Multi-Criteria Dimensionality Reduction with Applications to Fairness

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Abstract

Dimensionality reduction is a classical technique widely used for data analysis. One foundational instantiation is Principal Component Analysis (PCA), which minimizes the average reconstruction error. In this paper, we introduce the *multi-criteria dimensionality reduction* problem where we are given multiple objectives that need to be optimized simultaneously. As an application, our model captures several fairness criteria for dimensionality reduction such as our novel Fair-PCA problem and the Nash Social Welfare (NSW) problem. In Fair-PCA, the input data is divided into k groups, and the goal is to find a single d-dimensional representation for all groups for which the minimum variance of any one group is maximized. In NSW, the goal is to maximize the product of the individual variances of the groups achieved by the common low-dimensional space.

Our main result is an exact polynomial-time algorithm for the two-criterion dimensionality reduction problem when the two criteria are increasing concave functions. As an application of this result, we obtain a polynomial time algorithm for Fair-PCA for k=2 groups and a polynomial time algorithm for NSW objective for k=2 groups. We also give approximation algorithms for k>2. Our technical contribution in the above results is to prove new low-rank properties of extreme point solutions to semi-definite programs. We conclude with experiments indicating the effectiveness of algorithms based on extreme point solutions of semi-definite programs on several real-world data sets.

1 Introduction

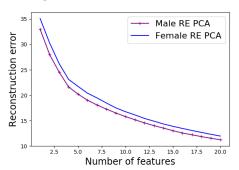
Dimensionality reduction is the process of choosing a low-dimensional representation of a large, high-dimensional data set. It is a core primitive for modern machine learning and is being used in image processing, biomedical research, time series analysis, etc. Dimensionality reduction can be used during the preprocessing of the data to reduce the computational burden as well as at the final stages of data analysis to facilitate data summarization and data visualization [72, 41]. Among the most ubiquitous and effective of dimensionality reduction techniques in practice are Principal Component Analysis (PCA) [68, 43, 39], multidimensional scaling [57], Isomap [78], locally linear embedding [73], and t-SNE [61].

One of the major obstacles to dimensionality reduction tasks in practice is complex high-dimensional data structures that lie on multiple different low-dimensional subspaces. For example, Maaten and Hinton [61] address this issue for low-dimensional visualization of images of objects from diverse classes seen from various viewpoints. Dimensionality reduction algorithms may optimize one data structure well while performs poorly on the others. In this work, we consider when those data structures lying on different

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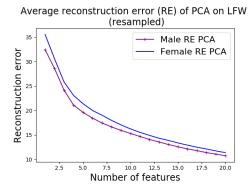


Figure 1: Left: average reconstruction error of PCA on labeled faces in the wild data set (LFW), separated by gender. Right: the same, but sampling 1000 faces with men and women equiprobably (mean over 20 samples).

low-dimensional subspaces are subpopulations partitioned by sensitive attributes, such as gender, race, and education level.

As an illustration, consider applying PCA on a high-dimensional data to do a visualization analysis in low dimensions. Standard PCA aims to minimize the single criteria of average reconstruction error over the whole data, but the reconstruction error on different parts of data can be different. In particular, we show in Figure 1 that PCA on the real-world labeled faces in the wild data set (LFW) [40] has higher reconstruction error for women than men, and this disparity in performance remains even if male and female faces are sampled with equal weight. We similarly observe difference in reconstruction errors of PCA in other real-world datasets. Dissimilarity of performance on different data structure, such as unbalanced average reconstruction errors we demonstrated, raises ethical and legal concerns whether outcomes of algorithms discriminate the subpopulations against sensitive attributes.

Relationship to fairness in machine learning. In recent years, machine learning community has witnessed an onslaught of charges that real-world machine learning algorithms have produced "biased" outcomes. The examples come from diverse and impactful domains. Google Photos labeled African Americans as gorillas [79, 75] and returned queries for CEOs with images overwhelmingly male and white [53], searches for African American names caused the display of arrest record advertisements with higher frequency than searches for white names [77], facial recognition has wildly different accuracy for white men than dark-skinned women [17], and recidivism prediction software has labeled low-risk African Americans as high-risk at higher rates than low-risk white people [4].

The community's work to explain these observations has roughly fallen into either "biased data" or "biased algorithm" bins. In some cases, the training data might under-represent (or over-represent) some group, or have noisier labels for one population than another, or use an imperfect proxy for the prediction label (e.g., using arrest records in lieu of whether a crime was committed). Separately, issues of imbalance and bias might occur due to an algorithm's behavior, such as focusing on accuracy across the entire distribution rather than guaranteeing similar false positive rates across populations, or by improperly accounting for confirmation bias and feedback loops in data collection. If an algorithm fails to distribute loans or bail to a deserving population, the algorithm won't receive additional data showing those people would have paid back the loan, but it will continue to receive more data about the populations it (correctly) believed should receive loans or bail.

Many of the proposed solutions to "biased data" problems amount to re-weighting the training set or adding noise to some of the labels; for "biased algorithms," most work has focused on maximizing accuracy

subject to a constraint forbidding (or penalizing) an unfair model. Both of these concerns and approaches have significant merit, but form an incomplete picture of the machine learning pipeline where unfairness might be introduced therein. Our work takes another step in fleshing out this picture by analyzing when dimensionality reduction might inadvertently introduce bias.

This work underlines the importance of considering fairness and bias at every stage of data science, not only in gathering and documenting a data set [33] and in training a model, but also in any interim data processing steps. Many scientific disciplines have adopted PCA as a default preprocessing step, both to avoid the curse of dimensionality and also to do exploratory/explanatory data analysis (projecting the data into a number of dimensions that humans can more easily visualize). The study of human biology, disease, and the development of health interventions all face both aforementioned difficulties, as do numerous economic and financial analysis. In such high-stakes settings, where statistical tools will help in making decisions that affect a diverse set of people, we must take particular care to ensure that we share the benefits of data science with a diverse community.

We also emphasize this work has implications for representational rather than just allocative harms, a distinction drawn by Crawford [25] between how people are represented and what goods or opportunities they receive. Showing primates in search results for African Americans is repugnant primarily due to its representing and reaffirming a racist painting of African Americans, not because it directly reduces any one person's access to a resource. If the default template for a data set begins with running PCA, and PCA does a better job representing men than women, or white people over minorities, the new representation of the data set itself may rightly be considered an unacceptable sketch of the world it aims to describe.

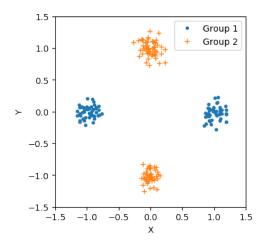
Remark 1.1. We focus on the setting where we ask for a single projection into d dimensions rather than separate projections for each group, because using distinct projections (or more generally distinct models) for different populations raises legal and ethical concerns.¹

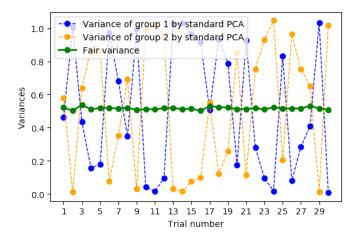
Instability of PCA. Disparity of performance in subpopulations of PCA is closely related to its instability. Maximizing total variance or equivalently minimizing total reconstruction errors is sensitive to a slight change of data, giving widely different outcomes even if data are sampled from the same distribution. An example is shown in Figure 2. Figure 2a shows the distribution of two groups lying in orthogonal dimensions. When the first group's variance in the x-axis is slightly higher than the second group's variance in the y-axis, PCA outputs the x-axis, otherwise it outputs the y-axis, and it rarely outputs something in between. The instability of performance can be shown in Figure 2b. Even though in each trial, data are sampled from the same distribution, PCA solutions are unstable and give oscillating variances to each group. However, solutions to one of our proposed formulations (FAIR-PCA, which is to be presented later) are stable and give the same, optimal variance to both groups in each trial.

Our work presents a novel general framework that addresses all aforementioned issues: data lying on different low-dimensional structure, unfairness, and instability of PCA. A common difficulty in those settings is that a single criteria for dimensionality reduction might not be sufficient to capture different structures in the data. This motivates our study of multi-criteria dimensionality reduction.

Multi-criteria dimensionality reduction. Multi-criteria dimensionality reduction could be used as an umbrella term with specifications changing based on the applications and the metrics that the machine learning researcher has in mind. Aiming for an output with a balanced error over different subgroups seems to be a natural choice, extending economic game theory literature. For example, this covers maximizing geometric mean of the variances of the groups, which is the well-studied Nash social welfare (NSW) objective [51, 63]. Motivated by these settings, the more general question that we would like to study is as follows.

¹Lipton et al. [59] have asked whether equal treatment requires different models for two groups.





- (a) A distribution of two groups where PCA into one dimension is unfair and unstable
- (b) Variances of two groups by PCA are unstable as data are resampled from the same distribution across many trials. Large gaps of two variances results from PCA favoring one group and ignoring the other. Fair-PCA equalizes two variances and is stable over resampling.

Figure 2: An example of the distribution of two groups which has very unstable and unfair PCA output

Question 1. How might one redefine dimensionality reduction to produce projections which optimize different groups' representation in a balanced way?

For simplicity of explanation, we first describe our framework for PCA, but the approach is general and applies to a much wider class of dimensionality reduction techniques. Consider the data points as rows of an $m \times n$ matrix A. For PCA, the objective is to find an $n \times d$ projection matrix P that maximizes the Frobenius norm $\|AP\|_F^2$ (this is equivalent to minimizing the reconstruction error $\|A - APP^T\|_F^2$). Suppose that the rows of P belong to different *groups* based on demographics or some other semantically meaningful clustering. The definition of these groups need not be a partition; each group could be defined as a different weighting of the data set (rather than a subset, which is a 0/1 weighting). Multi-criteria dimensionality reduction can then be viewed as simultaneously considering objectives on the different weightings of P, i.e., P in P is a projection P that maximizes the minimum objective value over each of the groups (weightings):

$$\max_{P \in \mathbb{R}^{n \times d}: P^T P = I_d} \min_{1 \le i \le k} \|A_i P\|_F^2 = \langle A_i^T A_i, P P^T \rangle.$$
 (FAIR-PCA)

More generally, let \mathcal{P}_d denote the set of all $n \times d$ projection matrices P, i.e., matrices with d orthonormal columns. For each group A_i , we associate a function $f_i: \mathcal{P}_d \to \mathbb{R}$ that denotes the group's objective value for a particular projection. We are also given an accumulation function $g: \mathbb{R}^k \to \mathbb{R}$. We define the (f,g)-multi-criteria dimensionality reduction problem as finding a d-dimensional projection P which optimizes

$$\max_{P\in\mathcal{P}_d}g(f_1(P),f_2(P),\ldots,f_k(P)). \tag{MULTI-CRITERIA-DIMENSION-REDUCTION}$$

In the above example of FAIR-PCA, g is simply the min function and $f_i(P) = ||A_iP||^2$ is the total squared norm of the projection of vectors in A_i . The central motivating questions of this paper are the following:

• What is the complexity of FAIR-PCA?

• More generally, what is the complexity of MULTI-CRITERIA-DIMENSION-REDUCTION?

Framed another way, we ask whether these multi-criteria optimization problems force us to incur substantial computational cost compared to optimizing g over A alone.

Summary of contributions. We summarize our contributions in this work as follows.

- 1. We introduce a novel definition of MULTI-CRITERIA-DIMENSION-REDUCTION.
- 2. We give polynomial-time algorithms for MULTI-CRITERIA-DIMENSION-REDUCTION with provable guarantees.
- 3. We analyze the complexity and show hardness of MULTI-CRITERIA-DIMENSION-REDUCTION.
- 4. We present empirical results to show efficacy of our algorithms in addressing fairness.

We have introduced MULTI-CRITERIA-DIMENSION-REDUCTION earlier, and we now present the technical contributions in this paper.

1.1 Summary of technical results

Let us first focus on FAIR-PCA for ease of exposition. The problem can be reformulated as the following mathematical program where we denote PP^T by X. A natural approach to solving this problem is to consider the SDP relaxation obtained by relaxing the rank constraint to a bound on the trace.

Exact FAIR-PCA	SDP relaxation of FAIR-PCA	
$\max z$	$\max z$	(1)
$\langle A_i^T A_i, X \rangle \ge z i \in \{1, \dots, k\}$	$\langle A_i^T A_i, X \rangle \ge z i \in \{1, \dots, k\}$	(2)
$\operatorname{rank}(X) \leq d$	$\operatorname{tr}(X) \leq d$	(3)
$0 \leq X \leq I$	$0 \leq X \leq I$	(4)

Our first main result is that the SDP relaxation is exact when there are *two* groups. Thus finding an extreme point of this SDP gives an exact algorithm for FAIR-PCA for two groups.

Theorem 1.2. Any optimal extreme point solution to the SDP relaxation for FAIR-PCA with two groups has rank at most d. Therefore, 2-group FAIR-PCA can be solved in polynomial time.

Given m data points partitioned into $k \leq n$ groups in n dimensions, the algorithm runs in $O(nm+n^{6.5})$ time. O(mnk) is from computing $A_i^TA_i$ and $O(n^{6.5})$ is from solving an SDP over $n \times n$ PSD matrices [10]. Alternative heuristics and their analyses are discussed in Section 7.2. Our results also hold for the MULTI-CRITERIA-DIMENSION-REDUCTION when g is monotone nondecreasing in any one coordinate and concave, and each f_i is an affine function of PP^T (and thus a special case of a quadratic function in P).

Theorem 1.3. There is a polynomial-time algorithm for 2-group MULTI-CRITERIA-DIMENSION-REDUCTION when g is concave and monotone nondecreasing for at least one of its two arguments and each f_i is linear in PP^T , i.e., $f_i(P) = \langle B_i, PP^T \rangle$ for some matrix $B_i(A)$.

As indicated in the theorem, the core idea is that extreme-point solutions of the SDP in fact have rank d, not just trace equal to d. For k>2, the SDP need not recover a rank-d solution. In fact, the SDP may be inexact even for k=3 (see Section 6.2). Nonetheless, we show that we can bound the rank of a solution to the SDP and obtain the following result. We state it for FAIR-PCA, although the same bound holds for MULTI-CRITERIA-DIMENSION-REDUCTION under the same assumptions as in Theorem 1.3. Note that this result generalizes Theorems 1.2 and 1.3.

Theorem 1.4. For any concave g that is monotone nondecreasing in at least one of its arguments, there exists a polynomial time algorithm for MULTI-CRITERIA-DIMENSION-REDUCTION with k groups that returns a $d + \left\lfloor \sqrt{2k + \frac{1}{4}} - \frac{3}{2} \right\rfloor$ -dimensional embedding whose objective value is at least that of the optimal d-dimensional embedding. If g is only concave, then the solution lies in at most d+1 dimensions.

We note that the iterative rounding framework for linear programs [58] would give a rank bound of d+k-1 for the FAIR-PCA problem (see [74] for details). Hence, we strictly improves the bound to $d+\left\lfloor\sqrt{2k+\frac{1}{4}}-\frac{3}{2}\right\rfloor$. Moreover, if the dimensionality of the solution is a hard constraint, instead of tolerating $s=O(\sqrt{k})$ extra dimension in the solution, one may solve FAIR-PCA for target dimension d-s to guarantee a solution of rank at most d. Thus, we obtain an approximation algorithm for FAIR-PCA of factor $1-\frac{O(\sqrt{k})}{d}$.

Corollary 1.5. Let A_1, \ldots, A_k be data sets of k groups and suppose $s := \left\lfloor \sqrt{2k + \frac{1}{4}} - \frac{3}{2} \right\rfloor < d$. Then there exists a polynomial-time approximation algorithm of factor $1 - \frac{s}{d} = 1 - \frac{O(\sqrt{k})}{d}$ to FAIR-PCA.

That is, the algorithm returns a projection $P \in \mathcal{P}_d$ of exact rank d with objective at least $1 - \frac{s}{d}$ of the optimal objective. More details on the approximation result are in Section 3.1. The runtime of Theorems 1.3 and 1.4 depends on the access to first order oracle to g, and standard application of the ellipsoid algorithm would take $\tilde{O}(n^2)$ oracle calls.

We also develop a general rounding framework for SDPs with eigenvalue upper bounds and k other linear constraints. This algorithm gives a solution of desired rank that violates each constraint by a bounded amount. It implies that for FAIR-PCA and some of its variants, the additive error is

$$\Delta(\mathcal{A}) := \max_{S \subseteq [m]} \sum_{i=1}^{\lfloor \sqrt{2|S|} + 1 \rfloor} \sigma_i(A_S)$$

where $A_S = \frac{1}{|S|} \sum_{i \in S} A_i$. The precise statement is Theorem 1.9 and full details are presented in Section 4. It is natural to ask whether FAIR-PCA is NP-hard to solve exactly. The following result implies that it is, even for target dimension d = 1.

Theorem 1.6. The FAIR-PCA problem for target dimension d = 1 is NP-hard when the number of groups k is part of the input.

This raises the question of the complexity for constant $k \geq 3$ groups. For k groups, we would have k constraints, one for each group, plus the eigenvalue constraint and the trace constraint; now the tractability of the problem is far from clear. In fact, as we show in Section 6.2, the SDP has an integrality gap even for k=3, d=1. We therefore consider an approach beyond SDPs, to one that involves solving nonconvex problems. Thanks to the powerful algorithmic theory of quadratic maps, developed by Grigoriev and Pasechnik [36], it is polynomial-time solvable to check feasibility of a set of quadratic constraints for any fixed k. As we discuss next, their algorithm can check for zeros of a function of a set of k quadratic functions, and can be used to optimize the function. Using this result, we show that for k = k = k0(1), there is a polynomial-time algorithm for rather general functions k0 of the values of individual groups.

Theorem 1.7. Let $g: \mathbb{R}^k \to \mathbb{R}$ where g is a degree- ℓ polynomial in some computable subring of \mathbb{R}^k , and let each f_i be quadratic for $1 \le i \le k$. Then there is an algorithm to solve (f,g)-MULTI-CRITERIA-DIMENSION-REDUCTION in time $(\ell dn)^{O(k+d^2)}$.

By choosing g to be the product polynomial over the usual $(\times, +)$ ring or the min function which is degree k in the $(\min, +)$ ring, this applies to FAIR-PCA discussed above and various other problems.

Techniques

SDP extreme points. For k=2, the underlying structural property we show is that extreme point solutions of the SDP have rank exactly d. First, for k=d=1, this is the largest eigenvalue problem, since the maximum obtained by a matrix of trace equal to 1 can also be obtained by one of the extreme points in the convex decomposition of this matrix. This extends to trace equal to any d, i.e., the optimal solution must be given by the top d eigenvectors of A^TA . Second, without the eigenvalue bound, for any SDP with k constraints, there is an upper bound on the rank of any extreme point, of $O(\sqrt{k})$, a seminal result of Pataki [67] (see also Barvinok [9]). However, we cannot apply this directly as we have the eigenvalue upper bound constraint. The complication here is that we have to take into account the constraint $X \leq I$ without increasing the rank.

Theorem 1.8. Let C and A_1, \ldots, A_m be $n \times n$ real matrices, $d \leq n$, and $b_1, \ldots b_m \in \mathbb{R}$. Suppose the *semi-definite program* $\mathbb{SDP}(\mathbb{I})$:

$$\min\langle C, X \rangle$$
 subject to (5)

$$\langle A_i, X \rangle \quad \triangleleft_i \quad b_i \quad \forall \ 1 \le i \le m$$
 (6)

$$\langle A_i, X \rangle \quad \lhd_i \quad b_i \quad \forall \ 1 \le i \le m$$
 (6)
 $\operatorname{tr}(X) \quad \le \quad d$ (7)

$$0 \le X \le I_n \tag{8}$$

where $\lhd_i \in \{\leq, \geq, =\}$, has a nonempty feasible set. Then, all extreme optimal solutions X^* to $\mathbb{SDP}(\mathbb{I})$ have rank at most $r^* := d + \left| \sqrt{2m + \frac{9}{4}} - \frac{3}{2} \right|$. Moreover, given a feasible optimal solution, an extreme optimal solution can be found in polynomial time.

To prove the theorem, we extend Pataki [67]'s characterization of rank of SDP extreme points with minimal loss in the rank. We show that the constraints $0 \le X \le I$ can be interpreted as a generalization of restricting variables to lie between 0 and 1 in the case of linear programming relaxations. From a technical perspective, our results give new insights into structural properties of extreme points of semi-definite programs and more general convex programs. Since the result of [67] has been studied from perspective of fast algorithms [16, 18, 19] and applied in community detection and phase synchronization [7], we expect our extension of the result to have further applications in many of these areas.

SDP iterative rounding. Using Theorem 1.8, we extend the iterative rounding framework for linear programs (see [58] and references therein) to semi-definite programs, where the 0, 1 constraints are generalized to eigenvalue bounds. The algorithm has a remarkably similar flavor. In each iteration, we fix the subspaces spanned by eigenvectors with 0 and 1 eigenvalues, and argue that one of the constraints can be dropped while bounding the total violation in the constraint over the course of the algorithm. While this applies directly to the FAIR-PCA problem, it is in fact a general statement for SDPs, which we give below.

Let $\mathcal{A} = \{A_1, \dots, A_m\}$ be a collection of $n \times n$ matrices. For any set $S \subseteq \{1, \dots, m\}$, let $\sigma_i(S)$ the i^{th} largest singular of the average of matrices $\frac{1}{|S|} \sum_{i \in S} A_i$. We let

$$\Delta(\mathcal{A}) := \max_{S \subseteq [m]} \sum_{i=1}^{\lfloor \sqrt{2|S|} + 1 \rfloor} \sigma_i(S).$$

Theorem 1.9. Let C be a real $n \times n$ matrix and $A = \{A_1, \ldots, A_m\}$ be a collection of real $n \times n$ matrices, $d \leq n$, and $b_1, \ldots b_m \in \mathbb{R}$. Suppose the semi-definite program \mathbb{SDP} :

$$\begin{array}{rcl} \min \langle C, X \rangle \ \textit{subject to} \\ & \langle A_i, X \rangle & \geq & b_i \ \forall \ 1 \leq i \leq m \\ & \operatorname{tr}(X) & \leq & d \\ & 0 \prec X & \prec & I_n \end{array}$$

has a nonempty feasible set and let X^* denote an optimal solution. The algorithm Iterative-SDP (see Algorithm 1 in Section 4) returns a matrix \tilde{X} such that

- 1. rank of \tilde{X} is at most d,
- 2. $\langle C, \tilde{X} \rangle \leq \langle C, X^* \rangle$, and
- 3. $\langle A_i, \tilde{X} \rangle \geq b_i \Delta(\mathcal{A})$ for each $1 \leq i \leq m$.

Moreover, ITERATIVE-SDP runs in polynomial time.

The time complexity of Theorems 1.8 and 1.9 is analyzed in Sections 2 and 4, respectively. Both algorithms introduce the rounding procedures that do not contribute significant computational cost; rather, solving the SDP is the bottleneck for running time both in theory and practice.

1.3 Organization

We present related work in Section 1.4. In Section 2, we prove Theorem 1.8 and apply the result to MULTI-CRITERIA-DIMENSION-REDUCTION to obtain Theorem 1.4. In Section 3, we present and motivate several fairness criteria for dimensionality reduction, including a novel one of our own, and apply Theorem 1.4 to get approximation algorithm of FAIR-PCA, thus proving Corollary 1.5. In Section 4, we give an iterative rounding algorithm and prove Theorem 1.9. In Section 5, we show the polynomial-time solvability of MULTI-CRITERIA-DIMENSION-REDUCTION when the number of groups k and the target dimension d are fixed, proving Theorem 1.7. In Section 6, we show NP-hardness and integrality gap of MULTI-CRITERIA-DIMENSION-REDUCTION for k > 2. In Section 7, we show the experimental results of our algorithms on real-world data sets, evaluated by different fairness criteria, and present additional algorithms with improved runtime. We present missing proofs in Appendix A. In Appendix B, we show that the rank of extreme solutions of SDPs in Theorem 1.8 cannot be improved.

1.4 Related work

Optimization. As mentioned earlier, Pataki [67] (see also Barvinok [9]) showed that low rank solutions to semi-definite programs with small number of affine constraints can be obtained efficiently. Restricting a feasible region of certain SDPs relaxations with low-rank constraints has been shown to avoid spurious local optima [7] and reduce the runtime due to known heuristics and analysis [18, 19, 16]. We also remark that methods based on Johnson-Lindenstrauss lemma can also be applied to obtain bi-criteria results for the FAIR-PCA problem. For example, So et al. [76] give algorithms that give low rank solutions for SDPs with affine constraints without the upper bound on eigenvalues. Here we have focused on the single criteria setting, with violation either in the number of dimensions or the objective but not both. We also remark that extreme point solutions to linear programming have played an important role in design of approximation algorithms [58], and our result add to the comparatively small, but growing, number of applications for utilizing extreme points of semi-definite programs.

A closely related area, especially to MULTI-CRITERIA-DIMENSION-REDUCTION, is multi-objective optimization which has a vast literature. We refer the reader to Deb [26] and references therein. We remark that properties of extreme point solutions of linear programs [70, 34] have also been utilized to obtain approximation algorithms to multi-objective problems. For semi-definite programming based methods, the closest works are on simultaneous max-cut [12, 13] that utilize sum of squares hierarchy to obtain improved approximation algorithms.

Fairness in machine learning. The applications of multi-criteria dimensionality reduction in fairness are closely related to studies on representational bias in machine learning [25, 65, 15], for which there have been various mathematical formulations studied [23, 22, 55, 56]. One interpretation of our work is that we suggest using multi-criteria dimensionality reduction rather than standard PCA when creating a lower-dimensional representation of a data set for further analysis. Two most relevant pieces of work take the posture of explicitly trying to reduce the correlation between a sensitive attribute (such as race or gender) and the new representation of the data. The first piece is a broad line of work [85, 11, 21, 62, 86] that aims to design representations which will be conditionally independent of the protected attribute, while retaining as much information as possible (and particularly task-relevant information for some fixed classification task). The second piece is the work by Olfat and Aswani [66], who also look to design PCA-like maps which reduce the projected data's dependence on a sensitive attribute. Our work has a qualitatively different goal: we aim not to hide a sensitive attribute, but to instead maintain as much information about each population after projecting the data. In other words, we look for representation with similar richness for population, rather than making each group indistinguishable.

Other work has developed techniques to obfuscate a sensitive attribute directly [69, 48, 20, 47, 60, 49, 50, 37, 30, 84, 31, 1]. This line of work diverges from ours in two ways. First, these works focus on representations which obfuscate the sensitive attribute rather than a representation with high fidelity regardless of the sensitive attribute. Second, most of these works do not give formal guarantees on how much an objective will degrade after their transformations. Our work gives theoretical guarantees including an exact optimality for two groups.

Much of other work on fairness for learning algorithms focuses on fairness in classification or scoring [27, 38, 54, 24], or in online learning settings [44, 52, 28]. These works focus on either statistical parity of the decision rule, or equality of false positives or negatives, or an algorithm with a fair decision rule. All of these notions are driven by a single learning task rather than a generic transformation of a data set, while our work focuses on a ubiquitous, task-agnostic preprocessing step.

Game theory applications. The applications of multi-criteria dimensionality reduction in fairness are closely related to studies on fair resource allocation in game theory [81, 29]. From the game theory literature, our model covers Nash social welfare objective [51, 63] and others [46, 45].

2 Low-rank solutions of MULTI-CRITERIA-DIMENSION-REDUCTION

In this section, we show that all extreme solutions of SDP relaxation of MULTI-CRITERIA-DIMENSION-REDUCTION have low rank, proving Theorem 1.2-1.4. Before we state the results, we make the following assumptions. In this section, we let $g: \mathbb{R}^k \to \mathbb{R}$ be a concave function, and mildly assume that g can be accessed with a polynomial-time subgradient oracle. We are explicitly given functions f_1, f_2, \ldots, f_k which are affine in PP^T , i.e., we are given real $n \times n$ matrices B_1, \ldots, B_k and constants $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$, and $f_i(P) = \langle B_i, PP^T \rangle + \alpha_i$.

We assume g to be G-Lipschitz. For functions f_1, \ldots, f_k, g that are L_1, \ldots, L_k, G -Lipschitz, we define an ϵ -optimal solution to (f, g)-MULTI-CRITERIA-DIMENSION-REDUCTION as a matrix $X \in \mathbb{R}^{n \times n}$ of rank

d with $0 \leq X \leq I_n$ whose objective value is at most $G\epsilon\left(\sum_{i=1}^k L_i^2\right)^{1/2}$ away from the optimum. IWhen an optimization problem has affine constraints $F_i(X) \leq b_i$ where F_i is L_i -Lipschitz for all $i \in \{1, \ldots, m\}$, we also define an ϵ -feasible solution as a projection matrix $X \in \mathbb{R}^{n \times n}$ of rank d with $0 \leq X \leq I_n$ that violates the ith affine constraint $F_i(X) \leq b_i$ by at most ϵL_i for all i. Note that the feasible region of the problem is implicitly bounded by the constraint $X \leq I_n$.

In this work, an algorithm may involve solving an optimization under a matrix linear inequality, whose exact optimal solutions may not be representable in finite bits of computation. However, we give algorithms that return an ϵ -feasible solution whose running time depends polynomially on $\log \frac{1}{\epsilon}$ for any $\epsilon > 0$. This is standard for computational tractability in convex optimization (see, for example, in [10]). Therefore, for ease of exposition, we omit the computational error dependent on this ϵ to obtain an ϵ -feasible and ϵ -optimal solution, and define polynomial time as polynomial in n, k and $\log \frac{1}{\epsilon}$.

We first prove Theorem 1.8 below. To prove Theorem 1.2-1.4, we first show that extreme point solutions in a general class of semi-definite cone under affine constraints and $X \leq I$ have low rank. The statement builds on a result of [67], and also generalizes to SDPs under a constraint $X \leq C$ for any given PSD matrix $C \in \mathbb{R}^{n \times n}$ by the transformation $X \mapsto C^{\frac{1}{2}}YC^{\frac{1}{2}}$ of the SDP feasible region. We then apply our result to MULTI-CRITERIA-DIMENSION-REDUCTION, which generalizes FAIR-PCA, and prove Theorem 1.4, which implies Theorem 1.2 and 1.3.

Proof of Theorem 1.8: Let X^* be an extreme point optimal solution to $\mathbb{SDP}(\mathbb{I})$. Suppose rank of X^* , say r, is more than r^* . Then we show a contradiction to the fact that X^* is extreme. Let $0 \leq l \leq r$ of the eigenvalues of X^* be equal to one. If $l \geq d$, then we have l = r = d since $\operatorname{tr}(X) \leq d$, and we are done. Thus we assume that $l \leq d-1$. In that case, there exist matrices $Q_1 \in \mathbb{R}^{n \times r - l}$, $Q_2 \in \mathbb{R}^{n \times l}$ and a symmetric matrix $\Lambda \in \mathbb{R}^{(r-l) \times (r-l)}$ such that

$$X^* = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} \Lambda & 0 \\ 0 & I_l \end{pmatrix} \begin{pmatrix} Q_1 & Q_2 \end{pmatrix}^\top = Q_1 \Lambda Q_1^\top + Q_2 Q_2^T$$

where $0 \prec \Lambda \prec I_{r-l}$, $Q_1^TQ_1 = I_{r-l}$, $Q_2^TQ_2 = I_l$, and that the columns of Q_1 and Q_2 are orthogonal, i.e. $Q = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix}$ has orthonormal columns. Now, we have

$$\langle A_i, X^* \rangle = \langle A_i, Q_1 \Lambda Q_1^\top + Q_2 Q_2^\top \rangle = \langle Q_1^\top A_i Q_1, \Lambda \rangle + \langle A_i, Q_2 Q_2^\top \rangle$$

and $\operatorname{tr}(X^*) = \langle Q_1^\top Q_1, \Lambda \rangle + \operatorname{tr}(Q_2 Q_2^\top)$ so that $\langle A_i, X^* \rangle$ and $\operatorname{tr}(X^*)$ are linear in Λ .

Observe that the set of $s \times s$ symmetric matrices forms a vector space of dimension $\frac{s(s+1)}{2}$ with the above inner product where we consider the matrices as long vectors. If $m+1 < \frac{(r-l)(r-l+1)}{2}$, then there exists a $(r-l) \times (r-l)$ -symmetric matrix $\Delta \neq 0$ such that $\langle Q_1^\top A_i Q_1, \Delta \rangle = 0$ for each $1 \leq i \leq m$ and $\langle Q_1^\top Q_1, \Delta \rangle = 0$.

But then we claim that $\tilde{X} = Q_1(\Lambda \pm \delta \Delta)Q_1^\top + Q_2Q_2^T$ is feasible for some small $\delta > 0$, which implies a contradiction to X^* being extreme. Indeed, \tilde{X} satisfies all the linear constraints by the construction of Δ . Thus it remains to check the eigenvalues of \tilde{X} . Observe that

$$Q_1(\Lambda \pm \delta \Delta)Q_1^{\top} + Q_2Q_2^{T} = Q \begin{pmatrix} \Lambda \pm \delta \Delta & 0 \\ 0 & I_l \end{pmatrix} Q^{\top}$$

with orthonormal Q. Thus it is enough to consider the eigenvalues of $\begin{pmatrix} \Lambda \pm \delta \Delta & 0 \\ 0 & I_l \end{pmatrix}$.

Observe that eigenvalues of the above matrix are exactly l ones and eigenvalues of $\Lambda \pm \delta \Delta$. Since eigenvalues of Λ are bounded away from 0 and 1, one can find a small $\delta>0$ such that the eigenvalues of $\Lambda \pm \delta \Delta$ are bounded away from 0 and 1 as well, so we are done. Therefore, we must have $m+1 \geq \frac{(r-l)(r-l+1)}{2}$ which implies $r-l \leq -\frac{1}{2} + \sqrt{2m+\frac{9}{4}}$. By $l \leq d-1$, we have $r \leq r^*$.

To obtain the algorithmic result, given feasible \bar{X} , we iteratively reduce r-l by at least one until $m+1\geq \frac{(r-l)(r-l+1)}{2}$. While $m+1<\frac{(r-l)(r-l+1)}{2}$, we obtain Δ by Gaussian elimination. Now we want to find the correct value of $\pm\delta$ so that $\Lambda'=\Lambda\pm\delta\Delta$ takes one of the eigenvalues to zero or one. First, determine the sign of $\langle C,\Delta\rangle$ to find the correct sign to move Λ that keeps the objective non-increasing, say it is in the positive direction. Since the feasible set of $\mathbb{SDP}(\mathbb{I})$ is convex and bounded, the ray $f(t)=Q_1(\Lambda+t\Delta)Q_1^\top+Q_2Q_2^\top,t\geq 0$ intersects the boundary of feasible region at a unique t'>0. Perform binary search for t' up to a desired accuracy, and set $\delta=t'$. Since $\langle Q_1^\top A_i Q_1,\Delta\rangle=0$ for each $1\leq i\leq m$ and $\langle Q_1^\top Q_1,\Delta\rangle=0$, the additional tight constraint from moving $\Lambda'\leftarrow\Lambda+\delta\Delta$ to the boundary of the feasible region must be an eigenvalue constraint $0\leq X\leq I_n$, i.e., at least one additional eigenvalue is now at 0 or 1, as desired. We apply eigenvalue decomposition to Λ' and update Q_1 accordingly, and repeat.

The algorithm involves at most n rounds of reducing r-l, each of which involves Gaussian elimination and several iterations of checking $0 \le f(t) \le I_n$ (from binary search) which can be done by eigenvalue value decomposition. Gaussian elimination and eigenvalue decomposition can be done in $O(n^3)$ time, and therefore the total runtime of SDP rounding is $\tilde{O}(n^4)$ which is polynomial.

One can initially reduce the rank of given feasible \bar{X} using an LP rounding in $O(n^{3.5})$ time [74] before our SDP rounding. This reduces the number of iterative rounding steps; particularly, r-l is further bounded by k-1. The runtime complexity is then $O(n^{3.5}) + \tilde{O}(kn^3)$.

The next corollary is another useful fact of the low-rank property and is used in the analysis of iterative rounding algorithm in Section 4. The corollary can be obtained from the bound $r-l \le -\frac{1}{2} + \sqrt{2m + \frac{9}{4}}$ in the proof of Theorem 1.8.

Corollary 2.1. The number of fractional eigenvalues in any extreme point solution X to $\mathbb{SDP}(\mathbb{I})$ is bounded by $\sqrt{2m + \frac{9}{4}} - \frac{1}{2} \leq \lfloor \sqrt{2m} \rfloor + 1$.

We are now ready to prove the main result that we can find a low-rank solution for MULTI-CRITERIA-DIMENSION-REDUCTION.

Proof of Theorem 1.4: Let $r^* := d + \left\lfloor \sqrt{2k + \frac{1}{4}} - \frac{3}{2} \right\rfloor$. Given assumptions on g, we write a relaxation of MULTI-CRITERIA-DIMENSION-REDUCTION as follows:

$$\max_{X \in \mathbb{R}^{n \times n}} g(\langle B_1, X \rangle + \alpha_1, \dots, \langle B_k, X \rangle + \alpha_k) \text{ subject to}$$
 (9)

$$tr(X) \le d \tag{10}$$

$$0 \le X \le I_n \tag{11}$$

Since g(x) is concave in $x \in \mathbb{R}^k$ and $\langle B_i, X \rangle + \alpha_i$ is affine in $X \in \mathbb{R}^{n \times n}$, we have that g as a function of X is also concave in X. By concavity of g and that the feasible set is convex and bounded, we can solve the convex program (9)-(11) in polynomial time, e.g. by ellipsoid method, to obtain a (possibly high-rank) optimal solution $\bar{X} \in \mathbb{R}^{n \times n}$. (In the case that g is linear, the relaxation is also an SDP and may be solved faster in theory and practice).

We first assume that g is monotonic in at least one coordinate, so without loss of generality, we let g be nondecreasing in the first coordinate. To reduce the rank of \bar{X} , we consider an $\mathbb{SDP}(\mathbb{II})$:

$$\max_{X \subset \mathbb{D}^{n \times n}} \langle B_1, X \rangle \text{ subject to}$$
 (12)

$$\langle B_i, X \rangle = \langle B_i, \bar{X} \rangle \quad \forall \ 2 \le i \le k$$
 (13)

$$tr(X) \leq d \tag{14}$$

$$0 \le X \le I_n \tag{15}$$

 $\mathbb{SDP}(\mathbb{II})$ has a feasible solution \bar{X} of objective $\langle B_1, X \rangle$, and note that there are k-1 constraints in (13). Hence, we can apply the algorithm in Theorem 1.8 with m=k-1 to find an extreme solution X^* of $\mathbb{SDP}(\mathbb{II})$ of rank at most r^* . Since g is nondecreasing in $\langle B_1, X \rangle$, an optimal solution to $\mathbb{SDP}(\mathbb{II})$ gives objective value at least the optimum of the relaxation and hence at least the optimum of the original MULTI-CRITERIA-DIMENSION-REDUCTION.

If the assumption that g is monotonic in at least one coordinate is dropped, the argument holds by indexing constraints (13) in $\mathbb{SDP}(\mathbb{H})$ for all k groups instead of k-1 groups.

Another way to state Theorem 1.4 is that the number of groups must reach $\frac{(s+1)(s+2)}{2}$ before additional s dimensions in the solution matrix P is required to achieve the optimal objective value. For k=2, no additional dimension in the solution is necessary to attain the optimum, which proves Theorem 1.3. In particular, it applies to FAIR-PCA with two groups, proving Theorem 1.2. We note that the rank bound in Theorem 1.8 (and thus also the bound in Corollary 2.1) is tight. An example of the problem instance follows from [14] with a slight modification. We refer the reader to Appendix B for details.

3 Applications of low-rank solutions in fairness

In this section, we show applications of low-rank solutions of Theorem 1.4 in fairness applications of dimensionality reduction. We describe existing fairness criteria and motivate our new fairness objective, summarized in Table 1. The new fairness objective appropriately addresses fairness when subgroups have different optimal variances in a low-dimensional space. We note that all fairness criteria in this section satisfy the assumption in Theorem 1.4 that g is concave and monotone nondecreasing in at least one (in fact, all) of its arguments, and thus these fairness objectives can be solved with Theorem 1.4. We also give approximation algorithm for FAIR-PCA, proving Corollary 1.5.

3.1 Application to FAIR-PCA

We prove Corollary 1.5 below. Recall that, by Theorem 1.4, $s:=\left\lfloor\sqrt{2k+\frac{1}{4}}-\frac{3}{2}\right\rfloor$ additional dimensions for the projection are required to achieve the optimal objective. One way to ensure that the algorithm outputs d-dimensional projection is to solve the problem in the lower target dimension d-s, and then apply the rounding algorithm described in Section 2.

Proof of Corollary 1.5. We find an extreme solution X^* of the FAIR-PCA problem of finding a projection from n to d-s target dimensions. By Theorem 1.4, the rank of X^* is at most d.

Denote OPT_d, X_d^* the optimal value and an optimal solution to exact FAIR-PCA with target dimension d, respectively. Note that $\frac{d-s}{d}X_d^*$ is a feasible solution to the FAIR-PCA relaxation (1)-(4) on target dimension d-s whose objective is at least $\frac{d-s}{d}\mathrm{OPT}_d$, since the FAIR-PCA relaxation objective scales linearly with X. Therefore, the optimal value of FAIR-PCA relaxation of target dimension d-s, which is achieved by X^* (Theorem 1.4), is at least $\frac{d-s}{d}\mathrm{OPT}_d$. Hence, we obtain the $(1-\frac{s}{d})$ -approximation.

3.2 Welfare economic and NSW

If we interpret $f_i(P) = \|A_iP\|^2$ in FAIR-PCA as individual utility, then standard PCA maximizes the total utility of individuals, also known as a utilitarian objective in welfare economic. One other objective is egalitarian, aiming to maximize the minimum utility [45], which is equivalent to FAIR-PCA in our setting. One other example, which lies between the two, is to choose the product function $g(y_1, \ldots, y_k) = \prod_i y_i$ for the accumulation function $g(y_1, \ldots, y_k) = \prod_i y_i$ for the accumulation function $g(y_1, \ldots, y_k) = \prod_i y_i$ for the bargaining problem [63, 51], and we call this objective Nash Social Welfare (NSW). The three objectives

Table 1: Examples of fairness criteria to which our results in this work apply. We are given A_i as the data matrix of group i in n dimensions for $i=1,\ldots,k$ and a target dimension d< n. We denote $\mathcal{P}_d=\left\{P\in\mathbb{R}^{n\times d}:P^TP=I_d\right\}$ the set of all $n\times d$ matrices with d orthonormal columns and $\beta_i=\max_{Q\in\mathcal{P}_d}\|A_iQ\|^2$ the variance of an optimal projection for group i alone.

Name	$f_i(P)$	g	MULTI-CRITERIA-DIMENSION-REDUCTION
Standard PCA	$ A_iP ^2$	sum	$\max_{P \in \mathcal{P}_d} \sum_{i \in [k]} A_i P ^2$
FAIR-PCA (MM-Var)	$ A_iP ^2$	min	$\max_{P \in \mathcal{P}_d} \min_{i \in [k]} A_i P ^2$
Nash social welfare (NSW)	$ A_iP ^2$	product	$\max_{P \in \mathcal{P}_d} \prod_{i \in [k]} A_i P ^2$
Marginal loss (MM-Loss)	$ A_iP ^2 - \beta_i$	min	$\max_{P \in \mathcal{P}_d} \min_{i \in [k]} \left(A_i P ^2 - \beta_i \right)$

are special cases of the pth power mean of individual utilities, i.e. $g(y_1, \ldots, y_k) = \left(\sum_{i \in [k]} y_i^p\right)^{1/p}$, with $p = 1, -\infty, 0$ giving standard PCA, FAIR-PCA, and NSW, respectively. Since the pth power mean is concave for $p \le 1$, the assumptions in Theorem 1.4 hold and our algorithms apply to these objectives.

Because the assumptions in Theorem 1.4 does not change under an affine transformation of g, we may also take any weighting and introduce additive constants on the square norm. For example, we can take the average squared norm of the projections rather than the total, replacing $\|A_iP\|^2$ by $\frac{1}{m_i}\|A_iP\|^2$ where m_i is the number of data points in A_i , in any of the discussed fairness criteria. This normalization equalizes weight of each group, which can be useful when groups are of very different sizes, and is also used in all of our experiments. More generally, one can weight each f_i by a positive constant w_i , where the appropriate weighting of k objectives often depends on the context and application. Another example is to replace $f_i(P) = \|A_iP\|^2$ by $\|A_iP\|^2 - \|A_i\|^2$, which optimizes the worst reconstruction error rather than the worst variance across all groups as in the FAIR-PCA definition.

3.3 Marginal loss objective

We now present a novel fairness criterion *marginal loss* objective. We first give a motivating example of two groups for this objective, shown in Figure 3.

In this example, two groups can have very different variances when projected onto one dimension: the first group has a perfect representation in the horizontal direction and enjoys high variance, while the second has lower variance for every projection. Thus, asking for a projection which maximizes the minimum variance might incur loss of variance on the first group while not improving the second group. So, minimizing the maximum reconstruction error of these two groups fails to account for the fact that two populations might have wildly different representation variances when embedded into d dimensions. Optimal solutions to such objective might behave in a counterintuitive way, preferring to exactly optimize for the group with smaller inherent representation variance rather than approximately optimizing for both groups simultaneously. We find this behaviour undesirable—it requires sacrifice in quality for one group for no improvement for the other group. In other words, it does not satisfy Pareto-optimality.

We therefore turn to finding a projection which minimizes the maximum deviation of each group from its optimal projection. This optimization asks that two groups suffer a similar *decrease* in variance for being projected together onto d dimensions compared to their individually optimal projections ("the marginal cost of sharing a common subspace"). Specifically, we set the utility $f_i(P) := \|A_iP\|_F^2 - \max_{Q \in \mathcal{P}_d} \|A_iQ\|_F^2$ as the change of variance for each group i and $g(f_1, f_2, \ldots, f_k) := \min\{f_1, f_2, \ldots, f_k\}$ in the MULTI-CRITERIA-DIMENSION-REDUCTION formulation. This gives an optimization problem

$$\min_{P \in \mathcal{P}_d} \max_{i \in [k]} \left(\max_{Q \in \mathcal{P}_d} \|A_i Q\|_F^2 - \|A_i P\|_F^2 \right)$$
 (16)

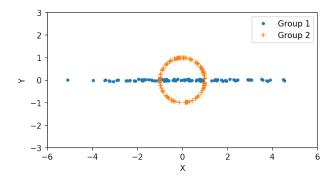


Figure 3: A distribution of two groups where, when projected onto one dimension, maximizing the minimum variance and minimizing the maximum reconstruction error are undesirable objectives.

We refer to $\max_{Q \in \mathcal{P}_d} ||A_i Q||_F^2 - ||A_i P||^2$ as the *loss* of group i by a projection P, and we call the objective (16) to be minimized the *marginal loss* objective.

For two groups, marginal loss objective prevents the optimization from incurring loss of variance for one subpopulation without improving the other as seen in Figure 3. In fact, we show that an optimal solution of marginal loss objective always gives the same loss for two groups. As a result, marginal loss objective not only satisfies Pareto-optimality, but also equalizes individual utilities, a property that none of the previously mentioned fairness criteria necessarily holds.

Theorem 3.1. Let P^* be an optimal solution to (16) for two groups. Then,

$$\max_{Q \in \mathcal{P}_d} \|A_1 Q\|_F^2 - \|A_1 P^*\|_F^2 = \max_{Q \in \mathcal{P}_d} \|A_2 Q\|_F^2 - \|A_2 P^*\|_F^2$$

Theorem 3.1 can be proved by a "local move" argument on the space of all d-dimensional subspaces equipped with a carefully defined distance metric. We define a new metric space since the move is not valid on the natural choice of space, namely the domain $\mathcal{P}_d \subseteq \mathbb{R}^{n \times n}$ with Euclidean distance, as the set \mathcal{P}_d is not convex. We also give a proof using the SDP relaxation of the problem. We refer the reader to Appendix A for the proofs of Theorem 3.1. In general, Theorem 3.1 does not generalize to more than two groups (see Section 6.2).

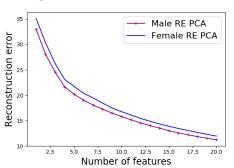
Another motivation of the marginal loss objective is to check bias in PCA performance on subpopulations. A data set may show a small gap in variances or reconstruction errors of different groups, but a significant gap in losses. An example is the labeled faces in the wild data set (LFW) [40] where we check both reconstruction errors and losses of male and female groups. As shown in Figure 5, the gap between male and female reconstruction errors is about 10%, while the marginal loss of female is about 5 to 10 times of the male group. This suggests that the difference in marginal losses is a primary source of bias, and therefore marginal losses rather than reconstruction errors should be equalized.

4 Iterative rounding framework with applications to FAIR-PCA

In this section, we give an iterative rounding algorithm and prove Theorem 1.9. The algorithm is specified in Algorithm 1. The algorithm maintains three subspaces of $\mathbb{R}^{n\times n}$ that are mutually orthogonal. Let F_0, F_1, F denote matrices whose columns form an orthonormal basis of these subspaces. We will also abuse notation and denote these matrices by sets of vectors in their columns. We let the rank of F_0, F_1 and F be F_0, F_1 and F respectively. We will ensure that F_0, F_1, F_2 and F_0, F_2, F_3 and F_0, F_3 and F_0, F_4 and F_0, F_3 and F_0, F_4 and F_0, F_4 and F_0, F_3 and F_0, F_4 and F_0, F_4 and F_0, F_5 and F_0, F_6 are F_0, F_6 and F_0, F_6 are F_0, F_6 and F_0, F_6 and F_0, F_6 and F_0, F_6 are F_0, F_6 and F_0, F_6 and F_0, F_6 and F_0, F_6 are F_0, F_6 and F_0, F_6 and F_0, F_6 are F_0, F_6 and F_0, F_6 and F_0, F_6 are F_0, F_6 and F_0, F_6 and F_0, F_6 are F_0, F_6 and F_0, F_6 and F_0, F_6 are F_0, F_6 and

We initialize $F_0 = F_1 = \emptyset$ and $F = I_n$. Over iterations, we increase the subspaces spanned by columns of F_0 and F_1 and decrease F while maintaining pairwise orthogonality. The vectors in columns of F_1 will be

Average reconstruction error (RE) of PCA on LFW



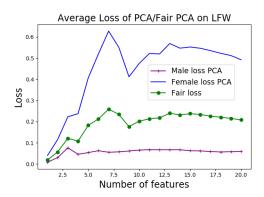


Figure 4: Left: reconstruction error of PCA on labeled faces in the wild data set (LFW), separated by gender. Right: marginal loss objective on the same data set. The fair loss is obtained by a solution to the marginal loss objective, which equalizes the two losses.

eigenvectors of our final solution with eigenvalue 1. In each iteration, we project the constraint matrices A_i orthogonal to F_1 and F_0 . We will then formulate a residual SDP using columns of F as a basis and thus the new constructed matrices will have size $r \times r$. To readers familiar with the iterative rounding framework in linear programming, this generalizes the method of fixing certain variables to 0 or 1 and then formulating the residual problem. We also maintain a subset of constraints indexed by S where S is initialized to $\{1, \ldots, m\}$.

In each iteration, we formulate the following $\mathbb{SDP}(r)$ with variables X(r) which will be a $r \times r$ symmetric matrix. Recall r is the number of columns in F.

$$\max \langle F^T C F, X(r) \rangle$$

$$\langle F^T A_i F, X(r) \rangle \ge b_i - F_1^T A_i F_1 \quad i \in S$$

$$\operatorname{tr}(X) \le d - \operatorname{rank}(F_1)$$

$$0 \le X(r) \le I_r$$

Algorithm 1 Iterative rounding algorithm ITERATIVE-SDP

Input: C a real $n \times n$ matrix, $A = \{A_1, \dots, A_m\}$ a set of real $n \times n$ matrices, $d \le n$, and $b_1, \dots b_m \in \mathbb{R}$. **Output:** A feasible solution \tilde{X} to \mathbb{SDP}

- 1: Initialize F_0, F_1 to be empty matrices and $F \leftarrow I_n, S \leftarrow \{1, \dots, m\}$.
- 2: If the \mathbb{SDP} is infeasible, declare infeasibility and stop.
- 3: **while** F is not the empty matrix **do**
- 4: Solve $\mathbb{SDP}(r)$ to obtain an extreme point $X^*(r) = \sum_{j=1}^r \lambda_j v_j v_j^T$ where λ_j are the eigenvalues and $v_j \in \mathbb{R}^r$ are the corresponding eigenvectors.
- 5: For any eigenvector v of $X^*(r)$ with eigenvalue 0, let $F_0 \leftarrow F_0 \cup \{Fv\}$.
- 6: For any eigenvector v of $X^*(r)$ with eigenvalue 1, let $F_1 \leftarrow F_1 \cup \{Fv\}$.
- 7: Let $X_f = \sum_{j:0 < \lambda_j < 1} \lambda_j v_j v_j^T$. If there exists a constraint $i \in S$ such that $\langle F^T A_i F, X_f \rangle < \Delta(\mathcal{A})$, then $S \leftarrow S \setminus \{i\}$.
- 8: For every eigenvector v of $X^*(r)$ with eigenvalue not equal to 0 or 1, consider the vectors Fv and form a matrix with these columns and use it as the new F.
- 9: end while
- 10: Return $\tilde{X} = F_1 F_1^T$.

It is easy to see that the semi-definite program remains feasible over all iterations if SDP is declared

feasible in the first iteration. Indeed the solution X_f defined at the end of any iteration is a feasible solution to the next iteration. We also need the following standard claim.

Claim 4.1. Let Y be a positive semi-definite matrix such that $Y \leq I$ with $tr(Y) \leq l$. Let B be a real matrix of the same size as Y and let $\lambda_i(B)$ denote the i^{th} largest singular value of B. Then

$$\langle B, Y \rangle \le \sum_{i=1}^{l} \lambda_i(B).$$

The following result follows from Corollary 2.1 and Claim 4.1. Recall that

$$\Delta(\mathcal{A}) := \max_{S \subseteq [m]} \sum_{i=1}^{\lfloor \sqrt{2|S|} + 1 \rfloor} \sigma_i(S).$$

where $\sigma_i(S)$ is the *i*'th largest singular value of $\frac{1}{|S|} \sum_{i \in S} A_i$. We let Δ denote $\Delta(A)$ for the rest of the section.

Lemma 4.2. Consider any extreme point solution X(r) of $\mathbb{SDP}(r)$ such that $\mathrm{rank}(X(r)) > \mathrm{tr}(X(r))$. Let $X(r) = \sum_{j=1}^r \lambda_j v_j v_j^T$ be its eigenvalue decomposition and $X_f = \sum_{0 < \lambda_j < 1} \lambda_j v_j v_j^T$. Then there exists a constraint i such that $\langle F^T A_i F, X_f \rangle < \Delta$.

Proof. Let l=|S|. From Corollary 2.1, it follows that the number of fractional eigenvalues of X(r) is at most $-\frac{1}{2}+\sqrt{2l+\frac{9}{4}}\leq \sqrt{2l}+1$. Observe that l>0 since $\mathrm{rank}(X(r))>\mathrm{tr}(X(r))$. Thus, we have $\mathrm{rank}(X_f)\leq \sqrt{2l}+1$. Moreover, $0\leq X_f\leq I$, so from Claim 4.1, we obtain that

$$\left\langle \sum_{j \in S} F^T A_j F, X_f \right\rangle \leq \sum_{i=1}^{\lfloor \sqrt{2l} + 1 \rfloor} \sigma_i \left(\sum_{j \in S} F^T A_j F \right) \leq \sum_{i=1}^{\lfloor \sqrt{2l} + 1 \rfloor} \sigma_i \left(\sum_{j \in S} A_j \right) \leq l \cdot \Delta$$

where the first inequality follows from Claim 4.1 and the second inequality follows since the sum of top l singular values reduces after projection. But then we obtain, by averaging, that there exists $j \in S$ such that

$$\langle F^T A_j F, X_f \rangle < \frac{1}{l} \cdot l\Delta = \Delta$$

as claimed. \Box

Now we complete the proof of Theorem 1.9. Observe that the algorithm always maintains that at the end of each iteration, $\operatorname{tr}(X_f) + \operatorname{rank}(F_1) \leq d$. Thus at the end of the algorithm, the returned solution has rank at most d. Next, consider the solution $X = F_1 F_1^T + F X_f F^T$ over the course of the algorithm. Again, it is easy to see that the objective value is non-increasing over the iterations. This follows since X_f defined at the end of an iteration is a feasible solution to the next iteration.

Now we argue a bound on the violation in any constraint i. While the constraint i remains in the SDP, the solution $X = F_1F_1^T + FX_fF^T$ satisfies

$$\langle A_i, X \rangle = \langle A_i, F_1 F_1^T \rangle + \langle A_i, F X_f F^T \rangle$$

= $\langle A_i, F_1 F_1^T \rangle + \langle F^T A_i F, X_f \rangle \le \langle A_i, F_1 F_1^T \rangle + b_i - \langle A_i, F_1 F_1^T \rangle = b_i.$

where the inequality again follows since X_f is feasible with the updated constraints.

When constraint i is removed, it might be violated by a later solution. At this iteration, $\langle F^T A_i F, X_f \rangle \leq \Delta$. Thus, $\langle A_i, F_1 F_1^T \rangle \geq b_i - \Delta$. In the final solution, this bound can only go up as F_1 might only become larger. This completes the proof of the theorem.

We now analyze the runtime of the algorithm which contains at most m iterations. First we note that we may avoid computing $\Delta(\mathcal{A})$ by deleting a constraint i from S with smallest $\langle F^TA_iF,X_f\rangle$ instead of checking $\langle F^TA_iF,X_f\rangle < \Delta(\mathcal{A})$ in step (7) of Algorithm 1. The guarantee still holds by Lemma 4.2. Each iteration requires solving an SDP and eigenvalue decompositions over $r\times r$ matrices, computing F_0,F_1,F , and finding $i\in S$ with the smallest $\langle F^TA_iF,X_f\rangle$. These can be done in $O(r^{6.5})$, $O(r^2n)$, and $O(rmn^2)$ time. However, the result in Section 2 shows that after solving the first $\mathbb{SDP}(r)$, we have $r\leq O(\sqrt{m})$, and hence the total runtime of iterative rounding after solving for an extreme solution of the SDP relaxation) is $O(m^{4.25}+m^{1.5}n^2)$.

Application to FAIR-PCA. For FAIR-PCA, iterative rounding recovers a rank-d solution whose variance goes down from the SDP solution by at most $\Delta(\{A_1^TA_1,\ldots,A_k^TA_k\})$. While this is no better than what we get by scaling (Corollary 1.5) for the max variance objective function, when we consider the marginal loss, i.e., the difference between the variance of the common d-dimensional solution and the best d-dimensional solution for each group, then iterative rounding can be much better. The scaling solution guarantee relies on the max-variance being a concave function, and for the marginal loss, the loss for each group could go up proportional to the *largest* max variance (largest sum of top k singular values over the groups). With iterative rounding applied to the SDP solution, the loss Δ is the sum of only $O(\sqrt{k})$ singular values of the average of some subset of data matrices, so it can be better by as much as a factor of \sqrt{k} .

5 Polynomial time algorithm for fixed number of groups

Functions of quadratic maps. We briefly summarize the approach of [36]. Let $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ be real-valued quadratic functions in n variables. Let $p : \mathbb{R}^k \to \mathbb{R}$ be a polynomial of degree ℓ over some subring of \mathbb{R}^k (e.g., the usual $(\times, +)$ or $(+, \min)$) The problem is to find all roots of the polynomial $p(f_1(x), f_2(x), \ldots, f_k(x))$, i.e., the set

$$Z = \{x : p(f_1(x), f_2(x), \dots, f_k(x)) = 0\}.$$

First note that the set of solutions above is in general not finite and is some manifold and highly non-convex. The key idea of Grigoriev and Paleshnik (see also Barvinok [8] for a similar idea applied to a special case) is to show that this set of solutions can be partitioned into a relatively small number of connected components such that there is an into map from these components to roots of a univariate polynomial of degree $(\ell n)^{O(k)}$; this therefore bounds the total number of components. The proof of this mapping is based on an explicit decomposition of space with the property that if a piece of the decomposition has a solution, it must be the solution of a linear system. The number of possible such linear systems is bounded as $n^{O(k)}$, and these systems can be enumerated efficiently.

The core idea of the decomposition starts with the following simple observation that relies crucially on the maps being quadratic (and not of higher degree).

Proposition 5.1. The partial derivatives of any degree d polynomial p of quadratic forms $f_i(x)$, where $f_i : \mathbb{R}^n \to \mathbb{R}$, is linear in x for any fixed value of $\{f_1(x), \ldots, f_k(x)\}$.

To see this, suppose $Y_i = f_i(x)$ and write

$$\frac{\partial p}{\partial x_i} = \sum_{j=1}^k \frac{\partial p(Y_1, \dots, Y_k)}{\partial Y_j} \frac{\partial Y_j}{\partial x_i} = \sum_{j=1}^k \frac{\partial p(Y_1, \dots, Y_k)}{\partial Y_j} \frac{\partial f_j(x)}{\partial x_i}.$$

Now the derivatives of f_j are linear in x_i as f_j is quadratic, and so for any fixed values of Y_1, \ldots, Y_k , the expression is linear in x.

The next step is a nontrivial fact about connected components of analytic manifolds that holds in much greater generality. Instead of all points that correspond to zeros of p, we look at all "critical" points of p defined as the set of points x for which the partial derivatives in all but the first coordinate, i.e.,

$$Z_c = \{x : \frac{\partial p}{\partial x_i} = 0, \quad \forall 2 \le i \le n\}.$$

The theorem says that Z_c will intersect every connected component of Z [35].

Now the above two ideas can be combined as follows. We will cover all connected components of Z_c . To do this we consider, for each fixed value of Y_1, \ldots, Y_k , the possible solutions to the linear system obtained, alongside minimizing x_1 . The rank of this system is in general at least n-k after a small perturbation (while [36] uses a deterministic perturbation that takes some care, we could also use a small random perturbation). So the number of possible solutions grows only as exponential in O(k) (and not n), and can be effectively enumerated in time $(\ell d)^{O(k)}$. This last step is highly nontrivial, and needs the argument that over the reals, zeros from distinct components need only to be computed up to finite polynomial precision (as rationals) to keep them distinct. Thus, the perturbed version still covers all components of the original version. In this enumeration, we check for true solutions. The method actually works for any level set of p, $\{x: p(x) = t\}$ and not just its zeros. With this, we can optimize over p as well. We conclude this section by paraphrasing the main theorem from [36].

Theorem 5.2. [36] Given k quadratic maps $q_1, \ldots, q_k : \mathbb{R}^k \to \mathbb{R}$ and a polynomial $p : \mathbb{R}^k \to \mathbb{R}$ over some computable subring of \mathbb{R} of degree at most ℓ , there is an algorithm to compute a set of points satisfying $p(q_1(x), \ldots, q_k(x)) = 0$ that meets each connected component of the set of zeros of p using at most $(\ell n)^{O(k)}$ operations with all intermediate representations bounded by $(\ell n)^{O(k)}$ times the bit sizes of the coefficients of p, q_1, \ldots, q_k . The minimizer, maximizer or infimum of any polynomial $r(q_1(x), \ldots, q_k(x))$ of degree at most ℓ over the zeros of p can also be computed in the same complexity.

5.1 Proof of Theorem 1.7

We apply Theorem 5.2 and the corresponding algorithm as follows. Our variables will be the entries of an $n \times d$ matrix P. The quadratic maps will be $f_i(P)$ plus additional maps for $q_{ii}(P) = ||P_i||^2 - 1$ and $q_{ij}(P) = P_i^T P_j$ for columns P_i, P_j of P. The final polynomial is

$$p(f_1, \dots, f_k, q_{11}, \dots, q_{dd}) = \sum_{i \le j} q_{ij}(P)^2.$$

We will find the maximum of the polynomial $r(f_1, \ldots f_k) = g(f_1, \ldots, f_k)$ over the set of zeros of p using the algorithm of Theorem 5.2. Since the total number of variables is dn and the number of quadratic maps is k + d(d+1)/2, we get the claimed complexity of $O(\ell dn)^{O(k+d^2)}$ operations and this times the input bit sizes as the bit complexity of the algorithm.

6 Hardness and integrality gap

6.1 NP-Hardness

In this section, we show NP-hardness of FAIR-PCA even for d=1, proving Theorem 1.6.

Theorem 6.1. *The* FAIR-PCA *problem:*

$$\max_{z \in \mathbb{R}, P \in \mathbb{R}^{n \times d}} z \qquad \text{subject to}$$

$$\left\langle B_i, PP^T \right\rangle \ge z \qquad , \forall i \in [k]$$

$$\tag{18}$$

$$\langle B_i, PP^T \rangle \ge z \qquad , \forall i \in [k]$$
 (18)

$$P^T P = I_d (19)$$

for arbitrary $n \times n$ symmetric real PSD matrices B_1, \ldots, B_k is NP-hard for d = 1 and k = O(n).

Proof of Theorem 6.1: We reduce another NP-hard problem MAX-CUT to FAIR-PCA with d=1. In MAX-CUT, given a simple graph G = (V, E), we optimize

$$\max_{S \subseteq V} e(S, V \setminus S) \tag{20}$$

over all subset S of vertices. Here, $e(S, V \setminus S) = |\{e_{ij} \in E : i \in S, j \in V \setminus S\}|$ is the size of the cut S in G. As common in NP-hard problems, the decision version of MAX-CUT:

$$\exists ?S \subseteq V : e(S, V \setminus S) \ge b \tag{21}$$

for an arbitrary b > 0 is also NP-hard. We may write MAX-CUT as an integer program as follows:

$$\exists ?v \in \{-1, 1\}^V : \frac{1}{2} \sum_{ij \in E} (1 - v_i v_j) \ge b.$$
 (22)

Here v_i represents whether a vertex i is in the set S or not:

$$v_i = \begin{cases} 1 & i \in S \\ -1 & i \notin S \end{cases} \tag{23}$$

and it can be easily verified that the objective represents the desired cut function.

We now show that this MAX-CUT integer feasibility problem can be formulated as an instance of FAIR-PCA (17)-(19). In particular, it will be formulated as a feasibility version of FAIR-PCA by checking if the optimum z of FAIR-PCA is at least b. We choose d=1 and n=|V| for this instance, and we write $P = [u_1; \dots; u_n] \in \mathbb{R}^n$. The rest of the proof is to show that it is possible to construct constraints in the form (18)-(19) to 1) enforce a discrete condition on u_i to take only two values, behaving similarly as v_i ; and 2) check an objective value of MAX-CUT.

Note that constraint (19) requires $\sum_{i=1}^n u_i^2 = 1$ but $\sum_{i=1}^n v_i^2 = n$. Hence, we scale the variables in MAX-CUT problem by writing $v_i = \sqrt{n}u_i$ and rearrange terms in (22) to obtain an equivalent formulation of MAX-CUT:

$$\exists ?u \in \left\{ -\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right\}^n : n \sum_{ij \in E} -u_i u_j \ge 2b - |E|$$
 (24)

We are now ready to give an explicit construction of $\{B_i\}_{i=1}^k$ to solve MAX-CUT formulation (24). Let k = 2n + 1. For each $j = 1, \ldots, n$, define

$$B_{2j-1} = bn \cdot \operatorname{diag}(\mathbf{e_j}), B_{2j} = \frac{bn}{n-1} \cdot \operatorname{diag}(\mathbf{1} - \mathbf{e_j})$$

where $\mathbf{e_j}$ and 1 denote vectors of length n with all zeroes except one at the jth coordinate, and with all ones, respectively. It is clear that B_{2j-1}, B_{2j} are PSD. Then for each $j=1,\ldots,n$, the constraints $\langle B_{2j-1}, PP^T \rangle \geq b$ and $\langle B_{2j}, PP^T \rangle \geq b$ are equivalent to

$$u_j^2 \ge \frac{1}{n}$$
, and $\sum_{i \ne j} u_j^2 \ge \frac{n-1}{n}$

respectively. Combining these two inequalities with $\sum_{i=1}^n u_i^2 = 1$ forces both inequalities to be equalities, implying that $u_j \in \left\{-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right\}$ for all $j \in [n]$, as we aim.

Next, we set

$$B_{2n+1} = \frac{bn}{2b - |E| + n^2} \cdot (nI_n - A_G)$$

where $A_G = (\mathbb{I}[ij \in E])_{i,j \in [n]}$ is the adjacency matrix of the graph G. Since the matrix $nI_n - A_G$ is diagonally dominant and real symmetric, B_{2n+1} is PSD. We have that $\langle B_{2n+1}, PP^T \rangle \geq b$ is equivalent to

$$\frac{bn}{2b - |E| + n^2} \left(n \sum_{i=1}^n u_i^2 - \sum_{ij \in E} u_i u_j \right) \ge b$$

which, by $\sum_{i=1}^{n} u_i^2 = 1$, is further equivalent to

$$n\sum_{ij\in E} -u_i u_j \ge 2b - |E|,$$

matching (24). To summarize, we constructed B_1, \ldots, B_{2n+1} so that checking whether an objective of FAIR-PCA is at least b is equivalent to checking whether a graph G has a cut of size at least b, which is NP-hard.

6.2 Integrality gap

We showed that FAIR-PCA for k=2 groups can be solved up to optimality in polynomial time using an SDP. For k>2, we used a different, non-convex approach to get a polynomial-time algorithm for any fixed k,d. We show that the SDP relaxation of FAIR-PCA has a gap even for k=3 and d=1 in the following lemma. Here, the constructed matrices B_i 's are also PSD, as required by $B_i=A_i^TA_i$ for a data matrix A_i in the FAIR-PCA formulation. A similar result on tightness of rank violation for larger k using real (non-PSD) matrices B_i 's is stated in Lemma B.1 in Appendix B.

Lemma 6.2. The FAIR-PCA SDP relaxation:

$$\max z$$

$$\langle B_i, X \rangle \ge z \quad i \in \{1, \dots, k\}$$

$$\operatorname{tr}(X) \le d$$

$$0 \le X \le I$$

for k=3, d=1, and arbitrary PSD $\{B_i\}_{i=1}^k$ contains a gap, i.e. the optimum value of the SDP relaxation is different from one of exact FAIR-PCA problem.

Proof of Lemma 6.2: Let $B_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $B_3 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. It can be checked that B_i are PSD. The optimum of the relaxation is 7/4 (given by the optimal solution $X = \begin{bmatrix} 1/2 & 1/8 \\ 1/8 & 1/2 \end{bmatrix}$). However, an optimal exact FAIR-PCA solution is $\hat{X} = \begin{bmatrix} 16/17 & 4/17 \\ 4/17 & 1/17 \end{bmatrix}$ which gives an optimum 26/17 (one way to solve for optimum rank-1 solution \hat{X} is by parameterizing $\hat{X} = v(\theta)v(\theta)^T$ for $v(\theta) = [\cos\theta; \sin\theta]$, $\theta \in [0, 2\pi)$). \square

The idea of the example in Lemma 6.2 is that an optimal solution is close to picking the first axis as a projection, whereas a relaxation solution splits two halves for each of the two axes. Note that the example also shows a gap for marginal loss objective (16). Indeed, the numerical computation shows that optimal marginal loss (which is to be minimized) for the exact problem is 1.298 by $X^* \approx \begin{bmatrix} 0.977 & 0.149 \\ 0.149 & 0.023 \end{bmatrix}$ and for relaxed problem is 1.060 by $\widehat{X} \approx \begin{bmatrix} 0.5 & 0.030 \\ 0.030 & 0.5 \end{bmatrix}$. This shows that equal losses for two groups in Theorem 3.1 cannot be extended to more than two groups. The same example also show a gap if the objective is to minimize the maximum reconstruction errors. Gaps for all three objectives remain even after normalizing the data by $B_i \leftarrow \frac{B_i}{\operatorname{tr}(B_i)}$ by numerical computation. The pattern of solutions across three objectives and in both unnormalized and normalized setting remains the same as mentioned: an exact solution is a projection close to the first axis, and the relaxation splits two halves for the two axes, i.e., picking X close to I_2 .

7 Experiments

7.1 Efficacy of our algorithms to fairness

We perform experiments using the algorithm as outlined in Section 2 on the Default Credit data set [83] for different target dimensions d, and evaluate the fairness performance based on marginal loss and NSW criteria (see Table 1 in Section 3 on definitions of these criteria). The data consists of 30K data points in 23 dimensions, partitioned into k=4,6 groups by education and gender, and then preprocessed to have mean zero and same variance over features. Our algorithms are set to optimize on either the marginal loss and NSW objective. The code is publicly available at https://github.com/uthaipon/multi-criteria-dimensionality-reduction.

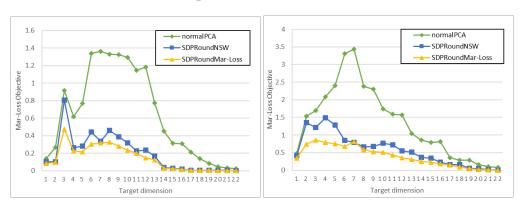


Figure 5: Marginal loss function of standard PCA compared to our SDP-based algorithms on Default Credit data. SDPRoundNSW and SDPRoundMar-Loss are two runs of the SDP-based algorithms maximizing NSW and minimizing marginal loss. Left: k = 4 groups. Right: k = 6.

Figure 5 shows the marginal loss by our algorithms compared to standard PCA on the entire data set. Our algorithms significantly reduce disparity of marginal loss of PCA that the standard PCA subtly introduces. We also assess the performance of PCA with NSW objective, summarized in Figure 6. With respect to NSW, standard PCA performs marginally worse (about 10%) compared to our algorithms. It is worth noting from Figures 5 and 6 that our algorithms which try to optimize either marginal loss or NSW also perform well on the other fairness objective, making these PCAs promising candidates for fairness applications.

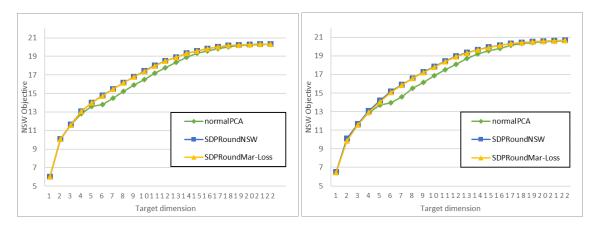


Figure 6: NSW objective of standard PCA compared to our SDP-based algorithms on Default Credit data. SDPRoundNSW and SDPRoundMar-Loss are two runs of the SDP-based algorithms maximizing NSW objective and minimizing marginal loss. Left: k=4 groups. Right: k=6.

Same experiments are done on the Adult Income data [80]. Some categorial features are preprocessed into integers vectors and some features and rows with missing values are discarded. The final preprocessed data contains m=32560 data points in n=59 dimensions and is partitioned into k=5 groups based on race. Figure 7 shows the performance of our SDP-based algorithms compared to standard PCA on marginal loss and NSW objectives. Similar to Credit Data, optimizing for either marginal loss or NSW gives a PCA solution that also performs well in another criterion and performs better than the standard PCA in both objectives.

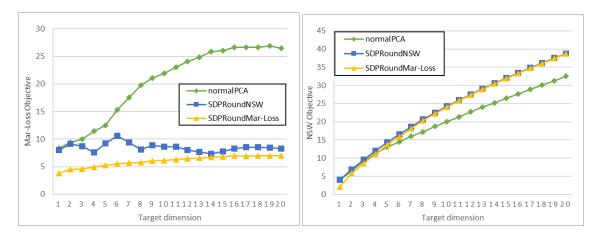


Figure 7: Marginal loss and NSW objective of standard PCA compared to our SDP-based algorithms on Adult Income data. SDPRoundNSW and SDPRoundMar-Loss are two runs of the SDP algorithms maximizing NSW objective and minimizing maximum marginal loss.

We note details of the labeled faces in the wild data set (LFW) [40] used in Figures 1 and 4 here. The

original data are in 1764 dimensions (42×42 images). We preprocess all data to have mean zero and we normalize each pixel value by multiplying $\frac{1}{255}$. The gender information for LFW was taken from Afifi and Abdelhamed [2], who manually verified the correctness of these labels. To obtain the fair loss in Figure 4, we solve using our SDP-based algorithm which, as our theory suggests, always give an exact optimal solution.

Rank violations in experiments. In all of the experiments, extreme point solutions from SDPs enjoy lower rank violation than our worst-case guarantee. Indeed, while the guarantee is that the numbers of additional rank are at most s=1,2 for k=4,6, almost all SDP solutions have *exact* rank, and in rare cases when the solutions are not exact, the rank violation is only one. As a result, we solve MULTI-CRITERIA-DIMENSION-REDUCTION in practice by solving the SDP relaxation targeting dimension d. If the solution is exact, then we are done. Else, we target dimension d-1 and check if the solution is of rank at most d. If not, we continue to target dimension $d-2, d-3, \ldots$ until the solution of the SDP relaxation has rank at most d. While our rank violation guarantee cannot be improved in general (due to the integrality gap in Section 6.2; also see Lemma B.1 for tightness of the rank violation bound), this opens a question whether the guarantee is better for instances that arise in practice.

Extreme property of SDP relaxation solutions in real data sets. One concern for solving an SDP is whether the solver will not return an extreme solution; if so, the SDP-based algorithm requires an additional time to round the solution. We found that a solution from SDP relaxation is, in fact, always already extreme in practice. This is because with probability one over random data sets, SDP is not degenerate, and hence have a unique optimal solution. Since any linear optimization over a compact, convex set must have an extreme optimal solution, this optimal solution is necessarily extreme. Therefore, in practice, it is not necessary to apply the SDP rounding algorithm to the solution of SDP relaxation. As an application, any faster algorithm or heuristics which can solve SDP relaxation to optimality in practice will always obtain a low-rank solution immediately. We discuss some useful heuristics in Section 7.2.

7.2 Runtime improvement

We found that the running time of solving SDP, which depends on n, is the bottleneck in all experiments. Each run (for one value of d) of the experiments is fast (< 0.5 seconds) on Default Credit data (n = 23), whereas a run on Adult Income data (n = 59) takes between 10 and 15 seconds on a personal computer. The runtime is not noticeably impacted by the numbers of data points and groups: larger m only increases the data preprocessing time (the matrix multiplication $B_i = A_i^T A_i$) to obtain $n \times n$ matrices, and larger k simply increases the number of constraints. SDP solver and rounding algorithms can handle moderate number of affine constraints efficiently. This observation is as expected from the theoretical analysis.

In this section, we show two heuristics for solving the SDP relaxation that run significantly faster in practice for large data sets: multiplicative weight update (MW) and Frank-Wolfe (FW). We also discuss several findings and suggestions for implementing our algorithms in practice. Both heuristics are publicly available at the same location as SDP-based algorithm experiments.

For the rest of this section, we assume that the utility of each group is $u_i(X) = \langle B_i, X \rangle$ for real $B_i \in \mathbb{R}^{n \times n}$, and that $g(z_1, \dots, z_k)$ is a concave function of z_1, \dots, z_k . When u_i is other linear function in X, we can model such different utility function by modifying g without changing the concavity of g. The

SDP relaxation of MULTI-CRITERIA-DIMENSION-REDUCTION can then be framed as the following SDP.

$$\max_{X \in \mathbb{R}^{n \times n}} g(z_1, z_2, \dots, z_k) \text{ subject to}$$
 (25)

$$z_i = \langle B_i, X \rangle \qquad \forall i = 1, 2, \dots, k$$
 (26)

$$tr(X) \le d \tag{27}$$

$$0 \le X \le I_n \tag{28}$$

7.2.1 Multiplicative Weight Update (MW)

One alternative method to solving (25)-(28) is multiplicative weight (MW) update [6]. Though this algorithm has theoretical guarantee, in practice the learning rate is tuned more aggressively and the algorithm becomes a heuristic without any certificate of optimality. We show the primal-dual derivation of MW which provides the primal-dual gap to certify optimality.

We take the Lagrangian with dual constraints in (26) to obtain that the optimum of the SDP equals to

$$\max_{\substack{X \in \mathbb{R}^{n \times n} \\ z \in \mathbb{R}^n \\ \text{tr}(X) = d \\ 0 \preceq X \preceq I}} \inf_{w \in \mathbb{R}^k} g(z) + \sum_{i=1}^k w_i \left(\langle B_i, X \rangle - z_i \right)$$

By strong duality, we may swap max and inf. After rearranging, the optimum of the SDP equals

$$\inf_{w \in \mathbb{R}^k} \left(\max_{\substack{X \in \mathbb{R}^{n \times n} \\ \operatorname{tr}(X) = d, 0 \le X \le I}} \sum_{i=1}^k w_i \langle B_i, X \rangle - \min_{z \in \mathbb{R}^n} \left(w^T z - g(z) \right) \right)$$
 (29)

The inner optimization

$$\max_{\substack{X \in \mathbb{R}^{n \times n} \\ \operatorname{tr}(X) = d, 0 \prec X \prec I}} \sum_{i=1}^{k} w_i \langle B_i, X \rangle \tag{30}$$

in (29) can easily be computed by standard PCA on weighted data $\sum_{i=1}^k w_i \cdot B_i$ projecting from n to d dimensions. The term (30) is also convex in w, as it is a maximum of (infinitely many) linear functions. The term $\min_{z \in \mathbb{R}^n} \left(w^T z - g(z) \right)$ is also known as concave conjugate of g, which we will denote by $g_*(w)$. Concave conjugate $g_*(w)$ is concave, as it is a minimum of linear functions. Hence, (29) is a convex optimization problem.

Solving (29) depends on the form of $g_*(w)$. For each fairness criteria outlined in this paper, we summarize the form of $g_*(w)$ below.

Max-Min Variance (FAIR-PCA or MM-Var): the fairness objective $g(z) = \min_{i \in [k]} z_i$ gives

$$g_*(w) = \begin{cases} 0 & \text{if } w \ge 0, \sum_{i=1}^k w_i = 1\\ -\infty & \text{otherwise} \end{cases}$$

Min-Max Loss (MM-Loss): the fairness objective (recall (16)) $g(z) = \min_{i \in [k]} z_i - \beta_i$, where $\beta_i = \max_{Q \in \mathcal{P}_d} \|A_i Q\|_F^2$ is the optimal variance of the group i, gives

$$g_*(w) = \begin{cases} \sum_{i=1}^k w_i \beta_i & \text{if } w \ge 0, \sum_{i=1}^k w_i = 1\\ -\infty & \text{otherwise} \end{cases}$$

More generally, the above form of $g_*(w)$ holds for any constants β_i 's. For example, this calculation also captures min-max reconstruction error: $g(X) = \min_{i \in [k]} \left\{ -\|A_i - A_i P\|_F^2 \right\} = \min_{i \in [k]} \{z_i - \operatorname{tr}(B_i)\}$ (recall that $X = PP^T$, $B_i = A_i^T A_i$, and $z_i = \langle B_i, X \rangle$).

Nash Social Welfare (NSW): the fairness objective $g(z) = \sum_{i=1}^{k} \log(z_i)$ gives

$$g_*(w) = \begin{cases} \sum_{i=1}^k (1 + \log w_i) & \text{if } w > 0\\ -\infty & \text{otherwise} \end{cases}$$

For fairness criteria of the "max-min" type, such as MM-Var and MM-Loss, solving the dual problem is an optimization over a simplex with standard PCA as the function evaluation oracle. This can be done using mirror descent [64] with negative entropy potential function $R(w) = \sum_{i=1}^k w_i \log w_i$. The algorithm is identical to multiplicative weight update (MW) by [6], described in Algorithm 2, and the convergence bounds from mirror descent and [6] are identical. However, with primal-dual formulation, the dual solution w_i obtained in each step of mirror descent can be used to calculate the dual objective in (29), and the optimum X in (30) is used to calculate the primal objective, which gives the duality gap. The algorithm runs iteratively until the duality gap is less than a set threshold. We summarize MW in Algorithm 2.

Algorithm 2 Multiplicative weight update (MW) for MULTI-CRITERIA-DIMENSION-REDUCTION

Input: PSD $B_1, \ldots, B_k \in \mathbb{R}^{n \times n}, \beta_1, \ldots, \beta_k \in \mathbb{R}$, learning rate $\eta > 0$, accuracy goal ϵ .

Output: an ϵ -additive approximate solution \hat{X} to

$$\max_{\substack{X \in \mathbb{R}^{n \times n} \\ \operatorname{tr}(X) = d, 0 \prec X \prec I}} \left(g(X) := \min_{i \in [k]} \langle B_i, X \rangle - \beta_i \right)$$

```
That is, g^* - g(\hat{X}) \leq \epsilon where g^* is the optimum of the above maximization.
1: Initialize w^0 \leftarrow (1/k, \dots, 1/k) \in \mathbb{R}^k; initialize dual bound y^{(0)} \leftarrow \infty
 2: t \leftarrow 0
 3: while true do
            P_t \leftarrow VV^T where V is n-by-d matrix of top d eigenvectors of \sum_{i \in [k]} w_i B_i
                                                                                                                                                                                                                ⊳ oracle of MW
 5: y_i^{(t)} \leftarrow \langle B_i, P_t \rangle - \beta_i \text{ for } i = 1, \dots, k

6: \hat{w}_i^{(t)} \leftarrow w_i^{(t-1)} e^{-\eta y_i^{(t)}} \text{ for } i = 1, \dots, k

7: w_i^{(t)} \leftarrow \hat{w}_i^{(t)} / (\sum_{i \in [k]} \hat{w}_i^{(t)}) \text{ for } i = 1, \dots, k

> Compute the duality gap
                X_t \leftarrow \frac{1}{t} \sum_{s \in [t]} P_s
               y^{(t)} \leftarrow \min \left\{ y^{(t-1)}, \sum_{i=1}^k w_i^{(t)} \cdot (\langle B_i, P_t \rangle - \beta_i) \right\}
                if y^{(t)} - g(X_t) \le \epsilon then
10:
                         break
11:
12:
                 end if
                 t \leftarrow t + 1
14: end while
15: return X^{(t)}
```

Runtime analysis. The convergence of MW is stated as follows and can be derived from the convergence of mirror descent [64]. A proof can be found in Appendix A.3.

Theorem 7.1. Given PSD $B_1, \ldots, B_k \in \mathbb{R}^{n \times n}$ and $\beta_1, \ldots, \beta_k \in \mathbb{R}$, we consider a maximization problem of the function

$$g(X) = \min_{i \in [k]} \langle B_i, X \rangle - \beta_i$$

over $\Omega = \{X \in \mathbb{R}^{n \times n} : \operatorname{tr}(X) = d, 0 \leq X \leq I\}$. For any $T \geq 1$, the T-th iterate X_T of multiplicative weight update algorithm with uniform initial weight and learning rate $\eta = \sqrt{\frac{\log k}{2T}}L$ where $L = \max_{i \in [k]} \operatorname{tr}(B_i)$ satisfies

$$g^* - g(X_T) \le \sqrt{\frac{2\log k}{T}} \max_{i \in [k]} \operatorname{tr}(B_i)$$
(31)

where g^* is the optimum of the maximization problem.

Theorem 7.1 implies that MW takes $O\left(\frac{\log k}{\epsilon^2}\right)$ iterations to obtain an additive error bound ϵ . For the MULTI-CRITERIA-DIMENSION-REDUCTION application, $\max_{i \in [k]} \operatorname{tr}(B_i)$ is a scaling of the data input and can be bounded if data are normalized. In particular, suppose $B_i = A_i^T A_i$ and each column of A_i has mean zero and variance at most one, then

$$\max_{i \in [k]} \operatorname{tr}(B_i) = \max_{i \in [k]} ||A_i||_F^2 \le n.$$

MW for two groups. For MM-Var and MM-Loss objectives in two groups, the simplex is a one-dimensional segment. The dual problem (29) reduces to

$$\inf_{w \in [0,1]} \left(h(w) := \max_{\substack{X \in \mathbb{R}^{n \times n} \\ \operatorname{tr}(X) = d, 0 \le X \le I}} \langle wB_1 + (1-w)B_2, X \rangle \right)$$
(32)

The function h(w) is a maximum of linear functions $\langle wB_1 + (1-w)B_2, X \rangle$ in w, and hence is convex on w. Instead of mirror descent, one can apply ternary search, a technique applicable to maximizing a general convex function in one dimension, to solve (32). However, we claim that binary search, which is faster than ternary search, is also a valid choice.

First, because h(w) is convex, we may assume that h achieves minimum at $w = w^*$ and that all subgradients $\partial h(w) \subseteq (-\infty, 0]$ for all $w < w^*$ and $\partial h(w) \subseteq [0, \infty)$ for all $w > w^*$. In the binary search algorithm with current iterate $w = w_t$, let

$$X_t \in \underset{\substack{X \in \mathbb{R}^{n \times n} \\ \operatorname{tr}(X) = d, 0 \preceq X \preceq I}}{\arg \max} \langle w_t B_1 + (1 - w_t) B_2, X \rangle$$

be any solution of the optimization (which can be implemented easily by standard PCA). Because a linear function $\langle wB_1+(1-w)B_2,X_t\rangle=\langle B_2,X_t\rangle+w\,\langle B_1-B_2,X_t\rangle$ is a lower bound of h(w) over $w\in[0,1]$ and h is convex, we have $\langle B_1-B_2,X_t\rangle\in\partial h(w_t)$. Therefore, the binary search algorithm can check the sign of $\langle B_1-B_2,X_t\rangle$ for a correct recursion. If $\langle B_1-B_2,X_t\rangle<0$, then $w^*>w_t$; if $\langle B_1-B_2,X_t\rangle>0$, then $w^*< w_t$; and the algorithm recurses in the left half or right half of the current segment accordingly. If $\langle B_1-B_2,X_t\rangle=0$, then w_t is an optimum dual solution.

Tuning in practice. In practice for MM-Var and MM-Loss objectives, we tune the learning rate of mirror descent much higher than in theory. In fact, we find that the last iterate of MW sometimes converges (certified by checking the duality gap), and in such case the convergence is much faster. For NSW, the dual is still a convex optimization, so standard technique such as gradient descent can be used. We found that in practice, however, the unboundedness of the feasible set is a challenge to tune MW for NSW to converge quickly.

7.2.2 Frank-Wolfe (FW)

Observe that while the original optimization (25)-(28), which is in the form

$$\max_{\substack{X \in \mathbb{R}^{n \times n} \\ \operatorname{tr}(X) = d, 0 \preceq X \preceq I}} g(z(X))$$

where the utility z is a function of X, is a nontrivial convex optimization, its linear counterpart

$$\max_{\substack{X \in \mathbb{R}^{n \times n} \\ \operatorname{tr}(X) = d, 0 \preceq X \preceq I}} \langle C, X \rangle$$

is easily solvable by standard PCA for any given matrix C. This motivates Frank-Wolfe (FW) algorithm [32] which requires a linear oracle (solving the same problem but with a linear objective) in each step. The instantiation of FW to MULTI-CRITERIA-DIMENSION-REDUCTION is summarized in Algorithm 3.

Algorithm 3 Frank-Wolfe Algorithm for Multi-Criteria Dimensionality Reduction

```
Input: B_1, \ldots, B_k \in \mathbb{R}^{n \times n}, d \leq n, concave g : \mathbb{R}^k \to \mathbb{R}, learning rate \eta_t, duality gap target \epsilon Output: A matrix X \in \mathbb{R}^{n \times n} that maximizes g(\langle B_1, X \rangle, \ldots, \langle B_k, X \rangle) subject to \operatorname{tr}(X) = d, 0 \leq d
X \leq I with additive error at most \epsilon from the optimum
```

- 1: Initialize a feasible X_0 (we use $X_0 = \frac{d}{n}I_n$), t = 0, and duality gap $g_0 = \infty$
- 2: while $q_t > \epsilon$ do
- $G_t \leftarrow \nabla_X g(X_t)$
- $S_t \leftarrow VV^T$ where V is n-by-d matrix of top d eigenvectors of G_t
- $X_{t+1} \leftarrow (1 \eta_t)x_t + \eta_t S_t$
- $g_t \leftarrow (S_t X_t) \cdot G_t$ Duality gap
- $t \leftarrow t + 1$ 7:
- 8: end while
- 9: Output X_t

One additional concern for implementing FW is obtaining gradient $\nabla_X g(X_t)$. For some objectives such as NSW, this gradient can be calculated analytically and efficiently (some small error may need to be added to stabilize the algorithm from exploding gradient when the variance is close to zero; see λ -smoothing below). Other objectives, such as MM-Var and MM-Loss, on the other hand, are not differentiable everywhere. Though one may try to still use FW and calculate gradients at the present point (which is differentiable with probability one), there is no theoretical guarantee for the FW convergence when the function is nondifferentiable (even when the feasible set is compact as in our SDP relaxation). Indeed, we find that FW does not converge in our experiment settings.

There are modifications of FW which has convergence guarantee for maximizing concave nondifferentiable functions. The algorithm by White [82] (also used by Ravi et al. [71]) requires a modified linear oracle, namely a maximization of $\min_{\delta \in T(X_t, \epsilon)} \delta \cdot (Y - X_t)$ over Y in SDP feasible set where $T(X_t, \epsilon)$ is the set of all subgradients at all points in the ϵ -neighborhood of X_t . In our setting, if the neighborhood has at least two distinct subgradients, then the oracle reduces to the form at least as hard as the original problem, making the method unhelpful. Another method is by smoothing the function, and the natural choice is by replacing $g(z) = \min_{i \in [k]} z_i$ by $g_{\mu}(z) = \mu \log \left(\sum_{i \in [k]} e^{\frac{z_i}{\mu}} \right)$ where $\mu > 0$ is the smoothing parameter. We find that the error from approximating g by g_{μ} is significant for even moderate $\mu \geq 10^{-2}$, and any smaller μ causes substantial numerical errors in computing the exponent. Hence, we do not find FW nor its modification useful in practice for solving MM-Var or MM-Loss objective.

Runtime analysis. The NSW objective can be ill-conditioned or even undefined when matrices B_i 's span low dimensions and those dimensions are distinct. This is due to $\log \langle B_i, X \rangle$ not defined at $\langle B_i, X \rangle = 0$ and having unbounded gradient when $\langle B_i, X \rangle$ is small. To stabilize the objective, we solve the λ -smoothed NSW objective

$$\max_{\substack{X \in \mathbb{R}^{n \times n} \\ \operatorname{tr}(X) = d, 0 \leq X \leq I}} g(X) := \sum_{i \in [k]} \log \left(\langle B_i, X \rangle + \lambda \cdot \|B_i\|_F \right) \tag{33}$$

Here, $\lambda > 0$ is a regularizer, and the regularization term $\lambda \cdot \|B_i\|_F$ normalizes the effect of each group when their variances are small, regardless of their original sizes $\|B_i\|_F$. Since g(X) is now also Lipschitz, FW has the convergence guarantee as follows, which follows from the standard FW convergence bound [42]. The proof can be found in Appendix A.4.

Theorem 7.2. Given n, k, d, λ and PSD $B_1, \ldots, B_k \in \mathbb{R}^{n \times n}$ as an input to λ -smooth NSW (33), the t-th iterate X_t of Frank-Wolfe with step sizes $\eta_s = 2/(s+1)$ satisfies

$$g^* - g(X_t) \le \frac{8kd}{\lambda(t+2)} \tag{34}$$

where g^* is the optimum of the optimization problem.

Theorem 7.2 implies that FW takes $O\left(\frac{kd}{\lambda\epsilon}\right)$ iterations to get an error bound ϵ for solving λ -smooth NSW.

Tuning in practice. In practice, we experiment with more aggressive learning rate schedule and line search algorithm. We found that FW converges quickly for NSW objective. However, FW does not converge for MM-Var and MM-Loss for any learning rate schedule, including the standard $\eta_t = \frac{1}{t+2}$ and line search. This is consistent with, as we mentioned, that FW does not have convergence guarantee for non-differentiable objectives.

7.2.3 Empirical runtime result on a large data set

We perform MW and FW heuristics on a large 1940 Colorado Census data set [5]. The data is preprocessed by one-hot encoding all discrete columns, ignoring columns with N/A, and normalizing the data to have mean zero and variance one on each feature. The preprocessed data set contains 661k data points and 7284 columns. Data are partitioned into 16 groups based on 2 genders and 8 education levels. We solve the SDP relaxation of MULTI-CRITERIA-DIMENSION-REDUCTION with MM-Var, MM-Loss, and NSW objectives until achieving the duality gap of no more than 0.1% (in the case of NSW, the product of variances, not the sum of logarithmic of variances, is used to calculate this gap). The runtime results, in seconds, are in shown in Table 2. When n increases, the bottleneck of the experiment becomes the standard PCA itself. Since speeding up the standard PCA is not in the scope of this work, we capped the original dimension of data by selecting the first n dimensions out of 7284, so that the standard PCA can still be performed in a reasonable amount of time. We note that the rank violation of solutions are almost always zero, and are exactly one when it is not zero, similarly to what we found for Adult Income and Credit data sets [80, 83].

Runtime of MW. We found that MM-Var and MM-Loss objectives are solved efficiently by MW, whereas MW with gradient descent on the dual of NSW does not converge quickly. It is usual that the solution of the relaxation has rank exactly d, and in all those cases we are able to tune learning rates so that the last iterate converges, giving a much faster convergence than the average iterate. For the Census data set, after parameter tuning, MW runs 100-200 iterations on both objectives. MW for both Credit and Income data sets (n=23,59) on 4-6 groups on both objectives finishes in 10-20 iterations whenever the last iterate converges, giving a total runtime of less than few seconds. Each iteration of MW takes 1x-2x of an SVD algorithm.

Table 2: Runtime of MW and FW for solving MULTI-CRITERIA-DIMENSION-REDUCTION on different fairness objectives and numbers of dimensions on the original 1940 Colorado Census data set. Runtimes of the standard PCA by SVD are included for comparison. All instances are solved to duality gap of at most 0.1%. Times are in second(s).

Original Dimensions	MM-Var (by MW)	MM-Loss (by MW)	NSW (by FW)	Standard PCA (by SVD)
n = 1000	77	65	11	0.22
n = 2000	585	589	69	1.5

Therefore, the price of fairness in PCA for MW-Var and MM-Loss objectives is 200-400x runtime for large data sets, and 20-40x runtime for medium data sets, as compared to the standard PCA without a fairness constraint.

Runtime of FW. FW converges quickly for NSW objective, and does not converge on MM-Var or MM-Loss. FW terminates in 10-20 iterations for Census Data. In practice, each iteration of FW has an overhead of 1.5x-3x of an SVD algorithm. We suspect codes can be optimized so that the constant overhead of each iteration is closer to 1x, as the bottleneck in each iteration is one standard PCA. Therefore, the price of fairness in PCA for NSW objective is 15-60x runtime compared to the standard PCA without a fairness constraint.

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A Proofs

A.1 Proof of Theorem 3.1 by local optimality

Before proving Theorem 3.1, we first state some notations and lemmas. We denote by $A_1 \in \mathbb{R}^{m_1 \times n}$, $A_2 \in \mathbb{R}^{m_2 \times n}$ data matrices of two groups, with rows as data points, in n dimensions that are to be projected onto d dimensions. Let \mathcal{S}_d denote the set of all d-dimensional subspaces of \mathbb{R}^n . For $U \in \mathcal{S}_d$, we let $P_U \in \mathcal{P}_d$ be a matrix which has an orthonormal basis of the subspace U as its columns.

For each matrix M, we let $P_M^* \in \arg\max_{Q \in \mathcal{P}_d} \|MQ\|_F^2$. We denote $loss(M,U) = \|MP_M^*\|_F^2 - \|MP_U\|_F^2$ be the loss of data M by a projection onto U. For data matrix A in n dimensions, we let $g_A(U) = \|AP_U\|_F^2$ for $U \in \mathcal{S}_d$. We let h(U) be the marginal loss objective (16) given by a projection matrix P_U to be minimized. We note that g_A , h, and loss are well-defined since their values do not change based on the orthonormal basis representation of U.

For a function $f:\mathcal{P}_d\to\mathbb{R}$, we call $P^*\in\mathcal{P}_d$ a local maximum of f if there exists $\epsilon>0$ such that $f(P^*)\geq f(P)$ for all $P\in\mathcal{P}_d$ with $\|P-P^*\|_F<\epsilon$ (we note that any $P\in\mathcal{P}_d$ always has an ϵ -neighbor by a slight change in one column while remaining in the orthogonal complement of the rest for all d< n). We define an ϵ -neighborhood of $U\in\mathcal{S}_d$ as the set of all d-dimensional subspaces $V\in\mathcal{S}_d$ such that there exist orthonormal bases P_U, P_V of U, V with $\|P_U-P_V\|_F<\epsilon$. (More generally, we let $d(U,V):=\inf\{\|P_U-P_V\|_F:P_U,P_V\}$ are orthonormal bases of $U,V\}$ be the metric on \mathcal{S}_d .) We then define the local optimality and continuity of g_A,h , and loss accordingly.

We state the a property of g_A which is a building block for the proof of Theorem 3.1.

Lemma A.1. Given a matrix $A \in \mathbb{R}^{a \times n}$, the value of the function g_A at any local maximum is the same.

Before we prove this lemma, we prove use it to prove Theorem 3.1.

Proof of Theorem 3.1: We prove by a contradiction. Let W be a global minimum of h and assume that

$$loss(A_1, W) > loss(A_2, W). \tag{35}$$

Since loss is continuous, there exists $\epsilon > 0$ such that for any W_{ϵ} in the ϵ -neighborhood of W, $h(W_{\epsilon}) = loss(A_1, W)$. Since W is a global minimum of h, it is a local minimum of $loss(A_1, W)$, or equivalently a local maximum of g_{A_1} .

Let $\{v_1,\ldots,v_n\}$ be an orthonormal basis of the eigenvectors of $A_1^TA_1$ corresponding to eigenvalues $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_n$. Let V^* be the subspace spanned by $\{v_1,\ldots,v_d\}$. Note that $loss(A_1,V^*)=0$. Since the loss is always non-negative for both A_1 and A_2 , (35) implies that $loss(A_1,W)>0$. Therefore, $W\neq V^*$ and $g_{A_1}(V^*)>g_{A_1}(W)$. By Lemma A.1, this is in contradiction with V^* being a global maximum and W being a local maximum of g_{A_1} .

We now prove Lemma A.1. We will also use the following formula: given a matrix $V = [v_1, \dots, v_d] \in \mathbb{R}^{n \times d}$ with orthonormal columns and $U = \operatorname{span}(V)$, we have

$$g_A(U) = ||AV||_F^2 = \sum_{i=1}^d ||Av_i||^2$$
(36)

Formula (36) is straightforward from the definitions of g_A and Frobenius norm.

Proof of Lemma A.1: We prove that the value of function g_A at its local maxima is equal to its value at its global maximum, which we know is the subspace spanned by a top d eigenvectors of A^TA . Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of eigenvectors of A^TA with corresponding eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ where ties are broken arbitrarily. Let V^* be the subspace spanned by $\{v_1, \ldots, v_d\}$ and let $U \in \mathcal{S}_d$ be a local optimum

of g_A . We assume for contradiction that $g_A(U) < g_A(V^*)$. We will show that there exists a small constant C > 0 such that for all $\epsilon \in (0, C)$, there exists U_{ϵ} in the ϵ -neighborhood of U which strictly increases g_A . This will contradict the local optimality of U.

As $g_A(U) < g_A(V^*)$, we have $U \neq V^*$. Let k be the smallest index such that $v_k \notin U$. Extend $\{v_1,\ldots,v_{k-1}\}$ to an orthonormal basis of $U:\{v_1,\ldots,v_{k-1},v_k',\ldots,v_d'\}$. Let $q\geq k$ be the smallest index such that $\|Av_q\|^2 > \|Av_q'\|^2$ (such an index q must exist because $g_A(U) < g_A(V^*)$). Without loss of generality, we can assume that q=1. Therefore, we assume that v_1 , the top eigenvector of A^TA , is not in U and that v_1 strictly maximizes the function $\|Au\|^2$ over the space of unit vectors u. Specifically, for any unit vector $u \in U$, $\|Au\|^2 < \|Av_1\|^2 = \lambda_1$. We distinguish two cases:

Case $v_1 \perp U$. Let $w_{\epsilon} = \sqrt{1 - \epsilon^2} u_1 + \epsilon v_1$. Then we have $\|w_{\epsilon}\| = 1$ and that $\{w_{\epsilon}, u_2, \dots, u_d\}$ is an orthonormal set of vectors. We set $U_{\epsilon} = \text{span}\{w_{\epsilon}, u_2, \dots, u_d\}$ and claim that $g_A(U_{\epsilon}) - g_A(U) > 0$ for all $\epsilon \in (0, 1)$. By (36), $g_A(U_{\epsilon}) = \|Aw_{\epsilon}\|^2 + \|Au_2\|^2 + \dots + \|Au_d\|^2$ and $g_A(U) = \|Au_1\|^2 + \|Au_2\|^2 + \dots + \|Au_d\|^2$. Hence, $g_A(U_{\epsilon}) - g_A(U) = \|Aw_{\epsilon}\|^2 - \|Au_1\|^2$. We have

$$||Aw_{\epsilon}||^{2} - ||Au_{1}||^{2} = ||A(\sqrt{1 - \epsilon^{2}}u_{1} + \epsilon v_{1})||^{2} - ||Au_{1}||^{2}$$

$$= (\sqrt{1 - \epsilon^{2}}u_{1}^{T} + \epsilon v_{1}^{T})A^{T}A(\sqrt{1 - \epsilon^{2}}u_{1} + \epsilon v_{1}) - ||Au_{1}||^{2}$$

$$= (1 - \epsilon^{2})u_{1}^{T}A^{T}Au_{1} + \epsilon^{2}v_{1}^{T}A^{T}Av_{1} + 2\sqrt{1 - \epsilon^{2}}\epsilon u_{1}^{T}A^{T}Av_{1} - ||Au_{1}||^{2}$$

$$= (1 - \epsilon^{2})||Au_{1}||^{2} + \epsilon^{2}\lambda_{1} + 2\epsilon\sqrt{1 - \epsilon^{2}}u_{1}^{T}A^{T}Av_{1} - ||Au_{1}||^{2}$$

$$= \epsilon^{2}(\lambda_{1} - ||Au_{1}||^{2}) + 2\epsilon\sqrt{1 - \epsilon^{2}}u_{1}^{T}A^{T}Av_{1}$$

Next, we have $u_1^T A^T A v_1 = u_1^T (\lambda_1 v_1) = \lambda_1 u_1^T v_1 = 0$ since v_1 is an eigenvector of $A^T A$ and $v_1 \perp u_1$. This and the fact that $||Au_1||^2 < \lambda_1$ give

$$||Aw_{\epsilon}||^{2} - ||Au_{1}||^{2} = \epsilon^{2}(\lambda_{1} - ||Au_{1}||^{2}) > 0$$

as desired.

Case $v_1 \not\perp U$. Let $v_1 = \sqrt{1-a^2}z_1 + az_2$ where $z_1 \in U$, $z_2 \perp U$, $a \in (0,1)$ and $||z_1|| = ||z_2|| = 1$, so the projection of v_1 to U is $\sqrt{1-a^2}z_1$. Note that $z_2 \neq 0$ and a > 0 since $v_1 \notin U$, and that a < 1 since $v_1 \not\perp U$. We extend $\{z_1\}$ to an orthonormal basis of U: $\{z_1, u_2, \ldots, u_k\}$.

Consider the unit vector $w_{\epsilon} = \sqrt{1 - \epsilon^2} z_1 + \epsilon z_2$ for $\epsilon \in (0, 1)$. Let $U_{\epsilon} := \text{span}\{w_{\epsilon}, u_2, \dots, u_d\}$. Note that $\{w_{\epsilon}, u_2, \dots, u_d\}$ is orthonormal for any ϵ since both z_1 and z_2 are orthogonal to all of u_2, \dots, u_d and w_{ϵ} is in the span of z_1, z_2 . Since the chosen orthonormal bases of U_{ϵ} and U differ only in w_{ϵ} and z_1 , by (36), $g_A(U_{\epsilon}) - g_A(U) = ||Aw_{\epsilon}||^2 - ||Az_1||^2$. We can write

$$w_{\epsilon} = \left(\sqrt{1 - \epsilon^2} - \frac{\epsilon\sqrt{1 - a^2}}{a}\right) z_1 + \frac{\epsilon}{a} \left(\sqrt{1 - a^2} z_1 + a z_2\right)$$
$$= \left(\sqrt{1 - \epsilon^2} - \frac{\epsilon\sqrt{1 - a^2}}{a}\right) z_1 + \frac{\epsilon}{a} v_1.$$

Thus, by $A^T A v_1 = \lambda_1 v_1$ (as v_1 is an eigenvector with eigenvalue λ_1) and $z_1^T v_1 = \sqrt{1 - a^2}$, we have

$$||Aw_{\epsilon}||^{2} = \left(\sqrt{1 - \epsilon^{2}} - \frac{\epsilon\sqrt{1 - a^{2}}}{a}\right)^{2} ||Az_{1}||^{2} + \frac{\epsilon^{2}}{a^{2}} ||Av_{1}||^{2} + 2\frac{\epsilon}{a} \left(\sqrt{1 - \epsilon^{2}} - \frac{\epsilon\sqrt{1 - a^{2}}}{a}\right) z_{1}^{T} A^{T} A v_{1}$$

$$= \left(1 - \epsilon^{2} + \frac{\epsilon^{2}(1 - a^{2})}{a^{2}} - 2\frac{\epsilon\sqrt{(1 - \epsilon^{2})(1 - a^{2})}}{a}\right) ||Az_{1}||^{2} + \frac{\epsilon^{2}}{a^{2}} \lambda_{1}$$

$$+ 2\frac{\epsilon}{a} \left(\sqrt{1 - \epsilon^{2}} - \frac{\epsilon\sqrt{1 - a^{2}}}{a}\right) \lambda_{1} z_{1}^{T} v_{1}$$

$$= \left(1 - 2\epsilon^{2} + \frac{\epsilon^{2}}{a^{2}} - 2\frac{\epsilon\sqrt{(1 - \epsilon^{2})(1 - a^{2})}}{a}\right) ||Az_{1}||^{2}$$

$$+ \left(\frac{\epsilon^{2}}{a^{2}} + 2\frac{\epsilon\sqrt{(1 - \epsilon^{2})(1 - a^{2})}}{a} - 2\frac{\epsilon^{2}(1 - a^{2})}{a^{2}}\right) \lambda_{1}$$

$$= ||Az_{1}||^{2} + (\lambda_{1} - ||Az_{1}||^{2}) \left(2\frac{\epsilon\sqrt{(1 - \epsilon^{2})(1 - a^{2})}}{a} + 2\epsilon^{2} - \frac{\epsilon^{2}}{a^{2}}\right) > ||Az_{1}||^{2}.$$

The last inequality follows since $\lambda_1 > \|Az_1\|^2$. Now, we let $0 < \epsilon < \frac{1}{1+b}$ for $b = 4a^2(1-a^2)$ so that $2\frac{\epsilon\sqrt{(1-\epsilon^2)(1-a^2)}}{a} > \frac{\epsilon^2}{a^2}$. Then, $\|Aw_\epsilon\|^2 > \|Az_1\|^2$ and therefore $g_A(U_\epsilon) > g_A(U)$ for all such ϵ .

A.2 Proof of Theorem 3.1 by SDP relaxation

We remark that Theorem 3.1 also follows from using SDP relaxation formulation of MULTI-CRITERIA-DIMENSION-REDUCTION for marginal loss objective, which is

$$\min_{X \in \mathbb{R}^{n \times n}} z \text{ subject to} \tag{37}$$

$$\beta_i - \langle A_i^T A_i, X \rangle \le z \quad , i \in \{1, 2\}$$
(38)

$$tr(X) \le d \tag{39}$$

$$0 \prec X \prec I \tag{40}$$

where $\beta_i = \max_{Q \in \mathcal{P}_d} \|A_i Q\|^2$. We provide a proof here as another application of the relaxation.

Proof of Theorem 3.1: Let X^* be an extreme solution of the SDP (37)-(40). Suppose the marginal loss of two groups are not equal; without loss of generality, we have

$$\beta_1 - \langle A_1^T A_1, X \rangle > \beta_2 - \langle A_2^T A_2, X \rangle.$$

Since the constraint $\beta_2 - \langle A_2^T A_2, X \rangle \leq z$ is not tight, we can delete it and the new SDP does not change the optimal solution. However, an optimal solution of the new SDP, which now has only one group, is a standard PCA solution of the first group. This solution gives a loss of zero, so the optimum of the new SDP is zero. Therefore, the optimum of original SDP (37)-(40) is also zero. However, since the losses of both groups are always non-negative, they must be zero and hence are equal.

A.3 Proof of MW convergence

Here we prove Theorem 7.1 using the mirror descent convergence [64]. When mirror descent is performed over a simplex, and the convergence guarantee from mirror descent is simplified as follows. We write $\Delta_k := \left\{ w \in \mathbb{R}^k : w \geq 0, \sum_{i \in [k]} w_i = 1 \right\}$.

Theorem A.2. ([64]) Consider a problem of maximizing a concave function h(w) over $w \in \Delta_k$ where $h(u) - h(v) \le L \|u - v\|_1$ for $u, v \in \Delta_k$. The t-th iterate $w^{(t)}$ of the mirror descent algorithm with negative entropy potential function $R(w) = \sum_{i=1}^k w_i \log w_i$, step size η , and the initial solution $w^{(0)} = (1/k, \ldots, 1/k)$ satisfies

$$h^* - h(w^{(t)}) \le \frac{\log k}{nt} + \frac{\eta}{2}L^2$$
 (41)

where h^* is the optimum of the maximization problem.

We apply Theorem A.2 to obtain the convergence bound for MULTI-CRITERIA-DIMENSION-REDUCTION for fairness criteria in the "max-min" type including MM-Var and MM-Loss.

Proof of Theorem 7.1. We showed earlier that the dual problem of maximizing g is

$$\inf_{w \in \Delta_k} \left(h(w) := \max_{X \in \Omega} \sum_{i=1}^k w_i \langle B_i, X \rangle \right)$$

and that MW algorithm is equivalent to mirror descent on h(w) over $w \in \Delta_k$. Applying Theorem A.2 and substituting $\eta = \sqrt{\frac{\log k}{2T}} L$, the desired convergence bound follows, so it remains to show that $h(u) - h(v) \le L \|u - v\|_1$ for $u, v \in \Delta_k$. Let $X_u^* \in \arg\max_{X \in \Omega} \sum_{i=1}^k u_i \langle B_i, X \rangle$. We have

$$h(u) = \sum_{i=1}^{k} u_i \langle B_i, X_u^* \rangle = \sum_{i=1}^{k} v_i \langle B_i, X_u^* \rangle + \sum_{i=1}^{k} (u_i - v_i) \cdot \langle B_i, X_u^* \rangle$$

$$\leq h(v) + \sum_{i=1}^{k} |u_i - v_i| \cdot \langle B_i, X_u^* \rangle$$

$$\leq h(v) + \sum_{i=1}^{k} |u_i - v_i| \cdot \text{tr}(B_i) \leq h(v) + ||u - v||_1 \cdot L$$

where the first inequality follows from $B_i, X_u^* \succeq 0$ so that their inner product is non-negative, and the second follows from $B_i \succeq 0$ and $X_u^* \preceq I$. This finishes the proof.

A.4 Proof of NSW convergence

We have a standard convergence of FW for differentiable and L-Lipschitz objective functions as follows.

Theorem A.3. ([42]) Consider a maximization problem of an L-Lipschitz concave function g(X) over a convex feasible set D of diameter $diam(D) := \max_{X,Y \in D} \|X - Y\|^2$. The t-th iterate X_t of Frank-Wolfe with step sizes $\eta_s = 2/(s+1)$ satisfies

$$g^* - g(X_t) \le \frac{2L \cdot \operatorname{diam}^2(D)}{t+2} \tag{42}$$

where g^* is the optimum of the maximization problem.

The theorem gives the proof of Theorem 7.2 as follow(s).

Proof of Theorem 7.2. We first show that g is L-Lipschitz for $L = \frac{k}{\lambda}$. We have

$$\left\|\nabla_X g(X)\right\|_F = \left\|\sum_{i \in [k]} \frac{B_i}{\langle B_i, X \rangle + \lambda \cdot \left\|B_i\right\|_F}\right\| \le \sum_{i \in [k]} \frac{\left\|B_i\right\|_F}{\lambda \cdot \left\|B_i\right\|_F} = \frac{k}{\lambda}$$

as claimed. We write $\Omega=\{X\in\mathbb{R}^{n\times n}:\operatorname{tr}(X)=d,0\preceq X\preceq I\}$. By the convergence of FW in Theorem A.3, it remains to show that $\operatorname{diam}(\Omega)\leq 2\sqrt{d}$. For any $X\in\Omega$, let $\sigma_i(X)$ be the eigenvalues of X in the descending order. Then, we have $\|X\|_F^2=\sum_{i\in[n]}\sigma_i^2(X), \sum_{i\in[n]}\sigma_i(X)\leq d$, and $\sigma_i(X)\in[0,1]$ for all i by the constraints in Ω . Since a function $h(x)=x^2$ is convex, $\sum_{i\in[n]}\sigma_i^2(X)$ is maximized subject to $\sum_{i\in[n]}\sigma_i(X)=1$ when $\sigma_1(X)=\ldots=\sigma_d(X)=1$ and $\sigma_{d+1}(X)=\ldots=\sigma_n(X)=0$. So, we have $\|X\|_F^2\leq d$, and therefore

$$\max_{X,Y\in\Omega}\|X-Y\|_F \le \max_{X\in\Omega}\|X\|_F + \max_{Y\in\Omega}\|Y\|_F \le 2\sqrt{d}$$
 (43)

as needed. \Box

B Tightness of the rank violation bound

Here, we show that the bound of rank of extreme solutions in Theorem 1.8 is tight by the following statement.

Lemma B.1. For any n,d, and s such that $d \le n$ and $1 \le s \le n-d$, there exist $m = \frac{(s+1)(s+2)}{2} - 1$ real matrices A_1, \ldots, A_m and a real matrix C, all of dimension $n \times n$, and $b_1, \ldots b_m \in \mathbb{R}$ such that $\mathbb{SDP}(\mathbb{I})$ is feasible and all of its solutions have rank at least d+s.

The example of instance is modified from [14] (and can also be found in [3]) to match our SDP that has the additional $X \leq I$ constraint.

Proof. We construct the constraints of $\mathbb{SDP}(\mathbb{I})$ so that the feasible set is

$$\left\{ X = \begin{bmatrix} \lambda I_{s+1} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \in \mathbb{R}^{n \times n} : \lambda \in \mathbb{R}, \text{ tr}(X) \le d, \ 0 \le X \le I_n \right\}$$
(44)

This can be done by m constraint of the form (6) (e.g. by using $\frac{s(s+1)}{2}$ equality constraints to set off-diagonal entries of top $(s+1)\times(s+1)$ submatrix to zero, and the rest to set diagonal entries to be identical). To finish the construction of the instance, we set

$$C = \operatorname{diag}(\underbrace{0, \dots, 0}_{s \text{ times}}, \underbrace{1, \dots, 1}_{d \text{ times}}, \underbrace{0, \dots, 0}_{n-s-d \text{ times}})$$

Then, we claim that $\mathbb{SDP}(\mathbb{I})$ has an extreme solution

$$X^* = \operatorname{diag}(\underbrace{1/s, \dots, 1/s}_{s+1 \text{ times}}, \underbrace{1, \dots, 1}_{d-1 \text{ times}}, \underbrace{0, \dots, 0}_{n-s-d \text{ times}})$$

$$\tag{45}$$

We first show that X^* is an optimal solution. Let x_1, \ldots, x_n be diagonal entries of X in the feasible set. By $0 \leq X \leq I$, we have $x_i = \mathbf{e_i}^T X \mathbf{e_i} \in [0,1]$ where $\mathbf{e_i}$ is the unit vector at coordinate i. This fact, combined with $\operatorname{tr}(X) = \sum_{i=1}^n x_i \leq d$, shows that $\langle C, X \rangle$ is maximized when $\operatorname{diag}(X)$ is as described in (45).

We now show that X^* is extreme. In fact, we show that X^* is the unique solution and hence necessarily extreme (since the feasible set is convex and compact). From the argument above, any solution $\bar{X} = [x_{ij}]_{i,j\in[n]}$ must satisfy $\mathrm{diag}(\bar{X}) = \mathrm{diag}(X^*)$, and it remains to show that off-diagonal entries must be zero. Because every 2×2 principle minor of \bar{X} is non-negative, we have $x_{ij}=0$ if $i\in\{s+d+1,\ldots,n\}$ or $j\in\{s+d+1,\ldots,n\}$. We know that $x_{ij}=0$ for all $i,j\leq s+1$ such that $i\neq j$ by the constraints in (44), so it remains to show that $x_{ij}=0$ for $s+2\leq i\leq d+s$ and $j\leq d+s$ such that $i\neq j$.

Let i,j be one of such pair. We claim that the bigger eigenvalue of the 2×2 submatrix $\begin{bmatrix} x_{ii} & x_{ij} \\ x_{ij} & x_{jj} \end{bmatrix}$ is at least max $\{x_{ii}, x_{jj}\}$, with a strict equality if and only if $x_{ij} \neq 0$. Let p(t) be the characteristic polynomial of the matrix. Observe that $p(x_{ii}), p(x_{jj}) \leq 0$ (< 0 if $x_{ij} \neq 0$ and = 0 if $x_{ij} = 0$) and that p(t') > 0 for a sufficiently large $t' > \max\{x_{ii}, x_{jj}\}$. Hence, there must be a root of p(t) in between max $\{x_{ii}, x_{jj}\}$ and t', and the claim follows. Now, for the pair i, j, we have $x_{jj} = 1$. By Cauchy's Interlacing Theorem, $X \leq I$ implies that any 2×2 submatrix of X has both eigenvalues at most 1. Therefore, the claim implies that $x_{ij} = 0$.