SYMMETRIC CYCLES

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The arrangement (1.1) *realizes* a *rank* 2 *oriented matroid* $\mathcal{N}$ in the following manner (see, e.g., Example 7.1.7 in Ref. [1]): For a vector $\boldsymbol{v} \in \mathbb{R}^2$, the ordered tuple of *signs* $X := (\mathrm{sign}(\langle \boldsymbol{a}_e, \boldsymbol{v}\rangle)\colon e \in E_t) \in \{1, 0, -1\}^t := \{\text{'+'}, \text{'0'}, \text{'−'}\}^t$ is called a *covector* of $\mathcal{N}$.

The covectors $T \in \mathcal{T} \subset \{1, -1\}^t$ of the oriented matroid $\mathcal{N}$ are called its *topes* (*maximal covectors*), and the set of topes $\mathcal{T}$ is in a one-to-one correspondence with the set of *regions* of the arrangement (1.1). The *cocircuits* $C^* \in \{1, 0, -1\}^t$ of the oriented matroid $\mathcal{N} := (E_t, \mathcal{T})$, on the *ground set* E_t, and with its set of topes $\mathcal{T}$, are the covectors that correspond to the *rays* emanating from the origin; *one* sign component of each cocircuit is 0.

– Given a vector $\boldsymbol{z} := (z_1, \dots, z_t) \in \mathbb{R}^t$, we denote its *support* $\{e \in E_t\colon\ z_e \neq 0\}$ by $\mathrm{supp}(\boldsymbol{z})$.

– Now consider the *rank t oriented matroid* $\mathcal{H} := (E_t, \{1, -1\}^t)$ on the *ground set* E_t, and with its set of *topes* (*maximal covectors*) $\mathcal{T} := \{1, -1\}^t$, *realizable* (see, e.g., Example 2.1.4 in Ref. [1]) as the *arrangement* of *coordinate hyperplanes*

$$\boxed{\Big\{\{\mathbf{x} := (\mathrm{x}_1, \dots, \mathrm{x}_t) \in \mathbb{R}^t\colon |\,\mathrm{supp}(\mathbf{x})| = t-1, \mathrm{x}_e = 0\} \cup \{\mathbf{0}\}\ : e \in E_t\Big\}} \tag{1.2}$$

in the space $\mathbb{R}^t$.

The hyperplanes of the arrangement are *oriented*: a vector $\boldsymbol{v} := (v_1, \dots, v_t) \in \mathbb{R}^t - \{\mathbf{0}\}$ lies on the *positive side* of a hyperplane $\boxed{\boldsymbol{H}_e := \{\ \mathbf{x} \in \mathbb{R}^t\colon |\,\mathrm{supp}(\mathbf{x})| = t-1, \mathrm{x}_e = 0\} \cup \{\mathbf{0}\},}$ if $v_e > 0$. Similarly, a *region* $\boldsymbol{T}$ of the arrangement (1.2), that is, a *connected component* of the *complement* $\mathbb{R}^t - \bigcup_{e \in E_t} \boldsymbol{H}_e$, lies on the *positive side* of the hyperplane $\boldsymbol{H}_e$ if $v_e > 0$, for an arbitrary vector $\boldsymbol{v} \in \boldsymbol{T}$. For a vector $\boldsymbol{v} \in \mathbb{R}^t$, the *sign* tuple $X := (\mathrm{sign}(v_e)\colon e \in E_t) \in \{1, 0, -1\}^t := \{\text{'+'}, \text{'0'}, \text{'−'}\}^t$ is a *covector* of the oriented matroid $\mathcal{H}$. The *cocircuits* $C^* \in \{1, 0, -1\}^t$ of $\mathcal{H}$ are the covectors that have *one* sign component different from 0.

– If $\mathcal{M}$ is one of the above oriented matroids $\mathcal{N} := (E_t, \mathcal{T})$ and $\mathcal{H} := (E_t, \mathcal{T} := \{1, -1\}^t)$, then a sign tuple $S := (S(1), \dots, S(t)) \in \{1, 0, -1\}^t$, with exactly *one* zero component $S(i) = 0$, is called a *subtope* of $\mathcal{M}$ if there are two topes, $T' := (T'(1), \dots, T'(t)) \in \mathcal{T}$, and $T'' \in \mathcal{T}$, such that the *Hamming distance* between the tuples T' and T'' is 1, that is, $|\{e \in E_t\colon\ T'(e) \neq T''(e)\}| = 1$, and $T'(i) \neq T''(i)$.

A.1 Ternary Smirnov Words

Let (θ, α, β) be an ordered three-letter alphabet, and let 'u', 'v' and 'w' be formal variables which mark the letters θ, α and β, respectively. Let $\mathrm{e}_{\bullet}(\cdot\,\cdot)$ denote elementary symmetric polynomials. The ordinary trivariate generating function of the set of ternary Smirnov words is

$$\frac{1}{1-\left(\frac{\mathrm{u}}{1+\mathrm{u}}+\frac{\mathrm{v}}{1+\mathrm{v}}+\frac{\mathrm{w}}{1+\mathrm{w}}\right)}=\frac{(1+\mathrm{u})(1+\mathrm{v})(1+\mathrm{w})}{1-\mathrm{e}_2(\mathrm{u},\mathrm{v},\mathrm{w})-2\mathrm{e}_3(\mathrm{u},\mathrm{v},\mathrm{w})}\;; \tag{A.1}$$

see Example III.24 in Ref. [118], and Section 2.4.16 of Ref. [27], on the general multivariate generating function of the Smirnov words.

In the theoretical framework of the breakthrough article [24], let us consider the system of generating functions

$$\begin{cases} f_\theta(\mathrm{u},\mathrm{v},\mathrm{w}) = \mathrm{u} + \mathrm{u}\, f_\alpha(\mathrm{u},\mathrm{v},\mathrm{w}) + \mathrm{u}\, f_\beta(\mathrm{u},\mathrm{v},\mathrm{w})\,, \\ f_\alpha(\mathrm{u},\mathrm{v},\mathrm{w}) = \qquad \mathrm{v}\, f_\theta(\mathrm{u},\mathrm{v},\mathrm{w}) + \mathrm{v}\, f_\beta(\mathrm{u},\mathrm{v},\mathrm{w})\,, \\ f_\beta(\mathrm{u},\mathrm{v},\mathrm{w}) = \qquad \mathrm{w}\, f_\theta(\mathrm{u},\mathrm{v},\mathrm{w}) + \mathrm{w}\, f_\alpha(\mathrm{u},\mathrm{v},\mathrm{w}) \end{cases}$$

rewritten, for short, as

$$\begin{cases} f_\theta = \mathrm{u} + \mathrm{u}\cdot(f_\alpha + f_\beta)\,, \\ f_\alpha = \qquad \mathrm{v}\cdot(f_\theta + f_\beta)\,, \\ f_\beta = \qquad \mathrm{w}\cdot(f_\theta + f_\alpha)\,. \end{cases} \tag{A.2}$$

For a letter $\mathfrak{s} \in (\theta, \alpha, \beta)$, the generating function $f_{\mathfrak{s}} := f_{\mathfrak{s}}(\mathrm{u},\mathrm{v},\mathrm{w})$ is meant to count the ternary Smirnov words starting with the letter θ and ending with the letter $\mathfrak{s}$.

The solutions to the system (A.2) are

$$f_\theta = \frac{\mathrm{u}(1-\mathrm{e}_2(\mathrm{v},\mathrm{w}))}{1-\mathrm{e}_2(\mathrm{u},\mathrm{v},\mathrm{w})-2\mathrm{e}_3(\mathrm{u},\mathrm{v},\mathrm{w})}\,,$$

$$f_\alpha = \frac{\mathrm{uv}(1+\mathrm{w})}{1-\mathrm{e}_2(\mathrm{u},\mathrm{v},\mathrm{w})-2\mathrm{e}_3(\mathrm{u},\mathrm{v},\mathrm{w})}$$

and

$$f_\beta = \frac{\mathrm{uw}(1+\mathrm{v})}{1-\mathrm{e}_2(\mathrm{u},\mathrm{v},\mathrm{w})-2\mathrm{e}_3(\mathrm{u},\mathrm{v},\mathrm{w})}\,,$$