

DISTINGUISHED SYMMETRIC CYCLES IN HYPERCUBE GRAPHS AND COMPUTATION-FREE VERTEX DECOMPOSITIONS

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Let t be a positive integer, $t \geq 3$. We call the set of integers $E_t := [t] := \{1, \dots, t\}$ the *ground set*.

The *vertex set* of the *hypercube graph* $\mathbf{H}(t, 2)$ by convention is the set $\{1, -1\}^t$, that is, the t -dimensional *discrete hypercube*. We regard the discrete hypercube $\{1, -1\}^t$ as a 2^t -subset of elements of the real Euclidean space \mathbb{R}^t of *row vectors* $T := (T(1), \dots, T(t))$.

The vertex $T^{(+)} := (1, \dots, 1)$ whose components are all 1's is called the *positive vertex*.

The *negative part* $\mathbf{n}(T)$ of a vertex $T \in \{1, -1\}^t$ is defined by

$$\mathbf{n}(T) := \{e \in E_t : T(e) = -1\} .$$

An unordered pair $\{T', T''\} \subset \{1, -1\}^t$ by convention is an *edge* of the hypercube graph $\mathbf{H}(t, 2)$ if and only if we have

$$|\{e \in E_t : T'(e) \neq T''(e)\}| = 1 ,$$

that is, the *Hamming distance* between the vertices T' and T'' equals 1.

We define the so-called *distinguished symmetric cycle* $\mathbf{R} := (R^0, R^1, \dots, R^{2^t-1}, R^0)$ in the graph $\mathbf{H}(t, 2)$ by

$$\begin{aligned} R^0 &:= T^{(+)} , \\ R^s &:= -_{[s]} R^0 , \quad 1 \leq s \leq t-1 , \end{aligned}$$

and

$$R^{t+k} := -R^k , \quad 0 \leq k \leq t-1 ,$$

where

$$-_{[s]} R^0 := -_{[s]} T^{(+)} := (\underbrace{-1, \dots, -1}_s, \underbrace{1, \dots, 1}_{t-s}) .$$

Since the square matrix

$$\mathbf{M} := \mathbf{M}(\mathbf{R}) := \begin{pmatrix} R^0 \\ R^1 \\ \vdots \\ R^{t-1} \end{pmatrix} \in \mathbb{R}^{t \times t} ,$$

whose rows and columns are indexed starting with 1, is *nonsingular*, the sequence $(R^0, R^1, \dots, R^{t-1})$ is an ordered *basis* of the space \mathbb{R}^t .

Given an arbitrary vertex $T \in \{1, -1\}^t$ of the graph $\mathbf{H}(t, 2)$, we thus have

$$T = \mathbf{x} \cdot \mathbf{M} ,$$

for the *unique* row ‘ \mathbf{x} -vector’

$$\mathbf{x} := \mathbf{x}(T, \mathbf{R}) := (x_1, \dots, x_t) := T \cdot \mathbf{M}^{-1} . \quad (0.1)$$

Recall that we have

$$\mathbf{x} \in \{-1, 0, 1\}^t .$$

The *decomposition set*

$$\mathbf{Q}(T, \mathbf{R}) := \{x_i \cdot R^{i-1} : x_i \neq 0\}$$

for the vertex T with respect to the cycle \mathbf{R} is the *unique inclusion-minimal* subset of the vertex set of the cycle \mathbf{R} such that

$$T = \sum_{Q \in \mathbf{Q}(T, \mathbf{R})} Q .$$

The subset $\mathbf{Q}(T, \mathbf{R}) \subset \mathbb{R}^t$, of *odd* cardinality, is *linearly independent*.

Given a subset $A \subseteq E_t$ of the ground set, we let $_{-A}\mathbf{T}^{(+)}$ denote the vertex of the discrete hypercube $\{1, -1\}^t$ such that its negative part is the set A , that is,

$$\mathbf{n}(_{-A}\mathbf{T}^{(+)}) := A .$$

In order to avoid the basic *linear algebraic* technique (0.1), we use the following ‘computation-free’ approach to finding the vectors $\mathbf{x}(_{-A}\mathbf{T}^{(+)}, \mathbf{R})$ and the decomposition sets $\mathbf{Q}(_{-A}\mathbf{T}, \mathbf{R})$, where $\sigma(e)$ denotes the e th standard unit vector of the space \mathbb{R}^t :

Proposition [1, Prop. 4.9]: *Let \mathbf{R} be the distinguished symmetric cycle in the hypercube graph $\mathbf{H}(t, 2)$.*

Let A be a nonempty subset of the ground set E_t , and let

$$A = [i_1, j_1] \dot{\cup} [i_2, j_2] \dot{\cup} \dots \dot{\cup} [i_\varrho, j_\varrho]$$

be its partition into inclusion-maximal intervals such that

$$j_1 + 2 \leq i_2, \quad j_2 + 2 \leq i_3, \quad \dots, \quad j_{\varrho-1} + 2 \leq i_\varrho ,$$

for some $\varrho := \varrho(A)$.

(i) *If $\{1, t\} \cap A = \{1\}$, then*

$$\begin{aligned} |\mathbf{Q}(_{-A}\mathbf{T}^{(+)}, \mathbf{R})| &= 2\varrho - 1 , \\ \mathbf{x}(_{-A}\mathbf{T}^{(+)}, \mathbf{R}) &= \sum_{1 \leq k \leq \varrho} \sigma(j_k + 1) - \sum_{2 \leq \ell \leq \varrho} \sigma(i_\ell) . \end{aligned}$$

(ii) *If $\{1, t\} \cap A = \{1, t\}$, then*

$$\begin{aligned} |\mathbf{Q}(_{-A}\mathbf{T}^{(+)}, \mathbf{R})| &= 2\varrho - 1 , \\ \mathbf{x}(_{-A}\mathbf{T}^{(+)}, \mathbf{R}) &= -\sigma(1) + \sum_{1 \leq k \leq \varrho-1} \sigma(j_k + 1) - \sum_{2 \leq \ell \leq \varrho} \sigma(i_\ell) . \end{aligned}$$

(iii) If $|\{1, t\} \cap A| = 0$, then

$$|Q(-_A T^{(+)}, \mathbf{R})| = 2\varrho + 1 ,$$

$$\mathbf{x}_{(-_A T^{(+)}, \mathbf{R})} = \sigma(1) + \sum_{1 \leq k \leq \varrho} \sigma(j_k + 1) - \sum_{1 \leq \ell \leq \varrho} \sigma(i_\ell) .$$

(iv) If $\{1, t\} \cap A = \{t\}$, then

$$|Q(-_A T^{(+)}, \mathbf{R})| = 2\varrho - 1 ,$$

$$\mathbf{x}_{(-_A T^{(+)}, \mathbf{R})} = \sum_{1 \leq k \leq \varrho-1} \sigma(j_k + 1) - \sum_{1 \leq \ell \leq \varrho} \sigma(i_\ell) .$$

REFERENCES

- [1] *Matveev A.O. Symmetric cycles: A 2D perspective on higher dimensional discrete hypercubes, the power sets of finite sets, and set families.* Leanpub: 2022.