DISTINGUISHED SYMMETRIC CYCLES IN HYPERCUBE GRAPHS AND COMPUTATION-FREE VERTEX DECOMPOSITIONS

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Let t be a positive integer, $t \geq 3$. We call the set of integers $E_t := [t] := \{1, \ldots, t\}$ the *ground set*.

The *vertex set* of the *hypercube graph* $\boldsymbol{H}(t,2)$ by convention is the set $\{1,-1\}^t$, that is, the t-dimensional *discrete hypercube*. We regard the discrete hypercube $\{1,-1\}^t$ as a 2^t-subset of elements of the real Euclidean space $\mathbb{R}^t$ of *row* vectors $T := (T(1), \ldots, T(t))$.

The vertex $\mathrm{T}^{(+)} := (1, \ldots, 1)$ whose components are all 1's is called the *positive vertex*.

The *negative part* $\mathfrak{n}(T)$ of a vertex $T \in \{1,-1\}^t$ is defined by

$$\mathfrak{n}(T) := \{e \in E_t \colon T(e) = -1\} \ .$$

An unordered pair $\{T', T''\} \subset \{1,-1\}^t$ by convention is an *edge* of the hypercube graph $\boldsymbol{H}(t,2)$ if and only if we have

$$|\{e \in E_t \colon T'(e) \neq T''(e)\}| = 1 \ ,$$

that is, the *Hamming distance* between the vertices T' and T'' equals 1.

We define the so-called *distinguished symmetric cycle* $\boldsymbol{R} := (R^0, R^1, \ldots, R^{2t-1}, R^0)$ in the graph $\boldsymbol{H}(t,2)$ by

$$\begin{aligned} R^0 &:= \mathrm{T}^{(+)} \ , \\ R^s &:= {}_{-[s]}R^0 \ , \quad 1 \leq s \leq t-1 \ , \end{aligned}$$

and

$$R^{t+k} := -R^k \ , \quad 0 \leq k \leq t-1 \ ,$$

where

$${}_{-[s]}R^0 := {}_{-[s]}\mathrm{T}^{(+)} := (\underbrace{-1, \ldots, -1}_{s}, \underbrace{1, \ldots, 1}_{t-s}) \ .$$

Since the square matrix

$$\mathbf{M} := \mathbf{M}(\boldsymbol{R}) := \begin{pmatrix} R^0 \\ R^1 \\ \vdots \\ R^{t-1} \end{pmatrix} \in \mathbb{R}^{t \times t} \ ,$$

whose rows and columns are indexed starting with 1, is *nonsingular*, the sequence $(R^0, R^1, \ldots, R^{t-1})$ is an ordered *basis* of the space $\mathbb{R}^t$.

Given an arbitrary vertex $T \in \{1,-1\}^t$ of the graph $\boldsymbol{H}(t,2)$, we thus have

$$T = \boldsymbol{x} \cdot \mathbf{M} \ ,$$

for the *unique* row '$\boldsymbol{x}$-vector'

$$\boldsymbol{x} := \boldsymbol{x}(T,\boldsymbol{R}) := (x_1,\ldots,x_t) := T \cdot \mathbf{M}^{-1} \ . \tag{0.1}$$

Recall that we have

$$\boldsymbol{x} \in \{-1,0,1\}^t \ .$$

The *decomposition set*

$$\boldsymbol{Q}(T,\boldsymbol{R}) := \{x_i \cdot R^{i-1} \colon x_i \neq 0\}$$

for the vertex T with respect to the cycle $\boldsymbol{R}$ is the *unique inclusion-minimal* subset of the vertex set of the cycle $\boldsymbol{R}$ such that

$$T = \sum_{Q \in \boldsymbol{Q}(T,\boldsymbol{R})} Q \ .$$

The subset $\boldsymbol{Q}(T,\boldsymbol{R}) \subset \mathbb{R}^t$, of *odd* cardinality, is *linearly independent.*

Given a subset $A \subseteq E_t$ of the ground set, we let ${}_{-A}\mathrm{T}^{(+)}$ denote the vertex of the discrete hypercube $\{1,-1\}^t$ such that its negative part is the set A, that is,

$$\mathfrak{n}({}_{-A}\mathrm{T}^{(+)}) := A \ .$$

In order to avoid the basic *linear algebraic* technique (0.1), we use the following 'computation-free' approach to finding the vectors $\boldsymbol{x}({}_{-A}\mathrm{T}^{(+)},\boldsymbol{R})$ and the decomposition sets $\boldsymbol{Q}({}_{-A}\mathrm{T},\boldsymbol{R})$, where $\boldsymbol{\sigma}(e)$ denotes the eth standard unit vector of the space $\mathbb{R}^t$:

Proposition [1, Prop. 4.9]: *Let $\boldsymbol{R}$ be the distinguished symmetric cycle in the hypercube graph $\boldsymbol{H}(t,2)$.*

Let A be a nonempty *subset of the ground set E_t, and let*

$$A = [i_1,j_1] \,\dot\cup\, [i_2,j_2] \,\dot\cup\, \cdots \,\dot\cup\, [i_\varrho,j_\varrho]$$

be its partition into inclusion-maximal intervals such that

$$j_1+2 \le i_2, \quad j_2+2 \le i_3, \quad \ldots, \quad j_{\varrho-1}+2 \le i_\varrho \ ,$$

for some $\varrho := \varrho(A)$.

(i) *If $\{1,t\} \cap A = \{1\}$, then*

$$|\boldsymbol{Q}({}_{-A}\mathrm{T}^{(+)},\boldsymbol{R})| = 2\varrho - 1 \ ,$$
$$\boldsymbol{x}({}_{-A}\mathrm{T}^{(+)},\boldsymbol{R}) = \sum_{1 \le k \le \varrho} \boldsymbol{\sigma}(j_k+1) - \sum_{2 \le \ell \le \varrho} \boldsymbol{\sigma}(i_\ell) \ .$$

(ii) *If $\{1,t\} \cap A = \{1,t\}$, then*

$$|\boldsymbol{Q}({}_{-A}\mathrm{T}^{(+)},\boldsymbol{R})| = 2\varrho - 1 \ ,$$
$$\boldsymbol{x}({}_{-A}\mathrm{T}^{(+)},\boldsymbol{R}) = -\boldsymbol{\sigma}(1) + \sum_{1 \le k \le \varrho-1} \boldsymbol{\sigma}(j_k+1) - \sum_{2 \le \ell \le \varrho} \boldsymbol{\sigma}(i_\ell) \ .$$

(iii) *If* $|\{1,t\}\cap A| = 0$, *then*

$$|\boldsymbol{Q}({}_{-A}\mathrm{T}^{(+)},\boldsymbol{R})| = 2\varrho\ +1\ ,$$

$$\boldsymbol{x}({}_{-A}\mathrm{T}^{(+)},\boldsymbol{R}) = \boldsymbol{\sigma}(1)\ +\sum_{1\leq k\leq\varrho}\boldsymbol{\sigma}(j_k+1)\ -\sum_{1\leq\ell\leq\varrho}\boldsymbol{\sigma}(i_\ell)\ .$$

(iv) *If* $\{1,t\}\cap A = \{t\}$, *then*

$$|\boldsymbol{Q}({}_{-A}\mathrm{T}^{(+)},\boldsymbol{R})| = 2\varrho-1\ ,$$

$$\boldsymbol{x}({}_{-A}\mathrm{T}^{(+)},\boldsymbol{R}) = \sum_{1\leq k\leq\varrho-1}\boldsymbol{\sigma}(j_k+1)-\sum_{1\leq\ell\leq\varrho}\boldsymbol{\sigma}(i_\ell)\ .$$

References

[1] *Matveev A.O.* Symmetric cycles, Jenny Stanford Publishing, 2023.