Model Explanations via the Axiomatic Causal Lens

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Abstract

Explaining the decisions of black-box models is a central theme in the study of trustworthy ML. Numerous measures have been proposed in the literature; however, none of them take an axiomatic approach to causal explainability. In this work, we propose three explanation measures which aggregate the set of all but-for causes — a necessary and sufficient explanation — into feature importance weights. Our first measure is a natural adaptation of Chockler and Halpern's notion of causal responsibility, whereas the other two correspond to existing game-theoretic influence measures. We present an axiomatic treatment for our proposed indices, showing that they can be uniquely characterized by a set of desirable properties. We also extend our approach to derive a new method to compute the Shapley-Shubik and Banzhaf indices for black-box model explanations. Finally, we analyze and compare the necessity and sufficiency of all our proposed explanation measures in practice using the Adult-Income dataset. Thus, our work is the first to formally bridge the gap between model explanations, game-theoretic influence, and causal analysis.

1 Introduction

The literature on model explanations has grown dramatically in recent years. While the idea of explaining the decisions of black-box models has indeed risen in popularity, it has roots in the theory of causality. Prior works on model explanations make preliminary connections between theories of causality and ML explainability; in particular, Datta et al. [2016] relate counterfactual causes to the Shapley value, currently one of the most popular methods of explaining black-box decisions. However, no formal connection between causal explanations, game theoretic influence and black-box model explanations has been made; this is where our work comes in.

1.1 Our Contributions

Our work formally relates causality, game theory and model explanations. In particular, while Datta et al. [2016] make a compelling argument for the Shapley value as the prime candidate for measuring aggregate causal counterfactual interventions over features, we show that an entirely different family of measures captures necessity, sufficiency and minimality in model explanations. Two of the measures we propose correspond to (somewhat obscure) game-theoretic influence measures [Deegan and Packel, 1978, Holler, 1982, Holler and Packel, 1983], whereas a newly introduced measure, which we term the responsibility index, captures the degree of responsibility in structural causal models, first described in Chockler and Halpern [2004].

Our results characterize measures that aggregate *minimal causes*, i.e. the smallest possible changes to model features that will result in a change in outcome. They differ, however, in the manner in which they aggregate them. The responsibility index (Section 3.1) emphasizes smaller causal sets, whereas the Holler-Packel index (Section 3.2) assigns more importance to features that appear in more minimal causes. On the other hand, the Deegan-Packel index (Section 3.3) does a

bit of both. Our work resolves a research question posed by Datta et al. [2016]. While their work sets the Shapley value as a prime method for model explanations (as it has indeed become), in an appendix to their work, Datta et al. argue that the Deegan-Packel index might be better suited for relating causality and explainability. We show that this is indeed the case: in Section 2.4, we relate causality and model explanations, with our proposed indices as the logical consequence of this analysis.

We also discuss measures which aggregate quasi-minimal causes — causes where the minimality condition is relaxed. We find that well-studied game theoretic explanation measures — the Shapley-Shubik (Section 4.1) and Banzhaf (Section 4.2) indices — fall into this category along with the relatively unknown Johnston index (Section 4.3). Our approach provides a new method to compute the Shapley-Shubik index using sets of counterfactual dependence. This new method also allows us to provide importance weights to categorical features directly as opposed to each of the features in their one hot encoded representation (as done by SHAP [Lundberg and Lee, 2017]).

We conclude with an empirical evaluation of our proposed explanation measures on the Adult-Income dataset.

1.2 Related Work

Recent years have seen a significant increase in model explanations research inspired by causality. Datta et al. [2016], Janzing et al. [2020] and Frye et al. [2021] use causality to assign value to sets of variables. However, they all stick to the well-established Shapley value to compute importance weights from the value function, as we argue in Section 2.4, is not the best choice. Mothilal et al. [2021], Watson et al. [2021] and Galhotra et al. [2021] use necessity and sufficiency to compute importance weights for each feature. Our work can also be seen as providing methods which compute necessity and sufficiency scores, via a different approach; instead of computing changes in probability like previous work, we aggregate minimal necessary sets and use an axiomatic approach to derive scores.

Other works propose variants of the Shapley value which incorporate the underlying causal graph. Frye et al. [2020] use causal information to ensure causal ancestors are assigned an assymetrically higher importance than their children; on the other hand, Singal et al. [2021] provide a flow based approach to capture the flow of importance through the causal network. Wang et al. [2021] also look into the model explanations problem in the presence of a causal graph, but their approach assigns importance weights to edges in a causal graph as opposed to nodes.

Our measures identify and aggregate minimal causes, a notion similar to prime implicants in SAT-based model explanations [Yu et al., 2020, Marques-Silva et al., 2020] and sparsity in counterfactual explanations [Mothilal et al., 2020, Karimi et al., 2020]. Minimality has also been explored in other works on model explanations: Watson et al. [2021] propose an algorithm to compute approximate minimal sufficient sets and Ribeiro et al. [2018] propose a method to compute a minimal set of rules to explain any model.

Causality has also been incorporated in other areas of explainable AI as well. Mahajan et al. [2019] propose a method of generating counterfactual explanations that preserve causal constraints and Karimi et al. [2021] reframe the problem of finding recourse as finding the best possible intervention.

2 Preliminaries

We denote vectors by \vec{x} and \vec{y} . We denote the *i*-th and *j*-th indices of the vector \vec{x} using x_i and x_j . Given a set S, we denote the restricted vector containing only the indices $i \in S$ using \vec{x}_S . We also use [k] to denote the set $\{1, 2, \ldots, k\}$. We refer readers to Halpern and Pearl [2005a] for a primer on causal models and causality.

Exogenous Variables

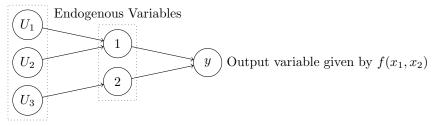


Figure 1: An example causal network of the machine learning model f. The model has two features whose values are determined by three unknown exogenous variables

2.1 Post-Hoc Explanations

We are given black box access to a machine learning model of interest f that predicts some binary output variable $y \in \{0,1\}$ using input features $N = \{1,2,\ldots,n\}$. This function f can be seen as part of an implicit causal model M with endogenous variables $V = N \cup \{y\}$ and some unknown set of exogenous variables U. The variable g in the causal model g depends only on the other endogenous variables g through the function g and the values of these endogenous variables (the features values) are determined by some unknown process which depends on the exogenous variables g. In addition to this, in line with Mothilal et al. [2021], we assume that none of the variables in g causally depend on each other in the causal model g. An example of such a causal model is given in Figure 1.

We are given a *point of interest* \vec{x} of values for the input features N and we would like to explain the outcome $f(\vec{x})$. More specifically, we assign importance weights (or *indices*) to each feature in N, proportional to their power in deciding the outcome y.

Remark 2.1. Our simplifying assumption that input features do not causally depend on each other does come with some loss of generality. It says that any correlation between features is not due to direct causal dependence but due to the effect of some unknown exogenous variable(s). We make this assumption mainly to make it easier to compute the effect of interventions. The underlying causal relations between variables are usually unknown in most applications, which significantly affects any explanation measure which assumes knowledge of these relations. This assumption, therefore, allows our work to be applied to a significantly larger set of ML models as compared to approaches which require knowledge of the underlying causal relations between features. This assumption is also made by Mothilal et al. [2021]. Janzing et al. [2020] use an approach similar in spirit in order to remain agnostic about causal dependencies between features — instead of assuming the features themselves do not causally depend on each other, they formally distinguish between features inputted to the model and the true feature values.

Remark 2.2. The restriction to a binary output variable does *not* result in a significant loss of generality: many domains which require explanations can be cast as classification problems. A model explanation answers a simple "why" question, e.g. "why was my loan accepted?". To answer these questions, the model of interest can be studied as a binary classifier, whose value is 1 when the event in question occurs and 0 otherwise (e.g. output 1 if the loan is approved and 0 otherwise). This can similarly be applied to several possible "why" questions, even when the model of interest is not a binary classifier.

2.2 Assigning Values to Sets of Variables

We define the value of a subset of features $S \subseteq N$ by the ability of an intervention on that set to change the outcome. We represent the value of every subset using a function v, defined as follows:

$$v(S, \vec{x}, f) = \max_{\vec{x}' : \vec{x}'_{N \setminus S} = \vec{x}_{N \setminus S}} |f(\vec{x}') - f(\vec{x})| \tag{1}$$

The function v is monotone in S for any fixed point of interest \vec{x} and model of interest f (see the proof of Proposition 2.3 in the appendix), and induces a *simple cooperative game* [Chalkiadakis et al., 2011] where the set of features corresponds to the set of players, and $v(S) \in \{0,1\}$ for all $S \subseteq N$.

Proposition 2.3. The function v as given by (1) is monotone in S for any fixed point of interest \vec{x} and model of interest f.

We omit \vec{x} and f when they are clear, focusing solely on the set S. The cooperative game formulation allows us to use prior research for estimating the power of players/features when deciding the output of a function at a particular point.

Note that while there are other ways to define the value of sets of features [Sundararajan and Najmi, 2020, Lundberg and Lee, 2017, Frye et al., 2021], none of them are guaranteed to be monotone. This means that the indices we propose become meaningless when used with the value functions defined by Sundararajan and Najmi [2020] and Lundberg and Lee [2017]; if v is not monotone, minimal causal sets are not well-defined.

2.3 Relation to Causality

For a subset of features $S \subseteq N$, the assignment $S = \vec{x}_S$ is a but-for cause of the output $y = f(\vec{x})$ (as defined by Mothilal et al. [2021]) in $(M, \vec{u}_{\vec{x}})$ if and only if v(S) = 1 and S is minimal, i.e no proper subset $T \subset S$ has v(T) = 1. This is a stronger definition of cause than the notion of actual cause (defined in Halpern and Pearl [2005a]). Sets with v(S) = 1 also correspond to sets of counterfactual dependence [Lewis, 1986].

We use the definition of but-for cause rather than actual cause mainly because it allows us to observe how features interact in the function f. Note that this is not possible with actual causes since they turn out to be singletons [Eiter and Lukasiewicz, 2002], masking the potential effects of other features on the outcome. We use the term cause to refer to but-for cause rather than actual cause.

2.4 Causes as Explanations

Consider a loan approval example where an algorithm approves a loan request if the applicant's credit score is over 700 and their account balance is over \$3000. Assume a stakeholder applies with a credit score of 750 and a bank balance of \$2500. The algorithm rejects the applicant and the fact that the applicant's bank account balance is \$2500 is the unique but-for cause of this outcome. This but-for cause (balance = \$2500) intuitively appears to be a good explanation of the outcome since it points out the reason for rejection: "your application would have been accepted, had it not been for your low bank balance".

Causes, more generally, make for good explanations [Miller, 2019, Halpern and Pearl, 2005b]. One reason for this is that the set of all but-for causes is a necessary and sufficient condition of the outcome y — two important qualities of an explanation [Watson et al., 2021, Miller, 2019, Mothilal et al., 2021]. An event a is a necessary condition of the outcome b if $b \to a$ (or $\neg a \to \neg b$); a is a sufficient condition of b if $a \to b$.

In the model explanations context, we define an assignment of variables $S = \vec{x}_S$ to be a necessary condition of the outcome $f(\vec{x})$ if there exists a change of values \vec{x}_S' to the variables in S such that

 $f(\vec{x}) \neq f(\vec{x}_S', \vec{x}_{N \setminus S})$. Similarly, an assignment of variables $S = \vec{x}_S$ is a sufficient condition if there is no change $\vec{x}_{N \setminus S}'$ that can be made to the features in $N \setminus S$ such that the value of the function changes i.e. $f(\vec{x}) \neq f(\vec{x}_S, \vec{x}_{N \setminus S}')$. Using the above definitions, we prove that the set of all but-for causes is a necessary and sufficient condition of the outcome $f(\vec{x})$.

Proposition 2.4. The set of all but-for causes of an outcome $f(\vec{x})$ is a necessary and sufficient condition of $f(\vec{x})$.

While the set of all but-for causes would make for a good explanation, they can unfortunately be exponential in number. We could handpick a few causes as an explanation, but this ignores a large chunk of information that the stakeholder would find useful. Instead, we aggregate them into indices which are both informative and easily understandable. Well-known aggregation methods like the Shapley index [Shapley and Shubik, 1954] and the Banzhaf index [Banzhaf, 1965] aggregate information about all sets, not just minimal ones; they, therefore, are not the most suitable candidates. We propose measures which aggregate causes in Section 3 and discuss measures which aggregate a weaker notion of minimal causality in Section 4.

Note that commonly used frameworks like LIME [Ribeiro et al., 2016] and SHAP [Lundberg and Lee, 2017] handle features differently. LIME is defined on an alternate binary 'interpretable' feature space. SHAP, on the other hand, does not assign an importance weight to every categorical feature, but rather assigns weights for each one-hot encoded feature. Therefore, the weights that SHAP and LIME output are for a different feature space altogether, and cannot be compared to any of our proposed indices. We believe that our proposed method for evaluating sets of features handles categorical features in a far more natural manner, overcoming a potential confounding factor in the SHAP and LIME frameworks.

3 Aggregating Causes

In this section, we still use the set function defined in Section 2.2, but only work with but-for causes. We denote the set of all causes by $\mathcal{M}(v)$ and the set of all causes that contain a feature i by $\mathcal{M}_i(v)$.

We also define a contraction operation on the set of features N where we replace a subset of features T by a single feature [T] corresponding to the set. This results in a reduced game defined on features $(N \setminus T) \cup [T]$ with the characteristic function $v_{[T]}$ defined as:

$$v_{[T]}(S) = \begin{cases} v((S \setminus [T]) \cup T) & [T] \in S \\ v(S) & [T] \notin S \end{cases}$$

The intuition behind this definition is straightforward — since the function v is monotone, if changing any variable in T helps change the output, then changing all the variables in T will still change the output. This definition is also used in Patel et al. [2021]. Lastly, given a permutation $\pi: N \mapsto N$, we define the permutation of a function $\pi v(S) = v(\{\pi(i) \mid i \in S\})$.

3.1 The Responsibility Index

When aggregating minimal causes, we could have our indices reflect the *number* of causes each element is in, or the *size* of these causes.

When we focus on the size of these causes, an intuitive property that we should require the indices to satisfy is *Minimal Size Monotonicity* which is defined for an index ρ as follows:

(MSM) Minimum Size Monotonicity: For any two monotone binary set functions v and v', if $\mathcal{M}_i(v)$ and $\mathcal{M}_i(v')$ are non-empty and $\min_{S \in \mathcal{M}_i(v)} |S| \leq \min_{T \in \mathcal{M}_i(v')} |T|$, then $\rho_i(v) \geq \rho_i(v')$. Moreover, equality holds if $\min_{S \in \mathcal{M}_i(v)} |S| = \min_{T \in \mathcal{M}_i(v')} |T|$.

In other words, the smaller the smallest cause that contains a feature, the higher that feature's importance.

Any index that satisfies Minimum Size Monotonicity should also satisfy these other basic properties.

- (UE) Unit Efficiency: If $\mathcal{M}_i(v) = \{i\}$, then $\rho_i(v) = 1$. This property upper bounds the value that the feature with the smallest possible cause can obtain.
- (S) Symmetry: For any permutation of features π , we have $\rho_{\pi i}(\pi v) = \rho_i(v)$. This ensures that features' indices are not a factor in determining influence.
- (NF) Null Feature: If $\mathcal{M}_i(v) = \emptyset$ then $\rho_i(v) = 0$: features that are never part of a cause should not be attributed any value. This property is equivalent to Dummy [Shapley, 1953] and similar in spirit to Sensitivity [Sundararajan et al., 2017].
- (C) Contraction: For any subset $T(|T| \geq 2)$ that does not contain a null feature, we have $\rho_{[T]}(v_{[T]}) \leq \sum_{i \in T} \rho_i(v)$. Moreover, equality holds if $T \in \{S \mid S \in \arg\min_{S' \in \mathcal{M}_i(v)} |S'|\}$ for all $i \in T$. In other words, equality holds if and only if T is the smallest cause for all the features in T. This property upper bounds the gain one gets by combining features and ensures that the total attribution that a set of features receives when combined does not exceed the sum of the individual attributions of each element in the set.

Our first result is that any index that satisfies these properties must output the degree of responsibility of a feature [Chockler and Halpern, 2004] for the event $y = f(\vec{x})$. We refer to this index as the responsibility index of a feature i (denoted by $\rho_i(v)$) whose value is given by

$$\rho_i(v) = \begin{cases} \frac{1}{k} & k = \min_{S \in \mathcal{M}_i(v)} |S| \\ 0 & \mathcal{M}_i(v) = \emptyset \end{cases}$$
 (2)

Theorem 3.1. For any monotone binary set function v, the only index which satisfies (MSM), (UE), (S), (NF) and (C) is the responsibility index $\rho(v)$.

3.2 The Holler-Packel Index

The other direction we take while developing indices is to focus solely on the number of causes that contain a feature, rather than their size. Any index that focuses on the number of causes should satisfy Minimal Monotonicity which is defined for an index η as

(MM) Minimal Monotonicity: For any two monotone binary set functions v and v', if $\mathcal{M}_i(v) \subseteq \mathcal{M}_i(v')$, then $\eta_i(v) \leq \eta_i(v')$. Equality holds if $\mathcal{M}_i(v) = \mathcal{M}_i(v')$.

In other words, belonging to more causes means more influence for i. In addition, we require that our index satisfies

(TMCE) Total Minimal Cause Efficiency:
$$\sum_{i \in N} \eta_i(v) = \frac{1}{2^{n-1}} \sum_{S \in \mathcal{M}(v)} |S|$$
.

Additionally, we require that the index satisfies the symmetry (S) and null feature (NF) properties we define in Section 3.1. The only difference between the properties presented here and the ones presented in Section 3.1 is that we replace (UE) and (C) with (TMCE). Both sets of properties bound the value the index can take, but are both better suited for their specific setting. When we focus on ensuring Minimal Size Monotonicity, features with smaller causes (in terms of size) can have an arbitrarily high attribution with respect to the other features. Therefore, we bound the index value by upper bounding the value the smallest possible minimal cause using (UE) and then relate this to other minimal causes using (C). However, when focusing on Minimal Monotonicity,

we upper bound the total attribution a feature with a large number of causes can obtain, via an efficiency condition that depends on the sum total of minimal causes each feature is in. This is a fairly common approach in the axiomatization of indices, including the Shapley value [Young, 1985, Shapley, 1953]. The division by 2^{n-1} in the efficiency axiom ensures that the index value remains in [0,1], and is commonly used for other power indices (e.g. the Banzhaf index [Banzhaf, 1965, Coleman, 1968]).

These properties are uniquely satisfied by the raw Holler Packel Index [Holler, 1982, Holler and Packel, 1983] (denoted $\eta_i(v)$) which is given by

$$\eta_i(v) = \frac{1}{2^{n-1}} |\mathcal{M}_i(v)|$$

Theorem 3.2. For any monotone binary set function, the only index satisfying (MM), (TMCE), (S) and (NF) is the raw Holler-Packel Index $\eta(v)$.

The raw Holler-Packel index satisfies a strictly stronger property than Minimal Monotonicity which we refer to as Count Monotonicity:

(CM) Count Monotonicity: If $|\mathcal{M}_i(v)| \leq |\mathcal{M}_i(v')|$, then $\eta_i(v) \leq \eta_i(v')$. Equality holds if $|\mathcal{M}_i(v)| = |\mathcal{M}_i(v')|$.

(CM) essentially says that the larger the size of the set of causes, the higher the attribution. However, we do not need this property to axiomatize the Holler-Packel index.

3.3 The Deegan-Packel Index

The efficiency property in Section 3.2 (TMCE) takes a very specific value that is not intuitively obvious. It exists primarily to upper bound the value that an index can take, replacing the value $\frac{1}{2^{n-1}} \sum_{S \in \mathcal{M}(v)} |S|$ by any positive value should achieve the same effect.

However, as we show in this section, simply changing the efficiency value can result in a significant change on the set of indices that can satisfy the properties in Section 3.2.

If we replace the value $\frac{1}{2^{n-1}} \sum_{S \in \mathcal{M}(v)} |S|$ in (TMCE) with $\frac{1}{2^{n-1}} |\mathcal{M}(v)|$, the properties uniquely characterize the raw Deegan-Packel index [Deegan and Packel, 1978] given by

$$\phi_i(v) = \frac{1}{2^{n-1}} \sum_{S \in \mathcal{M}_i(v)} \frac{1}{|S|}$$

More formally, let us define a new property

(MCE) Minimal Cause Efficiency: For any function $v, \sum_{i \in N} \phi_i(v) = \frac{1}{2^{n-1}} |\mathcal{M}(v)|$.

We have the following theorem

Theorem 3.3. For any monotone binary set function, the only index satisfying (MM), (MCE), (S) and (NF) is the raw Deegan-Packel Index $\phi(v)$.

The proof is very similar to that of Theorem 3.2 and the similar proof by Lorenzo-Freire et al. [2007], and is omitted.

The raw Deegan-Packel index accounts for both the size of minimal causes and their number. It does not satisfy the Count Monotonicity property that the Holler-Packel index satisfies. However, it satisfies another desirable property which prioritizes smaller minimal causes over larger ones. We refer to this property as Individual Set Monotonicity:

(ISM) Individual Set Monotonicity: Let $\mathcal{M}_i(v) = \{S_1, S_2, \dots, S_k\}$ and $\mathcal{M}_i(v') = \{T_1, T_2, \dots, T_k\}$. If $S_j \subseteq T_j$ for all $j \in [k]$, then $\phi_i(v) \ge \phi_i(v')$. Moreover, equality holds if and only if $S_j = T_j$ for all $j \in [k]$.

This property states that if you replace a cause with a proper subset of it, then the attribution of any feature in the subset will strictly increase. The Deegan-Packel index trivially satisfies this while the Holler-Packel index does not. This may result in a more informative aggregation of causes than aggregations which solely focus on the size or number of minimal causes.

Changing the efficiency value in the axioms defined in Section 3.2 seems to create very different indices. However, if we set the efficiency value to a constant, keeping the rest of the axioms the same as the ones used to axiomatize the Deegan-Packel index, there is no index that satisfies these new axioms. More formally, consider another axiom, which we term efficiency for an index $\beta(v)$.

(E) Efficiency: $\sum_{i \in N} \beta_i = 1$

Proposition 3.4. There exists no index which satisfies (MM), (E), (S) and (NF)

This impossibility result holds even when the efficiency value is a function of the number of features or v(N). In fact, as long as $\sum_{i \in N} \phi_i$ does not depend on $\mathcal{M}(v)$ or the number of causes feature i belongs to $(\mathcal{M}_i(v))$.

4 Aggregating Quasi Minimal Causes

From first impressions, the raw Deegan Packel index resolves the problem of aggregating causes by considering both the number and size of causes. However, upon closer inspection, this index is heavily biased towards the number of minimal causes. This can be more clearly seen in Example 4.1.

Example 4.1. Consider an example with five features, $N = \{1, 2, 3, 4, 5\}$. Consider two functions v and v' where $\mathcal{M}_1(v) = \{\{1\}\}$ $\mathcal{M}_1(v') = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$. Given these minimal causes, we get $\phi_1(v) = 1$ and $\phi_1(v') = \frac{3}{2}$. Even though 1 seems more important under v (it singlehandedly changes the outcome), its Deegan-Packel index is significantly lower under v than under v'.

This flaw mainly arises from the fact that the Deegan-Packel index only considers minimal causes. In the above example, the Deegan-Packel index does not consider the fact that the sets $\{1,2\},\{1,3\}$ and $\{1,4\}$ are sets of counterfactual dependence in v as well. One way to fix this flaw is to consider sets beyond minimal causes.

This relaxation however raises the question of which counterfactual dependence sets to consider, and which features to attribute each counterfactual dependence set to. When a feature i is not critical to a set S — removing i does not change anything $(S \setminus i)$ is still a counterfactual dependence set) — then attributing any importance to feature i seems unreasonable. Consider again Example 4.1: the set $\{1,2\}$ is a cause in v but it does not make much sense to assign 2 any importance: the set $\{1\}$ is a cause as well, and 2 adds no value.

This approach yields a well-defined set of counterfactual dependence sets to consider: we consider counterfactual dependence sets where at least one feature is *critical*. We refer to such sets as *quasi-minimal* causes and denote them by the set $\mathcal{G}(v)$. We denote the set of quasi-minimal causes where feature i is critical by $\mathcal{G}_i(v)$ and given a set S, we denote the set of critical features of S under v by $\chi^v(S)$. We ignore v when it is understood from context.

Any index (say β) that aggregates quasi-minimal causes should satisfy three important properties: symmetry (S) and null feature (NF) as defined in Section 3.1, as well as

(QMM) Quasi-minimal Monotonicity: For any two binary monotone set functions v and v', if $\mathcal{G}_i(v) \subseteq \mathcal{G}_i(v')$, then $\beta_i(v) \leq \beta_i(v')$. Moreover, equality holds if $\mathcal{M}_i(v) = \mathcal{M}_i(v')$.

Quasi-minimal Monotonicity is desirable since it considers both the size and the number of minimal causes that contain a feature, as well as the size and the number of minimal causes that contain other features. Any index which satisfies these properties will not face the issue that Deegan Packel did in Example 4.1.

To see how (QMM) works, consider a case where we have five features, $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{M}(v) = \{\{1, 2\}\}$. Every set S that contains $\{1, 2\}$ is a quasi minimal cause. Moreover, if we add an element to create a new value function $\mathcal{M}(v') = \{\{1, 2, 3\}\}$, every set which contains $\{1, 2, 3\}$ will be a quasi minimal cause; therefore, the number of quasi-minimal causes will decrease and we will have $\mathcal{G}_1(v') \subsetneq \mathcal{G}_1(v)$. On the other hand, if we add a new quasi minimal cause in v, e.g. $\mathcal{M}(v'') = \{\{1, 2\}, \{3\}\}$, then every superset of $\{1, 2\}$ which does not contain 3 will be a quasi minimal cause in v''. Therefore the number of quasi-minimal causes will decrease and we will have $\mathcal{G}_1(v'') \subsetneq \mathcal{G}_1(v)$. Therefore, any index which satisfies Quasi-minimal Monotonicity increases the attribution of feature i if the sets in $\mathcal{M}_i(v)$ reduce in size, the set $\mathcal{M}_i(v)$ increases in size or the set $\mathcal{M}(v) \setminus \mathcal{M}_i(v)$ reduces in size.

Furthermore, quasi-minimal monotonicity can be interpreted as Strong Monotonicity [Young, 1985] for the restricted setting of simple games. Strong monotonicity for TU cooperative games [Chalkiadakis et al., 2011] is defined as follows

(SM) Strong Monotonicity: For any two real valued set functions, v and v', if $v(S \cup \{i\}) - v(S) \ge v'(S \cup \{i\}) - v'(S)$ for all sets $S \subseteq N \setminus \{i\}$, then $\beta_i(v) \ge \beta_i(v')$. Moreover equality holds if $v(S \cup \{i\}) - v(S) = v'(S \cup \{i\}) - v'(S)$ for all $S \subseteq N \setminus \{i\}$

In this section we discuss three indices – the Shapley-Shubik power index Shapley and Shubik [1954], the Banzhaf index Banzhaf [1965] and the Johnston index Johnston [1978]. Each of these indices satisfies the above three properties. However, due to the complex definition of quasi-minimal monotonicity and the non-linear nature of monotone simple games, characterization results using these three properties for monotone simple games are non-existent. Instead, there exist characterization results for a more general class of cooperative games and we discuss them here.

4.1 The Shapley-Shubik index

The Shapley-Shubik index [Shapley and Shubik, 1954] is by far the most commonly used index in explainable AI research Lundberg and Lee [2017], Frye et al. [2020], Sundararajan and Najmi [2020], Datta et al. [2016]. It assigns each quasi-minimal cause S a weight of $\frac{(|S|-1)!(n-|S|)!}{n!}$. More formally, given a value function v, the Shapley-Shubik index (denoted by $\sigma(v)$) is defined as

$$\sigma_i(v) = \sum_{S \in \mathcal{G}_i(v)} \frac{(|S| - 1)!(n - |S|)!}{n!}$$

For TU Cooperative games, this definition is generalized to

$$\sigma_i(v) = \sum_{S \in N \setminus i} \frac{(|S|)!(n-|S|-1)!}{n!} (v(S \cup \{i\}) - v(S))$$

It clearly satisfies our three desirable properties. Furthermore, for TU Cooperative games, this index can be characterized by our three desirable properties and an efficiency property we call general efficiency; most standard textbooks refer to it simply as efficiency Chalkiadakis et al. [2011].

(GE) General Efficiency:
$$\sum_{i \in N} \sigma_i(v) = v(N)$$

Since the Null Feature property can only be applied to simple games, we redefine it for TU Cooperative games

(GNF) General Null Feature: If $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N \setminus \{i\}$, then $\sigma_i(v) = 0$

Theorem 4.2 (Young [1985]). For TU Cooperative games, the only index which satisfies (SM), (S), (GNF) and (GE) is the Shapley-Shubik index

4.2 The Banzhaf Index

The raw Banzhaf index [Banzhaf, 1965, Coleman, 1968], denoted by β , is given by

$$\beta_i(v) = \frac{1}{2^{n-1}} |\mathcal{G}_i(v)|$$

This is analogous to the Holler-Packel index. For TU Cooperative games, the Banzhaf index is given by

$$\beta_i(v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S))$$

The Banzhaf index can be characterized for TU cooperative games similarly to the Shapley value, with the only difference being that the Banzhaf index has a different efficiency value. Instead of General Efficiency, the Banzhaf index satisfies Total Power (TP):

(TP) Total Power:
$$\sum_{i \in N} \beta_i(v) = \frac{1}{2^{n-1}} \sum_{i \in N} \sum_{S \subset N \setminus \{i\}} v(S \cup \{i\}) - v(S)$$

Theorem 4.3. For TU Cooperative games, the only index which satisfies (SM), (S), (GNF) and (TP) is the raw Banzhaf index

This proof is derived via a similar reasoning to that of Young [1985].

4.3 The Johnston Index

The raw Johnston index [Johnston, 1978], denoted by $\psi(v)$ is given by

$$\psi_i(v) = \frac{1}{2^{n-1}} \sum_{S \in \mathcal{G}_i(v)} \frac{1}{|\chi^v(S)|}$$

Since this index depends on the critical set of a quasi-minimal cause, it cannot be extended to TU cooperative games: the notion of a critical set has no obvious analog in general TU cooperative games. Instead of showing uniqueness of the Johnston index in TU cooperative games, we generalize simple games along a different direction to show a uniqueness result.

There exist sets of quasi minimal causes which no monotone binary set function can define. However, our index is only defined for the sets of quasi minimal causes defined by a monotone binary set function. We characterize the Johnston index for the set of games where this constraint does not exist. We refer to this new index as the *alternate Johnston index*.

instead of taking as input a binary monotone set function v, this index takes as input a set of quasi minimal causes \mathcal{G} and a function $\chi: 2^N \mapsto 2^N$ mapping every quasi minimal set to the set of critical features of this set. We require that the mapping χ maps every non-empty set S to a subset of S, i.e. $\chi(S) \subseteq S$ and is non-empty if $S \in \mathcal{G}$. We refer to all functions that satisfy these restrictions as feasible mappings. We also denote the set of quasi-minimal causes where i is critical by \mathcal{G}_i . The alternate Johnston index (denoted by ω) is given by

$$\omega_i(v) = \frac{1}{2^{n-1}} \sum_{S \in \mathcal{G}_i} \frac{1}{|\chi(S)|}$$

To characterize the alternate Johnston index, we will need an alternate version of our desirable properties. We define the following alternate properties for an index ω :

- (AQM) Alternate Quasi-minimal Monotonicity: For any two set of quasi minimal causes \mathcal{G}, \mathcal{H} with feasble mappings χ_G and χ_H respectively, if $\mathcal{G}_i \subseteq \mathcal{H}_i$ and $\chi_G(S) = \chi_H(S)$ for all $S \in \mathcal{G}_i$, then we have $\omega_i(\mathcal{G}, \chi_G) \leq \omega_i(\mathcal{H}, \chi_H)$. Furthermore, equality holds if $\mathcal{G}_i = \mathcal{H}_i$.
- (AQCE) Alternate Quasi-minimal Cause Efficiency: For any set of quasi minimal sets \mathcal{G} and any feasble mapping χ , $\sum_{i \in N} \omega_i(\mathcal{G}, \chi) = \frac{1}{2^{n-1}} |\mathcal{G}|$.
- (AS) Alternate Symmetry: For any permutation of features π , we have $\omega_{\pi i}(\pi \mathcal{G}, \pi \chi) = \omega_i(\mathcal{G}, \chi)$
- (ANF) Alternate Null Feature: If $i \notin \chi(S)$ for all $S \in \mathcal{G}$, then $\omega_i(\mathcal{G}, \chi) = 0$.

These properties generalize our above mentioned desiderata and reduce to them when \mathcal{G} and χ correspond to the set of quasi minimal sets and the critical set function of a monotone simple game.

Theorem 4.4. For any set of quasi minimal causes \mathcal{G} and any feasble mapping χ , the only index which satisfies (AQM), (AQCE), (AS) and (ANF) is the alternate Johnston Index $\omega(\mathcal{G}, \chi)$.

5 Algorithmic Loan Approval: an Example

In this section, we discuss an example of algorithmic loan approval to show how the all the indices look like in practice. Consider a simple rule-based model f trained on the features 'Age', 'Purpose', 'Credit Score' and 'Bank Balance'. The rule based-model has the following closed form expression:

```
f(\vec{x}) = (\text{Age} < 20 \land \text{Purpose} = \text{Education})
\lor (\text{Age} > 30 \land \text{Purpose} = \text{Real Estate} \land \text{Credit } \not; 700)
\lor (\text{Credit} > 700 \land \text{Bank} > 300000)
\lor (\text{Age} > 25 \land \text{Bank} > 1000000)
```

Let the point of interest \vec{x} that we would like to explain be (Age = 22, Purpose = Real Estate, Credit = 0, Bank = 50000).

Since \vec{x} does not satisfy any of the rules, the model f rejects the applicant; the causes of the outcome are

```
{(Age, Purpose), (Age, Credit), (Bank, Credit), (Age, Bank)}.
```

We compute the feature highlighting indices for all features, presented in Table 1a. Table 1a offers several interesting observations. All six indices have the same weak ordering over the set of features. Age appears in three of four causes, and all indices (weakly) rank Age as the most important feature; however, the proportion of importance given to Age varies from index to index. On one hand, the responsibility index assigns Age the same importance as all other features as Age alone cannot change the outcome. On the other hand, the Johnston index assigns Age roughly the same importance as all other features combined. We do not argue in favor of any index over another, but believe that they all provide useful insights about the output of f.

Another use of explanation indices is that they allow developers to compare different functions via the importance each feature has on the outcome. To show how this can be done, we modify the function f to create a new function by removing the final two rules of f. More, formally g is defined as follows:

```
g(\vec{x}) = (\text{Age} < 20 \land \text{Purpose} = \text{Education})
 \lor (\text{Age} > 30 \land \text{Purpose} = \text{Real Estate} \land \text{Credit } \vec{x}, 700)
```

The applicant \vec{x} still does not satisfy any of the rules of g and is rejected. However, the causes of $g(\vec{x})$ — (Age, Purpose) and (Age, Credit) — are a subset of the causes of $f(\vec{x})$. Ideally, the explanation

Index	Purpose	Age	Bank	Credit
Responsibility	0.5	0.5	0.5	0.5
Holler-Packel	0.125	0.375	0.25	0.25
Deegan-Packel	0.062	0.187	0.125	0.125
Shapley-Shubik	0.083	0.417	0.25	0.25
Banzhaf	0.125	0.625	0.375	0.375
Johnston	0.062	0.437	0.187	0.187

Index	Purpose	Age	Bank	Credit
Responsibility	0.5	0.5	0	0.5
Holler-Packel	0.125	0.25	0	0.125
Deegan-Packel	0.062	0.125	0	0.062
Shapley-Shubik	0.166	0.666	0	0.166
Banzhaf	0.125	0.375	0	0.125
Johnston	0.062	0.25	0	0.062

(b) Index values explaining $g(\vec{x})$

Table 1: The explanations outputted for both $f(\vec{x})$ (Table 1a) and $g(\vec{x})$ (Table 1b), where \vec{x} equals (Age = 22, Purpose = Real Estate, Credit = 0, Bank = 50000). The model g uses a subset of the acceptance rules used in f, which changes feature importance in accordance to changes in the underlying causes.

indices should reflect this and assign features which are present in fewer causes less importance as compared to f. The indices explaining $g(\vec{x})$ are presented in Table 1b.

Some outputs are rather unsurprising. No index assigns a value to the Bank Balance since none of the causes contain it. However, even though the number of causes containing Age reduces, the responsibility index gives it the same amount of importance as f, while the Shapley index assigns it a higher importance than f. All other indices assign Age a lower importance compared to f. This suggests that indices which satisfy minimal monotonicity are better suited when comparing different functions.

6 Empirical Evaluation

We conclude our analysis with a comparison of our indices' practical performance. While our indices differ in the underlying desirable properties they satisfy, all of them ultimately aggregate causes (changes in feature values) for an event (a negative label for a point of interest). Indeed, our first empirical observation is that the indices output very similar results on real-world data. We apply

J	0.949 ± 0.020 R	HP	0.944 ± 0.038	SS SS	B
т	0.949 ± 0.026	0.785 ± 0.075	0.944 ± 0.038	0.999 ± 0.001	0.988 ± 0.005
В	0.983 ± 0.019	0.868 ± 0.057	0.983 ± 0.017	0.992 ± 0.004	
SS	0.959 ± 0.018	0.802 ± 0.072	0.953 ± 0.034		
DP	0.992 ± 0.022	0.940 ± 0.028			
HP	0.927 ± 0.039				

Table 2: Mean \pm Standard Deviation of the cosine similarities computed from 1000 randomly chosen datapoints. Glossary: HP = Holler Packel, DP = Deegan-Packel, SS = Shapley-Shubik, B = Banzhaf, J = Johnston, R = Responsibility

our indices on points in the Adult-Income dataset [Kohavi, 1996] and compare them using the cosine similarity — given two vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$, their cosine similarity is given by $\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}$. We also measure the necessity and sufficiency of the top k features of each of our indices. Cosine similarity compares the indices to one another, whereas necessity and sufficiency simply asks whether the features we identify as important are actually necessary/sufficient conditions of the outcome. Our code can be found at https://github.com/GaBi-1337/Causal-Explanations.

The Adult-Income dataset [Kohavi, 1996] consists of census data, and the classification task commonly associated with the Adult-Income Dataset is to predict whether an individual has an income higher than \$50,000 or not. We choose this dataset since the small number of features

⁽a) Index values explaining $f(\vec{x})$

allows us to exactly compute all the indices — an important requirement when comparing indices. While all the indices we discuss are computationally intractable, simple sampling algorithms can be used to obtain low error approximations of the indices — see Appendix A for complexity analysis and approximation algorithms for all the indices proposed in this paper.

To test our indices, we train a simple decision tree classifier with a depth of 5 on roughly 30,000 samples from the Adult-Income dataset, achieving $\sim 84\%$ test accuracy. The simplicity of our model allows us to easily compute the best possible intervention on a set of features S, i.e. v(S) as defined in (1). Computing v(S) for arbitrarily complex models is a challenge in its own right, which we hope will be explored in future work. The technical details of this experiment are presented in Appendix E.

We randomly sample (without replacement) 1000 points from the test dataset and compute all six indices for each point. Our similarity results are presented in Table 2. Each cell contains the mean and standard deviation of the cosine similarity scores computed for 1000 points. All six indices weigh features similarly most of the time. Most index pairs have an extremely high mean similarity (> 0.9) and low standard deviation (< 0.04). The only exception is the Holler-Packel index, which differs most significantly from the other indices. This is likely because of the difference in the definition of the Holler-Packel index — all other indices consider factors other than the number of causes a feature is present in, whereas the Holler-Packel index solely considers the number of causes an index is present in.

We also measure the quality of our indices by checking whether the top k ($k \in \{1, 2, 3, 5\}$) features are necessary and sufficient conditions of the outcome $f(\vec{x})$ for each of the 1000 points. As a baseline, we check for each datapoint, whether there exists a set of k features which is a necessary (resp. sufficient) condition of the outcome $f(\vec{x})$. We present our results in Table 3. Each cell in the table contains the fraction of points where the top k features of the index were necessary and sufficient. We refer to these values as necessity and sufficiency scores. The baseline computes the fraction of points that admit a necessary and sufficient condition consisting of only k features; it serves as an upper bound on the necessity and sufficiency scores. Ties were broken using the same order the features appeared in the dataset.

All indices other than the Holler-Packel index have a score within 0.11 of the baseline. The general trends as we vary k are unsurprising — when k is low, the sufficiency is 0 but as k increases, both the necessity and sufficiency scores increase till they reach 1 when k=5.

The Holler-Packel index has surprisingly lower necessity and sufficiency scores compared to other indices and the baseline. This is probably because of its definition: the Holler-Packel index counts the number of causes a feature belongs to, ignoring their size. Due to this, features which have a high Holler-Packel index are in many cases, not present in any small causes. This results in a low necessity and sufficiency score, especially when k=1. The Shapley-Shubik, Banzhaf and Johnston indices have the same necessity and sufficiency scores for all values of k. We attribute this to their incredibly high similarity scores in Table 2. This again is unsurprising since all three indices aggregate quasi-minimal causes in a way that satisfies Quasi-Minimal Monotonicity (Section 4). To avoid repetition throughout our analysis, we refer to these three indices as quasi-minimal indices.

There are a few other interesting trends we observe. When k is low $(k \in \{1,2\})$, the responsibility index has the highest necessity score, but as k increases $(k \in \{3,5\})$, the quasi-minimal indices have the highest necessity and sufficiency. The Deegan-Packel index has a slightly lower score than the quasi-minimal indices, which in turn have a slightly lower score than the baseline. We explain these trends using the indices of specific data points from the Adult-Income dataset.

The reason the responsibility index has a higher necessity score than the quasi-minimal indices is the tie breaking scheme we use. Consider the data point in Table 4a predicted to have income > \$50,000 by our model. The index values for this data point are presented in Table 4a; for ease of exposition, we remove all features with a 0 weight and ignore the normalization factor of $\frac{1}{2^{n-1}}$ when computing our indices.

As can be seen from the responsibility index, the most important features have a score of 0.5. This

	k	= 1	k = 2		k = 3		k = 5	
Index	Necessity	Sufficiency	Necessity	Sufficiency	Necessity	Sufficiency	Necessity	Sufficiency
Responsibility	0.980	0	0.995	0	0.995	0.804	1.000	0.984
Holler-Packel	0.305	0	0.792	0	1.000	0.403	1.000	0.877
Deegan-Packel	0.879	0	0.987	0	1.000	0.858	1.000	1.000
Shapley-Shubik	0.980	0	0.989	0	1.000	0.860	1.000	1.000
Banzhaf	0.980	0	0.989	0	1.000	0.860	1.000	1.000
Johnston	0.980	0	0.989	0	1.000	0.860	1.000	1.000
Baseline	0.980	0	1.000	0	1.000	0.862	1.000	1.000

Table 3: The fraction of points where the top k features of the index were necessary and sufficient.

implies that some subset of size 2 is a necessary set, but the values themselves do not indicate which feature pairs constitute a cause. Age and Employment are the top two features of the responsibility index (using our tie breaking scheme), and they are turn out to be a cause. However, the top two features of the quasi-minimal indices (Age and Capital Gain), which are present in the largest number of causes, are not a cause. The reason that the responsibility index does better when k=2 is that our tie breaking scheme coincidentally picks two features which form a cause: the features Age and Employment turn out to be a cause in several instances. On the other hand, choosing the two features which appear in the largest number of causes does not form a cause in a few instances. The tie breaking scheme unfortunately does not aid the responsibility index all the time, which is why it falls short of the baseline when k=2 and k=3.

When $k \geq 3$, the quasi-minimal indices have a higher necessity and sufficiency score than the responsibility index. This can be explained using the same data point from Table 4a. When k=3, the quasi-minimal indices choose the three features present in the largest number of causes (Age, Education and Capital Gain). Looking at the Holler-Packel and responsibility indices for this data point, we can infer that this set of features must contain a cause. Thus, the necessity condition is satisfied when k=3; as can be seen from Table 3, the quasi-minimal indices satisfy this condition for every point that we compute them for. The responsibility index, on the other hand, relies on the tie breaking scheme to break ties in a favorable manner. This does not always work and results in a slightly lower necessity score. Even when it does work in favor of the responsibility index, this results in the top k features being sufficient far less often than the quasi-minimal indices.

The Deegan-Packel index has roughly the same ranking as the quasi-minimal indices in Table 4a. However, for a few points, the rankings of the Deegan-Packel and the quasi-minimal indices differ. Consider the datapoint presented in Table 4b; this datapoint was also classified positively by our model. While the Deegan-Packel and the quasi-minimal indices (weakly) agree on the rankings of the top two features, they disagree on the third with the quasi-minimal indices choosing Age and Deegan-Packel choosing Marital Status. Marital Status is present in three causes, each of size 2. On the other hand, Age is present in three causes, two of them have size 2 and the third has a size of 3. These attributes can be determined by observing the Deegan-Packel and the Holler-Packel indices for the data point. Intuitively, this should give Marital Status a higher score: it is present in the same number of causes but one of its causes is smaller than that of Age. However, the quasi-minimal indices do the opposite. This difference results in the top k features of the Deegan-Packel and the quasi-minimal indices being necessary and sufficient on a slightly different set of points, which then results in slightly different necessity and sufficiency scores on the Adult-Income dataset. This example also illustrates the unique benefit of the Deegan-Packel index which existing model explanation measures like the Shapley and the Banzhaf index do not capture.

While necessity and sufficiency are indeed important metrics when evaluating model explanations, we believe that the lack of necessity does not necessarily imply that the explanation is uninformative. In our empirical evaluation, we find that the Holler-Packel has very low necessity and sufficiency

	Age	Employment	Education	Marital Status	Occupation	Relationship	Capital Gain	Weekly Hours
Point	49	'Self'	16	'Married'	'Prof'	'Husband'	15024	50
RI	0.5	0.5	0.5	0.5	0.33	0.5	0.5	0.5
HPI	3	1	3	2	1	1	3	1
DPI	1.33	0.5	1.33	1	0.33	0.5	1.5	0.5
SS	0.210	0.076	0.176	0.126	0.009	0.093	0.243	0.066
В	336	112	272	176	16	144	368	96
J	229.33	56	169.33	104	5.33	72	268	48

(a) Data point showing the differences between the responsibility index and the quasi-minimal indices

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	Age	Employment	Education	Marital Status	Occupation	Relationship	Capital Gain	Weekly Hours
Point	35	'Private'	13	'Married'	'Sales'	'Husband'	7298	48
RI	0.5	0.5	0.5	0.5	0.33	0.5	0.5	0.5
HPI	3	1	4	3	1	1	3	1
DPI	1.33	0.5	1.83	1.5	0.33	0.5	1.5	0.5
SS	0.176	0.076	0.226	0.176	0.009	0.059	0.209	0.066
В	272	112	336	240	16	80	304	96
J	173.33	56	233.33	144	5.33	40	204	48

⁽b) Data point showing the differences between the Deegan-Packel index and the quasi-minimal indices

Table 4: Data points from the Adult-Income data set along with all 6 indices. We ignore the normalization factor of 2^{n-1} for simplicity and we omit all features with an index value of 0 due to space constraints.

scores. However, in the same evaluation, the Holler-Packel index provides useful information when analysing data points and explaining why certain sets of features are necessity and sufficient sets of the outcome. Similarly, even though the explanation measures we propose have their differences, we do not argue for the use of one index over the other. We instead believe that all our measures are provably good in their own way and capture different important aspects of the outcome.

7 Conclusions and Future Work

In this work, we study methods which aggregate causes into easily understandable importance weights. We propose six explanation measures and compare them using the axioms that characterize them. More broadly, our work proposes a new approach to compute importance weights using necessity and sufficiency in a theoretically sound manner.

Several promising challenges remain for future work. The main challenge, we believe, is the efficient computation of the value function (described in (1)) for complex models and datasets. We also believe algorithms which efficiently compute the set of minimal causes for more specific function classes, e.g. random forests and neural networks, is an interesting objective for future work. Lastly, while we restrict our attention to linear explanations, other forms of model explanations, e.g. rule based or tree based explanations which aggregate causes warrants further exploration.

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A Computing Causal Model Explanations

In this section, we discuss computational aspects of our explanation frameworks. As we show in Section A.1, our explanation frameworks are computationally intractable to compute exactly. This result is rather unsurprising, as similar results have been shown for other indices; however, for the sake of completeness, we resolve the computational intractability of the indices we discuss in the following section. In Section A.2 we consider how one may estimate the indices from samples. While some of our results are similar to existing works on the estimation of the Banzhaf and Shapley indices, the estimation guarantees for our indices are different. This is mainly due to the fact that our approximation schemes need to estimate the number of causes, rather than simply the marginal contribution of players to a randomly sampled set.

A.1 Computational Complexity

In what follows, we assume that there exists an efficient oracle that will output v(S) given any input set $S \subseteq N$. This may not always be the case since f is not guaranteed to be efficiently computable, nor is the computation of the max function in v. While oracle access to v is a limiting assumption, computing indices when v is not efficiently computable is trivially guaranteed to be computationally intractable.

One class of set functions that fall in this category are the characteristic functions of weighted voting games. These functions are parameterized by a non-negative weight vector \vec{w} which assigns a weight to every feature $i \in N$ and a threshold q. The value of a set is given by $v(S) = \mathbb{I}\{\sum_{i \in S} w_i \geq q\}$. This class of set functions is a subset of the class of set functions we consider. Given a weight vector \vec{w} and a threshold q, we can generate the characteristic function of this weighted voting game by setting a binary vector $\vec{x} = \vec{0}$, and $f = \mathbb{I}\{\vec{x} \cdot \vec{w} \geq q\}$ where (\cdot) denotes the dot product of two vectors. Note that v can be computed in polynomial time.

It is well known that these indices are computationally intractable for weighted voting games. This stems from the fact that deciding whether a feature is a Null Feature is coNP-complete [Chalkiadakis et al., 2011, Chapter 5].

Moreover, the Banzhaf and the Shapley indices have been proven to be #P-complete [Prasad and Kelly, 1990, Deng and Papadimitriou, 1994] when the characteristic function v corresponds to that of a weighted voting game. The complexity of the Holler-Packel, Deegan-Packel and Johnston indices with respect to the class #P remains unresolved. In this section, we first prove that the Holler-Packel index is #P-complete, and that the Deegan-Packel and Johnston indices are #P-hard. All of these results hold even when v is a weighted voting game.

Theorem A.1. The raw Holler-Packel index given by $\eta_i(v) = \frac{1}{2^{n-1}} |\mathcal{M}_i(v)|$ is #P-Complete even when v corresponds to a weighted voting game.

Proof. This proof is very similar to that of Deng and Papadimitriou [1994].

It is easy to see that $2^{n-1}\eta_i(v)$ is in #P since it is easy to verify whether a set is a minimal cause and $2^{n-1}\eta_i(v)$ counts the number of minimal causes that contains i.

To show hardness, we reduce this to the NP-complete problem PARTITION:

```
Given a set of positive integers A = \{a_1, a_2, \dots, a_m\}, does there exist a set S \subseteq A such that \sum_{a \in S} a = \frac{W}{2} where W = \sum_{a \in A} a?
```

The counting version of this problem counts the number of such sets S that exist. This problem is #P-complete. We reduce this problem to computing the Holler-Packel index of a weighted voting game.

Given an instance of PARTITION $A = \{a_1, a_2, \dots, a_m\}$, we construct the following weighted voting game with the number of players n = m+1 and threshold q = (W+1)/2 where $W = \sum_{a \in A} a$.

For each $i \in [m]$, we set $w_i = a_i$ and $w_{m+1} = 1$. We assume W is even since we can output NO trivially if W is odd.

From the definition of a quasi-minimal cause, for any subset $S \subseteq N \setminus \{m+1\}$, $S \cup \{m+1\} \in \mathcal{G}_{m+1}(v)$ if and only if $\sum_{i \in S \cup \{m+1\}} w_i \geq (W+1)/2$ and $\sum_{i \in S} w_i < (W+1)/2$. Since $w_{m+1} = 1$ and W is even, this is the same as saying that $\sum_{i \in S} w_i = W/2$. This means S is a solution to the original instance of PARTITION. Therefore, $|\mathcal{G}_{m+1}(v)|$ is the number of solutions to the original partition instance.

Furthermore, note that $\mathcal{G}_{m+1}(v) = \mathcal{M}_{m+1}(v)$ since player m+1 has weight 1 and if it is critical in a set, all the other players in the set must be critical as well. Therefore, the raw Holler-Packel index $\frac{1}{2^{n-1}}|\mathcal{M}_{m+1}(v)|$ is a constant factor times the number of solutions to the original PARTITION instance. This completes the reduction and proves the raw Holler-Packel index is #P-complete. \square

Theorem A.2. The raw Deegan-Packel index is #P-Hard even when v corresponds to a weighted voting game.

Proof. This proof is very similar to the proof of Theorem A.1. The only difference is that we use the version of PARTITION where all solutions are guaranteed to have the same cardinality k. Note that this problem is NP-complete as well and the counting version of this problem is #P-complete Deng and Papadimitriou [1994], Papadimitriou [1994].

Given an instance of PARTITION $\{a_1,a_2,\ldots,a_m\}$ and an integer k which corresponds to the size of all the solutions, we construct the following weighted voting game with the number of players n=m+1 and threshold q=(W+1)/2 where $W=\sum_{i\in[m]}a_i$. For each $i\in[m]$, $w_i=a_i$ and $w_{m+1}=1$. We further assume that W is even since we can trivially output NO is W is odd.

Following a similar argument to the proof of Theorem A.1, we have $|\mathcal{G}_{m+1}(v)|$ is the number of solutions to the original partition instance and $\mathcal{G}_{m+1}(v) = \mathcal{M}_{m+1}(v)$.

solutions to the original partition instance and $\mathcal{G}_{m+1}(v) = \mathcal{M}_{m+1}(v)$. Therefore, the raw Deegan-Packel index $\frac{1}{2^{n-1}(k+1)}|\mathcal{M}_{m+1}(v)|$ is precisely $\frac{1}{2^{n-1}(k+1)}$ times the number of solutions to the original PARTITION instance. This completes the reduction proves that the raw Deegan-Packel index is #P-Hard.

Theorem A.3. The raw Johnston index is #P-Hard even when v corresponds to a weighted voting game.

Proof. This proof is very similar to the proof of Theorem A.2. For our reduction, we use the version of PARTITION where all the solutions are guaranteed to have the same cardinality k. This problem is NP-complete as well and the counting version of this problem is #P-complete Deng and Papadimitriou [1994], Papadimitriou [1994].

Given an instance of PARTITION $\{a_1, a_2, \ldots, a_m\}$ and an integer k which corresponds to the size of all the solutions, we construct the following weighted voting game with the number of players n=m+1 and threshold q=(W+1)/2 where $W=\sum_{i\in[m]}a_i$. For each $i\in[m]$, $w_i=a_i$ and $w_{m+1}=1$. We further assume that W is even since we can trivially output NO is W is odd.

Following a similar argument to the proof of Theorem A.1, we have $|\mathcal{G}_{m+1}(v)|$ is the number of solutions to the original partition instance and $\mathcal{G}_{m+1}(v) = \mathcal{M}_{m+1}(v)$.

The raw Johnston index for the feature m+1 is $\frac{1}{2^{n-1}(k+1)}|\mathcal{G}_{m+1}(v)|$ since each set in $\mathcal{G}_{m+1}(v)$ has size k+1 and every element in $\mathcal{G}_{m+1}(v)$ is critical $(\mathcal{G}_{m+1}(v) = \mathcal{M}_{m+1}(v))$. This value is precisely $\frac{1}{2^{n-1}(k+1)}$ times the number of solutions to the original PARTITION instance. This completes the reduction proves that the raw Johnston index is #P-Hard.

The complexity of causality has also been analyzed in Eiter and Lukasiewicz [2002] who show that deciding whether a feature i is an actual cause of $y = f(\vec{x})$ is NP-Complete. This is equivalent to saying that deciding whether a feature is not a Null Feature is coNP-complete.

A.2 Estimating Indices

These complexity results show that computing the actual index values is challenging in general. Instead, we try to approximate them. Apart from the Shapley and Banzhaf index Bachrach et al. [2008], there has not been much work discussing the approximate computation of the above indices. Matsui and Matsui [2000] and Uno [2012] propose dynamic programming algorithms to compute the Banzhaf and Deegan-Packel index when the characteristic function corresponds to that of a weighted voting game. However, as discussed in the previous section, the set function we use is much more general, rendering this approach ineffective.

Since the function we use is much more complex, we use sampling based approximations to compute our indices. Bachrach et al. [2008] provide ways to compute approximate values for the Shapley and Banzhaf indices via sampling. We extend these results to the Johnston, Deegan-Packel and Holler Packel index as well.

The high level intuition is as follows: if we sample sets from the uniform distribution over the power set of N and compute the index value for only these sets, we obtain an unbiased estimate of the index value we want to compute. We can then use the Hoeffding bound to bound the error of our estimate. The exact descriptions of the algorithm are given in Algorithms 1, 2 and 3.

It is also worth noting that this method of approximation still ensures that the approximate indices output by the algorithm are somewhat consistent with the axioms of the index being approximated. The indices we discuss are characterized by a desirable monotonicity property along with symmetry and null feature. Our estimation algorithms guarantee that these three axioms are satisfied with respect to the other features irrespective of the random samples used to compute the estimate. More specifically, we define relative properties of these indices for an index γ :

- (RQM) Relative Quasi-minimal Monotonicity: If we have two features i and j such that $\mathcal{G}_i(v) \subseteq \mathcal{G}_j(v)$, then $\gamma_i \leq \gamma_j$.
- (RMM) Relative Minimal Monotonicity: If we have two features i and j such that $\mathcal{M}_i(v) \subseteq \mathcal{M}_j(v)$, then $\gamma_i \leq \gamma_j$.
- (RS) Relative Symmetry: If we have two features i and j such that $\mathcal{M}_i(v) = \mathcal{M}_j(v)$, then $\gamma_i = \gamma_j$

Note that these properties are weaker than their non-relative counterpart. However, we show that our estimation algorithms are guaranteed to satisfy these properties whenever they apply. This ensures that the algorithms are robust and provide useful aggregations even when the samples are "bad".

More formally, we have the following result for the Johnston index estimation algorithm.

Proposition A.4. The output of Algorithm 1 satisfies (RQM), (RS) and (NF)

Proof. Assume we have two features i and j such that $\mathcal{G}_i(v) \subseteq \mathcal{G}_j(v)$. Let $\mathcal{S} \subseteq \mathcal{G}_i(v)$ be the set samples used to compute the index value for i ($\hat{\gamma}_i$). Since $\mathcal{G}_i(v) \subseteq \mathcal{G}_j(v)$, the set of samples \mathcal{S} will be used to compute $\hat{\gamma}_j$ as well. Therefore, we have

$$\hat{\gamma}_j \ge \sum_{S \in \mathcal{S}} \frac{2}{\chi(S)} = \hat{\gamma}_i \tag{3}$$

Therefore (RQM) is satisfied.

Note that if $\mathcal{M}_i(v) = \mathcal{M}_j(v)$, then $\mathcal{G}_i(v) = \mathcal{G}_j(v)$. If we have two features i and j such that $\mathcal{M}_i(v) = \mathcal{M}_j(v)$, then the same set of samples $\mathcal{S} \subseteq \mathcal{G}_i(v)$ used to compute $\hat{\gamma}_i$ will be used to compute $\hat{\gamma}_j$ as well. The only difference between this case and the previous case is that no sample outside \mathcal{S} will be used to compute $\hat{\gamma}_j$ since $\mathcal{G}_i(v) = \mathcal{G}_j(v)$. If such a sample existed, it would be used to

compute $\hat{\gamma}_i$ and we would have a contradiction. Therefore, the inequality in (3) becomes an equality and (RS) is satisfied.

When $\mathcal{M}_i(v) = \emptyset$, the $\mathcal{G}_i(v) = \emptyset$ as well. Since the index is computed using samples which are a subset of $\mathcal{G}_i(v)$, there will be no sample used to compute $\hat{\gamma}_i$ and the index does not change from its default value of 0. Therefore, (NF) is satisfied as well.

We prove a similar result for the Deegan-Packel and Holler-Packel estimation algorithms.

Proposition A.5. The output of Algorithms 2 and 3 satisfy (RMM), (RS) and (NF)

These results are mainly due to a subtle difference between our approach and that of Bachrach et al. [2008]. While Bachrach et al. [2008] use a different set of random samples for each feature, we use the same set of random samples for all features. This allows us to ensure symmetry and minimal monotonicity (or quasi-minimal monotonicity).

Algorithm 1 Sampling Algorithm for the Johnston Index

```
\begin{split} \hat{\gamma}_i &= 0 \quad \forall i \in N \\ \textbf{for } j \textbf{ in 1 to } m \textbf{ do} \\ &\quad \text{Sample } S_j \text{ uniformly from } 2^N \\ &\quad \hat{\gamma}_i = \hat{\gamma}_i + 2\mathbb{I}\{S_j \in \mathcal{G}_i(v)\}\frac{1}{|\chi(S_j)|} \quad \forall i \in N \\ \textbf{end for} \\ &\quad \hat{\gamma}_i = \hat{\gamma}_i/m \quad \forall i \in N \\ \textbf{Return } \hat{\gamma} \end{split}
```

We now turn to concentration bounds for our algorithms.

Theorem A.6. The output of Algorithm 1, $\hat{\gamma}_i$ is an unbiased estimate of the raw Johnston index γ_i for all i in N. Furthermore, for all $m \geq \frac{2}{\epsilon^2} \log \frac{2n}{\delta}$, we have with probability $1 - \delta$, for all $i \in N$,

$$|\hat{\gamma}_i - \gamma_i| \le \epsilon$$

Proof. All we need to do for the first part is that when a set S is sampled from the uniform distribution over 2^N , then $\mathbb{E}[2\mathbb{I}\{S \in \mathcal{G}_i(v)\}\frac{1}{|\chi(S)|}] = \gamma_i$. It then follows from the linearity of expectation that $\mathbb{E}[\hat{\gamma}_i] = \gamma_i$. We have

$$\mathbb{E}[2\mathbb{I}\{S \in \mathcal{G}_i(v)\}\frac{1}{|\chi(S)|}] = \sum_{T \subseteq N} \frac{1}{2^n} 2\mathbb{I}\{T \in \mathcal{G}_i(v)\}\frac{1}{|\chi(T)|}$$
$$= \frac{1}{2^{n-1}} \sum_{T \in \mathcal{G}_i(v)} \frac{1}{|\chi(T)|}$$
$$= \gamma_i$$

The second statement follows from the Hoeffding Inequality. From the hoeffding inequality, we have for all $i \in N$, $\Pr[|\hat{\gamma}_i - \gamma_i| \ge \epsilon] \le 2e^{-\frac{m\epsilon^2}{2}}$. When $m \ge \frac{2}{\epsilon^2}\log\frac{2}{\delta'}$ for some $\delta' \in [0,1]$, we have $\Pr[|\hat{\gamma}_i - \gamma_i| \ge \epsilon] \le \delta'$. Using the union bound and setting $\delta' = \delta/n$, we get the required bound. \square

We can prove similar results for the other two algorithms as well.

Theorem A.7. The output of Algorithm 2, $\hat{\phi}_i$ is an unbiased estimate of the raw Deegan-Packel index ϕ_i for all i in N. Furthermore, for all $m \geq \frac{2}{\epsilon^2} \log \frac{2n}{\delta}$, we have with probability $1 - \delta$, for all $i \in N$,

$$|\hat{\phi}_i - \phi_i| \le \epsilon$$

Algorithm 2 Sampling Algorithm for the Deegan-Packel Index

Theorem A.8. The output of Algorithm 3, $\hat{\eta}_i$ is an unbiased estimate of the raw Holler-Packel index η_i for all i in N. Furthermore, for all $m \geq \frac{2}{\epsilon^2} \log \frac{2n}{\delta}$, we have with probability $1 - \delta$, for all $i \in N$,

$$|\hat{\eta}_i - \eta_i| \le \epsilon$$

Algorithm 3 Sampling Algorithm for the Holler-Packel Index

```
\begin{array}{ll} \hat{\eta}_i = 0 & \forall i \in N \\ \textbf{for } j \textbf{ in } 1 \textbf{ to } m \textbf{ do} \\ & \text{Sample } S_j \text{ uniformly from } 2^N \\ & \hat{\eta}_i = \hat{\eta}_i + 2\mathbb{I}\{S_j \in \mathcal{M}_i(v)\} & \forall i \in N \\ \textbf{end for} \\ & \hat{\eta}_i = \hat{\eta}_i/m & \forall i \in N \\ \textbf{Return } \hat{\eta} \end{array}
```

The responsibility index cannot be computed using this approach because we cannot compute an unbiased estimate for it. Instead, we use a PAC style approach to estimate the responsibility index, described in Algorithm 4. This algorithm samples subsets uniformly from the power set of N; for each feature i, we check whether the set is a quasi-minimal cause where i is critical. The algorithm then compares all the different quasi-minimal causes from the samples and estimates the responsibility index of a particular feature as the inverse of the size of the smallest quasi-minimal cause observed. When the set of samples is the power set 2^N , the algorithm outputs the true responsibility index value for each feature. However, when the samples are random and polynomial in number, this algorithm has a simple theoretical guarantee: with $\mathcal{O}(\frac{1}{\epsilon}(\log(\frac{1}{\epsilon}) + \log(\frac{n}{\delta})))$ samples, the fraction of subsets S of N, which are both smaller than our estimate and causes where i is critical is upper bounded by ϵ with probability at least $1-\delta$. This essentially says that, with high probability, there is a very small number of minimal causes that are strictly smaller in size than the set Algorithm 4 uses to compute the responsibility index. More formally, we have the following theorem.

Theorem A.9. For all the indices $\rho_i(v)$ output by Algorithm 4, we have with $\mathcal{O}(\frac{1}{\epsilon}(\log(\frac{1}{\epsilon}) + \log(\frac{n}{\delta})))$ samples, $\Pr\left[\exists i \in N, \Pr_{S \in \mathcal{D}}\left[\frac{\mathbb{I}\{S \in \mathcal{G}_i(v)\}}{|S|} > \hat{\rho}_i(v)\right] > \epsilon\right] < \delta$ where \mathcal{D} is the uniform distribution over the power set of N.

Proof. Let $w_i(S) = \frac{\mathbb{I}\{S \in \mathcal{G}_i(v)\}}{|S|}$ for any set S and feature i. It is easy to see that given a set of samples, $\{S_1, S_2, \ldots, S_m\}$, Algorithm 4 sets $\hat{\rho}_i(v) = \max_{j \in [m]} w_i(S_j)$ for all $i \in N$. Jha and Zick [2020] show that the max function is learnable in $\mathcal{O}(\frac{1}{\epsilon}(\log(\frac{1}{\epsilon}) + \log(\frac{1}{\delta'})))$ samples and provides the guarantee $\Pr[\Pr_{S \in \mathcal{D}}[w_i(S) > \hat{\rho}_i(v)] > \epsilon] < \delta'$.

```
Taking the union bound for all i \in N with \delta' = \delta/n, we get \Pr[\exists i \in N, \Pr_{S \in \mathcal{D}}[\frac{\mathbb{I}\{S \in \mathcal{G}_i(v)\}}{|S|} > \rho_i(v)] > \epsilon] < \delta when m \geq \mathcal{O}(\frac{1}{\epsilon}(\log(\frac{1}{\epsilon}) + \log(\frac{n}{\delta}))).
```

Unlike the earlier algorithms, these estimates are not guaranteed to satisfy a relative version of Minimum Size Monotonicity but are guaranteed to satisfy the weaker notion of Relative Minimal Monotonicity along with Relative Symmetry and Null Feature.

Proposition A.10. The output of Algorithm 4 is guaranteed to satisfy, (RMM), (RS) and (NF)

Algorithm 4 Sampling Algorithm for the Responsibility Index

```
\begin{array}{ll} \hat{\rho}_i = 0 & \forall i \in N \\ \textbf{for } j \textbf{ in 1 to } m \textbf{ do} \\ & \text{Sample } S_j \text{ uniformly from } 2^N \\ & \hat{\rho}_i = \max\{\hat{\rho}_i, \frac{\mathbb{I}\{S_j \in \mathcal{G}_i(v)\}}{|S_j|}\} & \forall i \in N \\ \textbf{end for} \\ \textbf{Return } \hat{\rho} \end{array}
```

B Missing Proofs from Section 2

Proposition 2.3. The function v as given by (1) is monotone in S for any fixed point of interest \vec{x} and model of interest f.

Proof. Assume without loss of generality that $f(\vec{x}) = 0$. Let T be a set such that v(T) = 1. We need to show that for any set S such that $T \subseteq S$, f(S) = 1.

Let \vec{x}_T' be the best possible intervention on the set T such that $f(\vec{x}_T', \vec{x}_{N \setminus T}) = 1$. Note that such a point must exist by the definition of v. We construct an intervention on the set S, $\vec{x}_S'' = (\vec{x}_T', \vec{x}_{S \setminus T})$. We have

$$1 \ge v(S) \ge |f(\vec{x}_{N \setminus S}, \vec{x}_S'') - f(\vec{x})| = |f(\vec{x}_{N \setminus S}, \vec{x}_T', \vec{x}_{S \setminus T}) - f(\vec{x})| = |f(\vec{x}_T', \vec{x}_{N \setminus T}) - f(\vec{x})| = v(T) = 1$$

This completes the proof.

Proposition 2.4. The set of all but-for causes of an outcome $f(\vec{x})$ is a necessary and sufficient condition of $f(\vec{x})$.

Proof. The set of all but-for causes is trivially necessary by the definition of a but-for cause so we only prove sufficiency here.

Assume for contradiction there exists a set of features S not present in any but-for cause such that there exists an assignment \vec{x}_S' to the set of features in S that changes the outcome. If this was the case, the assignment $S = \vec{x}_S$ would be a but-for cause (or a superset of one) contradicting our original assumption. Therefore, the set of all but-for causes is a sufficient condition for the outcome as well.

C Missing Proofs from Section 3

Theorem 3.1. For any monotone binary set function v, the only index which satisfies (MSM), (UE), (S), (NF) and (C) is the responsibility index $\rho(v)$.

Proof. It is easy to see that the responsibility index satisfies (MSM), (UE), (NF) and (S). We show using the following Lemma that the responsibility index satisfies (C).

Lemma C.1. For any monotone binary set function v, the responsibility index $\rho(v)$ satisfies Contraction.

Proof. For any set T which does not contain a Null Feature, the responsibility index $\rho_{[T]}(v_{[T]})$ is non-zero and corresponds to the inverse of the size of some set $S_T \in \mathcal{M}_{[T]}(v_{[T]})$. This implies that there must be some set S in $\mathcal{M}(v)$ which contains some non-empty subset $T' \subseteq T$ such that $S_T \setminus [T] = S \setminus T'$. This is obtained from the definition of a contraction. From the definition of responsibility index, we have $\rho_{[T]}(v_{[T]}) = \frac{1}{k-|T'|+1}$ where k = |S|.

We first show that the total responsibility index of the elements in T' under v is weakly greater than the responsibility of [T] under $v_{[T]}$.

Let k be the size of S. Then, for all feasible |T'| and k, we have since $|T'| \in [1, k]$, the following inequality:

$$|T'|^{2} - (k+1)|T'| + k \leq 0$$

$$\Longrightarrow \frac{1}{k - |T'| + 1} \leq \frac{|T'|}{k}$$

$$\Longrightarrow \rho_{[T]}(v_{[T]}) \leq \sum_{i \in T'} \rho_{i}(v)$$
(4)

where (4) is true since the existence of S gives us a lower bound of $\frac{1}{k}$ on the responsibility indices of all the elements in T'. Since the responsibility index is always non-negative, from (4), we have

$$\rho_{[T]}(v_{[T]}) \le \sum_{i \in T} \rho_i(v)$$

which is the first part of the Contraction property.

To show the second part, assume that $T' = T \in \mathcal{M}_i(v)$ where none of the elements in T are present in a smaller minimal cause. They all have a responsibility of 1/|T|. We have $\rho_{[T]}(v_{[T]}) = 1$ since the set $[T] \in M_{[T]}(v_{[T]})$. It is easy to see that this satisfies the equality condition in (C).

We now show uniqueness via induction on the size of $|\mathcal{M}(v)|$. Let an arbitrary index which satisfies the above properties be denoted by $\gamma(v)$. When $|\mathcal{M}(v)| = 1$, let $\mathcal{M}(v) = \{T\}$. If $T = \{i\}$ for some $i \in N$, then by Unit Efficiency, $\gamma_i(v) = 1$ and by Null Feature, $\gamma_{i'}(v) = 0$ for $i' \neq i$. This coincides with the responsibility index.

When $|T| \geq 2$, using Contraction, we get $\gamma_{[T]}(v_{[T]}) = \sum_{i \in V} \gamma_i(v)$. Note that equality holds since T is a minimal cause and the smallest minimal cause for all the elements in T. Using Unit Efficiency, we get $\gamma_{[T]}(v_{[T]}) = 1$. Using symmetry, $\gamma_i(v) = \gamma_j(v)$ for all $i, j \in T$. Therefore $\gamma_i(v) = \frac{1}{|T|}$ for all $i \in T$. For all $i \in N \setminus T$, $\gamma_i(v) = 0$ because of the Null Feature property. This coincides with the responsibility index $\rho(v)$ for all $i \in N$.

Now assume $|\mathcal{M}(v)| = m$ i.e. $\mathcal{M}(v) = \{S_1, S_2, \dots S_m\}$ for some sets S_1, S_2, \dots, S_m . Let S be the set of of features which are present in at least one minimal cause. For any $i \in N$, let S_i be the smallest minimal cause that i is in (if there are multiple, we choose one arbitrarily). Define $\mathcal{M}(v') = S_i$ i.e. the cooperative game that v' induces is the unanimity game on S_i . By Minimum Size Monotonicity, $\gamma_i(v) = \gamma_i(v')$. Note that equality holds since the smallest minimal causes that contain i have the same size in both v and v'. By the inductive hypothesis, $\gamma_i(v')$ corresponds to the responsibility index for i under v'. Therefore $\gamma_i(v) = \gamma_i(v') = 1/|S_i|$ for all $i \in S$. This coincides with the degree of responsibility, since S_i is the smallest minimal cause that contains i.

For all $i \notin S$, we have $\gamma_i(v) = 0$ because of Null Feature and this coincides with the responsibility index as well.

Theorem 3.2. For any monotone binary set function, the only index satisfying (MM), (TMCE), (S) and (NF) is the raw Holler-Packel Index $\eta(v)$.

Proof. It is easy to see that the Holler Packel Index satisfies these properties so we go ahead and show uniqueness.

Let us denote the index that satisfies these properties by γ . We show uniqueness via induction on $|\mathcal{M}(v)|$. When $|\mathcal{M}(v)| = 1$, let the only minimal set be T. For all i in T, by Symmetry and Total Minimal Cause Efficiency, we get $\gamma_i(v) = \frac{1}{2^{n-1}}$. For all $i \in N \setminus T$, by Null Feature, we get $\gamma_i(v) = 0$. This coincides with the Holler Packel Index.

Now assume $|\mathcal{M}(v)| = m$ i.e. $\mathcal{M}(v) = \{S_1, S_2, \dots S_m\}$ for some sets S_1, S_2, \dots, S_m . Let $S = \bigcap_{j \in [m]} S_j$ be the set of of features where are present in all minimal causes. For any $i \notin S$, let $\mathcal{M}(v') = \mathcal{M}_i(v)$. Since $|\mathcal{M}(v')| < |\mathcal{M}(v)|$, we can apply the inductive hypothesis and $\gamma_i(v')$ coincides with the Holler-Packel index. Therefore $\gamma_i(v') = |\mathcal{M}_i(v')|$. Using Monotonicity, we get that $\gamma_i(v) = \gamma_i(v') = |\mathcal{M}_i(v')|$ as well. Equality holds since $\mathcal{M}_i(v') = \mathcal{M}_i(v)$. Therefore, we get $\gamma_i(v) = |\mathcal{M}_i(v)|$ which coincides with the Holler Packel index.

For all $i \in S$, using Symmetry, they all have the same value and using Total Minimal Cause Efficiency, this value is unique and since all the other features coincide with the Holler Packel index and the Holler Packel index satisfies these axioms, it must be the case that $\gamma_i(v)$ coincides with the Holler Packel index as well for all $i \in S$. This completes the proof.

Proposition 3.4. There exists no index which satisfies (MM), (E), (S) and (NF)

Proof. Consider a setting with 4 features $\{1,2,3,4\}$. Assume for contradiction that there exists an index $\gamma(v)$ that satisfies these properties. Consider a function v with $\mathcal{M}(v) = \{\{1,2\}\}$. Using Efficiency and Symmetry, we have that $\gamma_i(v) = 1/2$ for all $i \in \{1,2\}$. Now consider a v' with $\mathcal{M}(v') = \{\{1,2\},\{3,4\}\}$. Then from minimal monotonicity, we have $\gamma_i(v') = 1/2$ for all $i \in \{1,2\}$. Similarly, $\gamma_j(v') = 1/2$ for all $j \in \{3,4\}$. However, this clearly violates efficiency since $\sum_{i \in N} \gamma_i(v) = 2 \neq 1$. This is clearly a contradiction and therefore, such an index cannot exist.

D Missing Proofs from Section 4

Theorem 4.4. For any set of quasi minimal causes \mathcal{G} and any feasble mapping χ , the only index which satisfies (AQM), (AQCE), (AS) and (ANF) is the alternate Johnston Index $\omega(\mathcal{G}, \chi)$.

Proof. It is easy to see that the alternate Johnston Index satisfies these conditions, so proceed to show uniqueness.

Let us denote the index that satisfies these properties by γ . We show uniqueness via induction on $|\mathcal{G}|$. When $|\mathcal{G}| = 1$, let the only set in \mathcal{G} be T. For all $i \in N \setminus \chi(T)$, by Alternate Null Feature, we get $\gamma_i(\mathcal{G},\chi) = 0$. For all i in $\chi(T)$, by Alternate Symmetry and Alternate Quasi Minimal Cause Efficiency, we get $\gamma_i(\mathcal{G},\chi) = \frac{1}{2^{n-1}} \frac{1}{|\chi(T)|}$. This coincides with the Alternate Johnston Index. Now assume $|\mathcal{G}| = m$ i.e. $\mathcal{G} = \{S_1, S_2, \dots, S_m\}$ for some sets S_1, S_2, \dots, S_m . Let $S = \bigcap_{j \in [m]} \chi(S_j)$

Now assume $|\mathcal{G}| = m$ i.e. $\mathcal{G} = \{S_1, S_2, \dots S_m\}$ for some sets S_1, S_2, \dots, S_m . Let $S = \bigcap_{j \in [m]} \chi(S_j)$ be the set of of features which are critical in all the sets. For any $i \notin S$, let $\mathcal{H} = \mathcal{G}_i$ and χ_H be defined as follows:

$$\chi_H(S) = \begin{cases} \chi(S) & S \in G_i \\ \emptyset & \text{ow} \end{cases}$$

Since $|\mathcal{H}| < |\mathcal{G}|$, we can apply the inductive hypothesis and $\gamma_i(\mathcal{H}, \chi_H)$ coincides with the Alternate Johnston Index. Therefore $\gamma_i(\mathcal{H}, \chi_H) = \omega_i(\mathcal{H}, \chi_H)$. Using Quasi Minimal Monotonicity, we get that $\gamma_i(\mathcal{G}, \chi) = \gamma_i(\mathcal{H}, \chi_H) = \sum_{S \in \mathcal{G}_i} \frac{1}{|\chi(S)|}$ as well. Equality holds since $\mathcal{H}_i = \mathcal{G}_i$. Therefore, we get $\gamma_i(\mathcal{G}, \chi) = \omega_i(\mathcal{G}, \chi)$.

For all $i \in S$, using Symmetry, they all have the same value and using Alternate Quasi Minimal Cause Efficiency, this value is unique and since all the other features coincide with the Alternate Johnston Index and the Alternate Johnston index satisfies these axioms, it must be the case that

 $\gamma_i(\mathcal{G},\chi)$ coincides with the Alternate Johnston index as well for all $i \in S$. This completes the proof.

E Additional Experiment Details

In this section, we describe additional technical details important for the reproduction of our experiment.

E.1 Computing and Randomization Details

All our random computation uses the seed (or random_state) 0 and all the computation was done on a 2015 Macbook Pro with an Intel i5 processor and 8GB RAM.

E.2 Dataset Processing and the Model

We delete the features 'fnlwgt' and 'native-country' and replace 'education' with the simpler 'education_num'. We train a decision tree (using scikit-learn [Pedregosa et al., 2011]) on the one hot encoded feature set with a depth of 5 using data from the training dataset.

E.3 Value Function Computation

We maintain intervention values for each feature for each outcome. Given a set of features S, we test all combinations of intervention values for features in S to determine the value of the set S i.e. v(S). These values are presented in Table 5. They were chosen after careful examination of the decision tree with the main goal being to choose values which are capable of changing the output.

For example, if the set $S = \{Age, Employment\}$ and the outcome is 0, we evaluate the effect of the following interventions

We then output v(S) = 1 if any of these interventions change the outcome.

Feature	Intervention values when $f(\vec{x}) = 0$	Intervention values when $f(\vec{x}) = 1$
Age	$\{20, 30, 85\}$	$\{20, 30, 85\}$
Employment	'Private'	'Self-emp-not-inc'
Education_num	$\{1, 16\}$	$\{1, 16\}$
Marital Status	{'Never-Married', 'Married-civ-spouse'}	{'Never-Married', 'Married-civ-spouse'}
Occupation	'Exec-Managerial'	'Other-Service'
Relationship	'Husband'	'Other-relative'
Race	'White'	'White'
Sex	'Female'	'Female'
Capital Gain	$\{0, 99999\}$	$\{0, 99999\}$
Capital Loss	$\{1000, 2500\}$	$\{1000, 4536\}$
Hours Worked	99	1

Table 5: Intervention values for features in the Adult Dataset