On the price of explainability for some clustering problems

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Abstract

Machine learning models and algorithms are used in a number of systems that affect our daily life. Thus, in some settings, methods that are easy to explain or interpret may be highly desirable. The price of explainability can be thought of as the loss in terms of quality that is unavoidable if we restrict these systems to use explainable methods.

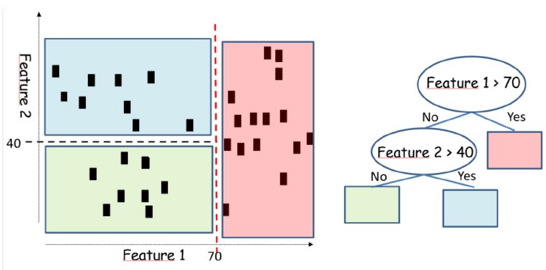
We study the price of explainability, under a theoretical perspective, for clustering tasks. We provide upper and lower bounds on this price as well as efficient algorithms to build explainable clustering for the k-means, k-medians, k-center and the maximum-spacing problems in a natural model in which explainability is achieved via decision trees.

1 Introduction

Machine learning models and algorithms have been used in a number of systems that take decisions that affect our lives. Thus, explainable methods are desirable so that people are able to have a better understanding about their behavior, which allows a more comfortable use of these systems or, eventually, the questioning of their applicability.

Although most of the work on the field of explainable machine learning has been focused on supervised learning [29, 23, 32], there has recently been some effort to devise explainable methods for unsupervised learning tasks, in particular for clustering [12, 6]. We investigate the framework discussed by [12], where an explainable clustering is obtained through the search of a partition that optimizes a given criterion, among the partitions that can be induced by the leaves of a decision tree (with one leaf per cluster).

Figure 1 shows a clustering with 3 groups induced by a decision tree with 3 leaves. As an example, the blue cluster can be explained as the set of points that satisfy `Feature 1` ≤ 70 and `Feature 2` > 40. Simple explanations as this one are usually not available for the partitions produced by popular methods such as k-means.



In order to achieve explainability, one may be forced to accept some loss in terms of the quality of the chosen criterion/objective function (e.g. sum of squared distances). In this sense, explainability has its price. [12] presents theoretical bounds on this price for the k-median and the k-means objective functions.

Here, we expand on their work by presenting new bounds for these objectives and also providing nearly tight bounds for two other goals that arise in relevant clustering problems, namely, the k-center and the maximum-spacing problems.

1.1 Problem definition

Let $\mathcal{X}$ be a set of n points in $\mathbb{R}^d$. We say that a decision tree is *standard* if each internal node v is associated with a test, specified by a coordinate $i_v \in [d]$ and a real value θ_v, that partitions the points in $\mathcal{X}$ that reach v into two sets: those having the coordinate i_v smaller than or equal to θ_v and those having it larger than θ_v. The leaves of a standard decision tree induce a partition of $\mathbb{R}^d$ into axis-aligned boxes and, naturally, a partition of $\mathcal{X}$ into clusters.

Let $k \geq 2$ be an integer. The clustering problems considered here consist of finding a partition of $\mathcal{X}$ into k groups that optimizes a given criterion, among the partitions that can be induced by a standard decision tree with k leaves. For k-means, k-medians and k-centers, in addition to the partition, a representative $\mu(C) \in \mathbb{R}^d$ for each group C must also be output.

For the k-means problem the criterion to be minimized is the Sum of the Squared Euclidean Distances (SSED) between each point $\mathbf{x} \in \mathcal{X}$ and the representative of the cluster where $\mathbf{x}$ lies. Mathematically, the cost (SSED) of a partition $\mathcal{C} = (C_1, \ldots, C_k)$ for $\mathcal{X}$ is given by

$$cost(\mathcal{C}) = \sum_{i=1}^{k} \sum_{\mathbf{x} \in C_i} ||\mathbf{x} - \mu(C_i)||_2^2.$$

The k-medians and the k-centers problems are also minimization problems. For the former, the cost of a partition $\mathcal{C} = (C_1, \ldots, C_k)$ is given by

$$cost(\mathcal{C}) = \sum_{i=1}^{k} \sum_{\mathbf{x} \in C_i} ||\mathbf{x} - \mu(C_i)||_1,$$

while for the latter it is given by

$$cost(\mathcal{C}) = \max_{i=1,\ldots,k} \max_{\mathbf{x} \in C_i} \{||\mathbf{x} - \mu(C_i)||_2\}.$$

All these objective functions are classified as intra-clustering criteria in the literature since they try to enforce the compactness of each group.

The maximum-spacing problem is a maximization problem for which the criterion to be maximized is the spacing $sp(\mathcal{C})$ of a partition $\mathcal{C}$, defined as

$$sp(\mathcal{C}) = \min\{||\mathbf{x} - \mathbf{y}||_2 : \mathbf{x} \text{ and } \mathbf{y} \text{ lie in different groups of } \mathcal{C}\}.$$

In contrast to the other criteria, the spacing is an inter-clustering criterion.

We note that an optimal solution of the unrestricted version of any of these problems, in which the decision tree constraint is not enforced, may be a partition that is hard to explain in terms of the input features. Thus, the motivation for using decision trees.

Along the lines of [12], we define the price of explainability $\rho(\mathcal{P})$ for a clustering problem $\mathcal{P}$, with a minimization objective function, as

$$\rho(\mathcal{P}) = \max_{I} \left\{ \frac{OPT_{exp}(I)}{OPT_{unr}(I)} \right\},$$

where I runs over all instances of $\mathcal{P}$; $OPT_{exp}(I)$ is the cost of an optimal explainable clustering (via standard decision trees) for instance I and $OPT_{unr}(I)$ is the cost of an optimal unrestricted clustering for I. If $\mathcal{P}$ has a maximization objective function, then it is defined as

$$\rho(\mathcal{P}) = \max_I \left\{ \frac{OPT_{unr}(I)}{OPT_{exp}(I)} \right\}.$$

1.2 Our Contributions

We provide bounds on the price of explainability as a function of the parameters k, d and n for the aforementioned objective functions.

First, we address the k-centers problem. We show that

$$\rho(k\text{-centers}) \in \begin{cases} \Omega(k^{1-1/d}), & \text{if } d \leq \frac{\ln k}{\ln \ln k} \\ \Omega\left(\sqrt{d} \cdot \frac{k \cdot \sqrt{\ln \ln k}}{\ln^{1.5} k}\right), & \text{otherwise} \end{cases}$$

and that $\rho(k$-centers) is $O(\sqrt{d}k^{1-1/d})$. Our bounds are tight for constant dimensions while, for arbitrary ones, there is a polylogarithmic gap between the upper and the lower bound.

For the k-medians it is known that the price of explainability is $O(k)$ and $\Omega(\log k)$ [12]. We contribute to the state of the art on this problem by showing that $O(d \log k)$ is also an upper bound – an exponential improvement for constant dimensions. For the k-means problem, we also improve, for low dimensions, the $O(k^2)$ bound from [12] by proving that $\rho(k$-means) is $O(kd \log k)$. These upper bounds are obtained by exploiting an interesting connection with the literature of binary searching in the presence of non-uniform testing costs [8, 20].

Finally, for maximum-spacing we provide a tight bound by showing that the price of explainability is $\Theta(n - k)$.

In order to derive our upper bounds we analyze polynomial-time algorithms that start with an optimal k-clustering and, then, transform it into an explainable one. Unfortunately, the unrestricted version of all the problems considered here, except for the maximum-spacing problem, are NP-Hard [24, 2]. However, all of them admit polynomial-time algorithms with constant approximation [33, 18] and, hence, if we take the partitions given by these algorithms instead of the optimal ones we obtain efficient algorithms with provable guarantees equal to the upper bounds proved for the price of explainability.

In summary, we provide bounds on the price of explainability and also efficient algorithms for clustering problems defined by a range of objective functions that include both intra- and inter-clustering measures as well as worst-case and average-case measures. We believe that our results are helpful for the construction of explainable clustering solutions as well as for guiding the choice of an objective function when explainability is required.

1.3 Related Work

In [12], an algorithm is presented that approximates a solution to the k-means or the k-medians problem through a decision tree that selects at each node the cut that minimizes the number of elements separated from their reference centers. Our approach for these problems, while similar, uses a significantly different strategy to build the final decision tree, based on decision trees that look at a single dimension of the data. The algorithm from [12] leads to a price of explainability that is $\Omega(\log k)$ and $O(k)$ for the k-medians algorithm; our strategy is $O(d \log k)$, an improvement for low dimensions.

Decision trees have long been associated to hierarchical agglomerative clustering (HAC), which produces a hierarchy of clusters that is usually represented by a dendrogram. Examples of models that explicitly use decision trees for HAC include [13, 9, 7, 4]. To our knowledge, the use of decision trees for non-hierarchical clustering was first suggested in [21], in which a standard classification tree is used to identify dense and sparse regions of data. In [14], unsupervised binary trees are used to create interpretable clusters in three steps: constructing a maximal tree, pruning according to a dissimilarity measure, and joining similar clusters (even if they don't share the same parent). More recently, an approach was presented in [6] using optimal classification trees [5], which are built in a single step by solving a mixed-integer optimization problem. For numerical databases, [22] presents a decision approach that decides on a split based on both the compactness of clusters and the separation between them.

The regions of space defined by decision-tree clustering will be hyper-rectangles (some of them may also be half-spaces if the overall region of interest is unbounded). Other approaches towards building hyper-rectangular clusters can be found in [27], with a generative model, and [11], with a discriminative one. Both models allow for probabilistic (soft) clustering, and [11] allows for incorporating previous knowledge to the model, but neither one guarantees that the resulting clusters can be represented by decision trees.

The main reason for using a (short) decision tree to build clusters is that the results of such algorithms are easily interpretable. Other avenues towards interpretable clustering have been explored in recent years. The technique presented in [28] is based on the information-theoretic concept of minimum description length. In [31], a tunable parameter (the fraction of elements in a cluster that share the same feature value) leverages the tradeoff between clustering performance and interpretability. The same tradeoff is explored in [15] by relaxing the requirement from [12] that the explainable clustering should be induced by a tree with no more than k leafs. In [17], a feature selection model from [16] is used for clustering interpretation in the field of wealth management compliance. [19] uses a two-step approach, rewriting k-means clustering models as neural networks and applying to these networks techniques for interpreting supervised learning models. More information regarding explainable clustering may be found in [10, 3].

Of all the works mentioned in this section, only [12] presents approximation guarantee with respect to the optimal unrestricted (i.e., potentially uninterpretable) solution. Two algorithms from [31] also have an approximation guarantee, but with respect to the optimal restricted (interpretable) solution, and the definition of interpretability in that work is quite different than ours (interpretable clusters are therein defined as those in which a given proportion of points share the same value for a predefined feature of interest).

Explainability and interpretability are topics of growing interest in the machine learning community [29, 23, 1, 30, 26, 25]. While there has been some focus on what [12] calls *post-modeling explainability*, or the ability to explain the output of a black-box model [29, 23, 19], the practice has also been criticized in contrast with *pre-modelling explainability*, or the use of interpretable models to begin with [30]. Our present work and [12] may be considered a middle-of-the-road approach, as the end result is a fully interpretable model (instead of, for instance, a model for locally interpreting the original model, or for explaining individual predictions) based on the output from a potentially black-box model.

2 On the Price of Explainability for the k-centers problem

In this section we address the k-centers problem. We first present a lower bound by constructing an instance for which the price of explainability is high.

2.1 Lower Bound

Let $p \leq \min\{d, \log_3 k\}$ be a positive integer whose exact value will be defined later in the analysis and let b be the largest integer for which $b^p \leq k$. Note that $b \geq 3$. Moreover, let $k' = b^p$.

Our instance I has $k + k' \cdot 2d$ points. We first discuss how to construct the k points, referred as centers, that will be set as representatives in an unrestricted k-clustering for I that has a low cost. The first k' centers will be obtained from the representation of the numbers $0, \ldots, k'-1$ in base b while the remaining $k - k'$ centers will be located sufficiently far from the others so that they will be isolated in the low-cost k-clustering. Let $\mathbf{c}^0, \ldots, \mathbf{c}^{k'-1}$ be the first k' centers.

For a number $i \in [k'-1]$ let $(i_{p-1}, \ldots, i_0)_b$ be its representation in base b. For $j \in [d]$, the value of the j-th component of center $\mathbf{c}^i$ is obtained by applying $(j-1)$ times a circular shift on $(i_{p-1}, \ldots, i_0)_b$. The values of the remaining $d-p$ components of $\mathbf{c}^i$ are obtained by copying the p first values d/p times so that $c^i_j = c^i_{j'}$ if $(j - j') \mod p = 0$.

As an example, if $b = 3$, $p = 3$ and $d = 9$ then $\mathbf{c}^{14} = (14, 22, 16, 14, 22, 16, 14, 22, 16)$. In fact, since $14 = (1, 1, 2)_3$ we have that $c^{14}_1 = (1, 1, 2)_3 = 14$; $c^{14}_2 = (2, 1, 1)_3 = 22$ and $c^{14}_3 = (1, 2, 1)_3 = 16$. The values of $c^{14}_4, \ldots, c^{14}_9$ are obtained by repeating the first 3 values.

The following observation is useful for our analysis.

Proposition 1. *For every $\ell \in [p]$, the values of the ℓ-th coordinate of the k' first centers are a permutation of the integers $0, \ldots, k'-1$.*

The remaining $k - k'$ centers, as mentioned above, should be far from each other and also far away from the k' first centers. We can achieve that by setting $\mathbf{c}^i = k^i \mathbf{1}$ for all $i > k' - 1$, where $\mathbf{1}$ is the unit vector in $\mathbb{R}^d$.

The next lemma gives a lower bound on the distance between any two centers.

Lemma 1. *For any two centers $\mathbf{c}^i$ and $\mathbf{c}^j$,*

$$||\mathbf{c}^i - \mathbf{c}^j||_2 \geq \sqrt{\lfloor d/p \rfloor} \cdot (b^{p-1}/2).$$

Proof. If one of the two centers is not among the k' first centers the result clearly holds. Thus, we assume that $i, j \leq k' - 1$.

It is enough to show that there is $\ell \in [p]$ for which $|c^i_\ell - c^j_\ell| \geq b^{p-1}/2$. In fact, if this inequality holds for some ℓ then $|c^i_{\ell'} - c^j_{\ell'}| \geq b^{p-1}/2$ for each ℓ' that is congruent to ℓ modulo p. Since there are $\lfloor d/p \rfloor$ of them, due to our construction, we get the desired bound.

Let $i = (i_{p-1}, \ldots, i_0)_b$ and $j = (j_{p-1}, \ldots, j_0)_b$ be the representations of i and j in base b, respectively. Let f be such that $|i_f - j_f|$ is maximum.

Thus, the difference between $\mathbf{c}^i$ and $\mathbf{c}^j$ in the coordinate $[(f+1) \mod p] + 1$ is at least

$$|i_f - j_f| \cdot \left(b^{p-1} - \sum_{g=0}^{p-2} b^g \right) \geq b^{p-1}/2,$$

where the last inequality holds because $|i_f - j_f| \geq 1$ and $b \geq 3$. $\square$

Now, we define the remaining points of instance I.

For each of the first k' centers we create $2d$ associated points: $\mathbf{x}^{i,1}, \ldots, \mathbf{x}^{i,2d}$. For $j = 1, \ldots, d$, the point $\mathbf{x}^{i,2j-1}$ is identical to $\mathbf{c}^i$ in all coordinates but on the j-th one, in which its value is $c^i_j - 3/4$. Similarly, the point $\mathbf{x}^{i,2j}$ is identical to $\mathbf{c}^i$ in all coordinates but in the j-th one, in which its value is $c^i_j + 3/4$. By considering the k-clustering for I where the k representatives are the k centers $\mathbf{c}^0, \ldots, \mathbf{c}^{k-1}$ and each point $\mathbf{x}^{i,j}$ lies in the group of $\mathbf{c}^i$, we obtain the following proposition.

Proposition 2. *There exists an urestricted k-clustering for instance I with cost $3/4$*

Now we analyse the cost of an optimal explainable clustering for I.

Proposition 3. *Let θ be an axis-aligned cut that separates at least two points from the set S that includes the k' first centers and its associated $k' \cdot 2d$ points. Then, θ separates one point from its associated center.*

Proof. Let θ be the cut that separates points with the j-th coordinate smaller than A from those with the j-th coordinate larger than A. Since θ separates at least two points from S then $A \in (-3/4, k' - 1 + 3/4)$.

If $A < 0$ then θ separates the center that has the j-th coordinate equal to 0 from its associated point that has coordinate j equal to $-3/4$. If $A > k' - 1$ then θ separates the center that has the j-th coordinate equal to 0 from its associated point that has coordinate j equal to $k' - 1 + 3/4$. Let z be an integer that satisfies $0 \le z \le k' - 2$ and such that $A \in (z, z + 1)$. If $A - z < 1/2$ (resp. $A - z > 1/2$), θ separates the center that has the j-th coordinate equal to z (resp. $z + 1$) from its associated point with j-th coordinate equal to $z + 3/4$ (resp. $z + 1 - 3/4$).

Note that the existence of centers with the aforementioned values for coordinate j is guaranteed by Proposition 1. □

Lemma 2. *Any explainable k-clustering for instance I has cost at least $\sqrt{\lfloor d/p \rfloor} \cdot (b^{p-1}/4) - 3/8$.*

Proof. Let $\mathcal{C}$ be an explainable k-clustering for instance I. It is enough to show that there is a cluster $C \in \mathcal{C}$ that contains two points, say $\mathbf{x}$ and $\mathbf{y}$, for which

$$||\mathbf{x} - \mathbf{y}||_2 \ge \sqrt{\lfloor d/p \rfloor} \cdot (b^{p-1}/2) - 3/4.$$

In fact, in this case, due to the triangle inequality, for any choice of the representative for C, either $\mathbf{x}$ or $\mathbf{y}$ will be at distance at least $\sqrt{\lfloor d/p \rfloor} \cdot (b^{p-1}/4) - 3/8$ from it.

If two centers lie in the same cluster of $\mathcal{C}$ then it follows from Lemma 1 that their distance is at least $\sqrt{\lfloor d/p \rfloor} \cdot (b^{p-1}/2)$.

On the other hand, if every center lies on a different cluster in $\mathcal{C}$ then let $\mathbf{x}$ be the point that was separated from its center, say $\mathbf{c}^i$, by a cut that satisfies the condition of Proposition 3. Then, $\mathbf{x}$ lies in the same cluster of $\mathbf{c}^j$, for some $j \neq i$. From the triangle inequality we have that

$$||\mathbf{c}^i - \mathbf{c}^j||_2 \le ||\mathbf{c}^i - \mathbf{x}||_2 + ||\mathbf{c}^j - \mathbf{x}||_2.$$

Hence, $||\mathbf{c}^j - \mathbf{x}||_2 \ge \sqrt{\lfloor d/p \rfloor} \cdot (b^{p-1}/2) - 3/4$. □

Theorem 1. *The price of explainability for the k-center problem satisfies*

$$\rho(k\text{-}center) \in \begin{cases} \Omega(k^{1-1/d}), & \text{if } d \le \frac{\ln k}{\ln \ln k} \\ \Omega\left(\sqrt{d} \cdot \frac{k \cdot \sqrt{\ln \ln k}}{\ln^{1.5} k}\right), & \text{otherwise} \end{cases}$$

Proof. Proposition 2 assures the existence of a k-clustering of cost $3/4$ for instance I. Let $\mathcal{C}$ be an explainable clustering for I and recall that $b^p = k'$. It follows from the previous lemma that

$$cost(\mathcal{C}) \ge \sqrt{\frac{d}{p}} \cdot \frac{b^{p-1}}{4} - 3/8 = \sqrt{\frac{d}{p}} \cdot \frac{(k')^{\frac{p-1}{p}}}{4} - 3/8.$$

Since $(b + 1)^p > k$ we have

$$k' > \frac{k}{(1 + 1/b)^p} > \frac{k}{\exp(p/b)}.$$

Thus,

$$cost(\mathcal{C}) \geq \sqrt{\frac{d}{p}} \cdot \frac{k^{\frac{p-1}{p}}}{4\exp((p-1)/b)} - 3/8.$$

Now we set $p = d$ if $d \leq \frac{\ln k}{\ln \ln k}$ and $p = \frac{\ln k}{\ln \ln k}$, otherwise. Since $b > k^{1/p} - 1$ we have that $b > \ln k - 1 > p - 1$ for both cases and, hence,

$$cost(\mathcal{C}) \geq \sqrt{\frac{d}{p}} \cdot \frac{k^{\frac{p-1}{p}}}{4} - 3/8.$$

By replacing p in the previous equation according to each of the cases we obtain the desired result. □

2.2 Upper bound

In this section we show that the price of explainability for the k-center problem is $O\left(\sqrt{d}k^{\frac{d-1}{d}}\right)$. Note that, for constant d, the upper bound matches the lower bound given by Theorem 1.

The upper bound is obtained by analyzing the cost of the explainable clustering induced by the decision tree built by the algorithm presented in Algorithm 1.

The algorithm receives the set of representatives of an optimal k-clustering $\mathcal{C}^*$ for $\mathcal{X}$. As long as it is possible, the algorithm applies axis-aligned cuts that do not separate the points in $\mathcal{X}$ from their reference centers. By the reference center of a point $\mathbf{x}$ we mean its representative in $\mathcal{C}^*$. When there is no such cut available for some node u, the algorithm partitions the bounding box of the points reaching u into boxes of the same dimensions using a decision tree that emulates a grid.

More precisely, let u be the current node of the decision tree under construction, let $\mathcal{X}^u$ be the set of points that reach u and let S^u be set of reference centers that reach u. Initially, u is the root of the tree, $\mathcal{X}^u = \mathcal{X}$ and S^u is the set containing all reference centers.

The algorithm uses the notion of a *clean cut*. We say that an axis-aligned cut is clean w.r.t $\mathcal{X}^u$ if it satisfies the following properties: (i) it separates at least two reference centers in S^u; (ii) it does not separate any point in $\mathcal{X}^u$ from its reference center.

When there is a clean cut with respect to $\mathcal{X}^u$, the algorithm applies it and then recurses on each of the two subsets of $\mathcal{X}^u$ induced by the cut. Otherwise, let H be the bounding box of $\mathcal{X}^u$, that is, the smallest box (hyper-rectangle) with axis-aligned sides that includes all points in $\mathcal{X}^u$. Moreover, let $p = \lfloor s^{1/d} \rfloor$, where $s = |S^u|$. If there is no clean cut with respect to $\mathcal{X}^u$, the algorithm employs a decision tree D^u to partition H into p^d identical axis-aligned boxes, so that the points of each of them are associated with a distinct leaf of D^u.

Algorithm 1 `Ex-kCenter`($\mathcal{X}'$: set of points)

$S \leftarrow$ reference centers of the points in $\mathcal{X}'$
if $|S| = 1$ **then**
  Return $\mathcal{X}'$ and the single reference center in S
else
  if there exists a clean cut w.r.t. $\mathcal{X}'$ **then**
    $(\mathcal{X}'_L, \mathcal{X}'_R) \leftarrow$ partition induced by a clean cut on $\mathcal{X}'$
    Create a node u
    u.`LeftChild` $\leftarrow$`Ex-kCenter`($\mathcal{X}'_L$)
    u.`RightChild` $\leftarrow$`Ex-kCenter`($\mathcal{X}'_R$)
    Return the tree rooted at u
  else
    $H \leftarrow$ bounding box for $\mathcal{X}'$
    $D^u \leftarrow$ decision tree that partitions H into $\lfloor |S|^{1/d} \rfloor^d$ identical axis-aligned boxes
    Return D^u as well as an arbitrarily chosen representative for each of its leaves
  end if
end if

Theorem 2. *The price of explainability for k-center is* $O\left(\sqrt{d} k^{1-1/d}\right)$.

Proof. We argue that for each leaf ℓ of the tree $\mathcal{D}$ built by `Ex-kCenter`($\mathcal{X}$), the maximum distance between a point in ℓ and its representative is $OPT\sqrt{d}k^{1-1/d}$, where OPT is the cost of the optimal unrestricted clustering.

We split the proof into two cases. The first case addresses the scenario in which only clean cuts are used in the path from the root of $\mathcal{D}$ to the leaf ℓ. The second case addresses the remaining scenarios.

Case 1. In this case all points that reach ℓ lie in the same cluster of the optimal unrestricted k-clustering $\mathcal{C}^*$. Thus, the maximum distance from a point in ℓ to the single reference center in S is upper bounded by OPT.

Case 2. Let u be the first node in the path from the root to ℓ for which a clean cut is not available. Moreover, let $\mathcal{X}^u$ be the set of points that reach u and let s be the total number of reference centers that reach u. In this case the algorithm splits the bounding box for $\mathcal{X}^u$ into boxes of dimensions

$$\frac{L_1}{\lfloor s^{1/d} \rfloor} \times \cdots \times \frac{L_d}{\lfloor s^{1/d} \rfloor},$$

where L_i is the difference between the maximum and minimum values of the i-th coordinate among points in $\mathcal{X}^u$.

The maximum distance between a point in ℓ and its representative can be upper bounded by the length of the diagonal of the axis-aligned box corresponding to ℓ. Let $m \in [d]$ be such that $L_m = \max\{L_1, \ldots, L_d\}$. Then, the length of the diagonal is upper bounded by $L_m\sqrt{d}/\lfloor s^{1/d} \rfloor \leq 2L_m\sqrt{d}/s^{1/d}$.

Thus, it suffices to show that $OPT \geq L_m/(2s)$. Let $\mathbf{c}^1, \ldots, \mathbf{c}^s$ be the s reference centers that reach node u. In addition, let $\mathbf{x}^j$ be a point in $\mathcal{X}^u$ with reference center $\mathbf{c}^j$ and such that $|x^j_m - c^j_m|$ is maximum, among the points in $\mathcal{X}^u$ with reference center $\mathbf{c}^j$. Then, we must have

$$\sum_{j=1}^{s} 2|x^j_m - c^j_m| \geq L_m,$$

for otherwise there would be a clean cut with coordinate m fixed. Hence, for some point $\mathbf{x}^j$, $|x^j_m - c^j_m| \geq L_m/(2s)$. Since $OPT \geq |x^j_m - c^j_m|$ we get that $OPT \geq L_m/(2s)$. □

3 Improved Bounds on k-medians and k-means for low dimensions

We first show that the price of explainability for k-medians is $O(d \log k)$, which improves the bound from [12] when $d = o(k/\log k)$. Then, in Section 3.3, we extend this result by proving an $O(dk \log k)$ upper bound for k-means.

As in the previous section we use an optimal unrestricted k-clustering $\mathcal{C}^*$ for $\mathcal{X}$ as a guide for building an explainable clustering. Again, by the reference center of a point $\mathbf{x} \in \mathcal{X}$ we mean its representative in $\mathcal{C}^*$.

We need some additional notation. For a decision tree $\mathcal{D}$ and a node $u \in \mathcal{D}$, let $diam(u)$ be the d-dimensional vector whose i-th coordinate $diam(u)_i$ is given by the difference between the maximum and the minimum values of coordinate i among the reference centers that reach u. Let t_u be the number of points that reach u and are separated from their reference centers by the cut employed in u. Note that a point $\mathbf{x} \in \mathcal{X}$ can only contribute to t_u if both $\mathbf{x}$ and its reference center reach u. Finally, we use OPT to denote the cost of the optimal unrestricted clustering $\mathcal{C}^*$.

The following lemma from [12], expressed in our notation, will be useful.

Lemma 3. *[12] Let $\mathcal{C}^*$ be an optimal unrestricted k-clustering for $\mathcal{X}$ and let $\mathcal{D}$ be a decision tree for $\mathcal{X}$ in which each representative of $\mathcal{C}^*$ lies in a distinct leaf of $\mathcal{D}$. Then, the clustering $\mathcal{C}$ induced by $\mathcal{D}$ satisfies*

$$cost(\mathcal{C}) \le OPT + \sum_{u \in \mathcal{D}} t_u ||diam(u)||_1. \tag{1}$$

In order to obtain a low-cost explainable clustering we focus on finding a decision tree $\mathcal{D}$ for which the rightmost term of the above inequality is small. This is the approach taken by [12], where a greedy strategy that at each node u selects the cut that yields the minimum possible value for t_u is proposed and analysed.

Although we follow the same approach, our strategy for building the tree is significantly different. For a node u in a decision tree $\mathcal{D}$, let $t_u^{\mathcal{X}}$ be the number of points in $\mathcal{X}$ that are separated from their reference centers by the cut associated with u, whether or not those points reached u. Note that $t_u \le t_u^{\mathcal{X}}$ since t_u only considers the points in $\mathcal{X}$ that reach u. Thus,

$$\sum_{u \in \mathcal{D}} t_u ||diam(u)||_1 \le \sum_{u \in \mathcal{D}} t_u^{\mathcal{X}} ||diam(u)||_1 =$$

$$\sum_{u \in \mathcal{D}} t_u^{\mathcal{X}} \cdot \left(\sum_{i=1}^{d} diam(u)_i \right) = \sum_{i=1}^{d} \sum_{u \in \mathcal{D}} t_u^{\mathcal{X}} diam(u)_i.$$

Motivated by the above inequality we define $UB_i(\mathcal{D})$ as the contribution of coordinate $i \in [d]$ to its rightmost term, that is,

$$UB_i(\mathcal{D}) = \sum_{u \in \mathcal{D}} t_u^{\mathcal{X}} diam(u)_i. \tag{2}$$

First, our strategy constructs d decision trees $\mathcal{D}_1, \ldots, \mathcal{D}_d$, where $\mathcal{D}_i$ is built with the aim of minimizing $UB_i()$, ignoring its impact on $UB_j()$ for $j \neq i$. Then, it constructs a decision tree $\mathcal{D}$ for $\mathcal{X}$ by picking nodes from these d trees. More precisely, to split a node v of $\mathcal{D}$ the strategy first selects a coordinate $i \in [d]$ for which $diam(v)_i$ is maximum. Next, it applies the cut that is associated with the node in $\mathcal{D}_i$ which is the least common ancestor (LCA) of the set of centers that reach v. A pseudo-code is given below.

Algorithm 2 `BuildTree`($\mathcal{X}'$: data points)

$S \leftarrow$ set of reference centers for points in $\mathcal{X}'$.
Create a node u and associate it with $\mathcal{X}'$ and S
if $|S| = 1$ **then**
  Return the leaf u
else
  Select $i \in [d]$ for which $diam(u)_i$ is maximum.
  $v \leftarrow$ node in $\mathcal{D}_i$ which is the LCA of the centers in S
  Split $\mathcal{X}'$ into $\mathcal{X}'_L$ and $\mathcal{X}'_R$ using the cut associated with v.
  u.`LeftChild` $\rightarrow$ `BuildTree`($\mathcal{X}'_L$)
  u.`RightChild` $\rightarrow$ `BuildTree`($\mathcal{X}'_R$)
  Return the decision tree rooted at u
end if

In order to fully specify the algorithm we need to explain how the decision trees $\mathcal{D}_i$ are built. Let $\mathbf{c}^1, \dots, \mathbf{c}^k$ be the reference centers sorted by coordinate i, that is, $c_i^j < c_i^{j+1}$ for $j = 1, \dots, k-1$. Moreover, let θ_i^j be the threshold cut that separates the points in $\mathcal{X}$ with the i-th coordinate smaller than or equal to $(c_i^j + c_i^{j+1})/2$ from the remaining ones.

For $1 \le a \le b \le k$, let $C_{a,b} = \{\mathbf{c}^a, \dots, \mathbf{c}^b\}$ and let $\mathcal{F}_{a,b}$ be the family of binary decision trees with $(b-a)$ internal nodes and $b-a+1$ leaves defined as follows:

(i) if $a = b$, then $\mathcal{F}_{a,b}$ has a single tree and this tree contains only one node.

(ii) if $a < b$, then every tree $\mathcal{D}'$ in $\mathcal{F}_{a,b}$ has the following structure: the root of $\mathcal{D}'$ is identified by a number $j \in \{a, \dots, b-1\}$ and associated with the cut θ_i^j; one child of the root of $\mathcal{D}'$ is a tree in the family $\mathcal{F}_{a,j}$ while the other is a tree in $\mathcal{F}_{j+1,b}$.

For our analysis, in the next sections, it will be convenient to view $\mathcal{F}_{a,b}$ as the family of binary search trees for the numbers in the set $\{a, \dots, b-1\}$.

Let T_j be the number of points in $\mathcal{X}$ that are separated from their centers by cut θ_i^j. Note that for every tree $\mathcal{D}' \in \mathcal{F}_{a,b}$

$$UB_i(\mathcal{D}') = \sum_{j=a}^{b-1} T_j \cdot diam(j)_i,$$

where $diam(j)$ is the diameter of the node identified by j.

The tree $\mathcal{D}_i$ is, then, defined as

$$\mathcal{D}_i = \text{argmin}\{UB_i(\mathcal{D}') \mid \mathcal{D}' \in \mathcal{F}_{1,k}\}.$$

We discuss how to construct it efficiently. Let $OPT_{a,b} = \min\{UB_i(\mathcal{D}') \mid \mathcal{D}' \in \mathcal{F}_{a,b}\}$, if $a < b$, and let $OPT_{a,b} = 0$ if $a = b$. Hence, $UB_i(\mathcal{D}_i) = OPT_{1,k}$. The following relation holds for all $a < b$

$$OPT_{a,b} = \min_{a \le j \le b-1} \left\{ T_j(c_i^b - c_i^a) + OPT_{a,j} + OPT_{j+1,b} \right\}. \tag{3}$$

Thus, given a set of k reference centers and the values T_j's, $\mathcal{D}_i$ can be computed in $O(k^3)$ time by solving equation (3), for $a = 1$ and $b = k$, via standard dynamic programming techniques.

3.1 Approximation Analysis: Overview

We prove that the cost of the clustering induced by $\mathcal{D}$ is $O(d \log k) \cdot OPT$. To reach this goal, we first we show that

$$UB_i(\mathcal{D}_i) \le 2 \log k \left(\sum_{j=1}^{k-1} (c_i^{j+1} - c_i^j) T_j \right) \tag{4}$$

To prove this bound we rely on the fact that $\mathcal{D}_i$ can be seen as a binary search tree with non-uniform testing costs and then we use properties of this kind of tree, in particular the one proved in [8] about its competitive ratio.

Let

$$OPT_i = \sum_{\mathbf{x} \in \mathcal{X}} |x_i - c(\mathbf{x})_i|$$

be the contribution of coordinate i to OPT, where $c(\mathbf{x})$ is the reference center of $\mathbf{x}$. Our second step consists of showing that

$$\left(\sum_{j=1}^{k-1} (c_i^{j+1} - c_i^j) T_j \right) /2 \le OPT_i. \tag{5}$$

Roughly speaking, the proof of this bound consists of projecting the points of $\mathcal{X}$ and the reference centers onto the axis i and then counting the number of times the interval $[c_i^j, c_i^{j+1}]$ appears in the segments that connect points in $\mathcal{X}$ to their reference centers. This is exactly the same line of reasoning employed to prove Lemma 6 presented in the supplementary version of [12].

At this point, from the two previous inequalities, we obtain

$$UB_i(\mathcal{D}_i) \le 4 \log k \cdot OPT_i. \tag{6}$$

Finally, we prove that a factor of d is incurred when we build the tree $\mathcal{D}$ from the nodes of the trees $\mathcal{D}_1, \ldots, \mathcal{D}_d$.

$$\sum_{v \in \mathcal{D}} t_v ||diam(v|)|_1 \le d \sum_{i=1}^{d} UB_i(\mathcal{D}_i). \tag{7}$$

From (6), (7) and the identity $OPT = \sum_{i=1}^{d} OPT_i$, we obtain

$$\sum_{v \in \mathcal{D}} t_v ||diam(v)||_1 \le 4d \log k \cdot OPT.$$

This together with Lemma 3 allows us to establish the main theorem of this section.

Theorem 3. *The price of explainability for k-medians is $O(d \log k)$.*

3.2 Approximation Analysis: Proofs

We start with the proof of inequality (4).

Lemma 4. *The tree $\mathcal{D}_i$ satisfies*

$$UB_i(\mathcal{D}_i) \le 2 \log k \left(\sum_{j=1}^{k-1} (c_i^{j+1} - c_i^j) T_j \right).$$

Proof. Let $\mathcal{D}'$ be a tree in $\mathcal{F}_{1,k}$. By construction, the set of centers that reach the node in $\mathcal{D}'$ identified by j is a contiguous subsequence of $\mathbf{c}^1, \dots, \mathbf{c}^k$. Let $r(j)$ and $s(j)$ be, respectively, the first and the last indexes of the centers of this subsequence. Thus,

$$UB_i(\mathcal{D}') = \sum_{j=1}^{k-1} T_j \cdot diam(j)_i = \sum_{j=1}^{k-1} T_j \sum_{\ell=r(j)}^{s(j)-1} (c_i^{\ell+1} - c_i^{\ell}). \tag{8}$$

We can show that the right-hand side of the above equation satisfies

$$\sum_{j=1}^{k-1} T_j \sum_{\ell=r(j)}^{s(j)-1} (c_i^{\ell+1} - c_i^{\ell}) = \sum_{j=1}^{k-1} (c_i^{j+1} - c_i^{j}) \cdot \sum_{u \in An(j,\mathcal{D}')} T_u, \tag{9}$$

where $An(j, \mathcal{D}')$ is the set of nodes that are ancestors (including j) of the node identified by j in $\mathcal{D}'$.

In fact, for every $j, \ell \in [k-1]$, the term $T_j(c_i^{\ell+1} - c_i^{\ell})$ contributes to the left-hand side of (9) if and only if both centers $\mathbf{c}^{\ell}$ and $\mathbf{c}^{\ell+1}$ reach the node j. This happens exactly for the nodes u that are ancestors of the node identified by j in $\mathcal{D}'$.

Now, we use Theorem 4.5 from [8]. It states that for any vector $(p_1, \dots, p_k)$ of k non-negative real numbers there exists a binary search tree B with k nodes, each of them associated with a number in $[k]$, and that for every node j of B

$$\sum_{\ell \in An(j,B)} p_{\ell} \leq (\log k + o(\log k)) p_j \leq 2 \log k \cdot p_j.$$

Let $\mathcal{D}_c$ be a tree obtained via the result of [8] for the vector $(T_1, \dots, T_{k-1})$. It satisfies

$$\sum_{\ell \in An(j,\mathcal{D}_c)} T_{\ell} \leq 2 \log k \cdot T_j.$$

By using this inequality, (8) and (9), we get that

$$UB_i(\mathcal{D}_c) = \sum_{j=1}^{k-1} (c_i^{j+1} - c_i^{j}) \sum_{\ell \in An(j,\mathcal{D}_c)} T_{\ell} \leq$$

$$2 \log k \sum_{j=1}^{k-1} (c_i^{j+1} - c_i^{j}) T_j.$$

The result follows because the minimality of $\mathcal{D}_i$ guarantees that $UB_i(\mathcal{D}_i) \leq UB_i(\mathcal{D}_c)$. □

Inequality (5) is formalized in the next lemma.

Lemma 5. *Let OPT_i be the contribution of the coordinate i for the cost of an optimal unrestricted clustering $\mathcal{C}^*$. Then,*

$$OPT_i = \sum_{\mathbf{x} \in \mathcal{X}} |x_i - c(\mathbf{x})_i| \geq \sum_{j=1}^{k-1} \frac{(c_i^{j+1} - c_i^{j}) T_j}{2}, \tag{10}$$

where $c(\mathbf{x})$ is the reference center of $\mathbf{x}$.

Proof. Let $\mathbf{c}^1, \ldots, \mathbf{c}^k$ be the reference centers sorted by increasing order of coordinate i and let I_j be the interval $[c_i^j, c_i^{j+1}]$.

Fix $\mathbf{x} \in \mathcal{X}$. We consider two cases.

Case 1: $x_i < c(x)_i$. Let r be the smallest integer j for which the midpoint of I_j has coordinate larger than x_i. Moreover, let s be the rank of $c(\mathbf{x})$ in the list of centers sorted by coordinate i, that is, $c(\mathbf{x}) = \mathbf{c}^s$. We have that

$$|x_i - c(x)_i| \geq \sum_{\ell=r}^{s} (c_i^{\ell+1} - c_i^{\ell})/2.$$

Case 2: $x_i \geq c(x)_i$. In this case, let $c(\mathbf{x}) = \mathbf{c}^r$ and let s be the largest integer j for which the midpoint of I_j has coordinate smaller than x_i. We have that

$$|x_i - c(x)_i| \geq \sum_{\ell=r}^{s} (c_i^{\ell+1} - c_i^{\ell})/2.$$

By adding the above inequalities for all $\mathbf{x} \in \mathcal{X}$ we conclude that the number of times that $(c_i^{j+1} - c_i^j)/2$ contributes to the right-hand side is exactly the number of times that the cut associated with interval I_j separates x and $c(\mathbf{x})$, that is, T_j. □

Finally, we present the proof of inequality (7).

Lemma 6. *Let $\mathcal{D}$ be the decision tree built by Algorithm 2. Then,*

$$\sum_{v \in \mathcal{D}} t_v ||diam(v)||_1 \leq d \sum_{i=1}^{d} UB_i(\mathcal{D}_i).$$

Proof. For a node $j \in \mathcal{D}_i$, let $S_{i,j}$ be the (possibly empty) set of nodes in the tree $\mathcal{D}$ that correspond to j, that is, the nodes that use the cut associated with the node j from $\mathcal{D}_i$. We have

$$\sum_{v \in \mathcal{D}} t_v ||diam(v)||_1 = \sum_{i=1}^{d} \sum_{j \in \mathcal{D}_i} \sum_{u \in S_{i,j}} t_u ||diam(u)||_1. \tag{11}$$

Moreover, we have that

$$\sum_{u \in S_{i,j}} t_u ||diam(u)||_1 \leq \sum_{u \in S_{i,j}} t_u d \cdot diam(u)_i \leq \tag{12}$$

$$\sum_{u \in S_{i,j}} t_u d \cdot \max_{u \in S_{i,j}} \{diam(u)_i\} \leq \sum_{u \in S_{i,j}} t_u d \cdot diam(j)_i, \tag{13}$$

where the first inequality in (12) holds because i is the coordinate for which the diameter of u is maximum and the inequality (13) holds because the set of centers in u is a subset of the set of centers that reach the node identified by j in $\mathcal{D}_i$.

Claim 1. *For a node $u \in S_{i,j}$, let $\mathcal{X}_u \subseteq \mathcal{X}$ be the set of points that reach u in $\mathcal{D}$. Then, $\mathcal{X}_u \cap \mathcal{X}_{u'} = \emptyset$ for every $u, u' \in S_{i,j}$, with $u \neq u'$.*

Proof. Let w be the least common ancestor of u' and u in $\mathcal{D}$. If $w \notin \{u, u'\}$ then the cut associated with w splits $\mathcal{X}_w$ into two disjoint regions, one of them containing $\mathcal{X}_u$ and the other containing $\mathcal{X}_{u'}$ so that $\mathcal{X}_u$ and $\mathcal{X}_{u'}$ are disjoint.

If $w \in \{u, u'\}$ let us assume w.l.o.g. that $w = u$. In this case, the cut associated with j splits $\mathcal{X}_u$ into two regions, one of them containing all the centers that reach u'. These reference centers are contained in the set of reference centers of one of the children of j in $\mathcal{D}_i$ and, hence, the LCA in $\mathcal{D}_i$ of the set of centers that reach u' is not j, that is, $u' \notin S_{i,j}$. This contradiction shows that this case cannot occur. □

From the previous claim we get that

$$\sum_{u \in S_{i,j}} t_u \leq T_j.$$

It follows from (12)-(13) and the above inequality that

$$\sum_{u \in S_{i,j}} t_u ||diam(u)||_1 \leq d \cdot T_j \cdot ||diam(j)||_1.$$

Hence, it follows from (11) that

$$\sum_{v \in \mathcal{D}} t_v ||diam(v)||_1 \leq \sum_{i=1}^{d} \sum_{j \in \mathcal{D}_i} d \cdot T_j ||diam(j)||_1 = d \sum_{i=1}^{d} UB_i(\mathcal{D}_i).$$ □

3.3 An upper bound for k-means

The result we obtained for the k-medians problem can be extended to the k-means problem:

Theorem 4. *The price of explainability for k-means is $O(dk \log k)$.*

From an algorithmic perspective, in order to establish the theorem, we only need to replace the definition of $UB_i(\mathcal{D}')$ for a tree $\mathcal{D}'$ in $\mathcal{F}_{a,b}$ with

$$UB'_i(\mathcal{D}') = \sum_{j=a}^{b-1} T_j \cdot (diam(j)_i)^2.$$

Note that the only difference is the replacement of $diam(j)_i$ with $(diam(j)_i)^2$. As a consequence, for the k-means problem, the tree $\mathcal{D}_i$ is defined as the tree $\mathcal{D}'$ in $\mathcal{F}_{1,k}$ for which $UB'_i(\mathcal{D}')$ is minimum. It can also be constructed via dynamic programming.

Then, Theorem 4 can be proved by using arguments similar to those employed to bound the price of explainability for k-medians. The following inequalities are, respectively, counterparts of the inequalities (1), (4), (5) and (7):

$$cost(\mathcal{C}) \leq OPT + \sum_{v \in \mathcal{D}} t_v ||diam(v)||_2^2, \tag{14}$$

$$UB'_i(\mathcal{D}_i) \leq 2k \log k \left(\sum_{j=1}^{k-1} (c_i^{j+1} - c_i^j)^2 \cdot T_j \right), \tag{15}$$

$$\left(\sum_{j=1}^{k-1}(c_i^{j+1}-c_i^j)^2 \cdot T_j\right)/2 \leq OPT_i, \tag{16}$$

$$\sum_{v \in \mathcal{D}} t_v ||diam(v)||_2^2 \leq d \sum_{i=1}^{d} UB'_i(\mathcal{D}_i). \tag{17}$$

From the three last inequalities and the identity $OPT = \sum_{i=1}^d OPT_i$, we obtain

$$\sum_{v \in \mathcal{D}} t_v ||diam(v)||_2^2 \leq 4dk \log k \cdot OPT.$$

This together with the inequality (14) allows us to establish Theorem 4.

Inequality (14) is proved in [12]. The validity of inequalites (16) and (17) can be established by using exactly the same arguments employed to prove their counterparts. More specifically, the proof of Lemma 5 can be used for the former while the proof of Lemma 6 can be used for the latter.

The inequality (15) incurs an extra factor of k with respect to its counterpart. In order to prove this inequality, we apply the arguments of the proof of Lemma 4. The only required adaptation consists of replacing Equation (8) with the inequality

$$UB'_i(\mathcal{D}_i) \leq k \sum_{j=1}^{k-1} T_j \cdot \sum_{\ell=r(j)}^{s(j)-1} (c_i^{\ell+1}-c_i^{\ell})^2. \tag{18}$$

Inequality (18) holds because

$$UB'_i(\mathcal{D}_i) = \sum_{j=1}^{k-1} T_j \cdot (c_i^{s(j)} - c_i^{r(j)})^2,$$

and a simple application of Jensen's inequality assures that

$$(c_i^{s(j)} - c_i^{r(j)})^2 \leq k \sum_{\ell=r(j)}^{s(j)-1} (c_i^{\ell+1}-c_i^{\ell})^2.$$

4 Maximum Spacing Clustering

We show that the price of explainability for the maximum-spacing problem is $\Theta(n-k)$.

4.1 Lower bound

The following simple construction shows that price of explainability is $\Omega(n-k)$.

Let $C_1 = \{(0,i) | 0 \leq i \leq p\} \cup \{(i,0) | 0 \leq i \leq p\}$. Moreover, for $i = 2, \ldots, k$, let $C_i = \{(i-1)(p-1), (p-1)\}$. The dataset $\mathcal{X}$ for our instance is given by $C_1 \cup \ldots \cup C_k$.

The unrestricted k-clustering $(C_1, \ldots, C_k)$ has spacing $p-1 = (n-k)/2-1$. On the other hand, every explainable k-clustering has spacing 1. To see that, note that we cannot have all the points of $C_1 \cup C_2$ in the same cluster, for otherwise we would have at most $k-1$ clusters. Thus, we need to separate at least 2 points from $C_1 \cup C_2$ and the only way to accomplish that, via axis-aligned cuts, forces the separation of 2 points in C_1 that are at distance 1 from each other. Thus, the spacing will be 1.

Lemma 7. *The price of explainability for the maximum spacing clustering problem is $\Omega(n-k)$.*

4.2 Upper Bound

We present an algorithm that always obtains an explainable clustering with spacing $O((n-k)OPT)$, where OPT is the spacing of the optimal unrestricted clustering. This, together with the previous lemma, implies that the price of explainability for the maximum-spacing problem is $\Theta(n-k)$.

Algorithm 3 receives an optimal k-clustering $\mathcal{C}^*$ as input and uses it as a guide to transform an initial single cluster containing all points of $\mathcal{X}$ into an explainable k-clustering. At each iteration, the algorithm finds a cluster C in the current clustering that contains two points that lie in distinct clusters in $\mathcal{C}^*$ and splits C into two new clusters by using the best possible axis-aligned cut. The method stops when a k-clustering is obtained.

Algorithm 3 `Ex-SingleLink`($\mathcal{X}$)

> $\mathcal{C}^* \leftarrow$ optimal unrestriced k-clustering for points in $\mathcal{X}$.
> $\mathcal{C} \leftarrow$ single cluster containing all points of $\mathcal{X}$
> **for** $i = 1, \ldots, k-1$ **do**
> Select a cluster $C \in \mathcal{C}$ that contains two points that lie in different clusters in $\mathcal{C}^*$. (*)
> Split C using an axis-aligned cut that yields a 2-clustering (C', C'') with maximum possible spacing.
> Remove C from $\mathcal{C}$ and then add both C' and C'' to $\mathcal{C}$
> **end for**

Lemma 8. *Given a set of points $\mathcal{X}$,* `Ex-SingleLink`*($\mathcal{X}$) obtains a k-clustering $\mathcal{C}$ with spacing at least $OPT/(n-k)$, where OPT is the spacing of an optimal unrestricted clustering.*

Proof. First, we observe that it is always possible to properly execute line (*) of `Ex-SingleLink`. In fact, if we pick k points covering all the k clusters of $\mathcal{C}^*$ then, by the pigeonhole principle, two of them will lie in the same group in $\mathcal{C}$ since $\mathcal{C}$ has less than k groups when line (*) is executed.

To establish the result it suffices to prove that there is always an axis-aligned cut that splits the selected cluster C into two clusters with spacing at least $OPT/(n-k)$.

Let $\mathbf{p}$ and $\mathbf{q}$ be two points in C that lie in distinct clusters in $\mathcal{C}^*$ and let $G=(V,E)$ be a graph, where V is the set of points in C and E connects points in V with distance smaller than $OPT/(n-k)$. Moreover, let $F=(T_1,\ldots,T_\ell)$ be a forest that is obtained by running Kruskal's MST algorithm on G.

Claim 2. *Points in C that belong to distinct clusters of $\mathcal{C}^*$ must also belong to different trees in forest F.*

Proof. For the sake of contradiction we assume that the claim does not hold. In this case, there would be a path from $\mathbf{p}$ to $\mathbf{q}$ in F and this path would have an edge joining two points that belong to different clusters in $\mathcal{C}^*$, which cannot occur since their distance is at least $OPT > OPT/(n-k)$. □

The previous claim implies that $\ell \geq 2$ since $\mathbf{p}$ and $\mathbf{q}$ belong to different clusters. We say that an axis-aligned cut is *good* with respect to a cluster C if it satisfies the following properties: (i) it separates the points in C into two non-empty clusters and (ii) it does not separate points that lie in the same tree of F. If a good cut exists, then we can use it to split C into two clusters with spacing at least $OPT/(n-k)$ since, by construction, points in different trees have distance at least $OPT/(n-k)$. For the sake of contradiction let us assume that such a cut does not exist.

For each $j \in [d]$ let $I_j^{\mathbf{pq}}$ be the real interval that starts in $\min\{p_j,q_j\}$ and ends in $\max\{p_j,q_j\}$, that is, $I_j^{\mathbf{pq}} = [\min\{p_j,q_j\}, \max\{p_j,q_j\}]$.

Moreover, for each tree T in F, let I_j^T be the interval that starts at $\min\{x_j|\mathbf{x}$ is a node in $T\}$ and ends at $\max\{x_j|\mathbf{x}$ is a node in $T\}$. Finally, for each edge $e = uv$ in F and each $j \in [d]$, let I_j^e be the real interval that starts at $\min\{u_j, v_j\}$ and ends at $\max\{u_j, v_j\}$. For a real interval I, let $len(I)$ be its length.

Since there are no good cuts, for $j = 1, \dots, d$, we have

$$\sum_{T \in F} len(I_j^T) \geq len(I_j^{\mathbf{pq}}).$$

From the triangle inequality we obtain

$$\sum_{T \in F} \sum_{e \in T} len(I_j^e) \geq \sum_{T \in F} len(I_j^T).$$

From the two previous inequalities we get

$$\sum_{e \in F} len(I_j^e) = \sum_{T \in F} \sum_{e \in T} len(I_j^e) \geq len(I_j^{\mathbf{pq}}).$$

A simple application of Jensen inequality shows that

$$\sum_{e \in F} len(I_j^e)^2 \geq \frac{(len(I_j^{\mathbf{pq}}))^2}{f},$$

where f is the number of edges in F. By adding the above inequality for all $j \in [d]$ we get

$$\sum_{e \in F} ||e||_2^2 \geq \frac{1}{f} ||\mathbf{p} - \mathbf{q}||_2^2 \geq \frac{OPT^2}{f},$$

where $||e||_2$ is the distance between the two endpoints of edge e.

The last inequality implies $||e||_2 \geq OPT/f$, for some edge e. Thus, to obtain a contradiction, it suffices to show that $f \leq n-k$, since we cannot have edges in F with distance $\geq OPT/(n-k)$

To see that $f \leq n-k$, let k' be the number of clusters in $\mathcal{C}$ that are singletons and let S' be the set of points in these clusters. Moreover, let $S \subseteq \mathcal{X} - S'$ be a set of $k - k'$ points with each of them belonging to a different cluster in $\mathcal{C}^*$. Note that cluster C is not a singleton since $\mathbf{p}, \mathbf{q} \in C$. Since both C and S are subsets of $\mathcal{X} - S'$ we have $|C \cup S| = |C| + |S| - |C \cap S| \leq n - k'$ so that $|C| - |C \cap S| \leq n - k$. It follows from Claim 2 that the number of trees in F is at least $|C \cap S|$ and, as a result, its number of edges f satisfies $f \leq |C| - |C \cap S| - 1 < n - k$ edges. $\square$

We can state the main result of this section.

Theorem 5. *The price of explainability for the maximum-spacing problem is $\Theta(n-k)$.*

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