\frac{1}{2})$ be the joint distribution over C,S,A decomposed into P and Q and the marginal $p(C)=\frac{1}{2}$. Under assumption A5, for any scorer $\bar{s}:\mathcal{X}\to\mathbb{R}$, regret $(\bar{s};\mathcal{D},\ell)=\frac{1}{2}\mathbb{E}_{X\sim Q}\left[B_{f^{\otimes}}(\rho,\hat{\rho})\right]$, where $f^{\otimes}(z)=(1+z)f\left(\frac{z}{1+z}\right)$.

The proof can be found in Menon and Ong (2016).

A.2 Proofs of technical results

Here, we provide technical proofs of our results.

A.2.1 Proof of Proposition 1

Proof.

$$\begin{split} & \|\mathbb{E}_{\pi_{0}} \left[\phi(a) \otimes \psi(s) \hat{\rho}\right] - \mathbb{E}_{\pi_{1}} \left[\phi(a_{i}) \otimes \psi(s)\right] \|_{1} \\ &= \|\mathbb{E}_{\pi_{0}} \left[\phi(a) \otimes \psi(s) \hat{\rho}(a,s)\right] - \mathbb{E}_{\pi_{0}} \left[\phi(a) \otimes \psi(s) \rho\right] \|_{1} \\ &= \left\|\sum_{i}^{N} \phi(a_{i}) \otimes \psi(s_{i}) \hat{\rho}(a_{i},s_{i}) \pi_{0}(a_{i},s_{i}) - \sum_{i}^{N} \phi(s_{i}) \otimes \psi(s_{i}) \frac{\pi_{1}(a_{i},s_{i})}{\pi_{0}(a,s)} \pi_{0}(a_{i},s_{i}) \right\|_{1} \\ &= \left\|\sum_{i}^{N} \phi(a_{i}) \psi(s_{i}) \pi_{0}(a_{i},s_{i}) \hat{\rho}(a_{i},s_{i}) - \phi(a_{i}) \otimes \psi(s_{i}) \pi_{1}(a_{i},s_{i}) \right\|_{1} \\ &= \left\|\sum_{i}^{N} \phi(a_{i}) \psi(s_{i}) p(a_{i},s_{i}) (\rho(a_{i},s_{i}) + (\hat{\rho}(a_{i},s_{i}) - \rho(a_{i},s_{i}))) - \phi(a_{i}) \otimes \psi(s_{i}) \pi_{1}(a_{i},s_{i}) \right\|_{1} \\ &= \left\|\sum_{i}^{N} \phi(a_{i}) \otimes \psi(s_{i}) \pi_{0}(a_{i},s_{i}) (\hat{\rho}(a_{i},s_{i}) - \rho(a_{i},s_{i})) \right\|_{1} \\ &= \left\|\mathbb{E}_{\pi_{0}(a,s)} \left[\phi(a) \otimes \psi(x) (\hat{\rho} - \rho)\right] \right\|_{1} \leq \left\|\mathbb{E}_{\pi_{0}(a,s)} \left[\phi(a) \otimes \psi(x) B(\hat{\rho}, \rho)\right] \right\|_{1} \end{split}$$

A.2.2 Proof of Proposition 2

Because the weights in the denominator $\hat{V}^{\text{B-OPE}}$ are each consistent for 1, we have that the sum is consistent for n. Therefore, by the continuous mapping theorem, we can consider the expectation of a single term in the $\hat{V}^{\text{B-OPE}}$ numerator.

Recall that $\rho(a,s) = \frac{\pi_1(a,s)}{\pi_0(a,s)}$ denotes the true density ratio and $\hat{\rho}(a,s)$ is the estimated density ratio. Further let $\delta(a,s) = \hat{\rho}(a,s) - \rho(a,s)$. First, we consider the discrete action setting. We can express the expectation as:

$$\begin{split} \mathbb{E}_{\pi_0} \left[\mathbb{1}_a(\pi_1(s)) \hat{\rho}(a,s) r(a,s) \right] &= \mathbb{E}_{\pi_0} \left[\mathbb{1}_a(\pi_1(s)) (\rho(a,s) + \delta(a,s)) r(a,s) \right] \\ &= \mathbb{E}_{\pi_0} \left[\mathbb{1}_a(\pi_1(s)) \rho(a,s) r(a,s) \right] + \mathbb{E}_{\pi_0} \left[\mathbb{1}_a(\pi_1(s)) \delta(a,s) r(a,s) \right] \end{split}$$

We can show that the first term is equal to the policy value of π_1 , while the second term provides the estimator's

bias. Considering the first term, we have:

$$\mathbb{E}_{\pi_0} \left[\mathbb{1}_a(\pi_1(s)) \rho(a, s) r(a, s) \right] = \sum_{(a, s)} \mathbb{1}_a(\pi_1(s)) \rho(a, s) r(a, s) \pi_0(a, s)$$

$$= \sum_{(a, s)} \mathbb{1}_a(\pi_1(s)) r(a, s) \pi_1(a, s)$$

$$= \sum_s r(\pi_1(s), s) \pi_1(\pi_1(s), s)$$

$$= \mathbb{E}_{\pi_1} \left[r_{\pi_1} \right],$$

where r_{π_1} denotes $r(\pi_1(s), s)$.

Now, considering the bias term, and bounding δ with the Bregman divergence between ρ and $\hat{\rho}$, we have:

$$\mathbb{E}_{\pi_0} \left[\mathbb{1}_a(\pi_1(s)) \delta(a, s) \right] = \sum_{(a, s)} \mathbb{1}_a(\pi_1(s)) \delta(a, s) r(a, s) \pi_0(a, s)$$

$$\leq \sum_{(a, s)} \mathbb{1}_a(\pi_1(s)) B(\rho, \hat{\rho}) r(a, s) \pi_0(a, s)$$

$$= \sum_s B(\rho, \hat{\rho}) r(\pi_1(s), s) \pi_0(\pi_1(s), s)$$

$$= \mathbb{E}_{\pi_0} \left[B(\rho, \hat{\rho}) r_{\pi_1} \right]$$

We now move on to the continuous action setting. We can express the expectation as:

$$\begin{split} &\mathbb{E}_{\pi_0}\left[\frac{1}{h}K\left(\frac{a-\pi_1(s)}{h}\right)\hat{\rho}(a,s)r(a,s)\right] \\ &= \int \frac{1}{h}K\left(\frac{a-\pi_1(s)}{h}\right)\left(\rho(a,s)+\delta(a,s)\right)r(a,s)\pi_0(a,s)d(a,s) \\ &= \int \frac{1}{h}K\left(\frac{a-\pi_1(s)}{h}\right)\rho(a,s)r(a,s)\pi_0(a,s)d(a,s) \\ &+ \int \frac{1}{h}K\left(\frac{a-\pi_1(s)}{h}\right)\delta(a,s)r(a,s)\pi_0(a,s)d(a,s) \end{split}$$

We can show that the first term is equal to the true counterfactual policy value, while the second term describes the bias induced from estimating the density ratio. Considering the first term, we have:

$$\int \frac{1}{h} K\left(\frac{a - \pi_1(s)}{h}\right) \frac{\pi_1(a, s)}{\pi_0(a, s)} r(a, s) \pi_0(s, a) d(s, a) = \int \frac{1}{h} K\left(\frac{a - \pi_1(s)}{h}\right) r(a, s) \pi_1(a, s) d(s, a)$$

Let $u = \frac{a - \pi_1(s)}{h}$. Thus, $a = \pi_1(s) + hu$ and da = hdu. Then, taking a second-order Taylor expansion of π_1

around $\pi_1(s)$:

$$\begin{split} &\int \frac{1}{h} K\left(\frac{a-\pi_1(s)}{h}\right) \frac{\pi_1(a,s)}{\pi_0(a,s)} r(a,s) \pi_0(s,a) d(s,a) \\ &= \int K\left(u\right) r(\pi_1(s) + hu, s) \pi_1(\pi_1(s) + hu, s) d(s,u) \\ &= \int K\left(u\right) r(\pi_1(s), s) \pi_1(\pi_1(s), s) d(s,u) + \int K\left(u\right) r(\pi_1(s), s) \pi_1'(\pi_1(s), s) (hu) d(s,u) \\ &+ \int K\left(u\right) r(\pi_1(s), s) \pi_1''(\pi_1(s), s) \frac{(hu)^2}{2} d(s,u) + \int K\left(u\right) o(h^2) r(\pi_1(s), s) d(s,u) \\ &= \int K\left(u\right) du \int r(\pi_1(s), s) \pi_1(\pi_1(s), s) ds + \int u K\left(u\right) du \int r(\pi_1(s), s) \pi_1'(\pi_1(s), s) hd(s,u) \\ &+ \int u^2 K\left(u\right) du \int \frac{h^2}{2} r(\pi_1(s), s) \pi_1''(\pi_1(s), s) ds + \int K\left(u\right) du \int o(h^2) r(\pi_1(s), s) ds \\ &= \int r(\pi_1(s), s) \pi_1(\pi_1(s), s) ds + o(h^2) \\ &= E_{\pi_1}[r_{\pi_1}] + o(h^2). \end{split}$$

This result follows similarly to those in Kallus and Zhou (2018), by properties of kernels, bounded rewards, and since $\pi_1(a, s)$ has a bounded second derivative with respect to a.

Now, considering the bias term, we use the same u-substitution and Taylor expansion as before. We also bound δ by the Bregman divergence between ρ and $\hat{\rho}$, yielding:

$$\begin{split} \int \frac{1}{h} K \left(\frac{a - \pi_1(s)}{h} \right) \delta(a, s) r(a, s) \pi_0(a, s) d(a, s) & \leq \int \frac{1}{h} K \left(\frac{a - \pi_1(s)}{h} \right) B(\rho, \hat{\rho}) r(a, s) \pi_0(a, s) d(a, s) \\ & = \int K \left(u \right) B(\rho, \hat{\rho}) r(\pi_1(s) + hu) \pi_0(\pi_1(s) + hu, s) d(u, s) \\ & = \int K \left(u \right) du \int B(\rho, \hat{\rho}) r(\pi_1(s), s) \pi_0(\pi_1(s), s) ds + Rem(h) \\ & = \int B(\rho, \hat{\rho}) r(\pi_1(s), s) \pi_0(\pi_1(s), s) ds + o(h^2) \\ & = E_{\pi_0} [B(\rho, \hat{\rho}) r_{\pi_1}] + o(h^2) \end{split}$$

A.2.3 Proof of Proposition 3

We consider the second moment of a single numerator term, and write the estimator in terms of ρ and δ as above. We first consider the discrete action setting.

$$\begin{split} \mathbb{E}_{\pi_0} \left[\mathbb{1}_a(\pi_1(s))^2 \hat{\rho}(a,s)^2 r(a,s)^2 \right] &= \mathbb{E}_{\pi_0} \left[\mathbb{1}_a(\pi_1(s)) (\rho(a,s) + \delta(a,s))^2 r(a,s)^2 \right] \\ &= \mathbb{E}_{\pi_0} \left[\mathbb{1}_a(\pi_1(s)) (\rho(a,s)^2 + \delta(a,s)^2 + 2\rho(a,s)\delta(a,s)) r(a,s)^2 \right] \\ &= \sum_{(a,s)} \mathbb{1}_a(\pi_1(s)) \rho(a,s)^2 r(a,s)^2 \pi_0(a,s) \\ &+ \sum_{(a,s)} \mathbb{1}_a(\pi_1(s)) \delta(a,s)^2 r(a,s)^2 \pi_0(a,s) \\ &+ \sum_{(a,s)} \mathbb{1}_a(\pi_1(s)) 2\rho(a,s) \delta(a,s) r(a,s)^2 \pi_0(a,s) \\ &\leq \sum_s \rho(\pi_1(s),s) r(\pi_1(s),s)^2 \pi_1(\pi_1(s),s) \\ &+ \sum_s 2B(\rho,\hat{\rho}) \rho(\pi_1(s),s) r(\pi_1(s),s)^2 \pi_0(\pi_1(s),s) \\ &+ \sum_s B(\rho,\hat{\rho})^2 r(\pi_1(s),s)^2 \pi_0(\pi_1(s),s) \\ &= \mathbb{E}_{\pi_1} [\rho(\pi_1(s),s) r_{\pi_1}^2] + \mathbb{E}_{\pi_0} [B(\rho,\hat{\rho})^2 r_{\pi_1}^2] + \mathbb{E}_{\pi_0} [2B(\rho,\hat{\rho}) \rho(\pi_1(s),s) r_{\pi_1}^2] \end{split}$$

Therefore, the variance of the estimator is bounded by:

$$\frac{1}{n} \left(\mathbb{E}_{\pi_1} [\rho(\pi_1(s), s) r_{\pi_1}^2] + \mathbb{E}_{\pi_0} [B(\rho, \hat{\rho})^2 r_{\pi_1}^2] + \mathbb{E}_{\pi_0} [2B(\rho, \hat{\rho}) \rho(\pi_1(s), s) r_{\pi_1}^2] \right)$$

Next, we consider the second moment of a term in the estimator in the continuous action setting:

$$\mathbb{E}_{\pi_0} \left[\left(\frac{1}{h} K \left(\frac{a - \pi_1(s)}{h} \right) \hat{\rho}(a, s) r(a, s) \right)^2 \right]$$

$$= \int \frac{1}{h^2} K \left(\frac{a - \pi_1(s)}{h} \right)^2 (\rho(a, s) + \delta(a, s))^2 r(a, s)^2 \pi_0(a, s) d(a, s)$$

We substitute $u = \frac{a - \pi_1(s)}{h}$ as before. Then, $a = \pi_1(s) + hu$ and da = hdu.

$$\mathbb{E}_{\pi_0} \left[\left(\frac{1}{h} K \left(\frac{a - \pi_1(s)}{h} \right) \hat{\rho}(a, s) r(a, s) \right)^2 \right]$$

$$= \int \frac{1}{h} K (u)^2 \left(\rho(\pi_1(s) + hu, s) + \delta(\pi_1(s) + hu, s) \right)^2 r(\pi_1(s) + hu)^2 \pi_0(\pi_1(s) + hu, s) d(s, u)$$

Next, we apply a second-order Taylor series expansion of ρ , δ , and π_0 around $\pi_1(s)$. Given that these functions have bounded second derivatives, we can bound the remainder by $o(h^{-1})$, as in Kallus and Zhou (2018). This

yields:

$$\begin{split} &\mathbb{E}_{\pi_0}\left[\left(\frac{1}{h}K\left(\frac{a-\pi_1(s)}{h}\right)\hat{\rho}(a,s)r(a,s)\right)^2\right] \\ &= \int \frac{1}{h}K\left(u\right)^2du \int \left(\rho(\pi_1(s),s) + \delta(\pi_1(s),s)\right)^2r(\pi_1(s),s)^2\pi_0(\pi_1(s),s)ds + o(h^{-1}) \\ &= \frac{R(K)}{h}\int \left(\rho(\pi_1(s),s) + \delta(\pi_1(s),s)\right)^2r(\pi_1(s),a)^2\pi_0(\pi_1(s),s)ds + o(h^{-1}) \\ &= \frac{R(K)}{h}\int \left(\rho(\pi_1(s),s)^2 + \delta(\pi_1(s),s)^2 + 2\rho(\pi_1(s),s)\delta(\pi_1(s),s)\right)r(\pi_1(s),a)^2\pi_0(\pi_1(s),s)ds + o(h^{-1}) \\ &= \frac{R(K)}{h}\left[\int \rho(\pi_1(s),s)^2r_{\pi_1}^2\pi_0(\pi_1(s),s)ds + \int \delta(\pi_1(s),s)^2r_{\pi_1}^2\pi_0(\pi_1(s),s)ds + \int 2\rho(\pi_1(s),s)\delta(\pi_1(s),s)r_{\pi_1}^2\pi_0(\pi_1(s),s)ds \right] \\ &+ o(h^{-1}) \\ &= \frac{R(K)}{h}\left[\int \rho(\pi_1(s),s)r_{\pi_1}^2\pi_1(\pi_1(s),s)ds + \int \delta(\pi_1(s),s)^2r_{\pi_1}^2\pi_0(\pi_1(s),s)ds + \int 2\delta(\pi_1(s),s)r_{\pi_1}^2\pi_1(\pi_1(s),s)ds \right] \\ &+ o(h^{-1}) \end{split}$$

where $R(K) := \int K(u)^2 du$ is some constant.

Then, bounding δ by the Bregman divergence B,

$$\mathbb{E}_{\pi_0} \left[\left(\frac{1}{h} K \left(\frac{a - \pi_1(s)}{h} \right) \hat{\rho}(a, s) r \right)^2 \right]$$

$$\leq \frac{R(K)}{h} \left[\mathbb{E}_{\pi_1} \left[\rho(\pi_1(s), s) r_{\pi_1}^2 \right] + \mathbb{E}_{\pi_0} \left[B(\rho, \hat{\rho})^2 r_{\pi_1}^2 \right] + \mathbb{E}_{\pi_1} \left[2B(\rho, \hat{\rho}) r_{\pi_1}^2 \right] \right] + o(h^{-1})$$

Therefore, the variance of our estimator is bounded by:

$$\frac{R(K)}{nh} \left(\mathbb{E}_{\pi_1} [\rho(\pi_1(s), s) r_{\pi_1}^2] + \mathbb{E}_{\pi_0} [B(\rho, \hat{\rho})^2 r_{\pi_1}^2] + \mathbb{E}_{\pi_1} [2B(\rho, \hat{\rho}) r_{\pi_1}^2] \right) + o\left(\frac{1}{nh}\right)$$

A.3 Evaluation details and full results

Table 1: Summary of datasets used in discrete reward experiments

Dataset	ecoli	glass	letters	optdigits	page-blocks	pendigits	satimage	vehicle	yeast
Classes (k)	5	6	26	10	5	10	6	4	9
Observations (n)	327	214	20000	5620	5473	10992	6435	846	1479
Covariates (p)	7	9	16	64	10	16	36	18	8

Table 2: Summary of datasets used in continuous reward experiments

Dataset	abalone	admissions	airfoil	auto	housing	power	wine
Observations (n)	4177	400	1503	392	10000	9568	1599
Covariates (p)	10	7	5	7	14	4	11

Table 3 shows the results of the discrete treatment simulations. Table 4 shows the results of the continuous treatment simulations.

A.4 Data sources

The sources for the datasets used in the experiments, along with necessary citations, can be found below.

Table 3: Discrete evaluation First row for each dataset is absolute bias, second is RMSE

			IPS	<i>O</i> 2			B-OPE	PE	
Dataset	Direct Method	IPS	Doubly Robust	SWITCH	SWITCH-DR	B-OPE	Doubly Robust	SWITCH	SWITCH-DR
ecoli	0.211	0.016	0.068	0.098	0.193	0.030	0.001	0.074	0.152
	0.220 ± 0.006	0.123 ± 0.012	0.189 ± 0.015	0.139 ± 0.010	0.203 ± 0.006	0.078 ± 0.003	0.112 ± 0.009	0.109 ± 0.007	0.166 ± 0.006
glass	0.300	0.033	0.092	0.207	0.290	0.121	0.128	0.167	0.235
	0.312 ± 0.008	0.207 ± 0.012	0.253 ± 0.016	0.267 ± 0.015	0.303 ± 0.009	0.174 ± 0.008	0.199 ± 0.012	0.214 ± 0.012	0.256 ± 0.010
letters	0.451	0.010	0.070	0.435	0.438	0.028	0.076	0.010	0.243
	0.451 ± 0.002	0.029 ± 0.002	0.097 ± 0.006	0.441 ± 0.005	0.439 ± 0.003	0.029 ± 0.001	0.081 ± 0.003	0.017 ± 0.001	0.245 ± 0.003
optdigits	0.217	0.007	0.047	0.023	0.179	0.017	0.051	0.005	0.128
	0.218 ± 0.002	0.031 ± 0.004	0.084 ± 0.006	0.027 ± 0.002	0.181 ± 0.002	0.018 ± 0.000	0.058 ± 0.003	0.010 ± 0.001	0.130 ± 0.002
pageblocks	0.016	0.024	0.034	0.003	0.005	0.010	0.011	0.009	0.003
	0.017 ± 0.001	0.035 ± 0.004	0.051 ± 0.004	0.008 ± 0.000	0.010 ± 0.001	0.015 ± 0.001	0.029 ± 0.005	0.012 ± 0.001	0.007 ± 0.001
$_{ m pendigits}$	0.130	0.001	0.047	0.008	0.081	0.008	0.019	0.000	0.064
	0.130 ± 0.001	0.010 ± 0.001	0.062 ± 0.004	0.010 ± 0.001	0.081 ± 0.001	0.008 ± 0.000	0.025 ± 0.001	0.004 ± 0.000	0.065 ± 0.001
sat	0.052	0.012	0.013	0.004	0.020	0.041	0.003	0.016	0.025
	0.053 ± 0.001	0.023 ± 0.001	0.033 ± 0.002	0.015 ± 0.001	0.024 ± 0.001	0.043 ± 0.001	0.023 ± 0.002	0.020 ± 0.001	0.027 ± 0.001
vehicle	0.056	0.000	0.031	0.017	0.043	0.112	0.041	0.014	0.018
	0.069 ± 0.003	0.119 ± 0.009	0.136 ± 0.011	0.058 ± 0.003	0.056 ± 0.003	0.126 ± 0.005	0.079 ± 0.005	0.047 ± 0.003	0.043 ± 0.003
yeast	0.110	0.037	0.053	0.060	0.104	0.083	0.024	0.039	0.071
	0.120 ± 0.005	0.152 ± 0.009	0.178 ± 0.012	0.092 ± 0.006	0.114 ± 0.005	0.114 ± 0.006	0.095 ± 0.007	0.075 ± 0.006	0.085 ± 0.005

Table 4: Continuous evaluation First row for each dataset is absolute bias, second is RMSE

IPS Doubly Robust SWITCH SWITCH-DR
1.107 0.090
$2.879 \pm$
0.029 0.318 0.294 0.343 0.029 $1.37 + 0.105$ $0.304 + 0.014$ $0.356 + 0.010$
0.723 0.000
0.128 ± 0.022 1.517 ± 0.124 0.045 ± 0.004 0.522 ± 0.016
0.315
$0.336 \pm$
0.411 0.057
$3.668 \pm 0.590 0.172 \pm 0.056$
0.060 ± 0.020 0.138 ± 0.012 0.015 ± 0.003 0.062 ± 0.003
0.253 1.502 0.282
0.825 ± 0.097 3.737 ± 0.524 0.487 ± 0.016 $0.478 \pm$

^a (Siebert, 1987) ^b(Acharya *et al.*, 2019) ^c(Kaya *et al.*, 2012; Tüfekci, 2014) ^d(Cortez *et al.*, 2009)