# Near Neighbor: Who is the Fairest of Them All?

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#### **Abstract**

In this work we study a fair variant of the near neighbor problem. Namely, given a set of n points P and a parameter r, the goal is to preprocess the points, such that given a query point q, any point in the r-neighborhood of the query, i.e.,  $\mathbb{B}(q,r)$ , have the same probability of being reported as the near neighbor.

We show that LSH based algorithms can be made fair, without a significant loss in efficiency. Specifically, we show an algorithm that reports a point in the r-neighborhood of a query q with almost uniform probability. The query time is proportional to  $O(\operatorname{dns}(q,r)\mathcal{Q}(n,c))$ , and its space is  $O(\mathcal{S}(n,c))$ , where  $\mathcal{Q}(n,c)$  and  $\mathcal{S}(n,c)$  are the query time and space of an LSH algorithm for c-approximate near neighbor, and  $\operatorname{dns}(q,r)$  is a function of the local density around q.

Our approach works more generally for sampling uniformly from a sub-collection of sets of a given collection and can be used in a few other applications. Finally, we run experiments to show performance of our approach on real data.

## 1. Introduction

Nowadays, many important decisions, such as college admissions, offering home loans, or estimating the likelihood of recidivism, rely on machine learning

algorithms. There is a growing concern about the fairness of the algorithms and creating bias toward a specific population or feature [HPS16, Cho17, MSP16, KLL<sup>+</sup>17]. While algorithms are not inherently biased, nevertheless, they may amplify the already existing biases in the data. Hence, this concern has led to the design of fair algorithms for many different applications, e.g., [DOBD<sup>+</sup>18, ABD<sup>+</sup>18, PRW<sup>+</sup>17, CKLV19, EJJ<sup>+</sup>19, OA18, CKLV17, BIO<sup>+</sup>19, BCN19, KSAM19].

Bias in the data used for training machine learning algorithms is a monumental challenge in creating fair algorithms [HGB<sup>+</sup>07, TE11, ZVGRG17, Cho17]. Here, we are interested in a somewhat different problem, of handling the bias introduced by the data-structures used by such algorithms. Specifically, data-structures may introduce bias in the data stored in them, and the way they answer queries, because of the way the data is stored and how it is being accessed. Such a defect leads to selection bias by the algorithms using such data-structures.

It is natural to want data-structures that do not introduce a selection bias into the data when handling queries.

The target as such is to derive data-structures that are bias-neutral. To this end, imagine a data-structure that can return, as an answer to a query, an item out of a set of acceptable answers. The purpose is then to return uniformly a random item out of the set of acceptable outcomes, without explicitly computing the whole set of acceptable answers (which might be prohibitively expensive).

Several notions of fairness have been studied, including *group fairness*<sup>1</sup> (where demographics of the population is preserved in the outcome) and *individual fairness* (where the goal is to treat individuals with similar conditions similarly) [DHP<sup>+</sup>12]. In this work, we study the near neighbor problem from the perspective of individual fairness.

Near Neighbor is a fundamental problem that has applications in many areas such as machine learning, databases, computer vision, information retrieval, and many others, see [SDI06, AI08] for an overview. The problem is formally defined as follows. Let  $(\mathcal{M}, d)$  be a metric space. Given a set  $P \subseteq \mathcal{M}$  of n points and a parameter r, the goal of the near neighbor problem is to preprocess P, such that for a query point  $q \in \mathcal{M}$ , one can report a point  $p \in P$ , such that  $d(p,q) \leq r$  if such a point exists. As all the existing algorithms for the exact variant of the problem have either space or query time that depends exponentially on the ambient dimension of  $\mathcal{M}$ , people have considered the approximate variant of the problem. In the c-approximate near neighbor (ANN) problem, the algorithm is allowed to report a point p whose distance to the query is at most cr if a point within distance r of the query exists, for some prespecified constant c > 1.

Perhaps the most prominent approach to get an ANN data structure is via Locality Sensitive Hashing (LSH) [IM98, HIM12], which leads to sub-linear query time and sub-quadratic space. In particular, for  $\mathcal{M} = \mathbb{R}^d$ , by using LSH one can get a query time of  $n^{\rho+o(1)}$  and space  $n^{1+\rho+o(1)}$  where for the  $L_1$  distance metric  $\rho = 1/c$  [IM98, HIM12], and for the  $L_2$  distance metric  $\rho = 1/c^2 + o_c(1)$  [AI08]. The idea of the LSH method is to hash all the points using several hash functions that are chosen randomly, with the property that closer points have a higher probability of collision than the far points. Therefore, the closer points to a query have a higher probability of falling into a bucket being probed than far points. Thus, reporting a random point from a random bucket computed for the query, produces a distribution that is biased by the distance to the query: closer points to the query have a higher probability of being chosen.

When random nearby is better than nearest. The bias mentioned above towards nearer points is usually a good property, but is not always desirable. Indeed, consider the following scenarios:

(I) The nearest neighbor might not be the best if the input is noisy, and the closest point might be viewed as an unrepresentative outlier. Any point in the neighborhood might be then considered to be equivalently beneficial. This is to some extent why k-NN classification [ELL09] is so effective in reducing the effect of noise.

<sup>&</sup>lt;sup>1</sup>The concept is denoted as statistical fairness too, e.g., [Cho17].

- (II) However, k-NN works better in many cases if k is large, but computing the k nearest-neighbors is quite expensive if k is large [HAAA14]. Computing quickly a random nearby neighbor can significantly speed-up such classification.
- (III) We are interested in annonymizing the query [Ada07], thus returning a random nearneighbor might serve as first line of defense in trying to make it harder to recover the query. Similarly, one might want to anonymize the nearest-neighbor [QA08], for applications were we are interested in a "typical" data item close to the query, without identifying the nearest item.
- (IV) If one wants to estimate the number of items with a desired property within the neighborhood, then the easiest way to do it is via uniform random sampling from the neighborhood. In particular, this is useful for density estimation [KLK12].
- (V) Another natural application is simulating a random walk in the graph where two items are connected if they are in distance at most r from each other. Such random walks are used by some graph clustering algorithms [HK01].

#### 1.1. Results

Our goal is to solve the near-neighbor problem, and yet be fair among "all the points" in the neighborhood. We introduce and study the **fair near neighbor** problem – where the goal is to report any point of N(q,r) with uniform distribution. That is, report a point within distance r of the query point with probability of  $\mathbb{P}(q,r) = 1/n(q,r)$ , where n(q,r) = |N(q,r)|. Naturally, we study the approximate fair near neighbor problem, where one can hope to get efficient data-structures. We have the following results:

- (I) **Exact neighborhood.** We present a data structure with space S(n,c), and query time  $\widetilde{O}(Q(n,c) \cdot \frac{n(q,cr)}{n(q,r)})$ , where S(n,c) and Q(n,c) are, respectively, the space and query time of the standard c-ANN data structure. Note that, the query time of the algorithm might be high if the approximate neighborhood of the query is much larger than the exact neighborhood. Guarantees of this data structure hold with high probability. See Lemma 4.9 for the exact statement.
- (II) **Approximate neighborhood.** This formulation reports an almost uniform distribution from an approximate neighborhood S of the query. We can provide such a data structure that uses space S(n,c) and whose query time is  $\widetilde{O}(\mathcal{Q}(n,c))$ , albeit in expectation. See Lemma 4.3 for the exact statement.

Moreover, the algorithm produces the samples independently of past queries. In particular, one can assume that an adversary is producing the set of queries and has full knowledge of the data structure. Even then the generated samples have the same (almost) uniform guarantees.

**Experiments.** Finally, we compare the performance of our algorithm with the algorithm that uniformly picks a bucket and reports a random point, on the MNIST data set. Our

As we show, the term  $Q(n,r) \cdot \frac{n(q,cr)}{n(q,r)}$  can also be replaced by  $Q(n,r) + |N(q,cr)\backslash N(q,r)|$  which can potentially be smaller.

empirical results show that while the standard LSH algorithm fails to fairly sample a point in the neighborhood of the query, our algorithm produces an empirical distribution which is much closer to the uniform distribution: it improves the statistical distance to the uniform distribution by a significant factor.

### 1.2. Data-structure: Sampling from a sub-collection of sets

We first study the more generic problem – given a collection  $\mathcal{F}$  of sets from a universe of n elements, a query is a sub-collection  $\mathcal{G} \subseteq \mathcal{F}$  of these sets and the goal is to sample (almost) uniformly from the union of the sets in this sub-collection. We do this by first sampling a set X in the sub-collection  $\mathcal{G}$  proportional to the size of the set X, and then sampling an element  $x \in X$  uniformly at random. This produces a distribution on elements in the sub-collection such that any element x is chosen proportional to its degree d(x) (i.e., the number of sets  $X \in \mathcal{G}$  that  $x \in X$ ). Therefore, we can use rejection sampling and only report x with probability 1/d(x).

We can compute the degree by checking if  $x \in X$  for all sets  $X \in \mathcal{G}$  in the collection, which takes time proportional to  $g = |\mathcal{G}|$ . Also because of the rejection sampling, we might need to repeat this process  $O(\deg_{avg})$  times which can be as large as g. This leads to expected runtime  $O(g^2)$  to generate a single sample, see Lemma 3.1.

As a first improvement, we approximate the degree using standard sampling techniques, which can be done in time  $\widetilde{O}(g/\mathsf{d}(x))$ . Although this can still be large for small degrees, however, those small values will also be rejected with a smaller probability. Using this, we can bound the runtime of a query by  $O(\varepsilon^{-2}g\log n)$ , see Lemma 3.3 (the sampling is  $(1 \pm \varepsilon)$ -uniform), where n is (roughly) the input size.

Our second improvement, which the authors believe to be quite interesting, follows by simulating this rejection sampling directly. This follows by first introducing a heuristic to approximate the degree, and then shortcutting it to get the desired simulation. Section 3.4 describes this modified algorithm. In particular, one can get uniform sampling with high probability. Specifically, one can sample uniformly in expected  $O(g \log \gamma^{-1})$  time, where the sampling succeeds with probability  $\geq 1 - \gamma$ . Alternatively, one can sample  $(1 \pm \varepsilon)$ -uniformly, with the expected running time being  $O(g \log(n/\varepsilon))$ . This is a significant improvement, in both simplicity and dependency on  $\varepsilon$ , over the previous scheme.

We also show how to modify that data-structure to handle outliers, as it is the case for LSH, as the sampling algorithm needs to ignore such points once they are reported as a sample.

**Applications.** Here are a few examples of applications of such a data-structure (for sampling from a union of sets):

- (A) Given a subset X of vertices in the graph, randomly pick (with uniform distribution) a neighbor to one of the vertices of X. This can be used in simulating disease spread [KE05].
- (B) Here, we use a variant of this data-structure to implement the fair ANN.

(C) Uniform sampling for range searching [HQT14, AW17, AP19]. Indeed, consider a set of points, stored in a data-structure for range queries. Using the above, we can support sampling from the points reported by several queries, even if the reported answers are not disjoint.

Being unaware of any previous work on this problem, we believe this data-structure is of independent interest.

#### 1.3. Paper organization

We describe some basic sampling and approximation tools in Section 2. We describe the sampling from union of set data-structure in Section 3. The application of the data-structure to LSH is described in Section 4. The experiments are described in Section 5.

#### 2. Preliminaries

Neighborhood, fair nearest-neighbor, and approximate neighborhood. Let  $(\mathcal{M}, d)$  be a metric space and let  $P \subseteq \mathcal{M}$  be a set of n points. Let  $\mathbb{B}(c, r) = \{x \in \mathcal{M} \mid d(c, x) \leq r\}$  be the (close) ball of radius r around a point  $c \in \mathcal{M}$ , and let  $N(c, r) = \mathbb{B}(c, r) \cap P$  be the r-neighborhood of c in P. The size of the r-neighborhood is n(c, r) = |N(c, r)|.

Definition 2.1 (FANN). Given a data set  $P \subseteq \mathcal{M}$  of n points and a parameter r, the goal is to preprocess P such that for a given query q, one reports each point  $p \in N(q, r)$  with probability  $\mu_p$  where  $\mu$  is an approximately uniform probability distribution:  $\mathbb{P}(q, r)/(1+\varepsilon) \leq \mu_p \leq (1+\varepsilon)\mathbb{P}(q, r)$ , where  $\mathbb{P}(q, r) = 1/n(q, r)$ .

Definition 2.2 (FANN with approximate neighborhood). Given a data set  $P \subseteq \mathcal{M}$  of n points and a parameter r, the goal is to preprocess them such that for a given query q, one reports each point  $p \in S$  with probability  $\mu_p$  where  $\varphi/(1+\varepsilon) \leqslant \mu_p \leqslant (1+\varepsilon)\varphi$ , where S is a point set such that  $N(q,r) \subseteq S \subseteq N(q,cr)$ , and  $\varphi = 1/|S|$ .

**Set representation.** Let  $\mathcal{U}$  be an underlying ground set of n objects (i.e., elements). In this paper, we deal with sets of objects. Assume that such a set  $X \subseteq \mathcal{U}$  is stored in some reasonable data-structure, where one can insert delete, or query an object in constant time. Querying for an object  $o \in \mathcal{U}$ , requires deciding if  $o \in X$ . Such a representation of a set is straightforward to implement using an array to store the objects, and a hash table. This representation allows random access to the elements in the set, or uniform sampling from the set.

If hashing is not feasible, one can just use a standard dictionary data-structure – this would slow down the operations by a logarithmic factor.

**Subset size estimation.** We need the following standard estimation tool, [BHR<sup>+</sup>17, Lemma 2.8].

**Lemma 2.3.** Consider two sets  $B \subseteq U$ , where n = |U|. Let  $\xi, \gamma \in (0, 1)$  be parameters, such that  $\gamma < 1/\log n$ . Assume that one is given an access to a membership oracle that, given an element  $x \in U$ , returns whether or not  $x \in B$ . Then, one can compute an estimate s, such that  $(1-\xi)|B| \le s \le (1+\xi)|B|$ , and computing this estimates requires  $O((n/|B|)\xi^{-2}\log \gamma^{-1})$  oracle queries. The returned estimate is correct with probability  $\ge 1 - \gamma$ .

Weighted sampling. We need the following standard data-structure for weighted sampling.

**Lemma 2.4.** Given a set of objects  $\mathcal{H} = \{o_1, \dots, o_t\}$ , with associated weights  $w_1, \dots, w_t$ , one can preprocess them in O(t) time, such that one can sample an object out of  $\mathcal{H}$ . The probability of an object  $o_i$  to be sampled is  $w_i / \sum_{j=1}^t w_j$ . In addition the data-structure supports updates to the weights. An update or sample operation takes  $O(\log t)$  time.

*Proof:* Build a balanced binary tree T, where the objects of  $\mathcal{G}$  are stored in the leaves. Every internal node u of T, also maintains the total weight w(u) of the objects in its subtree. The tree T has height  $O(\log t)$ , and weight updates can be carried out in  $O(\log t)$  time, by updating the path from the root to the leaf storing the relevant object.

Sampling is now done as follows – we start the traversal from the root. At each stage, when being at node u, the algorithm considers the two children  $u_1, u_2$ . It continues to  $u_1$  with probability  $w(u_1)/w(u)$ , and otherwise it continues into  $u_2$ . The object sampled is the one in the leaf that this traversal ends up at.

### 3. Data-structure: Sampling from the union of sets

**The problem.** Assume you are given a data-structure that contains a large collection  $\mathcal{F}$  of sets of objects. The sets in  $\mathcal{F}$  are not necessarily disjoint. The task is to preprocess the data-structure, such that given a sub-collection  $\mathcal{G} \subseteq \mathcal{F}$  of the sets, one can quickly pick uniformly at random an object from the set  $\bigcup \mathcal{G} = \bigcup_{X \in \mathcal{G}} X$ .

**Naive solution.** The naive solution is to take the sets under consideration (in  $\mathcal{G}$ ), compute their union, and sample directly from the union set  $\bigcup \mathcal{G}$ . Our purpose is to do (much) better – in particular, the goal is to get a query time that depends logarithmically on the total size of all sets in  $\mathcal{G}$ .

### 3.1. Preprocessing

For each set  $X \in \mathcal{F}$ , we build the set representation mentioned in the preliminaries section. In addition, we assume that each set is stored in a data-structure that enables easy random access or uniform sampling on this set (for example, store each set in its own array). Thus, for each set X, and an element, we can decide if the element is in X in constant time.

### 3.2. Uniform sampling via exact degree computation

The query is a family  $\mathcal{G} \subseteq \mathcal{F}$ , and define  $m = |\mathcal{G}| = \sum_{X \in \mathcal{G}} |X|$  (which should be distinguished from  $g = |\mathcal{G}|$  and from  $n = |\bigcup \mathcal{G}|$ ). The **degree** of an element  $x \in \bigcup \mathcal{G}$ , is the number of sets of  $\mathcal{G}$  that contains it – that is,  $d_{\mathcal{G}}(x) = |D_{\mathcal{G}}(x)|$ , where  $D_{\mathcal{G}}(x) = \{X \in \mathcal{G} \mid x \in X\}$ . The algorithm repeatedly does the following:

- (I) Picks one set from  $\mathcal{G}$  with probabilities proportional to their sizes. That is, a set  $X \in \mathcal{G}$  is picked with probability |X|/m.
- (II) It picks an element  $x \in X$  uniformly at random.
- (III) Computes the degree  $d = d_{\mathcal{G}}(x)$ .
- (IV) Outputs x and stop with probability 1/d. Otherwise, continues to the next iteration.

**Lemma 3.1.** Let  $n = |\bigcup \mathcal{G}|$  and  $g = |\mathcal{G}|$ . The above algorithm samples an element  $x \in \bigcup \mathcal{G}$  according to the uniform distribution. The algorithm takes in expectation  $O(gm/n) = O(g^2)$  time. The query time is takes  $O(g^2 \log n)$  with high probability.

*Proof:* Let  $m = |\mathcal{G}|$ . Observe that an element  $x \in \bigcup \mathcal{G}$  is picked by step (II) with probability  $\alpha = \mathsf{d}(x)/m$ . The element x is output with probability  $\beta = 1/\mathsf{d}(x)$ . As such, the probability of x to be output by the algorithm in this round is  $\alpha\beta = 1/|\mathcal{G}|$ . This implies that the output distribution is uniform on all the elements of  $\bigcup \mathcal{G}$ .

The probability of success in a round is n/m, which implies that in expectation m/n rounds are used, and with high probability  $O((m/n)\log n)$  rounds. Computing the degree  $d_{\mathcal{G}}(x)$  takes  $O(|\mathcal{G}|)$  time, which implies the first bound on the running time. As for the second bound, observe that an element can appear only once in each set of  $\mathcal{G}$ , which readily implies that  $d(y) \leq |\mathcal{G}|$ , for all  $y \in \bigcup \mathcal{G}$ .

### 3.3. Almost uniform sampling via degree approximation

The bottleneck in the above algorithm is computing the degree of an element. We replace this by an approximation.

Definition 3.2. Given two positive real numbers x and y, and a parameter  $\varepsilon \in (0,1)$ , the numbers x and y are  $\varepsilon$ -approximation of each other, denoted by  $x \approx_{\varepsilon} y$ , if  $x/(1+\varepsilon) \le y \le x(1+\varepsilon)$  and  $y/(1+\varepsilon) \le x \le y(1+\varepsilon)$ .

In the approximate version, given an item  $x \in \bigcup \mathcal{G}$ , we can approximate its degree and get an improved runtime for the algorithm.

**Lemma 3.3.** The input is a family of sets  $\mathcal{F}$  that one can preprocess in linear time. Let  $\mathcal{G} \subseteq \mathcal{F}$  be a sub-family and let  $n = |\bigcup \mathcal{G}|$ ,  $g = |\mathcal{G}|$ , and  $\varepsilon \in (0,1)$  be a parameter. One can sample an element  $x \in \bigcup \mathcal{G}$  with almost uniform probability distribution. Specifically, the probability of an element to be output is  $\approx_{\varepsilon} 1/n$ . After linear time preprocessing, the query time is  $O(g\varepsilon^{-2}\log n)$ , in expectation, and the query succeeds with high probability.

Proof: Let  $m = |\mathcal{G}|$ . Since  $d(x) = |D_{\mathcal{G}}(x)|$ , it follows that we need to approximate the size of  $D_{\mathcal{G}}(x)$  in  $\mathcal{G}$ . Given a set  $X \in \mathcal{G}$ , we can in constant time check if  $x \in X$ , and as such decide if  $X \in D_{\mathcal{G}}(x)$ . It follows that we can apply the algorithm of Lemma 2.3, which requires  $W(x) = O\left(\frac{g}{d(x)}\varepsilon^{-2}\log n\right)$  time, where the algorithm succeeds with high probability. The query algorithm is the same as before, except that it uses the estimated degree.

For  $x \in \bigcup \mathcal{G}$ , let  $\mathcal{E}_x$  be the event that the element x is picked for estimation in a round, and let  $\mathcal{E}'_x$  be the event that it was actually output in that round. Clearly, we have  $\mathbb{P}[\mathcal{E}'_x \mid \mathcal{E}_x] = 1/d$ , where d is the degree estimate of x. Since  $d \approx_{\varepsilon} \mathsf{d}(x)$  (with high probability), it follows that  $\mathbb{P}[\mathcal{E}'_x \mid \mathcal{E}_x] \approx_{\varepsilon} 1/\mathsf{d}(x)$ . Since there are  $\mathsf{d}(x)$  copies of x in  $\mathcal{G}$ , and the element for estimation is picked uniformly from the sets of  $\mathcal{G}$ , it follows that the probability of any element  $x \in \bigcup \mathcal{G}$  to be output in a round is

$$\mathbb{P}[\mathcal{E}_x'] = \mathbb{P}[\mathcal{E}_x' \mid \mathcal{E}_x] \, \mathbb{P}[\mathcal{E}_x] = \mathbb{P}[\mathcal{E}_x' \mid \mathcal{E}_x] \, \frac{\mathsf{d}(x)}{m} \, \approx_{\varepsilon} \, 1/m,$$

as  $\mathcal{E}'_x \subseteq \mathcal{E}_x$ . As such, the probability of the algorithm terminating in a round is  $\alpha = \sum_{x \in \bigcup \mathcal{G}} \mathbb{P}[\mathcal{E}'_x] \approx_{\varepsilon} n/m \geqslant n/2m$ . As for the expected amount of work in each round, observe that it is proportional to

$$W = \sum_{x \in \mathbb{I} \setminus \mathcal{G}} \mathbb{P}[\mathcal{E}_x] W(x) = \sum_{x \in \mathbb{I} \setminus \mathcal{G}} \frac{\mathsf{d}(x)}{m} \frac{g}{\varepsilon^2 \mathsf{d}(x)} \log n = O\Big(\frac{ng}{m} \varepsilon^{-2} \log n\Big).$$

Intuitively, since the expected amount of work in each iteration is W, and the expected number of rounds is  $1/\alpha$ , the expected running time is  $O(W/\alpha)$ . This argument is not quite right, as the amount of work in each round effects the probability of the algorithm to terminate in the round (i.e., the two variables are not independent). We continue with a bit more care – let  $L_i$  be the running time in the *i*th round of the algorithm if it was to do an *i*th iteration (i.e., think about a version of the algorithm that skips the experiment in the end of the iteration to decide whether it is going to stop), and let  $Y_i$  be a random variable that is 1 if the (original) algorithm had not stopped at the end of the first *i* iterations of the algorithm.

By the above, we have that  $y_i = \mathbb{P}[Y_i = 1] = \mathbb{P}[Y_i = 1 \mid Y_{i-1} = 1] \mathbb{P}[Y_{i-1} = 1] \leqslant (1 - \alpha)y_{i-1} \leqslant (1 - \alpha)^i$ , and  $\mathbb{E}[L_i] = O(W)$ . Importantly,  $L_i$  and  $Y_{i-1}$  are independent (while  $L_i$  and  $Y_i$  are dependent). We clearly have that the running time of the algorithm is  $O(\sum_{i=1}^{\infty} Y_{i-1}L_i)$  (here, we define  $Y_0 = 1$ ). Thus, the expected running time of the algorithm is proportional to

$$\mathbb{E}\left[\sum_{i} Y_{i-1} L_{i}\right] = \sum_{i} \mathbb{E}\left[Y_{i-1} L_{i}\right] = \sum_{i} \mathbb{E}\left[Y_{i-1}\right] \mathbb{E}\left[L_{i}\right] \leqslant W \sum_{i} y_{i-1} \leqslant W \sum_{i=1}^{\infty} (1 - \alpha)^{i-1} = \frac{W}{\alpha}$$
$$= O(g\varepsilon^{-2} \log n),$$

because of linearity of expectations, and since  $L_i$  and  $Y_{i-1}$  are independent.

Remark 3.4. The query time of Lemma 3.3 deteriorates to  $O(g\varepsilon^{-2}\log^2 n)$  if one wants the bound to hold with high probability. This follows by restarting the query algorithm if the query time exceeds (say by a factor of two) the expected running time. A standard application of Markov's inequality implies that this process would have to be restarted at most  $O(\log n)$  times, with high probability.

Remark 3.5. The sampling algorithm is independent of whether or not we fully know the underlying family  $\mathcal{F}$  and the sub-family  $\mathcal{G}$ . This means the past queries do not affect the sampled object reported for the query  $\mathcal{G}$ . Therefore, the almost uniform distribution property holds in the presence of several queries and independently for each of them.

#### 3.4. Almost uniform sampling via simulation

It turns out that one can avoid the degree approximation stage in the above algorithm. To this end, let x be the element picked. We need to simulate a process that accepts x with probability 1/d(x).

We start with the following natural idea for estimating  $\mathsf{d}(x)$  – probe the sets randomly (with replacement), and stop in the ith iteration if it is the first iteration where the probe found a set that contains x. If there are g sets, then the distribution of i is geometric, with probability  $p = \mathsf{d}(x)/g$ . In particular, in expectation,  $\mathbb{E}[i] = g/\mathsf{d}(x)$ , which implies that  $\mathsf{d}(x) = g/\mathbb{E}[i]$ . As such, it is natural to take g/i as an estimation for the degree of x. Thus, to simulate a process that succeeds with probability  $1/\mathsf{d}(x)$ , it would be natural to return 1 with probability i/g and 0 otherwise. Surprisingly, while this seems like a heuristic, it does work, under the right interpretation.

**Lemma 3.6.** Assume we have g urns, and exactly d > 0 of them, are non-empty. Furthermore, assume that we can check if a specific urn is empty in constant time. Then, there is a randomized algorithm, that outputs a number  $Y \ge 0$ , such that  $\mathbb{E}[Y] = 1/d$ . The expected running time of the algorithm is O(g/d).

*Proof:* The algorithm repeatedly probes urns (uniformly at random), until it finds a non-empty urn. Assume it found a non-empty urn in the ith probe. The algorithm outputs the value i/g and stops.

Setting p = d/g, and let Y be the output of the algorithm. we have that

$$\mathbb{E}\big[Y\big] = \sum_{i=1}^{\infty} \frac{i}{g} (1-p)^{i-1} p = \frac{p}{g(1-p)} \sum_{i=1}^{\infty} i (1-p)^i = \frac{p}{g(1-p)} \cdot \frac{1-p}{p^2} = \frac{1}{pg} = \frac{1}{\mathsf{d}},$$

using the formula  $\sum_{i=1}^{\infty} ix^i = x/(1-x)^2$ .

The expected number of probes performed by the algorithm until it finds a non-empty urn is 1/p = g/d, which implies that the expected running time of the algorithm is O(g/d).

The natural way to deploy Lemma 3.6, is to run its algorithm to get a number y, and then return 1 with probability y. The problem is that y can be strictly larger than 1,

which is meaningless for probabilities. Instead, we backoff by using the value  $y/\Delta$ , for some parameter  $\Delta$ . If the returned value is larger than 1, we just treat it at zero. If the zeroing never happened, the algorithm would return one with probability  $1/(d(x)\Delta)$  – which we can use to our purposes via, essentially, amplification. Instead, the probability of success is going to be slightly smaller, but fortunately, the loss can be made arbitrarily small by taking  $\Delta$  to be sufficiently large.

**Lemma 3.7.** There are g urns, and exactly d > 0 of them are not empty. Furthermore, assume one can check if a specific urn is empty in constant time. Let  $\gamma \in (0,1)$  be a parameter. Then one can output a number  $Z \ge 0$ , such that  $Z \in [0,1]$ , and  $\mathbb{E}[Z] \in I = \left[\frac{1}{d\Delta} - \gamma, \frac{1}{d\Delta}\right]$ , where  $\Delta = \left[\ln \gamma^{-1}\right] + 4 = \Theta(\log \gamma^{-1})$ . The expected running time of the algorithm is O(g/d).

Alternatively, the algorithm can output a bit X, such that  $\mathbb{P}[X=1] \in I$ .

*Proof:* We modify the algorithm of Lemma 3.6, so that it outputs  $i/(g\Delta)$  instead of i/g. If the algorithm does not stop in the first  $g\Delta + 1$  iterations, then the algorithm stops and outputs 0. Observe that the probability that the algorithm fails to stop in the first  $g\Delta$  iterations, for  $p = \mathsf{d}/g$ , is  $(1-p)^{g\Delta} \leqslant \exp\left(-\frac{\mathsf{d}}{g}g\Delta\right) \leqslant \exp(-\mathsf{d}\Delta) \leqslant \exp(-\Delta) \ll \gamma$ .

Let Z be the random variable that is the number output by the algorithm. Arguing as in Lemma 3.6, we have that  $\mathbb{E}[Z] \leq 1/(\mathsf{d}\Delta)$ . More precisely, we have  $\mathbb{E}[Z] = \frac{1}{\mathsf{d}\Delta} - \sum_{i=g\Delta+1}^{\infty} \frac{i}{g\Delta} (1-p)^{i-1} p$ . Let

$$\sum_{i=gj+1}^{g(j+1)} \frac{i}{g} (1-p)^{i-1} p \leqslant (j+1) \sum_{i=gj+1}^{g(j+1)} (1-p)^{i-1} p = (j+1)(1-p)^{gj} \sum_{i=0}^{g-1} (1-p)^{i} p$$

$$\leqslant (j+1)(1-p)^{gj} \leqslant (j+1) \left(1 - \frac{\mathsf{d}}{g}\right)^{gj} \leqslant (j+1) \exp(-\mathsf{d}j).$$

Let  $g(j) = \frac{j+1}{\Delta} \exp(-\mathsf{d}j)$ . We have that  $\mathbb{E}[Z] \geqslant \frac{1}{\mathsf{d}\Delta} - \beta$ , where  $\beta = \sum_{j=\Delta}^{\infty} g(j)$ . Furthermore, for  $j \geqslant \Delta$ , we have

$$\frac{g(j+1)}{g(j)} = \frac{(j+2)\exp(-\mathsf{d}(j+1))}{(j+1)\exp(-\mathsf{d}j)} \leqslant \left(1 + \frac{1}{\Delta}\right)e^{-\mathsf{d}} \leqslant \frac{5}{4}e^{-\mathsf{d}} \leqslant \frac{1}{2}.$$

As such, we have that

$$\beta = \sum_{j=\Delta}^{\infty} g(j) \leqslant 2g(\Delta) \leqslant 2\frac{\Delta+1}{\Delta} \exp(-\mathsf{d}\Delta) \leqslant 4 \exp(-\Delta) \leqslant \gamma,$$

by the choice of value for  $\Delta$ . This implies that  $\mathbb{E}[Z] \ge 1/(\mathsf{d}\Delta) - \beta \ge 1/(\mathsf{d}\Delta) - \gamma$ , as desired. The alternative algorithm takes the output Z, and returns 1 with probability Z, and zero otherwise.

**Lemma 3.8.** The input is a family of sets  $\mathcal{F}$  that one preprocesses in linear time. Let  $\mathcal{G} \subseteq \mathcal{F}$  be a sub-family and let  $n = |\bigcup \mathcal{G}|$ ,  $g = |\mathcal{G}|$ , and let  $\varepsilon \in (0,1)$  be a parameter. One can sample an element  $x \in \bigcup \mathcal{G}$  with almost uniform probability distribution. Specifically, the probability of an element to be output is  $\approx_{\varepsilon} 1/n$ . After linear time preprocessing, the query time is  $O(g \log(g/\varepsilon))$ , in expectation, and the query succeeds, with high probability (in g).

Proof: The algorithm repeatedly samples an element x using steps (I) and (II) of the algorithm of Section 3.2. The algorithm returns x if the algorithm of Lemma 3.7, invoked with  $\gamma = (\varepsilon/g)^{O(1)}$  returns 1. We have that  $\Delta = \Theta(\log(g/\varepsilon))$ . Let  $\alpha = 1/(\mathsf{d}(x)\Delta)$ . The algorithm returns x in this iteration with probability p, where  $p \in [\alpha - \gamma, \alpha]$ . Observe that  $\alpha \ge 1/(g\Delta)$ , which implies that  $\gamma \ll (\varepsilon/4)\alpha$ , it follows that  $p \approx_{\varepsilon} 1/(\mathsf{d}(x)\Delta)$ , as desired. The expected running time of each round is  $O(g/\mathsf{d}(x))$ .

Arguing as in Lemma 3.3, this implies that each round, in expectation takes O(ng/m) time, where  $m = |\mathcal{G}|$ . Similarly, the expected number of rounds, in expectation, is  $O(\Delta m/n)$ . Again, arguing as in Lemma 3.3, implies that the expected running time is  $O(g\Delta) = O(g\log(g/\varepsilon))$ .

Remark 3.9. Similar to Remark 3.4, the query time of Lemma 3.8 can be made to work with high probability with an additional logarithmic factor. Thus with high probability, the query time is  $O(g \log(g/\varepsilon) \log n)$ .

#### 3.5. Handling outliers

Imagine a situation where we have a marked set of outliers  $\mathcal{O}$ . We are interested in sampling from  $\bigcup \mathcal{G} \backslash \mathcal{O}$ . We assume that the total degree of the outliers in the query is at most  $m_{\mathcal{O}}$  for some prespecified parameter  $m_{\mathcal{O}}$ . More precisely, we have  $d_{\mathcal{G}}(\mathcal{O}) = \sum_{x \in \mathcal{O}} d_{\mathcal{G}}(x) \leqslant m_{\mathcal{O}}$ .

**Lemma 3.10.** The input is a family of sets  $\mathcal{F}$  that one can preprocess in linear time. A query is a sub-family  $\mathcal{G} \subseteq \mathcal{F}$ , a set of outliers  $\mathcal{O}$ , a parameter  $m_{\mathcal{O}}$ , and a parameter  $\varepsilon \in (0,1)$ . One can either

- (A) Sample an element  $x \in \bigcup \mathcal{G} \setminus \mathcal{O}$  with  $\varepsilon$ -approximate uniform distribution. Specifically, the probabilities of two elements to be output is the same up to a factor of  $1 \pm \varepsilon$ .
- (B) Alternatively, report that  $d_{\mathcal{G}}(\mathcal{O}) > m_{\mathcal{O}}$ . The expected query time is  $O(m_{\mathcal{O}} + g \log(N/\varepsilon))$ , and the query succeeds with high probability, where  $g = |\mathcal{G}|$ , and  $N = |\mathcal{F}|$ .

Proof: The main modification of the algorithm of Lemma 3.8, is that whenever we encounter an outlier (the assumption is that one can check if an element is an outlier in constant time), then we delete it from the set X where it was discovered. If we implement sets as arrays, this can be done by moving an outlier object to the end of the active prefix of the array, and decreasing the count of the active array. We also need to decrease the (active) size of the set. If the algorithm encounters more than  $m_{\mathcal{O}}$  outliers then it stops and reports that the number of outliers is too large.

Otherwise, the algorithm continues as before. The only difference is that once the query process is done, the active count (i.e., size) of each set needs to be restored to its original size, as is the size of the set. This clearly can be done in time proportional to the query time.

### 4. In the search for a fair near neighbor

In this section, we employ our data structure of Section 3 to show the two results on uniformly reporting a neighbor of a query point mentioned in Section 1.1. First, let us briefly give some preliminaries on LSH. We refer the reader to [HIM12] for further details. Throughout the section, we assume that our metric space, admits the LSH data structure.

### 4.1. Background on LSH

Locality Sensitive Hashing (LSH). Let  $\mathcal{D}$  denote the data structure constructed by LSH, and let c denote the approximation parameter of LSH. The data-structure  $\mathcal{D}$  consists of L hash functions  $g_1, \ldots, g_L$  (e.g.,  $L \approx n^{1/c}$  for a c-approximate LSH), which are chosen via a random process and each function hashes the points to a set of buckets. For a point  $p \in \mathcal{M}$ , let  $H_i(x)$  be the bucket that the point p is hashed to using the hash function  $g_i$ . The following are standard guarantees provided by the LSH data structure [HIM12].

**Lemma 4.1.** For a given query point q, let  $S = \bigcup_i H_i(q)$ . Then for any point  $p \in N(q, r)$ , we have that with a probability of least 1-1/e-1/3, we have (i)  $p \in S$  and (ii)  $|S \setminus \mathbb{B}(q, cr)| \leq 3L$ , i.e., the number of outliers is at most 3L. Moreover, the expected number of outliers in any single bucket  $H_i(q)$  is at most 1.

Therefore, if we take  $t = O(\log n)$  different data structures  $\mathcal{D}_1, \ldots, \mathcal{D}_t$  with corresponding hash functions  $g_i^j$  to denote the *i*th hash function in the *j*th data structure, we have the following lemma.

**Lemma 4.2.** Let the query point be q, and let p be any point in N(q,r). Then, with high probability, there exists a data structure  $\mathcal{D}_j$ , such that  $p \in S = \bigcup_i H_i^j(q)$  and  $|S \setminus \mathbb{B}(q,cr)| \leq 3L$ .

By the above, the space used by LSH is  $S(n,c) = \widetilde{O}(n \cdot L)$  and the query time is  $Q(n,c) = \widetilde{O}(L)$ .

### 4.2. Approximate Neighborhood

For  $t = O(\log n)$ , let  $\mathcal{D}_1, \ldots, \mathcal{D}_t$  be data structures constructed by LSH. Let  $\mathcal{F}$  be the set of all buckets in all data structures, i.e.,  $\mathcal{F} = \{H_i^j(p) \mid i \leq L, j \leq t, p \in P\}$ . For a query point q, consider the family  $\mathcal{G}$  of all buckets containing the query, i.e.,  $\mathcal{G} = \{H_i^j(q) \mid i \leq L, j \leq t\}$ , and thus  $|\mathcal{G}| = O(L \log n)$ . Moreover, we let  $\mathcal{O}$  to be the set of outliers, i.e., the points that

are farther than cr from q. Note that as mentioned in Lemma 4.1, the expected number of outliers in each bucket of LSH is at most 1. Therefore, by Lemma 3.10, we immediately get the following result.

**Lemma 4.3.** Given a set P of n points and a parameter r, we can preprocess it such that given query q, one can report a point  $p \in S$  with probability  $\mu_p$  where  $\varphi/(1+\varepsilon) \leq \mu_p \leq (1+\varepsilon)\varphi$ , where S is a point set such that  $N(q,r) \subseteq S \subseteq N(q,cr)$ , and  $\varphi = 1/|S|$ . The algorithm uses space S(n,c) and its expected query time is  $\widetilde{O}(\mathcal{Q}(n,c) \cdot \log(1/\varepsilon))$ .

*Proof:* Let  $S = \bigcup \mathcal{G} \setminus \mathcal{O}$ ; by Lemma 4.2, we know that  $N(q,r) \subseteq S \subseteq N(q,cr)$ , and moreover in expectation  $m_{\mathcal{O}} \leqslant L = |\mathcal{G}|$ . We apply the algorithm of Lemma 3.10. The runtime of the algorithm is in expectation  $\widetilde{O}(|\mathcal{G}|\log(1/\varepsilon)) = \widetilde{O}(L \cdot \log(1/\varepsilon)) = \widetilde{O}(\mathcal{Q}(n,c) \cdot \log(1/\varepsilon))$ , and the algorithm produces an almost uniform distribution over the points in S.

Remark 4.4. For the  $L_1$  distance, the runtime of our algorithm is  $\widetilde{O}(n^{(1/c^2)+o(1)})$  and for the  $L_2$  distance, the runtime of our algorithm is  $\widetilde{O}(n^{(1/c^2)+o(1)})$ . These matches the runtime of the standard LSH-based near neighbor algorithms up to polylog factors.

#### 4.3. Exact Neighborhood

As noted earlier, the result of the previous section only guarantees a query time which holds in expectation. Here, we provide an algorithm whose query time holds with high probability. Note that, here we cannot apply Lemma 3.10 directly, as the total number of outliers in our data structure might be large with non-negligible probability (and thus we cannot bound  $m_{\mathcal{O}}$ ). However, as noted in Lemma 4.2, with high probability, there exists a subset of these data structures  $J \subseteq [t]$  such that for each  $j \in J$ , the number of outliers in  $S_j = \bigcup_i H_i^j(q)$  is at most 3L, and moreover, we have that  $N(q,r) \subseteq \bigcup_{j \in J} S_j$ . Therefore, on a high level, we make a guess J' of J, which we initialize it to J' = [t], and start by drawing samples from  $\mathcal{G}$ ; once we encounter more than 3L outliers from a certain data structure  $\mathcal{D}_j$ , we infer that  $j \notin J$ , update the value of  $J' = J' \setminus \{j\}$ , and set the weights of the buckets corresponding to  $\mathcal{D}_j$  equal to 0, so that they will never participate in the sampling process. As such, at any iteration of the algorithm we are effectively sampling from  $\mathcal{G} = \{H_i^j(q) \mid i \leqslant L, j \in J'\}$ .

**Preprocessing.** We keep  $t = O(\log n)$  LSH data structures which we refer to as  $\mathcal{D}_1, \ldots, \mathcal{D}_t$ , and we keep the hashed points by the *i*th hash function of the *j*th data structure in the array denoted by  $H_i^j$ . Moreover, for each bucket in  $H_i^j$ , we store its size  $|H_i^j|$ .

Query Processing. We maintain the variables  $z_i^j$  showing the weights of the bucket  $H_i^j(q)$ , which is initialized to  $|H_i^j(q)|$  that is stored in the preprocessing stage. Moreover, we keep the set of outliers detected from  $H_i^j(q)$  in  $\mathcal{O}_i^j$  which is initially set to be empty. While running the algorithm, as we detect an outlier in  $H_i^j(q)$ , we add it to  $\mathcal{O}_i^j$ , and we further decrease  $z_i^j$  by one. Moreover, in order to keep track of J', for any data structure  $\mathcal{D}_j$ , whenever  $\sum_i |\mathcal{O}_i^j|$  exceeds 3L, we will ignore all buckets in  $\mathcal{D}_j$ , by setting all corresponding  $z_i^j$  to zero.

At each iteration, the algorithm proceeds by sampling a bucket  $H_i^j(q)$  proportional to its weight  $z_i^j$ , but only among the set of buckets from those data structures  $\mathcal{D}_j$  for which less than 3L outliers are detected so far, i.e.,  $j \in J'$ . We then sample a point uniformly at random from the points in the chosen bucket that have not been detected as an outlier, i.e.,  $H_i^j(q)\backslash\mathcal{O}_i^j$ . If the sampled point is an outlier, we update our data structure accordingly. Otherwise, we proceed as in Lemma 3.8.

Definition 4.5 (Active data structures and active buckets). Consider an iteration k of the algorithm. Let us define the set of active data structures to be the data structures from whom we have seen less than 3L outliers so far, and let us denote their indices by  $J'_k \subseteq [t]$ , i.e.,  $J'_k = \{j \mid \sum_i |\mathcal{O}_i^i| < 3L\}$ .

Moreover, let us define the *active buckets* to be all buckets containing the query in these active data structures, i.e.,  $\mathcal{G}_k = \{H_i^j(q) \mid i \leq L, j \in J_k'\}.$ 

**Observation 4.6.** Lemma 4.2 implies that with high probability at any iteration k of the algorithm  $N(q,r) \subseteq \bigcup \mathcal{G}_k$ .

Definition 4.7 (active size). For an active bucket  $H_i^j(q)$ , we define its active size to be  $z_i^j$  which shows the total number of points in the bucket that have not yet been detected as an outlier, i.e.,  $|H_i^j(q) \setminus \mathcal{O}_i^j|$ .

**Lemma 4.8.** Given a set P of n points and a parameter r, we can preprocess it such that given a query q, one can report a point  $p \in P$  with probability  $\mu_p$ , so that there exists a value  $\rho \in [0,1]$  where

- For  $p \in N(q, r)$ , we have  $\frac{\rho}{(1+O(\varepsilon))} \le \mu_p \le (1+O(\varepsilon))\rho$ .
- For  $p \in N(q, cr) \backslash N(q, r)$ , we have  $\mu_p \leq (1 + O(\varepsilon))\rho$ .
- For  $p \notin N(q, cr)$ , we have  $\mu_p = 0$ .

The space used is  $\widetilde{O}(\mathcal{S}(n,c))$  and the query time is  $\widetilde{O}(\mathcal{Q}(n,c) \cdot \log(1/\varepsilon))$  with high probability.

Proof: First note that the algorithm never outputs an outlier, and thus the third item is always satisfied. Next, let K be a random variable showing the number of iterations of the algorithm, and for an iteration k, define the random variable  $M_k = N(q, cr) \cap \bigcup \mathcal{G}_k$  as the set of non-outlier points in the set of active buckets. Conditioned on K = k, by Lemma 3.8, we know that the distribution of the output is almost uniform on  $M_k$ . Moreover, we know that for all k we have  $M_k \subseteq M_{k-1}$ , and that by Observation 4.6,  $N(q,r) \subseteq M_k$ . Therefore, for all points in N(q,r) their probability of being reported as the final output of the algorithm is equal, and moreover, for all points in  $N(q,cr)\backslash N(q,r)$ , their probability of being reported is lower (as at some iteration, some of these points might go out of the set of active buckets). This proves the probability condition.

To bound the query time, let us consider the iterations where the sampled point p is an outlier, and not an outlier, separately. The total number of iterations where an outlier point is sampled is at most  $3L \cdot t = \widetilde{O}(L) = \widetilde{O}(\mathcal{Q}(n,c))$  for which we only pay  $\widetilde{O}(1)$  cost. For non-outlier points, their total cost can be bounded using Lemma 3.8 and Remark 3.9 by  $\widetilde{O}(|\mathcal{G}_1|\log(1/\varepsilon)) = \widetilde{O}(L \cdot \log(1/\varepsilon)) = \widetilde{O}(\mathcal{Q}(n,c) \cdot \log(1/\varepsilon))$ .

**Lemma 4.9.** Given a set P of n points and a parameter r, we can preprocess it such that given a query q, one can report a point  $p \in S$  with probability  $\mu_p$  where  $\mu$  is an approximately uniform probability distribution:  $\varphi/(1+\varepsilon) \leq \mu_p \leq \varphi(1+\varepsilon)$ , where  $\varphi = 1/|N(q,r)|$ . The algorithm uses space S(n,c) and has query time of  $\widetilde{O}(\mathcal{Q}(n,c) \cdot \frac{|N(q,cr)|}{|N(q,r)|} \cdot \log(1/\varepsilon))$  with high probability.

*Proof:* We run Algorithm of Lemma 4.8, and while its output is outside of N(q,r), we ignore it and run the algorithm again. By Lemma 4.8, the output is guaranteed to be almost uniform on N(q,r). Moreover, by Lemma 4.8, and because with high probability, we only need to run the algorithm  $\widetilde{O}(\frac{|N(q,cr)|}{|N(q,r)|})$  times, we get the desired bound on the query time.

# 5. Experiments

In this section, we consider the task of retrieving a random point from the neighborhood of a given query point, and evaluate the effectiveness of our proposed algorithm empirically on a real data set.

Data set and Queries. Our data set contains the first 10000 points in the MNIST training data set [LBBH98]<sup>3</sup>. The full data set contains 60000 images of hand-written digits, where each image is of size 28 by 28. For the query, we use the first 100 (out of 10000) images of the MNIST test data set. Therefore, each of our points lie in a 784 dimensional Euclidean space and each coordinate is in [0, 255]. We use the  $L_2$  Euclidean distance to measure the distance between the points.

**LSH** data structure and parameters. We use the locality sensitive hashing data structure for the  $L_2$  Euclidean distance [AI08]. That is, each of the L hash functions  $g_i$ , is a concatenation of k unit hash functions  $h_i^1 \oplus \cdots \oplus h_i^k$ . Each of the unit hash functions  $h_i^j$  is chosen by selecting a point in a random direction (by choosing every coordinate from a Gaussian distribution with parameters (0,1)). Then all the points are projected onto this one dimensional direction. Then we put a randomly shifted one dimensional grid of length w along this direction. The cells of this grid are considered as buckets of the unit hash function. We tune the parameters of LSH as follows.

The average distance of a query to its nearest neighbor in the our data set is around 4.5. Thus we choose the near neighbor radius r=5. Consequently, as we observe, the r-neighborhood of at least half of the queries are non-empty. As suggested in [DIIM04] to set the value of w=4, we tune it between 3 and 5 and set its value to w=3.1. We tune k and k so that the false negative rate (the near points that are not retrieved by LSH) is less than 10%, and moreover the cost of hashing (proportional to k) balances out the cost of scanning. We thus get k=15 and k=100. Note that we use a single LSH data structure as opposed to taking k=15 and k=100 instances.

<sup>&</sup>lt;sup>3</sup>The dataset is available here: http://yann.lecun.com/exdb/mnist/

**Algorithms.** Given a query point q, we retrieve all L buckets corresponding to the query. We then implement the following algorithms and compare their performance in returning a neighbor of the query point.

- Uniform/Uniform: Picks bucket uniformly at random and picks a random point in bucket.
- Weighted/Uniform: Picks bucket according to its size, and picks uniformly random point inside bucket.
- Optimal: Picks bucket according to size, and then picks uniformly random point p inside bucket. Then it computes p's degree exactly and rejects p with probability 1 1/deg(p).
- **Degree approximation**: Picks bucket according to size, and picks uniformly random point p inside bucket. It approximates p's degree and rejects p with probability 1 1/deg'(p).

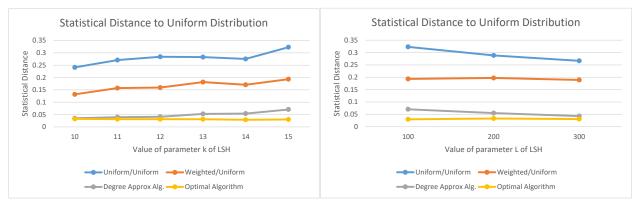
**Degree approximation method.** We use the algorithm of Section 3.4 for the degree approximation: we implement a variant of the sampling algorithm which repeatedly samples a bucket uniformly at random and checks whether p belongs to the bucket. If the first time this happens is at iteration i, then it outputs the estimate as deg'(p) = L/i.

**Experiment Setup.** In order to compare the performance of different algorithms, for each query q, we compute M(q): the set of neighbors of q which fall to the same bucket as q by at least one of the L hash functions. Then for 100|M(q)| times, we draw a sample from the neighborhood of the query, using all four algorithms. We compare the empirical distribution of the reported points on |M(q)| with the uniform distribution on it. More specifically, we compute the total variation distance (statistical distance)<sup>4</sup> to the uniform distribution. We repeat each experiment 10 times and report the average result of all 10 experiments over all 100 query points.

**Results.** Figure 5.1 shows the comparison between all four algorithms. To compare their performance, we compute the total variation distance of the empirical distribution of the algorithms to the uniform distribution. For the tuned parameters (k = 15, L = 100), we see that our proposed degree approximation based algorithm performs only 2.5 times worse than the optimal algorithm. This is true while we see that other standard sampling methods perform 6.5 times and 10.8 times worse than the optimal algorithm.

For this setting of parameters we get that the range of degrees are in [1,33]. To show that our algorithm also performs well for higher ranges of the degree, we vary the parameters of LSH. Namely, first we decrease k (the number of unit hash functions used in each of the L hash function); this will result in more collisions. Second, we increase L (the total number of hash functions). These are two ways to increase the degree of points, and by doing so we get degrees in the range [1,99]. The results are shown in Figure 5.1.

<sup>&</sup>lt;sup>4</sup>For two discrete distributions  $\mu$  and  $\nu$  on a finite set X, the total variation distance is  $\frac{1}{2} \sum_{x \in X} |\mu(x) - \nu(x)|$ .



- (a) Varying the parameter k of LSH
- (b) Varying the parameter L of LSH

Figure 5.1: Comparison of the performance of the four algorithms is measured by computing the statistical distance of their empirical distribution to the uniform distribution.

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