complex numbers

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$\cos(\theta) = (e^{i\theta} + e^{-i\theta})/2$

$\sin(\theta) = (e^{i\theta} - e^{-i\theta})/(2i)$

$(-1)^n = (e^{i\pi})^n$

$(i)^n = (e^{i\frac{\pi}{2}})^n$

$z^* = e^{-i\theta}$ $zz^* = x^2 + y^2$

$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$ $|z| = \sqrt{x^2 + y^2}$

polar coords

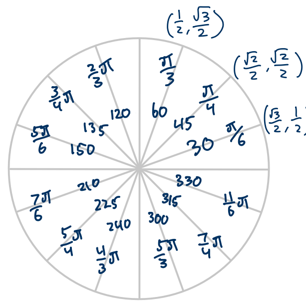
for $z^n = 1$ $\theta = 2k\pi/n,\ k = 0, 1, \dots n-1$

$z^n = r^n e^{in\theta} = 1 e^{i2k\pi}$

$z = re^{i\theta} = x + iy$

$r = |z| = \sqrt{x^2 + y^2}$

$\angle z = \theta = \arg(z) = \tan^{-1}(y/x)$



ex: $\frac{1}{2-2i}$

$2-2i = 2\sqrt{2}\, e^{-i\frac{\pi}{4}}$

$\frac{1}{2\sqrt{2}e^{-i\frac{\pi}{4}}} = \frac{1}{2\sqrt{2}} e^{i\frac{\pi}{4}}$

inner products

$\langle x, y+z \rangle = x^T(y+z)$

$= x^Ty + x^Tz$

$= \langle x, y \rangle + \langle x, z \rangle$

$\langle x+y, x+y \rangle = \langle x, x+y \rangle + \langle y, x+y \rangle$

$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$

$= \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle$

$\langle \alpha v, v \rangle = \alpha \langle v, v \rangle$ $\langle v, v \rangle = \|v\|^2$

vector spaces

10 axioms:

- vector add.
  - associative: $u + (v + w) = (u + v) + w$
  - commutative: $u + v = v + u$
  - additive identity: exists $0 \rightarrow v + 0 = v$
  - additive inverse: exists $-v \rightarrow v + (-v) = 0$
  - closure: $v + u \in \mathbb{V}$
- scalar mult.
  - associative: $\alpha(\beta v) = (\alpha\beta)v$
  - multiplicative identity: exists $1 \rightarrow 1 \cdot v = v$
  - distributive in vector add: $\alpha(u + v) = \alpha v + \alpha v$
  - distributive in scalar add: $(\alpha + \beta)v = \alpha v + \beta v$
  - closure: $\alpha v \in \mathbb{V}$

subspace conditions:

1. contains the 0 vector
2. closed under vector add.
3. closed under scalar mult.

ex: $\mathbb{V} = \{ p(t) = \sum_{i=0}^{n} p_i t^i \mid p(0) = 0,\ p_i \in \mathbb{R},\ i = 0, \dots, n \}$

① set $\mathbb{V}$ contains the 0 vector, $p(0) = 0$

②+③ consider $p(t) = p_0 + p_1 t + \dots + p_n t^n$

$q(t) = q_0 + q_1 t + \dots + q_n t^n$

$\alpha, \beta \in \mathbb{R}$

and $r(t) = \alpha p(t) + \beta q(t)$ for all $t \in \mathbb{R}$

$r(t) = (\alpha p_0 + \beta q_0) + (\alpha p_1 + \beta q_1)t + \dots + (\alpha p_n + \beta q_n)t^n$

Thus, $r(t)$ is a polynomial of deg n. and satisfies $r(t) = 0$, it belongs in $\mathbb{V}$.

Thus, $\mathbb{V}$ contains the 0 vector and is closed under linear combination, $\mathbb{V}$ is a subspace of P_n.

norms

general norms: any real-valued function $f: \mathbb{V} \rightarrow \mathbb{R}$ that satisfies:

① nonnegative homogeneity: $\|\beta v\| = |\beta|\, \|v\|$

② triangle inequality: $\|v + u\| \leq \|v\| + \|u\|$

③ nonnegativity: $\|v\| \geq 0$, with $\|v\| = 0$ iff $v = 0$

Euclidean norm, $\|x\| = \sqrt{x_1^2 + \dots + x_n^2} = \sqrt{x^Tx}$

ℓ^2-norm, $\|x\|_1 = |x_1| + \dots + |x_n|$

ℓ^∞-norm, $\|x\|_\infty = \max(|x_1|, \dots |x_n|)$

norm-squared of block vector, $d = (\vec{a}, \vec{b}, \vec{c})$: $\|d\|^2 = d^Td = a^Ta + b^Tb + c^Tc = \|a\|^2 + \|b\|^2 + \|c\|^2$

$\hookrightarrow$ for norm: $\|(a, b, c)\| = \sqrt{a^Ta + b^Tb + c^Tc} = \sqrt{\|a\|^2 + \|b\|^2 + \|c\|^2} = \|(\|a\|, \|b\|, \|c\|)\|$

$\mathrm{dist}(a, b) = \|a - b\|$

linear (in)dependence

list w/ single vector is lin. dep. iff $v = 0$

any list w/ zero vector is lin. dep.

list w/ 2 vectors is lin. dep. iff $v = \alpha u$

formulas

cauchy-schwarz: $|\langle x, y \rangle| \leq \|x\|\,\|y\|$

triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$

angle between vectors: $\theta = \arccos\left(\frac{\langle x, y \rangle}{\|x\|\,\|y\|}\right)$

random proofs!

pythagorean theorem proof

$\boxed{\|v_1 + v_2 + \dots + v_n\|^2 = \|v_1\|^2 + \|v_2\|^2 + \dots + \|v_n\|^2 \ \Big|\ \langle v_i, v_j \rangle = 0}$

$\|v_1 + v_2 + \dots + v_n\|^2 = \langle v_1 + v_2 + \dots + v_n \rangle$

$= \sum_{i=1}^{n} \langle v_i, v_i \rangle + \sum_{i \neq j}^{n} \langle v_i, v_j \rangle$

$= \sum_{i=1}^{n} \langle v_i, v_i \rangle = \sum_{i=1}^{n} \|v_i\|^2$

$= \|v_1\|^2 + \|v_2\|^2 + \dots + \|v_n\|^2$

linear independence of orthogonal vectors

$\alpha u + \beta v = 0$

$\langle (\alpha u + \beta v), u \rangle = \langle 0, u \rangle$

$\langle \alpha u, u \rangle + \langle \beta v, u \rangle = \langle 0, u \rangle$

$\alpha \langle u, u \rangle + \beta \langle v, u \rangle = 0$

$\alpha \|u\|^2 = 0$

$$\left\{ w_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, w_1 = \begin{bmatrix} 1 \\ e^{i(2\pi/p)} \\ e^{i(2\pi/p)2} \\ \vdots \\ e^{i(2\pi/p)(p-1)} \end{bmatrix}, \dots, w_k = \begin{bmatrix} 1 \\ e^{i(2\pi/p)k} \\ e^{i(2\pi/p)2k} \\ \vdots \\ e^{i(2\pi/p)k(p-1)} \end{bmatrix}, \dots, w_{p-1} = \begin{bmatrix} 1 \\ e^{i(2\pi/p)(p-1)} \\ e^{i(2\pi/p)2(p-1)} \\ \vdots \\ e^{i(2\pi/p)(p-1)^2} \end{bmatrix} \right\}$$

$$(w_k)_n = e^{i\frac{2\pi}{p}nk} = \left(e^{i\frac{2\pi}{p}}\right)^{nk}$$

$$X_k = \frac{1}{p}\sum_{n=0}^{p-1} x(n)e^{-iw_0 nk} \qquad X_k = \frac{1}{\sqrt{p}}\sum_{n=\langle p\rangle} x(n)e^{-iw_0 kn}$$

$$x(n) = \sum_{k=0}^{p-1} X_k e^{i\frac{2\pi}{p}nk} \qquad x(n) = \frac{1}{\sqrt{p}}\sum X_k e^{iw_0 nk}$$

$$\langle x, \psi_k\rangle = X_k \cdot p \qquad X_k = \langle x, w_k\rangle \qquad x = \sum X_k w_k$$

$$e_1 = \frac{v_1}{\|v_1\|}$$

$$e_2 = \frac{v_2 - \langle v_2, e_1\rangle e_1}{\|v_2 - \langle v_2, e_1\rangle e_1\|}$$

$$e_3 = \frac{v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2}{\|\text{numerator}\|}$$

x is real and even $\Rightarrow$ X_k are real & E

$$X_0 = 1/9, \quad \sum X_k \cos\left(k\frac{2\pi}{5}\right) = 2/9$$

$$y(n) = x(n-2), \quad \sum Y_k = 1/9$$

$$X_0 = \frac{1}{9} \rightarrow X_0 = \frac{1}{5}\sum_{n=-2}^{2} x(n)e^{-i\cancel{w_0 0 n}}$$

$$X_0 = \frac{1}{5}\left(x(-2) + x(-1) + x(0) + x(1) + x(2)\right) = \frac{1}{9}$$

$$\sum_{k\in\langle 5\rangle} X_k \cos\left(k\frac{2\pi}{5}\right) = \sum_{k\in\langle 5\rangle} X_k\left(\frac{1}{2}e^{ik\frac{2\pi}{5}} + \frac{1}{2}e^{-ik\frac{2\pi}{5}}\right)$$

$$= \frac{1}{2}\sum_{k\in\langle 5\rangle} X_k e^{ik\frac{2\pi}{5}(1)} + \frac{1}{2}\sum_{k\in\langle 5\rangle} X_k e^{-ik\frac{2\pi}{5}} \qquad w_0 = \frac{2\pi}{p}$$

$$= \frac{1}{2}x(1) + \frac{1}{2}x(-1) = \frac{2}{9} \qquad x(1) = x(-1)$$

$$x(1) = \sum_{k\in\langle 5\rangle} X_k e^{iw_0 k(1)} \qquad x(1) = \frac{2}{9} = x(-1)$$

$$\sum_{k\in\langle 5\rangle} Y_k = \frac{1}{9} \rightarrow y(0) = \frac{1}{9} \rightarrow x(-2) = \frac{1}{9} = x(2)$$

$$y(0) = \sum_{k\in\langle 5\rangle} Y_k e^{\cancel{iw_0(0)k}} = \sum_{k\in\langle 5\rangle} Y_k$$

Kronecker Delta function

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

$$x(n) = \sum_{l=-\infty}^{\infty} \delta(n - 4l)$$

$$X_k = \frac{1}{p}\sum_{n=0}^{p-1} x(n)e^{-ikw_0 n} = \frac{1}{p}$$

$Mx = d$

$M\alpha v = 0$

$M(x + \alpha v) = d$

matrix a is orthonormal

$A^{-1} = A^T$

$$x(n) = \frac{1}{p}\sum_{k=0}^{p-1} e^{i\frac{2\pi}{p}nk} = \frac{1}{p}\left(1 + e^{i\frac{2\pi}{p}n(1)} + e^{i\frac{2\pi}{p}n(2)} + \dots e^{i\frac{2\pi}{p}n(p-1)}\right)$$

Invertible Matrix

A is invertible

A^T is invertible

rank $A = n$

null $A = \{0\}$

cols form a linearly independent set $\rightarrow$ Gram-Schmidt succeeds

rows form a linearly independent set

$\det(A) \neq 0$

$A \in \mathbb{R}^{m \times n}$

$r = \text{rank}(A) = \dim(\text{col}(A))$

	dim
row space is subspace of $\mathbb{R}^n$	r
col space is subspace of $\mathbb{R}^m$	r
null space is subspace of $\mathbb{R}^n$	$n-r$
left null space is subspace of $\mathbb{R}^m$	$m-r$

$\text{null}(A) \perp \text{col}(A^T)$

$\text{null}(A^T) \perp \text{col}(A)$

Let the vector representation of signal x be $x = \begin{bmatrix} x(0) \\ \vdots \\ x(p-1) \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}$

x can also be expressed in the DFT basis with coefficients $X_0, \dots, X_{p-1}$

V = orthonormal basis that spans $\mathbb{C}^n$

vector $y \in \mathbb{R}$

$$y = b_1 v_1 + b_2 v_2 + \dots + b_n v_n = a_1 e_1 + \dots + a_n e_n = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\langle y, y\rangle = \langle b_1 v_1 + \dots + b_n v_n,\ b_1 v_1 + \dots + b_n v_n\rangle$$

$$= b_1 b_1^* \langle v_1, v_1\rangle + \dots + b_n b_n^* \langle v_n, v_n\rangle$$

$$= |b_1|^2 + |b_2|^2 + \dots + |b_n|^2$$

$$= \sum_{k=1}^{n} |b_k|^2 = \sum_{k=1}^{n} a_k^2$$