Causal Spatial and Spatio-Temporal Models

“An application to flux estimation"

Andrew Zammit-Mangion
and
Noel Cressie

National Institute for Applied Statistics Research Australia
University of Wollongong

Outline

Section 1

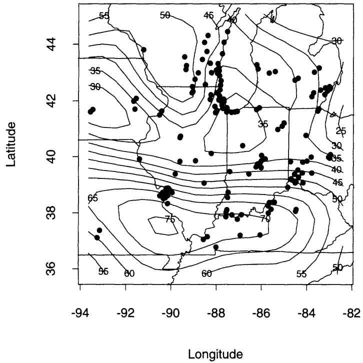
Introduction

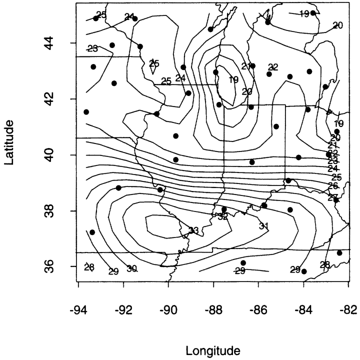
Introduction

- **Univariate** spatial model.

- **Multivariate** spatial model.
  - Two or more interacting spatial variables.
  - Improve prediction on one of the variates by observing the others: **Cokriging**.
  - Determine which variate caused the observed phenomenon: **Source separation**.

Royle and Berliner (1999), Midwestern USA.





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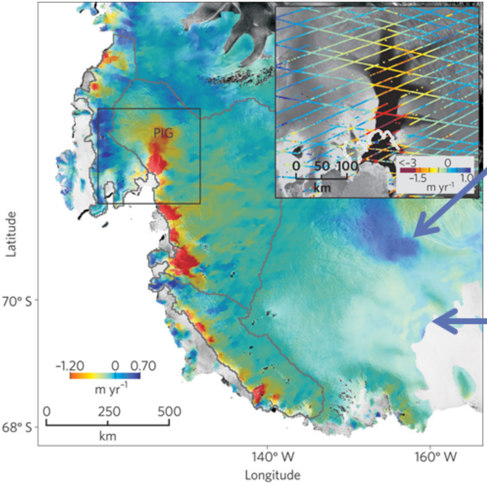
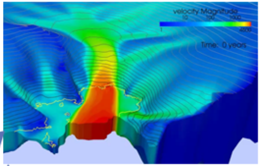
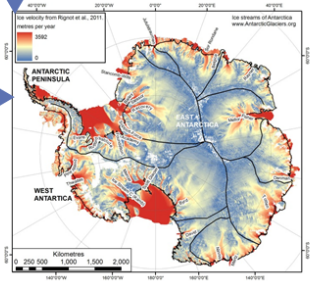
ICESAT data (elevation change per year)

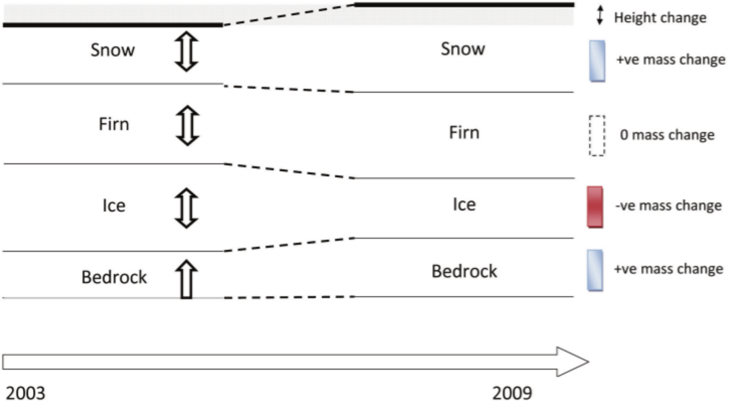


Elmer/Ice (LGGE) model





- Zammit-Mangion et al. (2014, 2015b,a), Antarctica.



The challenge

- **Modelling**: Given a bivariate process $(Y_1(\cdot), Y_2(\cdot))$, what is a valid *cross-covariance function matrix* (CCFM)

$$\begin{pmatrix} C_{11}(\cdot,\cdot) & C_{12}(\cdot,\cdot) \\ C_{21}(\cdot,\cdot) & C_{22}(\cdot,\cdot) \end{pmatrix}, \tag{1}$$

  such that **any** covariance matrix derived from it is positive-definite?

- **Computational**: Sometimes we struggle with univariate models – how do our algorithms scale to multivariate models?

- **Linear model of co-regionalisation** (LMC, Wackernagel, 1995): Define

$$Y_1(\cdot) = a_{11}\widetilde{Y}_1(\cdot) + a_{12}\widetilde{Y}_2(\cdot), \tag{2}$$
$$Y_2(\cdot) = a_{21}\widetilde{Y}_1(\cdot) + a_{22}\widetilde{Y}_2(\cdot), \tag{3}$$

where, independently,

$$\widetilde{Y}_1(\cdot) \sim \mathcal{N}(\mu_1(\cdot), C_1(\cdot,\cdot)), \tag{4}$$
$$\widetilde{Y}_2(\cdot) \sim \mathcal{N}(\mu_2(\cdot), C_2(\cdot,\cdot)). \tag{5}$$

- $C_{ij}(\cdot,\cdot) = a_{i1}a_{j1}C_1(\cdot,\cdot) + a_{i2}a_{j2}C_2(\cdot,\cdot)$.
- CCFM is positive-definite for any $\{a_{ij} : i,j = 1,\ldots,2\}$.

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- **Bivariate parsimonious Matérn model** (Gneiting et al., 2010): Let $C^o(\cdot)$ be a stationary, isotropic covariance function. Define

$$C^o_{ij}(\cdot) \equiv \beta_{ij} M(\cdot; \nu_{ij}, \kappa_{ij}), \qquad (6)$$

  where $M(\cdot)$ is a Matérn covariance function. Let $\kappa_{ii} = \kappa_{ij} = \kappa$ and set $\nu_{ij} = (\nu_{ii} + \nu_{jj})/2$. Then if $(\beta_{ij} : i, j = 1, 2)$ is positive-definite, the CCFM is positive-definite.

- **Bivariate full Matérn model**: Relaxes assumptions on smoothness and scales, but finding valid parameters is much more involved.

Current approaches (limitations)

- Stuck with homogeneous models (e.g., convolution methods).
- Stuck with fixed scales (parsimonious Matérn).
- Stuck with Matérn models (e.g., full Matérn models).
- **Stuck with symmetry (e.g., LMC)**.

Asymmetry

- $Y_1(\cdot)$: precipitation at present.
- $Y_2(\cdot)$: precipitation in 5 minutes time.

Section 2

Causal spatial models

Specification:

$$E\left(Y_2(\mathbf{s}) \mid Y_1(\cdot)\right) = \int_D b(\mathbf{s},\mathbf{v})\, Y_1(\mathbf{v})\, \mathrm{d}\mathbf{v}; \quad \mathbf{s} \in D, \tag{7}$$

$$\operatorname{cov}\left(Y_2(\mathbf{s}), Y_2(\mathbf{u}) \mid Y_1(\cdot)\right) = C_{2|1}(\mathbf{s},\mathbf{u}); \quad \mathbf{s},\mathbf{u} \in \mathbb{R}^d. \tag{8}$$

Building blocks:

- $C_{11}(\cdot,\cdot)$,
- $C_{2|1}(\cdot,\cdot)$,
- $b(\cdot,\cdot)$ (interaction function).

Properties of causal spatial models

- CCFM is easy to find:

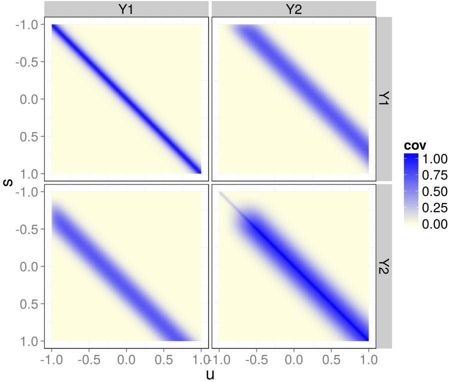
$$\begin{bmatrix} C_{11}(\mathbf{s},\mathbf{u}) & \int_D C_{11}(\mathbf{s},\mathbf{v})b(\mathbf{u},\mathbf{v})\mathrm{d}\mathbf{v} \\ \int_D b(\mathbf{s},\mathbf{v})C_{11}(\mathbf{v},\mathbf{u})\mathrm{d}\mathbf{v} & C_{22}(\mathbf{s},\mathbf{u}) \end{bmatrix}; \tag{9}$$

$$C_{22}(\mathbf{s},\mathbf{u}) = C_{2|1}(\mathbf{s},\mathbf{u}) + \int_D \int_D b(\mathbf{s},\mathbf{v})C_{11}(\mathbf{v},\mathbf{w})b(\mathbf{w},\mathbf{u})\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}, \tag{10}$$

and is always valid (we will outline the proof soon).

- Asymmetry (i.e., $C_{12}(\mathbf{s},\mathbf{u}) \neq C_{21}(\mathbf{s},\mathbf{u})$) is guaranteed if $b(\cdot,\cdot)$ is not symmetric.

- Assume $b^o(\cdot) = b(\cdot, \cdot)$ and that it is off-centre.
- $\mathbf{s}, \mathbf{u} \in \{-1, -0.9, \ldots, 1\}$.



- Heterogeneity, since $C_{11}(\cdot,\cdot)$, $C_{2|1}(\cdot,\cdot)$ need not be homogeneous and $b(\mathbf{s},\mathbf{u})$ need not be symmetric.

- We are not restricted to Matérn fields. The bivariate parsimonious Matérn field is a **special case**.

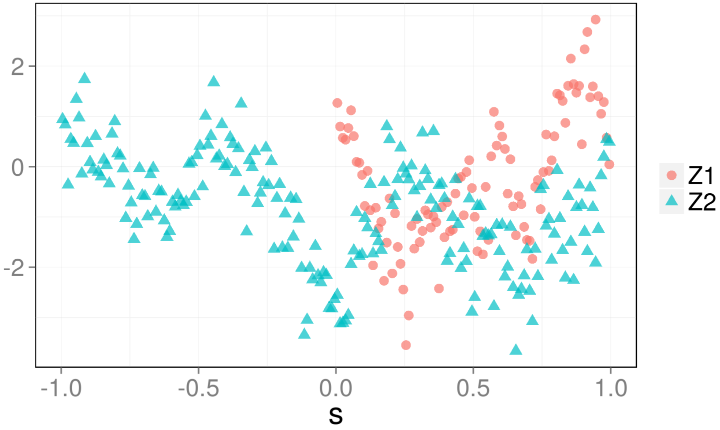
- $Y_2(\cdot)$ can be arbitrarily smoother than $Y_1(\cdot)$ **and** have a different scale.

Example

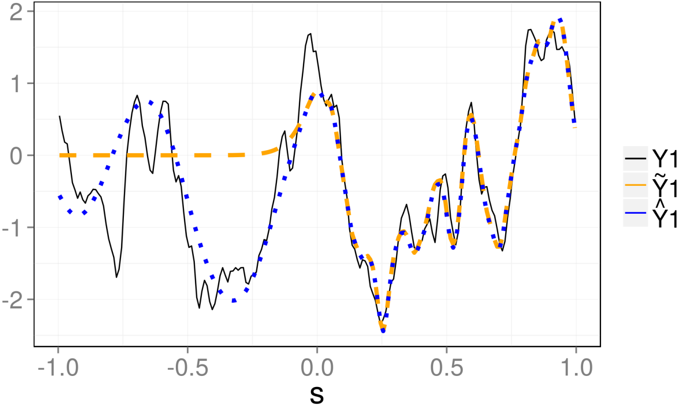
- Assume all parameters are known and $Y_1(\cdot)$ is only partially observed.
- Use simple cokriging **or** simple kriging to estimate $Y_1(\cdot)$:

$$\begin{aligned} \hat{Y}_1(\mathbf{s}_0) &\equiv E(Y_1(\mathbf{s}_0) \mid \mathbf{Z}_1, \mathbf{Z}_2) \quad \text{simple cokriging predictor,} \\ \widetilde{Y}_1(\mathbf{s}_0) &\equiv E(Y_1(\mathbf{s}_0) \mid \mathbf{Z}_1) \qquad\quad \text{simple kriging predictor.} \end{aligned}$$

Example



Example



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- If $C_{11}(\mathbf{s},\mathbf{u})$ and $C_{2|1}(\mathbf{s},\mathbf{u})$ are positive-definite, then $C_{22}(\cdot,\cdot)$ is positive-definite (recall quadratic form).
- $C_{12}(\mathbf{s},\mathbf{u}) = C_{21}(\mathbf{u},\mathbf{s})$.
- CCFM is positive-definite if, for any n_1, n_2 such that $n_1 + n_2 > 0$, any locations $\{\mathbf{s}_{1k}\}, \{\mathbf{s}_{2l}\}$ and any real numbers $\{a_{1k}\}, \{a_{2l}\}$,

$$\begin{aligned}
&\mathrm{var}\left(\sum_{k=1}^{n_1} a_{1k} Y_1^0(\mathbf{s}_{1k}) + \sum_{l=1}^{n_2} a_{2l} Y_2^0(\mathbf{s}_{2l})\right) \\
&= \sum_{k=1}^{n_1}\sum_{k'=1}^{n_1} a_{1k}a_{1k'} C_{11}^0(\mathbf{s}_{1k},\mathbf{s}_{1k'}) + \sum_{l=1}^{n_2}\sum_{l'=1}^{n_2} a_{2l}a_{2l'} C_{22}^0(\mathbf{s}_{2l},\mathbf{s}_{2l'}) \\
&+ \sum_{k=1}^{n_1}\sum_{l'=1}^{n_2} a_{1k}a_{2l'} C_{12}^0(\mathbf{s}_{1k},\mathbf{s}_{2l'}) + \sum_{l=1}^{n_2}\sum_{k'=1}^{n_1} a_{2l}a_{1k'} C_{21}^0(\mathbf{s}_{2l},\mathbf{s}_{1k'}) \;\geq 0.
\end{aligned}$$

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- It can be shown that

$$\mathrm{var}\left(\sum_{k=1}^{n_1} a_{1k} Y_1^0(\mathbf{s}_{1k}) + \sum_{l=1}^{n_2} a_{2l} Y_2^0(\mathbf{s}_{2l})\right)$$
$$= \sum_{l=1}^{n_2}\sum_{l'=1}^{n_2} a_{2l}a_{2l'} C_{2|1}(\mathbf{s}_{2l}, \mathbf{s}_{2l'}) + \int_D \int_D a(\mathbf{s})a(\mathbf{u})C_{11}(\mathbf{s}, \mathbf{u})\,\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{u},$$

where

$$a(\mathbf{s}) \equiv \sum_{k=1}^{n_1} a_{1k}\delta(\mathbf{s} - \mathbf{s}_{1k}) + \sum_{l=1}^{n_2} a_{2l} b(\mathbf{s}_{2l}, \mathbf{s}); \quad \mathbf{s} \in \mathbb{R}^d.$$

- $[Y_1(\cdot), \ldots, Y_p(\cdot)]$ can be decomposed as,

$$[Y_p(\cdot) \mid Y_{p-1}(\cdot), Y_{p-2}(\cdot), \ldots, Y_1(\cdot)] \ldots [Y_1(\cdot)].$$

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- The conditional expectation is

$$E(Y_q(\mathbf{s}) \mid \{Y_r(\cdot) : r = 1, \ldots, (q-1)\}) \equiv \sum_{r=1}^{q-1} \int_D b_{qr}(\mathbf{s}, \mathbf{v}) Y_r(\mathbf{v}) \mathrm{d}\mathbf{v};$$

$$\mathbf{s} \in D.$$

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- The conditional covariance is

$$\mathrm{cov}(Y_q(\mathbf{s}), Y_q(\mathbf{u}) \mid \{Y_r(\cdot) : r = 1, \ldots, (q-1)\}) \equiv C_{q|(r<q)}(\mathbf{s}, \mathbf{u}); \quad \mathbf{s}, \mathbf{u} \in \mathbb{R}^d,$$

where $\{b_{qr}(\cdot, \cdot) : r = 1, \ldots, (q-1);\ q = 2, \ldots, p\}$ are integrable.

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We need to show that the p-variate process is well defined. The proof is by induction:

- We know that the bivariate process is well defined.
- Assume that the $(p-1)$-variate process is well defined.
- Show that the p-variate process is well defined.

$$\mathrm{var}\left(\sum_{q=1}^{p}\sum_{m=1}^{n_q} a_{qm} Y_q(\mathbf{s}_{qm})\right) = \sum_{m=1}^{n_p}\sum_{m'=1}^{n_p} a_{pm}a_{pm'} C_{p|(q<p)}(\mathbf{s}_{pm}, \mathbf{s}_{pm'}) + \sum_{q=1}^{p-1}\sum_{r=1}^{p-1}\int_D\int_D a_q(\mathbf{s})a_r(\mathbf{u})C_{qr}(\mathbf{s},\mathbf{u})\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{u},$$

where

$$a_q(\mathbf{s}) \equiv \left(\sum_{k=1}^{n_q} a_{qk}\delta(\mathbf{s}-\mathbf{s}_{qk}) + \sum_{m=1}^{n_p} a_{pm} b_{pq}(\mathbf{s}_{pm}, \mathbf{s})\right).$$

Generalisations

The following can all be shown to be special cases of causal spatial models:

- The parsimonious Matérn model of Gneiting et al. (2010),
- The full Matérn model of Gneiting et al. (2010),
- The linear model of coregionalisation, used for example by Wackernagel (1995),
- The moving average model of Ver Hoef and Barry (1998).

Graphical structure

- No restriction on graphical structure. Starting with a well defined joint distribution, the structure could be undirected, directed, or a chain graph (Lauritzen, 1996).
- Computationally-efficient algorithms available for some structures.

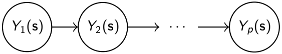
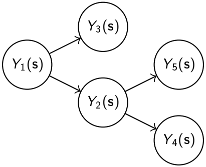
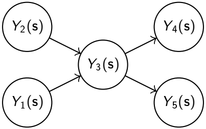


Figure : Ordered nodes





Model flexibility

$C_{11}(\cdot,\cdot)$	
$C_{21}(\cdot,\cdot)$	$C_{22}(\cdot,\cdot)$

Figure : Bivariate system: Need to specify three marginal/cross-covariance functions.

Available building blocks: Three functions, $C_{11}(\cdot,\cdot)$, $C_{2|1}(\cdot,\cdot)$, $b(\cdot,\cdot)$.

Model flexibility

$C_{11}(\cdot,\cdot)$		
$C_{21}(\cdot,\cdot)$	$C_{22}(\cdot,\cdot)$	
$C_{31}(\cdot,\cdot)$	$C_{32}(\cdot,\cdot)$	$C_{33}(\cdot,\cdot)$

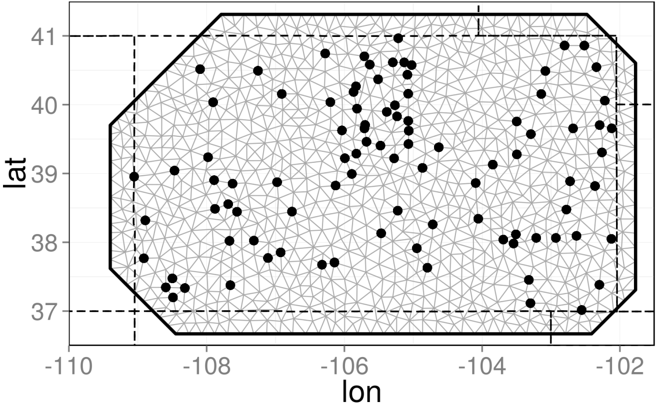
Figure : Trivariate system: Need to specify six marginal/cross-covariance functions.

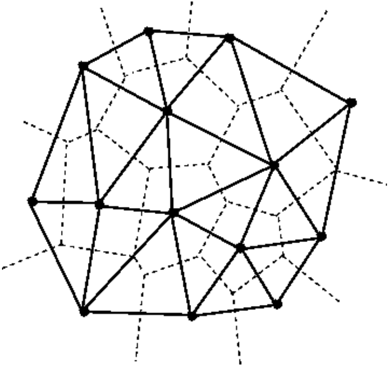
Available building blocks: Six functions, $C_{11}(\cdot,\cdot)$, $C_{2|1}(\cdot,\cdot)$, $C_{3|1,2}(\cdot,\cdot)$, $b_{21}(\cdot,\cdot)$, $b_{31}(\cdot,\cdot)$, $b_{32}(\cdot,\cdot)$.

- Minimum and maximum temperatures taken on September 19, 2004 in the state of Colorado, USA.
- 94 measurement stations (collocated measurements); residuals are obtained by subtraction of statewide mean.
- Maximum-temperature residual later in the afternoon ($Y_2(\cdot)$) highly dependent on minimum-temperature residual in the early morning hours ($Y_1(\cdot)$).
- Fit three models and compare using DIC:

$$\begin{array}{ll}
\text{Model 1:} & b_o(\mathbf{h}) \equiv 0, \\
\text{Model 2:} & b_o(\mathbf{h}) \equiv A\delta(\mathbf{h}), \\
\text{Model 3:} & b_o(\mathbf{h}) \equiv \begin{cases} A\{1 - (\|\mathbf{h} - \boldsymbol{\Delta}\|/r)^2\}^2, & \|\mathbf{h} - \boldsymbol{\Delta}\| \leq r \\ 0, & \text{otherwise.} \end{cases}
\end{array}$$

- Consider a discretisation of $Y_1(\cdot)$ and $Y_2(\cdot)$, $\mathbf{Y}_1$ and $\mathbf{Y}_2$ respectively, and let $\mathbf{Y} \equiv (\mathbf{Y}_1, \mathbf{Y}_2)'$, $\mathbf{Z} \equiv (\mathbf{Z}_1, \mathbf{Z}_2)'$.





$$E\left(Y_2(\mathbf{s}) \mid Y_1(\cdot)\right) = \int_D b(\mathbf{s}, \mathbf{v}) Y_1(\mathbf{v})\, \mathrm{d}\mathbf{v}; \quad \mathbf{s} \in D.$$

$$E(Y_2(\mathbf{s}_l) \mid Y_1(\cdot)) \simeq \sum_{k=1}^{n} A_k b(\mathbf{s}_l, \mathbf{v}_k) Y_1(\mathbf{v}_k),$$

where $\{A_k\}$ are the tessellation areas.

Full model

- Observation model:

$$\begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{pmatrix} \Bigg| \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}, \boldsymbol{\theta} \sim \mathcal{N}\left(\begin{pmatrix} \mathbf{D}\mathbf{Y}_1 \\ \mathbf{D}\mathbf{Y}_2 \end{pmatrix}, \sigma_\varepsilon^2 \begin{pmatrix} \mathbf{I} & \rho_\varepsilon \mathbf{I} \\ \rho_\varepsilon \mathbf{I} & \mathbf{I} \end{pmatrix} \right),$$

  where $\mathbf{D}$ is an incidence matrix and $\boldsymbol{\theta}$ includes σ_ε^2 and ρ_ε.

- Process model:

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \Bigg| \boldsymbol{\theta} \sim \mathcal{N}\left(\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{11}\mathbf{B}' \\ \mathbf{B}\boldsymbol{\Sigma}_{11} & \mathbf{B}\boldsymbol{\Sigma}_{11}\mathbf{B}' + \boldsymbol{\Sigma}_{2|1} \end{pmatrix} \right),$$

  where $\boldsymbol{\Sigma}_{11}, \boldsymbol{\Sigma}_{2|1}$ and $\mathbf{B}$ depend on parameters in $\boldsymbol{\theta}$.

We have many unknown parameters!

- Assume that $C_{11}(\cdot)$ and $C_{2|1}(\cdot)$ are Matérn covariance functions with smoothness parameter $\nu = 3/2$.

Parameter	Model 1	Model 2	Model 3
σ^2_ε	x	x	x
ρ_ε	x	x	x
σ^2_{11}	x	x	x
$\sigma^2_{2\|1}$	x	x	x
κ_{11}	0.98 [0.76, 1.22]	1 [0.8, 1.26]	1.03 [0.83, 1.25]
$\kappa_{2\|1}$	0.76 [0.56, 1]	0.62 [0.46, 0.81]	3.65 [1.16, 6.72]
A		x	x
r			x
Δ_1			x
Δ_2			x
DIC	992.45	985.17	982.45

Median fields

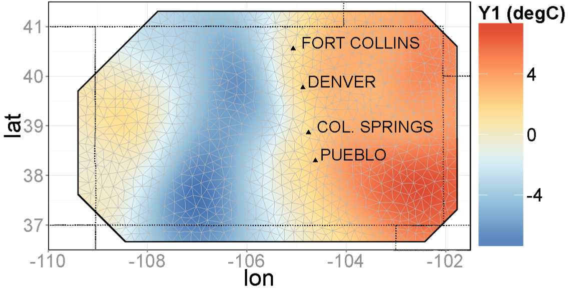


Figure : Interpolated map in degrees Celsius (degC) of $E(\mathbf{Y}_1 \mid \mathbf{Z}_1, \mathbf{Z}_2)$.

Median fields

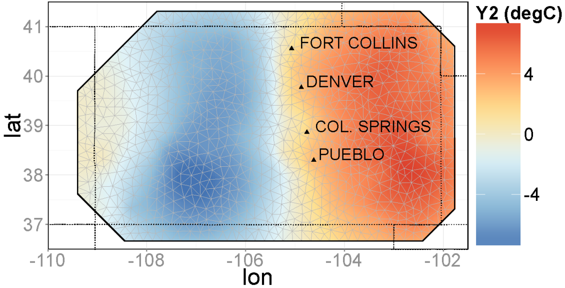


Figure : Interpolated map in degrees Celsius (degC) of $E(\mathbf{Y}_2 \mid \mathbf{Z}_1, \mathbf{Z}_2)$.

Interaction function

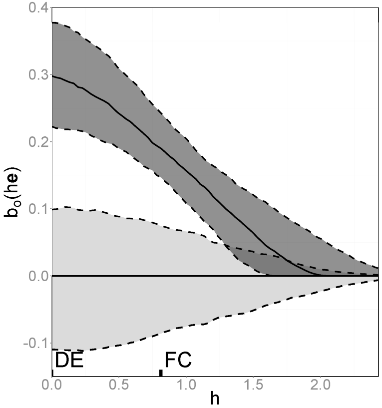


Figure : Prior (light grey) and posterior (dark grey) median (solid line) and inter-quartile ranges (enclosed by dashed lines) of the interaction function $b_o(\cdot)$ of Model 3, along a unit vector **e** originating at Denver (DE) in the direction of Fort Collins (FC)

Section 3

Conclusions

Conclusions

- Bivariate and multivariate models often appear in environmental studies. Usually, one or more of these are 'explained away' prior to commencing the analysis.
- Causal models allow for a (very) flexible model class through interaction functions that can be arbitrarily complex.
- Computation is key: For large, non-Gaussian systems, approximate message passing + variational techniques are probably needed (Cseke et al., 2014).
- Slides and reproducible code available at `https://github.com/andrewzm/bicon`.
- Thanks to Anita Ganesan and Matthew Rigby (University of Bristol) for help with the case study of CH_4 fluxes.

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