Corrected proof

$$B(\underline{\omega}) = \frac{\Gamma(v_b+1)\,K_b^{2v_b}}{\Gamma(v_b)\,\pi}\;\frac{\sigma_b^2}{(K^2+\|\underline{\omega}\|^2)^{v_b+1}}$$

$$\Gamma_{11}(\underline{\omega}) = \dots$$

$$\Gamma_{22}(\underline{\omega}) = \dots$$

$$\sigma_{22}^2\,\frac{\Gamma(v_{22}+1)\,K_{22}^{2v_{22}}}{\Gamma(v_{22})\,\cancel{\pi}}\;\frac{1}{(K^2+\|\underline{\omega}\|^2)^{v_{22}+1}} >$$

$$\sigma_b^4\,\frac{\Gamma(v_b+1)^2\,K_b^{4v_b}}{\Gamma(v_b)^2\,\pi^2}\cdot\sigma_{11}^2\,\frac{\Gamma(v_{11}+1)\,K_{11}^{2v_{11}}}{\Gamma(v_{11})\,\cancel{\pi}}\,(K^2+\|\omega\|^2)^{-2v_b-2-v_{11}-1}$$

$$\Rightarrow \frac{\Gamma(v_b)^2}{\sigma_b^4\,\Gamma(v_b+1)^2\,K^{4v_b}} > \frac{\Gamma(v_{11}+1)\,\Gamma(v_{22})}{\pi^2\,\Gamma(v_{22}+1)\,\Gamma(v_{11})}\,\frac{K^{2v_{11}}}{K^{2v_{22}}}\,(K^2+\|\underline{\omega}\|^2)^{-2v_b-2-v_{11}\cancel{-1}+v_{22}\cancel{+1}}\;\frac{\sigma_{11}^2}{\sigma_{22}^2}$$

$$\Rightarrow \sigma_b^4 < \frac{\sigma_{22}^2\,\pi^2}{\sigma_{11}^2}\cdot\frac{\Gamma(v_b)^2}{\Gamma(v_b+1)^2\,K^{4v_b}}\;\frac{\Gamma(v_{22}+1)\,K^{2v_{22}}}{\Gamma(v_{22})\,K^{2v_{11}}}\cdot\frac{\Gamma(v_{11})}{\Gamma(v_{11}+1)}\,(K^2+\|\underline{\omega}\|^2)^{2v_b+2+v_{11}-v_{22}}$$

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Now: $\Gamma(v+1) = v\,\Gamma(v)$

$$\Rightarrow \frac{\Gamma(v+1)}{\Gamma(v)} = v$$

$$\Rightarrow \sigma_b^4 < \frac{\sigma_{22}^2 \pi^2}{\sigma_{11}^2} \frac{1}{\nu_b^2 \kappa^{4\nu_b}} \cdot \frac{\nu_{22} \kappa^{2\nu_{22}}}{\nu_{11} \kappa^{2\nu_{11}}} \left(\kappa^2 + \|\underline{\omega}\|^2\right)^{2\nu_b + 2 + \nu_{11} - \nu_{22}}$$

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$$\therefore \; 2\nu_b + 2 + \nu_{11} - \nu_{22} \geq 0$$

$$\Rightarrow \; \nu_b \geq \frac{\nu_{22} - \nu_{11} - 2}{2}$$

and $$\sigma_b^4 < \frac{\sigma_{22}^2 \pi^2}{\sigma_{11}^2} \frac{1}{\nu_b^2 \kappa^{4\nu_b}} \frac{\nu_{22} \kappa^{2\nu_{22}}}{\nu_{11} \kappa^{2\nu_{11}}}$$

$$< \frac{\sigma_{22}^2 \pi^2}{\sigma_{11}^2} \frac{4}{(\nu_{22} - \nu_{11} - 2)^2 \kappa^{4\nu_b}} \frac{\nu_{22} \kappa^{2\nu_{22}}}{\nu_{11} \kappa^{2\nu_{11}}}$$

$$\sigma_b^2 < \frac{\sigma_{22} \pi}{\sigma_{11}} \frac{2}{(\nu_{22} - \nu_{11} - 2) \kappa^{2\nu_b}} \frac{\kappa^{\nu_{22}}}{\kappa^{\nu_{11}}} \sqrt{\frac{\nu_{22}}{\nu_{11}}}$$

Now: $C_{12}(\underline{\omega}) = B(\underline{\omega}) C_{11}(\underline{\omega})$

$$= \frac{\sigma_b^2 \nu_b \kappa^{2\nu_b}}{\pi} \sigma_{11}^2 \frac{\nu_{11} \kappa^{2\nu_{11}}}{\pi} \cdot \left(\kappa^2 + \|\underline{\omega}\|^2\right)^{-\nu_b - \nu_{11} - 2}$$

$$= \frac{\sigma_b^2 \sigma_{11}^2}{\pi} \left[\frac{\nu_b \nu_{11}}{\pi} \kappa^{2(\nu_b + \nu_{11})} \left(\kappa^2 + \|\underline{\omega}\|^2\right)^{-(\nu_b + \nu_{11} + 1) - 1} \right]$$

$\Rightarrow$ smoothness $(C_{12}) = \nu_b + \nu_{11} + 1$

$$\text{variance}(C_{12}) = \frac{\frac{1}{\pi^2} \nu_b \nu_{11} \kappa^{2(\nu_b + \nu_{11})} \cdot \sigma_b^2 \sigma_{11}^2}{\frac{\nu_b + \nu_{11} + 1}{\pi} \cdot \kappa^{2(\nu_b + \nu_{11} + 1)}}$$

d C: b n C

$$= \frac{1}{\pi} \frac{\nu_b \nu_{11}}{\nu_b + \nu_{11} + 1} \cdot \kappa^{-2} \cdot \sigma_b^2 \sigma_{11}^2$$

Now, let ~~K = 1 and~~ $\nu_b = \dfrac{\nu_{22} - \nu_{11} - 2}{2}$

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$$\text{Then } \operatorname{var}(C_{12}) = \frac{1}{\pi}\,\frac{\nu_b \nu_{11}}{\nu_b + \nu_{11} + 1}\,\sigma_b^2 \sigma_{11}^2\, K^{-2}$$

$$= \frac{1}{2\pi}\,\frac{(\nu_{22} - \nu_{11} - 2)\,\nu_{11}}{\frac{\nu_{22} - \nu_{11} - 2}{2} + \nu_{11} + 1}\,\sigma_b^2 \sigma_{11}^2\, K^{-2}$$

$$= \underset{\sim}{\frac{1}{\pi}}\,\frac{\nu_{11}(\nu_{22} - \nu_{11} - 2)}{\nu_{22} + \nu_{11}}\,\underset{\sim}{\sigma_b^2}\,\underset{\sim}{\sigma_{11}^2}\, K^{-2}$$

$$< \frac{\sigma_{11}^{\cancel{2}}}{\cancel{\pi}} \cdot \frac{\sigma_{22}\cancel{\pi}}{\cancel{\sigma_{11}}} \cdot \frac{2}{\cancel{\nu_{22} - \nu_{11} - 2}} \sqrt{\frac{\nu_{22}}{\nu_{11}}} \cdot \frac{\nu_{11}(\cancel{\nu_{22} - \nu_{11} - 2})}{\nu_{22} + \nu_{11}} \cdot \frac{K^{\cancel{-2}}K^{\cancel{\nu_{11}}}}{K^{\cancel{\nu_{11}+\nu_{22}} \cancel{-\nu_{11}/2}}}$$

$$= \frac{\sigma_{11}\sigma_{22} \cdot 2\sqrt{\nu_{22}\nu_{11}}}{\nu_{22} + \nu_{11}}$$

$$= \frac{\sigma_{11}\sigma_{22}\sqrt{\nu_{22}\nu_{11}}}{\frac{(\nu_{22} + \nu_{11})}{2}}$$

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So we retrieve the parsimonious Mutein model.
But the really cool thing is that $\nu_{12} \geq \dfrac{\nu_1 + \nu_2}{2}$ and not =

What about different length scales? From #

$$\sigma_b^4 < \frac{\sigma_{22}^2}{\sigma_{11}^2} \frac{\gamma^2 \Gamma(\nu_b)^2}{\Gamma(\nu_b+1)^2 K_b^{4\nu_b}} \frac{\Gamma(\nu_{22}+1) K_{22}^{2\nu_{22}}}{\Gamma(\nu_{22}) K_{11}^{2\nu_{11}}} \frac{\Gamma(\nu_{11})}{\Gamma(\nu_{11}+1)}$$

$$\cdot (K_b^2 + \|\underline{\omega}\|^2)^{2\nu_b+2} \cdot (K_{11}^2 + \|\underline{\omega}\|^2)^{\nu_{11}+1} \cdot$$

$$(K_{22}^2 + \|\underline{\omega}\|^2)^{-\nu_{22}-1}$$

$\therefore$ What we need is that

$$(K_b^2 + \|\underline{\omega}\|^2)^{2\nu_b+2} (K_{11}^2 + \|\underline{\omega}\|^2)^{\nu_{11}+1} (K_{22}^2 + \|\underline{\omega}\|^2)^{-\nu_{22}-1}$$

$$\geq 1 \quad \forall \ \underline{\omega}$$

Simplify: $(\theta_1 + x^2)^{\gamma_1} (\theta_2 + x^2)^{\gamma_2} \geq 1 \quad \forall \ x$

$$\Rightarrow (K_b^2 + \|\underline{\omega}\|^2)^{2\nu_b+2} \geq \frac{(K_{22}^2 + \|\omega\|^2)^{\nu_{22}+1}}{(K_{11}^2 + \|\omega\|^2)^{\nu_{11}+1}} \quad \forall \ \underline{\omega}$$

x_1