complex numbers

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$
$$\cos(\theta) = (e^{i\theta} + e^{-i\theta})/2$$

$$z^* = e^{-i\theta} \qquad zz^* = x^2 + y^2 \qquad \sin(\theta) = (e^{i\theta} - e^{-i\theta})/(2i)$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

polar coords

for $z^n = 1$ \qquad $\theta = 2k\pi/n$, $k = 0, 1, \ldots n-1$
$$z^n = r^n e^{in\theta} = 1 e^{i2k\pi}$$

$$z = re^{i\theta} = x + iy$$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\angle z = \theta = \arg(z) = \tan^{-1}(y/x)$$

ex: $\dfrac{1}{2 - 2i}$

$2 - 2i = 2\sqrt{2}\, e^{-i\frac{\pi}{4}}$

$$\frac{1}{2\sqrt{2}\, e^{-i\frac{\pi}{4}}} = \frac{1}{2\sqrt{2}} e^{i\frac{\pi}{4}}$$

$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

(polar circle diagram with angles: $\frac{\pi}{3}$ 60, $\frac{\pi}{4}$ 45, $\frac{\pi}{6}$ 30, $\frac{2\pi}{3}$ 120, $\frac{3\pi}{4}$ 135, $\frac{5\pi}{6}$ 150, $\frac{7\pi}{6}$ 210, $\frac{5\pi}{4}$ 225, $\frac{4\pi}{3}$ 240, $\frac{5\pi}{3}$ 300, $\frac{7\pi}{4}$ 315, $\frac{11\pi}{6}$ 330, 330)

inner products

$$\langle x, y+z \rangle = x^T(y+z)$$
$$= x^T y + x^T z$$
$$= \langle x, y \rangle + \langle x, z \rangle$$

$$\langle x+y, x+y \rangle = \langle x, x+y \rangle + \langle y, x+y \rangle$$
$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$
$$= \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle$$

$$\langle \alpha v, v \rangle = \alpha \langle v, v \rangle \qquad \langle v, v \rangle = \|v\|^2$$
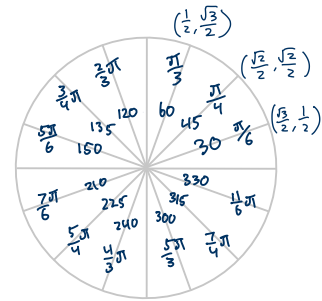
vector spaces

10 axioms:

- vector add.
  - associative: $u + (v+w) = (u+v) + w$
  - commutative: $u + v = v + u$
  - additive identity: exists $0 \to v + 0 = v$
  - additive inverse: exists $-v \to v + (-v) = 0$
  - closure: $v + u \in \mathbb{V}$
- scalar mult.
  - associative: $\alpha(\beta v) = (\alpha\beta)v$
  - multiplicative identity: exists $1 \to 1 \cdot v = v$
  - distributive in vector add: $\alpha(u+v) = \alpha v + \alpha v$
  - distributive in scalar add: $(\alpha + \beta)v = \alpha v + \beta v$
  - closure: $\alpha v \in \mathbb{V}$

subspace conditions:

① contains the 0 vector
② closed under vector add.
③ closed under scalar mult.

ex:
$$\mathbb{V} = \left\{ p(t) = \sum_{i=0}^{n} p_i t^i \,\middle|\, p(0) = 0, \ p_i \in \mathbb{R}, \ i = 0, \ldots, n \right\}$$

① set $\mathbb{V}$ contains the 0 vector, $p(0) = 0$

②+③ consider $p(t) = p_0 + p_1 t + \ldots + p_n t^n$
$$q(t) = q_0 + q_1 t + \ldots + q_n t^n$$
$$\alpha, \beta \in \mathbb{R}$$
and $r(t) = \alpha p(t) + \beta q(t)$ for all $t \in \mathbb{R}$
$$r(t) = (\alpha p_0 + \beta q_0) + (\alpha p_1 + \beta q_1)t + \ldots + (\alpha p_n + \beta q_n)t^n$$

Thus, $r(t)$ is a polynomial of deg n. and satisfies $r(0) = 0$, it belongs in $\mathbb{V}$.

Thus $\mathbb{V}$ contains the 0 vector and is closed under linear combination, $\mathbb{V}$ is a subspace of P_n.

norms

general norms: any real-valued function $f: \mathbb{V} \to \mathbb{R}$ that satisfies:

① nonnegative homogeneity: $\|\beta v\| = |\beta| \|v\|$
② triangle inequality: $\|v + u\| \leq \|v\| + \|u\|$
③ nonnegativity: $\|v\| \geq 0$, with $\|v\| = 0$ iff $v = 0$

Euclidean norm, $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2} = \sqrt{x^T x}$

l^2-norm, $\|x\|_1 = |x_1| + \ldots + |x_n|$

l^∞-norm, $\|x\|_\infty = \max(|x_1|, \ldots |x_n|)$

norm-squared of block vector, $d = (\vec{a}, \vec{b}, \vec{c})$: $\|d\|^2 = d^T d = a^T a + b^T b + c^T c = \|a\|^2 + \|b\|^2 + \|c\|^2$

$\hookrightarrow$ for norm: $\|(a,b,c)\| = \sqrt{a^T a + b^T b + c^T c} = \sqrt{\|a\|^2 + \|b\|^2 + \|c\|^2} = \left\| \left(\|a\|, \|b\|, \|c\| \right) \right\|$

$\text{dist}(a,b) = \|a - b\|$

linear (in)dependence

list w/ single vector is lin. dep. iff $v = 0$

any list w/ zero vector is lin. dep.

list w/ 2 vectors is lin. dep. iff $v = \alpha u$

formulas

cauchy-schwarz: $|\langle x, y \rangle| \leq \|x\| \|y\|$

triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$

angle between vectors: $\theta = \arccos\left(\dfrac{\langle x, y \rangle}{\|x\| \|y\|} \right)$