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Farey Sequences

Duality and Maps Between Subsequences

DE GRUYTER

that is,

$$\begin{aligned} & \min\left\{\frac{m+h_{j+2}}{h_{j+1}}, \frac{n+k_{j+2}}{k_{j+1}}, \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}}\right\} \\ &= \begin{cases} \frac{m+h_{j+2}}{h_{j+1}} & \text{if } h_{j+1}n - k_{j+1}m \geq 1, \\ \frac{m+h_{j+2}}{h_{j+1}} = \frac{n+k_{j+2}}{k_{j+1}} = \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}} & \text{if } h_{j+1}n - k_{j+1}m = 1, \\ \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}} & \text{if } h_{j+1}n - k_{j+1}m \leq 1 \end{cases} \\ &= \min\left\{\frac{m+h_{j+2}}{h_{j+1}}, \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}}\right\}. \end{aligned}$$

On the one hand, if $\min\left\{\frac{m+h_{j+2}}{h_{j+1}}, \frac{n+k_{j+2}}{k_{j+1}}\right\} = \frac{m+h_{j+2}}{h_{j+1}}$, that is, $h_{j+1}n - k_{j+1}m \geq 1$, then we have

$$\left\lfloor \frac{m+h_{j+2}}{h_{j+1}} \right\rfloor = \left\lfloor \frac{h_j + h_{j+2} + (m-h_j)}{h_{j+1}} \right\rfloor = \frac{h_j + h_{j+2}}{h_{j+1}} + \left\lfloor \frac{m-h_j}{h_{j+1}} \right\rfloor.$$

But the summand $\left\lfloor \frac{m-h_j}{h_{j+1}} \right\rfloor$ vanishes, since $h_j + h_{j+1} > m$, in view of Remark 1.22 (i). As a consequence, we have

$$\frac{h_j + h_{j+2}}{h_{j+1}} = \left\lfloor \frac{m+h_{j+2}}{h_{j+1}} \right\rfloor.$$

On the other hand, if

$$\min\left\{\frac{n+k_{j+2}}{k_{j+1}}, \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}}\right\} = \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}},$$

that is, $h_{j+1}n - k_{j+1}m \leq 1$, then

$$\begin{aligned} \left\lfloor \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}} \right\rfloor &= \left\lfloor \frac{(k_j - h_j) + (k_{j+2} - h_{j+2}) + ((n-m) - (k_j - h_j))}{k_{j+1} - h_{j+1}} \right\rfloor \\ &= \frac{(k_j - h_j) + (k_{j+2} - h_{j+2})}{k_{j+1} - h_{j+1}} + \left\lfloor \frac{(n-m) - (k_j - h_j)}{k_{j+1} - h_{j+1}} \right\rfloor. \end{aligned}$$

But $\left\lfloor \frac{(n-m)-(k_j-h_j)}{k_{j+1}-h_{j+1}} \right\rfloor = 0$, since $(k_j - h_j) + (k_{j+1} - h_{j+1}) > n - m$, in view of Remark 1.22 (ii). As a consequence, we have

$$\frac{(k_j - h_j) + (k_{j+2} - h_{j+2})}{k_{j+1} - h_{j+1}} = \left\lfloor \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}} \right\rfloor.$$

Thus, we have

$$\begin{aligned} \frac{h_j + h_{j+2}}{h_{j+1}} &= \frac{k_j + k_{j+2}}{k_{j+1}} = \frac{(k_j - h_j) + (k_{j+2} - h_{j+2})}{k_{j+1} - h_{j+1}} \\ &=: \gcd(h_j + h_{j+2}, k_j + k_{j+2}) \\ &= \left\lfloor \min\left\{\frac{m+h_{j+2}}{h_{j+1}}, \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}}\right\} \right\rfloor. \end{aligned} \quad (1.115)$$

Hence, (1.111) and (1.112) follow.

Table 2.7. The neighbors of the fraction $\frac{1}{3}$ in Farey subsequences; see also Table 2.6.

Sequence	The predecessor of $\frac{1}{3}$	See
$\mathcal{F}(\mathbb{B}(2m), m), m > 1$	$\frac{\lfloor (m-1)/2 \rfloor}{3\lfloor (m-1)/2 \rfloor + 1}$	Remark 2.24
$\mathcal{F}(\mathbb{B}(n), m), n < 2m, (n-m) > 1$	$\frac{\lfloor (n-m-1)/2 \rfloor}{3\lfloor (n-m-1)/2 \rfloor + 1}$	Remark 2.42 (i)
$\mathcal{F}(\mathbb{B}(n), m), n > 2m, m > 1$	$\frac{m}{3m+1}$ if $n - 3m \geq 1$ $\frac{\lfloor (n-m-1)/2 \rfloor}{3\lfloor (n-m-1)/2 \rfloor + 1}$ if $n - 3m \leq 1$	Remark 2.42 (ii)
Sequence	The successor of $\frac{1}{3}$	See
$\mathcal{F}(\mathbb{B}(2m), m), m > 1$	$\frac{\lceil m/2 \rceil}{3\lceil m/2 \rceil - 1}$	Remark 2.24
$\mathcal{F}(\mathbb{B}(n), m), n < 2m, (n-m) > 1$	$\frac{\lceil (n-m)/2 \rceil}{3\lceil (n-m)/2 \rceil - 1}$	Remark 2.42 (i)
$\mathcal{F}(\mathbb{B}(n), m), n > 2m, m > 1$	$\frac{\lceil (n-m)/2 \rceil}{3\lceil (n-m)/2 \rceil - 1}$ if $3m - n \geq 1$ $\frac{m}{3m-1}$ if $3m - n \leq 1$	Remark 2.42 (ii)

precedes $\frac{1}{3}$ in $\mathcal{F}(\mathbb{B}(2m), m)$, and the fraction

$$\frac{\lceil m/2 \rceil}{3\lceil m/2 \rceil - 1} = \frac{\lfloor (m-1)/2 \rfloor + 1}{3\lfloor (m-1)/2 \rfloor + 2}$$

succeeds $\frac{1}{3}$ in $\mathcal{F}(\mathbb{B}(2m), m)$.

Thus, if m is even, then the fractions

$$\frac{(m-2)/2}{(3m-4)/2} < \frac{1}{3} < \frac{m/2}{(3m-2)/2}$$

are consecutive in $\mathcal{F}(\mathbb{B}(2m), m)$. If m is odd, then the fractions

$$\frac{(m-1)/2}{(3m-1)/2} < \frac{1}{3} < \frac{(m+1)/2}{(3m+1)/2}$$

are consecutive in $\mathcal{F}(\mathbb{B}(2m), m)$.

2.2.7.4 The neighbors of $\frac{2}{3}$

Now that we have found in the previous subsection the neighbors of the fraction $\frac{1}{3}$ in the Farey subsequences $\mathcal{F}(\mathbb{B}(2m), m)$, it is natural to ask which fractions are the neighbors of the fraction $\frac{2}{3}$. Let us apply the order-reversing bijection $\mathcal{F}(\mathbb{B}(2m), m) \rightarrow \mathcal{F}(\mathbb{B}(2m), m)$ defined in (1.8) to the observations made in Remark 2.24.

Remark 2.25 (see also Table 2.8). If $m > 1$, then the fraction

$$\frac{2\lceil m/2 \rceil - 1}{3\lceil m/2 \rceil - 1} = \frac{2\lfloor (m-1)/2 \rfloor + 1}{3\lfloor (m-1)/2 \rfloor + 2}$$

precedes $\frac{2}{3}$ in $\mathcal{F}(\mathbb{B}(2m), m)$, and the fraction

$$\frac{2\lfloor (m-1)/2 \rfloor + 1}{3\lfloor (m-1)/2 \rfloor + 1} = \frac{2\lceil m/2 \rceil - 1}{3\lceil m/2 \rceil - 2}$$

succeeds $\frac{2}{3}$ in $\mathcal{F}(\mathbb{B}(2m), m)$.

Table 2.8. The neighbors of the fraction $\frac{2}{3}$ in Farey subsequences; see also Table 2.6.

Sequence	The predecessor of $\frac{2}{3}$	See
$\mathcal{F}(\mathbb{B}(2m), m), m > 1$	$\frac{2\lfloor m/2 \rfloor - 1}{3\lfloor m/2 \rfloor - 1}$	Remark 2.25
$\mathcal{F}(\mathbb{B}(n), m), n < 2m, (n - m) > 1$	$\frac{2\lfloor m/2 \rfloor - 1}{3\lfloor m/2 \rfloor - 1}$ if $2n - 3m \geq 1$ $\frac{2(n-m)-1}{3(n-m)-1}$ if $2n - 3m \leq 1$	Remark 2.43 (i)
$\mathcal{F}(\mathbb{B}(n), m), n > 2m, m > 1$	$\frac{2\lfloor m/2 \rfloor - 1}{3\lfloor m/2 \rfloor - 1}$	Remark 2.43 (ii)

Sequence	The successor of $\frac{2}{3}$	See
$\mathcal{F}(\mathbb{B}(2m), m), m > 1$	$\frac{2\lceil (m-1)/2 \rceil + 1}{3\lceil (m-1)/2 \rceil + 1}$	Remark 2.25
$\mathcal{F}(\mathbb{B}(n), m), n < 2m, (n - m) > 1$	$\frac{2(n-m)+1}{3(n-m)+1}$ if $3m - 2n \geq 1$ $\frac{2\lceil (m-1)/2 \rceil + 1}{3\lceil (m-1)/2 \rceil + 1}$ if $3m - 2n \leq 1$	Remark 2.43 (i)
$\mathcal{F}(\mathbb{B}(n), m), n > 2m, m > 1$	$\frac{2\lceil (m-1)/2 \rceil + 1}{3\lceil (m-1)/2 \rceil + 1}$	Remark 2.43 (ii)

Thus, if m is even, then the fractions

$$\frac{m-1}{(3m-2)/2} < \frac{2}{3} < \frac{m-1}{(3m-4)/2}$$

are consecutive in $\mathcal{F}(\mathbb{B}(2m), m)$. If m is odd, then the fractions

$$\frac{m}{(3m+1)/2} < \frac{2}{3} < \frac{m}{(3m-1)/2}$$

are consecutive in $\mathcal{F}(\mathbb{B}(2m), m)$.

2.2.8 More on neighboring fractions in \mathcal{F}_m^ℓ and \mathcal{G}_m^ℓ

In this subsection we will describe explicitly the neighbors of fractions of the form $\frac{1}{j}$, $\frac{j-1}{j}$, $\frac{2}{j}$ and $\frac{j-2}{j}$ in the Farey subsequences \mathcal{F}_m^ℓ and \mathcal{G}_m^ℓ , with the help of the formulas provided by Lemmas 2.13 and 2.15. Recall that we have already found in Section 2.2.6 the neighbors of these fractions in the Farey sequences \mathcal{F}_m . See Table 2.6 on the selected pairs of neighboring fractions that are considered in this book.

2.2.8.1 The neighbors of $\frac{1}{j}$

Let us find the neighbors of a fraction $\frac{1}{j}$ in the Farey subsequences \mathcal{F}_m^ℓ and \mathcal{G}_m^ℓ .

Corollary 2.26. (i) Consider a fraction $\frac{1}{j} \in \mathcal{F}_m^\ell$, where $0 < \ell < m$.

(a) If $m - j\ell \geq 1$, then the fraction

$$\frac{\ell}{j\ell + 1} \tag{2.51}$$

precedes the fraction $\frac{1}{j}$ in \mathcal{F}_m^ℓ .

Thus, if $3m - n \geq 1$ and $(n - m)$ is even, then

$$\frac{(n - m)/2}{(3(n - m) - 2)/2}$$

succeeds $\frac{1}{3}$ in $\mathcal{F}^{\leq \frac{1}{2}}(\mathbb{B}(n), m)$. If $3m - n \geq 1$ and $(n - m)$ is odd, then

$$\frac{(n - m + 1)/2}{(3(n - m) + 1)/2}$$

succeeds $\frac{1}{3}$ in $\mathcal{F}^{\leq \frac{1}{2}}(\mathbb{B}(n), m)$.

If $3m - n \leq 1$, then $\frac{m}{3m - 1}$ succeeds $\frac{1}{3}$ in $\mathcal{F}^{\leq \frac{1}{2}}(\mathbb{B}(n), m)$.

2.2.9.10 The neighbors of $\frac{2}{3}$

In order to find the neighbors of the fraction $\frac{2}{3}$ in the Farey subsequences $\mathcal{F}(\mathbb{B}(n), m)$, with $n \neq 2m$, let us apply the order-reversing bijections $\mathcal{F}(\mathbb{B}(n), m) \leftrightarrow \mathcal{F}(\mathbb{B}(n), n - m)$ defined in (1.8) to the observations, made in Remark 2.42, on the neighbors of the fraction $\frac{1}{3}$.

Remark 2.43 (see also Table 2.8). (i) Consider a Farey subsequence $\mathcal{F}(\mathbb{B}(n), m)$ such that $n < 2m$ and $n - m > 1$.

If $2n - 3m \geq 1$, then

$$\frac{2\lceil m/2 \rceil - 1}{3\lceil m/2 \rceil - 1} = \frac{2\lfloor (m - 1)/2 \rfloor + 1}{3\lfloor (m - 1)/2 \rfloor + 2}$$

precedes $\frac{2}{3}$ in $\mathcal{F}^{\geq \frac{1}{2}}(\mathbb{B}(n), m)$.

Thus, if $2n - 3m \geq 1$ and m is even, then

$$\frac{m - 1}{(3m - 2)/2}$$

precedes $\frac{2}{3}$ in $\mathcal{F}^{\geq \frac{1}{2}}(\mathbb{B}(n), m)$. If $2n - 3m \geq 1$ and m is odd, then

$$\frac{m}{(3m + 1)/2}$$

precedes $\frac{2}{3}$ in $\mathcal{F}^{\geq \frac{1}{2}}(\mathbb{B}(n), m)$.

If $2n - 3m \leq 1$, then

$$\frac{2(n - m) - 1}{3(n - m) - 1}$$

precedes $\frac{2}{3}$ in $\mathcal{F}^{\geq \frac{1}{2}}(\mathbb{B}(n), m)$.

If $3m - 2n \geq 1$, then

$$\frac{2(n - m) + 1}{3(n - m) + 1}$$

succeeds $\frac{2}{3}$ in $\mathcal{F}^{\geq \frac{1}{2}}(\mathbb{B}(n), m)$. If $3m - 2n \leq 1$, then

$$\frac{2\lfloor (m - 1)/2 \rfloor + 1}{3\lfloor (m - 1)/2 \rfloor + 1} = \frac{2\lceil m/2 \rceil - 1}{3\lceil m/2 \rceil - 2}$$

succeeds $\frac{2}{3}$ in $\mathcal{F}^{\geq \frac{1}{2}}(\mathbb{B}(n), m)$.