Fairness in ad auctions through inverse proportionality

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We study the tradeoff between social welfare maximization and fairness in the context of ad auctions. We study an ad auction setting where users arrive in an online fashion, k advertisers submit bids for each user, and the auction assigns a distribution over ads to the user. Following the works of Dwork and Ilvento [10] and Chawla et al. [12], our goal is to design a truthful auction that satisfies multiple-task fairness in its outcomes: informally speaking, users that are similar to each other should obtain similar allocations of ads.

We develop a new class of allocation algorithms that we call inverse-proportional allocation. These allocation algorithms are truthful, online, and do not explicitly need to know the fairness constraint over the users. In terms of fairness, they guarantee fair outcomes as long as every advertiser's bids are non-discriminatory across users. In terms of social welfare, inverse-proportional allocation achieves a constant factor approximation in social welfare against the optimal (unfair) allocation, independent of the number of advertisers in the system. In this respect, these allocation algorithms greatly surpass the guarantees achieved in previous work; in fact, they achieve the optimal tradeoffs between fairness and social welfare in some contexts. We also extend our results to broader notions of fairness that we call subset fairness.

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1 INTRODUCTION

Algorithms play an increasingly important role in today's society and can arguably have far-ranging social, economic, and political ramifications in many different contexts. One such context is access to sponsored information such as ads on social media, search pages, and other websites. It has been well documented that online ads can exhibit significant biases with respect to the viewers' sensitive features like race, gender, age, religion, and national origin. For example, certain employment jobs on Facebook have been found to be targeted exclusively towards men or excluded older people; and housing ads have been found to exclude viewers based on race [2].

There are two main sources of unfairness in digital ads. The first is explicit or implicit targeting of users based on sensitive attributes by the advertisers. This sort of targeting is relatively easy to spot and correct for. Indeed, in response to a lawsuit brought on by the U.S. Department of Housing and Urban Development and ProPublica against Facebook on discrimination in housing ads, Facebook's first action was to disallow advertisers to target users based on sensitive attributes [15]. However it has been further documented that unfairness can persist even in the absence of inappropriate targeting of users [1]. The second, more subtle, source of unfairness is the ad delivery mechanism itself that determines for each user which ad to display based on advertisers' bids, budgets, relevance of the ads to the user, etc. Recent empirical studies [1, 16] have observed, in fact, that unfairness can arise in outcomes of ad auctions simply as a consequence of different levels of competition among advertisers for different users.

In this paper, we address the second source of unfairness: **we study the design of ad auctions with an explicit fairness constraint**. Informally, our goal is to design a truthful auction with the property that if all advertisers bid fairly then the outcome of the auction is also fair. In other words, the auction itself does not introduce any further

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unfairness than what may be present already in advertisers' bids. Fairness by itself is easy to achieve: for example, by displaying a random ad for every user regardless of the features (bid, relevance, etc.) of the ad. Can we achieve fairness while also ensuring high quality of the outcome? In this work, we focus on the objective of social welfare maximization and study the tradeoffs between the strength of the fairness constraint the auction satisfies and the social welfare it achieves.

Fairness of bids and outcomes. What does it mean for bids and ad allocations to be fair? We use the notion of *individual fairness* developed by Dwork et al. [9], which endows users with a measure of similarity that captures all relevant and non-sensitive attributes of the users that can be used to differentiate treatment. For example, for employment ads, the measure of similarity may represent the educational level and job experience of a user. For credit ads, it may represent the users' creditworthiness and annual income. Individual fairness requires that users that are similar to each other should be treated similarly, but allows for some small room for variation. Applying this notion to our context, we require that every advertiser's bids on two similar users should be within a small factor of each other; the less similar two users are, the greater variation in bids is permitted.

On the other hand, we require the ad auction to assign a similar mix of ads to similar users. Formally, we employ the notion of *multiple-task fairness* of Dwork and Ilvento [10], that extends individual fairness to multi-dimensional settings: viewing the outcome of the auction for a particular user as a distribution over ads, we require that any two similar users are assigned any particular ad with roughly the same probability. In other words, the distributions assigned to the two users are close under ℓ_∞ distance.

One criticism of the notion of individual fairness is its reliance on an appropriate similarity metric on users that captures exactly what features can be used to differentiate between users. Where does this metric come from and who has the authority to certify that this is the right metric to use in a given context? We sidestep these questions in our work. We aim to develop auctions that are oblivious to the underlying metric in the following sense: if the bids submitted by the advertisers respect the fairness constraint imposed by the metric, then our auctions return fair outcomes with respect to the same metric *without explicitly knowing what the metric is*. However, if the bids are unfair, then our auctions provide no fairness guarantees. On the other hand, our performance guarantees with respect to social welfare hold regardless of whether the bids are fair. This approach separates the responsibilities of the auctioneer from that of an auditor, whose job it is to check whether advertisers are following fairness guidelines. In doing so, it potentially simplifies the design of both of these components of the ad auction pipeline.

The tension between fairness and optimality. Observe that standard optimal auction formats do not satisfy fairness even when the advertisers' bids are fair. Consider the highest bid wins auction, for example. Suppose that two users have very similar but not identical job qualifications, and an employment ad bids slightly different amounts on them. Then, depending on other bids, the highest bid wins auction may display this ad exclusively to one user and never to the other. In other words, the highest bid wins auction greatly exaggerates minor, subtle differences in input into huge swings in output. Can we design an auction to avoid such behavior?

Chawla et al. [12] previously studied the tradeoff of social welfare and fairness in the context of ad auctions. They developed a class of proportional allocation mechanisms that assign an ad to a user with probability proportional to some increasing function of the advertiser's bid. The choice of the increasing function depends on the strength of the fairness constraint satisfied by the bids. The mechanisms of [12] are metric oblivious, multiple-task fair, and achieve approximately optimal social welfare. However, their approximation guarantee for social welfare degrades polynomially with the number of advertisers. Is it possible to do better?

Our contribution: Inverse proportional allocation. We develop a new family of allocation rules that we call *inverse proportional allocation*. Informally, our auction begins by allocating each ad fully to the user. It then "takes away" the over-assignment from the ads in proportion to some decreasing function of the advertisers' bids. The choice of the decreasing function depends on the strength of the fairness constraint satisfied by the bids. As in the work of [12], our mechanisms are truthful (for an appropriate choice of payments made by the advertisers), metric oblivious, and multiple-task fair. Also similar to the previous work, our auctions are anonymous across advertisers, online, and history independent. The advantage of our auctions is that they achieve a *constant factor approximation* to the unfair optimal social welfare, in a significant improvement over the previous work. In particular, the constant is independent of the number of advertisers, but depends in a mild manner on the strength of the bid fairness constraint. Moreover, our auctions achieve the optimal tradeoff between social welfare and fairness in some contexts.

Total variation and subset fairness. The family of proportional allocation mechanisms designed by [12] in fact satisfy a particularly strong fairness guarantee that they call *total variation fairness*: they guarantee that the distributions over ads assigned to two similar users are not only close under ℓ_∞ distance, but also close under ℓ_1 or total variation distance. One implication of this stronger guarantee is that users are not discriminated against even when considering sets of related ads rather than just a single ad. To take an example, suppose that two users Alice and Bob are similarly qualified for jobs, but that all high paying job ads bid slightly lower for Alice than for Bob. Then, an ad auction that satisfies multiple-task fairness may show every high paying job ad with slightly less probability to Alice than to Bob. However, considering the set of all high paying jobs together, Alice may see a job in this set with far lower probability than Bob. Total variation fairness disallows this kind of discrimination.

We observe that our family of auctions does not always satisfy total variation fairness. Indeed it appears to be challenging to satisfy this stronger fairness property while also guaranteeing a constant factor approximation in social welfare. We show, however, that an intermediate fairness guarantee can be achieved. We consider a setting where an authority identifies a restricted collection of sets of advertisers over which fairness must be guaranteed. This may include, for example, all job ads belonging to a particular category or a particular compensation bracket, etc. We then develop auctions that guarantee fairness with respect to every set in the given family, while obtaining approximation ratios that degrade with the complexity of the set family.

We consider three kinds of set families, for each of which we provide subset fairness in conjunction with approximate optimality. The first kind of set family is composed of small sets. Here we show that inverse-proportional allocation coupled with a minimum guaranteed allocation to every advertiser obtains subset fairness. Our second setting consists of set families where each set corresponds to a union over few disjoint groups of advertisers. This captures, for example, settings where advertisers can be partitioned into a few subcategories (of arbitrary sizes) and fairness is desired over arbitrary combinations of these subcategories. Once again a "smoothed" version of inverse-proportional allocation provides fairness and good social welfare. Our last setting is complementary to the second one. Here we assume that advertisers are partitioned into small subcategories and fairness is desired over arbitrary subsets within any single subcategory. This captures settings where only one subcategory of ads may be relevant to any particular user, for example, ads for local services can be geographically categorized. In this setting, we combine inverse-proportional allocation with the proportional allocation algorithm of [12] to obtain guarantees.

Fairness across different types of ads. Throughout our work we assume that different advertisers participating in the system face the same similarity metric over users with respect to which they are required to bid fairly. This assumption makes sense when all ads belong to the same category of product. What if we have different kinds of ads competing, such as a job ad competing against a credit ad? Chawla et al. [12] argue that multiple-task fairness is too strong in such a context and propose a compositional notion of fairness that combines envy-freeness among users across different categories of ads along with multiple-task fairness within each category. We note that the algorithms we develop for a single category of ads combine in a straightforward fashion with [12]'s envy-free algorithms for across-category fairness to produce allocations that satisfy their compositional fairness constraint. In doing so, we significantly improve upon the competitive ratios they achieve, as our guarantees from within-category allocations carry over to the compositional setting.

1.1 Other related work

Fairness in ad auctions. Apart from [12], other recent work has also studied algorithmic fairness in ad auctions. Celis et al. [7] design auctions that provide group fairness guarantees. Nasr and Tschantz [17] consider the complementary approach of modifying advertising bidding strategies to preemptively correct for unfairness introduced by the platform mechanism. These papers consider group fairness guarantees, in contrast to our work, which considers individual fairness guarantees.

Algorithmic fairness and social welfare. Recent work [7, 11, 17] has also considered the relationship between algorithmic fairness notions and social welfare. The fairness in ad auctions papers [7, 17] empirically consider the relationship between social welfare and group fairness guarantees. Hu and Chen [11] consider the relation between algorithmic fairness and social welfare from a theoretical perspective, though with a focus on classification. In contrast, our work provides a theoretical study of the relationship between individual fairness and social welfare in ad auctions.

Fair division. Complementary to algorithmic fairness, there has been an extensive literature on fair division [5] that considers notions including "envy-freeness". In fact, the tension between fairness and social welfare has been considered in the context of fair division [4]. However, a major difference between the algorithmic fairness and envy-freeness is that algorithmic fairness generally focuses on individual qualifications, while envy-freeness generally focuses on individual preferences, though recent work [3, 12, 14, 18] has proposed definitions which combine aspects of envy-freeness and algorithmic fairness notions.

1.2 Outline for the rest of the paper

In Section 2, we present the details of our model for ad auctions along with fairness definitions. In Section 3, we describe our new family of allocation rules that we call inverse-proportional allocation algorithms. We analyze their social welfare and prove that they achieve multiple-task fairness, as well as show that they achieve the optimal tradeoff between social welfare and fairness in some contexts. In Section 4, we consider the stronger notion of set fairness, and develop fair allocation algorithms with social welfare that degrades with the complexity of the set family.

2 MODEL AND DEFINITIONS

We follow the model for ad auctions described in [12]. Users drawn from a universe U arrive in an online fashion. There are k advertisers competing for one ad slot per user. When a user u arrives, the auctioneer runs an auction over the bids submitted by the advertisers and returns a distribution over the advertisers. We use b_u to denote the vector of bids for this user and p_u to denote the distribution returned. An advertiser is then picked from the distribution p_u and the corresponding ad is displayed.

We use the notation p_u^i to denote the ith coordinate of the vector p_u or the probability of allocation to advertiser i for user u. b_u^i likewise denotes the bid of advertiser i for user u. We now describe some desired properties the auction should satisfy, as well as a measure of performance for the quality of allocation.

Incentive compatibility. Interpreting p_u as an allocation function that maps the bid vector b_u for user u to a distribution over advertisers, we require that the function p_u is weakly monotone in bids: for every user u and advertiser i, fixing the bids of other advertisers b_u^{-i}, the allocation to advertiser i, p_u^i, should be a weakly increasing function of the bid of advertiser i, b_u^i. Using the standard payment identity, weak monotonicity implies that the auction can be implemented in an incentive compatible fashion.

Individual and multiple-task fairness. We assume that the universe U is endowed with a distance metric d that captures similarities between different users: the shorter the distance between two users, the more similar they are to each other. We use the notion of individual fairness from [9] to formalize the fairness properties of our auctions. Informally, individual fairness requires that similar users should receive similar outcomes.

Definition 2.1 (Paraphrased from [9]). *A function $f : U \to O$ assigning users to outcomes from a set O is said to be **individually fair** with respect to distance metrics d over U and D over O, if for all $u, v \in U$ we have $D(f(u), f(v)) \leq d(u, v)$.*

In our setting, the space of outcomes is the set of all distributions over advertisers: $O = \Delta([k])$. Accordingly, we need to define fairness with respect to an appropriate distance function over probability distributions. Different choices of the distance function provide different types of guarantees. Dwork and Ilvento [10] proposed using the ℓ_∞ distance over the probability vectors to define fairness. Their **multiple-task fairness** guarantees that the allocations to any single advertiser across two similar users are close. In the following definition, we reinterpret allocations as functions mapping users to distributions over advertisers.

Definition 2.2 (Paraphrased from [10]). *An allocation function $p : U \to \Delta([k])$ satisfies **multiple-task fairness** with respect to distance metric d if for all $u, v \in U$ and $i \in [k]$, we have $|p_u^i - p_v^i| \leq d(u, v)$.*

We observe that the notion of multiple-task fairness can be extended to the case where different advertisers may face different similarity metrics over users based on the features of their ads. For example, the notion of similarity over users for a jobs ad can be very different from the notion of similarity with respect to a housing ad. In such a situation multiple-task fairness requires every advertiser's allocation to be close across similar users with respect to their specific metric. In this paper we focus on the case where all ads competing against each other belong to the same category and therefore face the same distance metric. We refer the reader to [12] for a discussion of the different categories setting. We note that our results in the single category case compose in a straightforward manner with the across-category mechanisms of [12] to provide hybrid guarantees.

For advertisers in a single category, Chawla et al. [12] propose a much stronger fairness guarantee than multiple-task fairness, based on using the ℓ_1 or total variation distance over the probability vectors. **Total variation fairness** guarantees that the allocation to *any set* of advertisers across two similar users are close.

Definition 2.3 (Total-Variation Fairness [12]). *An allocation function $p : U \to \Delta([k])$ satisfies **total-variation fairness** with respect to distance metric d if for all $u, v \in U$ we have $|p_u - p_v|_1 \leq 2d(u, v)$. Equivalently, for all $u, v \in U$ and $S \subseteq [k]$, we have $|\sum_{i \in S} p_u^i - \sum_{i \in S} p_v^i| \leq d(u, v)$.*

Metric obliviousness. Following the work of [12], we require our auctions to be oblivious to the underlying metric d over users. The metric d is implicitly encoded in advertisers' bids through a bid fairness constraint to be discussed in the following subsection. We require that the outcomes of our auctions satisfy fairness if the input bids are fair without explicitly checking or ensuring the corresponding fairness constraints.

This approach enables a separation of responsibility: the advertisers are held accountable for bidding fairly, while the platform is responsible for ensuring the allocation is fair as long as the bids are fair. In particular, our work calls for a separation between the tasks of (1) determining the suitability/relevance of different ads for a users; (2) auditing advertisers for fair bidding; and (3) running an auction to match an ad to an ad slot. The first and second tasks require knowledge of fairness requirements and should ensure that the input to the third task is fair in itself. We focus on the third task. While allowing the auction to consider the fairness constraints explicitly may allow for better outcomes, we believe it is undesirable for the auctioneer (or ad exchange) to possess and utilize such information. In fact in some contexts, the similarity metric over users may itself be sensitive information that an auctioneer is not allowed to explicitly use in decision making.

Symmetry and history obliviousness. In the spirit of designing simple mechanisms, we advocate for auctions that are history-oblivious: the allocation to every users is independent of the users that have previously arrived in the system and allocations made to those users. Another desirable property is symmetry across advertisers: permuting the bids of the advertisers permutes their allocations.

Social welfare. In this work we focus on the objective of social welfare maximization. Since we focus on the design of incentive compatible mechanisms, we may assume that advertisers bid their true values. The maximum social welfare achievable for any single user u is therefore $\max_i b_u^i$. On the other hand, the social welfare achieved by the allocation p_u is given by $p_u \cdot b_u$. We say that an auction achieves a competitive ratio of $\alpha < 1$ for social welfare if for any sequence of users u and bids b_u, we have:

$$\sum_u p_u \cdot b_u \geq \alpha \sum_u \max_i b_u^i.$$

2.1 Fairness constraint on advertiser values.

Informally, we require that for any two users u and v that are similar to each other (close under the distance metric d), every advertiser should bid similar amounts on the two users. If the distance between u and v is 0, the users should be treated identically and every advertiser should submit exactly the same bid on the two. On the other hand, if the distance between u and v is 1, the maximum possible, then no constraint is imposed: the bids on the two users can be completely arbitrary. For intermediate distances, we follow the approach of [12] and require that the ratio of the bids b_u^i/b_v^i is bounded by some function f of the distance between u and v.

Definition 2.4 ([12]). *A* **bid ratio constraint** *is a continuous function $f : [0,1] \to [1,\infty]$ with $f(0) = 1$ and $f(1) = \infty$. We say that the bid function b^i of advertiser i satisfies the bid ratio constraint f with respect to metric d if we have for all $u, v \in U$: $\frac{1}{f(d(u,v))} \leq \frac{b_u^i}{b_v^i} \leq f(d(u,v))$.*

Observe that the smaller the function f is, the tighter is the constraint on the bids, and the easier it should be for the auction to achieve fairness. Following [12], we characterize the strength of the fairness constraint on bids by considering an explicit family $\mathcal{F}$ of functions parameterized by a parameter $\ell \in (0, \infty)$: $f_\ell(d) = (1-d)^{-1/\ell}$.[1]

Observe that larger values of ℓ correspond to stronger conditions on bids; see Figure 1a. Furthermore, for any continuous function $h : [0,1] \to [1,\infty]$ with $h(0) = 1$, the family $\mathcal{F}$ contains a function f_ℓ such that $h(d) \le f_\ell(d)$ for all $d \in [0,1]$. In this respect, the family $\mathcal{F}$ is fully expressive. We will describe our results in terms of the largest (strongest) parameter ℓ for which bids satisfy the bid ratio constraint f_ℓ. Accordingly, we define the degree of fairness of an advertiser's bids as follows.

Definition 2.5. *We say that the bid function of an advertiser i satisfies* **degree-ℓ fairness** *if ℓ is the largest value for which the bid function satisfies the bid ratio constraint f_ℓ.*

2.2 Subset fairness

The mechanisms we design satisfy multiple-task fairness but not total-variation fairness. Chawla et al. [12] observe that in some settings multiple-task fairness is too weak: two users may receive similar allocations of any individual advertisement, but if many advertisements of the same kind are clubbed together, the two users may see an ad from this group with vastly different probabilities. This behavior can be considered discriminatory. Chawla et al.'s approach is to impose fairness with respect to arbitrary such groups of users. We take a more nuanced approach and consider notions of fairness that interpolate between multiple-task fairness and total-variation fairness. Specifically, we require the algorithm to satisfy fairness with respect to a restricted family of sets over advertisers. We also consider allowing for Lipschitz relaxations of the fairness constraints.

Definition 2.6. *Let $\mathcal{C}$ be a collection of subsets of $[k]$. An allocation p satisfies* ***subset fairness*** *with respect to $\mathcal{C}$ if for all $C \in \mathcal{C}$, it holds that*

$$\sup_{C \in \mathcal{C}} \left| \left(\sum_{i \in C} p_u^i \right) - \left(\sum_{i \in C} p_v^i \right) \right| \le d(u,v).$$

More generally, for a parameter $\alpha \ge 1$, we say that p satisfies α-subset fairness if

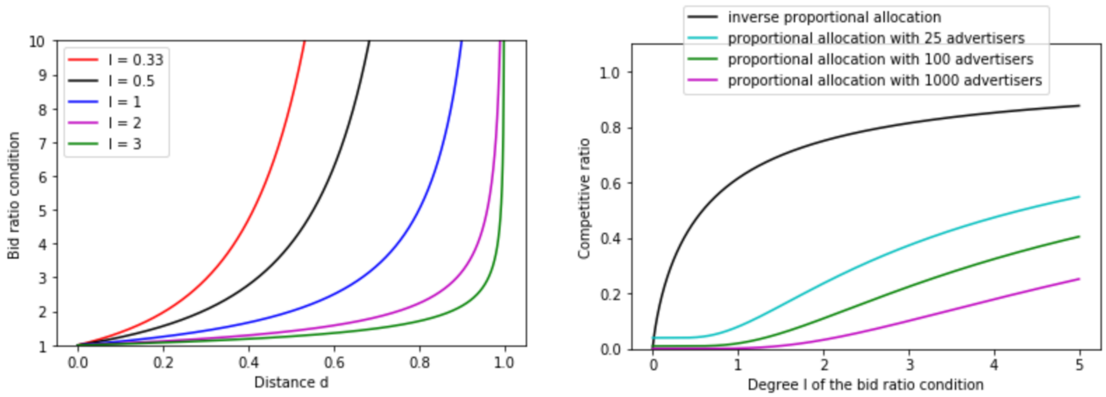
$$\sup_{C \in \mathcal{C}} \left| \left(\sum_{i \in C} p_u^i \right) - \left(\sum_{i \in C} p_v^i \right) \right| \le \alpha d(u,v).$$

Notice that when $\mathcal{C}$ contains all subsets of $[k]$, then this definition is identical to total-variation fairness. When $\mathcal{C}$ only consists of subsets of size 1, then this definition reduces to multiple-task fairness. For intermediate collections $\mathcal{C}$, the definition is stronger than multiple-task fairness but not as strong as total-variation fairness. Observe that we do not include singleton sets by default in $\mathcal{C}$. Our intention is for the algorithm to satisfy subset fairness in addition to multiple-task fairness.

3 INVERSE PROPORTIONAL ALLOCATION ALGORITHMS

In this section, we introduce a new family of auctions that achieve a good tradeoff between fairness and social welfare. To our knowledge, the class of allocation rules we introduce has not been studied previously. The allocation rule we design is metric-oblivious but requires knowing the degree ℓ (i.e. strength) of the fairness constraint imposed on the

[1] We remark that the precise form of the bid ratio function in [12] is $\left(\frac{1+d}{1-d}\right)^{1/\ell}$. The two functions are related to each other within small constant factors, and the form we use is mathematically more convenient for our proofs.



(a) Degree ℓ bid ratio condition

(b) Competitive ratio of inverse proportional allocation algorithms and of proportional allocation algorithms

Fig. 1. Bid ratio conditions and competitive ratio

advertisers' bids. We introduce the family of inverse-proportional allocations in Section 3.1. In Section 3.2 we analyze the social welfare of inverse-proportional allocations. Importantly, the competitive ratios we achieve depend only on the degree ℓ of the fairness constraint, and do not decay with the number of advertisers k. In Section 3.3 we prove that inverse-proportional allocations achieve multiple-task fairness. In Section 3.4 we provide upper bounds showing that inverse-proportional allocation achieves the optimal tradeoff between fairness and social welfare in some contexts.

3.1 Definition and basic properties of inverse-proportional allocation algorithms

The main contribution of this paper is a new class of allocation functions that we call inverse-proportional allocation. Let us begin with an informal description of our mechanism. When a user u arrives to the platform, we first assign every advertiser an allocation of 1 (i.e. the full slot). Of course, this results in a total allocation of k units rather than the required total allocation of 1 unit, so we need to take away $k-1$ units of allocation. The core intuition is that we take away these units *proportionally with a decreasing function of the bids*. We compute weights $w_u^1, \ldots, w_u^k$ that are functions of the bids on user u, and then set p_u^i to be $1 - w_u^i$. In terms of fairness, this means that if two users u and v are close, then for any given advertiser i, the bids b_u^i and b_v^i will be close, so the weights w_u^i and w_v^i will be close, and thus the allocations p_u^i and p_v^i will be close. In terms of utility, relatively high bids will receive low weights, so a large portion of the allocation will be assigned to the highest bids.

More specifically, our allocation algorithm is parameterized by a decreasing function $g : \mathbb{R}^{\geq 0} \to (0, \infty]$, where $g(0) = \infty$ and $\lim_{x\to\infty} g(x) = 0$. We then define the weights as $w_u^i \propto g(b_u^i)$ with $\sum_i w_u^i = k-1$. However, there is a problem: when b_u^i is very small, it may turn out to be the case that $w_u^i > 1$, and $1 - w_u^i$ is negative. In this case, we assign an allocation of 0 to the corresponding advertiser, and recurse to find an allocation over the remaining advertisers.

Remark 3.1. *Notice that when $|S| = 2$, every advertiser $i \in S$ has $w_i = \frac{g(b_u^i)}{\sum_{j\in S} g(b_u^j)} < 1$. So both advertisers will receive nonzero allocations. Likewise, whenever $k \geq 2$, at least two advertisers will be assigned nonzero allocation probabilities. More specifically, the inverse-proportional allocation algorithm assigns nonzero allocation probabilities to exactly the K advertisers with the highest bids, for some* $\min(k, 2) \leq K \leq k$.

Algorithm 1: Inverse-Proportional Allocation with parameter g

Input : Function $g : \mathbb{R}^{\geq 0} \to (0, \infty]$ with $g(0) = \infty$ and $\lim_{x \to \infty} g(x) = 0$. Bids $b_u^1, \ldots, b_u^k$.

Initialize $S = [k]$.

For each $i \in S$, set $w_i = (|S| - 1) \frac{g(b_u^i)}{\sum_{j \in S} g(b_u^j)}$.

while $\exists j \in S$ *with* $w_j > 1$ **do**

  Let $J = \{j \in S : w_j > 1\}$.

  Set $S = S \setminus J$. Set $w_j = 1$ for all $j \in J$.

  For each $i \in S$, set $w_i = (|S| - 1) \frac{g(b_u^i)}{\sum_{j \in S} g(b_u^j)}$.

end

Set $p_u^i = 1 - w_i$ for all $i \in [k]$.

The inverse-proportional allocation algorithm satisfies all of the desirable properties discussed in Section 2. First, it is immediate from the definition of the algorithm that it is metric oblivious, history oblivious, and symmetric. We will now formally prove incentive compatibility. The following lemma shows that the allocation assigned to any advertiser is weakly monotone increasing in her own bid and weakly monotone decreasing in other advertisers' bids.

Lemma 3.2. *Consider any advertiser $1 \leq i \leq k$, and let b_u and b_v be bid vectors such that $b_u^j = b_v^j$ for all $j \neq i$ and $b_u^i > b_v^i$. Then it holds that $p_u^i \geq p_v^i$ and $p_u^j \leq p_v^j$ for all $j \neq i$.*

Lemma 3.2, by guaranteeing monotonicity, implies that the algorithm can be implemented as an incentive-compatible mechanism using an appropriately designed payment rule.

3.2 The social welfare of inverse-proportional allocation

We now show that the inverse-proportional allocation algorithm achieves good social welfare.

Theorem 3.3. *For any $\ell \in (0, \infty)$, the inverse-proportional allocation algorithm with k advertisers and parameter $g(x) = x^{-\ell}$ obtains a competitive ratio for social welfare of at least*

$$1 - \frac{1}{\ell + 1} \left(\frac{\ell}{\ell + 1} \right)^{\ell}.$$

Observe that at $\ell = 1$, this gives us a competitive ratio of $3/4$.[2] As $\ell \to \infty$, the competitive ratio goes to 1, while as $\ell \to 0$, the ratio goes to 0. Observe also that the competitive ratio is independent of the number of advertisers k. This is a marked improvement over the proportional allocation algorithms in [12], whose competitive ratio degrades with the number of advertisers, approaching 0 in the limit as k goes to ∞. (See Figure 1b for a depiction of the growth rate as a function of ℓ.) Our improvement makes these algorithms practical in real-life settings where platforms service a large number of advertisers in a given category (e.g. employment).

The intuition for the bound $1 - \frac{1}{\ell+1} \left(\frac{\ell}{\ell+1} \right)^{\ell}$ can be seen by considering a bid vector of the form $[1, b, b, \ldots, b]$ for $b < 1$. For the inverse-proportional allocation algorithm with parameter $g(x) = x^{-\ell}$, a simple calculation shows that in the limit as $k \to \infty$, the probability assigned to the bid of 1 approaches $1 - b^{\ell}$. Thus, the competitive ratio becomes

[2] In the case of $\ell = 1$, the inverse-proportional allocation algorithm coincidentally arises as a full-information Nash equilibrium of an algorithm that allocates proportionally to bids with an all-pay payment rule. These mechanisms are considered in [6, 8, 13], and, in this context, the competitive ratio was shown to be at least $3/4$ (for the case of $\ell = 1$). Our bound in Theorem 3.3 is more general in that it holds for all $\ell \in (0, \infty)$.

$1 - b^\ell + b^{\ell+1}$. A simple calculation shows that $1 - \frac{1}{\ell+1}\left(\frac{\ell}{\ell+1}\right)^\ell$ is the minimum value that $1 - b^\ell + b^{\ell+1}$ can take over the interval $[0, 1]$. The proof of Theorem 3.3 (deferred to Appendix A) formalizes this intuition by showing that $[1, b, b, \ldots, b]$ is the "worst-case" for bids vectors in terms of competitive ratio.

3.3 Fairness of inverse-proportional allocation algorithms

We will now establish the fairness properties of the inverse-proportional allocation algorithm. Recall that we may assume that the bids provided as input to the algorithm satisfy degree-ℓ fairness. We will show that for an appropriate choice of g, the algorithm returns allocations satisfying multiple-task fairness. We choose $g(x) = x^{-\ell/2}$.

Theorem 3.4. *Suppose that there are k advertisers. Then, for any $\ell > 0$, the inverse-proportional allocation algorithm with parameter $g(x) = x^{-\ell/2}$ satisfies multiple-task fairness when every advertiser's bid function satisfies degree-ℓ fairness.*

Our high-level approach to prove Theorem 3.4 is as follows. Our goal is to show that for any users u and v, if the bid vectors b_u and b_v satisfy degree-ℓ fairness, then for every advertiser i we have $p_u^i - p_v^i \leq d(u, v)$. To prove this, let us fix a single user u, a distance $d \in (0, 1)$, and an advertiser i. We will construct a worst-case bid vector b_v such that p_v^i is as small as possible while respecting the bid ratio constraint for every $j \in [k]$ at distance $d = d(u, v)$. The monotonicity properties given in Lemma 3.2 enable us to identify such a bid vector: we decrease the bid of advertiser i by a factor of $f_\ell(d)$ and increase the bids of all other advertisers by a factor of $f_\ell(d)$.

Now, we need to bound $p_u^i - p_v^i$ between these two users. The challenge is that the sets of advertisers that remain after the removal process in Step (5) of the algorithm might be different for these two users. Nonetheless, we show that we can reduce to the case where the users have the same set S of advertisers in the weight-assignment procedure, using the intuition that reintroducing the advertisers that were removed weakly increases the allocation on the other advertisers. We then handle the weight-assignment procedure directly.

More formally, we show the following two propositions. In the first proposition, we show that reintroducing advertisers that were removed in the removal process weakly increases the allocation of the remaining advertisers. (The proof of this proposition can be found in Appendix A.)

Proposition 3.5. *Consider a user u, and suppose that $b_u^1 \leq b_u^2 \leq \ldots \leq b_u^k$. Let R_1 be the set of advertisers with nonzero allocation returned by the inverse-proportional allocation algorithm with parameter g on the bid vector b_u. Let R_2 be any subset of $[k]$ of the form $\{z, z+1, \ldots, k\}$ for some $z \in [k]$. If $R_1 \subseteq R_2$, then for any $i \in S$:*

$$1 - (|R_1| - 1)\frac{g(b_u^i)}{\sum_{j \in R_1} g(b_u^j)} \leq 1 - (|R_2| - 1)\frac{g(b_u^i)}{\sum_{j \in R_2} g(b_u^j)}.$$

In the second proposition, we show the fairness of the inverse proportional assignment on any fixed set of advertisers in the absence of the removal process.

Proposition 3.6. *Let $\ell \in (0, \infty)$, $d \in (0, 1)$, and $g(x) = x^{-\ell/2}$ for all $x \geq 0$. Let $\alpha = f_\ell(d) = (1-d)^{-1/\ell}$. Consider bid vectors b_u and b_v, and a set $S \subseteq [k]$ such that $b_v^i/\alpha \leq b_u^i \leq b_v^i \alpha$ for all $i \in S$. Then for every $i \in S$, we have:*

$$\max\left(0, 1 - (|S| - 1)\frac{g(b_u^i)}{\sum_{j \in S} g(b_u^j)}\right) - \max\left(0, 1 - (|S| - 1)\frac{g(b_v^i)}{\sum_{j \in S} g(b_u^j)}\right) \leq d.$$

Proof of Proposition 3.6. Let $D := \max\left(0, 1 - \frac{(|S|-1)g(b_u^i)}{\sum_{j\in S} g(b_u^j)}\right) - \max\left(0, 1 - \frac{(|S|-1)g(b_v^i)}{\sum_{j\in S} g(b_u^j)}\right)$. We need to show that, if $b_v^i/\alpha \leq b_u^i \leq b_v^i\alpha$, it holds that $D \leq d$. If $|S| = 1$, notice that $\max\left(0, 1 - \frac{(|S|-1)g(b_u^i)}{\sum_{j\in S} g(b_u^j)}\right) - \max\left(0, 1 - \frac{(|S|-1)g(b_v^i)}{\sum_{j\in S} g(b_u^j)}\right) = 0$, so the statement trivially holds. Thus, we can assume that $|S| \geq 2$ for the remainder of the proof.

Let $W_u^j = g(b_u^j)/g(b_u^i)$ and $W_v^j = g(b_u^j)/g(b_u^i)$. Then we have that:

$$\begin{aligned}
D &= \max\left(0, 1 - \frac{(|S|-1)g(b_u^i)}{\sum_{j\in S} g(b_u^j)}\right) - \max\left(0, 1 - \frac{(|S|-1)g(b_v^i)}{\sum_{j\in S} g(b_u^j)}\right)\\
&= \max\left(0, 1 - \frac{|S|-1}{\sum_{j\in S} W_u^j}\right) - \max\left(0, 1 - \frac{|S|-1}{\sum_{j\in S} W_v^j}\right)\\
&= \max\left(0, 1 - \frac{|S|-1}{1+\sum_{j\in S, j\neq i} W_u^j}\right) - \max\left(0, 1 - \frac{|S|-1}{1+\sum_{j\in S, j\neq i} W_v^j}\right)\\
&= (|S|-1)\left[\min\left(\frac{1}{|S|-1}, \frac{1}{1+\sum_{j\in S, j\neq i} W_v^j}\right) - \min\left(\frac{1}{|S|-1}, \frac{1}{1+\sum_{j\in S, j\neq i} W_u^j}\right)\right].
\end{aligned}$$

Let $B = \sum_{j\in S, j\neq i} W_u^j$. Notice that if $\min\left(\frac{1}{|S|-1}, \frac{1}{1+B}\right) = \frac{1}{|S|-1}$, then $D \leq 0$, so the statement follows trivially. Thus, we can assume that $\min\left(\frac{1}{|S|-1}, \frac{1}{1+B}\right) = \frac{1}{1+B}$ for the remainder of the proof. Equivalently, we can assume that $B \geq |S|-2$. In this notation, we can express:

$$D = (|S|-1)\left[\min\left(\frac{1}{|S|-1}, \frac{1}{1+\sum_{j\in S, j\neq i} W_v^j}\right) - \frac{1}{1+B}\right].$$

From the conditions on the bid vector, it follows that $\frac{W_v^j}{W_u^j} \geq \frac{g(b_v^j)/g(b_v^i)}{g(b_u^j)/g(b_u^i)} = \frac{g(b_v^j)}{g(b_u^j)}\frac{g(b_u^i)}{g(b_v^i)} \geq \alpha^{-\ell}$. Thus, we have that $W_v^j \geq W_u^j\alpha^{-\ell}$, and so $\sum_{j\in S, j\neq i} W_v^j \geq B\alpha^{-\ell}$, so:

$$\begin{aligned}
D &\leq (|S|-1)\left[\min\left(\frac{1}{|S|-1}, \frac{1}{1+B\alpha^{-\ell}}\right) - \frac{1}{1+B}\right]\\
&= (|S|-1)\left[\frac{1}{1+\max(B\alpha^{-\ell}, |S|-2)} - \frac{1}{1+B}\right].
\end{aligned}$$

Let $D'(B) := (|S|-1)\left[\frac{1}{1+\max(B\alpha^{-\ell}, |S|-2)} - \frac{1}{1+B}\right]$. Then, we know that $D \leq \max_{|B|\geq|S|-2} D'(B)$. We show that $\max_{|B|\geq|S|-2} D'(B) \leq d$.

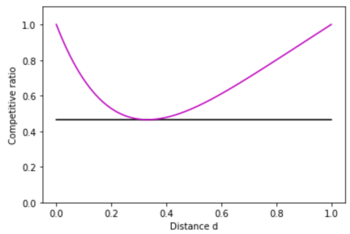
First, let's handle the case where $|S| = 2$. In this case, we have that $D'(B) = \frac{1}{1+\max(B\alpha^{-\ell},0)} - \frac{1}{1+B} = \frac{1}{1+B\alpha^{-\ell}} - \frac{1}{1+B}$. Applying Proposition A.2, we see $\frac{1}{1+B\alpha^{-\ell}} - \frac{1}{1+B}$ is maximized at $B^* = \alpha^{\ell/2}$, so $\max_{|B|\geq|S|-2} D'(B) = D'(B^*)$. At this value, we obtain:

$$D'(B^*) = \frac{1}{1+B^*\alpha^{-\ell}} - \frac{1}{1+B^*} = \frac{1}{1+\alpha^{-\ell/2}} - \frac{1}{\alpha^{\ell}+1} = \frac{\alpha^{\ell/2}}{1+\alpha^{\ell/2}} - \frac{1}{\alpha^{\ell/2}+1} = \frac{\alpha^{\ell/2}-1}{1+\alpha^{\ell/2}}.$$

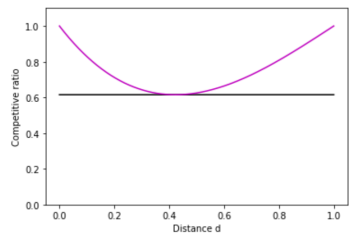
Notice that when $\alpha = f_\ell(d) \leq f_{\ell/2}(d)$, this expression will be upper bounded by d, as desired.

Now, let's handle the case where $|S| \geq 3$. Suppose first that $B \leq \alpha^{\ell}(|S|-2)$. In this case, we know that $\max(B\alpha^{-\ell}, |S|-2) = |S|-2$. Thus:

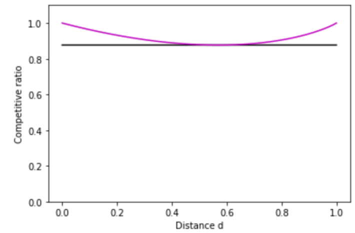
$$D'(B) = (|S|-1)\left[\frac{1}{|S|-1} - \frac{1}{1+\max(B, |S|-2)}\right].$$



(a) $\ell = 0.5$



(b) $\ell = 1$



(c) $\ell = 5$

Fig. 2. Comparison of inverse-proportional allocation algorithms with multiple-task fairness upper bounds as $k \to \infty$ for bid ratio conditions f_ℓ

In this regime, notice that $D'(B)$ is increasing as a function of B. Thus $\max_{|S|-2 \leq |B| \leq \alpha^\ell(|S|-2)} D'(B) \leq D'(\alpha^\ell(|S|-2))$.

Thus, it suffices to consider $\max_{|B| \geq \alpha^\ell(|S|-2)} D'(B)$. Now, using Proposition A.2, we know that $D'(B) = (|S| - 1)\left[\frac{1}{1+B\alpha^{-\ell}} - \frac{1}{1+B}\right]$ is non-increasing on $[\alpha^\ell(|S|-2), \infty)$, so $\max_{|B| \geq \alpha^\ell(|S|-2)} D'(B) \leq D'(\alpha^\ell(|S|-2))$. Hence, it suffices to show that $D'(\alpha^\ell(|S|-2)) \leq d$. Notice that:

$$\begin{aligned} D'(\alpha^\ell(|S|-2)) &= (|S|-1)\left[\frac{1}{1+\max(B^*\alpha^{-\ell}, |S|-2)} - \frac{1}{1+B^*}\right] \\ &\leq (|S|-1)\left[\frac{1}{1+|S|-2} - \frac{1}{1+(|S|-2)\alpha^\ell}\right] \\ &= 1 - \frac{|S|-1}{1+(|S|-2)\alpha^\ell}. \end{aligned}$$

Notice that when $\alpha \leq f_\ell(d)$, this expression is bounded by d as desired. □

Propositions 3.5 and 3.6 along with Lemma 3.2, provide us with the necessary ingredients to prove Theorem 3.4, as we described earlier. We defer the proof to Appendix A.

3.4 Optimality of inverse-proportional allocation

Finally we show that the inverse-proportional allocation algorithm with parameter $g(x) = x^{-\ell/2}$ achieves the optimal competitive ratio achievable by any online allocation algorithm that given degree-ℓ fair bids satisfies multiple-task fairness.

To state our upper bound, let us fix the fairness parameter ℓ, the number of advertisers k, the universe U, and a metric d over U. We may then design an adversary that determines the order in which users arrive, as well as bid vectors assigned to each user that collectively respect degree-ℓ fairness with respect to the metric d, so as to force any fair online allocation algorithm to obtain low social welfare. Let $\mathcal{U}(\ell, k, d)$ denote the maximum competitive ratio achievable by any algorithm given such an adversary. This quantity is an upper bound on the performance of any online allocation algorithm for the fair ad auction problem. Since we are interested in designing metric oblivious allocation algorithms, we further consider the worst case such bound across all possible distance metrics over U: $\mathcal{U}(\ell, k, d) = \inf_{d \in [0,1]} \mathcal{U}(\ell, k)$.

We show that the performance of inverse-proportional allocation essentially matches this upper bound for *every* value of ℓ. In particular, denoting the worst case competitive ratio of inverse-proportional allocation over k advertisers by $\mathcal{L}(\ell, k)$, we prove that $\mathcal{L}(\ell, k) \geq \mathcal{U}(\ell, k) - 1/k$. Further, recalling that our bound on the performance of inverse-proportional allocation does not depend on the number of advertisers k, this implies that $\mathcal{L}(\ell) = \lim_{k \to \infty} \mathcal{U}(\ell, k)$.

Theorem 3.7. *For $0 < \ell < \infty$, let $\mathcal{L}(\ell)$ denote the competitive ratio achieved by the inverse-proportional allocation algorithm with parameter $g(x) = x^{-\ell/2}$ on any instance satisfying degree-ℓ fairness. Let $\mathcal{U}(\ell, k, d)$ denote the maximum competitive ratio achievable by any multiple-task fair online allocation algorithm over an instance with k advertisers and uniform distance metric d where the bids satisfy degree-ℓ fairness . Let $\mathcal{U}(\ell, k) = \inf_{d\in[0,1]} \mathcal{U}(\ell, k, d)$ denote the upper bound corresponding to the worst-case metric. Then we have:*

$$\mathcal{L}(\ell) \geq \mathcal{U}(\ell, k) - 1/k \quad \forall k, \tag{1}$$

$$\mathcal{L}(\ell) = \lim_{k\to\infty} \mathcal{U}(\ell, k). \tag{2}$$

Our main technical ingredient for this optimality result is an upper bound from [12] on the competitive ratio of any multiple-task fair online algorithm. Chawla et al. [12] establish their upper bounds on instances with the uniform metric, where for all u, v, we have $d(u, v) = d$ for some parameter $d \in (0, 1)$.

Lemma 3.8 ([12]). *Let $d \in [0, 1 - 1/k]$ and $\alpha = f_\ell(d)$. Then there exists an instance of the fair ad auction problem over the uniform metric with $d(u, v) = d$ for all $u, v \in U$, along with bid vectors satisfying degree-ℓ fairness, such that no online allocation mechanism satisfying multiple-task fairness can obtain a competitive ratio better than $\left(\frac{1}{k} + d + \alpha^{-2}\left(1 - \frac{1}{k} - d\right)\right)$.*

Figure 2 illustrates the upper bound $\lim_{k\to\infty} \mathcal{U}(\ell, k, d)$ provided by Lemma 3.8 as k tends to ∞ over a uniform distance metric and for different values of ℓ. We observe that at each value of ℓ, there is some distance d for which $\lim_{k\to\infty} \mathcal{U}(\ell, k, d)$ exactly coincides with the competitive ratio achieved by inverse-proportional allocation. The proof of Theorem 3.7 formalizes this observation.

A natural question is whether it is possible to obtain better guarantees if bids satisfy a different form of the bid ratio constraint. We show that this is not possible using inverse proportional allocation. Specifically, we show in Appendix C that in order for any inverse-proportional allocation algorithm (with a different choice of the function g) to match the fairness and welfare guarantees we achieve, bids must essentially satisfy the bid ratio constraint $f_{\ell/c}$ for some constant c.

4 SUBSET FAIRNESS

We now turn to the notion of subset fairness defined in Section 2. We first argue that subset fairness comes at a cost: the inverse-proportional allocation algorithm defined in Section 3 satisfies multiple-task fairness, but we show that it does not satisfy total-variation fairness. In fact, there is a performance gap between algorithms that satisfy total-variation fairness and those that only satisfy multiple-task fairness, as we show in Section 4.1. In Section 4.2 we develop variants of the inverse-proportional allocation algorithm so as to satisfy fairness with respect to nice classes of subsets while also maintaining a good competitive ratio for social welfare.

4.1 A gap between multiple-task fairness and total-variation fairness

We begin by showing that the class of inverse-proportional allocation algorithms described in Section 3 does not satisfy total-variation fairness.

Example 4.1. *Let k be any even number, and consider two users u and v at some distance d from each other. Suppose that for user u the algorithm receives bids of 1 from advertisers $1, \cdots, k/2$ and b from advertisers $k/2 + 1, \cdots, k$; whereas for user v the algorithm receives bids of b from advertisers $1, \cdots, k/2$ and 1 from advertisers $k/2 + 1, \cdots, k$. Then, setting $b = 1/f_\ell(d)$ ensures that the bid vectors satisfy degree-ℓ fairness.*

We now observe that for some (not too small) value of b, the inverse-proportional allocation algorithm with parameter $\ell/2$ will assign $p_v^i = 0$ for all $1 \leq i \leq k/2$ and $p_u^j = 0$ for all $k/2+1 \leq j \leq k$. As result, it would hold that $\sum_{i=1}^{k/2-1} p_u^i = 1$ and $\sum_{i=1}^{k/2-1} p_v^i = 0$. In fact it suffices to set $d = 1 - \left(\frac{k/2-1}{k/2}\right)^2 \approx 4/k$ and correspondingly set $b = \left(\frac{k/2-1}{k/2}\right)^{2/\ell}$.[3]

Example 4.1 shows that even for two users u and v that are very close to each other, there may exist a subset of advertisers on which u receives an allocation of 1 while v receives an allocation of 0. Thus, total-variation fairness is violated with respect to any reasonable Lipschitz relaxation. Is it possible to construct a different allocation algorithm that is total-variation fair but performs as well as inverse-proportional allocation on social welfare? We show that this is not possible for small values of ℓ: there is a gap between the competitive ratio achievable by any total-variation fair algorithm and the competitive ratio of the inverse-proportional algorithm.

Theorem 4.2. *For any $0 < \ell < \infty$, let $T(k,\ell)$ be the optimal competitive ratio achievable by any total-variation fair online allocation algorithm (that is also metric-oblivious, history-oblivious, and symmetric) over an instance with k advertisers and bids satisfying degree-ℓ fairness. Let $\mathcal{L}(\ell)$ denote the competitive ratio achieved by the inverse-proportional allocation algorithm with parameter $g(x) = x^{-\ell/2}$ on any instance satisfying degree-ℓ fairness. Then, it holds that* $\limsup_{k\to\infty} T(k,\ell) \leq \frac{\ell}{\ell+1}$ *and* $\liminf_{k\to\infty} \frac{\mathcal{L}(\ell)}{T(k,\ell)} \geq \frac{1}{\ell}\left(1 - \left(1 - \frac{1}{\ell/2+1}\right)^{\ell/2}\right)$. *In particular,* $\liminf_{\ell\to 0}\left(\liminf_{k\to\infty} \frac{\mathcal{L}(\ell)}{T(k,\ell)}\right) = \infty$.

4.2 Subset fairness over structured set families

Given the gap in Theorem 4.2, we ask whether good competitive ratios can be achieved if fairness is only required over some nice structured family of sets of advertisers as opposed to arbitrary subsets of advertisers. We show in the remainder of this section that such improved guarantees are indeed possible. In this section, we describe and motivate three kinds of set families that enable positive results.

As a warm-up, one might ask whether the challenge with ensuring total-variation fairness is that there are exponentially many fairness constraints to satisfy – one for each possible subset of advertisers. This turns out to not be the case. Indeed, the proof of Theorem 4.2 relies on requiring subset fair allocation with respect to a collection of only $\Theta(\log k)$ subsets of advertisers.

Our first observation is that subset fairness over sets of small size can be achieved without much loss in performance. To formalize this, we define the “bandwidth” of a family of sets—the size of the largest set in the family—as a measure of complexity of the set system:

Definition 4.3. *The* width *of a collection $\mathcal{C} \subset 2^{[k]}$ of subsets of advertisers, denoted ω, is defined as* $\max_{C\in\mathcal{C}} |C|$.

While families of small sets may not be very interesting in themselves, this definition leads us to a more nuanced notion of complexity of set systems. Given a set system $\mathcal{C} \subset 2^{[k]}$, we will say that two advertisers i and j are equivalent if they belong to exactly the same sets in $\mathcal{C}$: for all $C \in \mathcal{C}$, $i \in C \iff j \in C$. This partitions advertisers into equivalence classes or clusters. Let $L_1, L_2, \cdots$ denote these clusters. Every set $C \in \mathcal{C}$ is then the union of some subset of the clusters. We will define the “cluster bandwidth” of a family of sets, denoted ω_{cluster}, as the maximum over all sets $C \in \mathcal{C}$ of the number of clusters contained in C.

Definition 4.4. *Given the partition $\{L_1, L_2, \cdots\}$ of advertisers into clusters, as described above, the* cluster bandwidth *of a collection $\mathcal{C} \subset 2^{[k]}$ of subsets of advertisers, denoted $\omega_{cluster}(\mathcal{C})$, is defined as* $\max_{C\in\mathcal{C}} |\{i : L_i \subset C\}|$.

[3]With these values, we have $(k-1)\frac{(k/2)/(k/2-1)}{k/2+(k/2)((k/2)/(k/2-1))} = 1$, and so the advertisers with bids b will get dropped.

Many interesting set families can have low cluster bandwidth. For example, when the sets $C \in \mathcal{C}$ are disjoint, the cluster bandwidth of the family is simply 1. Likewise, when advertisers have few relevant attributes, the cluster bandwidth is no larger than the number of different values the attributes can take. For example, suppose that we classify job ads according to whether they are high pay, medium pay, or low pay jobs, and whether they are tech sector, or finance, or academic jobs. Then there are nine possible kinds of ads and the sets in our collection may correspond to some subset of these nine types. Then, the cluster bandwidth of this set family is no more than nine.

Finally, we consider settings where the subset fairness constraints apply only over small sub-categories of advertisers. For example, we may be interested in providing fairness guarantees across arbitrary sets of job ads relevant to a single geographical area. In this case, we would partition the set of all advertisers according to geographic location, and then enforce fairness over arbitrary subsets that lie entirely within a single component of the partition. The performance of our allocation algorithm will then depend on the sizes of the components of the partition.

Definition 4.5. *Let $\{L_1, L_2, \cdots\}$ be a partition of advertisers into clusters. Let $\mathcal{C} \subset 2^{[k]}$ be a set family where for each set $C \in \mathcal{C}$, there exists an index i with $C \subset L_i$. We define the* partitioned width *of $\mathcal{C}$, denoted $\omega_{part}(\mathcal{C})$, as $\max_i |L_i|$.*

4.2.1 Algorithms achieving subset fairness. We now construct variants of inverse-proportional allocation that are competitive for social welfare while satisfying subset fairness over small-width set families as defined in the previous subsection. We use three building blocks for our constructions, put together in novel ways: the inverse-proportional allocation algorithm, the proportional allocation algorithm of [12], and uniform allocation that assigns equal allocation probability to all advertisers. As described previously, inverse-proportional allocation aggressively favors high bidders but hurts advertisers with low bids when these are numerous. Proportional allocation, on the other hand, hurts advertisers with high bids when there are too many low bids. Uniform allocation allows for evening out the differences in allocation probabilities at the expense of losing out on performance. Accordingly, the three types of allocation rules address different aspects of fair allocation. However, putting them requires care so as to not magnify differences in allocation across different users.

We first describe a hybrid allocation algorithm designed to achieve subset fairness across small sets.

Algorithm 4.6. *The **modified inverse-proportional allocation algorithm with parameters** g **and** T applies the inverse-proportional algorithm with parameter g with probability $1/T$ and uniformly assigns with allocation across all advertisers with probability $1 - 1/T$.*

We show that the modified inverse-proportional allocation satisfies subset fairness on all subsets of size T and achieves at least a $1/T$ fraction of the competitive ratio of the corresponding inverse proportional allocation algorithm.

Theorem 4.7. *Let $\mathcal{C}$ be a collection such that $\omega(\mathcal{C}) \le T$. For any $0 < \ell < \infty$, the modified inverse-proportional allocation algorithm with parameters $g(x) = x^{-\ell/2}$ and T achieves subset fairness with respect to $\mathcal{C}$ as well as multiple-task fairness when bids satisfy degree-ℓ fairness, and achieves a competitive ratio of at least $\frac{1}{T}\left(1 - \frac{1}{(\ell/2)+1}\left(\frac{(\ell/2)}{(\ell/2)+1}\right)^{\ell/2}\right)$.*

Proof. Consider two users u and v, and suppose the bid ratio condition f_ℓ is satisfied. For fairness, by Theorem 3.4, we know that the inverse-proportional allocation algorithm with parameter $g(x) = x^{-\ell/2}$ assigns allocations p_u^i and p_v^i that satisfy $|p_u^i - p_v^i| \le d(u, v)$. Now, notice that the modified inverse-proportional allocation algorithm assigns allocations $P_u^i = \frac{p_u^i}{T} + \frac{1}{k}\left(1 - \frac{1}{T}\right)$ and $P_v^i = \frac{p_v^i}{T} + \frac{1}{k}\left(1 - \frac{1}{T}\right)$. We see that $|P_u^i - P_v^i| = \frac{1}{T}|p_u^i - p_v^i| \le \frac{1}{T} d(u, v)$. Now, let S be a subset in $\mathcal{C}$. We know that $|S| \le T$, so $\left|\sum_{i \in S} P_u^i - \sum_{i \in S} P_v^i\right| \le \sum_{i \in S}\left|P_u^i - P_v^i\right| \le |S| d(u, v)/T \le d(u, v)$.

For the competitive ratio, we note that the competitive ratio of the modified inverse proportional allocation algorithm is at least a $\frac{1}{T}$ fraction of the inverse-proportional allocation algorithm with parameter $g(x) = x^{-\ell/2}$. Applying Theorem 3.3 gives the desired result. □

Next we modify Algorithm 4.6 to handle set families with small cluster width. At a high level, we use the Algorithm 4.6 to first allocate probabilities to each cluster of advertisers as defined in Definition 4.4, and then use the inverse-proportional allocation algorithm to further subdivide the allocation to advertisers within each cluster. More formally:

Algorithm 4.8. *The algorithm with parameters ℓ and T operates as follows:*

(1) Partition advertisers into clusters $\mathcal{L} = \{L_1, L_2, \cdots\}$, as defined in Definition 4.4.
(2) Use the modified inverse-proportional allocation algorithm with parameters $g(x) = x^{-\ell}$ and T on $\{L_1, L_2, \cdots\}$, treating each set L_i as an advertiser with bid $\max_{j \in L_i} b_j^u$. Obtain an allocation $\left\{q_u^{L_i}\right\}_{L_i \in \mathcal{L}}$.
(3) For each $L_i \in \mathcal{L}$, run the inverse-proportional allocation algorithm with parameter $g(x) = x^{-\ell}$ to obtain an allocation $\left\{P_u^j\right\}_{j \in L_i}$. Now, set $p_u^j = P_u^j \cdot q_u^{L_i}$.

This algorithm satisfies nice properties in terms of fairness and competitive ratio:

Theorem 4.9. *Consider collection of subsets C such that $\omega_{cluster}(C) \leq T$. Then, given as input bid vectors that satisfy degree-ℓ fairness, Algorithm 4.8 with parameters $\ell/2$ and T satisfies subset fairness with respect to C as well as 2-multiple-task fairness. Furthermore, the algorithm achieves a competitive ratio of at least $\frac{1}{T}\left(1 - \frac{1}{(\ell/2)+1}\left(\frac{\ell/2}{(\ell/2)+1}\right)^{\ell/2}\right)^2$.*

Proof. Suppose the bids satisfy the bid ratio condition f_ℓ. Then, using Theorem 4.7, the allocation $\left\{q_u^{L_i}\right\}_{L_i \in \mathcal{L}}$, and by extension the final allocation $\left\{p_u^j\right\}$ satisfies subset fairness with respect to C, since every set in C is a union of at most T sets in $\mathcal{L}$. To show 2-multiple-task fairness, we use the fact that the allocations $\left\{P_u^j\right\}_{j \in L_i}$ satisfy multiple-task fairness by Theorem 3.4 and the allocations $\left\{q_u^{L_i}\right\}_{L_i \in \mathcal{L}}$ satisfy multiple-task fairness by Theorem 4.7. Thus, we know that $|p_u^j - p_v^j| = |P_u^j \cdot q_u^{L_i} - P_v^j q_v^{L_i}| \leq |P_u^j - P_v^j| q_u^{L_i} + |q_u^{L_i} - q_v^{L_i}| P_v^j \leq 2d(u, v)$. The competitive ratio follows from Theorem 4.7 (on the first-level allocation) and Theorem 3.3 (on the second-level allocation). □

Finally, we show that a similar two-step composition provides fairness for set families with small partitioned width. For this setting, we compose the inverse-proportional allocation algorithm with the proportional allocation algorithm in [12]. In particular, denoting the partition over advertisers as $L_1, L_2, \cdots$, we use Algorithm 4.6 to perform the allocation across different L_is, and then use the inverse-proportional allocation algorithm to divide the allocation within each L_i. More formally:

Algorithm 4.10. *The algorithm with parameters ℓ and $\mathcal{L} = (L_1, L_2 \ldots)$ operates as follows:*

(1) Use the inverse-proportional allocation algorithm with parameter $g(x) = x^{-\ell}$ on $(L_1, L_2 \ldots)$, treating each set L_i as an advertiser with bid $\max_{j \in L_i} b_j^u$. Obtain an allocation $\left\{q_u^{L_i}\right\}_{L_i \in \mathcal{L}}$.
(2) For each $L_i \in \mathcal{L}$, run the proportional allocation algorithm with parameter $g(x) = x^{2\ell}$ to obtain an allocation $\left\{P_u^j\right\}_{j \in L_i}$. Now, set $p_u^j = P_u^j \cdot q_u^{L_i}$.

This algorithm achieves set fairness over the family $C = \cup_i 2^{P_i}$ and competitive ratio guarantees that degrade with $\omega_{\text{part}}(C)$.

THEOREM 4.11. *Let $\mathcal{L} = (L_1, L_2, \ldots)$ be a partition of $[k]$. Consider a collection $C = \cup_i 2^{L_i}$ such that $\omega_{part}(C) = T$. Given as input bid vectors that satisfy degree-ℓ fairness, Algorithm 4.10 with parameters $\ell/2$ and $\mathcal{L}$ satisfies 2-subset fairness with respect to C as well as 2-multiple task fairness and achieves a competitive ratio of at least $T^{-\frac{1}{\ell}}\left(1 - \frac{1}{(\ell/2)+1}\left(\frac{\ell/2}{(\ell/2)+1}\right)^{\ell/2}\right)$.*

PROOF. Suppose the bids satisfy the bid ratio condition f_ℓ. To show 2-subset fairness and 2-multiple-task fairness on C, we use the fact that the allocations $\left\{P_u^j\right\}_{j \in L_i}$ satisfy total-variation fairness by the properties of the proportional allocation algorithm shown in [12] and the allocations $\left\{q_u^{L_i}\right\}_{L_i \in \mathcal{L}}$ satisfy multiple-task fairness by Theorem 3.4. For every $S \subseteq L_i$, we know that $\left|\sum_{j \in S} p_u^j - \sum_{j \in S} p_v^j\right| = \left|\left(\sum_{j \in S} P_u^j\right) \cdot q_u^{L_i} - \left(\sum_{j \in S} P_v^j\right) q_v^{L_i}\right| \leq \left|\sum_{j \in S} P_u^j - \sum_{j \in S} P_v^j\right| q_u^{L_i} + |q_u^{L_i} - q_v^{L_i}| \left(\sum_{j \in S} P_v^j\right) \leq 2d(u, v)$. The competitive ratio follows from Theorem 3.3 (on the first-level allocation) and the competitive ratio bound in [12] (on the second-level allocation). □

5 FUTURE WORK

In this work we address the tradeoffs between fairness and social welfare through a simplistic model that assumes that advertisers bid to maximize their utility in every auction. In reality, advertisers have budgets and optimize for their long term returns. Incorporating fairness constraints into these more general models of ad auctions is an interesting avenue for future work. For the model we study, our work presents an alternative to the proportional allocation mechanism of [12]. The two classes of allocation algorithms perform well under different contexts, and it would be interesting to understand whether one can interpolate between the two to obtain fairness guarantees as strong as those of proportional allocation while at the same time achieving performance commensurate with that of inverse proportional allocation.

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A PROOFS FOR SECTION 3

We prove the following auxiliary lemma.

LEMMA A.1. *Consider the inverse-proportional allocation algorithm with parameter g on k advertisers. Let b_u be a bid vector, and suppose that $b_u^1 \leq b_u^2 \leq \ldots \leq b_u^k$. For each $1 \leq i \leq k$, the allocation $p_u^i = 0$ if and only if $(k-i-1)g(b_u^i) \geq \sum_{j=i+1}^k g(b_u^j)$. In particular, this condition only depends on the advertisers who bid higher than i.*

PROOF OF LEMMA A.1. For the first direction, we show that if $p_u^i = 0$, then $(k-i-1)g(b_u^i) \geq \sum_{j=i+1}^k g(b_u^j)$. If $p_u^i = 0$, then at some iteration of the removal process (Step 5 of Algorithm 1), i is assigned a weight of 1. Consider the first such iteration. At this iteration, the set S in Step 5 of Algorithm 1 contains i at the beginning of this iteration. Hence, S is of the form $[b, k]$ for some $1 \leq b \leq i$. We claim that $(k-i-1)g(b_u^i) \geq \sum_{j=i+1}^k g(b_u^j)$. The weight on advertiser i is exactly $|k-b| \frac{g(b_u^i)}{\sum_{j=b}^k g(b_u^j)}$. Thus, if $w_i = 1$, then $(k-b-1)g(b_u^i) \geq \sum_{b \leq j \leq k, j \neq i} g(b_u^j)$. This can written as

$$(k-i-1)g(b_u^i) + (i-b)g(b_u^i) \geq \sum_{i+1 \leq j \leq k} g(b_u^j) + \sum_{b \leq j \leq i-1} g(b_u^j).$$

Since $(i-b)g(b_u^i) \leq \sum_{b \leq j \leq i-1} g(b_u^j)$, this implies that $(k-i-1)g(b_u^i) \geq \sum_{j=i+1}^k g(b_u^j)$, as desired.

Now, we show the opposite direction. Suppose that $(k-i-1)g(b_u^i) \geq \sum_{j=i+1}^k g(b_u^j)$. We show that $p_u^i = 0$. Assume for sake of contradiction that $p_u^i > 0$. This means that the set S of advertisers that receive nonzero allocation is of the form $[b, k]$ for some $1 \leq b \leq i$, and $w_i < 1$. In particular, $(k-b)\frac{g(b_u^b)}{\sum_{j=b}^k g(b_u^j)} < 1$, which implies that $(k-b-1)g(b_u^b) < \sum_{j=b+1}^k g(b_u^j)$. We can write this as:

$$(k-i-1)g(b_u^b) + \sum_{j=b+1}^{i} g(b_u^b) < \sum_{j=i+1}^{k} g(b_u^j) + \sum_{j=b+1}^{i} g(b_u^j)$$

which we can write as:

$$(k-i-1)g(b_u^b) + \sum_{j=b+1}^{i} (g(b_u^b) - g(b_u^j)) < \sum_{j=i+1}^{k} g(b_u^j).$$

Now, notice that $(k-i-1)g(b_u^i) \leq (k-i-1)g(b_u^b) + \sum_{j=b+1}^i (g(b_u^j) - g(b_u^j))$. This implies that $(k-i-1)g(b_u^i) < \sum_{j=i+1}^k g(b_u^j)$, which is a contradiction. ◻

We prove an auxiliary proposition used in the proof of Theorem 3.4.

PROPOSITION A.2. *Let $C < 1$ be a constant. Consider the function $F(X) = \frac{1}{1+XC} - \frac{1}{1+X}$ on $X \geq 0$. Thus function is increasing when $X \leq \frac{1}{\sqrt{C}}$ and decreasing when $X \geq \frac{1}{\sqrt{C}}$. Thus, the maximum is attained when $X = \frac{1}{\sqrt{C}}$. (When $C = 1$, $F(X)$ is always 0.)*

PROOF. Let's take a derivative. Notice that the derivative is $-\frac{C}{(1+XC)^2} + \frac{1}{(1+X)^2}$. We wish to show $\frac{C}{(1+XC)^2} \geq \frac{1}{(1+X)^2}$. This is equivalent to $C + 2BC + B^2C \geq 1 + B^2C^2 + 2BC$. Rearranging yields $C - 1 \geq B^2C(C-1)$. Since $C < 1$, this is equivalently $1 - C \leq X^2C(1-C)$, which is equivalently $X \geq \frac{1}{\sqrt{C}}$. ◻

We prove Lemma 3.2.

Proof of Lemma 3.2. Let's consider an inverse-proportional allocation with parameter g. Consider a user u with bid vector b_u. WLOG, consider advertiser 1, and the impact of changing bids on the allocation p_u^1. WLOG suppose that $b_u^2 \leq \ldots b_u^k$. Let S be the set of advertisers who receive nonzero allocation.

Suppose that b_u^1 increases to some value c_u^1. Consider the bid vector $[c_u^1, b_u^2, \ldots, b_u^k]$. Now, suppose that when the inverse-proportional allocation algorithm with parameter g is applied to $[c_u^1, b_u^2, \ldots, b_u^k]$, the set S' is the set of advertisers who receive nonzero allocation. Since $g(b_u^1) > g(c_u^1)$, using Lemma A.1, it is easy to verify that, for every advertiser $j \neq 1$, the condition for $p_u^j = 0$ becomes weakly weaker. Thus, $(S' \setminus \{1\}) \subseteq (S \setminus \{1\})$. On the other hand, by Lemma A.1, we know that the condition for $p_u^1 = 0$ becomes weakly stronger, so if $1 \in S$, then $1 \in S'$.

Advertiser 1's allocation weakly increases. First, we show that the allocation on advertiser 1 will weakly increase from $[b_u^1, b_u^2, \ldots, b_u^k]$ to $[c_u^1, b_u^2, \ldots, b_u^k]$. If $1 \notin S$, then it follows trivially that the allocation on u weakly increases as desired. Thus, we can assume that $1 \in S$ (and as argued above, this means that $1 \in S'$). Thus, $S' \subseteq S$. We break down the increase of b_u^1 into a series of increments $b_u^1 \leq (b_u^1)_1 < (b_u^1)_2, (b_u^1)_3 < \ldots < (b_u^1)_c \leq c_u^1$, where the $(b_u^1)_i$ are exactly chosen to the be values where the set of advertisers with nonzero allocation changes as the bid on advertiser 1 changes from b_u^1 to c_u^1. Let S^i be the set of advertisers who receive nonzero allocations on $[b, b_u^2, \ldots, b_u^k]$ when $b \leq (b_u^1)_i$ and let S^{i+1} be the set of advertisers who receive nonzero allocations on $[b, b_u^2, \ldots, b_u^k]$ when $b > (b_u^1)_i$. The arguments thus far show that $S = S^1 \supsetneq S^2 \supsetneq S^3 \ldots \supsetneq S^c = S'$. Since $1 - (|S^i| - 1)\frac{g((b_u^1)_i)}{g((b_u^1)_i) + \sum_{j \in S^i} g(b_u^j)} = 1 - (|S^{i+1}| - 1)\frac{g((b_u^1)_i)}{g((b_u^1)_i) + \sum_{j \in S^{i+1}} g(b_u^j)}$, we know that the advertiser 1's allocation is continuous in its bid even at the values $(b_u^1)_1, (b_u^1)_2, (b_u^1)_3, \ldots, (b_u^1)_c$. Thus, it suffices to show that *within* each of these intervals (where the set of advertisers receiving nonzero allocations is fixed), as the bid increases, advertiser 1's allocation increases. Notice that on the bid vector $[b, b_u^2, \ldots, b_u^k]$, the allocation on advertiser 1 is $1 - (|S| - 1)\frac{g(b)}{g(b) + \sum_{j \in S^{i+1}} g(b_u^j)} = 1 + (|S| - 1)\left(\frac{\sum_{j \in S^{i+1}} g(b_u^j)}{g(b) + \sum_{j \in S^{i+1}} g(b_u^j)} - 1\right)$, which is clearly an increasing function in b.

Other advertisers' allocations weakly decrease. Now, we show that the allocation on every advertiser $i \neq 1$ will weakly decrease from $[b_u^1, b_u^2, \ldots, b_u^k]$ to $[c_u^1, b_u^2, \ldots, b_u^k]$. If $i \notin S'$, then the statement trivially holds. We can thus assume that $i \in S'$. Notice that this implies that $i \in S$.

Suppose that $1 \notin S$. Using that advertiser 1's allocation is monotonic in their bid, we know that there exists a maximum bid b' so that advertiser 1 receives a 0 allocation on the bid vector $[b', b_u^2, \ldots, b_u^k]$. (Thus, $b_u^1 \leq b'$.) We claim that the set of advertiser who receive a nonzero allocation on $[b', b_2^2, \ldots, b_2^k]$ is also S. Note that $b' \leq \min_{i \in S} b_u^i$, and so by Lemma A.1, all of the advertisers in S will continue to receive a nonzero allocation. Moreover, for advertisers $j \notin S$, we know that $b_u^j \leq b'$, and so they will continue to receive a zero allocation. Thus, the allocation on advertiser i is the same for all bid vectors $[b, b_u^2, \ldots, b_u^k]$ for $b \leq b'$.

Hence, it suffices to handle the case where $b' \leq b_u^1 < c_u^1$. Since $1 - (|S| - 1)\frac{g(b_u^i)}{\sum_{j \in S} g(b_u^j)} = 1 - |S|\frac{g(b_u^i)}{g(b') + \sum_{j \in S} g(b_u^j)}$, we also know that advertiser i's allocation is continuous in advertiser 1's bid at b'. So, we can actually just handle the case where $b' < b_u^1 < c_u^1$. In this case, we have that $1 \in S$, so $S' \subseteq S$. By Remark 3.1, we satisfy the necessary conditions to apply Proposition 3.5. Thus, we have that:

$$
\begin{aligned}
1 - (|S'| - 1)\frac{g(b_u^i)}{g(c_u^1) + \sum_{j \in S', j \neq 1} g(b_u^j)} &\leq 1 - (|S| - 1)\frac{g(b_u^i)}{g(c_u^1) + \sum_{j \in S, j \neq 1} g(b_u^j)} \\
&\leq 1 - (|S| - 1)\frac{g(b_u^i)}{g(b_u^1) + \sum_{j \in S'} g(b_u^j)},
\end{aligned}
$$

as desired. □

We prove Proposition 3.5.

Proof of Proposition 3.5. By Remark 3.1, we know that the inverse-proportional allocation algorithm assigns nonzero allocation probabilities to the advertisers with the K highest bids, for some $\min(2,k) \le K \le k$. Since $b_u^1 \le b_u^2 \le \ldots \le b_u^k$, it is straightforward to verify that S is of the form $[a,k]$ for some $a \in [k]$. Moreover, since R_2 is given to be of the form $\{z, z+1, \ldots, k\}$ and since $R_1 \subseteq R_2$, this means that $a \ge z$. Let's suppose that we run the inverse-proportional allocation algorithm on the advertisers in R_2 (i.e. we do not include the advertisers in $[k] \setminus R_2$ in the algorithm). Lemma A.1 enables us to relate the inverse-proportional allocation algorithm run on advertisers in R_2 to the inverse-proportional allocation algorithm run on advertisers in $[k]$. In fact, by Lemma A.1, we know that the inverse-proportional allocation algorithm run with advertisers in R_2 will also result in R_1 being exactly the set of advertisers who receive nonzero allocation.

Informally, we reverse the removal process for the inverse-proportional allocation algorithm on the advertisers in R_2 in order to go from R_1 to R_2. Let c be the number of times that Algorithm 1 goes through Step 5, and let J_i be the set that is removed in the ith iteration. It is easy to verify that the sets $J_1, \ldots, J_c$ are disjoint, adjacent intervals that satisfy $J_1 \cup \ldots \cup J_c = R_2 \setminus R_1$.

We utilize a set S, that is initialized to R_1 and will eventually become R_2. This set S will track the behavior of the variable S in Algorithm 1 (in reverse order). We perform the following c step procedure. At step $0 \le s \le c-1$, we add advertisers J_{c-s}. It suffices to show that, for every $i \in R_1$, every step weakly increases $1 - \frac{(g(b_u^i))(|S|-1)}{\sum_{j\in S} 1/b_u^j}$.

Consider the state of S prior to step s. At this step, J_{c-s} is about to be added. It suffices to show that for any $i \in R_1$:

$$1 - \frac{(g(b_u^i))(|S| + |J_{c-s}| - 1)}{\sum_{j\in S} g(b_u^j)} \le 1 - \frac{(g(b_u^i))(|S|-1)}{\sum_{n\in J_{c-s}} g(b_u^n) + \sum_{j\in S} g(b_u^j)}$$

Manipulating this inequality several times, we see it is equivalent to showing that:

$$0 \le 1 - \frac{(g(b_u^i))(|S|-1)}{\sum_{n\in J_{c-s}} g(b_u^n) + \sum_{j\in S} g(b_u^j)} - 1 + \frac{(g(b_u^i))(|S| + |J_{c-s}| - 1)}{\sum_{j\in S} g(b_u^j)}$$

$$0 \le \frac{(g(b_u^i))(|S|-1)}{\sum_{j\in S} g(b_u^j)} - \frac{(g(b_u^i))(|S| + |J_{c-s}| - 1)}{\sum_{n\in J_{c-s}} g(b_u^n) + \sum_{j\in S} g(b_u^j)}$$

$$0 \le (|S|-1)\left(\sum_{n\in J_{c-s}} g(b_u^n) + \sum_{j\in S} g(b_u^j)\right) - (|S| + |J_{c-s}| - 1)\left(\sum_{j\in S'} g(b_u^j)\right)$$

$$|J_{c-s}|\left(\sum_{n\in J_{c-s}} g(b_u^n) + \sum_{j\in S} g(b_u^j)\right) \le (|S| + |J_{c-s}| - 1)\left(\sum_{n\in J_{c-s}} g(b_u^n)\right)$$

$$|J_{c-s}| \le \sum_{n\in J_{c-s}} \left((|S| + |J_{c-s}| - 1)\frac{g(b_u^n)}{\sum_{n\in J_{c-s}} g(b_u^n) + \sum_{j\in S} g(b_u^j)}\right).$$

Now, notice that $(|S| + |J_{c-s}| - 1)\frac{g(b_u^n)}{\sum_{n\in J_{c-s}} g(b_u^n) + \sum_{j\in S} g(b_u^j)}$, is exactly the weight assigned to advertiser n at Step 5 of the algorithm, in the iteration when the set J_{c-s} has advertisers with weight bigger than 1. Since $n \in J_{c-s}$, we know that $\left((|S| + |J_{c-s}| - 1)\frac{g(b_u^n)}{\sum_{n\in J_{c-s}} g(b_u^n) + \sum_{j\in S'} g(b_u^j)}\right) \ge 1$, so this yields the desired statement. □

We prove Theorem 3.4.

Proof of Theorem 3.4. Let $\alpha = f_l(d)$. Take an arbitrary bid vector $[b_u^1, \ldots, b_u^k]$. WLOG assume that $b_u^2 \leq b_u^3 \leq \ldots \leq b_u^k$. Suppose that the allocation to advertiser 1 is p_u^1. Take any other sequence of bids $[b_v^1, \ldots, b_v^k]$ where each b_v^j is within an α ratio of b_u^j. Suppose that the allocation to advertiser 1 is p_v^1. It suffices to show that $p_v^1 \geq p_u^1 - d$.

Take the sequence of values $[b_w^1, \ldots, b_w^k] = [b_u^1/\alpha, b_u^2 \cdot \alpha, \ldots, b_u^k \cdot \alpha]$. Suppose that the allocation to advertiser 1 is p_w^1. We claim that $p_w^1 \leq p_v^1$. Since $b_w^1 \leq b_v^1$ and $b_w^i \geq b_v^i$ for $i \neq 1$, this follows from Lemma 3.2. Thus it suffices to show that $p_w^1 \geq p_u^1 - d$.

Let S_u be the set of advertisers receiving nonzero allocation on $[b_u^1, \ldots, b_u^k]$. Let S_w be the set of advertisers receiving nonzero allocation on $[b_w^1, \ldots, b_w^k]$. These sets will both have size at least 2. If $1 \notin S_u$, we know that $p_u^1 - d \leq 0$, so the statement trivially holds. Thus, we can assume $1 \in S_u$. We break into two cases: $1 \in S_w$ and $1 \notin S_w$.

Case 1: $1 \in S_w$**.** First, we show that $S_u \subseteq S_w$. It suffices to show that if $i \notin S_w$ and $i \neq 1$, then $i \notin S_u$. By Remark 3.1, if $i \notin S_w$, this means that $b_w^i < b_w^1$. Let $R_u^i = \left\{j \mid b_u^j > b_u^i\right\}$ and let $R_w^i = \left\{j \mid b_w^j > b_w^i\right\}$. The fact that $b_u^2 \leq \ldots \leq b_u^k$ and $b_w^2 \leq \ldots \leq b_w^k$ immediately tells us that $R_u^i \setminus \{1\} = R_w^i \setminus \{1\}$. Moreover, $1 \in R_w^i$ and $\alpha b_u^i = b_w^i < b_w^1 = \alpha^{-1} b_u^1 = \alpha^{-2}(\alpha b_u^1)$, so $1 \in R_u^i$. This implies that $R_w^i = R_u^i$. Now, the fact that $i \notin S_w$, using Lemma A.1, tells us that

$$(|R_w^i| - 1)g(b_w^i) - \sum_{j \in R_w^i} g(b_u^j) \geq 0$$

$$\alpha^{\ell/2}(|R_w^i| - 1)g(b_w^i) - \alpha^{\ell/2} \sum_{j \in R_w^i} g(b_w^j) \geq 0$$

$$(|R_w^i| - 1)g(\alpha^{-1} b_w^i) - \sum_{j \in R_w^i} g(\alpha^{-1} b_w^j) \geq 0$$

Notice that $g(\alpha^{-1} b_w^i) = g(b_u^i)$ and $g(\alpha^{-1} b_w^i) \geq g(\alpha^{-1} b_u^1)$. Thus, $\sum_{j \in R_w^i} g(\alpha^{-1} b_u^j) \geq \sum_{j \in R_w^i} g(b_u^j)$. This means that

$$(|R_u^i| - 1)g(b_u^i) - \sum_{j \in R_u^i} g(b_u^j) = (|R_w^i| - 1)g(b_u^i) - \sum_{j \in R_w^i} g(\alpha^{-1} b_u^j) \geq (|R_w^i| - 1)g(\alpha^{-1} b_w^i) - \sum_{j \in R_w^i} g(\alpha^{-1} b_w^j) \geq 0,$$

so applying Lemma A.1 again, we see that $i \notin S_u$.

Now, we can apply Proposition 3.5 to obtain:

$$p_u^1 = 1 - \frac{(g(b_u^1))(|S_u| - 1)}{\sum_{j \in S_u} g(b_u^j)} \leq 1 - \frac{(g(b_u^1))(|S_w| - 1)}{\sum_{j \in S_w} g(b_u^j)}.$$

Let the RHS be p_*^1. Now, we can apply Proposition 3.6 to see that $p_w^1 \geq p_*^1 - d \geq p_u^1 - d$ as desired.

Case 2: $1 \notin S_w$**.** Now, let's consider the case where $1 \notin S_w$. Let $R_w = \left\{i \mid b_w^i \geq b_w^1\right\}$ (so $1 \in R_w$). We claim that $S_u \subseteq R_w$. It suffices to show that if $i \notin R_w$ and $i \neq 1$, then $i \notin S_u$. Now, the same argument as in Case 1 shows that $i \notin S_u$ (with the additional observation that if $i \notin R_w$, then using that $1 \notin S_w$ and Remark 3.1, we can deduce that $i \notin S_w$).

Thus, we can apply Proposition 3.5 to obtain:

$$p_u^1 = 1 - \frac{(g(b_u^i))(|S_u| - 1)}{\sum_{j \in S_u} g(b_u^j)} \leq 1 - \frac{(g(b_u^i))(|R_w| - 1)}{\sum_{j \in R_w} g(b_u^j)} =: p_*^1.$$

Moreover, observe that $1 - \frac{g(b_w^1)(|S_w|-1)}{\sum_{j\in S_w} g(b_w^j)} \leq 0 = p_w^1$ since advertiser 1 is removed at this stage. Now, we can apply Proposition 3.6 to see that $p_w^1 \geq p_*^1 - d \geq p_u^1 - d$ as desired. □

We prove Theorem 3.3.

Proof of Theorem 3.3. Let $g(x) = x^{-l}$. We bound the competitive ratio of the inverse-proportional allocation algorithm with parameter g. Consider a sequence of bids $b_u^1 \geq b_u^2 \geq \ldots \geq b_u^k$. Consider $\left\{i \mid 1 \leq i \leq k, p_u^i > 0\right\}$. We know that this set is of the form $\{1, \ldots, K\}$ for some $1 \leq K \leq k$. We can thus just consider bids $b_u^1 \geq b_u^2 \geq \ldots \geq b_u^K$ for the remainder of the calculation.

Since multiplicative scaling does not affect the competitive ratio, we can scale the bids so that the maximum bid is 1. This conveniently makes the competitive ratio equal to the social welfare. For $1 \leq i \leq K$, let $B_u^i = \frac{b_u^i}{b_u^1}$. Now, the competitive ratio is:

$$\sum_{i=1}^{K} B_u^i \left(1 - (k-1)\frac{g(B_u^i)}{\sum_{j=1}^K g(B_u^i)}\right) = \sum_{i=1}^{K} B_u^i - (k-1)\frac{\sum_{i=1}^K g(B_u^i)B_u^i}{\sum_{i=1}^K g(B_u^i)}.$$

Our goal is to lower bound this expression. In particular, we wish to compute c independent of $[B_u^2, \ldots, B_u^k]$ such that:

$$\sum_{i=1}^{K} B_u^i - (k-1)\frac{\sum_{i=1}^K g(B_u^i)B_u^i}{\sum_{i=1}^K g(B_u^i)} \geq c.$$

Notice that we can rewrite the desired equality as:

$$\left(\sum_{i=1}^{K} g(B_u^i)\right)\left(\sum_{i=1}^{K} B_u^i\right) - (K-1)\sum_{i=1}^{K} g(B_u^i)B_u^i \geq c\left(\sum_{i=1}^{K} g(B_u^i)\right).$$

We use the fact that $B_u^1 = 1$ and $g(B_u^1) = 1$. Thus, we can write this as:

$$\left(1 + \sum_{i=2}^{K} g(B_u^i)\right)\left(1 + \sum_{i=2}^{K} B_u^i\right) - (K-1) - (K-1)\sum_{i=2}^{K} g(B_u^i)B_u^i \geq c + c\left(\sum_{i=2}^{K} g(B_u^i)\right)$$

$$1 + \sum_{i=2}^{K} g(B_u^i) + \sum_{i=2}^{K} g(B_u^i) + \left(\sum_{i=2}^{K} g(B_u^i)\right)\left(\sum_{i=2}^{K} B_u^i\right) - (K-1) - (K-1)\sum_{i=2}^{K} g(B_u^i)B_u^i \geq c + c(\sum_{i=2}^{K} p_i)$$

$$1 - c + (1-c)\sum_{i=2}^{K} g(B_u^i) + \sum_{i=2}^{K} B_u^i + \left(\sum_{i=2}^{K} g(B_u^i)\right)\left(\sum_{i=2}^{K} B_u^i\right) - (K-1)\sum_{i=2}^{K} g(B_u^i)B_u^i \geq K-1$$

We claim that $\left(\sum_{i=2}^K g(B_u^i)\right)\left(\sum_{i=2}^K B_u^i\right) \geq (K-1)\sum_{i=2}^K g(B_u^i)B_u^i$. We can write $\left(\sum_{i=2}^K g(B_u^i)\right)\left(\sum_{i=2}^K B_u^i\right)$ as $\sum_{i=1}^{K-1}\left(\sum_{j=2}^K g(B_u^j)B_u^{i+j-1}\right)$, where the indices are modulo $K-1$. We apply the rearrangement inequality on the inner expression, using the fact that $g(B_u^2) \leq g(B_u^3) \leq \ldots \leq g(B_u^k)$ and $B_u^2 \geq B_u^3 \geq \ldots \geq B_u^k$. This means that $\sum_{j=2}^K g(B_u^j)B_u^{i+j-1} \geq \sum_{j=2}^K g(B_u^j)B_u^j$, so $\sum_{i=1}^{K-1}\left(\sum_{j=2}^K g(B_u^j)B_u^{i+j-1}\right) \geq (K-1)\sum_{j=2}^K g(B_u^j)B_u^j$. As a result, we have proved the desired statement. (Implicitly, in this step, we partially reduced to the case where $B_u^2 = \ldots = B_u^K$.)

Thus, we can lower bound $1 - c + (1-c)\sum_{i=2}^K g(B_u^i) + \sum_{i=2}^K B_u^i + \left(\sum_{i=2}^K g(B_u^i)\right)\left(\sum_{i=2}^K B_u^i\right) - (K-1)\sum_{i=2}^K g(B_u^i)B_u^i$ as the following expression:

$$\sum_{i=2}^{K}\left((1-c)g(B_u^i) + B_u^i\right).$$

Hence, it suffices to choose c such that $(1-c)g(B_u^i)+B_u^i \geq 1$ for all $0 < B_u^i < 1$. We can write this as $(B_u^i)^{-l}(1-c)+B_u^i-1 \geq 0$, which can be rewritten as $(B_u^i)^{\ell+1} - (B_u^i)^{\ell} + 1 \geq c$. Thus, we can set $c = \min_{0<B_u^i<1}\left((B_u^i)^{\ell+1} - (B_u^i)^{\ell} + 1\right)$ always. (Implicitly, in this step, we finished reducing to the case where $B_u^2 = \ldots = B_u^K$.) We can take a derivative to see that the minimum occurs at $B_u^i = \frac{\ell}{\ell+1}$, and obtain the desired bound. □

Proof of Theorem 3.7. We wish to show that $\mathcal{L}(\ell,k) \geq \mathcal{U}(\ell,k) - 1/k$. For every k, we know that $\mathcal{L}(\ell,k) \geq 1 - \frac{1}{\ell/2+1}\left(\frac{\ell/2}{\ell/2+1}\right)^{\ell/2}$. We know use the bound $\mathcal{U}(\ell,k)$:

$$\begin{aligned}
\mathcal{U}(\ell,k) - \frac{1}{k} &\leq \min_{0<d<1}\left(\frac{1}{k} + d + \left(\frac{1}{1-d}\right)^{-2/\ell}\left(1 - \frac{1}{k} - d\right)\right) - \frac{1}{k} \\
&= \min_{0<d<1}\left(d + \left(\frac{1}{1-d}\right)^{-2/\ell}\left(1 - \frac{1}{k} - d\right)\right) \\
&\leq \min_{0<d<1}\left(d + \left(\frac{1}{1-d}\right)^{-2/\ell}(1-d)\right) \\
&= \min_{0<d<1}\left(d + (1-d)^{1+2/\ell}\right).
\end{aligned}$$

At $d = 1 - \left(\frac{\ell/2}{\ell/2+1}\right)^{\ell/2}$, notice that $d + (1-d)^{1+2/\ell} = 1 - \left(\frac{\ell/2}{\ell/2+1}\right)^{\ell/2} + \left(\frac{\ell/2}{\ell/2+1}\right)^{\ell/2+1} = 1 - \frac{1}{\ell/2}\left(\frac{\ell/2}{\ell/2+1}\right)^{\ell/2}$, which is equal to $\mathcal{L}(\ell,k)$. Thus, we have shown that $\mathcal{L}(\ell,k) \geq \mathcal{U}(\ell,k) - \frac{1}{k}$.

Let's show the result as $k \to \infty$. Moreover, taking a limit as $k \to \infty$, we obtain that $\mathcal{L}(\ell) \geq \limsup_{k\to\infty}\mathcal{U}(\ell,k)$. To obtain equality, we use the fact that the inverse-proportional allocation algorithm for k advertisers with parameter $g(x) = x^{-\ell/2}$ achieves multiple-task fairness (by Theorem 3.4), so $\mathcal{L}(\ell,k) \leq \mathcal{U}(\ell,k) - \frac{1}{k}$, which means that $\mathcal{L}(\ell) \leq \liminf_{k\to}\mathcal{U}(\ell,k)$. In particular, this implies that $\lim_{k\to\infty}\mathcal{U}(\ell,k)$ exists, and $\mathcal{L}(\ell) = \lim_{k\to\infty}\mathcal{U}(\ell,k)$. □

B PROOFS FOR SECTION 4

In the proof of Theorem 4.2, we use the following construction of users.

Example B.1. *Let $0 < \gamma < 1$, $0 < \ell < \infty$, and $n \in \mathbb{N}$ be parameters. We construct $k = 2^n$ advertisers with as follows. All of the user we consider will have the same multi-set of bids in their bid vectors, but the ordering across the advertisers will be different. We define this set of bids B as follows: B will have 2^{n-1} bids of γ^n, 2^{n-2} bids of γ^{n-1}, 2^{n-3} bids of γ^{n-2}, ..., 1 bid of γ and 1 bid of 1.*

We now design a sequence of users $u_0, u_1, \ldots, u_n$ as follows. The multi-set of bids for all of these users will be B, but the ordering will differ from user to user. User u_0 will organize the bids in B so that $b_{u_0}^1 \geq b_{u_0}^2 \geq \ldots \geq b_{u_0}^k$. For $1 \leq i \leq n$, user u_i will organize the bids so that $b_{u_i}^j = b_{u_0}^j$ for $j > 2^i$, and $b_{u_i}^j = b_{u_i}^{2^i+1-j}$ for $j \leq 2^i$ (i.e. the first 2^{i-1} bids are flipped with the next set of 2^{i-1} bids, and everything else stays the same.)

We define a partial fairness metric so that $d(u_0,u_i) = f_\ell^{-1}(\gamma^{-i})$. It is easy to verify that the bid vectors satisfy the bid ratio condition f_ℓ.

Proof of Theorem 4.2. To compute an upper bound on $T(k,\ell)$, we use the construction in Example B.1. The function of the bids structure of the algorithm requires that the allocation assigned to each bid is always the same. Suppose that the algorithm places a total allocation of p_i on the bids γ^i for each $0 \leq i \leq n$. We know that the competitive ratio is thus:

$$\sum_{i=0}^{n}(p_n \cdot \gamma^n).$$

Now, let's require subset fairness on the collection of sets $S_i := \left\{1, \dots, 2^{i-1}\right\}$ for $1 \leq i \leq n$. Notice that this collection only has $\Theta(n) = \Theta(\log k)$ sets. In particular, we enforce set fairness on S_i between users u_0 and u_i. This means that it must be true that:

$$p_0 + \dots + p_{i-1} - p_i \leq f^{-1}(\gamma^{-i}).$$

The other constraint is that $p_0 + \dots + p_n \leq 1$ since the total allocation cannot exceed 1. Notice that the utility is maximized when $p_0 = \frac{1}{2^n} + \sum_{i=1}^{n} \frac{f^{-1}(\gamma^{-i})}{2^i}$ and for $1 \leq j \leq n$, $p_j = \frac{1}{2^{n-j+1}} + \left(\sum_{i=j+1}^{n} \frac{f^{-1}(\gamma^{-i})}{2^{i-j+1}}\right) - \frac{f^{-1}(\gamma^{-j})}{2}$. Notice that this allocation is always nonnegative (and thus valid) because f is an increasing function.

This means that:

$$T(k,\ell) \leq \left(\frac{1}{2^n} + \sum_{i=1}^{n} \frac{f^{-1}(\gamma^{-i})}{2^i}\right) + \sum_{K=1}^{n} \gamma^K \left(\frac{1}{2^{n-K+1}} + \left(\sum_{i=K+1}^{n} \frac{f^{-1}(\gamma^{-i})}{2^{i-K+1}}\right) - \frac{f^{-1}(\gamma^{-K})}{2}\right).$$

Now, let's consider the bid ratio condition f_ℓ for some parameter $0 < \ell < \infty$. Let's use the fact that $f_\ell^{-1}(\gamma^{-K}) = 1 - \gamma^{K\ell}$. Now, we can conclude that:

$$T(k,\ell) \leq \left(1 - \sum_{i=1}^{n} \frac{\gamma^{i\cdot\ell}}{2^i}\right) + \sum_{k=1}^{n} \gamma^K \left(-\left(\sum_{i=K+1}^{n} \frac{\gamma^{i\cdot\ell}}{2^{i-K+1}}\right) + \frac{\gamma^{K\cdot\ell}}{2}\right).$$

We can write this as:

$$T(k,\ell) \leq \left(1 - \sum_{i=1}^{n} \frac{\gamma^{i\cdot l}}{2^i}\right) + \frac{1}{2}\left(\sum_{K=1}^{n} \gamma^{K(\ell+1)}\right) - \left(\sum_{K=1}^{n} \gamma^K \left(\sum_{i=K+1}^{n} \frac{\gamma^{i\cdot l}}{2^{i-K+1}}\right)\right).$$

Let the RHS of this expression be $E(k, \ell, \gamma)$.

Now, we use this expression to show that $\lim_{k\to\infty} E(k, \ell, \gamma) = \frac{\ell}{\ell+1}$. Notice that taking a limit as $k \to \infty$ is the same as taking a limit as $n \to \infty$. Let's handle each term in the RHS separately.

For the first term, notice that the limit as $n \to \infty$ of $\left(1 - \sum_{i=1}^{n} \frac{\gamma^{i\ell}}{2^i}\right)$ is $\left(1 - \sum_{i=1}^{\infty} \frac{\gamma^{i\ell}}{2^i}\right) = 1 - \left(\sum_{i=0}^{\infty} \frac{\gamma^{i\ell}}{2^i} - 1\right) = 2 - \frac{1}{1-\gamma^\ell/2}$.

For the second term, notice that the limit as $n \to \infty$ is $\frac{1}{2}\left(\sum_{K=1}^{n} \gamma^{(\ell+1)K}\right) = \frac{\gamma^{\ell+1}}{2}\left(\sum_{K=0}^{n-1} \gamma^{(\ell+1)K}\right)$. Taking a limit as $n \to \infty$, we obtain $\frac{\gamma^{\ell+1}}{2}\frac{1}{1-\gamma^{\ell+1}}$.

For the third term, notice that $\frac{1}{2}\left(\sum_{K=1}^{n} \gamma^K \cdot 2^K \left(\sum_{i=K+1}^{n} \frac{\gamma^{i\cdot l}}{2^i}\right)\right) = \frac{1}{2}\left(\sum_{K=1}^{n} \gamma^K \cdot 2^K \frac{\gamma^{(K+1)\cdot l}}{2^{K+1}}\left(\sum_{i=0}^{n-K-1} \frac{\gamma^{i\cdot l}}{2^i}\right)\right)$. We can write this as: $\frac{1}{4\gamma}\left(\sum_{K=1}^{n} \gamma^{(K+1)(\ell+1)}\left(\sum_{i=0}^{n-K-1} \frac{\gamma^{i\cdot l}}{2^i}\right)\right)$. Now, we can write this as:

$$\frac{1}{4\gamma}\left(\sum_{K=1}^{n} \gamma^{(K+1)(\ell+1)}\left(\frac{1-\gamma^{(n-K)l}/2^{n-K}}{1-\gamma^\ell/2}\right)\right).$$

We can write this as:

$$\frac{1}{4\gamma}\left[\left(\sum_{K=1}^{n} \gamma^{(K+1)(\ell+1)}\left(\frac{1}{1-\gamma^\ell/2}\right)\right) - \left(\sum_{k=1}^{n} \gamma^{(K+1)(\ell+1)}\left(\frac{\gamma^{(n-K)l}/2^{n-K}}{1-\gamma^\ell/2}\right)\right)\right].$$

For the first term, notice that it can be written as: $\left(\sum_{K=1}^{n} \gamma^{(K+1)(\ell+1)} \left(\frac{1}{1-\gamma^{\ell}/2}\right)\right) = \left(\frac{1}{1-\gamma^{\ell}/2}\right) \sum_{K=1}^{n} \gamma^{(K+1)(\ell+1)} = \left(\frac{\gamma^{2(\ell+1)}}{1-\gamma^{\ell}/2}\right) \sum_{K=0}^{n-1} \gamma^{K(\ell+1)}$. Taking a limit as $n \to \infty$, we obtain $\left(\frac{\gamma^{2(\ell+1)}}{1-\gamma^{\ell}/2}\right) \frac{1}{1-\gamma^{\ell+1}}$.

For the second term, notice that

$$\frac{1}{1-\gamma^{\ell}/2}\left(\sum_{K=1}^{n} \gamma^{K+1}\gamma^{(n+1)l}/2^{n-K}\right) = \frac{2}{1-\gamma^{\ell}/2}\frac{\gamma^2\gamma^{(n+1)l}}{2^n}\left(\sum_{K=1}^{n}\gamma^{K-1}2^{K-1}\right) = \frac{1}{1-\gamma^{\ell}/2}\frac{\gamma^2\gamma^{(n+1)l}}{2^n}\frac{1-\gamma^n 2^n}{1-2\gamma}.$$

We can write this as: $\frac{\gamma^{2+l}}{(1-\gamma^{\ell}/2)(1-2\gamma)}\gamma^{nl}(1/2^n - \gamma^n)$, which goes to 0 as $n \to \infty$.

So, our third term overall becomes:

$$\frac{1}{4\gamma}\left(\frac{\gamma^{2(\ell+1)}}{1-\gamma^{\ell}/2}\right)\frac{1}{1-\gamma^{\ell+1}} = \left(\frac{\gamma^{2\ell+1}}{4(1-\gamma^{\ell}/2)}\right)\frac{1}{1-\gamma^{\ell+1}}.$$

Let $E(\ell,\gamma) = \lim_{k\to\infty} E(\ell,k,\gamma)$. Putting it all together, we obtain that for every $\gamma < 1$, it holds that:

$$E(\ell,\gamma) = 2 - \frac{1}{1-\gamma^{\ell}/2} + \frac{\gamma^{\ell+1}}{2}\frac{1}{1-\gamma^{\ell+1}} - \left(\frac{\gamma^{2\ell+1}}{4(1-\gamma^{\ell}/2)}\right)\frac{1}{1-\gamma^{\ell+1}}.$$

Let $E(\ell) = \limsup_{\gamma\to 1} E(\ell,\gamma) = \lim_{\gamma\to 1} E(\ell,\gamma)$. Notice that

$$E(\ell) = \lim_{\gamma\to 1}\left(2 - \frac{1}{1-\gamma^{\ell}/2} + \frac{\gamma^{\ell+1}}{2}\frac{1}{1-\gamma^{\ell+1}} - \left(\frac{\gamma^{2\ell+1}}{4(1-\gamma^{\ell}/2)}\right)\frac{1}{1-\gamma^{\ell+1}}\right).$$

We claim that $\lim_{\gamma\to 1}\left(2 - \frac{1}{1-\gamma^{\ell}/2} + \frac{\gamma^{\ell+1}}{2}\frac{1}{1-\gamma^{\ell+1}} - \left(\frac{\gamma^{2\ell+1}}{4(1-\gamma^{\ell}/2)}\right)\frac{1}{1-\gamma^{\ell+1}}\right) = \frac{l}{\ell+1}$. We see that $2 - \frac{1}{1-\gamma^{\ell}/2} \to 0$ as $\gamma \to 1$. Let's look at the second and third terms. Notice that:

$$\frac{\gamma^{\ell+1}}{4(1-\gamma^{\ell+1})(1-\gamma^{\ell}/2)}\left(2(1-\gamma^{\ell}/2)-\gamma^{l}\right) = \frac{\gamma^{\ell+1}}{4(1-\gamma^{\ell+1})(1-\gamma^{\ell}/2)}\left(2-2\gamma^{\ell}\right) = \frac{\gamma^{\ell+1}}{2(1-\gamma^{\ell}/2)}\frac{1-\gamma^{\ell}}{1-\gamma^{\ell+1}}.$$

We see that $\frac{\gamma^{\ell+1}}{2(1-\gamma^{\ell}/2)} \to 1$. Notice that $\frac{1-\gamma^{\ell}}{1-\gamma^{\ell+1}} \to \frac{\ell}{\ell+1}$, as desired. Since $T(\ell,k) \le E(\ell,k,\gamma)$ for every γ and every k, this means that $\limsup_{k\to\infty} T(\ell,k) \le \limsup_{k\to\infty} E(\ell,k,\gamma) = E(\ell,\gamma)$ for every $\gamma < 1$, and thus $\limsup_{k\to\infty} T(\ell,k) \le \lim_{\gamma\to 1} E(\ell,\gamma) = E(\ell) = \frac{l}{l+1}$.

Now, observe that $\frac{\mathcal{L}(\ell)}{T(\ell,k)} \ge \frac{1-\frac{1}{\ell/2+1}\left(\frac{\ell/2}{\ell/2+1}\right)^{\ell/2}}{E(\ell,k,\gamma)}$. In particular, we have that:

$$\begin{aligned}
\liminf_{k\to\infty}\frac{\mathcal{L}(\ell)}{T(\ell,k)} &\ge \liminf_{k\to\infty}\frac{1-\frac{1}{\ell/2+1}\left(\frac{\ell/2}{\ell/2+1}\right)^{\ell/2}}{E(\ell,k,\gamma)} \\
&= \lim_{k\to\infty}\frac{1-\frac{1}{\ell/2+1}\left(\frac{\ell/2}{\ell/2+1}\right)^{\ell/2}}{E(\ell,k,\gamma)} \\
&= \frac{1-\frac{1}{\ell/2+1}\left(\frac{\ell/2}{\ell/2+1}\right)^{\ell/2}}{\lim_{k\to\infty}E(\ell,k,\gamma)} \\
&= \frac{1-\frac{1}{\ell/2+1}\left(\frac{\ell/2}{\ell/2+1}\right)^{\ell/2}}{\lim_{k\to\infty}E(\ell,\gamma)}
\end{aligned}$$

Since this holds for every $\gamma > 1$, this holds for:

$$\begin{aligned}
\liminf_{k\to\infty} \frac{\mathcal{L}(\ell)}{T(\ell,k)} &\geq \lim_{\gamma\to 1} \frac{1-\frac{1}{\ell/2+1}\left(\frac{\ell/2}{\ell/2+1}\right)^{\ell/2}}{E(\ell,\gamma)} \\
&= \frac{1-\frac{1}{\ell/2+1}\left(\frac{\ell/2}{\ell/2+1}\right)^{\ell/2}}{\lim_{\gamma\to 1} E(\ell,\gamma)} \\
&= \frac{1-\frac{1}{\ell/2+1}\left(\frac{\ell/2}{\ell/2+1}\right)^{\ell/2}}{E(\ell)} \\
&= \frac{1-\frac{1}{\ell/2+1}\left(\frac{\ell/2}{\ell/2+1}\right)^{\ell/2}}{\frac{l}{\ell+1}} \\
&= \frac{\ell+1}{\ell/2+1} \frac{\ell/2+1-\left(\frac{\ell/2}{\ell/2+1}\right)^{\ell/2}}{\ell} \\
&= \frac{\ell+1}{\ell/2+1} \left(\frac{1}{2} + \frac{1-\left(\frac{\ell/2}{\ell/2+1}\right)^{l/2}}{\ell} \right) \\
&= \frac{1-\left(\frac{\ell/2}{\ell/2+1}\right)^{l/2}}{l} \\
&= \frac{1}{(\ell/2+1)^{\ell/2}} \frac{(\ell/2+1)^{\ell/2}-(\ell/2)^{\ell/2}}{\ell}.
\end{aligned}$$

Now, let's a limit as $l \to 0$ to obtain:

$$\liminf_{l\to 0} \left(\liminf_{k\to\infty} \frac{\mathcal{L}(\ell)}{T(\ell,k)} \right) \geq \liminf_{\ell\to 0} \frac{(\ell/2+1)^{\ell/2}-(\ell/2)^{\ell/2}}{\ell} = \lim_{\ell\to 0} \frac{(\ell/2+1)^{\ell/2}-(\ell/2)^{\ell/2}}{\ell} = \infty.$$

□

C POINTWISE NEAR-OPTIMALITY WITHIN FAMILY OF INVERSE-PROPORTIONAL ALLOCATION ALGORITHMS

We consider the family of inverse-proportional allocation rules with decreasing functions $g(x)$ that take positive values to positive values. We show that for a large portion of the parameter range, the inverse-proportional allocation rules that we select are near-optimal within this family of allocation algorithms. For ease of analysis, we only consider optimality in the limit as $k \to \infty$. In order to compare properly, we show the following simplified bound on the proportional allocation mechanisms with $g(v) = 1/v^l$.

Corollary C.1. *If $l \geq \frac{e^{-1}}{1-r}$, then the allocation rule with $g(v) = 1/v^l$ achieves a competitive ratio of at least r. A sufficient value ratio condition is $f(d) = \left(\frac{1}{1-d}\right)^{1/(2l)}$.*

Now, we show for a competitive ratio of r, this bound is pointwise near-optimal (up to a constant factor on l). (We ignore the dependence on k.)

Lemma C.2. *Suppose that the inverse-proportional allocation mechanism with a decreasing function g (mapping positive values to positive values) achieves a competitive ratio of $r \geq 0.56$ and multiple-task fairness for all $k \geq 2$. Then, for $l = \frac{\ln 2}{5(1-r)}$, we have that $f(d) \leq \left(\frac{1}{1-d}\right)^{1/(2l)}$ for infinitely many points on f, i.e. for d in*

$$D := \left\{d \mid f(d) = \left(e^{5(1-r)}\right)^m, m \in \{n, 1/n \mid n \text{ a natural number}\}.\right\}.$$

First, we prove Corollary C.1.

Proof of Corollary C.1. Applying Theorem 3.3, it suffices to have:

$$f \leq 1 - \frac{1}{l+1}\left(\frac{l}{l+1}\right)^l.$$

Note that $\left(\frac{l}{l+1}\right)^{l+1} = \left(1 - \frac{1}{l+1}\right)^{l+1} \leq e^{-1}$ for all $l \geq 0$. Thus, we have that

$$1 - \frac{1}{l+1}\left(\frac{l}{l+1}\right)^l = 1 - \frac{\frac{1}{l+1}}{1 - \frac{1}{l+1}}\left(1 - \frac{1}{l+1}\right)^{l+1} \geq 1 - e^{-1}\frac{1}{l}.$$

Thus, if we have $l \geq \frac{e^{-1}}{1-r}$, then we would know that

$$1 - e^{-1}\frac{1}{l} \geq r$$

as desired. The bid ratio condition follows from Theorem 3.4 followed by slightly strengthening the value ratio condition. □

Now, we prove Lemma C.2. First, for the competitive ratio, we claim

Proposition C.3. *For $m < M$, let $R_b = M/m$ and let $R_g = g(m)/g(M)$. Suppose that g achieves a competitive ratio of $r \geq 0.56$. Let $s = e^{5(1-r)}$ and suppose that $\log_s(R_b)$ is a positive integer. Then, it must be true that $R_g \geq R_b^{\frac{\ln 2}{5(1-r)}}$.*

Proof. Let $m < M$ be positive values. Let's consider the competitive ratio on bid vector $[m', m, \ldots, m]$. As $k \to \infty$, we see that it is:

$$1 - \frac{g(m')}{g(m)} + \frac{g(m')}{g(m)}\frac{m}{M}.$$

The requirement that the competitive ratio is r means that this expression must be lower bounded by r. Solving yields:

$$\frac{g(m)}{g(m')} \geq \frac{1 - m/m'}{1 - r}.$$

Let s be a positive value such that $\log_s(R_b)$ is an integer. Let's apply the above fact to $m' = s^i \cdot m$ for each $1 \leq i \leq \log_s(R_b)$, so that m' takes on exponentially spaced values between m and M. For each $0 \leq i \leq \log_s(R_b) - 1$, we will have that:

$$\frac{g(s^i \cdot m)}{g(s^{i+1} \cdot m)} \geq \frac{1 - 1/s}{1 - r}.$$

If we multiply these expressions together, the left hand side telescopes, and we obtain that:

$$\frac{g(m)}{g(M)} \geq \left(\frac{1 - 1/s}{1 - r}\right)^{\log_s(R_b)} = R_b^{\frac{\log\left(\frac{1-1/s}{1-r}\right)}{\log s}}.$$

When $s = e^{5(1-r)}$, the denominator becomes $5(1-r)$. It is straightforward to see that $\frac{1-e^{-5(1-r)}}{1-r} \geq 2$ for $r \geq 0.56$. This proves the desired result. □

Using Proposition C.3, we prove Lemma C.2.

Proof of Lemma C.2. First, we prove two auxiliary facts, one based on fairness and one based on competitive ratio.

Fairness. Let $m \leq M$ be values, and consider the bid vectors $b_u = [1, \ldots, 1]$ and $b_v = [M, m, \ldots, m]$. Notice that these two bid vectors could belong to users u and v such that $d(u, v) = d$ where $f(d)^2 = \frac{M}{m}$. Observe that the allocation p_v^1 on the advertiser who bid M on user v is $1 - \frac{k-1}{1-(k-1)g(m)/g(M)}$. As $k \to \infty$, this becomes $1 - \frac{g(M)}{g(m)}$. In comparison, the allocation $p_u^1 \to 0$. Thus, we must have that $1 - \frac{k-1}{1-(k-1)g(m)/g(M)} \leq d$. Solving yields that $\frac{g(m)}{g(M)} \leq \frac{1}{1-d}$.

competitive ratio. Now, we show that there m, M such that $\frac{g(m)}{g(M)}$ must be large if the competitive ratio is large. Moreover, specifically suppose the competitive ratio is r, and let $s = e^{5(1-r)}$. Suppose that R_b is such that $\log_s(R_b)$ is a positive integer or the reciprocal of a positive integer. We claim that there exist m, M such that $\frac{M}{m} = R_b$ and $\frac{g(m)}{g(M)} \geq R_b^{\frac{\ln 2}{5(1-r)}}$. Proposition C.3 already gives us this when $\log_s(R_b)$ is a positive integer. Now, suppose that $\log_s(R_b)$ is the reciprocal of a positive integer, say $1/n$. Let m and M be values such that $\frac{M}{m} = s$. We show that there exists $0 \leq i \leq n-1$ such that $\frac{g(m \cdot s^{i/n})}{g(m \cdot s^{(i+1)/n})} \geq R_b^{\frac{\ln 2}{5(1-r)}}$. Let's apply Proposition C.3 to m and M to obtain:

$$\left(\frac{M}{m}\right)^{\frac{\ln 2}{5(1-r)}} \leq \frac{g(m)}{g(M)}.$$

We can write this as:

$$(R_b^{\frac{\ln 2}{5(1-r)}})^n \leq \prod_{i=1}^{n} \frac{g(m \cdot \left(e^{5(1-r)}\right)^{(i-1)/n})}{g(m \cdot \left(e^{5(1-r)}\right)^{i/n})}.$$

This implies that there exists $1 \leq i \leq n$ such that

$$\frac{g(m \cdot \left(e^{5(1-r)}\right)^{(i-1)/n})}{g(m \cdot \left(e^{5(1-r)}\right)^{i/n})} \geq R_b^{\frac{\ln 2}{5(1-r)}}.$$

as desired.

Now, we are ready to prove the theorem statement. Suppose that d is such that $f(d) = s^m$, where m is an integer or the reciprocal of an integer. Then, we know that there exist M and m such that $\frac{M}{m} = f(d)^2$ and $\frac{g(m)}{g(M)} \geq R_b^{\frac{\ln 2}{5(1-r)}}$ by the second fact. By the first fact, we know that $\frac{g(m)}{g(M)} \leq \frac{1}{1-d}$. Combining these facts, we know that

$$\frac{1}{1-d} \geq f(d)^{\frac{2\ln 2}{5(1-r)}}$$

which gives us the desired result. □