

$$x_1 = r_1 \cos(\theta_1) \tag{1}$$

$$y_1 = r_1 \sin(\theta_1) \tag{2}$$

$$x_2 = l_1 \cos(\theta_1) + r_2 \cos(\theta_1 + \theta_2) \tag{3}$$

$$y_2 = l_1 \sin(\theta_1) + r_2 \sin(\theta_1 + \theta_2) \tag{4}$$

$$\dot{x}_1 = -r_1 \sin(\theta_1) \dot{\theta}_1 \tag{5}$$

$$\dot{y}_1 = r_1 \cos(\theta_1) \dot{\theta}_1 \tag{6}$$

$$\dot{x}_2 = -l_1 \sin(\theta_1) \dot{\theta}_1 - r_2 \left( \dot{\theta}_1 + \dot{\theta}_2 \right) \sin(\theta_1 + \theta_2) \tag{7}$$

$$\dot{y}_2 = l_1 \cos(\theta_1) \dot{\theta}_1 + r_2 \left( \dot{\theta}_1 + \dot{\theta}_2 \right) \cos(\theta_1 + \theta_2) \tag{8}$$

$$\tag{9}$$

$$K = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} m_1 v_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} m_2 v_2^2 \tag{10}$$

$$= \frac{I_1}{2} \dot{\theta}_1^2 + \frac{I_2}{2} \dot{\theta}_2^2 + \frac{m_1 v_1^2}{2} \dot{\theta}_1^2 + \frac{m_2}{2} \left( \dot{\theta}_1^2 \dot{\theta}_1^2 + 2 l_1 r_2 \cos(\theta_2) \dot{\theta}_1 \dot{\theta}_2 + r_2^2 \dot{\theta}_1^2 + 2 r_2^2 \dot{\theta}_1 \dot{\theta}_2 + r_2^2 \dot{\theta}_2^2 \right) \tag{11}$$

$$\tag{12}$$

$$U = m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \tag{13}$$

$$= g \left( l_1 m_1 \sin(\theta_1) + m_2 \left( l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \right) \right) \tag{14}$$

$$\tag{15}$$

$$L = K - U \tag{16}$$

$$= \frac{I_1}{2} \dot{\theta}_1^2 + \frac{I_2}{2} \dot{\theta}_2^2 - g l_1 m_1 \sin(\theta_1) - g m_2 \left( l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \right) + \frac{m_1 v_1^2}{2} \dot{\theta}_1^2 + \frac{m_2}{2} \left( \dot{\theta}_1^2 \dot{\theta}_1^2 + 2 l_1 r_2 \cos(\theta_2) \dot{\theta}_1 \dot{\theta}_2 + r_2^2 \dot{\theta}_1^2 + 2 r_2^2 \dot{\theta}_1 \dot{\theta}_2 + r_2^2 \dot{\theta}_2^2 \right) \tag{17}$$

$$\tag{18}$$

$$\frac{\partial L}{\partial \theta_1} = l_1 \dot{\theta}_1 + m_1 r_1^2 \dot{\theta}_1 + m_2 \left( \dot{\theta}_1 \dot{\theta}_1 + 2 l_1 r_2 \cos(\theta_2) \dot{\theta}_1 \dot{\theta}_2 + l_1 r_2 \cos(\theta_2) \dot{\theta}_2 + r_2^2 \dot{\theta}_1 + r_2^2 \dot{\theta}_2 \right) \tag{19}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_1} \right) = l_1 \ddot{\theta}_1 + m_1 r_1^2 \ddot{\theta}_1 + m_2 \left( \dot{\theta}_1 \ddot{\theta}_1 + 2 l_1 r_2 \sin(\theta_2) \dot{\theta}_1 \ddot{\theta}_2 - l_1 r_2 \sin(\theta_2) \dot{\theta}_2^2 + 2 l_1 r_2 \cos(\theta_2) \dot{\theta}_1 \ddot{\theta}_2 + l_1 r_2 \cos(\theta_2) \dot{\theta}_2 + r_2^2 \ddot{\theta}_1 + r_2^2 \ddot{\theta}_2 \right) \tag{20}$$

$$\frac{\partial L}{\partial \theta_1} = -g \left( l_1 m_1 \cos(\theta_1) + l_1 m_2 \cos(\theta_1) + l_2 m_2 \cos(\theta_1 + \theta_2) \right) \tag{21}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = l_1 \ddot{\theta}_1 + g \left( l_1 m_1 \cos(\theta_1) + l_1 m_2 \cos(\theta_1) + l_2 m_2 \cos(\theta_1 + \theta_2) \right) + m_1 r_1^2 \ddot{\theta}_1 + m_2 \left( \dot{\theta}_1 \ddot{\theta}_1 + 2 l_1 r_2 \sin(\theta_2) \dot{\theta}_1 \ddot{\theta}_2 - l_1 r_2 \sin(\theta_2) \dot{\theta}_2^2 + 2 l_1 r_2 \cos(\theta_2) \dot{\theta}_1 \ddot{\theta}_2 + l_1 r_2 \cos(\theta_2) \dot{\theta}_2 + r_2^2 \ddot{\theta}_1 + r_2^2 \ddot{\theta}_2 \right) \tag{22}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = l_2 \ddot{\theta}_2 + m_2 r_2 \left( l_1 \cos(\theta_2) \dot{\theta}_1 + r_2 \dot{\theta}_1 + r_2 \dot{\theta}_2 \right) \tag{23}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) = l_2 \ddot{\theta}_2 + m_2 r_2 \left( -l_1 \sin(\theta_2) \dot{\theta}_1 \ddot{\theta}_2 + l_1 \cos(\theta_2) \dot{\theta}_1 + r_2 \ddot{\theta}_1 + r_2 \ddot{\theta}_2 \right) \tag{24}$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 \left( g l_2 \cos(\theta_1 + \theta_2) + l_1 r_2 \left( \dot{\theta}_1 + \dot{\theta}_2 \right) \sin(\theta_2) \dot{\theta}_1 \right) \tag{25}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = l_2 \ddot{\theta}_2 + g l_2 m_2 \cos(\theta_1 + \theta_2) + l_1 m_2 r_2 \sin(\theta_2) \dot{\theta}_1^2 + l_1 m_2 r_2 \cos(\theta_2) \dot{\theta}_1 + m_2 r_2^2 \ddot{\theta}_1 + m_2 r_2^2 \ddot{\theta}_2 \tag{26}$$

$$\tag{27}$$

$$= l_1 \ddot{\theta}_1 - b_1 \dot{\theta}_1 - f_1 + g l_1 m_1 \cos(\theta_1) + g l_1 m_2 \cos(\theta_1) + g l_2 m_2 \cos(\theta_1 + \theta_2) + \dot{\theta}_1^2 m_2 \ddot{\theta}_1 - 2 l_1 m_2 r_2 \sin(\theta_2) \dot{\theta}_1 \ddot{\theta}_2 - l_1 m_2 r_2 \sin(\theta_2) \dot{\theta}_2^2 + 2 l_1 m_2 r_2 \cos(\theta_2) \dot{\theta}_1 \ddot{\theta}_2 + l_1 m_2 r_2 \cos(\theta_2) \dot{\theta}_2 + m_1 r_1^2 \ddot{\theta}_1 + m_2 r_2^2 \ddot{\theta}_1 + m_2 r_2^2 \ddot{\theta}_2 \tag{28}$$

$$\tag{29}$$