# CSC484 Homework #3

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## Exercise 1.1

$\Omega = \{-n, -(n-1), ..., -1, 0, 1, ...n-1, n\}$.
Transition probabilities:
$P(x,x) = \dfrac{1}{2}, \forall x \in \Omega$.
$P(-n, -(n-1)) = P(n, n-1) = \dfrac{1}{2}$
$P(0,1) = P(0,-1) = \dfrac{1}{4}$
$\forall x \in \{1, ..., n-1\}$ :

$$P(-x, -(x+1)) = P(x, x+1) = \frac{1}{3}$$
$$P(-x, -(x-1)) = P(x, x-1) = \frac{1}{6}$$

Transition matrix:

$$T = \begin{bmatrix}
1/2 & 1/2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
1/3 & 1/2 & 1/6 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 1/3 & 1/2 & 1/6 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 1/3 & 1/2 & 1/6 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\hdotsfor{14} \\
0 & 0 & 0 & 0 & 0 & \dots & 1/3 & 1/2 & 1/6 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 1/4 & 1/2 & 1/4 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1/6 & 1/2 & 1/3 & \dots & 0 & 0 \\
\hdotsfor{14} \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 1/3 & 0 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 1/2 & 1/3 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 1/2 & 1/2
\end{bmatrix}$$

Let's find stationary distribution $\pi$ for this MC: $\pi T = \pi$.
$\dfrac{1}{2}\pi(-n) + \dfrac{1}{3}\pi(-n+1) = \pi(-n) \Rightarrow \pi(-n+1) = \dfrac{3}{2}\pi(-n)$.

$\frac{1}{2}\pi(-n) + \frac{1}{2}\pi(-n+1) + \frac{1}{3}\pi(-n+2) = \pi(-n+1) \Rightarrow \pi(-n+2) = \frac{3}{2}\pi(-n+1) - \frac{3}{2}\pi(-n) = \frac{3}{4}\pi(-n)$.

And then for every $k \in \{2, ..., n-1\}$ we have that:

$$\frac{1}{6}\pi(-n+k-1) + \frac{1}{2}\pi(-n+k) + \frac{1}{3}\pi(-n+k+1) = \pi(-n+k) \Rightarrow$$
$$\Rightarrow \pi(-n+k+1) = \frac{3}{2}\pi(-n+k) - \frac{1}{2}\pi(-n+k-1).$$

It can be easily solved using initial conditions for $\pi(-n+1), \pi(-n+2)$ stated above($\pi(n)$ we treat as a constant), and then we have the following: $\forall k \in \{2, ..., n-1\} : \pi(-n+k) = \frac{3\pi(-n)}{2^k}$.
$\frac{1}{2}\pi(n) + \frac{1}{3}\pi(n-1) = \pi(n) \Rightarrow \pi(n-1) = \frac{3}{2}\pi(n)$.
$\frac{1}{2}\pi(n) + \frac{1}{2}\pi(n-1) + \frac{1}{3}\pi(n-2) = \pi(n-1) \Rightarrow \pi(n-2) = \frac{3}{2}\pi(n-1) - \frac{3}{2}\pi(n) = \frac{3}{4}\pi(n)$.

And then for every $k \in \{1, ..., n-2\}$ we have that:

$$\frac{1}{6}\pi(n-k-1) + \frac{1}{2}\pi(n-k) + \frac{1}{3}\pi(n-k+1) = \pi(n-k) \Rightarrow$$
$$\Rightarrow \pi(n-k+1) = \frac{3}{2}\pi(n-k) - \frac{1}{2}\pi(n-k-1).$$

It can be easily solved using initial conditions for $\pi(n-1), \pi(n-2)$ stated above, and then we have the following: $\forall k \in \{1, ..., n-2\} : \pi(n-k) = \frac{3\pi(n)}{2^k}$. It‘s quite obvious that would get same result here.

Currently the only state where we have not defined $\pi$ is 0. In this case we have $\frac{1}{6}\pi(-2) + \frac{1}{2}\pi(-1) + \frac{1}{4}\pi(0) = \pi(-1)$, for example as equation for the undefined state. Then $\pi(0) = 2^{2-n}\pi(-n)$. And this value would agree if $\pi(n) = \pi(-n)$ with solution for $\pi(0)$ derived from equation of $\pi(2), \pi(1)$. Let‘s define $\pi(-n) = \pi(n) = c$, where $c$ is some constant piked that probabilities sum up to one. We do not care about it‘s value, really.

We know that conductance for chain is defined in the following way $\Phi = \min_{S \subset \Omega: \pi(S) \leq 1/2} \frac{Q(S, \overline{S})}{\pi(S)}$, where $Q(S, \overline{S}) = \sum_{(x,y) \in S \times \overline{S}} \pi(x)T(x,y)$. Let‘s denote by $\Phi_S := \frac{Q(S, \overline{S})}{\pi(S)}$ for some $S \subset \Omega$. Obviously, we have $\Phi \leq \Phi_S$.

Now consider $S = \{-n, -n+1, ..., -1\}$.
$\pi(S) = c\left(1 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + ...\right) = c(1 + 3 - 3 \cdot 2^{-n+1}) = c\frac{4 \cdot 2^{n-1} - 3}{2^{n-1}}$.
$Q(S, \overline{S}) = \pi(-1) \cdot T(-1, 0) = \frac{3c}{6 \cdot 2^{n-1}} = \frac{c}{2^n}$, as there is only one pair $(x, y) \in S \times \overline{S}$, such that $T(x, y) \neq 0$.
Then we have $\Phi_S = \frac{c}{2^n} \frac{2^{n-1}}{c(2^{n+1} - 3)} = \frac{1}{2} \frac{1}{2^{n+1} - 3}$.

From theory we have the following estimation for second largest eigenvalue $\lambda_2$ of transition matrix $T$:

$$1 - 2\Phi \leq \lambda_2 \leq 1 - \frac{\Phi^2}{2}.$$

From this we have: $1 - \lambda_2 \leq 2\Phi \leq 2\Phi_S = \frac{1}{2^{n+1} - 3}$.

Also know lower bound for mixing time is $\tau(\epsilon) \geq \frac{1}{2} \frac{\lambda_2}{1 - \lambda_2} \log(2\epsilon)^{-1}$.

We can now have the following:

$$\tau(\epsilon) \;\geq\; \frac{1}{2} \frac{\lambda_2}{1 - \lambda_2} \log(2\epsilon)^{-1} \;\geq\; \frac{1}{2} \frac{1}{2} (2^{n+1} - 3) \frac{1}{\log 2\epsilon} \;=\; (2^{n+1} - 3) \frac{1}{4 \log 2\epsilon}$$

Note: in the upper part of fraction we used obvious inequality(in this case) $\lambda_2 \geq 1/2$, which is not the best possible estimation, but it does not changes the situation really.

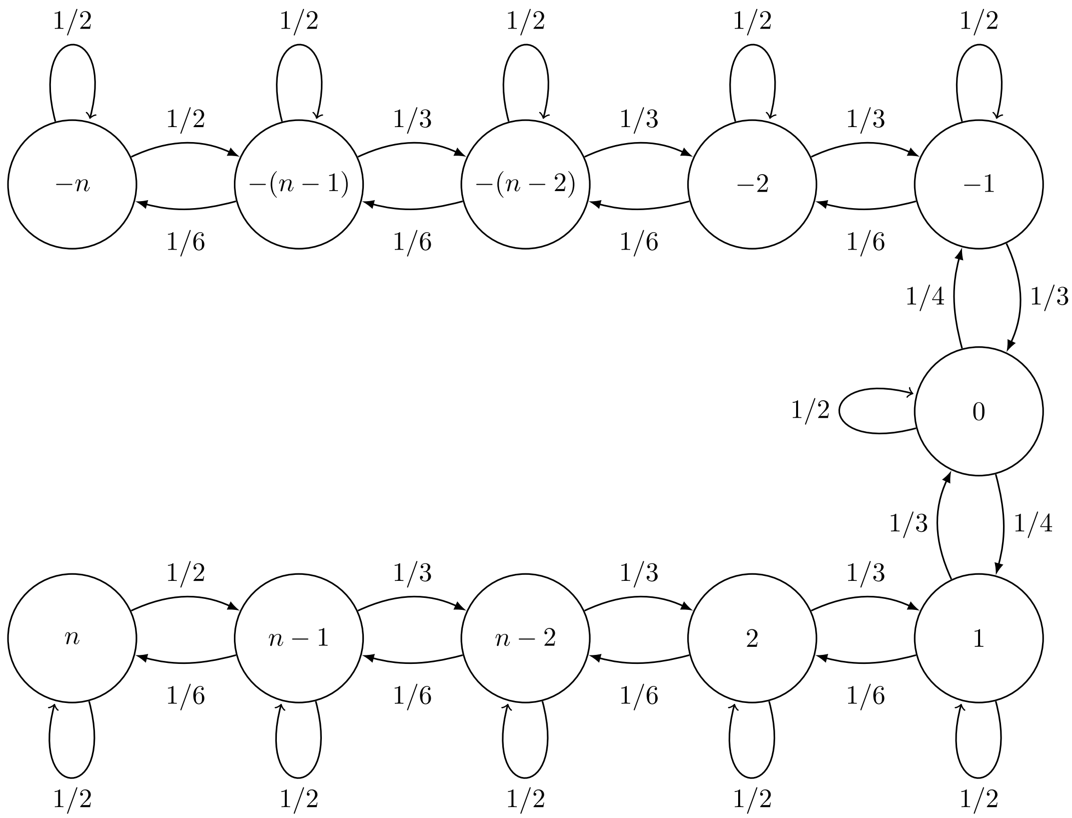
Conclusion: mixing time for this MC bounded from below by exponential function, therefore it mixes very very very slow.

## Exercise 1.2

$\Omega = \{-n, -(n-1), ..., -1, 0, 1, ...n-1, n\}$.
Transition probabilities:
$P(x, x) = \frac{1}{2}, \forall x \in \Omega$.
$P(-n, -(n-1)) = P(n, n-1) = \frac{1}{2}$
$P(0, 1) = P(0, -1) = \frac{1}{4}$
$\forall x \in \{1, ..., n-1\}$ :

$$P(-x, -(x+1)) = P(x, x+1) = \frac{1}{6}$$
$$P(-x, -(x-1)) = P(x, x-1) = \frac{1}{3}$$

Chain states graph:




Now consider coupling $(X, Y)$, where $X$ and $Y$ is same chain. Lets look at undirected graph $(\Omega, S)$, where $S \subseteq \Omega \times \Omega$, such that $(a, b) \in \Omega \times \Omega$ is in $S$ if for given MC it is possible that for some $t$ $X_t = a$ and $X_{t+1} = b$. It's exactly graph drawn above. Now we want to prove that there exists $\beta < 1$ such that $\forall (X_t, Y_t) \in S : E[dist(X_{t+1}, Y_{t+1})|X_t, Y_t] \leq \beta dist(X_t, Y_t)$, where $dist(a, b)$ is simple distance in the graph. We will solve this inequality for the following cases(everywhere assume $X_t \neq Y_t$):

- $X_t, Y_t \in \{-(n-1), -(n-2), ..., -1, 1, ..., n-2, n-2\}$ and $X_tY_t < 0$. In other words both initial states are not endpoints($n$ and $-n$) and are not zero and lie on different sides from 0. Assume $d = dist(X_t, Y_t)$ and $d' = dist(X_{t+1}, Y_{t+1})$. In this case we have $d' \in \{d-2, d-1, 0, d+1, d+2\}$. Let's compute probability for each case:
  $P(d' = d) = \frac{1}{2}\frac{1}{2} + \frac{1}{3}\frac{1}{6} + \frac{1}{6}\frac{1}{6} = \frac{12}{36}$, which corresponds to the following possibilities of next step $(X_{t+1}, Y_{t+1})$: both stay on same state, both go to the left(state + 1), both go to the right(state -1).
  $P(d' = d + 1) = \frac{1}{2}\frac{1}{6} + \frac{1}{6}\frac{1}{2} = \frac{6}{36}$: one stays, other goes one step further from other point and opposite.

$P(d' = d+2) = \frac{1}{6}\frac{1}{6} = \frac{1}{36}$: both go opposite directions(in the direction of the endpoints).
$P(d' = d-1) = \frac{1}{2}\frac{1}{3} + \frac{1}{3}\frac{1}{2} = \frac{12}{36}$: one remains on place, the other goes towards 0(as they are on the opposite sides this causes distance decrease by 2).
$P(d' = d-2) = \frac{1}{3}\frac{1}{3} = \frac{4}{36}$: both go towards 0.
Now we can calculate $E[d'|d] = \frac{12d + 6d + 6 + d + 2 + 12d - 12 + 4d - 8}{36} = d - \frac{1}{3}$.
Also, as both initial states have deifferent sign, we can easily see that $d \geq 2$. Then we want $d - \frac{1}{3} \geq \beta d$ to hold, therefore $\beta \geq 1\frac{1}{6}$ and $\beta = 1 - \frac{1}{6}$ satisfies this case.
$P(d' \neq d|d) = \frac{2}{3}$.

- $X_t, Y_t \in \{-(n-1), -(n-2), ..., -1, 1, ..., n-2, n-2\}$ and $X_tY_t > 0$. In other words both initial states are not endpoints($n$ and $-n$) and are not zero and lie on same side from 0. Assume $d = dist(X_t, Y_t)$ and $d' = dist(X_{t+1}, Y_{t+1})$. In this case we have $d' \in \{d-2, d-1, 0, d+1, d+2\}$. Let's compute probability for each case:
$P(d' = d) = \frac{1}{2}\frac{1}{2} + \frac{1}{3}\frac{1}{3} + \frac{1}{6}\frac{1}{6} = \frac{14}{36}$: either both stay on same states or move in same direction.
$P(d' = d+1) = \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{6} = \frac{9}{36}$: one remains in same state, other moves further.
$P(d' = d+2) = \frac{1}{3}\frac{1}{6} = \frac{2}{36}$: both move in opposite directions.
$P(d' = d-1) = \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{6} = \frac{9}{36}$: one remains in same state, other moves closer.
$P(d' = d-2) = \frac{1}{3}\frac{1/6}{=}\frac{2}{36}$: both move in opposite directions.
$E[d'|d] = \frac{14d + 9d + 9 + 2d + 4 + 9d - 9 + 2d - 4}{36} = d$, which means inequality holds with $\beta = 1$.
$P(d' \neq d|d) = \frac{22}{36}$.

- $X_t = -n, Y_t \in \{-(n-1), -(n-2), ..., -1\}$: endpoint and some other state on same side from 0. Really does not matter which of them is endpoint. Assume $d = dist(X_t, Y_t)$ and $d' = dist(X_{t+1}, Y_{t+1})$. In this case we have $d' \in \{d-2, d-1, 0, d+1\}$. Let's compute probability for each case:
$P(d' = d) = \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{3} = \frac{15}{36}$: both remain on same states, or both move in the direction of 0.
$P(d' = d+1) = \frac{1}{2}\frac{1}{3} = \frac{6}{36}$: endpoint remains, other moves towards 0.

$P(d' = d-1) = \frac{1}{2}\frac{1}{6} + \frac{1}{2}\frac{1}{2} = \frac{12}{36}$: endpoint remains, other moves towards it, endpoint mover towards 0, other remains in same state.
$P(d' = d-2) = \frac{1}{2}\frac{1}{6} = \frac{3}{36}$: both move towards each other.
$E[d'|d] = d - \frac{1}{3}$ and therefore goal inequality holds with $\beta = 1 - \frac{1}{3}$.
$P(d' \neq d|d) = \frac{21}{36}$.

- $X_t = n, Y_t \in \{1, 2, ..., n\}$. Same case as before.
- $X_t = -n, Y_t = n$: two endpoints. $d' \in \{d, d-1, d-2\}$. Obviously, $E[d'|d] \leq d$ and $\beta = 1$ is good in this case.
  $P(d' \neq d|d) = \frac{3}{4}$.
- $X_t = -n, Y_t = 0$: endpoint and 0. $d' \in \{d, d-1, d-2, d+1\}$.
  $P(d' = d) = \frac{1}{2}\frac{1}{2} + \frac{1/2}{}$

  $14 = \frac{3}{8}$: both remain in same states, or both move same direction.
  $P(d' = d+1) = \frac{1}{2}\frac{1}{4} = \frac{1}{8}$: endpoint remains, zero moves.
  $P(d' = d-1) = \frac{1}{2}\frac{1}{4} + \frac{1}{2}\frac{1}{2} = \frac{3}{8}$: endpoint remains and zero mover towards it or zero remains and endpoint moves towards it.
  $P(d' = d-2) = \frac{1}{2}\frac{1}{4} = \frac{1}{8}$: both move towards each other.
  $E[d'|d] = d - \frac{1}{2}$. Easy to see that $\beta = 1$ satisfies this case.
  $P(d' \neq d|d) = \frac{5}{8}$.

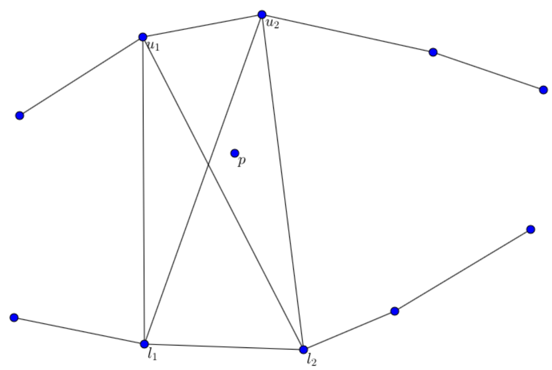
As we can change $X$ nd $Y$ places, we have already covered all possible cases and have now that for all of them $E[d'|d] \leq \beta d$ holds for $\beta = 1$. And also we have that $P(d' \neq d) \geq \frac{21}{36} =: \alpha$.
Now we can use **Lemma 4.6 (Bubley and Dyer [1])** from lecture notes. $T_{mix}(\epsilon) \leq \lceil \frac{ed_{max}^2}{\alpha} \rceil \lceil \log \frac{1}{\epsilon} \rceil$, where $d_{max}$ is diameter of the graph and we have $d_{max} = 2n$, obviously. Therefore we get:

$$T_{mix}(\epsilon) \leq \lceil \frac{144 \cdot e \cdot n^2}{21} \rceil \lceil \log \frac{1}{\epsilon} \rceil.$$

# Exercise 1.3

$S = \{(x_1, y_1), ..., (x_n, y_n)\}$.
Geometric understanding of the problem: if point is inside a triangle with vertexes from $S$ then it is inside a convex hull of $S$. Then we can simplify problem

a bit: if it is inside convex hull then we will try to find triangle with vertexes from convex hull of $S$, not from the all set $S$.
Preprocessing: build upper $U$ and lower $L$ parts of convex hull for set $S$. In both these arrays we have our points sorted.
Preprocessing time complexity: $O(n \log n)$.




*Notation:* by $x(p)$ we will denote coordinate $x$ of point $p$.
Query answering: given point $p = (p_x, p_y)$ we do the following:

1. Using binary search find two consecutive points $u_1, u_2 \in U$ such that $x(u_1) \leq x(p) \leq x(u_2)$.
   **Note:** if we can not find pair of consecutive point to satisfy this condition then the point $p$ lies to the left of all points from set $S$ or to the right and therefore is not inside the convex hull. In such case return that no triangle exists.

2. Using binary search find two consecutive points $l_1, l_2 \in L$ such that $x(l_1) \leq x(p) \leq x(l_2)$.

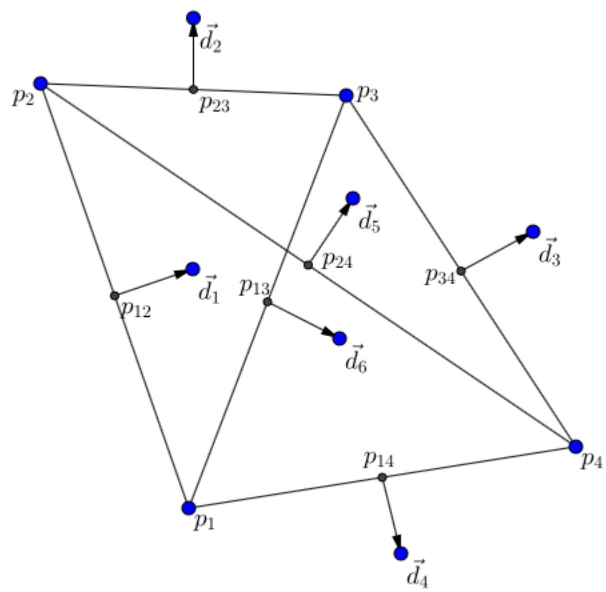
3. Check if point is inside convex hull: it must be to the right from line through points $u_1, u_2$ with direction $\vec{u_2} - \vec{u_1}$;
   and to the left from line through points $l_1, l_2$ with direction $\vec{l_2} - \vec{l_1}$.
   **Note:**If point is not inside convex hull, in other words, this statement does not hold then report that no such triangle exists.

4. If previous statement holds then point is inside $u_1u_2l_2l_1$. We can write $u_1u_2l_2l_1 = l_1u_1l_2 \cup u_1l_2u_2$ - this figure is just union of two triangles, therefore point $p$ must be in one of them.
   Check if $p$ is to the right from the line through $u_1, l_2$ with direction $\vec{l_2} - \vec{u_1}$.
   If this condition holds then output $u_1, l_1, u_2$, otherwise output $l_1, v_1, l_2$.

Let denote by $h$ size of a convex hull of a set $S$. Then we have the following time complexity for query answering: $O(\log h)$ to find $u_1, u_2, l_1, l_2$ and $O(1)$ for all the rest operations. Therefore we have $O(\log h)$, or in worst case $O(\log n)$.

## Exercise 2.1

Given points $P = \{p_1, ..., p_n\}$, where $p_i = (x_i, y_i)$.
We will say that line with direction $(a, b)$ is line which is collinear with vector $\vec{v} = (a, b)$. Easy to see that if we want to write an equation of some line with direction $(a, b)$ we can just take $bx - ay + c = 0$, where $c$ is any number. And easy to see that line with direction $\vec{v}$ and line with direction $-\vec{v}$ are collinear and therefore when we talk about line directions we assume $\vec{v}$ and $-\vec{v}$ as equal. Same works for $\alpha\vec{v}$ for any $\alpha \in \mathbb{R} - \{0\}$.
We want pairs of points be reflections of each other therefore line $l$ goes through midpoint of a segment between the point in the pair and it is perpendicular to that segment. Therefore line $l$ must have same direction as bisection of a segment.




Algorithm:

1. Compute direction $\vec{d_{mk}}$ of a bisection of segment $p_m p_k$ for each possible distinct pairs $p_m, p_k \in P, m \neq k$ and store this result in array $D$ in the form $(\vec{d_{mk}}, p_m, p_k)$. To avoid some chaos in array we want first coordinate of $d_{mk}$ non-negative and norm of $\vec{d_{mk}}$ be equal 1. As was mentioned before, multiplying direction by non zero constant does not change the direction in the meaning we are interested in it. Therefore we can easily

adjust this vector.

Formula for $\vec{d_{mk}}$: $\vec{d_{mk}} = \dfrac{sign(y_m - y_k)}{\sqrt{(y_m - y_k)^2 + (x_k - x_m)^2}}(y_m - y_k, x_k - x_m)$.

Time complexity: $O(n^2)$.

2. Next step is to sort array $D$. Each element in $D$ has 3 components, we would sort by the first component - direction vector. More precisely we would sort by second coordinate of the vector only. As with our requirements for $\vec{d_{mk}}$ all the vectors have angle from $-\pi/2$ to $\pi/2$ with the x axe and their ends - just point on a half-circle with radius 1 and center in (0,0).
   Motivation to sort: we assume that there may be same direction vectors for some pairs of points and want such pairs to be consecutive in the array $D$.
   Time complexity: $O(n^2 \log n^2) = O(2n^2 \log n) = O(n^2 \log n)$, as size of $D$ is $O(n^2)$.

3. Now we loop through already sorted array $D$. Let $T$ be some temporary array, $\vec{d}_{current}$ direction vector of current element in $D$ we are looking at and $\vec{d}_{prev}$ direction of element we looked previously. Algorithm is the following:

   - if $\vec{d}_{current} = \vec{d}_{prev}$ then compute $x_{midpoint}, y_{midpoint}$ for points of current entry in $D$ and add to array $T$ value $\vec{d}_{current}(2)x_{midpoint} - \vec{d}_{current}(1)y_{midpoint}$.
     *Note:* $\vec{d}(i)$ means $i^{th}$ coordinate of vector $\vec{d}$.
     Motivation: we know that line with this direction has form $\vec{d}_{current}(2)x - \vec{d}_{current}(1)y + c = 0$, where c is some constant. If we want this line to pass through current midpoint it should have $c = -(\vec{d}_{current}(2)x_{midpoint} - \vec{d}_{current}(1)y_{midpoint})$.
   - Otherwise, we have array $T$ filled with values of coefficient $c$ for line, multiplied by $-1$. If some value appears in array $T$ $k$ times then there are $k$ pairs that are reflections if $l$ is line with this value as third coefficient and other two coefficient defined from current direction.
     We have to find the most popular value in array $T$, let's denote it by $max_d$. This takes $O(r_d)$ times, where $r_d$ is number of pairs of points that have $\vec{d}$ as direction for bisection of segment formed by pair of points.
     After we found $max_d$ we set $T = \emptyset$ and fill it in same way as mentioned in the first step.

   Number of times we would look for the most popular value in $T$ is number of distinct directions from all possible pairs of points. We have that each pair have one direction, so $\sum\limits_d r_d \leq n^2$. Time complexity for part stated above is: $\sum\limits_d O(r_d) = O(\sum\limits_d r_d) = O(n^2)$. After we have $max_d$ for every direction $d$ we output maximum of them, which takes $O(n^2)$.

Therefore total runtime for the algorithm is $O(n^2 \log n + n^2) = O(n^2 \log n)$.