Andrey O. Matveev

Farey Sequences

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Duality and Maps Between Subsequences

DE GRUYTER

Mathematics Subject Classification 2010
Primary: 11B57; secondary: 05-01, 11-01

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ISBN 978-3-11-054662-0
e-ISBN (PDF) 978-3-11-054766-5
e-ISBN (EPUB) 978-3-11-054665-1
Set-ISBN 978-3-11-054767-2

Library of Congress Cataloging-in-Publication Data
A CIP catalog record for this book has been applied for at the Library of Congress.

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available on the Internet at http://dnb.dnb.de.

© 2017 Walter de Gruyter GmbH, Berlin/Boston
Cover image: A Perspective on Farey Duality, 2017, by Andrey L. Kopyrin, Ekaterinburg, Russia
Typesetting: Dimler & Albroscheit, Müncheberg
Printing and binding: CPI books GmbH, Leck
♾ Printed on acid-free paper
Printed in Germany

www.degruyter.com

To the memory of my mother

Irina M. Matveeva

1934–2014

To my father

Oleg S. Matveev

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DOI 10.1515/9783110547665-203

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