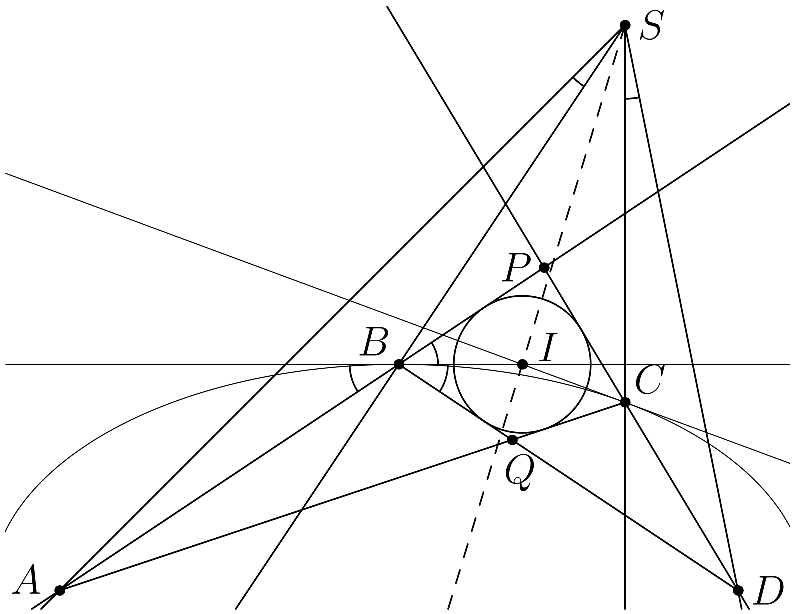
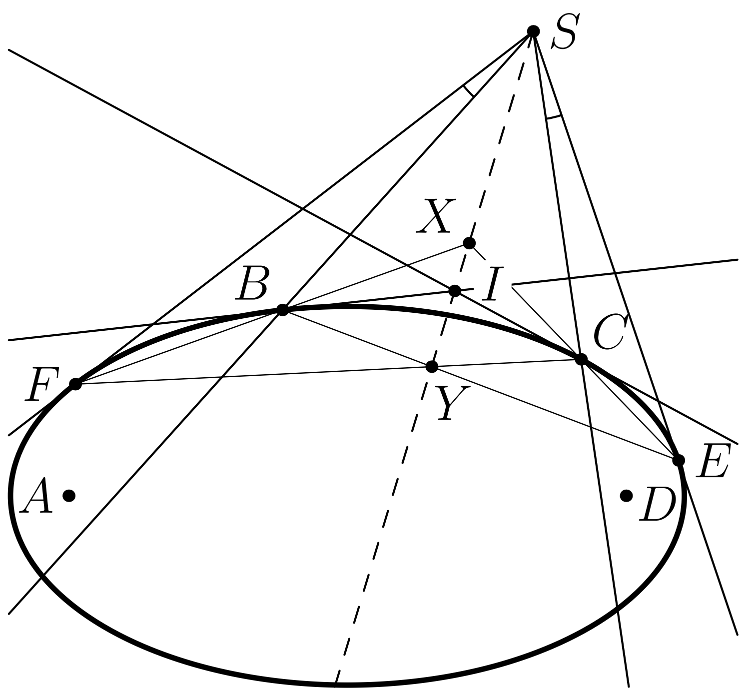
Problem. Consider a points B and C, lying inside the given angle ASD, such that $\angle BSA = \angle CSD$. Lines BD and AC meet at Q and lines AB and CD meet at P. It turns out, that quadrilateral $BPCQ$ has inscribed circle ω. Prove that center I of ω lies on the angle bisector of angle $\angle ASD$. (Author – Nikita Kolesnikov)

Solution.



Let me remind you, that there exists an ellipse $\boldsymbol{\mathcal{P}}$ with focuses A and D, passing through points B and C. It's easy to see, that tangents to ellipse $\boldsymbol{\mathcal{P}}$ at points B and C are the angle bisectors of angles $\angle PBQ$ and $\angle QCP$ respectively. (it can be obtainted by applying optical property of ellipse to these tangents)

Let ℓ be the angle bisector of angle $\angle ASD$. So it suffices to prove, that tangents to $\boldsymbol{\mathcal{P}}$ at B and at C intersect on ℓ.



Let E and F be the points on $\boldsymbol{\mathcal{P}}$ such that SE and SF are tangents to the ellipse $\boldsymbol{\mathcal{P}}$. Since $\angle FSA = \angle ESD$ (isogonal property of ellipse) and $\angle ASB = \angle DSC$ it follows that $\angle FSB = \angle ESC$. Let X be the intersection point of lines FB and CE. Let Y be the intersection point of lines BE and CF. Applying Isogonal's theorem (to the points F, B, C, E), we get $\angle FSX = \angle ESY$. On the other hand, Pascal's theorem for quadrilateral $FBCE$, gives us that S, X, Y, I are collinear. Since $\angle FSX = \angle ESY$ this implies that X and Y lies on ℓ. It follows that I also lies on ℓ, as desired.