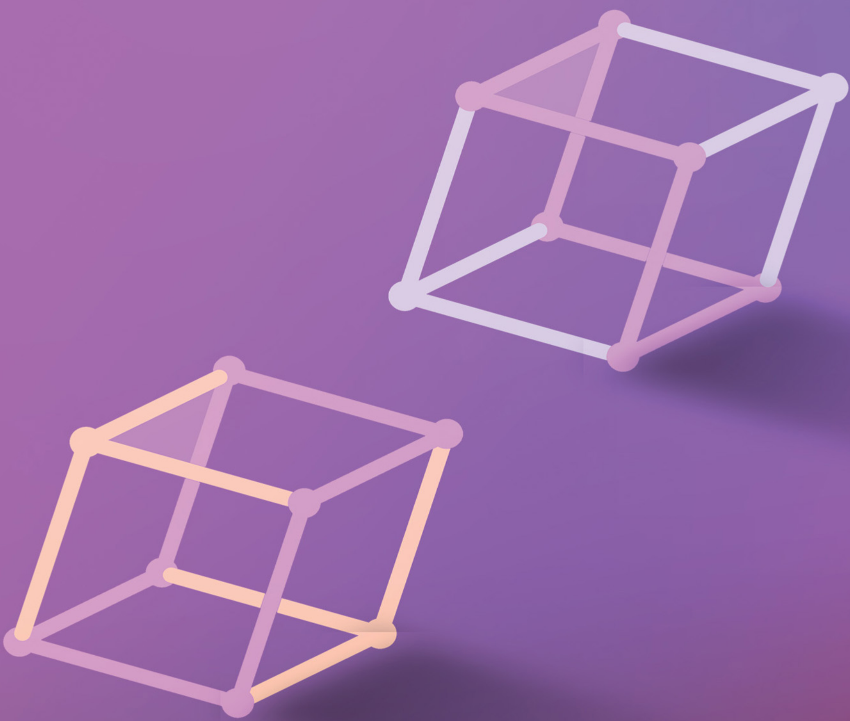

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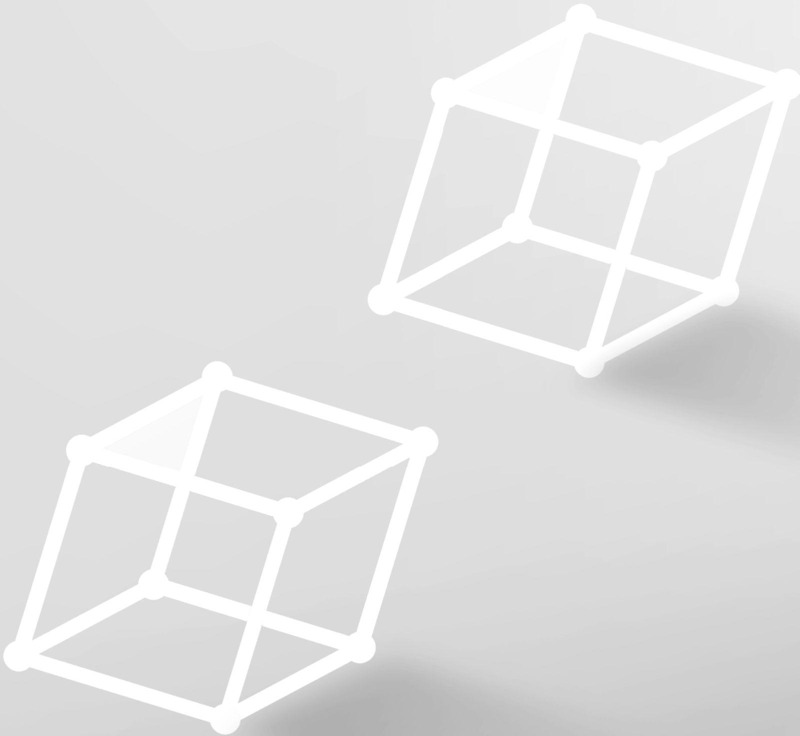

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Andrey O. Matveev

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To the memory of my parents

Irina M. Matveeva

1934–2014

Oleg S. Matveev

1931–2022

To the memory of

Dr. Damir N. Gainanov

1954–2022

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Contents

Preface

The present monograph begins, in a sense, where the book *Pattern Recognition on Oriented Matroids*, published with *De Gruyter* in 2017, left off.

Given an ordered *two-letter alphabet* (θ, α) and an integer *dimensional* parameter t, the *discrete hypercube* $\{\theta, \alpha\}^t$ is the *vertex set* of the *hypercube graph* $\boldsymbol{\Gamma}(t, 2; \theta, \alpha)$, a central discrete mathematical construct with countless applications in theoretical computer science, Boolean function theory, combinatorics, combinatorial optimization, coding theory, discrete and computational geometry, etc.

Let $E_t := [t] := \{1, \ldots, t\}$ denote a *ground set* of indices. A pair $\{X, Y\}$ of vertices $X := (X(1), \ldots, X(t))$ and Y of the hypercube $\{\theta, \alpha\}^t$ by definition is an *edge* of the graph $\boldsymbol{\Gamma}(t, 2; \theta, \alpha)$ if the *Hamming distance* $|\{e \in E_t\colon\ X(e) \neq Y(e)\}|$ between the ordered tuples X and Y is 1.

A vertex $Z \in \{\theta, \alpha\}^t$ of the hypercube graph $\boldsymbol{\Gamma}(t, 2; \theta, \alpha)$ has its '*positive part*' $\mathfrak{p}(Z) := Z^+$, defined to be the subset $\{e \in E_t\colon Z(e) = \theta\} \subseteq E_t$, and its '*negative part*' $\mathfrak{n}(Z) := Z^- := \{e \in E_t\colon Z(e) = \alpha\} = E_t - \mathfrak{p}(Z)$.

Recall that the *power set* $\mathbf{2}^{[t]}$ of the set E_t is defined to be the family $\{A\colon A \subseteq E_t\}$ of all subsets of E_t. The partial ordering of this family by inclusion turns the power set $\mathbf{2}^{[t]}$ into the *Boolean lattice* $\mathbb{B}(t)$ of rank t. Throughout the monograph, by convention we interpret the power set

$$\mathbf{2}^{[t]} = \left\{\mathfrak{n}(Z)\colon Z \in \{\theta, \alpha\}^t\right\}$$

of the ground set E_t as the family of the negative parts of vertices of a particular *discrete hypercube*, $\{1, -1\}^t \subset \mathbb{R}^t$, or $\{0, 1\}^t \subset \mathbb{R}^t$, associated with the two-letter alphabet $(\theta, \alpha) := (1, -1)$, or $(\theta, \alpha) := (0, 1)$, respectively, where the letters θ and α are regarded as *real numbers*.

In the monograph, we consider both the *hypercube graphs*

$$\boldsymbol{H}(t, 2) := \boldsymbol{\Gamma}(t, 2; 1, -1) , \quad \text{and} \quad \widetilde{\boldsymbol{H}}(t, 2) := \boldsymbol{\Gamma}(t, 2; 0, 1) ,$$

though our main interest lies in the graph $\boldsymbol{H}(t, 2)$ on its vertex set $\{1, -1\}^t$.

A *symmetric cycle* $\boldsymbol{D} := (D^0, D^1, \ldots, D^{2t-1}, D^0)$ in the hypercube graph $\boldsymbol{H}(t, 2)$ is defined to be a $2t$-cycle, with its *vertex set* $\mathrm{V}(\boldsymbol{D}) := \{D^0, D^1, \ldots, D^{2t-1}\}$, such that

$$D^{k+t} = -D^k , \quad 0 \le k \le t - 1 .$$

The vertex set $\mathrm{V}(\boldsymbol{D})$ of the symmetric cycle $\boldsymbol{D}$ is a *maximal positive basis* of the space $\mathbb{R}^t$. For any vertex $T \in \{1, -1\}^t$ of the graph $\boldsymbol{H}(t, 2)$ there exists a *unique inclusion-minimal* and *linearly independent* subset (of *odd* cardinality) $\boldsymbol{Q}(T, \boldsymbol{D}) \subset \mathrm{V}(\boldsymbol{D})$ such that

$$T = \sum_{Q \in \boldsymbol{Q}(T, \boldsymbol{D})} Q .$$

In particular, that linear algebraic decomposition describes how the members of the family of subsets $\{\mathfrak{n}(Q) \colon Q \in \boldsymbol{Q}(T, \boldsymbol{D})\} \subset \mathbf{2}^{[t]}$ *vote* for, or against, the elements of the ground set E_t, thus arriving at their *collective decision* on the subset $\mathfrak{n}(T) \subseteq E_t$.

Informally speaking, we discuss in the monograph various aspects of how (based on the above decomposition) all the vertices of the discrete hypercube $\{1, -1\}^t$, as well as the power set of the ground set E_t, emerge from a *rank* 2 *oriented matroid*, from an underlying *rank* 2 *system of linear inequalities*, and thus literally from an *arrangement* of t distinct *straight lines* crossing a common point on a piece of paper.

In the introductory Chapter 2, we briefly recall basic properties of vertex decompositions in hypercube graphs $\boldsymbol{H}(t, 2)$ with respect to their symmetric cycles. We then establish a connection between *coherent decompositions* in the hypercube graphs $\boldsymbol{H}(t, 2)$ and $\widetilde{\boldsymbol{H}}(t, 2)$.

In Chapter 3, we recall some enumerative results on rank 2 infeasible systems of linear inequalities related to arrangements of oriented lines in the plane. *Dehn–Sommerville type relations* are presented that concern the numbers of faces of abstract simplicial complexes associated with large-size decomposition sets for vertices of hypercube graphs.

Chapter 4 concerns additional *Dehn–Sommerville type relations* that are valid for large-size decomposition sets for vertices in hypercube graphs. We present a common *orthogonality relation* that establishes a connection between enumerative properties of large-size decomposition sets in the graphs $\boldsymbol{H}(s, 2)$ and $\boldsymbol{H}(t, 2)$ with specific dimensional parameters s and t.

In Chapter 5, we give a few comments on certain *distinguished* symmetric cycles in hypercube graphs. The main results of the chapter relate to the *interval* structure of the negative parts of vertices of hypercube graphs, to *computation-free decompositions* with respect to the distinguished symmetric cycles, and to statistics on decompositions. We also discuss *equinumerous decompositions* of vertices. We conclude the chapter by mentioning that vector descriptions of vertex decompositions with respect to arbitrary symmetric cycles are *valuations* on the Boolean lattices of subsets of the vertex sets of hypercube graphs.

In Chapter 6, we touch on the question on a structural connection between the decomposition sets for vertices whose negative parts are *comparable* by inclusion. Further enumerative results concern statistics on *partitions* of the negative parts of vertices of hypercube graphs and on decompositions of vertices. The key computational tool that allows us to present quite fine statistics is an approach to enumeration of *ternary Smirnov words* (i.e., words over a three-letter alphabet, such that adjacent letters in the words never coincide) discussed in Appendix A.

An even more involved analysis, based on enumeration of *Smirnov words* over *four-letter* alphabets (also discussed in Appendix A), leads us in Chapter 7 to statistics on *unions* of the negative parts of vertices of hypercube graphs and on decompositions of vertices.

An innocent-looking transformation (with serious applications given later in Chapter 9) of vertices of a discrete hypercube $\{1, -1\}^t$, that turns a vertex of the hypercube graph $\boldsymbol{H}(t, 2)$ into its '*relabeled opposite*', is presented and discussed in Chapter 8.

Recall that a nonempty family of nonempty subsets $\mathcal{A} := \{A_1, \ldots, A_\alpha\} \subset \mathbf{2}^{[t]}$ of the ground set E_t is called a *clutter* (*Sperner* *family*), if no set from the family $\mathcal{A}$ contains another. One says that a subset $B \subseteq E_t$ is a *blocking set* of the clutter $\mathcal{A}$ if the set B has a nonempty intersection with each member A_i of $\mathcal{A}$. The *blocker* $\mathfrak{B}(\mathcal{A})$ of the clutter $\mathcal{A}$ is defined to be the family of all inclusion-minimal blocking sets of $\mathcal{A}$. In Chapter 9, we drastically change the dimensionality of our research constructs from t to 2^t, since we indirectly represent the families of blocking sets of clutters as the negative parts of relevant vertices of the hypercube graphs $\boldsymbol{H}(2^t, 2)$ and $\widetilde{\boldsymbol{H}}(2^t, 2)$ associated with the discrete hypercubes $\{1, -1\}^{2^t}$ and $\{0, 1\}^{2^t}$, respectively. We describe in detail a '*blocking/voting*' connection between the families of blocking sets of the clutters $\mathcal{A}$ and $\mathfrak{B}(\mathcal{A})$.

In Appendix A, Smirnov words over three-letter and four-letter alphabets are enumerated.

In Appendix B we investigate the enumerative properties of the subset families generated by the *self-dual clutters* $\mathcal{A} = \mathfrak{B}(\mathcal{A})$.

Andrey O. Matveev
February 2023

Chapter 1

Preliminaries and Notational Conventions

Throughout the monograph, ‘$:=$’ means equality by definition.

– Consider an *arrangement* of distinct *straight lines*

$$\{\mathbf{x} \in \mathbb{R}^2 \colon \langle \boldsymbol{a}_e, \mathbf{x} \rangle = 0,\ e \in E_t\} \tag{1.1}$$

in the plane $\mathbb{R}^2$ (throughout the monograph, $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ means the standard scalar product $\sum_e v_e w_e$ of real vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ of relevant dimension), with the set of corresponding normal vectors

$$\boldsymbol{A} := \{\boldsymbol{a}_e \colon e \in E_t\} .$$

One associates with the arrangement (1.1) the *rank* 2 *system* of *homogeneous strict linear inequalities*

$$\{\langle \boldsymbol{a}_e, \mathbf{x} \rangle > 0 \colon \mathbf{x} \in \mathbb{R}^2,\ \boldsymbol{a}_e \in \boldsymbol{A}\} .$$

The lines of the arrangement are *oriented*: a vector $\boldsymbol{v} := (v_1, v_2) \in \mathbb{R}^2 - \{\mathbf{0}\}$ lies on the *positive side* of a line $\boldsymbol{L}_e := \{\mathbf{x} \in \mathbb{R}^2 \colon \langle \boldsymbol{a}_e, \mathbf{x} \rangle = 0\}$ if we have $\langle \boldsymbol{a}_e, \boldsymbol{v} \rangle > 0$. Similarly, a *region* $\boldsymbol{T}$ of the arrangement (1.1), that is, a *connected component* of the *complement* $\mathbb{R}^2 - \bigcup_{e \in E_t} \boldsymbol{L}_e$, lies on the *positive side* of the line $\boldsymbol{L}_e$ if we have $\langle \boldsymbol{a}_e, \boldsymbol{v} \rangle > 0$, for an arbitrary vector $\boldsymbol{v} \in \boldsymbol{T}$.

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The arrangement (1.1) *realizes* a *rank* 2 *oriented matroid* $\mathcal{N}$ in the following manner (see, e.g., Example 7.1.7 in Ref. [1]): For a vector $\boldsymbol{v} \in \mathbb{R}^2$, the ordered tuple of *signs* $X := (\mathrm{sign}(\langle \boldsymbol{a}_e, \boldsymbol{v}\rangle) \colon e \in E_t) \in \{1, 0, -1\}^t := \{\text{'}+\text{'}, \text{'}0\text{'}, \text{'}-\text{'}\}^t$ is called a *covector* of $\mathcal{N}$.

The covectors $T \in \mathcal{T} \subset \{1, -1\}^t$ of the oriented matroid $\mathcal{N}$ are called its *topes* (*maximal covectors*), and the set of topes $\mathcal{T}$ is in a one-to-one correspondence with the set of *regions* of the arrangement (1.1). The *cocircuits* $C^* \in \{1, 0, -1\}^t$ of the oriented matroid $\mathcal{N} := (E_t, \mathcal{T})$, on the *ground set* E_t, and with its set of topes $\mathcal{T}$, are the covectors that correspond to the *rays* emanating from the origin; *one* sign component of each cocircuit is 0.

– Given a vector $\boldsymbol{z} := (z_1, \dots, z_t) \in \mathbb{R}^t$, we denote its *support* $\{e \in E_t \colon\ z_e \neq 0\}$ by $\mathrm{supp}(\boldsymbol{z})$.

– Now consider the *rank t oriented matroid* $\mathcal{H} := (E_t, \{1, -1\}^t)$ on the *ground set* E_t, and with its set of *topes* (*maximal covectors*) $\mathcal{T} := \{1, -1\}^t$, *realizable* (see, e.g., Example 2.1.4 in Ref. [1]) as the *arrangement* of *coordinate hyperplanes*

$$\begin{aligned}\big\{\{\mathbf{x} := (\mathrm{x}_1, \dots, \mathrm{x}_t) \in \mathbb{R}^t \colon |\,\mathrm{supp}(\mathbf{x})| \\ = t-1,\ \mathrm{x}_e = 0\} \colon e \in E_t\big\}\end{aligned} \tag{1.2}$$

in the space $\mathbb{R}^t$.

The hyperplanes of the arrangement are *oriented*: a vector $\boldsymbol{v} := (v_1, \dots, v_t) \in \mathbb{R}^t - \{\mathbf{0}\}$ lies on the *positive side* of a hyperplane $\boldsymbol{H}_e := \{\mathbf{x} \in \mathbb{R}^t \colon\ |\,\mathrm{supp}(\mathbf{x})| = t-1,\ \mathrm{x}_e = 0\}$ if $v_e > 0$. Similarly, a *region* $\boldsymbol{T}$ of the arrangement (1.2), that is, a *connected component* of the *complement* $\mathbb{R}^t - \bigcup_{e \in E_t} \boldsymbol{H}_e$, lies on the *positive side* of the hyperplane $\boldsymbol{H}_e$ if $v_e > 0$, for an arbitrary vector $\boldsymbol{v} \in \boldsymbol{T}$. For a vector $\boldsymbol{v} \in \mathbb{R}^t$, the *sign* tuple $X := (\mathrm{sign}(v_e) \colon e \in E_t) \in \{1, 0, -1\}^t := \{\text{'}+\text{'}, \text{'}0\text{'}, \text{'}-\text{'}\}^t$ is a *covector* of the oriented matroid $\mathcal{H}$. The *cocircuits* $C^* \in \{1, 0, -1\}^t$ of $\mathcal{H}$ are the covectors that have *one* sign component different from 0.

– If $\mathcal{M}$ is one of the above oriented matroids $\mathcal{N} := (E_t, \mathcal{T})$ and $\mathcal{H} := (E_t, \mathcal{T} := \{1, -1\}^t)$, then a sign tuple $S := (S(1), \dots, S(t)) \in \{1, 0, -1\}^t$, with exactly *one* zero component $S(i) = 0$, is called a *subtope* of $\mathcal{M}$ if there are two topes, $T' := (T'(1), \dots, T'(t)) \in \mathcal{T}$, and $T'' \in \mathcal{T}$, such that the *Hamming distance* between the tuples T' and T'' is 1, that is, $|\{e \in E_t \colon\ T'(e) \neq T''(e)\}| = 1$, and $T'(i) \neq T''(i)$.

Note that the subtopes of the rank 2 oriented matroid $\mathcal{N}$ are its cocircuits.

The *separation set* $\mathbf{S}(T', T'')$ of topes T' and T'' is defined by

$$\mathbf{S}(T', T'') := \{e \in E_t \colon T'(e) \neq T''(e)\} ,$$

and the *graph distance* between them is

$$d(T', T'') := |\mathbf{S}(T', T'')| ,$$

that is, $d(T', T'')$ is the *Hamming distance* between the tuples T' and T''. Recall that

$$\begin{aligned} d(T', T'') &= t - \frac{1}{4}\|T'' + T'\|^2 = \frac{1}{4}\|T'' - T'\|^2 \\ &= \frac{1}{2}(t - \langle T'', T'\rangle) , \end{aligned}$$

where $\|X\|^2 := \langle X, X\rangle$. If t is *even*, then we have[1]

$$\langle T'', T'\rangle = 0 \quad \Longleftrightarrow \quad d(T', T'') = \frac{t}{2} .$$

– The vertex set of the *tope graph* of an oriented matroid $\mathcal{M}$ by definition is its set of topes $\mathcal{T}$. Topes T' and T'' are *adjacent* in the tope graph if they have a *common subtope*.

Note that the tope graph of the oriented matroid $\mathcal{N}$ is a *cycle* on its vertex set $\mathcal{T}$. The tope graph of the oriented matroid $\mathcal{H}$ is the *hypercube graph* on its vertex set $\{1, -1\}^t$.

The tope graph turns into the *Hasse diagram* of the *tope poset* of the oriented matroid $\mathcal{M}$, *based* at a tope $B \in \mathcal{T}$, if we partially order the set of topes as follows:

$$T' \preceq T'' \quad \Longleftrightarrow \quad \mathbf{S}(B, T') \subseteq \mathbf{S}(B, T'') .$$

The graph distance $d(B, \cdot)$ becomes in this situation the *rank function* of the tope poset of the oriented matroid $\mathcal{M}$.

– In the present monograph, we use almost everywhere the nonstandard representation of *sign components* '$+$', '0' and '$-$' of covectors/vertices by the *real numbers* 1, 0 and -1, respectively. Nevertheless, most figures involve the traditional notation.

---

[1] Interesting subsets of vertices of discrete hypercubes $\{1, -1\}^t$, with *zero* pairwise *scalar products*, are the rows of *Hadamard matrices*, see, e.g., Refs. [2, 3, 4].

As a common rule, no matter what the notation is used, covectors/vertices are always thought of as elements of the *real Euclidean space* $\mathbb{R}^t$ (or $\mathbb{R}^{2^t}$, in Chapter 9) of *row vectors*.

– For readability, we prefer the nonstandard notation $\mathfrak{p}(X)$ and $\mathfrak{n}(X)$ for the *positive parts* and *negative parts* of tuples $X \in \{1, -1\}^t$, instead of the traditional notation X^+ and X^-, respectively. Thus, by convention we define the parts by

$$\mathfrak{p}(X) := \{e \in E_t \colon X(e) = \quad 1\} =: X^+ ,$$

and

$$\mathfrak{n}(X) := \{e \in E_t \colon X(e) = -1\} =: X^- .$$

– We denote by $\mathrm{T}^{(+)}$ the row tuple of all one's

$$\mathrm{T}^{(+)} := (+, \dots, +) := (1, \dots, 1) \in \mathbb{R}^t .$$

If $\mathrm{T}^{(+)}$ is a (maximal) covector of our oriented matroid $\mathcal{M}$, then it is called the *positive tope* of $\mathcal{M}$. We say that $\mathrm{T}^{(+)}$ is the *positive vertex* of the discrete hypercube $\{1, -1\}^t$, and that it is the *positive vertex* of the hypercube graph $\boldsymbol{H}(t, 2)$. The row tuple $\mathrm{T}^{(-)}$ is the element $\mathrm{T}^{(-)} := -\mathrm{T}^{(+)} := (-, \dots, -) := (-1, \dots, -1)$ of the space $\mathbb{R}^t$; if the tuple $\mathrm{T}^{(-)}$ is a (maximal) covector of the oriented matroid $\mathcal{M}$, then $\mathrm{T}^{(-)}$ is called the *negative tope* of $\mathcal{M}$.

– A subset $A \subseteq E_t$ of the ground set of the oriented matroid $\mathcal{N}$ is called *acyclic* if there is a tope $F \in \mathcal{T}$ of $\mathcal{N}$ such that $\mathfrak{p}(F) \supseteq A$.

– Given a tuple $T \in \{1, -1\}^t$, and a subset $A \subseteq E_t$, the notation ${}_{-A}T$ is used to denote the tuple obtained from T by *sign reversal* or *reorientation* on the set A:

$$({}_{-A}T)(e) := \begin{cases} -T(e) \, , & \text{if } e \in A, \\ T(e) \, , & \text{if } e \notin A. \end{cases}$$

Thus, by convention we have $\mathfrak{n}({}_{-A}\mathrm{T}^{(+)}) := A$.

One says that the oriented matroid ${}_{-A}\mathcal{N}$, whose set of topes by definition is the set ${}_{-A}\mathcal{T} := \{{}_{-A}T : T \in \mathcal{T}\}$, is obtained from the oriented matroid $\mathcal{N}$ by *reorientation* on the subset A.

Since we have ${}_{-A}\{1, -1\}^t = \{1, -1\}^t$ and, as a consequence, ${}_{-A}\mathcal{H} = \mathcal{H}$, the oriented matroid $\mathcal{H}$ is insensitive to reorientations.

– The unit vectors of the standard basis of the space $\mathbb{R}^t$ are denoted by $\boldsymbol{\sigma}(i) := (0, \ldots, \underset{\substack{\uparrow \\ i}}{1}, \ldots, 0)$, $1 \leq i \leq t$.

– We denote by $\#\mathcal{A}$ the number α of sets in a family $\mathcal{A} := \{A_1, \ldots, A_\alpha\}$. The cardinality of a finite set A is denoted by $|A|$.

– An *abstract simplicial complex* Δ on its *vertex* set E_t is defined to be a family $\Delta \subseteq \mathbf{2}^{[t]}$, such that $\{e\} \in \Delta$, for every vertex $e \in E_t$, and the following implications hold:

$$A, B \subseteq E_t\ , \quad A \subseteq B \in \Delta \quad \Longrightarrow \quad A \in \Delta\ .$$

Members of the family Δ are called *faces*. The inclusion-maximal faces are the *facets* of the complex Δ.

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