SYMMETRIC CYCLES

A 2D Perspective on Higher Dimensional Discrete Hypercubes, the Power Sets of Finite Sets, and Set Families

Andrey O. Matveev

Mathematics Subject Classification 2020
Primary: 05-02, 68-02;
secondary: 52C30, 52C40, 68R01, 68R05, 68R10, 68R15

Author
Dr. Andrey O. Matveev
Ekaterinburg
Russia
andrey.o.matveev@gmail.com

Cover image: Cubes and Symmetric Cycles, 2022, by Andrey L. Kopyrin,
Ekaterinburg, Russia

To the memory of my parents

Irina M. Matveeva
1934 – 2014

Oleg S. Matveev
1931 – 2022

To the memory of

Dr. Damir N. Gainanov
1954 – 2022

Preface

The present monograph begins, in a sense, where the book *Pattern Recognition on Oriented Matroids*, published with *De Gruyter* in 2017, left off.

Given an ordered *two-letter alphabet* (θ, α) and an integer *dimensional* parameter t, the *discrete hypercube* $\{\theta, \alpha\}^t$ is the *vertex set* of the *hypercube graph* $\boldsymbol{\Gamma}(t, 2; \theta, \alpha)$, a central discrete mathematical construct with countless applications in theoretical computer science, Boolean function theory, combinatorics, combinatorial optimization, coding theory, discrete and computational geometry, etc.

Let $E_t := [t] := \{1, \ldots, t\}$ denote a *ground set* of indices. A pair $\{X, Y\}$ of vertices $X := (X(1), \ldots, X(t))$ and Y of the hypercube $\{\theta, \alpha\}^t$ by definition is an *edge* of the graph $\boldsymbol{\Gamma}(t, 2; \theta, \alpha)$ if the *Hamming distance* $|\{e \in E_t \colon X(e) \neq Y(e)\}|$ between the ordered tuples X and Y is 1.

A vertex $Z \in \{\theta, \alpha\}^t$ of the hypercube graph $\boldsymbol{\Gamma}(t, 2; \theta, \alpha)$ has its '*positive part*' $\mathfrak{p}(Z) := Z^+$, defined to be the subset $\{e \in E_t \colon Z(e) = \theta\} \subseteq E_t$, and its '*negative part*' $\mathfrak{n}(Z) := Z^- := \{e \in E_t \colon Z(e) = \alpha\} = E_t - \mathfrak{p}(Z)$.

Recall that the *power set* $\mathbf{2}^{[t]}$ of the set E_t is defined to be the family $\{A \colon A \subseteq E_t\}$ of all subsets of E_t. The partial ordering of this family by inclusion turns the power set $\mathbf{2}^{[t]}$ into the *Boolean lattice* $\mathbb{B}(t)$ of rank t. Throughout the monograph, by convention we interpret the power set

$$\mathbf{2}^{[t]} = \big\{\mathfrak{n}(Z) \colon Z \in \{\theta, \alpha\}^t\big\}$$

of the ground set E_t as the family of the negative parts of vertices of a particular *discrete hypercube*, $\{1, -1\}^t \subset \mathbb{R}^t$, or $\{0, 1\}^t \subset \mathbb{R}^t$, associated with the two-letter alphabet $(\theta, \alpha) := (1, -1)$, or $(\theta, \alpha) := (0, 1)$, respectively, where the letters θ and α are regarded as *real numbers*.

In the monograph we consider both the *hypercube graphs*

$$\boldsymbol{H}(t, 2) := \boldsymbol{\Gamma}(t, 2; 1, -1) , \qquad \text{and} \qquad \widetilde{\boldsymbol{H}}(t, 2) := \boldsymbol{\Gamma}(t, 2; 0, 1) ,$$

though our main interest lies in the graph $\boldsymbol{H}(t, 2)$ on its vertex set $\{1, -1\}^t$.

A *symmetric cycle* $\boldsymbol{D} := (D^0, D^1, \ldots, D^{2t-1}, D^0)$ in the hypercube graph $\boldsymbol{H}(t, 2)$ is defined to be a $2t$-cycle, with its *vertex set* $\mathrm{V}(\boldsymbol{D}) := \{D^0, D^1, \ldots, D^{2t-1}\}$, such that

$$D^{k+t} = -D^k , \quad 0 \leq k \leq t - 1 .$$

The vertex set $\mathrm{V}(\boldsymbol{D})$ of the symmetric cycle $\boldsymbol{D}$ is a *maximal positive basis* of the space $\mathbb{R}^t$. For any vertex $T \in \{1, -1\}^t$ of the graph $\boldsymbol{H}(t, 2)$ there exists a *unique inclusion-minimal* and *linearly independent* subset (of *odd* cardinality) $\boldsymbol{Q}(T, \boldsymbol{D}) \subset \mathrm{V}(\boldsymbol{D})$ such that

$$T = \sum_{Q \in \boldsymbol{Q}(T, \boldsymbol{D})} Q .$$

In particular, that linear algebraic decomposition describes how the members of the family of subsets $\{\mathfrak{n}(Q)\colon Q \in \boldsymbol{Q}(T,\boldsymbol{D})\} \subset \mathbf{2}^{[t]}$ *vote* for, or against, the elements of the ground set E_t, thus arriving at their *collective decision* on the subset $\mathfrak{n}(T) \subseteq E_t$.

Informally speaking, we discuss in the monograph various aspects of how (based on the above decomposition) all the vertices of the discrete hypercube $\{1,-1\}^t$, as well as the power set of the ground set E_t, emerge from a *rank* 2 *oriented matroid*, from an underlying *rank* 2 *system of linear inequalities*, and thus literally from an *arrangement* of t distinct *straight lines* crossing a common point on a piece of paper.

In the introductory Chapter 1 we briefly recall basic properties of vertex decompositions in hypercube graphs $\boldsymbol{H}(t,2)$ with respect to their symmetric cycles. We then establish a connection between *coherent decompositions* in the hypercube graphs $\boldsymbol{H}(t,2)$ and $\widetilde{\boldsymbol{H}}(t,2)$.

In Chapter 2 we recall some enumerative results on rank 2 infeasible systems of linear inequalities related to arrangements of oriented lines in the plane. *Dehn–Sommerville type relations* are presented that concern the numbers of faces of abstract simplicial complexes associated with large-size decomposition sets for vertices of hypercube graphs.

Chapter 3 concerns additional *Dehn–Sommerville type relations* that are valid for large-size decomposition sets for vertices in hypercube graphs. We present a common *orthogonality relation* that establishes a connection between enumerative properties of large-size decomposition sets in the graphs $\boldsymbol{H}(s,2)$ and $\boldsymbol{H}(t,2)$ with specific dimensional parameters s and t.

In Chapter 4 we give a few comments on certain *distinguished* symmetric cycles in hypercube graphs. The main results of the chapter relate to the *interval* structure of the negative parts of vertices of hypercube graphs, to *computation-free decompositions* with respect to the distinguished symmetric cycles, and to statistics on decompositions. We also discuss *equinumerous decompositions* of vertices. We conclude the chapter by mentioning that vector descriptions of vertex decompositions with respect to arbitrary symmetric cycles are *valuations* on the Boolean lattices of subsets of the vertex sets of hypercube graphs.

In Chapter 5 we touch on the question on a structural connection between the decomposition sets for vertices whose negative parts are *comparable* by inclusion. Further enumerative results concern statistics on *partitions* of the negative parts of vertices of hypercube graphs and on decompositions of vertices. The key computational tool that allows us to present quite fine statistics is an approach to enumeration of *ternary Smirnov words* (i.e., words over a three-letter alphabet, such that adjacent letters in the words never coincide) discussed in Appendix A.

An even more involved analysis, based on enumeration of *Smirnov words* over *four-letter* alphabets (also discussed in Appendix A) leads us in Chapter 6 to statistics on *unions* of the negative parts of vertices of hypercube graphs and on decompositions of vertices.

An innocent-looking transformation (with serious applications given later in Chapter 8) of vertices of a discrete hypercube $\{1,-1\}^t$, that turns a vertex of the hypercube graph $\boldsymbol{H}(t,2)$ into its '*relabeled opposite*', is presented and discussed in Chapter 7.

Recall that a nonempty family of nonempty subsets $\mathcal{A} := \{A_1,\ldots,A_\alpha\} \subset \mathbf{2}^{[t]}$ of the ground set E_t is called a *clutter* (*Sperner family*), if no set from the family $\mathcal{A}$ contains another. One says that a subset $B \subseteq E_t$ is a *blocking set* of the clutter $\mathcal{A}$ if the set B has a nonempty intersection with each member A_i of $\mathcal{A}$. The *blocker* $\mathfrak{B}(\mathcal{A})$ of the clutter $\mathcal{A}$ is defined to be the family of all inclusion-minimal blocking sets of $\mathcal{A}$. In Chapter 8 we drastically change the dimensionality of our research constructs from t to 2^t, since we indirectly represent the families of blocking sets of clutters as the negative parts of relevant vertices of the hypercube graphs $\boldsymbol{H}(2^t,2)$ and $\widetilde{\boldsymbol{H}}(2^t,2)$ associated with the discrete hypercubes $\{1,-1\}^{2^t}$ and $\{0,1\}^{2^t}$, respectively. We describe in detail a '*blocking/voting*' connection between the families of blocking sets of the clutters $\mathcal{A}$ and $\mathfrak{B}(\mathcal{A})$.

In Appendix A, Smirnov words over three-letter and four-letter alphabets are enumerated.

In Appendix B we investigate the enumerative properties of the subset families generated by the *self-dual clutters* $\mathcal{A} = \mathfrak{B}(\mathcal{A})$.

May 2022

Andrey O. Matveev
Ekaterinburg

Contents

Preface v

Preliminaries and Notational Conventions 1

Chapter 1. A 2D Perspective on Higher Dimensional Discrete Hypercubes and Power Sets of Finite Sets 5
- 1.1. Symmetric Cycles in Hypercube Graphs, and Vertex Decompositions 5
- 1.2. Coherent Vertex Decompositions in Hypercube Graphs $\widetilde{\boldsymbol{H}}(t,2)$ and $\boldsymbol{H}(t,2)$ 26

Chapter 2. Vertex Decompositions in Hypercube Graphs, and Dehn–Sommerville Type Relations 29
- 2.1. Dehn–Sommerville Equations for the Feasible Subsystems of Rank 2 Infeasible Systems of Homogeneous Strict Linear Inequalities 29
- 2.2. Dehn–Sommerville Type Relations for Vertex Decompositions in Hypercube Graphs. I 32

Chapter 3. Vertex Decompositions in Hypercube Graphs, and Orthogonality Relations 37
- 3.1. Dehn–Sommerville Type Relations for Vertex Decompositions in Hypercube Graphs. II 37
- 3.2. Orthogonality Relations for Vertex Decompositions in Hypercube Graphs 39

Chapter 4. Distinguished Symmetric Cycles in Hypercube Graphs and Computation-free Vertex Decompositions 43
- 4.1. Distinguished Symmetric Cycles in Hypercube Graphs 44
- 4.2. Separation Sets, Negative Parts, and Decompositions of Vertices of Hypercube Graphs 46
- 4.3. The Sizes of Decompositions 62
- 4.4. Equinumerous Decompositions of Vertices 63
- 4.5. Decompositions, Inclusion–Exclusion and Valuations 65

Chapter 5. Distinguished Symmetric Cycles in Hypercube Graphs and Pairwise Decompositions of Vertices: Two-member Families of Disjoint Sets 67
- 5.1. Partitions of the Negative Parts of Vertices Into Two Subsets, and Decompositions of Vertices 67
- 5.2. Inclusion of the Negative Parts of Vertices, and Decompositions of Vertices 70

5.3. Statistics on Partitions of the Negative Parts of Vertices of Hypercube Graphs and on Decompositions of Vertices 72

Chapter 6. Distinguished Symmetric Cycles in Hypercube Graphs and Pairwise Decompositions of Vertices: Arbitrary Two-member Clutters 89
6.1. Statistics on Unions of the Negative Parts of Vertices of Hypercube Graphs and on Decompositions of Vertices 91

Chapter 7. Vertices, Their Relabeled Opposites, and Distinguished Symmetric Cycles in Hypercube Graphs 161
7.1. Vertices, Their Relabeled Opposites, and Decompositions 162

Chapter 8. Set Families, Blocking Sets, Blockers, and Distinguished Symmetric Cycles in Hypercube Graphs 171
8.1. Blocking Sets and Blockers 175
8.2. Increasing Families of Blocking Sets, and Blockers: Set Covering Problems 177
8.3. (Sub)set Families: Characteristic Vectors and Characteristic Covectors 179
8.4. Increasing Families of Blocking Sets, and Blockers: Characteristic Vectors and Characteristic Covectors 182
8.5. Decompositions of the Characteristic (Co)vectors of Families 194

Chapter 9. Vertex Decompositions and Subtope Decompositions in Hypercube Graphs 203
9.1. Symmetric Cycles in Hypercube Graphs: Vertices, Edges and Subtopes 204
9.2. Vertex Decompositions and Subtope Decompositions in Hypercube Graphs with Respect to the Edges of Their Symmetric Cycles 206
9.3. Vertex Decompositions in Hypercube Graphs with Respect to the Edges of Their Distinguished Symmetric Cycles 209

Appendix A. Enumeration of Smirnov Words over Three-letter and Four-letter Alphabets 213
A.1. Ternary Smirnov Words 213
A.2. Smirnov Words over a Four-letter Alphabet 215

Appendix B. The Increasing Families of Sets Generated by Self-dual Clutters 219
B.1. Long f- and h-vectors of Set Families 220
B.2. Abstract Simplicial Complexes Δ, Such That $\Delta^* = \Delta$, on Their Ground Sets E_t of Even Cardinality t 223
B.3. The Increasing Families of Sets Generated by Self-dual Clutters on Their Ground Sets E_t of Even Cardinality t 225
B.4. Remarks on Set Families $\mathcal{F}$ Such That $\mathcal{F}^* = \mathcal{F}$ 226

Bibliography 231

List of Notation 239

Index 243