

An Axiomatic Theory of Provably-Fair Welfare-Centric Machine Learning

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Abstract

We address an inherent difficulty in welfare-theoretic fair machine learning by proposing an equivalently axiomatic-justified alternative and studying the resulting computational and statistical learning questions. Welfare metrics quantify *overall wellbeing* across a population of one or more groups, and welfare-based objectives and constraints have recently been proposed to incentivize *fair machine learning methods* to produce satisfactory solutions that consider the diverse needs of multiple groups. Unfortunately, many machine-learning problems are more naturally cast as *loss minimization* tasks, rather than *utility maximization*, which complicates direct application of welfare-centric methods to fair machine learning. In this work, we define a complementary measure, termed *malfare*, measuring overall societal harm (rather than wellbeing), with axiomatic justification via the standard axioms of cardinal welfare.

We then cast fair machine learning as *malfare minimization* over the *risk values* (expected losses) of each group. Surprisingly, the axioms of cardinal welfare (malfare) dictate that this is not equivalent to simply defining utility as negative loss. Building upon these concepts, we define *fair-PAC learning*, where a fair-PAC learner is an algorithm that learns an ϵ - δ malfare-optimal model with bounded sample complexity, for *any data distribution*, and for *any* (axiomatic-justified) malfare concept. Finally, we show broad conditions under which, with appropriate modifications, standard PAC-learners may be converted to fair-PAC learners. This places fair-PAC learning on firm theoretical ground, as it yields *statistical* and *computational* efficiency guarantees for many well-studied machine-learning models, and is also practically relevant, as it democratizes fair machine learning by providing concrete training algorithms and rigorous generalization guarantees for these models.

Keywords: Fair Machine Learning ♦ Cardinal Welfare Theory ♣ PAC-Learning
Uniform Convergence ♤ Computational Learning Theory ♠ Statistical Learning Theory

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1 Introduction

It is now well-understood that contemporary machine learning systems for facial recognition (Buolamwini and Gebru, 2018; Cook et al., 2019; Cavazos et al., 2020), medical settings (Mac Namee et al., 2002; Ashraf et al., 2018), and many others exhibit *differential accuracy* across gender, race, and other protected-group membership. This causes *accessibility issues* to users of such systems, and can lead to *direct discrimination*, e.g., facial recognition in policing yields disproportionate false-arrest rates, and machine learning in medical technology yields disproportionate health outcomes, thus exacerbating existing structural and societal inequalities impacting many minority groups. In welfare-centric machine learning methods, both *accuracy* and *fairness* are encoded in a single *welfare function* defined on a *collection of subpopulations*. Welfare is then directly optimized (Rolf et al., 2020) or constrained (Speicher et al., 2018; Heidari et al., 2018) to promote *fair learning* across *all groups*. This addresses differential performance and bias issues across groups by ensuring that (1), each group is *seen* and *considered* during training, and (2), an outcome is incentivized that is desirable overall, ideally according to some mutually-agreed-upon welfare function. Unfortunately, welfare based metrics require a notion of (positive) utility, and we argue that this is not natural to many machine learning tasks, where we instead *minimize* some *negatively connote*d risk value (expected loss). We thus define a complementary measure to *welfare*, termed *malfare*, measuring societal harm (rather than wellbeing). In particular, malfare arises naturally when one applies the standard *axioms of cardinal welfare* (with appropriate modifications) to *risk*, rather than *utility*. With this framework, we cast fair machine learning as a direct *malfare minimization* problem over the *risk values* of each *group*.

Perhaps surprisingly, defining and minimizing a *malfare function* is *not equivalent* to defining and maximizing some *welfare function* while taking utility to be negative loss (except in the trivial cases of egalitarian and utilitarian malfare). This is essentially because nearly every function satisfying the standard axioms of cardinal welfare requires *nonnegative* inputs, and it is not in general possible to contort a loss function into a utility function while satisfying this requirement. For example, while minimizing the 0-1 loss, which simply counts the number of mistakes a classifier makes, is isomorphic to maximizing the 1-0 gain, which counts number of correct classifications, minimizing some *malfare function* defined on 0-1 loss over groups *is not* in general equivalent to maximizing any *welfare function* defined on 1-0 gain. More strikingly, for learning problems with unbounded loss functions (i.e., absolute or square error in regression problems, or cross entropy in logistic regression), it is in general not even possible to define a complementary nonnegative gain function without changing the optimal solution.

Building upon these concepts, we develop a *mathematically precise* concept of generic fair machine learning, termed *fair probably-approximately-correct* (FPAC) learning, wherein a model class is FPAC-learnable if an ε - δ malfare-optimal model can be learned with *uniformly-bounded sample complexity*, w.r.t. any *fair malfare concept* and per-group *instance distributions*. In other words, it must be possible to learn a model that, with probability at least $1 - \delta$, has ε -additively optimal malfare, from a *finite sample* whose size depends only on ε , δ , the *group count*, and the *model class*, *but not* on the *instance distributions*, nor on the *malfare concept*. This definition extends Valiant’s (1984) PAC-learning formalization of machine learning beyond a single group, and we show that, with appropriate modifications, many (standard) PAC-learners may be converted to FPAC learners. We argue that FPAC-learners are intuitive and easy to use, as one must only select a malfare concept (encoding their desired fairness concept), model class, and error tolerance, and then one receives a provably ε - δ optimal model. Crucially, the class of “fair malfare concepts” considered in FPAC learning is not arbitrary, but rather arises from our natural axiomatization, and thus should contain every fair malfare objective that one would want to minimize.

The *uniformly-bounded sample complexity* requirement of FPAC-learnability is substantially stronger than classical concepts of statistical estimability. In particular, although *consistent* estimators of (dis)utility values generally imply consistent estimators of welfare or malfare functions, we show that the *rate* at which a consistent estimator converges, and thus *sample complexity*, is strongly impacted by the choice of welfare or malfare function, as well as the instance distributions. Consequently, a class may not be FPAC-learnable, even if there exist consistent estimators for per-group risk values for each model in the class. This is essentially due to the order of existential quantifiers: *uniform* sample complexity requires a convergence rate to hold *uniformly* over a family of related estimation tasks. Despite this difficulty, we show via a constructive polynomial reduction that *realizable FPAC-learning* and *realizable PAC-learning* are equivalent, and furthermore, we show, non-constructively, that for learning problems where PAC-learnability implies uniform convergence, it is equivalent to FPAC-learnability. We also show that when training is possible via *convex optimization*, or by efficient-enumeration of an *approximate cover* of the space of models, then training ε - δ malfare-optimal models, like risk-optimal models, requires polynomial time.

We argue that our axiomatization of malfare is quite natural, and the resulting family of malfare functions admits uniform sample-complexity guarantees. Section 4.2 explores the alternative *additive separability* axiom, under which the resulting welfare and malfare families are isomorphic to ours under comparison, however *uniform* sample complexity bounds are unsatisfying (and often impossible), essentially because additive-error guarantees are less meaningful, as the scale, and even the units, of additively separable malfare functions vary wildly across the family. Our alternative axiomatization essentially nonlinearly normalizes this variation in scale, and also standardizes malfare

units to match disutility units; under it, uniform sample complexity guarantees for welfare are possible and meaningful. It should be noted that uniform sample complexity bounds for *welfare functions* are generally impossible, due to the statistical instability of estimating some welfare functions, such as the *geometric mean* (or *Nash social welfare*), thus we argue that, compared to welfare maximization, welfare minimization is not only often more natural, but also more statistically tractable.

1.1 Related Work

Constraint-based notions of algorithmic fairness (Dwork et al., 2012) have risen to prominence in fair machine learning, with the potential to ensure demographic-parity (e.g., equality of opportunity, equality of outcome, or equalized odds), thus correcting for some forms of data or algorithmic bias. While noble in intent and intuitive by design, fairness by demographic-parity constraints has several prominent flaws: most notably, several popular parity constraints are mutually unsatisfiable (Kleinberg et al., 2017), and their constraint-based formulation inherently puts *accuracy* and *fairness* at odds, where additional *tolerance parameters* are required to strike a balance between the two. Furthermore, recent works (Hu and Chen, 2020; Kasy and Abebe, 2021) have shown that *welfare* and even *disadvantaged group utility* can decrease even as fairness constraints are tightened, calling into question whether demographic parity constraints are even beneficial to those they purport to aid.

Perhaps in response to these issues, some recent work has trended toward welfare-based fairness-concepts, wherein both *accuracy* and *fairness* are encoded in a *welfare function* defined on a *group of subpopulations*. Welfare is then directly optimized (Hu and Chen, 2020; Rolf et al., 2020; Siddique et al., 2020) or constrained (Speicher et al., 2018; Heidari et al., 2018) to promote *fair learning* across *all groups*. Perhaps the most similar to our work is a method of Hu and Chen (2020), wherein they *directly maximize* empirical welfare over linear (halfspace) classifiers; however as with other previous works, an appropriate utility function must be selected. We argue that *empirical welfare maximization* is an effective strategy when a measure of *utility* is available, but in machine learning contexts, there is no “correct” or clearly neutral way to convert loss to utility. Our strategy avoids this issue by working directly in terms of welfare and risk.

The above works, and even their criticisms, largely focus on fairness concepts in-and-of-themselves, and sparsely treat the issue of showing that a given fairness concept *generalizes* from *training* to *underlying task*. The history of machine learning is fraught with the consequences of ignoring overfitting (as after all, it is human nature to perceive patterns, even where none exist), and we argue they are particularly dire in fairness sensitive settings. We argue that *overfitting to fairness* is manifest not only in *generalization error*, but also as models *appearing fair* in training, but failing to be so on the underlying task. This can mean fairness constraints are satisfied in the training set but violated on the underlying distribution, or that a model overfits to *small* or *poorly studied* groups (for which a dearth of data may be available). More complicated issues may arise; with data-dependent constraints, the feasible model space is data-dependent, and thus learning may exhibit instability, sample complexity depends on these constraints in complicated ways, and in some cases it may not even be possible to satisfy all constraints.

Rothblum and Yona (2018) argue that the *individual-level* metric-fair constraints of Dwork et al. (2012) can’t be expected to generalize, so they introduce a relaxed notion for which they can show generalizability. Thomas et al. (2019) make similar criticisms, and introduce the *Seldonian learner* framework, which can be thought of as extending PAC-learning to learning problems with both *arbitrary constraints* and *arbitrary nonlinear objectives*. While very useful from a practical perspective to codify the desiderata of fair learning algorithms, the authors investigate individual Seldonian learners of interest, rather than studying the class of Seldonian learners as a *mathematical object*. Such study is difficult, due to the extreme generality of the class,¹ and also due to the difficulty of bounding sample complexity for *constrained objectives*.²

In contrast to the above methods, the FPAC-learning framework considers optimizing a single (unconstrained) cardinal welfare objective. No fairness tolerance parameters, demographic parity constraints, or explicit utility function definitions are required, and, although nonlinear, all fair welfare objectives, *unlike some fair welfare objectives*, are Lipschitz continuous. This simplicity also leads naturally to straightforward *statistical analysis* and *generalization guarantees* for welfare objectives, and such generalization guarantees are particularly significant, as with welfare, they control for overfitting of both accuracy and fairness. Consequently, in many cases, the *sample complexity* (statistical hardness), and often the *computational complexity* (algorithmic hardness) of training welfare-optimal models is comparable to standard (fairness-agnostic) machine-learning methods.

¹The Seldonian learner concept generalizes earlier fair-learnability concepts, such as *probably approximately correct and fair learning* for approximate metric-fairness (Rothblum and Yona, 2018), as well as the standard PAC concept, and indeed, the FPAC concept presented here.

²Note that the sample complexity of determining whether constraints are even feasible is, in general, unbounded.

1.2 Contributions

This manuscript is split into two main parts; we first define welfare and derive its properties in sections 2 to 4, and subsequently we define and explore FPAC learning (and learnability) in sections 5 to 7. We briefly summarize our contributions as follows.

1. We derive in section 2 the *malfare concept*, extending welfare to measure *negatively-connoted* sentiments, and show that *malfare-minimization* naturally generalizes *risk-minimization* to produce *fairness-sensitive* machine-learning objectives that consider multiple protected groups.
2. We show in section 3 that in many cases, while empirical estimates of welfare and malfare are *statistically biased*, they are *consistent*, and malfare may be *sharply estimated* using finite-sample concentration-of-measure bounds.
3. In section 4, we examine the decisions made in section 2, and explore what would change under alternative axioms and other counterfactuals. We also contrast malfare minimization with welfare maximization, and relate both to fairness constraints on *inequality indices*. This section contextualizes the work as a whole, but may be skipped without impeding understanding of the sequel.
4. Section 5 extends PAC-learning to fair-PAC (FPAC) learning, where we consider minimization not only of *risk* (expected loss) objectives, but also of *malfare* objectives. Both PAC and FPAC learning are parameterized by a *learning task* (model space and loss function), and we explore the rich learnability-hierarchy under variations of these concepts. In particular, we show that
 - (a) for many loss functions, PAC and FPAC learning are *statistically equivalent* (i.e., PAC-learnability implies FPAC-learnability) in section 6; and
 - (b) standard convexity and coverability conditions sufficient for PAC-learnability are also sufficient for FPAC-learnability in section 7.

While we explore the basic relationships between various learnability classes, many open questions remain, and we hope future work will further characterize these practically interesting and theoretically deep problems. For brevity, longer, more technical proofs are presented in the appendix.

2 Aggregating Sentiment within Populations

A generic *aggregator function* function $M(\mathcal{S}; \mathbf{w})$ quantifies some *sentiment value* \mathcal{S} in aggregate across a population Ω weighted by \mathbf{w} . In particular, $\mathcal{S} : \Omega \rightarrow \mathbb{R}_{0+}$ describes the *values* over which we aggregate, and \mathbf{w} , a probability measure over Ω , describes their *weights*. We assume throughout the *nondegeneracy condition* that $\text{Support}(\mathbf{w}) = \Omega$; this ensures no part of the population is ignored, and simplifies the algebra and presentation. We also often assume $|\Omega| > 1$, and usually Ω is finite, in which case \mathcal{S} and \mathbf{w} may be represented as a *sentiment vector* and *probability vector*, respectively.

When \mathcal{S} measures a *desirable quantity*, generally termed *utility*, the aggregator function is a measure of *cardinal welfare* (Moulin, 2004), and thus quantifies overall *wellbeing*. We also consider the inverse-notion, that of overall *illbeing*, termed *malfare*, in terms of an *undesirable* \mathcal{S} , generally *loss* or *risk*, which naturally extends the concept. We show an equivalent *axiomatic justification* for malfare, and argue that its use is more natural in many situations, particularly when considering or optimizing *loss functions* in machine learning.

Definition 2.1 (Aggregator Functions: Welfare and Malfare). An *aggregator function* function $M(\mathcal{S}; \mathbf{w})$ measures the *overall sentiment* of population Ω , measured by *sentiment function* $\mathcal{S} : \Omega \rightarrow \mathbb{R}_{0+}$, weighted by *probability measure* \mathbf{w} over Ω (with full support). If \mathcal{S} denotes a desirable quantity (e.g., utility), we call $M(\mathcal{S}; \mathbf{w})$ a *welfare function*, written $W(\mathcal{S}; \mathbf{w})$, and inversely, if it is undesirable (e.g., disutility, loss, or risk), we call $M(\mathcal{S}; \mathbf{w})$ a *malfare function*, written $M(\mathcal{S}; \mathbf{w})$.

For now, think of the term *aggregator function* as signifying that an entire population, with diverse and subjective desiderata, is considered and summarized, as opposed to an individual's objective viewpoint (sentiment value). Note that we use the term *sentiment* to refer to \mathcal{S} with neutral connotation, but when discussing welfare or malfare, we often refer to \mathcal{S} as *utility* or *risk*, respectively, as in these cases, \mathcal{S} describes a well-understood preexisting concept. Coarsely speaking, the three notions are identical, all being functions of the form³ $(\Omega \rightarrow \mathbb{R}_{0+}) \times \text{MEASURE}(\Omega, 1) \rightarrow \bar{\mathbb{R}}$, however, we shall see that in order to promote fairness, the axioms of malfare and welfare functions differ slightly. The notation reflects this; $M(\mathcal{S}; \mathbf{w})$ is an M for *mean*, whereas $W(\mathcal{S}; \mathbf{w})$ is a W for *welfare*, and $M(\mathcal{S}; \mathbf{w})$ is an M (*inverted W*), to emphasize its inverted nature.

³Ideally, aggregator functions would have domain $\mathbb{R}_{0+} = [0, \infty)$ (the nonnegative reals), rather than $\bar{\mathbb{R}} = [-\infty, \infty]$ (the extended reals), to match that of the sentiment value function, but infinite and/or negative aggregates are sometimes required, particularly in the *additively separable form* (see section 4.2).

Often we are interested in *unweighted* aggregator functions of finite discrete populations, where the *sentiment function* may be represented as a *sentiment vector* $\mathcal{S} \in \mathbb{R}_{0+}^g$. Unweighted aggregators may then be defined in terms of weighted aggregators as

$$M(\mathcal{S}) \doteq M(i \mapsto \mathcal{S}_i; i \mapsto \frac{1}{g}) ,$$

abusing notation to concisely express the *uniform measure*. Indeed, it may seem antithetical to fairness to allow for weights in welfare and welfare definitions; consider however that weights can represent *differential population sizes*, and thus ensure that the welfare or malice of *weight-preserving decompositions* of groups into subgroups with equal risk or utility remains constant.

Example 2.2 (Utilitarian Welfare). Suppose individuals reside in some space \mathcal{X} , where *distributions* $\mathcal{D}_{1:g}$ over domain \mathcal{X} describe the distribution over individuals in *each group*. Suppose also *utility function* $U(x) : \mathcal{X} \rightarrow \mathbb{R}_{0+}$, describing the *level of satisfaction* of an individual, w.r.t., e.g., some *situation, allocation, or classifier*. We now take the *sentiment function* to be the *arithmetic mean utility* (per-group), i.e.,

$$\mathcal{S}(\omega_i) \doteq \mathbb{E}_{x \sim \mathcal{D}_i}[U(x)] = \mathbb{E}_{\mathcal{D}_i}[U] .$$

Now, given a *weights vector* \mathbf{w} , describing the *relative frequencies* of membership in each of the g groups, we define the *utilitarian welfare* as

$$W_1(\mathcal{S}; \mathbf{w}) \doteq \sum_{i=1}^g \mathbf{w}(\omega_i) \mathcal{S}(\omega_i) = \mathbb{E}_{\omega \sim \mathbf{w}}[\mathcal{S}(\omega)] = \mathbb{E}_{\mathbf{w}}[\mathcal{S}] .$$

Of course, in statistical, sampling, and machine learning contexts, $\mathcal{D}_{1:g}$ and \mathbf{w} may be unknown, so we now discuss an *empirical analog* of utilitarian welfare. Section 3 is then devoted to showing how and when empirical aggregator functions well-approximate their true counterparts.

Example 2.3 (Empirical Utilitarian Welfare). Now suppose $\mathcal{D}_{1:g}$ are unknown, but instead, we are given a *sample* $\mathbf{x}_{1:g, 1:m} \in \mathcal{X}^{g \times m}$, where $\mathbf{x}_{i, 1:m} \sim \mathcal{D}_i$. We define the *empirical analog* of the utilitarian welfare as

$$\hat{\mathcal{S}}(\omega_i) \doteq \hat{\mathbb{E}}_{x \in \mathbf{x}_i}[U(x)] \quad \& \quad \hat{W}_1(\hat{\mathcal{S}}, \mathbf{w}) \doteq \mathbb{E}_{\mathbf{w}}[\hat{\mathcal{S}}] .$$

Similarly, if \mathbf{w} is unknown, but we may sample from some \mathcal{D} over $\Omega \times \mathcal{X}$, we can use *empirical frequencies* $\hat{\mathbf{w}}$ in place of *true frequencies* \mathbf{w} , and define $\hat{\mathcal{S}}(\omega_i)$ as *conditional averages* over the subsample associated with group i .

2.1 Axioms of Cardinal Welfare and Malfare

In this section, we describe various desiderata for aggregator functions, and in particular for fair malice and welfare functions. We shall see that the utilitarian welfare is the only aggregator function that is both a fair malice and welfare function (due to the opposite sense of utility and disutility, *egalitarian* welfare and malice are analogous, but do not share a functional form, being the *minimum* or *maximum* sentiment value, respectively). In general, with our axioms, all aggregator functions belong to the single-parameter *power-mean* family (section 2.2), but if an alternative, *additive separability* axiom is instead taken, we get a similar family (section 4.2). The axioms are generally referred to as the axioms of cardinal welfare, though nearly all work equally well as malice axioms. Typically, they are stated for *positive, unweighted, and finite* populations, rather than non-negative, weighted, and measurable populations, but the technical impact of this distinction is quite minor.⁴

Definition 2.4 (Axioms of Cardinal Welfare and Malfare). We define the *aggregator function axioms* for aggregator function $M(\mathcal{S}; \mathbf{w})$ below. For each item, assume (if necessary) that the axiom applies $\forall \mathcal{S}, \mathcal{S}' \in \Omega \rightarrow \mathbb{R}_{0+}$, scalars $\alpha, \beta \in \mathbb{R}_{0+}$, and probability measures \mathbf{w} over Ω .

1. (Strict) Monotonicity: If $0 \notin \mathcal{S}(\Omega)$, then $\forall \epsilon : \Omega \rightarrow \mathbb{R}_{0+}$ s.t. $\int_{\omega} \epsilon(\omega) d(\omega) > 0$: $M(\mathcal{S}; \mathbf{w}) < M(\mathcal{S} + \epsilon; \mathbf{w})$.
2. Symmetry: \forall permutations π over Ω : $M(\mathcal{S}; \mathbf{w}) = M(\pi(\mathcal{S}); \pi(\mathbf{w}))$.
3. Continuity: $\{\mathcal{S}' \mid M(\mathcal{S}'; \mathbf{w}) \leq M(\mathcal{S}; \mathbf{w})\}$ and $\{\mathcal{S}' \mid M(\mathcal{S}'; \mathbf{w}) \geq M(\mathcal{S}; \mathbf{w})\}$ are closed sets.
4. Independence of Unconcerned Agents (IOUA): Suppose subpopulation $\Omega' \subseteq \Omega$. Then

$$M\left(\begin{cases} \omega \in \Omega' : \alpha \\ \omega \notin \Omega' : \mathcal{S}(\omega) \end{cases}; \mathbf{w}\right) \leq M\left(\begin{cases} \omega \in \Omega' : \alpha \\ \omega \notin \Omega' : \mathcal{S}'(\omega) \end{cases}; \mathbf{w}\right) \implies M\left(\begin{cases} \omega \in \Omega' : \beta \\ \omega \notin \Omega' : \mathcal{S}(\omega) \end{cases}; \mathbf{w}\right) \leq M\left(\begin{cases} \omega \in \Omega' : \beta \\ \omega \notin \Omega' : \mathcal{S}'(\omega) \end{cases}; \mathbf{w}\right) .$$
5. Independence of Common Scale (IOCS): $M(\mathcal{S}; \mathbf{w}) \leq M(\mathcal{S}'; \mathbf{w}) \implies M(\alpha \mathcal{S}; \mathbf{w}) \leq M(\alpha \mathcal{S}'; \mathbf{w})$.

⁴In particular, allowing \mathcal{S} to attain 0 values (by including the appropriate limit sequences) may violate *strict monotonicity*, so the condition is relaxed around 0.

6. Multiplicative Linearity: $M(\alpha \mathcal{S}; \mathbf{w}) = \alpha M(\mathcal{S}; \mathbf{w})$.
7. Unit Scale: $M(\mathbf{1}; \mathbf{w}) = M(\omega \mapsto 1; \mathbf{w}) = 1$.
8. Pigou-Dalton Transfer Principle: Suppose $\mu = \mathbb{E}_{\mathbf{w}}[\mathcal{S}] = \mathbb{E}_{\mathbf{w}}[\mathcal{S}']$, and $\forall \omega \in \Omega : |\mu - \mathcal{S}'(\omega)| \leq |\mu - \mathcal{S}(\omega)|$. Then $W(\mathcal{S}'; \mathbf{w}) \geq W(\mathcal{S}; \mathbf{w})$.
9. Anti Pigou-Dalton Transfer Principle: Suppose as in axiom 8, and conclude $M(\mathcal{S}'; \mathbf{w}) \leq M(\mathcal{S}; \mathbf{w})$.

We take a moment to comment on each of these axioms, to preview their purpose and assure the reader of their necessity. Axioms 1-5 are the standard *axioms of cardinal welfarism* (1-4 are discussed by Sen (1977); Roberts (1980), and 5 by Debreu (1959); Gorman (1968)). Together, they imply (via the Debreu-Gorman theorem) that any aggregator function can be decomposed as a *monotonic function of a sum* (over groups) of *logarithm* or *power* functions. Axiom 6 is a natural and useful property, and ensures that *dimensional analysis* on aggregator functions is possible; in particular, the *units* of aggregator functions match those of sentiment values. Note that axiom 6 implies axiom 5, and it is thus a simple strengthening of a traditional cardinal welfare axiom. We will also see that it is essential to show convenient *statistical* and *learnability* properties. Axiom 7 furthers this theme, as it ensures that not only do *units* of aggregates match those of \mathcal{S} , but *scale* does as well (making comparisons like “ \mathcal{S}_i is above the welfare (of the population)” meaningful), and also enabling comparison *across populations*, in the sense that comparing *averages* is more meaningful than *sums*. Finally, axiom 8 (the *Pigou-Dalton transfer principle*. see Pigou (1912); Dalton (1920)) is also standard in cardinal welfare theory as it ensures fairness, in the sense that welfare is higher when utility values are more uniform, i.e., incentivizing *equitable redistribution of “wealth”* in welfare. Its antithesis, axiom 9, encourages the opposite; in the context of welfare, this perversely incentivizes an expansion of inequality, but for malfare, which we generally wish to *minimize*, the opposite occurs, thus this axiom characterizes *fairness for malfare*.

Axioms 6 & 7 are novel to this work, and are key in strengthening the Debreu-Gorman theorem to ensure that all welfare and malfare functions are *power means* in the sequel. Axiom 9 is also novel, as it is necessary to flip the inequality of axiom 8 when the sense of the aggregator function is inverted from welfare to malfare; in particular, the semantic meaning shifts from requiring that “redistribution of utility is *desirable*” to “redistribution of disutility is *not undesirable*.”

2.2 The Power Mean

We now define the *p-power mean*⁵ $M_p(\cdot)$, for any $p \in \bar{\mathbb{R}}$, and the weighted *p-power-mean* $M_p(\cdot; \cdot)$, which we shall use to quantify both malfare and welfare. We shall see that power means exhibit many convenient properties (theorem 2.6), and arise often (theorem 2.7) when analyzing aggregator functions obeying the various axioms of definition 2.4.

Definition 2.5 (Power-Mean Welfare and Malfare). Suppose $p \in \bar{\mathbb{R}}$. We first define the *unweighted power-mean* of sentiment vector $\mathcal{S} \in \mathbb{R}_{0+}^g$ as

$$M_p(\mathcal{S}) \doteq \begin{cases} p \in \mathbb{R} \setminus \{0\} & \sqrt[p]{\frac{1}{g} \sum_{i=1}^g \mathcal{S}_i^p} \\ p = -\infty & \min_{i \in 1, \dots, g} \mathcal{S}_i \\ p = 0 & \sqrt[g]{\prod_{i=1}^g \mathcal{S}_i} = \exp\left(\frac{1}{g} \sum_{i=1}^g \ln(\mathcal{S}_i)\right) \\ p = \infty & \max_{i \in 1, \dots, g} \mathcal{S}_i . \end{cases}$$

We now define the *weighted power-mean*, given *sentiment value function* $\mathcal{S} : \Omega \rightarrow \mathbb{R}_{0+}$ and *probability measure* \mathbf{w} over Ω , as

$$M_p(\mathcal{S}; \mathbf{w}) \doteq \begin{cases} p \in \mathbb{R} \setminus \{0\} & \sqrt[p]{\int_{\mathbf{w}} \mathcal{S}^p(\omega) d(\omega)} = \sqrt[p]{\mathbb{E}_{\omega \sim \mathbf{w}} [\mathcal{S}^p(\omega)]} \\ p = -\infty & \inf_{\omega \in \Omega} \mathcal{S}(\omega) \\ p = 0 & \exp\left(\int_{\mathbf{w}} \ln \mathcal{S}(\omega) d(\omega)\right) = \exp\left(\mathbb{E}_{\omega \sim \mathbf{w}} [\ln \mathcal{S}(\omega)]\right) \\ p = \infty & \sup_{\omega \in \Omega} \mathcal{S}(\omega) . \end{cases}$$

⁵The *p*-power-mean is referred to by some authors as the *generalized mean* or Hölder mean, and is itself a generalization of the *Pythagorean* (arithmetic, geometric, and harmonic) means.

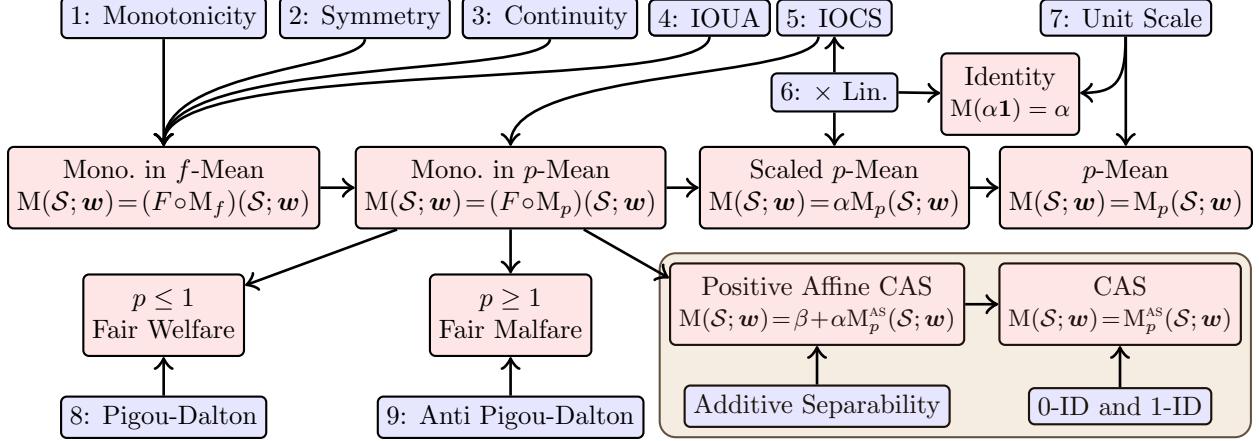


Figure 1: Relationships between aggregator function axioms and properties. Assumptions and axioms are shown in pastel blue, and properties shown in pastel red. These results are stated as theorem 2.7, except for the *additive separability* results (brown box), which are derived in section 4.2.

In both the weighted and unweighted cases, $p \in \{-\infty, 0, \infty\}$ resolve as their (unique) limits, and for all $p \in \mathbb{R}$, *power means* are special cases of the (weighted) *generalized f-mean* (a.k.a. the *f*-mean or Kolmogorov mean), defined for strictly monotonic f as

$$M_f(\mathcal{S}; \mathbf{w}) \doteq f^{-1} \left(\mathbb{E}_{\omega \sim \mathbf{w}} [(f \circ \mathcal{S})(\omega)] \right).$$

Note also that, as always, we assume nondegeneracy condition $\text{Support}(\mathbf{w}) = \Omega$; otherwise the $p \in \pm\infty$ cases would need to restrict their attention to $\text{Support}(\mathbf{w})$, rather than all of Ω . Finally, note that, if care is not taken, $M_p(\mathcal{S}; \mathbf{w})$ is $p \leq 0$ is undefined when some $\mathcal{S}(\omega) = 0$, as this creates $\log(0)$ or $\frac{1}{0}$ expressions. We resolve this issue by taking the above definitions for positive-valued \mathcal{S} , and extending to the general case by taking

$$\lim_{\varepsilon \rightarrow 0^+} M_p(\mathcal{S} + \varepsilon; \mathbf{w}).$$

Theorem 2.6 (Properties of the Power-Mean). Suppose $\mathcal{S}, \mathcal{S}'$ are sentiment functions in $\Omega \rightarrow \mathbb{R}_{0+}$, and \mathbf{w} is a probability measure over Ω . The following then hold.

1. Monotonicity: $M_p(\mathcal{S}; \mathbf{w})$ is weakly-monotonically-increasing in p , and strictly so if $\exists \omega, \omega' \in \Omega$ s.t. $\mathcal{S}(\omega) \neq \mathcal{S}(\omega')$.
2. Subadditivity: $\forall p \geq 1 : M_p(\mathcal{S} + \mathcal{S}'; \mathbf{w}) \leq M_p(\mathcal{S}; \mathbf{w}) + M_p(\mathcal{S}'; \mathbf{w})$.
3. Contraction: $\forall p \geq 1 : |M_p(\mathcal{S}; \mathbf{w}) - M_p(\mathcal{S}'; \mathbf{w})| \leq M_p(|\mathcal{S} - \mathcal{S}'|; \mathbf{w}) \leq \|\mathcal{S} - \mathcal{S}'\|_\infty$.
4. Curvature: $M_p(\mathcal{S}; \mathbf{w})$ is concave in \mathcal{S} for $p \in [-\infty, 1]$ and convex for $p \in [1, \infty]$.

2.3 Properties of Welfare and Malfare Functions

We now show that the axioms of definition 2.4 are sufficient to characterize many properties of welfare and malfare functions.

Theorem 2.7 (Aggregator Function Properties). Suppose aggregator function $M(\mathcal{S}; \mathbf{w})$. If $M(\cdot; \cdot)$ satisfies (subsets of) the aggregator function axioms (see definition 2.4), we have that $M(\cdot; \cdot)$ exhibits the following properties. For each, assume arbitrary sentiment-value function $\mathcal{S} : \Omega \rightarrow \mathbb{R}_{0+}$ and weights measure \mathbf{w} over Ω . The following then hold.

1. *Identity*: Axioms 6-7 imply $M(\omega \mapsto \alpha; \mathbf{w}) = \alpha$.
2. *Linear Factorization*: Axioms 1-3 imply strictly-monotonically-increasing continuous $F, f : \mathbb{R} \rightarrow \mathbb{R}$, s.t.

$$M(\mathcal{S}; \mathbf{w}) = F \left(\int_{\mathbf{w}} f(\mathcal{S}(\omega)) d(\omega) \right) = F \left(\mathbb{E}_{\omega \sim \mathbf{w}} [f(\mathcal{S}(\omega))] \right).$$

3. *Debreu-Gorman*: Axioms 1-5 imply that, for some $p \in \mathbb{R}$, $f(x) = f_p(x) \doteq \begin{cases} p = 0 & \ln(x) \\ p \neq 0 & \text{sgn}(p)x^p \end{cases}$.
4. *Power Mean*: Axioms 1-7 imply $F(x) = f_p^{-1}(x)$, thus $M(\mathcal{S}; \mathbf{w}) = M_p(\mathcal{S}; \mathbf{w})$.

5. *Fair Welfare*: Axioms 1-5 and 8 imply $p \in (-\infty, 1]$.
6. *Fair Malfare*: Axioms 1-5 and 9 imply $p \in [1, \infty)$.

Taken together, the items of theorem 2.7 tell us that the mild conditions of axioms 1-5 (generally assumed for welfare), along with the *multiplicative linearity* axiom (6), imply that welfare and utility, or malfare and loss, are measured in the *same units* (e.g., *nats* or *bits* for *cross-entropy loss*, square- \mathcal{Y} -units for *square error*, or *dollars* for *income utility*). Furthermore, the entirely milquetoast *unit scale* axiom (7) implies that sentiment values and aggregator functions have the same *scale*, imbuing meaning to comparisons like “the risk of group i is above (or below) the population malfare.” Finally, as far as fairness goes, the Pigou-Dalton transfer principle (axiom 8) leads to the conclusion that $p \in [-\infty, 1)$ incentivize redistribution of utility from better-off groups to worse-off groups, and similarly, the corresponding principle for malfare (axiom 9) yields the conclusion that $p \in (1, \infty]$ incentivize redistribution of harm⁶ from worse-off groups to better-off groups.

We may also conclude that the power-mean is effectively the only reasonable family of welfare or malfare functions. Even without axioms 6-7, axioms 1-5 imply (via the Debreu-Gorman theorem) that all aggregator functions are still *monotonic transformations* of power-means. These and other results relating various aggregator functions to the relevant axioms are summarized in figure 1.

3 Statistical Estimation of Welfare and Malfare Values

We now show that for countable populations, consistent estimators for sentiment values imply consistent estimators for aggregator functions (via the plugin estimator). Despite this promising first step, in general, aggregator functions don’t preserve *unbiasedness* or even *asymptotic unbiasedness* of sentiment value estimators, and furthermore, the *rate of convergence* of consistent estimators to the true aggregator function depends intricately on the aggregator function in question. The following lemma requires only the *monotonicity axiom*, and allows us to bound *aggregator functions* in terms of *estimated sentiment values*.

Lemma 3.1 (Statistical Estimation). Suppose probability distribution \mathcal{D} over \mathcal{X} , sample $\mathbf{x} \sim \mathcal{D}^m$, and some function $f : \mathcal{X} \times \Omega \rightarrow \mathbb{R}_{0+}$. Let *sentiment value* function $\mathcal{S}(\omega) \doteq \mathbb{E}_{x \sim \mathcal{D}}[f(x; \omega)]$, and *empirical sentiment value estimate* $\hat{\mathcal{S}}(\omega) \doteq \hat{\mathbb{E}}_{x \in \mathbf{x}}[f(x; \omega)]$. If it holds for some $\varepsilon > 0$ that, with probability at least $1 - \delta$ over choice of \mathbf{x} , $\forall \omega \in \Omega : \hat{\mathcal{S}}(\omega) - \varepsilon(\omega) \leq \mathcal{S}(\omega) \leq \hat{\mathcal{S}}(\omega) + \varepsilon(\omega)$, then with said probability, for all aggregator functions $M(\cdot; \cdot)$ obeying the *monotonicity axiom* (definition 2.4 item 1) and weights measures \mathbf{w} over Ω , we have that

$$M_p(\mathbf{0} \vee (\hat{\mathcal{S}} - \varepsilon); \mathbf{w}) \leq M_p(\mathcal{S}; \mathbf{w}) \leq M_p(\hat{\mathcal{S}} + \varepsilon; \mathbf{w}),$$

where $\mathbf{a} \vee \mathbf{b}$ denotes the (elementwise) maximum.

Proof. This result follows from the assumption, and the *monotonicity* axiom (i.e., adding/subtracting ε can not decrease/increase the aggregate, respectively). The minimum with 0 on the LHS is *valid* simply because, by definition, sentiment values are nonnegative, and is *necessary*, since $M_p(\cdot; \cdot)$ is in general undefined on negative sentiment values. \square

The principal question we are interested in however is not merely *whether* an estimator is consistent, but rather *how rapidly* it converges to the true aggregator function. In particular, an ε - δ additive-error guarantee allows us to solve for the sample complexity of estimating a particular aggregator function to within ε - δ error. Furthermore, we are interested in *uniform* sample complexity bounds, which need to hold *uniformly* over a family of probability distribution and aggregator functions. This is even trickier than showing simple single-function sample complexity bounds, because it can be the case that while any individual function in the family admits a sample complexity bound, the entire family has unbounded sample complexity.⁷ The following result shows such a uniform guarantee for fair malfare functions, by applying the well-known Hoeffding (1963) and Bennett (1962) bounds to show concentration, and derive an explicit form for ε .

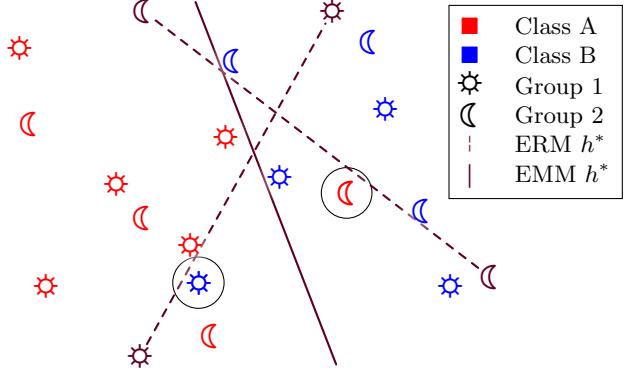
Corollary 3.2 (Statistical Estimation with Hoeffding and Bennett Bounds). Suppose fair power-mean malfare $M(\cdot; \cdot)$ (i.e., $p \geq 1$), discrete *weights measure* \mathbf{w} over g groups, *probability distributions* $\mathcal{D}_{1:g}$, *samples* $\mathbf{x}_i \sim \mathcal{D}_i^m$, and *loss function* $\ell : \mathcal{X} \rightarrow [0, r]$ s.t. $\mathcal{S}_i = \mathbb{E}_{\mathcal{D}_i}[\ell]$ and $\hat{\mathcal{S}}_i \doteq \hat{\mathbb{E}}_{\mathbf{x}_i}[\ell]$. Then, with probability at least $1 - \delta$ over choice of \mathbf{x} ,

$$\left| M_p(\mathcal{S}; \mathbf{w}) - M_p(\hat{\mathcal{S}}; \mathbf{w}) \right| \leq r \sqrt{\frac{\ln \frac{2g}{\delta}}{2m}}.$$

⁶Note that, mathematically speaking, it is entirely *valid* to quantify welfare with $p > 1$ or malfare with $p < 1$, and indeed such characterizations may arise in the analysis of unfair systems; however we generally advocate against *intentionally creating* such unfair systems.

⁷This is essentially due to the order of existential quantifiers: each quantity in the family may admit bounded sample complexity, even while the entire family has unbounded sample complexity.

Figure 2: Empirical welfare minimization on a *linear classifier* family in \mathbb{R}^2 (with affine offset) over two groups. Note that classification is realizable for both groups *individually*, in the sense that both are linearly separable, thus there exists a 0-risk classifier for each, though *jointly*, they are not realizable. Risk-optimal classifiers are shown for both groups (dashed lines), as is a welfare-optimal classifier (solid line). Note that exactly which classifier is optimal depends on the weighting and welfare metric, but the selected welfare-minimizer compromises fairly in the sense that each group suffers one error (circled).



Alternatively, again with probability at least $1 - \delta$ over choice of \mathbf{z} , we have

$$|\mathbb{M}_p(\mathcal{S}; \mathbf{w}) - \mathbb{M}_p(\hat{\mathcal{S}}; \mathbf{w})| \leq \frac{r \ln \frac{2g}{\delta}}{3m} + \max_{i \in 1, \dots, g} \sqrt{\frac{2 \mathbb{V}_{\mathcal{D}_i}[\ell] \ln \frac{2g}{\delta}}{m}}.$$

Corollary 3.2 follows directly from lemma 3.1, with Hoeffding and Bennett inequalities applied to derive ϵ bounds, and similar results are immediately possible with arbitrary concentration inequalities. In particular, similar *data-dependent* bounds may be shown, e.g., with *empirical Bennett bounds*, removing dependence on *a priori known variance*. Furthermore, while such bounds may be used for evaluating the welfare or welfare of a *particular* classifier or mechanism (through \mathcal{S} and $\hat{\mathcal{S}}$), in machine-learning contexts, \mathcal{S} may be a function of some model, so we must consider the entire *space of possible models*, represented by some *hypothesis class* \mathcal{H} . Via the union bound, corollary 3.2 is sufficient for *learning* over *finite* \mathcal{H} , as the exponential tail bounds allow \mathcal{H} to grow *exponentially*, at *linear* cost to sample complexity. As in standard uniform convergence analysis (generally discussed in the context of *empirical risk minimization*), we can easily handle infinite hypothesis classes, and obtain much sharper bounds by considering *data-dependent uniform-convergence bounds* over the family, e.g., with Rademacher averages (Bartlett and Mendelson, 2002), *localized Rademacher averages* (Bartlett et al., 2005), or *empirically-centralized Rademacher averages* (Cousins and Riondato, 2020).

3.1 The Empirical Welfare Minimization Principle

In learning contexts, minimizing the *welfare* among all groups generalizes minimizing *risk* of a single group. These statistical estimation bounds immediately imply that the *empirical welfare-optimal* solution is a reasonable proxy for the true welfare-optimal solution, as we now formalize. Figure 2 illustrates empirical welfare minimization in action with a *linear classifier* on two groups.

Definition 3.3 (The Empirical Welfare Minimization (EMM) Principle). Suppose hypothesis class $\mathcal{H} \subseteq \mathcal{X} \rightarrow \mathcal{Y}$, training samples $\mathbf{z}_{1:g}$ drawn from distributions $\mathcal{D}_{1:g}$ over $\mathcal{X} \times \mathcal{Y}$, loss function $\ell : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}_{0+}$, welfare function \mathbb{M} , and group weights \mathbf{w} . The *empirical welfare minimizer* is then defined as

$$\hat{h} \doteq \operatorname{argmin}_{h \in \mathcal{H}} \mathbb{M}(i \mapsto \hat{R}(h; \ell, \mathbf{z}_i); \mathbf{w}),$$

and the EMM principle states that \hat{h} is a reasonable proxy for the *true welfare minimizer*

$$h^* \doteq \operatorname{argmin}_{h \in \mathcal{H}} \mathbb{M}(i \mapsto R(h; \ell, \mathcal{D}_i); \mathbf{w}).$$

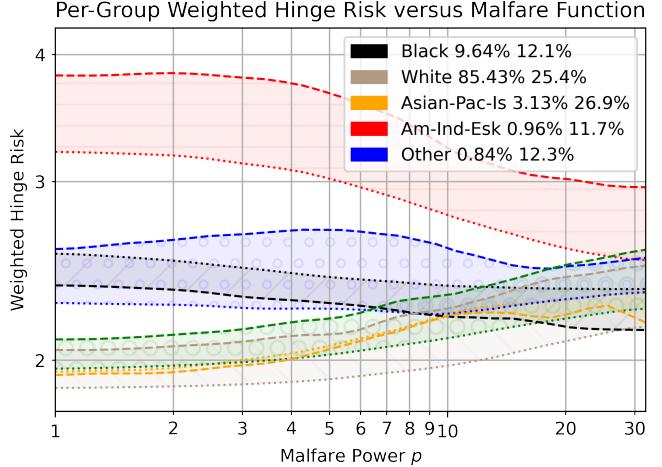
3.2 Experimental Validation of Empirical Welfare Minimization

Figure 3 presents a brief experiment on the lauded `adult` dataset, where the task is to predict whether income is above or below \$50k/year. We train $\mathbb{M}_p(\cdot; \mathbf{w})$ -minimizing SVM, and find significant variation in model performance (as measured by risk) between groups; in general, we observe that the classifier is most accurate for the *white* and *Asian-Pacific-Islander* groups, and generally less accurate for the *native American* and *other* groups. The $p = 1$ model is a standard weighted SVM, with poor performance for small and traditionally marginalized groups, as expected in an 85.43% majority-white population. As p increases (towards egalitarianism), we observe interesting fairness

Figure 3: We minimize malfare on a *weighted hinge-loss* SVM, with $g = 5$ racial groups, listed in the figure legend with *group weight* \mathbf{w}_i (population frequency) and *class bias* \mathbf{b}_i (proportion with income $\geq \$50,000$ per annum). Due to existing societal inequity, *class imbalance* varies widely by group, so we weight all risk values as $\frac{1}{\mathbf{b}_i} \hat{\mathbf{R}}(h; \ell_{\text{hinge}}, \mathbf{z}_i)$. We report per-group training (dotted) and test (dashed) hinge risk, along with the $\Lambda_p(\cdot; \mathbf{w})$ value (green) of the EMM solution

$$\hat{h} \doteq \operatorname{argmin}_{h \in \mathcal{H}} \Lambda_p \left(i \mapsto \frac{1}{\mathbf{b}_i} \hat{\mathbf{R}}(h; \ell_{\text{hinge}}, \mathbf{z}_i); \mathbf{w} \right),$$

as a function of $p \in [1, 32]$. The experimental setup is fully detailed in appendix B.1.



tradeoffs; training malfare increases monotonically, and in general (but not monotonically⁸), *white* and *Asian* training risks increase, as the remaining risks decrease, and greater equity is achieved. At first, most improvement is in the relatively-large (9.64%), high-risk Black group, but for larger p , the much smaller (0.96%), but even higher-risk, native American group sharply improves.

Both training and test performance generally improve for high-risk groups, but significant overfitting occurs in small groups and malfare. This is unsurprising, as although SVM generalization error is well-understood (see Shalev-Shwartz and Ben-David, 2014, Chapter 26), bounds are generally vacuous for tiny subpopulations of ≈ 400 individuals. In general, overfitting increases with p , due to higher relative importance of small high-risk groups on \hat{h} . This experiment validates EMM as a fair-learning technique, with the capacity to specify tradeoffs between majority and marginalized groups, while demonstrating *overfitting to fairness*, which we formally treat in the sequel. We observe similar fairness tradeoffs in our supplementary experiments (appendix B.2), on weighted and unweighted SVM and logistic regressors with race and gender groups.

4 Comparative Analysis of Welfare, Malfare, and Inequality Indices

This section serves as an interlude between the concept and axiomatic derivation malfare, and the statistical and machine learning applications of malfare minimization. Here we examine some of our core decisions, and explore the differences that arise under alternative axioms and other counterfactuals.

In particular, section 4.1 shows that malfare and welfare functions are not equivalent, and describes salient differences that arise when trying to estimate them from sampled (dis)utility values. Section 4.2 then shows that under an alternative axiomatization, i.e., that of additive separability, the concept of uniform sample complexity is generally ill-behaved. Finally, section 4.3 explores the relationships between *inequality indices* and welfare or malfare functions, deriving deep connections between the power mean and the Atkinson, Theil, and generalized entropy indices.

4.1 The Non-Equivalence of Welfare and Malfare Functions

We now take a moment to comment on the surprising dissimilarity between welfare and malfare functions. In particular, we show that intuition from univariate optimization, where maximization and minimization are symmetric, breaks down for welfare maximization and malfare minimization, and furthermore, from the perspective of estimation, except for the egalitarian and utilitarian cases, no fair welfare function is equivalent to any fair malfare function.

We would like to show that there *does not exist* some mapping between utility and disutility values, such that under said mapping, welfare and malfare are equivalent. Furthermore we adopt a weak notion of equivalence, requiring only that they induce the same partial ordering. In other words, given fair welfare and malfare functions $W(\cdot; \cdot)$ and $\Lambda(\cdot; \cdot)$, we now study the existence of mappings $f_S(\cdot)$ and $F_M(\cdot)$ such that

$$F_M \circ W(f_S \circ S; \mathbf{w}) = \Lambda(S; \mathbf{w}), \quad (\text{or equivalently, } W(S; \mathbf{w}) = F_M \circ \Lambda(f_S \circ S; \mathbf{w})),$$

⁸Note that for continuous loss functions and $g = 2$ groups, group training risks are monotonic in p , as seen in the supplementary *gender-group* experiments.

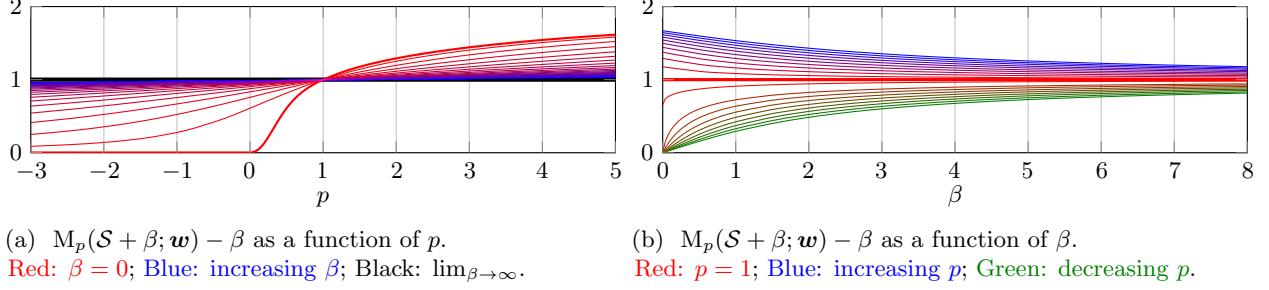


Figure 4: Plots of the affine-transformed $M_p(\mathcal{S} + \beta; \mathbf{w}) - \beta$ aggregator function, for various values of β and p . All plots use $\mathcal{S} \doteq (0, 1, 2)$ and $\mathbf{w} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ (i.e., unweighted power-means).

where, as usual, \mathcal{S} must be positively connoted in $W(\cdot, \cdot)$ and negatively connoted in $\Lambda(\cdot, \cdot)$.

Of course, such a function pair exists in general; for $W_p(\cdot, \cdot)$ and $\Lambda_p(\cdot, \cdot)$, we may take $f_S(u) \doteq u^{p/q}$ and $F_M(u) \doteq f_S^{-1}(u) = u^{q/p}$. However, from the perspective of *estimation* (and thus from the perspective of *machine learning*), this relationship is unsatisfying, as we want to exploit a relationship between $\mathbb{E}_{\mathcal{D}}[\ell]$ and an empirical estimate $\hat{\mathbb{E}}_{\mathbf{x}}[\ell]$, for some loss function ℓ , where $\mathbf{x} \sim \mathcal{D}^m$. In general, $\hat{\mathbb{E}}_{\mathbf{x}}[\ell^{p/q}]$ is a biased estimator of $\mathbb{E}_{\mathcal{D}}^{p/q}[\ell]$, as is any nonlinear function; we thus restrict our attention to *affine functions*, i.e., we require $f_S(u) \doteq \beta + \alpha u$.

At first glance, this seems promising, as in univariate optimization, we have

$$\max_{x \in \mathcal{X}} f(x) = -\min_{x \in \mathcal{X}} -f(x) ,$$

which would seem to suggest we take $F_M(u) = -u$ and $f_S(u) = -u$, yielding

$$W(\mathcal{S}; \mathbf{w}) = F_M \circ \Lambda(f_S \circ \mathcal{S}; \mathbf{w}) - \Lambda(-\mathcal{S}; \mathbf{w}) ?$$

Unfortunately, except in the *egalitarian* ($p = -\infty$ welfare, $p = \infty$ malfare) and *utilitarian* ($p = 1$) cases, or when $|\Omega| = 1$, we have the necessary requirements that sentiment values be nonnegative, as otherwise key properties (various cardinal welfare axioms) of the power-mean break down. Thus the attempt to pattern-match the univariate case has failed; a more sophisticated strategy is required.

The sophomoric approach is then to preserve nonnegativity, by taking $f_S(u) \doteq \beta - u$, and $F_M(u) \doteq f_S^{-1}(u) = \beta - u$, where we must choose β to exceed the maximum utility value. Of course, the choice of β is rather arbitrary, and this strategy is fruitless with *unbounded sentiment values* (e.g., the cross entropy loss or square loss). Furthermore this strategy fails to ensure fairness, in the sense that the original fairness concept is not preserved, and in particular, the status quo (utilitarianism) is preserved as β is taken to infinity, i.e.,

$$\forall p \in \mathbb{R} : \lim_{\beta \rightarrow \infty} M_p(\beta \pm \mathcal{S}, \mathbf{w}) - \beta = \pm M_1(\mathcal{S}; \mathbf{w}) .$$

We thus conclude that, in general, there is no way to contort a *loss function* into a *utility function* such that any welfare function of *expected utility* preserves the fairness trade-offs made by some malfare function on the *expected loss* (nor vice versa).

The Statistical Inestimability of Welfare Functions In this work, we focus primarily on fair learning and statistical estimation with malfare functions. Much of what we accomplish is not possible for fair welfare functions, primarily because $W_p(\cdot, \cdot)$ for $p \in [0, 1]$ are not Lipschitz continuous. Leveraging this idea, we now construct welfare estimation tasks for which *sample complexity* is significantly larger than mean estimation, and may even be *unbounded*.

We first show that even the *unweighted Nash social welfare* of two groups is surprisingly difficult to estimate. This result is best appreciated in light of the fact that the sample complexity of estimating the bias p of a Bernoulli coin is $\Omega(\frac{\ln \frac{1}{\delta}}{\varepsilon})$, which is sharp as $p \rightarrow 0$, $\varepsilon \rightarrow 0$, $\delta \rightarrow 0$, yet we find that the sample-complexity of welfare estimation is substantially larger. Note also that the construction is quite natural, utilizing only two (unweighted) groups, with utility samples of *bounded range*.

Example 4.1 (Estimating Nash Social Welfare). Suppose utility samples for groups 1 and 2 are $\text{BERNOULLI}(1)$ and $\text{BERNOULLI}(p)$ distributed, respectively, for some $p \in [0, 1]$. Clearly $W_0(\mathcal{S}; \mathbf{w}) = \sqrt{p}$, and given a size m sample for group 2, the probability of observing all 0 values is $(1-p)^m$. As this always occurs for $p=0$, in this case, we must predict $W_0(\mathcal{S}; \mathbf{w}) \leq \varepsilon$. However, if $\sqrt{p} > 2\varepsilon \Leftrightarrow p > 4\varepsilon^2$, we must predict $W_0(\mathcal{S}; \mathbf{w}) > \varepsilon$, which is mutually exclusive with the above. Now let δ denote the probability of this event, and note that no mean-estimator can disambiguate the

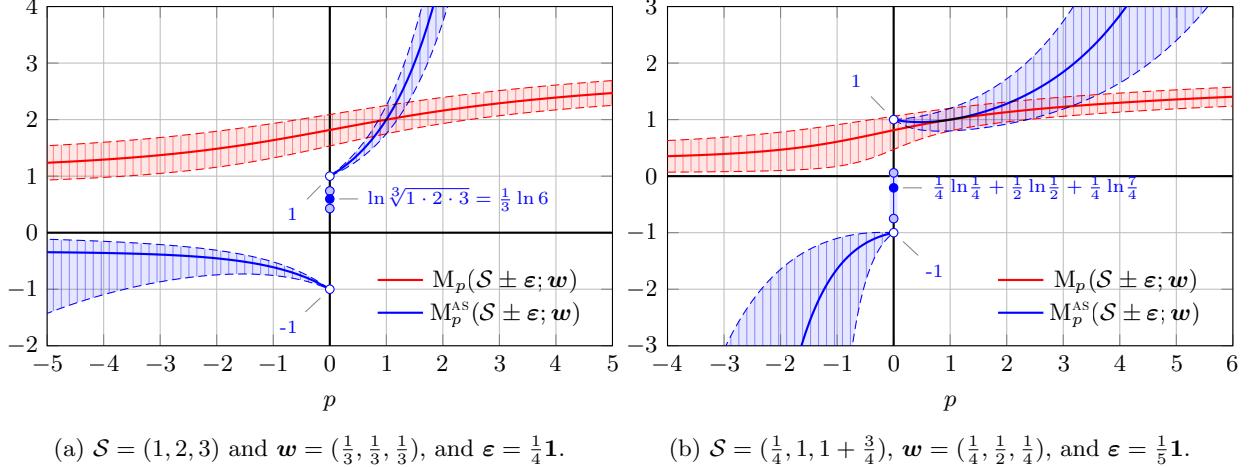


Figure 5: A comparison of the p -power-mean and p -CAS aggregator function families, both as a function of p , for an unweighted (5a) and weighted (5b) three-member population. We plot the aggregates themselves, as well as *upper and lower bounds* on means based on an uncertainty interval $M(\mathcal{S} \pm \varepsilon; \mathbf{w})$, shown as *shaded regions*. The power mean is continuous in p , but the p -CAS is discontinuous at $p = 0$, where the value is distinct from both the left and right limits. These discontinuities are plotted in the usual manner, with upper and lower bounds at $p = 0$ shaded.

above cases, and thus δ lower-bounds the failure rate of any mean estimator (or welfare estimator). We now conclude that for any $p > 0$, any $(\varepsilon < \frac{\sqrt{p}}{2}, \delta \leq (1-p)^m)$ approximation of $W_0(\mathcal{S}; \mathbf{w})$ requires a necessary sample of size

$$m \geq \frac{\ln(\delta)}{\ln(1-p)} > \frac{\ln(\delta)}{\ln(1-4\varepsilon^2)} \geq \frac{\ln \frac{1}{\delta}}{4\varepsilon^2} .$$

We now find that the situation is infinitely worse when we are allowed to weight the welfare function. The next example shows that the sample complexity of welfare estimation then becomes unbounded.

Example 4.2 (Estimating Weighted Nash Social Welfare). Suppose as in example 4.1. We now consider the *weighted Nash social welfare*, letting $\mathbf{w} \doteq (1-w, w)$, for $w \in (0, 1)$. We then have

$$W_0(\mathcal{S}; \mathbf{w}) = \exp((1-w)\ln(1) + w\ln(p)) = p^w .$$

Again, when we observe all 0 values, we must predict $W_0(\mathcal{S}; \mathbf{w}) \leq \varepsilon$, but now if $p^w > 2\varepsilon$, we must predict $W_0(\mathcal{S}; \mathbf{w}) > \varepsilon$, which are again mutually exclusive predictions. Now, for any $\varepsilon < \frac{1}{2}$, we may take $p^w > 2\varepsilon$, which implies $w > \frac{\ln(2\varepsilon)}{\ln(p)}$, and $p > (2\varepsilon)^{\frac{1}{w}}$. Thus for any $p > 0$, we require

$$m \geq \frac{\ln(\delta)}{\ln(1-p)} > \frac{\ln(\delta)}{\ln(1-(2\varepsilon)^{\frac{1}{w}})} \geq \frac{\ln \frac{1}{\delta}}{(2\varepsilon)^{\frac{1}{w}}} .$$

As w was a free variable (for any $w \in (0, 1)$, the constraint $w > \frac{\ln(2\varepsilon)}{\ln(p)}$ is satisfied for sufficiently small p), we may thus conclude that for fixed ε, δ , there exist problem instances (parameterized by w, p) in this class for which the sample complexity of welfare estimation is arbitrarily large.

These results should be contrasted with lemma 3.1 and corollary 3.2, where we show that estimation of *any fair malice function* is essentially no harder than estimation of risk values. Thus despite their apparent similarity, we conclude that welfare and malice functions are not isomorphic, and furthermore they have substantially different properties, where malice is generally more amenable to statistical estimation.

4.2 A Comparison with the Additively Separable Form

For context, we present an additional axiom; that of *additive separability*. We do not assume this axiom henceforth; rather we present it for comparison purposes, as it is commonly assumed in welfare economics.

Definition 4.3 (Additive Separability). An aggregator function $M(\cdot, \cdot)$ is *additively separable* if there exists a function $f : \mathbb{R}_{0+} \times \Omega \rightarrow \bar{\mathbb{R}}$ such that for any $\mathcal{S} : \Omega \rightarrow \mathbb{R}_{0+}$ and weights measure \mathbf{w} over Ω , $M(\mathcal{S}, \mathbf{w})$ may be decomposed as

$$M(\mathcal{S}; \mathbf{w}) = \int_{\omega} f(\mathcal{S}(\omega); \omega) d(\omega) = \mathbb{E}_{\omega \sim \mathbf{w}} [f(\mathcal{S}(\omega); \omega)] .$$

Definition 4.4. Suppose $\mathcal{S} : \Omega \rightarrow \mathbb{R}_{0+}$ and weights measure \mathbf{w} . For any $p \in \mathbb{R}$, we define the *p-canonical-additively-separable* (*p*-CAS) aggregator function as

$$M_p^{\text{AS}}(\mathcal{S}; \mathbf{w}) \doteq \lim_{\varepsilon \rightarrow 0^+} \int_{\omega} f_p(\mathcal{S}(\omega) + \varepsilon) d(\omega) = \lim_{\varepsilon \rightarrow 0^+} \mathbb{E}_{\omega \sim \mathbf{w}} [f_p(\mathcal{S}(\omega) + \varepsilon)] , \text{ with } \begin{cases} p = 0 & f_0(x) \doteq \ln(x) \\ p \neq 0 & f_p(x) \doteq \text{sgn}(p)x^p \end{cases} . \quad (1)$$

Here f_p is defined as in theorem 2.7 item 3, and again $\mathcal{S} : \Omega \rightarrow \mathbb{R}_+$ is extended to $\Omega \rightarrow \mathbb{R}_{0+}$ via the right limit. These limits are simpler than in the power-mean, and we could equivalently take the limits in f_p , which results in $\ln(0^+) = -\frac{1}{0^+} = -\infty$. Note that the Debreu-Gorman theorem is often stated in this form; i.e., it is theorem 2.7 item 3, taking $F(x) = x$. It is thus closely related to the power-mean, as

$$M_p^{\text{AS}}(\mathcal{S}; \mathbf{w}) = \int_{\omega} f_p^{-1}(\text{sgn}(p)M_p(\mathcal{S}; \mathbf{w})) d(\omega) = M_p^p(\mathcal{S}; \mathbf{w}) . \quad (2)$$

If we assume the additive separability axiom, as well as axioms 1-5, it then holds that any aggregator function $M(\cdot, \cdot)$ can be expressed as

$$M(\mathcal{S}; \mathbf{w}) = \beta + \alpha M_p^{\text{AS}}(\mathcal{S}; \mathbf{w}) ,$$

for some $p \in \mathbb{R}$, $\alpha \in \mathbb{R}_+$, $\beta \in \mathbb{R}$, i.e., $M(\cdot, \cdot)$ is a positive affine transform of the CAS family (this identity follows essentially from theorem 2.7 item 3). It is a rather subtle matter to axiomatically restrict this family to the CAS family, but the following pair of axioms suffice.

Definition 4.5 (Canonical Additive Separability Restriction Identity Axioms). Suppose aggregator function $M(\cdot, \cdot)$ and probability measure \mathbf{w} . We define the following axioms.

1. 0-Identity: $M(\omega \mapsto 0; \mathbf{w}) \in \{-\infty, 0\}$.

2. 1-Identity: $M(\omega \mapsto 1; \mathbf{w}) \in \{-1, 0, 1\}$.

Unlike the entirely natural *unit scale* and *multiplicative linearity* axioms (for the power-mean), the *0-identity* and *1-identity* axioms read as quite arbitrary, and only through a detailed six-way case analysis can it be seen that they just-so-happen to restrict the family appropriately. Furthermore, both are incompatible with the *unit scale* axiom (consider $p \leq 0$) and the *identity property* (theorem 2.7 item 1). Indeed, even the additive-separability axiom itself seems rather heavy-handed, assuming something very specific that is supposedly convenient for the economist, with little justification as to why and how it serves as a fundamental property of *cardinal welfare* itself.

We note that from a classical perspective, there is very little difference between the p -power-mean and p -CAS families. They are isomorphic under the comparison operator, thus they defined the same ordering over preferences, and we shall see that it is easy to construct consistent estimators from either from consistent estimators for sentiment values. However, the same cannot be said for *uniform sample complexity*, therefore our results for FPAC-learnability would be quite different under this alternative axiomatization. Corollary 3.2 describes such a bound for the $p \geq 1$ power-means, and figure 5 directly contrasts these families, wherein it is clear that the difficulty of estimating the p -CAS family varies wildly as a function of p .

In closing, we remark that in many ways, the power-mean is more intuitive as a generalization of the mean-concept, and its convenient dimensional-analysis properties, and the potential for direct comparisons between aggregator functions and sentiment values, do not extend to the p -CAS family.

4.3 Relating Power Means and Inequality Indices

We now discuss and define *relative inequality indices* $I(\mathcal{S}; \mathbf{w})$, which have been employed in the literature (Sen, 1997) to construct *welfare functions* of the form

$$W(\mathcal{S}; \mathbf{w}) = W_1(\mathcal{S}; \mathbf{w})(1 - I(\mathcal{S}; \mathbf{w})) .$$

This characterization intuitively starts with the *utilitarian welfare*, which measures *overall satisfaction* and then *downweights* based on how unfairly distributed utility is amongst the population. The “relative” in relative inequality indices connotes the fact that they are restricted to domain $[0, 1]$, thus the welfare metric matches the utilitarian under perfect equality, and is 0 under maximal inequality.

We show that a large class of such functions are actually power means, which both gives them axiomatic justification, and shows prior support in the literature for the power mean. In particular, we first consider the *Atkinson index* (1970) relative inequality measure family.

Definition 4.6 (Atkinson Index). For all $\varepsilon \in \mathbb{R}$, we define the Atkinson index as

$$\text{Atk}_\varepsilon(\mathcal{S}; \mathbf{w}) \doteq 1 - \frac{\text{M}_{1-\varepsilon}(\mathcal{S}; \mathbf{w})}{\text{M}_1(\mathcal{S}; \mathbf{w})} .$$

Note that often the Atkinson index is restricted to $\varepsilon \in [0, 1]$; outside this range, it may exceed 1. Furthermore, the Atkinson index is generally stated without weights, and in a mathematically equivalent form, in which the resemblance to the power mean is less obvious, but for our purposes the above form is clearer. From it, we immediately have the following lemma.

Lemma 4.7 (Relating Atkinson Indices and Power Means). Suppose some $\varepsilon \in \mathbb{R}$, and take $p = 1 - \varepsilon$. It then holds that

$$\text{M}_p(\mathcal{S}; \mathbf{w}) = \text{M}_1(\mathcal{S}; \mathbf{w})(1 - \text{Atk}_\varepsilon(\mathcal{S}; \mathbf{w})) .$$

Proof. This is a direct consequence of definition 4.6, noting $p = 1 - \varepsilon \Leftrightarrow \varepsilon = 1 - p$. \square

This result is not particularly surprising in light of the welfare-centric derivation of Atkinson (1970), but nonetheless it yields a valuable alternative way to think about power means and inequality-weighted welfare functions. In particular, it gives a direct *axiomatic justification* of the welfare function $\text{W}(\mathcal{S}; \mathbf{w}) = \text{W}_1(\mathcal{S}; \mathbf{w})(1 - \text{Atk}_\varepsilon(\mathcal{S}; \mathbf{w}))$ (see theorem 2.7), and also gives an alternative intuitive interpretation of power-mean welfare (as inequality-weighted utilitarian welfare).

Furthermore, lemma 4.7 casts light on the relationship between *inequality-index constrained* ($\leq c$) fair learning methods and power-mean welfare (or malice) optimization. In particular, if \mathcal{S} is a function of some parameter $\theta \in \Theta$, assuming *strong duality* holds, the *Lagrangian dual* yields

$$\sup_{\theta \in \Theta: \text{Atk}_\varepsilon(\mathcal{S}(\theta); \mathbf{w}) \geq c} \text{W}_1(\mathcal{S}(\theta); \mathbf{w}) = \inf_{\lambda \geq 0} \sup_{\theta \in \Theta} \text{W}_1(\mathcal{S}(\theta); \mathbf{w}) - \lambda(\text{Atk}_\varepsilon(\mathcal{S}(\theta); \mathbf{w}) - c) .$$

Now, consider that for the power-mean, by lemma 4.7, we have

$$\sup_{\theta \in \Theta} \text{W}_p(\mathcal{S}(\theta); \mathbf{w}) = \sup_{\theta \in \Theta} \text{W}_1(\mathcal{S}(\theta); \mathbf{w}) - \text{W}_1(\mathcal{S}(\theta); \mathbf{w}) \text{Atk}_\varepsilon(\mathcal{S}(\theta); \mathbf{w}) .$$

The similarity between these forms is immediately clear, though they may make different trade-offs between equality (Atkinson index) and total utility (utilitarian welfare). However, note that given a sufficiently rich parameter space Θ , there always exists some c such that the infimum and supremum of the Lagrangian dual are realized by some (λ, θ) , such that θ also realizes the supremum of the power-mean. In this sense, we may think of maximizing the power mean welfare as maximizing inequality-constrained welfare, while automatically selecting an appropriate value of the constraint c to balance utility and equality.

Another advantage of direct welfare or malice optimization over inequality-constrained optimization is that it can be quite difficult to accurately estimate inequality indices from a finite sample, e.g., Rongve and Beach (1997) show asymptotic normality and variance analysis, but not finite-sample guarantees. However, it's worth noting that even *uniform sample complexity bounds* on both the objective and the constraints do not imply uniform sample complexity for *constrained maximization*. This is in general a very difficult problem, and makes the analysis of *Seldonian learners* (Thomas et al., 2019) quite challenging, as the sample complexity of constrained optimization may be uniformly bounded only for particular choices of constraint and objective. The following example sharply portrays the issue.

Example 4.8 (Unbounded Sample Complexity of Constrained Optimization). Suppose we wish to select between three classifiers with utility vectors $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$, utilitarian welfares $(1, 2, 3)$, and inequality indices $(0, c - \gamma, c + \gamma)$. In particular, we wish to select the \mathcal{S}_i to maximize utilitarian welfare under the constraint that some inequality index does not exceed c over the data distribution. Clearly \mathcal{S}_1 satisfies the c -inequality constraint, but we require a γ -estimate of the inequality indices to determine whether \mathcal{S}_2 and \mathcal{S}_3 satisfy the inequality constraints. Thus for any $\varepsilon < \frac{1}{2}$, the sample complexity of this welfare maximization problem is actually independent of the additive error ε , but depends on γ , which may be taken arbitrarily close to 0, yielding unbounded sample complexity.

Note that similar relationships and impossibility results may be shown for isomorphic inequality measures, including the *Theil indices* (1967) and *generalized entropy indices* (Shorrocks, 1980), although in this context their forms are generally less pleasing. On the other hand, there exist inequality indices with no relation to the power mean. For example, many such inequalities measures based on the Lorenz curve, such as the generalized Gini index, can't be expressed as a function of power mean and utilitarian welfare. Such indices are instead naturally related to other welfare functions, e.g., the generalized Gini social welfare function (Weymark, 1981). We don't directly consider such welfare functions (as they necessarily violate one or more axioms of definition 2.4), but many of our results and constructions can be adapted to them with little difficulty.

5 Statistical and Computational Learning-Efficiency Guarantees

In this section, we define a formal notion of fair-learnability, termed *fair-PAC (FPAC) learning*, where a loss function and hypothesis class are FPAC-learnable essentially if any distribution can be learned to *approximate welfare-optimality* from a *finite sample* (w.h.p.). We then construct various FPAC learners, and relate the concept to standard PAC learning (Valiant, 1984), with the understanding that this allows the vast breadth of research of PAC-learning algorithms, and quite saliently, necessary and sufficient conditions, to be applied to FPAC learning. In particular, we show a hierarchy of fair-learnability via generic statistical and computational learning theoretic bounds and reductions.

Hypothesis Classes and Sequences We now define *hypothesis class sequences*, which allow us to distinguish statistically-easy problems, like learning hyperplanes in finite-dimensional \mathbb{R}^d , from statistically-challenging problems, like learning hyperplanes in \mathbb{R}^∞ . It is also used to analyze the *computational complexity* of learning algorithms as d increases. This definition is adapted from definition 8.1 of Shalev-Shwartz and Ben-David (2014), which treats only *binary classification*.

Definition 5.1 (Hypothesis Class Sequence). A *hypothesis class* is a family of functions mapping domain \mathcal{X} to codomain \mathcal{Y} , and a *hypothesis class sequence* $\mathcal{H} = \mathcal{H}_1, \mathcal{H}_2, \dots$ is a *concentric* (nondecreasing) sequence of *hypothesis classes*, each mapping $\mathcal{X} \rightarrow \mathcal{Y}$. In other words, $\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \dots$.

Usually, each \mathcal{H}_d is easily derived from \mathcal{H}_{d-1} . For instance, *linear classifiers* naturally form a sequence of families using their *dimension*:

$$\mathcal{H}_d \doteq \left\{ \vec{x} \mapsto \text{sgn}(\vec{x} \cdot (\vec{w} \circ \vec{0})) \mid \vec{w} \in \mathbb{R}^d \right\} .$$

Here each \mathcal{H}_d is defined over domain $\mathcal{X} = \mathbb{R}^\infty$, but it is often more natural to discuss each \mathcal{H}_d as a family over $\mathcal{X}_d = \mathbb{R}^d$. In such cases, $\mathcal{X} = \lim_{n \rightarrow \infty} \mathcal{X}_d$, where the set-theoretic limit always exists (this essentially follows from nondecreasing monotonicity of the sequence \mathcal{H}). Similarly, unit-scale *univariate polynomial regression* naturally decomposes as

$$\mathcal{H}_d \doteq \left\{ x \mapsto (x, x^2, \dots, x^d) \cdot \vec{w} \mid \vec{w} \in [-1, 1]^d \right\} .$$

For context, we first present a generalized notion of PAC-learnability, which we then generalize to FPAC-learnability. Standard presentations consider only classification under 0-1 loss, but following the generalized learning setting of Vapnik (2013), some authors consider generalized notions for other learning problems (see, e.g., Shalev-Shwartz and Ben-David, 2014, definition 3.4)

Definition 5.2 (PAC-Learnability). Suppose *hypothesis class sequence* $\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \dots$, all over $\mathcal{X} \rightarrow \mathcal{Y}$, and *loss function* $\ell : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}_{0+}$. We say \mathcal{H} is *PAC-learnable* w.r.t. ℓ if there exists a (randomized) algorithm \mathcal{A} , such that for all

1. sequence indices d ;
2. instance distributions \mathcal{D} over $\mathcal{X} \times \mathcal{Y}$;
3. additive approximation errors $\varepsilon > 0$; and
4. failure probabilities $\delta \in (0, 1)$;

it holds that \mathcal{A} can identify a hypothesis $\hat{h} \in \mathcal{H}$, i.e., $\hat{h} \leftarrow \mathcal{A}(\mathcal{D}, \varepsilon, \delta, d)$, such that

1. there exists some *sample complexity* function $m(\varepsilon, \delta, d) : (\mathbb{R}_+ \times (0, 1) \times \mathbb{N}) \rightarrow \mathbb{N}$ s.t. $\mathcal{A}(\mathcal{D}, \varepsilon, \delta, d)$ consumes no more than $m(\varepsilon, \delta, d)$ samples from \mathcal{D} (i.e., has finite sample complexity); and
2. with probability at least $1 - \delta$ (over randomness of \mathcal{A}), \hat{h} obeys

$$R(\hat{h}; \ell, \mathcal{D}) \leq \inf_{h^* \in \mathcal{H}} R(h^*; \ell, \mathcal{D}) + \varepsilon .$$

The class of such learning problems is denoted PAC, thus we write $(\mathcal{H}, \ell) \in \text{PAC}$ to denote PAC-learnability.

Furthermore, if for all d , the space of \mathcal{D} is restricted such that

$$\exists h \in \mathcal{H}_d \text{ s.t. } R(h; \ell, \mathcal{D}) = 0 ,$$

then (\mathcal{H}, ℓ) is *realizable-PAC-learnable*, written $(\mathcal{H}, \ell) \in \text{PAC}^{\text{Rlz}}$.

Observation 5.3 (On Realizable Learning). Our definition of realizability appears to differ from the standard form, in which \mathcal{D} is a distribution over only \mathcal{X} , and y is simply computed as $h^*(x)$, for some $h^* \in \mathcal{H}$. We instead constrain \mathcal{D} such that there exists a 0-risk $h^* \in \mathcal{H}$, which is equivalent for any loss function ℓ such that $\ell(y, \hat{y}) = 0 \Leftrightarrow y = \hat{y}$, e.g., the 0-1 classification loss, or the absolute or square error regression losses. With our definition, it is much clearer

that realizable learning is a special case of agnostic learning, and furthermore, we handle a much broader class of problems, for which there may be some amount of *noise*, or wherein a ground truth may not even exist.

For example, in a *recommender system*, y may represent the *set of items* that x will like, and h may predict a singleton set, and thus we take $\ell(y, \{\hat{y}\}) = \mathbb{1}_y(\hat{y})$. There is no ground-truth here, but rather we seek a *compatible solution* that recommends appropriate items to everyone. Similarly, in *multiclass classification*, often the classifier output \hat{y} is a *ranked list* of predictions, and the *top-k loss* is taken to be $\ell(y, \hat{y}) = \mathbb{1}_{\hat{y}_{1:k}}(y)$. We don't necessarily have $h^* \in \mathcal{H}$, but 0-risk learning is still possible if there is not “too much” ambiguity (e.g., foxes and dogs can be confused, as long as they are ranked above horses and zebras). Finally, the task of an *interval estimator* is to predict an *interval* \hat{y} for every x in which y must lie, thus again $\ell(y, \hat{y}) = \mathbb{1}_{\hat{y}}(y)$. Under bounded noise conditions, the *interval estimation* problem can easily be realizable, even if it is impossible to exactly recover the *ground truth* from noisy labels.

5.1 Fair Probably Approximately Correct Learning

We now generalize PAC-learnability to *fair-PAC (FPAC) learnability*. In particular, we replace the *univariate risk-minimization* task with a *multivariate welfare-minimization* task. Following the theory of section 2.3, we do not commit to any particular objective, but instead require that a FPAC-learner is able to minimize *any* fair welfare function satisfying the standard axioms. As we move from a univariate task to a multivariate (over g groups) task, problem instances grow not just in problem complexity d , but also in the number of groups g , as it stands to reason that both sample complexity and computational complexity may increase with additional groups.

Definition 5.4 (FPAC-Learnability). Suppose *hypothesis class sequence* $\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \dots \subseteq \mathcal{X} \rightarrow \mathcal{Y}$, and *loss function* $\ell : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}_{0+}$. We say \mathcal{H} is *fair PAC-learnable* w.r.t. ℓ if there exists a (randomized) algorithm \mathcal{A} , such that for all

1. sequence indices d ;
2. group counts g ;
3. per-group instance distributions $\mathcal{D}_{1:g}$ over $(\mathcal{X} \times \mathcal{Y})^g$;
4. group weights measures \mathbf{w} over group indices $\{1, \dots, g\}$;
5. welfare concepts $\Lambda(\cdot; \cdot)$ satisfying axioms 1-7 and 9;
6. additive approximation errors $\varepsilon > 0$; and
7. failure probabilities $\delta \in (0, 1)$;

it holds that \mathcal{A} can identify a hypothesis $\hat{h} \in \mathcal{H}$, i.e., $\hat{h} \leftarrow \mathcal{A}(\mathcal{D}_{1:g}, \mathbf{w}, \Lambda, \varepsilon, \delta, d)$, such that

1. there exists some *sample complexity* function $m(\varepsilon, \delta, d, g) : (\mathbb{R}_+ \times (0, 1) \times \mathbb{N} \times \mathbb{N}) \rightarrow \mathbb{N}$ s.t. $\mathcal{A}(\mathcal{D}_{1:g}, \mathbf{w}, \Lambda, \varepsilon, \delta, d)$ consumes no more than $m(\varepsilon, \delta, d, g)$ samples (finite sample complexity); and
2. with probability at least $1 - \delta$ (over randomness of \mathcal{A}), \hat{h} obeys

$$\Lambda(i \mapsto R(\hat{h}; \ell, \mathcal{D}_i); \mathbf{w}) \leq \inf_{h^* \in \mathcal{H}} \Lambda(i \mapsto R(h^*; \ell, \mathcal{D}_i); \mathbf{w}) + \varepsilon .$$

The class of such fair-learning problems is denoted FPAC, thus we write $(\mathcal{H}, \ell) \in \text{FPAC}$ to denote fair-PAC-learnability.

Finally, if for all d , the space of \mathcal{D} is restricted such that

$$\exists h \in \mathcal{H}_d \text{ s.t. } \max_{i \in 1, \dots, g} R(h; \ell, \mathcal{D}_i) = 0 ,$$

then (\mathcal{H}, ℓ) is *realizable-FPAC-learnable*, written $(\mathcal{H}, \ell) \in \text{FPAC}^{\text{Rlz}}$.

We now observe that a few special cases are familiar learning problems, though we argue that all cases are of interest, and simply represent different ideals of fairness, which may be situationally appropriate.

Observation 5.5 (Melfare Functions and Special Cases). By assumption, $\Lambda(\cdot; \cdot)$ must be $\Lambda_p(\cdot; \cdot)$ for some $p \in [1, \infty]$. Taking $g = 1$ implies $\mathbf{w} = (1)$, and $\Lambda_p(\mathcal{S}; \mathbf{w}) = \mathcal{S}_1$, thus reducing the problem to standard PAC-learning (risk minimization). Similarly, taking $p = 1$ converts the problem to *weighted risk minimization* (weights determined by \mathbf{w}), and $p = \infty$ yields a *minimax optimization problem*, where the maximum is over groups, as commonly encountered in adversarial and robust learning settings.

An Aside: The Flexibility of FPAC-Learning Note that the generalized definition of (fair) PAC-learnability is sufficiently broad so as to include many *supervised*, *semi-supervised*, and *unsupervised* learning problems. While this is not immediately apparent, consider that, for instance, *k-means clustering* can be expressed as a *learning problem*, where the task is to identify a set of k cluster centers, each of which are vectors in \mathbb{R}^d . In particular, the hypothesis class is isomorphic to $\mathbb{R}^{k \times d}$, it operates by mapping a given vector \vec{x} onto the nearest cluster center, and the loss function is the *square distance* to said cluster center. This is a surprisingly natural fairness issue when cast as a *resource allocation problem*. For example, if each cluster center represents a cellphone tower, then we seek to place towers to serve *all groups*, and to avoid serving one or more groups particularly well at the expense of the others.

On Computational Efficiency Some authors consider not just the *statistical* but also the *computational* performance of learners, generally requiring that \mathcal{A} have *polynomial time complexity* (thus implicitly polynomial sample complexity). In other words, they require that $\mathcal{A}(\mathcal{D}, \varepsilon, \delta, d)$ terminates in $m(\varepsilon, \delta, d) \in \text{Poly}(\frac{1}{\varepsilon}, \frac{1}{\delta}, d)$ steps. A similar concept of *polynomial-time FPAC-learnability* is equally interesting, where here we assume $\mathcal{A}(\mathcal{D}_{1:g}, \mathbf{w}, \mathbf{M}, \varepsilon, \delta, d)$ may be computed by a Turing machine (with access to *sampling* and *entropy* oracles) in $m(\varepsilon, \delta, d, g) \in \text{Poly}(\frac{1}{\varepsilon}, \frac{1}{\delta}, d, g)$ steps. We denote these concepts $\text{PAC}_{\text{Poly}}^{\text{Agn}}$, $\text{PAC}_{\text{Poly}}^{\text{Rlz}}$, $\text{FPAC}_{\text{Poly}}^{\text{Agn}}$, and $\text{FPAC}_{\text{Poly}}^{\text{Rlz}}$.

Some trivial reductions We first observe (immediately from definitions 5.2 and 5.4) that PAC-learning is a special case of FPAC-learning. In particular, taking $g = 1$ implies $M_p(\mathcal{S}) = M_1(\mathcal{S}) = \mathcal{S}_1$, thus *magnitude minimization* coincides with *risk minimization*. The more interesting question, which we seek to answer in the remainder of this document, is *when* and *whether* the *converse* holds. Furthermore, when possible, we would like to show practical, sample-and-compute-efficient *constructive reductions*.

Realizability We first show that in the *realizable case*, *PAC-learnability* implies *FPAC-learnability*. In particular, we employ a simple and practical *constructive* polynomial-time reduction. Our reduction simply takes a sufficiently number of samples from the *uniform mixture distribution* over all g groups, and PAC-learns on this distribution. More efficient reductions are possible for particular values of p, g , and \mathbf{w} , but our polynomial reduction suffices to show the desideratum. As the reduction is constructive (and polynomial), this gives us generic algorithms for (polynomial-time) realizable FPAC-learning in terms of algorithms for (polynomial-time) realizable PAC-learning.

Theorem 5.6 (Realizable Reductions). Suppose loss function ℓ and hypothesis class \mathcal{H} . Then

1. $(\mathcal{H}, \ell) \in \text{PAC}^{\text{Rlz}} \implies (\mathcal{H}, \ell) \in \text{FPAC}^{\text{Rlz}}$; and
2. $(\mathcal{H}, \ell) \in \text{PAC}_{\text{Poly}}^{\text{Rlz}} \implies (\mathcal{H}, \ell) \in \text{FPAC}_{\text{Poly}}^{\text{Rlz}}$.

In particular, we construct a (polynomial-time) FPAC-learner for (\mathcal{H}, ℓ) by noting that there exists some \mathcal{A}' with sample-complexity $m_{\mathcal{A}'}(\varepsilon, \delta, d)$ and time complexity $t_{\mathcal{A}'}(\varepsilon, \delta, d)$ to PAC-learn (\mathcal{H}, ℓ) , and taking $\mathcal{A}(\mathcal{D}_{1:g}, \mathbf{w}, \mathbf{M}, \varepsilon, \delta, d) \doteq \mathcal{A}'(\text{mix}(\mathcal{D}_{1:g}), \frac{\varepsilon}{g}, \delta, d)$. Then \mathcal{A} FPAC-learns (\mathcal{H}, ℓ) , with sample-complexity $m_{\mathcal{A}}(\varepsilon, \delta, d, g) = m_{\mathcal{A}'}(\frac{\varepsilon}{g}, \delta, d)$, and time-complexity $t_{\mathcal{A}}(\varepsilon, \delta, d, g) = t_{\mathcal{A}'}(\frac{\varepsilon}{g}, \delta, d)$.

Proof. We first show the *correctness* of \mathcal{A} . Suppose $\hat{h} \leftarrow \mathcal{A}'(p, \mathbf{w}, \mathcal{D}_{1:g}, \varepsilon, \delta, d)$. Then, with probability at least $1 - \delta$ (by the guarantee of \mathcal{A}'), we have

$$\begin{aligned} M_p(i \mapsto R(h; \ell, \mathcal{D}_i)), \mathbf{w} &\leq M_\infty(i \mapsto R(h; \ell, \mathcal{D}_i), i \mapsto \frac{1}{g}) \\ &\leq g M_1(i \mapsto R(h; \ell, \mathcal{D}_i), i \mapsto \frac{1}{g}) \\ &= g R(h; \ell, \text{mix}(\mathcal{D}_{1:g})) \leq g \frac{\varepsilon}{g} = \varepsilon . \end{aligned}$$

We thus may conclude that (\mathcal{H}, ℓ) is *realizable-PAC-learnable* by \mathcal{A}' , with *sample complexity* $m_{\mathcal{A}}(\varepsilon, \delta, d, g) = m_{\mathcal{A}'}(\frac{\varepsilon}{g}, \delta, d)$, which by the nature of \mathcal{A}' , is finite. Similarly, if \mathcal{A} has polynomial runtime, then so too does \mathcal{A}' , thus we may also conclude efficiency. \square

While mathematically correct, if somewhat trivial, unfortunately, this argument does not extend to the agnostic case, essentially because it is not in general possible to simultaneously satisfy all groups. Some authors (e.g., Krasanakis et al., 2018; Jiang and Nachum, 2020) have addressed related fair-learning problems by optimizing the *risk* of a mixture over groups, *iteratively reweighting* the mixture during training. This strategy generalizes our algorithm for the realizable case, wherein we begin with the uniform mixture, and terminate at an ε - δ optimum before executing a single reweighting. It is tempting to think it could be adapted to FPAC-learn in the agnostic setting, however the following example shows this is not the case. Suppose $\mathcal{Y} \doteq \{a, b\}$, group A always wants a , and group B always wants b , with symmetric preferences, and we wish to optimize egalitarian welfare. For any reweighting, the utility-optimal solution is always to produce all a or all b , except when $\mathbf{w}_1 = \mathbf{w}_2 = \frac{1}{2}$, in which case all solutions are equally good. In

this example, for no reweighting do all reweighted-risk solutions even *approximate* the egalitarian-optimal solution (which is evenly split between a and b). We thus conclude that simple constructive reductions using PAC-learners as subroutines are not likely to solve the FPAC-learning problem.

In addition to the argument being inextensible to the agnostic case, we note that, philosophically speaking, realizable FPAC learning is rather uninteresting, essentially because in a world where all parties may be satisfied completely, the obvious solution is to do so (and this solution is in fact an equilibrium). Thus unfairness and bias issues logically only arise in a world of *conflict* (e.g., in zero-sum settings, or under limited resources constraints, which foster *competition* between groups). We henceforth focus our efforts on the more interesting agnostic-learning setting.

6 Characterizing Fair Statistical Learnability with FPAC-Learners

We first consider only questions of *statistical learning*. In other words, we ignore computation for now, and show only that *there exist* FPAC-learning algorithms. In particular, we show a generalization of the *fundamental theorem of statistical learning* to fair learning problems. The aforementioned result relates *uniform convergence* and *PAC-learnability*, and is generally stated for binary classification only. We define a natural generalization of uniform convergence to arbitrary learning problems within our framework, and then show conditions under which a generalized fundamental theorem of (fair) statistical learning holds. In particular, we show that, neglecting computational concerns, PAC-learnability and FPAC-learnability are equivalent for learning problems where PAC-learnability implies uniform convergence (e.g., binary classification). For problems where this relationship does not hold, it remains an open question whether $(\mathcal{H}, \ell) \in \text{PAC} \implies (\mathcal{H}, \ell) \in \text{FPAC}$.

6.1 A Generalized Concept of Uniform Convergence

We now define a *generalized notion* of *uniform convergence*. In particular, our definition applies to *any bounded loss function*,⁹ thus greatly generalizing the standard notion for binary classification (see, e.g., Shalev-Shwartz and Ben-David, 2014).

Definition 6.1 (Uniform Convergence). Suppose $\ell : \mathcal{Y} \times \mathcal{Y} \rightarrow [0, r] \subseteq \mathbb{R}$ and hypothesis class $\mathcal{H} \subseteq \mathcal{X} \rightarrow \mathcal{Y}$. We say $(\mathcal{H}, \ell) \in \text{UC}$ if

$$\lim_{m \rightarrow \infty} \sup_{\mathcal{D} \text{ over } \mathcal{X} \times \mathcal{Y}} \mathbb{E}_{z \sim \mathcal{D}^m} \left[\sup_{h \in \mathcal{H}} |\hat{R}(h; \ell, z) - R(h; \ell, \mathcal{D})| \right] = 0 .$$

We stress that this definition is both uniform over ℓ composed with the *hypothesis class* \mathcal{H} and uniform over *all possible distributions* \mathcal{D} . The classical definition of *uniform convergence in probability* applies to a singular \mathcal{D} , however it is standard in PAC-learning and VC theory to assume uniformity over \mathcal{D} , so we adopt this latter convention. Standard uniform convergence definitions also consider only the convergence of *empirical frequencies* of events to their *true frequencies*, whereas we generalize to consider uniform convergence of the *empirical means* of functions to their *expected values*.

In discussing uniform convergence, it is often necessary to consider not the loss function or hypothesis class *in isolation*, but rather their *composition*, defined as

$$\forall h \in \mathcal{H} : (\ell \circ h)(x, y) \doteq \ell(y, h(x)) \quad \& \quad \ell \circ \mathcal{H} \doteq \{\ell \circ h \mid h \in \mathcal{H}\} .$$

It is also helpful to consider the *sample complexity* of ε - δ uniform-convergence, where we take

$$m_{\text{UC}}(\ell \circ \mathcal{H}, \varepsilon, \delta) \doteq \operatorname{argmin} \left\{ m \left| \sup_{\mathcal{D} \text{ over } \mathcal{X} \times \mathcal{Y}} \mathbb{P} \left(\sup_{h \in \mathcal{H}} \left| \mathbb{E}_{\mathcal{D}}[\ell \circ h] - \hat{\mathbb{E}}_{z \sim \mathcal{D}^m}[\ell \circ h] \right| > \varepsilon \right) \leq \delta \right. \right\} ,$$

i.e., the minimum sufficient sample size to ensure ε - δ uniform-convergence over the *loss family* $\ell \circ \mathcal{H}$.

It is in general true that *uniform convergence* implies *PAC-learnability*; this is well-known for binary classification, but we show the generalized result for completeness. The converse is true for some learning problems, but not for others, which we shall use in the consequent subsection as a powerful tool to characterize when PAC-learnability implies FPAC-learnability.

⁹Boundedness should not be strictly necessary for learnability even uniform convergence, but vastly simplifies all aspects of the analysis. In many cases, it can be relaxed to moment-conditions, such as *sub-Gaussian* or *sub-exponential* assumptions.

6.2 The Fundamental Theorem of (Fair) Statistical Learning

The following result, generally termed the *fundamental theorem of statistical learning*, relates *uniform convergence*, *combinatorial dimensions* and *PAC-learnability*. It is often stated for *binary classification* (Shalev-Shwartz and Ben-David, 2014, theorem 6.2), wherein the relevant combinatorial dimension is the *Vapnik-Chervonenkis dimension*, though we state the multi-class variant (Shalev-Shwartz and Ben-David, 2014, theorem 29.3), in terms of the *Natarajan dimension*.

Theorem 6.2 (Fundamental Theorem of Statistical Learning [Classification]). Suppose ℓ is the 0-1 loss for k -class classification, where $k < \infty$. Then the following are equivalent.

1. $\forall d \in \mathbb{N}$: \mathcal{H}_d has finite Natarajan-dimension (= VC dimension for $k = 2$ classes).
2. $\forall d \in \mathbb{N}$: (ℓ, \mathcal{H}_d) has the uniform convergence property.
3. Any ERM rule is a successful agnostic-PAC learner for \mathcal{H} .
4. \mathcal{H} is agnostic-PAC learnable.
5. Any ERM rule is a successful realizable-PAC learner for \mathcal{H} .
6. \mathcal{H} is realizable-PAC learnable.

It is somewhat subtle to generalize this result to arbitrary learning problems. In particular, there are PAC-learnable problems for which uniform convergence *does not hold*. However, Alon et al. (1997) show similar results for various regression problems, with the (scale-sensitive) γ -*fat-shattering dimension* playing the role of the Vapnik-Chervonenkis or Natarajan dimensions in classification. We now show that essentially the same result holds for *fair statistical learning*, i.e., malfare minimization.

Theorem 6.3 (Fundamental Theorem of Fair Statistical Learning). Suppose ℓ such that $\forall \mathcal{H} : (\mathcal{H}, \ell) \in \text{PAC}^{\text{Rlz}} \implies (\mathcal{H}, \ell) \in \text{UC}$. Then, for any hypothesis class sequence \mathcal{H} , the following are equivalent:

1. $\forall d \in \mathbb{N}$: (ℓ, \mathcal{H}_d) has the (generalized) uniform convergence property.
2. Any EMM rule is a successful agnostic-FPAC learner for (ℓ, \mathcal{H}) .
3. (ℓ, \mathcal{H}) is agnostic-FPAC learnable.
4. Any EMM rule is a successful realizable-FPAC learner for (ℓ, \mathcal{H}) .
5. (ℓ, \mathcal{H}) is realizable-FPAC learnable.

Proof. First note that $1 \implies 2$ is a rather straightforward consequence of the definition of uniform convergence and the contraction property of fair malfare functions (theorem 2.6 item 3). In particular, take $m \doteq m_{\text{UC}}(\ell \circ \mathcal{H}_d, \frac{\varepsilon}{2}, \frac{\delta}{g})$. By union bound, this implies that with probability at least $1 - \delta$, taking samples $\mathbf{z}_{1:g, 1:m} \sim \mathcal{D}_1^m \times \cdots \times \mathcal{D}_g^m$, we have

$$\forall i \in \{1, \dots, g\} : \sup_{h \in \mathcal{H}_d} \left| R(h; \ell, \mathcal{D}_i) - \hat{R}(h; \ell, \mathbf{z}_i) \right| \leq \frac{\varepsilon}{2} .$$

Consequently, as $\Lambda(\cdot; \mathbf{w})$ is $1\|\cdot\|_\infty + |\cdot|$ -Lipschitz in risk (see lemma 3.1), it holds with probability at least $1 - \delta$ that

$$\forall h \in \mathcal{H}_d : \left| \Lambda(i \mapsto \hat{R}(h; \ell, \mathbf{z}_i); \mathbf{w}) - \Lambda(i \mapsto R(h; \ell, \mathcal{D}_i); \mathbf{w}) \right| \leq \frac{\varepsilon}{2} .$$

Now, for EMM-optimal \hat{h} , and malfare-optimal h^* , we apply this result twice to get

$$\begin{aligned} \Lambda(i \mapsto R(\hat{h}; \ell, \mathcal{D}_i); \mathbf{w}) &\leq \Lambda(i \mapsto \hat{R}(\hat{h}; \ell, \mathbf{z}_i); \mathbf{w}) + \frac{\varepsilon}{2} \\ &\leq \Lambda(i \mapsto \hat{R}(h^*; \ell, \mathbf{z}_i); \mathbf{w}) + \frac{\varepsilon}{2} \\ &\leq \Lambda(i \mapsto R(h^*; \ell, \mathcal{D}_i); \mathbf{w}) + \varepsilon . \end{aligned}$$

Therefore, under uniform convergence, the EMM algorithm agnostic FPAC learns (\mathcal{H}, ℓ) with finite sample complexity $m_{\mathcal{A}}(\varepsilon, \delta, d, g) = g \cdot m_{\text{UC}}(\ell \circ \mathcal{H}_d, \frac{\varepsilon}{2}, \frac{\delta}{g})$, completing $1 \implies 2$.

Now, observe that $2 \implies 3$ and $4 \implies 5$ are almost tautological: the existence of (agnostic / realizable) FPAC learning algorithms imply (agnostic / realizable) FPAC learnability.

Now, $2 \implies 4$ and $3 \implies 5$ hold, as realizable learning is a special case of agnostic learning.

As 1 implies 2-4, which in turn each imply 5, it remains only to show that $5 \implies 1$, i.e., if \mathcal{H} is realizable FPAC learnable, then \mathcal{H} has the uniform convergence property. In general, the question is rather subtle, but here

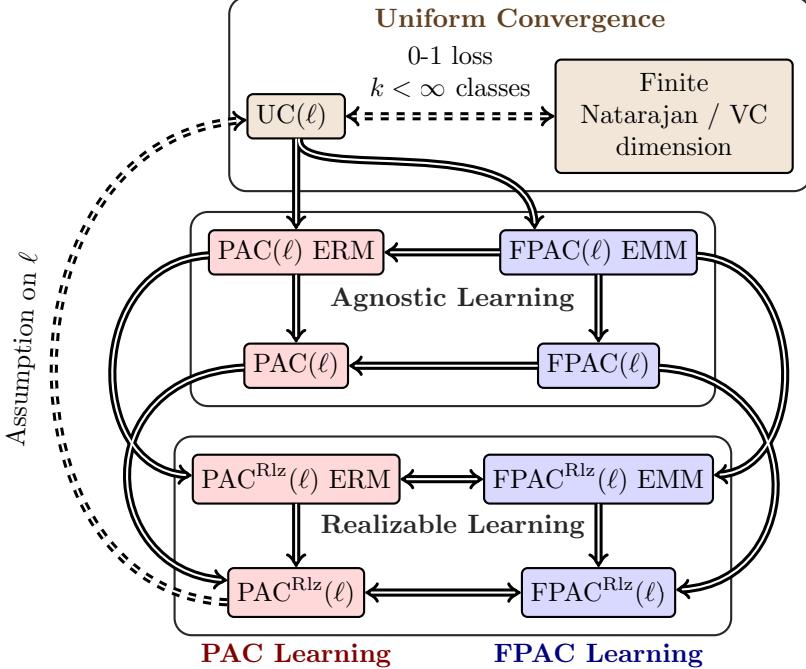


Figure 6: Implications between membership in PAC and FPAC classes. In particular, for arbitrary fixed ℓ , implication denotes *implication of membership* of some \mathcal{H} (i.e., containment); see theorem 5.6 and 6.3. Dashed implication arrows hold conditionally on ℓ . Note that when the assumption on ℓ (see theorem 6.3) holds, the hierarchy collapses, and in general, under realizability, some classes are known to coincide.

the assumption “suppose ℓ such that $(\mathcal{H}, \ell) \in \text{PAC}^{\text{Rlz}} \implies (\mathcal{H}, \ell) \in \text{UC}$ ” does most of the work. In particular, as PAC-learning is a special case of FPAC-learning, we have

$$(\mathcal{H}, \ell) \in \text{FPAC}^{\text{Agn}} \implies (\mathcal{H}, \ell) \in \text{PAC}^{\text{Agn}},$$

then applying the assumption yields $(\mathcal{H}, \ell) \in \text{UC}$. \square

The reductions and equivalences that compose this result are graphically depicted in figure 6.

Observation 6.4 (The Gap between Uniform Convergence and (Fair) PAC-Learnability). Note that the assumption “suppose ℓ such that $(\mathcal{H}, \ell) \in \text{PAC}^{\text{Rlz}} \implies (\mathcal{H}, \ell) \in \text{UC}$ ” does not in general hold. In many cases of interest, it is known to hold, e.g., finite-class classification under 0-1 loss, and bounded regression under square and absolute loss (Alon et al., 1997). In general, verifying this condition is a rather subtle task that must be repeated for each learning problem (loss function). We fully characterize the relationship between PAC and FPAC learnability when they are equivalent to uniform convergence, but in the remaining cases, while clearly FPAC implies PAC, it remains an open question whether PAC implies FPAC.

7 Characterizing Computational Fair-Learnability

In this section, we consider the more granular question of whether FPAC learning is computationally harder than PAC learning. In other words, where previously we showed conditions under which $\text{PAC} = \text{FPAC}$, here we focus on the subset of models with polynomial time training efficiency guarantees, i.e., we ask the question, when does $\text{PAC}_{\text{Poly}} = \text{FPAC}_{\text{Poly}}$ hold? Theorem 5.6 has already characterized the computational complexity of *realizable* FPAC-learning, so we now focus on the *agnostic* case. Here we show neither a generic reduction or non-constructive proof that $\text{PAC}_{\text{Poly}} = \text{FPAC}_{\text{Poly}}$, nor do we show a counterexample; rather we leave this question for future work. We do, however, show that under conditions commonly leveraged as sufficient for polynomial-time PAC-learning, so too is polynomial-time FPAC-learning possible. In particular, section 7.1 provides an efficient constructive reduction (i.e., an algorithm) for efficient FPAC-learning under standard *convex optimization* settings, and section 7.2 shows

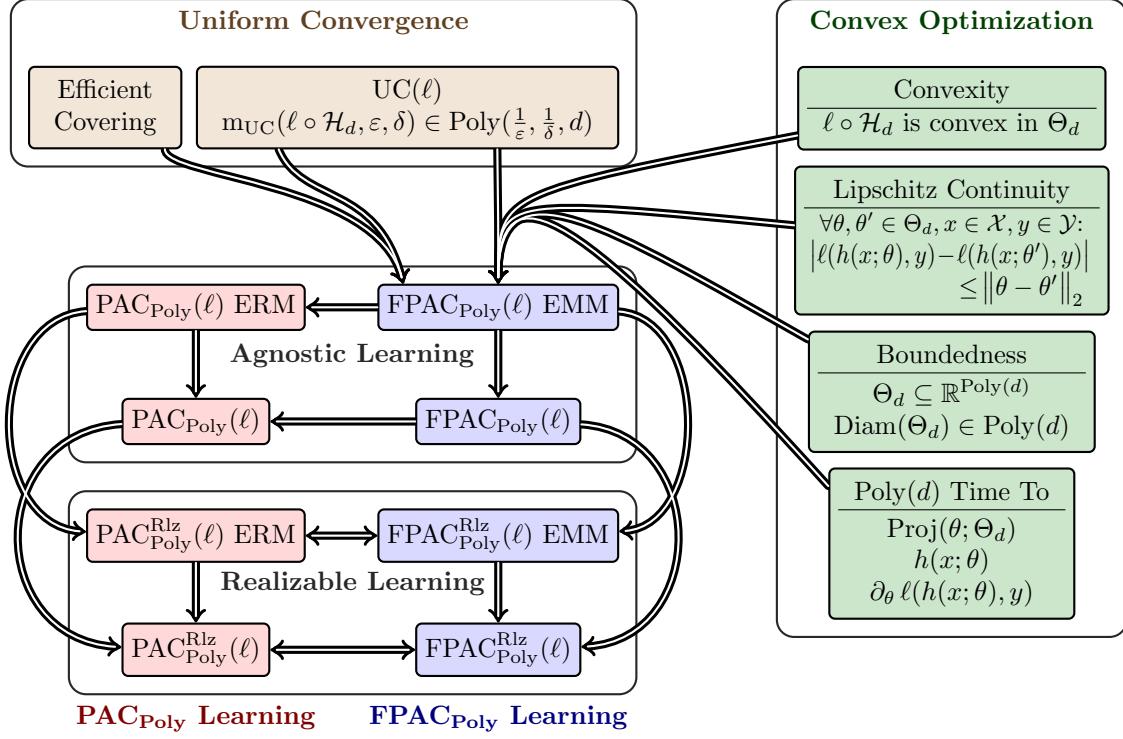


Figure 7: Implications between membership in various poly-time PAC and FPAC classes. In particular, for arbitrary but fixed ℓ , implication denotes *implication of membership* of some \mathcal{H} (i.e., containment). See theorems 7.1 and 7.2.

the same when \mathcal{H} may be approximated by a small cover, and said cover may be efficiently enumerated. The computation-theoretic results of this section are summarized graphically in figure 7.

In both the convex optimization and efficient enumeration settings, the proofs take the same general form: we show that ε -approximate EMM on m total samples is computationally efficient (in $\text{Poly}(m, \varepsilon, d)$ time), and then argue that so long as *sample complexity* $m_{\text{UC}}(\ell \circ \mathcal{H}_d, \varepsilon, \delta) \in \text{Poly}(\frac{1}{\varepsilon}, \frac{1}{\delta}, d)$ of *uniform convergence* is polynomial, i.e., in $\text{Poly}(\frac{1}{\varepsilon}, \frac{1}{\delta}, d)$, then we may construct an FPAC-learner using ε -approximate EMM with polynomial *time complexity*. In particular, the proofs simply account for *optimization* and *sampling* error, and in both cases construct polynomial-time FPAC-learners. Furthermore, as our training meta-algorithms can be applied to various hypothesis classes, we discuss specific instantiations for well-known machine learning models throughout, and these and others are summarized in table 1.

7.1 Efficient FPAC Learning with Convex Optimization

Here we present algorithm 1, which constructs a polynomial-time FPAC-learner under standard convex-optimization assumptions via the *subgradient method*¹⁰ (Shor, 2012), with constants fully derived. Sharper analyses are of course possible, and potential improvements are discussed subsequently, but our result is immediately practical, and can be applied verbatim to problems like *generalized linear models* (Nelder and Wedderburn, 1972) and many *kernel methods* with little analytical effort (under appropriate regularity conditions). Further details on several such models are presented in table 1.

Theorem 7.1 (Efficient FPAC Learning via Convex Optimization). Suppose each hypothesis space $\mathcal{H}_d \in \mathcal{H}$ is indexed by $\Theta_d \subseteq \mathbb{R}^{\text{Poly}(d)}$, i.e., $\mathcal{H}_d = \{h(\cdot; \theta) \mid \theta \in \Theta_d\}$, s.t. (Euclidean) $\text{Diam}(\Theta_d) \in \text{Poly}(d)$, and $\forall x \in \mathcal{X}, \theta \in \Theta_d, h(x; \theta)$ can be evaluated in $\text{Poly}(d)$ time, and $\tilde{\theta} \in \mathbb{R}^{\text{Poly}(d)}$ can be Euclidean-projected onto Θ_d in $\text{Poly}(d)$ time. Suppose also ℓ

¹⁰The *subgradient* $\partial_\theta f(\theta)$ generalizes the *gradient* $\nabla_\theta f(\theta)$ of a function f evaluated at θ , and the two are coincident for *differentiable convex functions*, i.e., $\partial_\theta f(\theta) = \nabla_\theta f(\theta)$. We adopt this setting since the subgradient method yields optimization convergence guarantees even for *nondifferentiable* convex functions, and we assume throughout that a subgradient $\partial_\theta \ell(h(x; \theta), y)$ may be evaluated in $\text{Poly}(d)$ time.

Table 1: A Menagerie of Malfare-Minimizing Model-Classes

Model Name	Model Class				Training Details		
	\mathcal{X}_d	Θ_d	\mathcal{Y}	ℓ	Sample Complexity	Learner	PAC _{Poly}
$\lambda\ \cdot\ _1$ Linear SVM [◊]	\mathcal{B}_{∞}^d	$\lambda\mathcal{B}_1^d$	\mathbb{R}	ℓ_{hinge}	$\mathbf{O}\left(\frac{g\lambda^2 \log \frac{d}{\delta}}{\varepsilon^2}\right)$	Algorithm 1	✓
$\lambda\ \cdot\ _2$ Linear SVM [◊]	\mathcal{B}_2^d	$\lambda\mathcal{B}_2^d$	\mathbb{R}	ℓ_{hinge}	$\mathbf{O}\left(\frac{g\lambda^2 \log \frac{g}{\delta}}{\varepsilon^2}\right)$		
$\lambda\ \cdot\ _2$ Logistic Regr. [◊]	\mathcal{B}_2^d	$\lambda\mathcal{B}_2^d$	\mathbb{R}	ℓ_H	$\mathbf{O}\left(\frac{g\lambda^2 \log \frac{g}{\delta}}{\varepsilon^2}\right)$		
$\lambda\ \cdot\ _2\text{-}\Phi$ SVM / LR [◊]	\mathcal{B}_2^d	$\lambda\Phi(\mathcal{B}_2^d)$	\mathbb{R}	$\ell_{\text{hinge}}/\ell_H$	$\mathbf{O}\left(\frac{g\lambda^2 \text{Diam}^2(\Theta_d) \log \frac{g}{\delta}}{\varepsilon^2}\right)$	with kernel trick*	
Decision Stump [♡]	\mathbb{R}^d	T_d	\mathcal{Y}	ℓ	$\mathbf{O}\left(\frac{g\ \ell\ _{\infty}^2 \log \frac{d}{\delta}}{\varepsilon^2}\right)$	Algorithm 2	✓
Depth- k Decision Tree [♡]	\mathbb{R}^d	$T_d^{2^k-1} \times \mathbb{R}^{2^k}$	\mathcal{Y}	ℓ	$\mathbf{O}\left(\frac{gk2^k\ \ell\ _{\infty}^2 \log \frac{d}{\delta}}{\varepsilon^2}\right)$		✓
Hyperplane Classifier	\mathbb{R}^d	\mathbb{R}^{d+1}	± 1	ℓ_{0-1}	$\mathbf{O}\left(\frac{gd \log \frac{g}{\delta}}{\varepsilon^2}\right)$		✗
McCulloch-Pitts NN [♣]	\mathbb{R}^d	$\mathbb{R}^{dxh} \times \mathbb{R}^{h \times k}$	$1, \dots, k$	ℓ_{0-1}	$\tilde{\mathbf{O}}\left(\frac{gh(\bar{d}+k)}{\varepsilon^2}\right)$		✗

Here $\mathcal{B}_q^d \doteq \{x \in \mathbb{R}^d \mid \|x\|_q \leq 1\}$ denotes the ℓ_q -unit ball in \mathbb{R}^d , and $T_d \doteq (\{1, \dots, d\} \times \{\pm 1\} \times \mathbb{R})$ denotes a univariate threshold function, which consists of a *feature index*, a *direction*, and a *threshold value*. Furthermore, $\ell_{\text{hinge}}(\cdot, \cdot)$ denotes the *hinge loss*, $\ell_H(\cdot, \cdot)$ the *cross entropy loss*, and $\ell_{0-1}(\cdot, \cdot)$ the *0-1 loss*.

◊ Sample complexity bounds via standard Rademacher average bounds for linear families (see, e.g., Shalev-Shwartz and Ben-David, 2014, Chapter 26), leveraging the *boundedness* and *Lipschitz continuity* of this construction.

♣ Training efficiency via the kernel trick requires additional assumptions on the projection $\Phi(\cdot)$ and a compatible kernel $K(\cdot, \cdot)$.

♡ VC-theoretic sample complexity bounds for decision trees and stumps are as derived by Leboeuf et al. (2020).

♣ The McCulloch-Pitts (1943) neural network uses the *threshold activation function*. We analyze a 3-layer model, with hidden layer width h , for which the Natarajan dimension is $\tilde{\mathbf{O}}(H(d+k))$.

Algorithm 1 Approximate Empirical Malfare Minimization via the Subgradient Method

- 1: **procedure** $\mathcal{A}_{\text{PSG}}(\ell, \mathcal{H}, \theta_0, \text{m}_{\text{UC}}(\cdot, \cdot), \mathcal{D}_{1:g}, \mathbf{w}, \mathbf{M}(\cdot, \cdot), \varepsilon, \delta)$
 - 2: **Input:** λ_{ℓ} -Lipschitz loss function ℓ , $\lambda_{\mathcal{H}}$ -Lipschitz hypothesis class \mathcal{H} with parameter space Θ s.t. $\ell \circ \mathcal{H}$ is convex, initial guess $\theta_0 \in \Theta$, uniform-convergence sample-complexity bound $\text{m}_{\text{UC}}(\cdot, \cdot)$, group distributions $\mathcal{D}_{1:g}$, group weights \mathbf{w} , malfare function $\mathbf{M}(\cdot, \cdot)$, and optimality guarantee ε, δ
 - 3: **Output:** ε, δ - $\mathbf{M}(\cdot, \cdot)$ -optimal $\hat{h} \in \mathcal{H}$
 - 4: $\text{m}_{\mathcal{A}} \leftarrow \text{m}_{\text{UC}}\left(\frac{\varepsilon}{3}, \frac{\delta}{g}\right)$ ▷ Determine sufficient sample size
 - 5: $\mathbf{z}_{1:g, 1:\text{m}_{\mathcal{A}}} \sim \mathcal{D}_1^{\text{m}_{\mathcal{A}}} \times \dots \times \mathcal{D}_g^{\text{m}_{\mathcal{A}}}$ ▷ Draw training sample for each group
 - 6: $n \leftarrow \left\lceil \left(\frac{3 \text{Diam}(\Theta) \lambda_{\ell} \lambda_{\mathcal{H}}}{\varepsilon} \right)^2 \right\rceil$ ▷ Iteration count
 - 7: $\alpha \leftarrow \frac{\text{Diam}(\Theta)}{\lambda_{\ell} \lambda_{\mathcal{H}} \sqrt{n}}$ ▷ Learning rate ($\approx \frac{\varepsilon}{3\lambda_{\ell}^2 \lambda_{\mathcal{H}}^2}$)
 - 8: $f(\theta) : \Theta \mapsto \mathbb{R}_{0+} \doteq \mathbf{M}(i \mapsto \hat{R}(h(\cdot; \theta); \ell, \mathbf{z}_i); \mathbf{w})$ ▷ Define empirical malfare objective
 - 9: $\hat{\theta} \leftarrow \text{PROJECTEDSUBGRADIENT}(f, \Theta, \theta_0, n, \alpha)$ ▷ Run PSG algorithm on empirical malfare
 - 10: **return** $h(\cdot; \hat{\theta})$ ▷ Return ε, δ optimal model
 - 11: **end procedure**
-

such that $\forall x \in \mathcal{X}, y \in \mathcal{Y} : \theta \mapsto \ell(y, h(x; \theta))$ is a *convex function*, and suppose Lipschitz constants $\lambda_\ell, \lambda_{\mathcal{H}} \in \text{Poly}(d)$ and some norm $\|\cdot\|_{\mathcal{Y}}$ over \mathcal{Y} s.t. ℓ is $\lambda_\ell\|\cdot\|_{\mathcal{Y}}\|\cdot\|$ -Lipschitz in \hat{y} , i.e.,

$$\forall y, \hat{y}, \hat{y}' \in \mathcal{Y} : |\ell(y, \hat{y}) - \ell(y, \hat{y}')| \leq \lambda_\ell \|\hat{y} - \hat{y}'\|_{\mathcal{Y}},$$

and also that each \mathcal{H}_d is $\lambda_{\mathcal{H}}\|\cdot\|_2\|\cdot\|_{\mathcal{Y}}$ -Lipschitz in θ , i.e.,

$$\forall x \in \mathcal{X}, \theta, \theta' \in \Theta_d : \|h(x; \theta) - h(x; \theta')\|_{\mathcal{Y}} \leq \lambda_{\mathcal{H}} \|\theta - \theta'\|_2.$$

Finally, assume $\ell \circ \mathcal{H}_d$ exhibits ε - δ uniform convergence with sample complexity $m_{UC}(\varepsilon, \delta, d) \in \text{Poly}(\frac{1}{\varepsilon}, \frac{1}{\delta}, d)$.

It then holds that, for arbitrary initial guess $\theta_0 \in \Theta_d$, given any group distributions $\mathcal{D}_{1:g}$, group weights \mathbf{w} , fair welfare function $\Lambda(\cdot; \cdot)$, ε , δ , and d , the algorithm (see algorithm 1)

$$\mathcal{A}(\mathcal{D}_{1:g}, \mathbf{w}, \Lambda(\cdot; \cdot), \varepsilon, \delta, d) \doteq \mathcal{A}_{PSG}(\ell, \mathcal{H}_d, \theta_0, m_{UC}(\cdot, \cdot, d), \mathcal{D}_{1:g}, \mathbf{w}, \Lambda(\cdot; \cdot), \varepsilon, \delta)$$

FPAC-learns (\mathcal{H}, ℓ) with sample complexity $m(\varepsilon, \delta, d, g) = g \cdot m_{UC}(\frac{\varepsilon}{3}, \frac{\delta}{g}, d)$, and (training) time-complexity $\in \text{Poly}(\frac{1}{\varepsilon}, \frac{1}{\delta}, d, g)$, thus $(\mathcal{H}, \ell) \in \text{FPAC}_{\text{Poly}}^{\text{Agn}}$.

It is of course possible to show similar guarantees under relaxed conditions, and with sharper sample complexity and time complexity bounds; theorem 7.1 merely characterizes a simple and standard convex optimization setting under which standard convex-optimization guarantees for *risk minimization* readily translate to *malfare minimization*. In particular, we note that the Lipschitz assumptions can also be weakened without sacrificing (polynomial) time guarantees, and that more sophisticated optimization methods may yield (polynomially) more efficient optimization routines. Furthermore, in risk minimization, stronger conditions like *strong convexity* and *self-concordancy* yield substantial improvements to optimization time complexity; future work shall determine whether and when such properties are preserved in composition with power-mean malfare functions, and thus whether the relevant highly-efficient specialized optimization methods are applicable.

Indeed we remark now that for $p \approx 1$, and when per-group samples have similar empirical risk values for all models \mathbf{h} encountered in the traversal through parameter space, then

$$\forall h \in \mathbf{h} : \Lambda_p(i \mapsto \hat{R}(h; \ell, \mathbf{z}_i); \mathbf{w}) \approx \Lambda_1(i \mapsto \hat{R}(h; \ell, \mathbf{z}_i); \mathbf{w}),$$

thus the optimization aspects of the problem mimic a standard (weighted) loss minimization problem. In contrast, as $p \rightarrow \infty$, the task becomes a minimax optimization problem (see, e.g., the adversarial learning setting of (Mazzetto et al., 2021)), so more specific methods for such tasks, such as the *mirror-prox* algorithm of Juditsky et al. (2011), as employed to great effect in a similar minimax setting by Cortes et al. (2020), may exhibit better (smoother, less oscillatory) behavior when multiple groups are near-tied for maximal empirical risk.

7.2 Uniform Convergence and Efficient Covering

Algorithm 2 Approximate Empirical Malfare Minimization via Empirical Cover Enumeration

- 1: **procedure** $\mathcal{A}_{\hat{\mathcal{C}}}(\ell, \mathcal{H}, \hat{\mathcal{C}}(\cdot, \cdot), \mathcal{N}(\cdot, \cdot), \mathcal{D}_{1:g}, \mathbf{w}, \Lambda(\cdot; \cdot), \varepsilon, \delta)$
 - 2: **Input:** Loss function ℓ , hypothesis class \mathcal{H} , empirical covering routine $\hat{\mathcal{C}}(\cdot, \cdot)$, uniform covering number bound $\mathcal{N}(\cdot, \cdot)$, group distributions $\mathcal{D}_{1:g}$, group weights \mathbf{w} , malfare function $\Lambda(\cdot; \cdot)$, solution optimality guarantee ε - δ .
 - 3: **Output:** ε - δ - $\Lambda(\cdot; \cdot)$ -optimal $\hat{h} \in \mathcal{H}$
 - 4: $m_{UC}(\varepsilon, \delta) \doteq \left\lceil \frac{8\|\ell\|_{\infty}^2 \ln(\sqrt{\frac{2g}{\delta}} \mathcal{N}(\ell \circ \mathcal{H}, \frac{\varepsilon}{4}))}{\varepsilon^2} \right\rceil$ ▷ Bound sample complexity (see lemma A.1 item 3)
 - 5: $m_{\mathcal{A}} \leftarrow m_{UC}(\frac{\varepsilon}{3}, \delta)$ ▷ Determine sufficient training sample size
 - 6: $\mathbf{z}_{1:g, 1:m_{\mathcal{A}}} \sim \mathcal{D}_1^{m_{\mathcal{A}}} \times \cdots \times \mathcal{D}_g^{m_{\mathcal{A}}}$ ▷ Draw training sample for each group
 - 7: $\gamma \doteq \frac{\varepsilon}{3\sqrt{g}}$ ▷ Select cover resolution
 - 8: $\mathcal{H}_{\gamma} \leftarrow \hat{\mathcal{C}}(\mathcal{H}, \bigcup_{i=1}^g \mathbf{z}_i, \gamma)$ ▷ Enumerate empirical cover of concatenated samples
 - 9: $\hat{h} \leftarrow \underset{h_{\gamma} \in \mathcal{H}_{\gamma}}{\operatorname{argmin}} \Lambda(i \mapsto \hat{R}(h_{\gamma}; \ell, \mathbf{z}_i); \mathbf{w})$ ▷ Perform EMM over \mathcal{H}_{γ}
 - 10: **return** \hat{h} ▷ Return ε - δ - $\Lambda(\cdot; \cdot)$ optimal model
 - 11: **end procedure**
-

As we have seen in section 6, uniform convergence implies, and is often equivalent to, (fair) PAC-learnability. However, these results all consider only *statistical learning*, and to analyze *computational learning* questions, we must introduce a strengthening of uniform convergence that considers *computation*. We now show sufficient conditions for polynomial-time FPAC-learnability via *covering numbers*, which we use both to show uniform convergence and to construct an efficient training algorithm. In particular, we show that if a *polynomially-large cover* of each $\ell \circ \mathcal{H}_d$ exists, and can be efficiently enumerated, then $(\mathcal{H}, \ell) \in \text{FPAC}_{\text{Poly}}$.

In what follows, an ℓ_2 - γ -empirical-cover of loss family $(\ell \circ \mathcal{H}_d) \subseteq \mathcal{X} \mapsto \mathbb{R}_{0+}$ on a sample $\mathbf{z} \in (\mathcal{X} \times \mathcal{Y})^m$ is any $\mathcal{H}_{d,\gamma}$ such that

$$\forall h \in \mathcal{H}_d : \min_{h_\gamma \in \mathcal{H}_{d,\gamma}} \sqrt{\frac{1}{m} \sum_{i=1}^m ((\ell \circ h)(\mathbf{z}_i) - (\ell \circ h_\gamma)(\mathbf{z}_i))^2} \leq \gamma .$$

We take $\mathcal{C}(\ell \circ \mathcal{H}_d, \mathbf{z}, \gamma)$ to denote such a cover, and $\mathcal{C}^*(\ell \circ \mathcal{H}_d, \mathbf{z}, \gamma)$ to denote such a cover of *minimum cardinality*. Finally, we define the *uniform covering numbers*

$$\mathcal{N}(\ell \circ \mathcal{H}_d, m, \gamma) \doteq \sup_{\mathbf{z} \in (\mathcal{X} \times \mathcal{Y})^m} |\mathcal{C}^*(\ell \circ \mathcal{H}_d, \mathbf{z}, \gamma)| \quad \& \quad \mathcal{N}(\ell \circ \mathcal{H}_d, \gamma) \doteq \sup_{m \in \mathbb{N}} \mathcal{N}(\ell \circ \mathcal{H}_d, m, \gamma) .$$

This concept is crucial to both our *uniform convergence* and *optimization efficiency* guarantees. In particular, our construction ensures that $\mathcal{N}(\ell \circ \mathcal{H}_d, \gamma)$ is sufficiently small so as to ensure *polynomial training time* on a *polynomially-large* training sample is sufficient to FPAC-learn (ℓ, \mathcal{H}) .

With this exposition complete, we present algorithm 2, which performs EMM on an empirical cover $\hat{\mathcal{C}}(\ell \circ \mathcal{H}_d, \mathbf{z}, \gamma)$. We now show that, under appropriate conditions, such a cover exists, is not superpolynomially larger than $\mathcal{N}(\ell \circ \mathcal{H}_d, \gamma)$, and may be efficiently enumerated. Furthermore, we show that algorithm 2 requires only a polynomially-large training sample, and thus is an FPAC-learner.

Theorem 7.2 (Efficient FPAC-Learning by Covering). Suppose loss function ℓ of bounded codomain (i.e., $\|\ell\|_\infty$ is bounded), and hypothesis class sequence \mathcal{H} , s.t. $\forall m, d \in \mathbb{N}, \mathbf{z} \in (\mathcal{X} \times \mathcal{Y})^m$, there exist

1. a γ - ℓ_2 cover $\mathcal{C}^*(\ell \circ \mathcal{H}_d, \mathbf{z}, \gamma)$, where $|\mathcal{C}^*(\ell \circ \mathcal{H}_d, \mathbf{z}, \gamma)| \leq \mathcal{N}(\ell \circ \mathcal{H}_d, \gamma) \in \text{Poly}(\frac{1}{\gamma}, d)$; and
2. an algorithm to enumerate a γ - ℓ_2 cover $\hat{\mathcal{C}}(\ell \circ \mathcal{H}_d, \mathbf{z}, \gamma)$ of size $\text{Poly} \mathcal{N}(\ell \circ \mathcal{H}_d, m, \gamma)$ in $\text{Poly}(m, \frac{1}{\gamma}, d)$ time.

It then holds that, given any group distributions $\mathcal{D}_{1:g}$, group weights \mathbf{w} , fair welfare function $\mathbb{M}(\cdot; \cdot)$, ε , δ , and d , the algorithm (see algorithm 2)

$$\mathcal{A}(\mathcal{D}_{1:g}, \mathbf{w}, \mathbb{M}(\cdot; \cdot), \varepsilon, \delta, d) \doteq \mathcal{A}_{\hat{\mathcal{C}}}(\ell, \mathcal{H}_d, \hat{\mathcal{C}}(\cdot, \cdot), \mathcal{N}(\ell \circ \mathcal{H}_d, \cdot), \mathcal{D}_{1:g}, \mathbf{w}, \mathbb{M}(\cdot; \cdot), \varepsilon, \delta)$$

FPAC-learns (ℓ, \mathcal{H}) in polynomial time. In particular, (1) is sufficient to show that (ℓ, \mathcal{H}) is FPAC learnable with polynomial *sample complexity*, and (2) is required only to show polynomial *training time complexity*.

This immediately implies that fixed \mathcal{H} that are *finite*, or of bounded *VC-dimension*, *Natarajan dimension*, *pseudodimension*, or γ -*fat-shattering dimension*¹¹ are FPAC-learnable. For instance, this includes *classifiers* such as *all possible languages* of *Boolean formulae* over (constant) d variables, or *halfspaces* (i.e., linear hard classifiers $\mathcal{H} \doteq \{\vec{x} \mapsto \text{sgn}(\vec{x} \cdot \vec{w}) \mid \vec{w} \in \mathbb{R}^d\}$), as well as GLM, subject to regularity constraints to appropriately control the loss function. However, it is perhaps not as powerful as it appears; it applies to *fixed hypothesis classes*, thus each of the above linear models over \mathbb{R}^d is polynomial-time FPAC learnable, but it says nothing about their performance as $d \rightarrow \infty$.

This is essentially because the *statistical analysis* to show polynomial *sample complexity* requires only that $\ln \mathcal{N}(\ell \circ \mathcal{H}_d, \gamma) \in \text{Poly}(\frac{1}{\gamma}, d)$, whereas our *training algorithm* must actually *enumerate* an empirical cover, which yields the (exponentially) stronger requirement that $\mathcal{N}(\ell \circ \mathcal{H}_d, \gamma) \in \text{Poly}(\frac{1}{\gamma}, d)$ for polynomial *time complexity*. Indeed, we see that while the covering numbers we assume imply uniform convergence with *sample complexity* polynomial in d , when covering numbers grow exponentially in d , then our algorithm yields only *exponential* time complexity in d . Consequently, the result only implies polynomial-time algorithms w.r.t. sequences that grow slowly in complexity; e.g., sequences of linear classifiers that grow only *logarithmically* in dimension, i.e., $\mathcal{H}_d \doteq \{\vec{x} \mapsto \text{sgn}(\vec{x} \cdot \vec{w}) \mid \vec{w} \in \mathbb{R}^{[\ln d]}\}$. Further details on when optimizing such models via covering is computationally efficient are presented in table 1.

Note also that theorem 7.2 leverages *covering arguments* in both their *statistical* and *computational* capacities. Statistical bounds based on covering are generally well-regarded, particularly when strong analytical bounds on covering numbers are available, although sharper results are possible (e.g., through the *entropy integral* or *majorizing measures*). Furthermore, while we do construct a *polynomial time* training algorithm, in many cases, specific optimization methods (e.g., stochastic gradient descent or Newton's method) exist to perform EMM *more efficiently* and with *higher accuracy*. Worse yet, *efficient enumerability* of a cover may be non-trivial in some cases; while most covering arguments in the wild are either constructive, or compositional to the point where each component can easily be constructed, it may hold for some problems that computing or enumerating a cover is computationally prohibitive.

¹¹The reader is invited to consult (Anthony and Bartlett, 2009) for an encyclopedic overview of various combinatorial dimensions, associated covering-number and shattering-coefficient concepts, and their applications to statistical learning theory.

On Compositionality and Coverability Conditions The covers and covering numbers discussed above are of course properties of each $\ell \circ \mathcal{H}_d$, rather than ℓ and each \mathcal{H}_d individually. This creates proof obligation for each loss function of interest, in contrast to theorem 7.1, wherein only Lipschitz continuity of ℓ is assumed, and the remaining analysis is on \mathcal{H} . Fortunately, in many cases it is still possible to analyze covers of each \mathcal{H}_d in isolation, and then draw conclusions across a broad family of ℓ composed with each \mathcal{H}_d . In particular, via standard properties of covering numbers, if ℓ is Lipschitz continuous w.r.t. some pseudonorm $\|\cdot\|_{\mathcal{Y}}$ over \mathcal{Y} , and $\gamma\ell_2$ covering numbers of each \mathcal{H}_d w.r.t. $\|\cdot\|_{\mathcal{Y}}$ are well-behaved, it can be shown that the conditions of theorem 7.2 are met. This is useful as, for example, regression losses like *square error*, *absolute error*, and *Huber loss* are all Lipschitz continuous on bounded domains, and thus analysis on each \mathcal{H}_d alone is sufficient to apply theorem 7.2 with each such loss function.

8 Conclusion

This work introduces *malfare minimization* as a fair learning task, and shows relationships between the *statistical* and *computational* issues of malfare and risk minimization. In particular, we argue that our method is more in line with welfare-centric machine learning theory than demographic-parity theory, however in section 4.3 we do show deep connections between welfare or malfare optimization and inequality-constrained loss minimization, which to some extent bridge this divide. We also find that malfare is better aligned to address machine learning tasks cast as loss minimization problems than is welfare, both due to convenient statistical properties, and the greater simplicity of such constructions. As such, the first half of this manuscript is dedicated to deriving and motivating malfare minimization, while the latter half defines the *fair-PAC learning* formalism, and studies the problem from statistical and computational learning theoretic perspectives.

Before further detailing our contributions in these areas, we reiterate that malfare itself, as well as its axiomatic characterization as the $p \geq 1$ power-mean family, is indeed the main contribution of this work. The remainder of the paper explores the *consequences* of this axiomatic definition, some rather simplistic, and others more sophisticated, but we stress that the natural parallels between the statistical and computational aspects of risk minimization and malfare minimization stem from this key definitional decision.

We see this as a measure of the appropriateness of the malfare definition and its use as a fair learning objective, as indeed, other fair-learning formalizations would not behave as such. What may seem straightforward in hindsight was not, in a sense, predestined to be so; for instance had we adopted the *additive separability* axiom instead of multiplicative linearity (as discussed in section 4.2), malfare would be characteristically $M_p^{\text{AS}}(\mathcal{S}; \mathbf{w})$ ($= M_p^p(\mathcal{S}; \mathbf{w})$ for $p > 0$), rather than $M_p(\mathcal{S}; \mathbf{w})$. The FPAC-learnability definition, which requires uniform sample complexity *over all fair malfare functions* (all $p \geq 1$) would then be fundamentally flawed, as risk values above 1 would explode, while risk values below 1 would vanish, as $p \rightarrow \infty$ (see figure 5). Similarly, section 4.1 outlines the difficulties that arise should we instead seek to *maximize* any fair *welfare* function. For a third example, the Seldonian learner (Thomas et al., 2019) framework, which treats arbitrary constrained nonlinear objectives, also seems not to be amenable to uniform sample complexity analysis, due to the generally unbounded sample complexity of determining whether even very simple constraints are satisfied (as discussed in section 4.3).

8.1 Contrasting Malfare and Welfare

With our framework now fully laid out and initial results presented, we now contrast our malfare-minimization framework with traditional welfare-maximization approaches in greater detail. We do not claim that malfare is a better or more useful concept than welfare; but rather we argue only that it is *significantly different* (with surprising non-equivalence results between power-mean welfare and malfare functions), stands on an equal axiomatic footing, and it stands to reason that the right tool (malfare) should be used for the tasks at hand (fair risk-minimization).

With this said, we acknowledge that some learning tasks, e.g., bandit problems and reinforcement learning tasks, are more naturally phrased as *maximizing* utility or (discounted) reward. However, with a few exceptions, e.g., the *spherical scoring rule* from decision theory, most supervised learning problems are naturally cast as minimizing nonnegative *loss functions* (arguably via cross-entropy or KL-divergence minimization through maximum-likelihood, either as explicitly intended (Nelder and Wedderburn, 1972), or *ex-post-facto* through subsequent analysis (Cousins and Riondato, 2019)).

We are highly interested in exploring a parallel theory of fair welfare optimization, however some key malfare properties do not hold for welfare. In particular, fair welfare functions $W_p(\cdot; \cdot)$ for $p \in [0, 1)$ are not Lipschitz continuous; for example, the *Nash social welfare* (a.k.a. *unweighted geometric welfare*) $W_0(\mathcal{S}; \omega \mapsto \frac{1}{g}) = \sqrt[g]{\prod_{i=1}^g \mathcal{S}_i}$ is unstable to perturbations of each \mathcal{S}_i around 0, which causes difficulty in both the *statistical* and *computational* aspects of learning. In section 4.1, we leverage this fact to construct seemingly trivial welfare estimation problems that actually exhibit *unbounded sample complexity*. In particular, in these problems, we must only estimate a single group's *Bernoulli-distributed* utility, which is quite straightforward, but welfare estimation remains intractable.

This impossibility result makes straightforward translation of our FPAC framework into a welfare setting rather vacuous, except in contrived, trivial, or degenerate cases. This difference between malpractice and welfare stems from the fact that although lemma 3.1 holds for both welfare and malpractice, it does not imply *uniform sample-complexity* bounds, whereas, such bounds are trivial for fair malpractice (see corollary 3.2), due to the *contraction property* (theorem 2.6 item 3). It thus seems that such a theory of welfare optimization would need either to either impose additional assumptions to avoid non-Lipschitz behavior (e.g., artificially limit the permitted range of p), or otherwise provide weaker (non-uniform) learning guarantees.

8.2 FPAC Learning: Contributions and Open Questions

After motivating the malpractice-minimization machine learning task, we introduce fair-PAC-learning to study the statistical and computational difficulty of malpractice minimization. As a generalization of PAC-learning, known hardness results (e.g., lower-bounds on computational and sample complexity of loss minimization) immediately apply, thus, coarsely speaking, the interesting question is whether, for some tasks, malpractice minimization is harder than risk minimization. Theorem 5.6 answers this question in the negative *under realizability*, as does theorem 6.3 for *sample complexity*, under appropriate conditions on the loss function. However, as far as sample complexity goes, it remains an open question whether agnostic FPAC-learning and PAC-learning are equivalent for loss functions where *uniform convergence* and *PAC-learnability* are not equivalent. Furthermore, the question of their computational equivalence in the agnostic setting is also open, although section 7 at least shows that many conditions sufficient for PAC-learnability are also sufficient for FPAC-learnability.

We are optimistic that our FPAC-learning definitions will motivate the community to further pursue the deep connections between various PAC and FPAC learning settings, as well as promote cross-pollination between computational learning theory and fair machine learning research. We believe that deeper inquiry into these questions will lead to both a better understanding of what is and is not FPAC-learnable, as well as more practical and efficient reductions and FPAC-learning algorithms.

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A A Compendium of Missing Proofs

Here we present all missing proofs of results stated in the main text.

A.1 Welfare and Malfare

We now show theorem 2.6.

Theorem 2.6 (Properties of the Power-Mean). Suppose $\mathcal{S}, \mathcal{S}'$ are sentiment functions in $\Omega \rightarrow \mathbb{R}_{0+}$, and \mathbf{w} is a probability measure over Ω . The following then hold.

1. Monotonicity: $M_p(\mathcal{S}; \mathbf{w})$ is weakly-monotonically-increasing in p , and strictly so if $\exists \omega, \omega' \in \Omega$ s.t. $\mathcal{S}(\omega) \neq \mathcal{S}(\omega')$.
2. Subadditivity: $\forall p \geq 1 : M_p(\mathcal{S} + \mathcal{S}'; \mathbf{w}) \leq M_p(\mathcal{S}; \mathbf{w}) + M_p(\mathcal{S}'; \mathbf{w})$.
3. Contraction: $\forall p \geq 1 : |M_p(\mathcal{S}; \mathbf{w}) - M_p(\mathcal{S}'; \mathbf{w})| \leq M_p(|\mathcal{S} - \mathcal{S}'|; \mathbf{w}) \leq \|\mathcal{S} - \mathcal{S}'\|_\infty$.
4. Curvature: $M_p(\mathcal{S}; \mathbf{w})$ is concave in \mathcal{S} for $p \in [-\infty, 1]$ and convex for $p \in [1, \infty]$.

Proof. We omit proof of item 1, as this is a standard property of power-means, generally termed the *power mean inequality* (Bullen, 2013, Chapter 3).

We first show item 2. By the triangle inequality (for $p \geq 1$), we have

$$M_p(\mathcal{S} + \mathcal{S}'; \mathbf{w}) \leq M_p(\mathcal{S}; \mathbf{w}) + M_p(\mathcal{S}'; \mathbf{w}) .$$

We now show item 3 First take $\boldsymbol{\varepsilon} \doteq \mathcal{S} - \mathcal{S}'$, and let $\boldsymbol{\varepsilon}_+ \doteq \mathbf{0} \vee \boldsymbol{\varepsilon}$, where $\mathbf{a} \vee \mathbf{b}$ denotes the (elementwise) maximum. Now consider

$$\begin{aligned} M_p(\mathcal{S}; \mathbf{w}) &= M_p(\mathcal{S}' + \boldsymbol{\varepsilon}; \mathbf{w}) && \text{DEFINITION OF } \boldsymbol{\varepsilon} \\ &\leq M_p(\mathcal{S}' + \boldsymbol{\varepsilon}_+; \mathbf{w}) && \text{MONOTONICITY} \\ &\leq M_p(\mathcal{S}'; \mathbf{w}) + M_p(\boldsymbol{\varepsilon}_+; \mathbf{w}) && \text{ITEM 2} \\ &\leq M_p(\mathcal{S}'; \mathbf{w}) + M_p(|\mathcal{S} - \mathcal{S}'|; \mathbf{w}) , && \text{MONOTONICITY} \end{aligned}$$

where here MONOTONICITY refers to monotonicity of $M_p(\mathcal{S}; \mathbf{w})$ in each $\mathcal{S}(\omega)$. By symmetry, we then have $M_p(\mathcal{S}', \mathbf{w}) \leq M_p(\mathcal{S}, \mathbf{w}) + M_p(|\mathcal{S} - \mathcal{S}'|; \mathbf{w})$, which implies the result.

We now show item 4. First note the special cases of $p \in \pm\infty$ follow by convexity of the maximum ($p = \infty$) and concavity of the minimum ($p = -\infty$).

Now, note that for $p \geq 1$, by concavity of $\sqrt[p]{\cdot}$, Jensen's inequality gives us

$$M_1(\mathcal{S}; \mathbf{w}) = \underbrace{\mathbb{E}_{\omega \sim \mathbf{w}} [\mathcal{S}(\omega)]}_{\text{DEFINITION OF CONVEXITY}} = \underbrace{\sqrt[1]{\mathbb{E}_{\omega \sim \mathbf{w}} [\mathcal{S}^1(\omega)]}}_{\text{DEFINITION OF CONVEXITY}} = M_p(\mathcal{S}; \mathbf{w}) ,$$

i.e., convexity, and similarly, for $p \leq 1, p \neq 0$, by convexity of $\sqrt[p]{\cdot}$, we have

$$M_1(\mathcal{S}; \mathbf{w}) = \underbrace{\mathbb{E}_{\omega \sim \mathbf{w}} [\mathcal{S}(\omega)]}_{\text{DEFINITION OF CONCAVITY}} = \underbrace{\sqrt[1]{\mathbb{E}_{\omega \sim \mathbf{w}} [\mathcal{S}^1(\omega)]}}_{\text{DEFINITION OF CONCAVITY}} = M_p(\mathcal{S}; \mathbf{w}) .$$

Similar reasoning, now by convexity of $\ln(\cdot)$, shows the case of $p = 0$. □

We now show theorem 2.7.

Theorem 2.7 (Aggregator Function Properties). Suppose aggregator function $M(\mathcal{S}; \mathbf{w})$. If $M(\cdot; \cdot)$ satisfies (subsets of) the aggregator function axioms (see definition 2.4), we have that $M(\cdot; \cdot)$ exhibits the following properties. For each, assume arbitrary sentiment-value function $\mathcal{S} : \Omega \rightarrow \mathbb{R}_{0+}$ and weights measure \mathbf{w} over Ω . The following then hold.

1. *Identity*: Axioms 6-7 imply $M(\omega \mapsto \alpha; \mathbf{w}) = \alpha$.
2. *Linear Factorization*: Axioms 1-3 imply strictly-monotonically-increasing continuous $F, f : \mathbb{R} \rightarrow \mathbb{R}$, s.t.

$$M(\mathcal{S}; \mathbf{w}) = F \left(\int_{\mathbf{w}} f(\mathcal{S}(\omega)) d(\omega) \right) = F \left(\mathbb{E}_{\omega \sim \mathbf{w}} [f(\mathcal{S}(\omega))] \right) .$$

3. *Debreu-Gorman*: Axioms 1-5 imply that, for some $p \in \mathbb{R}$, $f(x) = f_p(x) \doteq \begin{cases} p = 0 & \ln(x) \\ p \neq 0 & \operatorname{sgn}(p)x^p \end{cases}$.

4. *Power Mean*: Axioms 1-7 imply $F(x) = f_p^{-1}(x)$, thus $M(\mathcal{S}; \mathbf{w}) = M_p(\mathcal{S}; \mathbf{w})$.
5. *Fair Welfare*: Axioms 1-5 and 8 imply $p \in (-\infty, 1]$.
6. *Fair Malware*: Axioms 1-5 and 9 imply $p \in [1, \infty)$.

Proof. Item 1 is an immediate consequence of axioms 6 & 7 (multiplicative linearity and unit scale).

We now note that item 3 is the celebrated Debreu-Gorman theorem (Debreu, 1959; Gorman, 1968), extended by continuity and measurability of \mathcal{S} to the weighted case, and item 2 is a simple corollary thereof.

We now show item 4. This result is essentially a corollary of item 3, hence the dependence on axioms 1-4. Suppose $S(\cdot) = 1$. By item 1, for all $p \neq 0$, we have

$$\alpha = \alpha M(\mathcal{S}; \mathbf{w}) = M(\alpha \mathcal{S}; \mathbf{w}) = F\left(\mathbb{E}_{\omega \sim \mathbf{w}}[f_p(\alpha \mathcal{S}(\omega))]\right) = F\left(\mathbb{E}_{\omega \sim \mathbf{w}}[f_p(\alpha)]\right) = F(\operatorname{sgn}(p)\alpha^p) .$$

From here, we have $\alpha = F(\operatorname{sgn}(p)\alpha^p)$, thus $F^{-1}(u) = \operatorname{sgn}(p)u^p$, and consequently, $F(v) = \sqrt[p]{\operatorname{sgn}(p)v}$.

Taking $p = 0$ gets us

$$\alpha = \alpha M(\mathcal{S}; \mathbf{w}) = F\left(\mathbb{E}_{\omega \sim \mathbf{w}}[\ln(\alpha \mathcal{S}(\omega))]\right) = F(\ln \alpha) ,$$

from which it is clear that $F^{-1}(u) = \ln(u) \implies F(v) = \exp(v)$.

For all values of $p \in \mathbb{R}$, substituting the values of f_p and $F(\cdot)$ into item 3 yields $M(\mathcal{S}; \mathbf{w}) = M_p(\mathcal{S}; \mathbf{w})$ by definition.

We now show 5 and 6. These properties follow directly from 3, wherein f_p are defined, and Jensen's inequality. \square

We now show corollary 3.2.

Corollary 3.2 (Statistical Estimation with Hoeffding and Bennett Bounds). Suppose fair power-mean welfare $M(\cdot; \cdot)$ (i.e., $p \geq 1$), discrete weights measure \mathbf{w} over g groups, probability distributions $\mathcal{D}_{1:g}$, samples $\mathbf{x}_i \sim \mathcal{D}_i^m$, and loss function $\ell : \mathcal{X} \rightarrow [0, r]$ s.t. $\mathcal{S}_i = \mathbb{E}_{\mathcal{D}_i}[\ell]$ and $\hat{\mathcal{S}}_i \doteq \mathbb{E}_{\mathbf{x}_i}[\ell]$. Then, with probability at least $1 - \delta$ over choice of \mathbf{x} ,

$$\left|M_p(\mathcal{S}; \mathbf{w}) - M_p(\hat{\mathcal{S}}; \mathbf{w})\right| \leq r \sqrt{\frac{\ln \frac{2g}{\delta}}{2m}} .$$

Alternatively, again with probability at least $1 - \delta$ over choice of \mathbf{x} , we have

$$\left|M_p(\mathcal{S}; \mathbf{w}) - M_p(\hat{\mathcal{S}}; \mathbf{w})\right| \leq \frac{r \ln \frac{2g}{\delta}}{3m} + \max_{i \in 1, \dots, g} \sqrt{\frac{2 \mathbb{V}_{\mathcal{D}_i}[\ell] \ln \frac{2g}{\delta}}{m}} .$$

Proof. This result is a corollary of lemma 3.1, applied to ε , where we note that for $p \geq 1$, by theorem 2.6 item 3 (contraction) it holds that

$$M_p(\hat{\mathcal{S}} + \varepsilon; \mathbf{w}) \leq M_p(\hat{\mathcal{S}}; \mathbf{w}) + \|\varepsilon\|_\infty \quad \& \quad M_p(\mathbf{0} \vee (\hat{\mathcal{S}} - \varepsilon); \mathbf{w}) \leq M_p(\hat{\mathcal{S}}; \mathbf{w}) - \|\varepsilon\|_\infty .$$

Now, for the first bound, note that we take $\varepsilon_i \doteq r \sqrt{\frac{\ln \frac{2g}{\delta}}{2m}}$, and by Hoeffding's inequality and the union bound, for $\Omega = \{1, \dots, n\}$, we have $\forall \omega : \mathcal{S}'(\omega) - \varepsilon(\omega) \leq \mathcal{S}(\omega) \leq \mathcal{S}'(\omega) + \varepsilon(\omega)$ with probability at least $1 - \delta$. The result then follows via the power-mean contraction (theorem 2.6 item 3) property.

Similarly, for the second bound, note that we take $\varepsilon_i \doteq \frac{r \ln \frac{2g}{\delta}}{3m} + \sqrt{\frac{2 \mathbb{V}_{\mathcal{D}_i}[\ell] \ln \frac{2g}{\delta}}{m}}$, which this time follows via Bennett's inequality and the union bound. Now, we again apply lemma 3.1, noting that $M(\varepsilon) \leq M_\infty(\varepsilon) = \|\varepsilon\|_\infty$ (by power-mean monotonicity, theorem 2.6 item 1), and the rest follows as in the Hoeffding case. \square

A.2 Efficient FPAC-Learning

We now show theorem 7.1.

Theorem 7.1 (Efficient FPAC Learning via Convex Optimization). Suppose each hypothesis space $\mathcal{H}_d \in \mathcal{H}$ is indexed by $\Theta_d \subseteq \mathbb{R}^{\text{Poly}(d)}$, i.e., $\mathcal{H}_d = \{h(\cdot; \theta) \mid \theta \in \Theta_d\}$, s.t. (Euclidean) $\operatorname{Diam}(\Theta_d) \in \text{Poly}(d)$, and $\forall x \in \mathcal{X}, \theta \in \Theta_d$, $h(x; \theta)$ can be evaluated in $\text{Poly}(d)$ time, and $\tilde{\theta} \in \mathbb{R}^{\text{Poly}(d)}$ can be Euclidean-projected onto Θ_d in $\text{Poly}(d)$ time. Suppose also ℓ such that $\forall x \in \mathcal{X}, y \in \mathcal{Y} : \theta \mapsto \ell(y, h(x; \theta))$ is a convex function, and suppose Lipschitz constants $\lambda_\ell, \lambda_{\mathcal{H}} \in \text{Poly}(d)$ and some norm $\|\cdot\|_{\mathcal{Y}}$ over \mathcal{Y} s.t. ℓ is λ_ℓ - $\|\cdot\|_{\mathcal{Y}}$ -Lipschitz in \hat{y} , i.e.,

$$\forall y, \hat{y}, \hat{y}' \in \mathcal{Y} : |\ell(y, \hat{y}) - \ell(y, \hat{y}')| \leq \lambda_\ell \|\hat{y} - \hat{y}'\|_{\mathcal{Y}} ,$$

and also that each \mathcal{H}_d is $\lambda_{\mathcal{H}} \|\cdot\|_2 + \|\cdot\|_{\mathcal{Y}}$ -Lipschitz in θ , i.e.,

$$\forall x \in \mathcal{X}, \theta, \theta' \in \Theta_d : \|h(x; \theta) - h(x; \theta')\|_{\mathcal{Y}} \leq \lambda_{\mathcal{H}} \|\theta - \theta'\|_2 .$$

Finally, assume $\ell \circ \mathcal{H}_d$ exhibits ε - δ uniform convergence with sample complexity $m_{UC}(\varepsilon, \delta, d) \in \text{Poly}(\frac{1}{\varepsilon}, \frac{1}{\delta}, d)$.

It then holds that, for arbitrary initial guess $\theta_0 \in \Theta_d$, given any group distributions $\mathcal{D}_{1:g}$, group weights \mathbf{w} , fair welfare function $\Lambda(\cdot; \cdot)$, ε , δ , and d , the algorithm (see algorithm 1)

$$\mathcal{A}(\mathcal{D}_{1:g}, \mathbf{w}, \Lambda(\cdot; \cdot), \varepsilon, \delta, d) \doteq \mathcal{A}_{PSG}(\ell, \mathcal{H}_d, \theta_0, m_{UC}(\cdot, \cdot; d), \mathcal{D}_{1:g}, \mathbf{w}, \Lambda(\cdot; \cdot), \varepsilon, \delta)$$

FPAC-learns (\mathcal{H}, ℓ) with sample complexity $m(\varepsilon, \delta, d, g) = g \cdot m_{UC}(\frac{\varepsilon}{3}, \frac{\delta}{g}, d)$, and (training) time-complexity $\in \text{Poly}(\frac{1}{\varepsilon}, \frac{1}{\delta}, d, g)$, thus $(\mathcal{H}, \ell) \in \text{FPAC}_{\text{Poly}}^{\text{Agn}}$.

Proof. We now show that this subgradient-method construction of \mathcal{A} requires $\text{Poly}(\frac{1}{\varepsilon}, \frac{1}{\delta}, d, g)$ time to identify an ε - δ - $\Lambda_p(\cdot; \cdot)$ -optimal $\tilde{\theta} \in \Theta_d$, and thus fair-PAC-learns (\mathcal{H}, ℓ) . This essentially boils down to showing that (1) the empirical welfare objective is *convex* and *Lipschitz continuous*, and (2) that algorithm 1 runs sufficiently many subgradient-update steps, with appropriate step size, on a sufficiently large training set, to yield the appropriate guarantees, and that each step of the subgradient method, of which there are polynomially many, itself requires polynomial time.

First, note that by theorem 2.7 items 4 and 6, we may assume that $\Lambda(\cdot; \cdot)$ can be expressed as a p -power mean with $p \geq 1$; thus henceforth we refer to it as $\Lambda_p(\cdot; \cdot)$. Now, recall that the empirical welfare objective (given $\theta \in \Theta_d$ and training sets $\mathbf{z}_{1:g}$) is defined as

$$\Lambda_p(i \mapsto \hat{R}(h(\cdot; \theta); \ell, \mathbf{z}_i); \mathbf{w}) .$$

We first show that empirical welfare is convex in Θ_d . By assumption and positive linear closure, $\hat{R}(h(\cdot; \theta'); \ell, \mathbf{z}_i)$ is convex in $\theta \in \Theta_d$. The objective of interest is the composition of $\Lambda_p(\cdot; \mathbf{w})$ with this quantity evaluated on each of g training sets. By theorem 2.6 item 4, $\Lambda_p(\cdot; \mathbf{w})$ is convex $\forall p \in [1, \infty]$ in \mathbb{R}_{0+}^g , and by the monotonicity axiom, it is monotonically increasing. Composition of a monotonically increasing convex function on \mathbb{R}_{0+}^g with convex functions on Θ_d yields a convex function, thus we conclude the empirical welfare objective is convex in Θ_d .

We now show that empirical welfare is Lipschitz continuous. Now, note that for any $p \geq 1$, \mathbf{w} ,

$$\forall \mathcal{S}, \mathcal{S}' : |\Lambda_p(\mathcal{S}; \mathbf{w}) - \Lambda_p(\mathcal{S}'; \mathbf{w})| \leq 1 \|\mathcal{S} - \mathcal{S}'\|_{\infty} ,$$

i.e., $\Lambda_p(\cdot; \mathbf{w})$ is $1\|\cdot\|_{\infty} + \|\cdot\|$ -Lipschitz in *empirical risks* (see theorem 2.6 item 3), and thus by Lipschitz composition, we have Lipschitz property

$$\forall \theta, \theta' \in \Theta_d : |\Lambda_p(i \mapsto \hat{R}(h(\cdot; \theta); \ell, \mathbf{z}_i); \mathbf{w}) - \Lambda_p(i \mapsto \hat{R}(h(\cdot; \theta'); \ell, \mathbf{z}_i); \mathbf{w})| \leq \lambda_{\ell} \lambda_{\mathcal{H}} \|\theta - \theta'\|_2 .$$

We now show that algorithm 1 FPAC-learns (\mathcal{H}, ℓ) . As above, take $m \doteq m_{UC}(\frac{\varepsilon}{3}, \frac{\delta}{g}, d)$. Our algorithm shall operate on a training sample $\mathbf{z}_{1:g, 1:m} \sim \mathcal{D}_1^m \times \cdots \times \mathcal{D}_g^m$.

First note that evaluating a subgradient (via forward finite-difference estimation or automated subdifferentiation) requires $(\dim(\Theta_d) + 1)m$ evaluations of $h(\cdot; \cdot)$, which by assumption is possible in $\text{Poly}(d, m) = \text{Poly}(\frac{1}{\varepsilon}, \frac{1}{\delta}, d, g)$ time.

The subgradient method produces $\tilde{\theta}$ approximating the empirically-optimal $\hat{\theta}$ such that (see Shor, 2012)

$$f(\tilde{\theta}) \leq f(\hat{\theta}) + \frac{\|\theta_0 - \hat{\theta}\|_2^2 + \Lambda^2 \alpha^2 n}{2\alpha n} \leq \frac{\text{Diam}^2(\Theta_d) + \Lambda^2 \alpha^2 n}{2\alpha n} ,$$

for $\Lambda \|\cdot\|_2 + \|\cdot\|$ -Lipschitz objective f , thus taking $\alpha \doteq \frac{\text{Diam}(\Theta_d)}{\Lambda \sqrt{n}}$ yields

$$f(\tilde{\theta}) - f(\hat{\theta}) \leq \frac{\text{Diam}(\Theta_d) \Lambda}{\sqrt{n}} .$$

As shown above, $\Lambda = \lambda_{\ell} \lambda_{\mathcal{H}}$, thus we may guarantee *optimization error*

$$\varepsilon_{\text{opt}} \doteq f(\hat{\theta}) - f(\theta^*) \leq \frac{\varepsilon}{3}$$

if we take iteration count

$$n \geq \frac{9 \text{Diam}^2(\Theta_d) \lambda_{\ell}^2 \lambda_{\mathcal{H}}^2}{\varepsilon^2} = \left(\frac{3 \text{Diam}(\Theta_d) \lambda_{\ell} \lambda_{\mathcal{H}}}{\varepsilon} \right)^2 \in \text{Poly}(\frac{1}{\varepsilon}, d) .$$

As each iteration requires $m \cdot \text{Poly}(d) \subseteq \text{Poly}(\frac{1}{\varepsilon}, \frac{1}{\delta}, d, g)$ time, the subgradient method identifies an $\frac{\varepsilon}{3}$ -empirical-magnitude-optimal $\tilde{\theta} \in \Theta_d$ in $\text{Poly}(\frac{1}{\varepsilon}, \frac{1}{\delta}, d, g)$ time.

As m was selected to ensure $\frac{\varepsilon}{3} - \frac{\delta}{g}$ uniform convergence, we thus have that by uniform convergence, and union bound (over g groups), with probability at least $1 - \delta$ over choice of $\mathbf{z}_{1:g}$, we have

$$\forall i \in \{1, \dots, g\}, \theta \in \Theta_d : \left| \Lambda_p(i \mapsto \hat{R}(h(\cdot; \theta); \ell, \mathbf{z}_i); \mathbf{w}) - \Lambda_p(i \mapsto R(h(\cdot; \theta); \ell, \mathcal{D}_i); \mathbf{w}) \right| \leq \frac{\varepsilon}{3} .$$

Combining *estimation* and *optimization* errors, we get that with probability at least $1 - \delta$, the approximate-EMM-optimal $h(\cdot; \tilde{\theta})$ obeys

$$\begin{aligned} \Lambda_p(i \mapsto R(h(\cdot; \tilde{\theta}); \ell, \mathcal{D}_i); \mathbf{w}) &\leq \Lambda_p(i \mapsto \hat{R}(h(\cdot; \tilde{\theta}); \ell, \mathbf{z}_i); \mathbf{w}) + \frac{\varepsilon}{3} \\ &\leq \Lambda_p(i \mapsto \hat{R}(h(\cdot; \hat{\theta}); \ell, \mathbf{z}_i); \mathbf{w}) + \frac{2\varepsilon}{3} \\ &\leq \Lambda_p(i \mapsto \hat{R}(h(\cdot; \theta^*); \ell, \mathbf{z}_i); \mathbf{w}) + \frac{2\varepsilon}{3} \\ &\leq \Lambda_p(i \mapsto R(h(\cdot; \theta^*); \ell, \mathcal{D}_i); \mathbf{w}) + \varepsilon . \end{aligned}$$

We may thus conclude that \mathcal{A} fair-PAC learns \mathcal{H} with sample complexity $gm = g \cdot \text{muc}(\frac{\varepsilon}{3}, \frac{\delta}{g}, d)$. Furthermore, as the entire operation requires polynomial time, we have $(\mathcal{H}, \ell) \in \text{PAC}_{\text{Poly}}^{\text{Agn}}$. \square

We now work towards proof of theorem 7.2. We begin with a technical lemma deriving relevant properties of the cover employed in the main result.

Lemma A.1 (Group Cover Properties). Suppose loss function ℓ of bounded codomain (i.e., $\|\ell\|_\infty$ is bounded), hypothesis class $\mathcal{H} \subseteq \mathcal{X} \rightarrow \mathcal{Y}$, and per-group samples $\mathbf{z}_{1:g, 1:m} \in (\mathcal{X} \times \mathcal{Y})^{g \times m}$, letting $\bigcup_{i=1}^g \mathbf{z}_i$ denote their *concatenation*. Now define

$$\hat{\mathcal{C}}_{\cup(1:g)} \doteq \bigcup_{i=1}^g \hat{\mathcal{C}}(\ell \circ \mathcal{H}, \mathbf{z}_i, \gamma) \quad \& \quad \hat{\mathcal{C}}_{\circ(1:g)} \doteq \hat{\mathcal{C}}\left(\ell \circ \mathcal{H}, \bigcup_{i=1}^g \mathbf{z}_i, \frac{\gamma}{\sqrt{g}}\right) .$$

Then, letting $\hat{\mathcal{C}}$ refer generically to either $\hat{\mathcal{C}}_{\cup(1:g)}$ or $\hat{\mathcal{C}}_{\circ(1:g)}$, the following hold.

1. If $\hat{\mathcal{C}}$ is of *minimal cardinality*, then

$$\left| \hat{\mathcal{C}}_{\cup(1:g)} \right| \leq g \mathcal{N}(\ell \circ \mathcal{H}, m, \gamma) \quad \& \quad \left| \hat{\mathcal{C}}_{\circ(1:g)} \right| \leq \mathcal{N}(\ell \circ \mathcal{H}, gm, \frac{\gamma}{\sqrt{g}}) .$$

$$2. \sup_{\mathcal{D} \text{ over } \mathcal{X} \times \mathcal{Y}} \mathfrak{R}_m(\ell \circ \mathcal{H}, \mathcal{D}) \leq \inf_{\gamma \geq 0} \gamma + \|\ell\|_\infty \sqrt{\frac{\ln \mathcal{N}(\ell \circ \mathcal{H}, \gamma)}{2m}} .$$

3. Suppose $\ln \mathcal{N}(\ell \circ \mathcal{H}, \gamma) \in \text{Poly}(\frac{1}{\gamma})$. Then the uniform-convergence sample-complexity of $\ell \circ \mathcal{H}$ over g groups obeys

$$\begin{aligned} \text{muc}(\ell \circ \mathcal{H}, \varepsilon, \delta, g) &\doteq \operatorname{argmin} \left\{ m \left| \sup_{\mathcal{D}_{1:g}} \sup_{(\mathcal{X} \times \mathcal{Y})^g} \mathbb{P} \left(\max_{i \in 1, \dots, g} \sup_{h \in \mathcal{H}} \left| \mathbb{E}_{\mathcal{D}_i}[\ell \circ h] - \hat{\mathbb{E}}_{\mathcal{D}_i^m}[\ell \circ h] \right| > \varepsilon \right) \leq \delta \right\} \\ &\leq \left\lceil \frac{8\|\ell\|_\infty^2 \ln \left(\sqrt{\frac{2g}{\delta}} \mathcal{N}(\ell \circ \mathcal{H}, \frac{\varepsilon}{4}) \right)}{\varepsilon^2} \right\rceil \\ &\in \mathbf{O}\left(\frac{\ln \frac{g \mathcal{N}(\ell \circ \mathcal{H}, \varepsilon)}{\delta}}{\varepsilon^2}\right) \subset \text{Poly}\left(\frac{1}{\varepsilon}, \exp \frac{1}{\delta}, \exp g\right) . \end{aligned}$$

4. For the sample \mathbf{z}_i associated with each group $i \in 1, \dots, g$, $\hat{\mathcal{C}}$ is a γ -uniform-approximation of *empirical risk* $\hat{R}(h; \ell, \mathbf{z}_i)$, and a γ - ℓ_2 cover of the *loss family* $\ell \circ \mathcal{H}$, as

$$\max_{i \in 1, \dots, g} \min_{h_\gamma \in \hat{\mathcal{C}}} \left| \hat{R}(h; \ell, \mathbf{z}_i) - \hat{R}(h_\gamma; \ell, \mathbf{z}_i) \right| \leq \max_{i \in 1, \dots, g} \min_{h_\gamma \in \hat{\mathcal{C}}} \sqrt{\frac{1}{m} \sum_{j=1}^m ((\ell \circ h)(\mathbf{z}_{i,j}) - (\ell \circ h_\gamma)(\mathbf{z}_{i,j}))^2} \leq \gamma .$$

5. $\hat{\mathcal{C}}_{\circ(1:g)}$, but not necessarily $\hat{\mathcal{C}}_{\cup(1:g)}$, *simultaneously* (across all groups) γ -uniformly-approximates *empirical risk*, and is a γ - ℓ_2 cover of the *loss family* $\ell \circ \mathcal{H}$, as

$$\min_{h_\gamma \in \hat{\mathcal{C}}_{\circ(1:g)}} \max_{i \in 1, \dots, g} \left| \hat{R}(h; \ell, \mathbf{z}_i) - \hat{R}(h_\gamma; \ell, \mathbf{z}_i) \right| \leq \min_{h_\gamma \in \hat{\mathcal{C}}_{\circ(1:g)}} \max_{i \in 1, \dots, g} \sqrt{\frac{1}{m} \sum_{j=1}^m ((\ell \circ h)(\mathbf{z}_{i,j}) - (\ell \circ h_\gamma)(\mathbf{z}_{i,j}))^2} \leq \gamma .$$

Proof. We first show items 1 to 3, followed by a key intermediary relating risk values and ℓ_2 distances, and close by showing items 4 and 5.

We begin with item 1. Both bounds follow directly from the definition of uniform covering numbers.

We now show item 2. This result follows via a standard sequence of operations over the Rademacher average. In particular, observe

$$\begin{aligned}
\sup_{\mathcal{D} \text{ over } \mathcal{X} \times \mathcal{Y}} \mathfrak{R}_m(\ell \circ \mathcal{H}, \mathcal{D}) &= \sup_{\mathcal{D} \text{ over } \mathcal{X} \times \mathcal{Y}} \mathbb{E}_{\mathbf{z} \sim \mathcal{D}^m} [\hat{\mathfrak{R}}_m(\ell \circ \mathcal{H}, \mathbf{z})] && \text{DEFINITION OF } \mathfrak{R} \\
&\leq \sup_{\mathcal{D} \text{ over } \mathcal{X} \times \mathcal{Y}} \mathbb{E}_{\mathbf{z} \sim \mathcal{D}^m} \left[\inf_{\gamma \geq 0} \gamma + \hat{\mathfrak{R}}_m(\mathcal{C}^*(\ell \circ \mathcal{H}, \mathbf{z}, \gamma), \mathbf{z}) \right] && \text{DISCRETIZATION} \\
&\leq \inf_{\gamma \geq 0} \gamma + \sup_{\mathcal{D} \text{ over } \mathcal{X} \times \mathcal{Y}} \mathbb{E}_{\mathbf{z} \sim \mathcal{D}^m} \left[\|\ell\|_\infty \sqrt{\frac{\ln |\mathcal{C}^*(\ell \circ \mathcal{H}, \mathbf{z}, \gamma)|}{2m}} \right] && \text{MASSART'S INEQUALITY} \\
&\leq \inf_{\gamma \geq 0} \gamma + \|\ell\|_\infty \sqrt{\frac{\ln \mathcal{N}(\ell \circ \mathcal{H}, \gamma)}{2m}}, && \text{DEFINITION OF } \mathcal{N}
\end{aligned}$$

where the MASSART'S INEQUALITY step follows via *Massart's finite class inequality* (Massart, 2000, lemma 1), and the DISCRETIZATION step via *Dudley's discretization argument*.

We now show item 3. By the *symmetrization inequality*, and a 2-tailed application of McDiarmid's bounded difference inequality (McDiarmid, 1989), where changing any $\mathbf{z}_{i,j}$ has bounded difference $\frac{\|\ell\|_\infty}{m}$, we have that

$$\forall i : \mathbb{P} \left(\sup_{h \in \mathcal{H}} \left| \mathbb{E}_{\mathcal{D}_i}[\ell \circ h] - \hat{\mathbb{E}}_{\mathbf{z}_i \sim \mathcal{D}_i^m}[\ell \circ h] \right| > 2\mathfrak{R}_m(\ell \circ \mathcal{H}, \mathcal{D}_i) + \|\ell\|_\infty \sqrt{\frac{\ln \frac{2}{\delta}}{2m}} \right) \leq \delta$$

thus by union bound over g groups, we have

$$\mathbb{P} \left(\max_{i \in 1, \dots, g} \sup_{h \in \mathcal{H}} \left| \mathbb{E}_{\mathcal{D}_i}[\ell \circ h] - \hat{\mathbb{E}}_{\mathbf{z}_i \sim \mathcal{D}_i^m}[\ell \circ h] \right| > \sup_{\mathcal{D} \text{ over } \mathcal{X} \times \mathcal{Y}} 2\mathfrak{R}_m(\ell \circ \mathcal{H}, \mathcal{D}) + \|\ell\|_\infty \sqrt{\frac{\ln \frac{2g}{\delta}}{2m}} \right) \leq \delta .$$

Now, let *estimation error* bound $\epsilon_{\text{est}} \doteq \sup_{\mathcal{D} \text{ over } \mathcal{X} \times \mathcal{Y}} 2\mathfrak{R}_m(\ell \circ \mathcal{H}, \mathcal{D}) + \|\ell\|_\infty \sqrt{\frac{\ln \frac{2g}{\delta}}{2m}}$, and observe that via item 2,

$$\epsilon_{\text{est}} \leq \inf_{\gamma \geq 0} 2\gamma + 2\|\ell\|_\infty \sqrt{\frac{\ln \mathcal{N}(\ell \circ \mathcal{H}, \gamma)}{2m}} + \|\ell\|_\infty \sqrt{\frac{\ln \frac{2g}{\delta}}{2m}} .$$

From here, we solve for an upper-bound on sample-size m to get

$$\begin{aligned}
m_{\text{UC}}(\ell \circ \mathcal{H}, \varepsilon, \delta, g) &\leq \left\lceil \inf_{\gamma \geq 0} \frac{4\|\ell\|_\infty^2 \ln \mathcal{N}(\ell \circ \mathcal{H}, \gamma) + \|\ell\|_\infty^2 \ln \frac{2g}{\delta}}{2(\varepsilon - 2\gamma)^2} \right\rceil = \left\lceil \frac{2\|\ell\|_\infty^2 \ln \left(\sqrt[4]{\frac{2g}{\delta}} \mathcal{N}(\ell \circ \mathcal{H}, \gamma) \right)}{(\varepsilon - 2\gamma)^2} \right\rceil \\
&\leq \left\lceil \frac{8\|\ell\|_\infty^2 \ln \left(\sqrt[4]{\frac{2g}{\delta}} \mathcal{N}(\ell \circ \mathcal{H}, \frac{\varepsilon}{4}) \right)}{\varepsilon^2} \right\rceil && \text{SET } \gamma = \frac{\varepsilon}{4} \\
&\in \mathbf{O} \left(\frac{\ln \frac{g \mathcal{N}(\ell \circ \mathcal{H}, \varepsilon)}{\delta}}{\varepsilon^2} \right) \subset \text{Poly} \left(\frac{1}{\varepsilon}, \exp \frac{1}{\delta}, \exp g \right) . && \mathcal{N}(\ell \circ \mathcal{H}, \varepsilon) \in \text{Poly} \frac{1}{\varepsilon}
\end{aligned}$$

We now show an intermediary which immediately implies the left inequalities of both items 4 and 5. In particular, we may relate these empirical risk gaps to (size-normalized) ℓ_2 distance, as (for each i) we have $\forall h \in \mathcal{H}, h_\gamma \in \hat{\mathcal{C}}$ that

$$|\hat{R}(h; \ell, \mathbf{z}_i) - \hat{R}(h_\gamma; \ell, \mathbf{z}_i)| \leq \frac{1}{m} \sum_{j=1}^m |(\ell \circ h)(\mathbf{z}_{i,j}) - (\ell \circ h_\gamma)(\mathbf{z}_{i,j})| \leq \sqrt{\frac{1}{m} \sum_{j=1}^m ((\ell \circ h)(\mathbf{z}_{i,j}) - (\ell \circ h_\gamma)(\mathbf{z}_{i,j}))^2} .$$

Here the last inequality holds since we divide m inside the $\sqrt{\cdot}$. The opposite inequality holds for standard ℓ_1 and Euclidean distance, where m is not divided, essentially because the ℓ_1 and ℓ_2 distances differ by up to a factor \sqrt{m} , but this form may be familiar as the relationship between the *mean* and *root mean square* errors. The unconvinced reader may note that this size-normalized ℓ_2 distance is in fact the (unweighted) $p = 2$ power-mean, and thus this step follows via theorem 2.6 item 1.

We now show the right inequality of item 4. Note that for the case of $\hat{\mathcal{C}}_{\cup(1:g)}$, the result is almost tautological, as it holds per group by the union-based construction of $\hat{\mathcal{C}}_{\cup(1:g)}$. The case of $\hat{\mathcal{C}}_{\circ(1:g)}$ is more subtle, but we defer its proof to the final item, as it then follows as an immediate consequence of the *max-min inequality*, i.e., $\forall h \in \mathcal{H}$,

$$\max_{i \in 1, \dots, g} \min_{h_\gamma \in \hat{\mathcal{C}}_{\circ(1:g)}} \sqrt{\frac{1}{m} \sum_{j=1}^m ((\ell \circ h)(\mathbf{z}_{i,j}) - (\ell \circ h_\gamma)(\mathbf{z}_{i,j}))^2} \leq \min_{h_\gamma \in \hat{\mathcal{C}}_{\circ(1:g)}} \max_{i \in 1, \dots, g} \sqrt{\frac{1}{m} \sum_{j=1}^m ((\ell \circ h)(\mathbf{z}_{i,j}) - (\ell \circ h_\gamma)(\mathbf{z}_{i,j}))^2}.$$

We now show item 5. Suppose (by way of contradiction) that there exists some $h \in \mathcal{H}$ such that

$$\min_{h_\gamma \in \hat{\mathcal{C}}_{\circ(1:g)}} \max_{i \in 1, \dots, g} \sqrt{\frac{1}{m} \sum_{j=1}^m ((\ell \circ h)(\mathbf{z}_{i,j}) - (\ell \circ h_\gamma)(\mathbf{z}_{i,j}))^2} > \gamma.$$

One then need only consider the summands associated with a maximal i to observe that this implies

$$\min_{h_\gamma \in \hat{\mathcal{C}}_{\circ(1:g)}} \sqrt{\frac{1}{mg} \sum_{i=1}^g \sum_{j=1}^m ((\ell \circ h)(\mathbf{z}_{i,j}) - (\ell \circ h_\gamma)(\mathbf{z}_{i,j}))^2} > \frac{\gamma}{\sqrt{g}},$$

thus $\hat{\mathcal{C}}_{\circ(1:g)}$ is not a $\frac{\gamma}{\sqrt{g}}$ - ℓ_2 cover of $\bigcup_{i=1}^g \mathbf{z}_i$, which contradicts its very definition. We thus conclude

$$\forall h \in \mathcal{H} : \min_{h_\gamma \in \hat{\mathcal{C}}_{\circ(1:g)}} \max_{i \in 1, \dots, g} \sqrt{\frac{1}{m} \sum_{j=1}^m ((\ell \circ h)(\mathbf{z}_{i,j}) - (\ell \circ h_\gamma)(\mathbf{z}_{i,j}))^2} \leq \gamma.$$

□

With lemma A.1 in hand, we are now ready to show theorem 7.2.

Theorem 7.2 (Efficient FPAC-Learning by Covering). Suppose loss function ℓ of bounded codomain (i.e., $\|\ell\|_\infty$ is bounded), and hypothesis class sequence \mathcal{H} , s.t. $\forall m, d \in \mathbb{N}, \mathbf{z} \in (\mathcal{X} \times \mathcal{Y})^m$, there exist

1. a γ - ℓ_2 cover $\mathcal{C}^*(\ell \circ \mathcal{H}_d, \mathbf{z}, \gamma)$, where $|\mathcal{C}^*(\ell \circ \mathcal{H}_d, \mathbf{z}, \gamma)| \leq \mathcal{N}(\ell \circ \mathcal{H}_d, \gamma) \in \text{Poly}(\frac{1}{\gamma}, d)$; and
2. an algorithm to enumerate a γ - ℓ_2 cover $\hat{\mathcal{C}}(\ell \circ \mathcal{H}_d, \mathbf{z}, \gamma)$ of size $\text{Poly} \mathcal{N}(\ell \circ \mathcal{H}_d, m, \gamma)$ in $\text{Poly}(m, \frac{1}{\gamma}, d)$ time.

It then holds that, given any group distributions $\mathcal{D}_{1:g}$, group weights \mathbf{w} , fair welfare function $\Lambda(\cdot; \cdot)$, ε , δ , and d , the algorithm (see algorithm 2)

$$\mathcal{A}(\mathcal{D}_{1:g}, \mathbf{w}, \Lambda(\cdot; \cdot), \varepsilon, \delta, d) \doteq \mathcal{A}_{\hat{\mathcal{C}}}(\ell, \mathcal{H}_d, \hat{\mathcal{C}}(\cdot, \cdot), \mathcal{N}(\ell \circ \mathcal{H}_d, \cdot), \mathcal{D}_{1:g}, \mathbf{w}, \Lambda(\cdot; \cdot), \varepsilon, \delta)$$

FPAC-learns (ℓ, \mathcal{H}) in polynomial time. In particular, (1) is sufficient to show that (ℓ, \mathcal{H}) is FPAC learnable with polynomial *sample complexity*, and (2) is required only to show polynomial *training time complexity*.

Proof. We now constructively show the existence of a fair-PAC-learner \mathcal{A} for (ℓ, \mathcal{H}) over domain \mathcal{X} and codomain \mathcal{Y} . As in theorem 7.1, we first note that by theorem 2.7 items 4 and 6, under the conditions of FPAC learning, this reduces to showing that we can learn any welfare concept $\Lambda_p(\cdot; \cdot)$ that is a p -power mean with $p \geq 1$.

We first assume a training sample $\mathbf{z}_{1:g, 1:m} \sim \mathcal{D}_1^m \times \cdots \times \mathcal{D}_g^m$, i.e., a collection of m draws from each of the g groups. In particular, we shall select m to guarantee that the *estimation error* for the welfare does not exceed $\frac{\varepsilon}{3}$ with probability at least $1 - \delta$, i.e., we require that with said probability,

$$\epsilon_{\text{est}} \doteq \left| \Lambda_p(i \mapsto \hat{R}(h; \ell, \mathbf{z}_i); \mathbf{w}) - \Lambda_p(i \mapsto R(h; \ell, \mathcal{D}_i); \mathbf{w}) \right| \leq \frac{\varepsilon}{3}.$$

Now, note that by theorem 2.6 item 3 (contraction), we have

$$\left| \Lambda_p(i \mapsto \hat{R}(h; \ell, \mathbf{z}_i); \mathbf{w}) - \Lambda_p(i \mapsto R(h; \ell, \mathcal{D}_i); \mathbf{w}) \right| \leq \max_{i \in 1, \dots, g} \sup_{h \in \mathcal{H}_d} \left| R(h; \ell, \mathcal{D}_i); \mathbf{w} \right| - \hat{R}(h; \ell, \mathbf{z}_i); \mathbf{w} \right|,$$

and by lemma A.1 item 3, a sample of size

$$m = \left\lceil \frac{81\|\ell\|_\infty^2 \ln(\sqrt{\frac{2g}{\delta}} \mathcal{N}(\ell \circ \mathcal{H}, \frac{\varepsilon}{12}))}{\varepsilon^2} \right\rceil \in \mathbf{O}\left(\frac{\ln \frac{g\mathcal{N}(\ell \circ \mathcal{H}, \varepsilon)}{\delta}}{\varepsilon^2}\right) \subset \text{Poly}\left(\frac{1}{\varepsilon}, \exp \frac{1}{\delta}, \exp g\right)$$

suffices to ensure that

$$\mathbb{P}\left(\max_{i \in 1, \dots, g} \sup_{h \in \mathcal{H}_d} \left| R(h; \ell, \mathcal{D}_i); \mathbf{w} - \hat{R}(h; \ell, \mathbf{z}_i); \mathbf{w} \right| > \frac{\varepsilon}{3}\right) \leq \delta ,$$

thus guaranteeing the stated estimation error bound.

With our sample size and *estimation error* guarantee, we now define the *learning algorithm* and bound its *optimization error*. Take *cover precision* $\gamma \doteq \frac{\varepsilon}{3\sqrt{g}}$. By assumption, for each $d \in \mathbb{N}$, we may enumerate a γ -cover $\hat{\mathcal{C}}(\ell \circ \mathcal{H}_d, \bigcup_{i=1}^g \mathbf{z}_i, \gamma)$, where $\bigcup_{i=1}^g \mathbf{z}_i$ denotes the *concatenation* of each \mathbf{z}_i , in $\text{Poly}(gm, \frac{1}{\gamma}, d) = \text{Poly}(\frac{1}{\varepsilon}, \frac{1}{\delta}, d, g)$ time. For the remainder of this proof, we refer to this cover as $\hat{\mathcal{C}}$.

Now, we take the learning algorithm to be *empirical welfare minimization* over $\hat{\mathcal{C}}$. Let

$$\hat{h} \doteq \underset{h_\gamma \in \hat{\mathcal{C}}}{\operatorname{argmin}} \Lambda_p(i \mapsto \hat{R}(h_\gamma; \ell, \mathbf{z}_i); \mathbf{w}) \quad \& \quad \tilde{h} \doteq \underset{h \in \mathcal{H}_d}{\operatorname{argmin}} \Lambda_p(i \mapsto \hat{R}(h; \ell, \mathbf{z}_i); \mathbf{w}) ,$$

where ties may be broken arbitrarily. Note that via standard covering properties, that the *optimization error* is bounded as

$$\begin{aligned} \varepsilon_{\text{opt}} &\doteq \Lambda_p(i \mapsto \hat{R}(\hat{h}; \ell, \mathbf{z}_i); \mathbf{w}) - \Lambda_p(i \mapsto \hat{R}(\tilde{h}; \ell, \mathbf{z}_i); \mathbf{w}) && \text{DEFINITION} \\ &= \sup_{h \in \mathcal{H}_d} \min_{h_\gamma \in \hat{\mathcal{C}}} \Lambda_p(i \mapsto \hat{R}(h_\gamma; \ell, \mathbf{z}_i); \mathbf{w}) - \Lambda_p(i \mapsto \hat{R}(h; \ell, \mathbf{z}_i); \mathbf{w}) && \text{PROPERTIES OF SUPREMA} \\ &\leq \sup_{h \in \mathcal{H}_d} \min_{h_\gamma \in \hat{\mathcal{C}}} \max_{i \in \{1, \dots, g\}} \left| \hat{R}(h_\gamma; \ell, \mathbf{z}_i) - \hat{R}(h; \ell, \mathbf{z}_i) \right| && \text{THEOREM 2.6 ITEM 3 (CONTRACTION)} \\ &\leq \sqrt{g}\gamma = \frac{\varepsilon}{3} . && \text{LEMMA A.1 ITEM 5} \end{aligned}$$

This controls for *optimization error* between the true and approximate EMM solutions \tilde{h} and \hat{h} .

We now combine the optimization and estimation error inequalities, letting

$$h^* \doteq \underset{h \in \mathcal{H}_d}{\operatorname{argmin}} \Lambda_p(i \mapsto R(h; \ell, \mathcal{D}_i); \mathbf{w}) ,$$

denote the true *melfare optimal* solution, over *distributions* rather than *samples*, breaking ties arbitrarily. We then derive

$$\begin{aligned} \Lambda_p(i \mapsto R(h^*; \ell, \mathcal{D}_i); \mathbf{w}) - \Lambda_p(i \mapsto R(\hat{h}; \ell, \mathcal{D}_i); \mathbf{w}) &= \left(\Lambda_p(i \mapsto R(h^*; \ell, \mathcal{D}_i); \mathbf{w}) - \Lambda_p(i \mapsto \hat{R}(h^*; \ell, \mathbf{z}_i); \mathbf{w}) \right) && \leq \epsilon_{\text{est}} \\ &+ \left(\Lambda_p(i \mapsto \hat{R}(h^*; \ell, \mathbf{z}_i); \mathbf{w}) - \Lambda_p(i \mapsto \hat{R}(\tilde{h}; \ell, \mathbf{z}_i); \mathbf{w}) \right) && \leq 0 \\ &+ \left(\Lambda_p(i \mapsto \hat{R}(\tilde{h}; \ell, \mathbf{z}_i); \mathbf{w}) - \Lambda_p(i \mapsto \hat{R}(\hat{h}; \ell, \mathbf{z}_i); \mathbf{w}) \right) && \leq \epsilon_{\text{opt}} \\ &+ \left(\Lambda_p(i \mapsto \hat{R}(\hat{h}; \ell, \mathbf{z}_i); \mathbf{w}) - \Lambda_p(i \mapsto R(\hat{h}; \ell, \mathcal{D}_i); \mathbf{w}) \right) && \leq \epsilon_{\text{est}} \\ &\leq \epsilon_{\text{opt}} + 2\epsilon_{\text{est}} = \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon . && \text{SEE ABOVE} \end{aligned}$$

We thus conclude that this algorithm produces an ε - $\Lambda_p(\cdot, \cdot)$ optimal solution with probability at least $1 - \delta$, and furthermore both the sample complexity and time complexity of this algorithm are $\text{Poly}(\frac{1}{\varepsilon}, \frac{1}{\delta}, g, d)$. Hence, as we have constructed a polynomial-time fair-PAC learner for (\mathcal{H}, ℓ) , we may conclude $(\mathcal{H}, \ell) \in \text{FPAC}_{\text{Poly}}$. \square

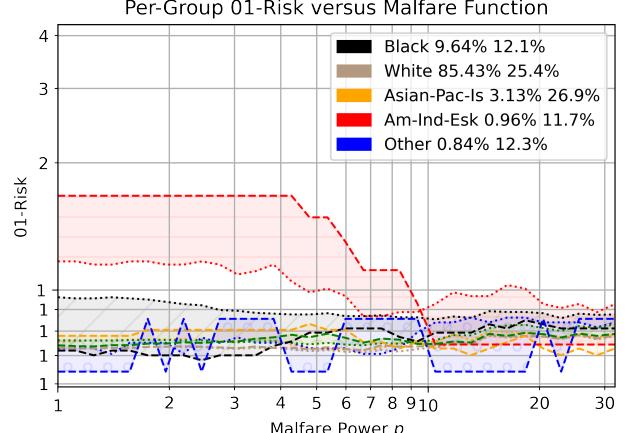


Figure 8: Training and test 0-1 risk, per-group and malfare (green), of `adult` experiment, as a function of malfare power p . The model is optimized for weighted malfare of weighted hinge-risk, and is thus identical to that of figure 3. Here hinge-risk is optimized as a *proxy* for the 0-1 risk, so only the reported risk function changes in this figure.

B Experimental Setup and Extensions

B.1 Data, Preprocessing, and Experimental Setup

All experiments are conducted on the `adult` dataset, derived from the 1994 US Census database, and obtained from the UCI repository (Dua and Graff, 2021), where it was donated by Ronny Kohavi and Barry Becker. This dataset has $m = 48842$ instances, and we used a 90% : 10% training:test split. The task has binary target variable `income`, 6 numeric features, and 8 categorical features, including `race` split into 5 ethnoracial groups, and `gender` split into 2 gender groups. In each experiment, the target and protected group are omitted from the feature set, the remaining categorical features are 1-hot encoded, and all d features are z -score normalized.

All experiments are with λ - ℓ_2 -norm constrained linear predictors, i.e., the hypothesis class is

$$\mathcal{H} \doteq \left\{ h(\vec{x}; \vec{\theta}) = \vec{x} \cdot \vec{\theta} \mid \vec{\theta} \in \mathbb{R}^d, \|\vec{\theta}\|_2 \leq \lambda \right\} .$$

The output of this hypothesis class is real-valued, but for this binary classification task, we take $\mathcal{Y} = \pm 1$, so the loss function is selected to reify this value with a semantic classification interpretation. The 0-1 loss (for hard classification) is defined as

$$\ell_{01}(y, h(\vec{x}; \vec{\theta})) = 1 - y \operatorname{sgn}(\vec{x} \cdot \vec{\theta}) ,$$

which is readily interpreted in a decision-theoretic sense, but is generally computationally intractable to optimize. The SVM objective is generally stated in terms of the *hinge loss*, which acts as a convex relaxation of the 0-1 loss. The hinge-loss is defined as

$$\ell_{\text{hinge}}(y, h(\vec{x}; \vec{\theta})) = \max(0, 1 - y(\vec{x} \cdot \vec{\theta})) ,$$

which is of course *convex* in $\vec{\theta}$, and obeys $\ell_{01}(y, h(\vec{x}; \vec{\theta})) \leq \ell_{\text{hinge}}(y, h(\vec{x}; \vec{\theta}))$. Finally, the logistic-regression cross-entropy loss (measured in nats) is (see, e.g., ch. 9.3 of (Shalev-Shwartz and Ben-David, 2014))

$$\ell_{\text{LRCE}}(y, h(\vec{x}; \vec{\theta})) = \ln(1 + \exp(-y(\vec{x} \cdot \vec{\theta}))) ,$$

which interprets the model output as a *probabilistic classification* $\mathbb{P}(y = 1 | \hat{y}) = \frac{1}{1 + \exp(-\hat{y})}$. Note that for $\hat{y} \not\approx 0$, $\ell_{\text{LRCE}}(y, \hat{y}) \approx \ell_{\text{hinge}}(y, \hat{y})$, and logistic regression may also be viewed as a *convex relaxation* of hard classification, as $\ell_{01}(y, \hat{y}) \leq \frac{1}{\ln(2)} \ell_{\text{LRCE}}$ (perhaps more naturally, the $\frac{1}{\ln(2)}$ constant vanishes if we measure cross entropy in *bits* rather than *nats*).

In all experiments with *weighted risk values*, we use regularity constraint $\lambda = 4$, and in the experiments with *unweighted risk values*, we take $\lambda = 10$.

Implementation and Computational Resources Computation was not a concern on these simple convex linear models; all experiments were run on a low-end laptop with no GPU acceleration.

Theorem 7.1 analytically quantifies the computational complexity of ε -EMM, but in our experiments, we simply used standard out-of-the-box first-order methods (adaptive projected gradient descent and SLSQP), as well as derivative-free methods (COBYLA) to train all models.

B.2 Supplementary Experiments

0-1 Risk of Weighted SVM Figure 8 complements figure 3, reporting the same per-group and malfare statistics, except now on the (similarly weighted) 0-1 risk, rather than the weighted hinge risk. Here, the interpretation is that

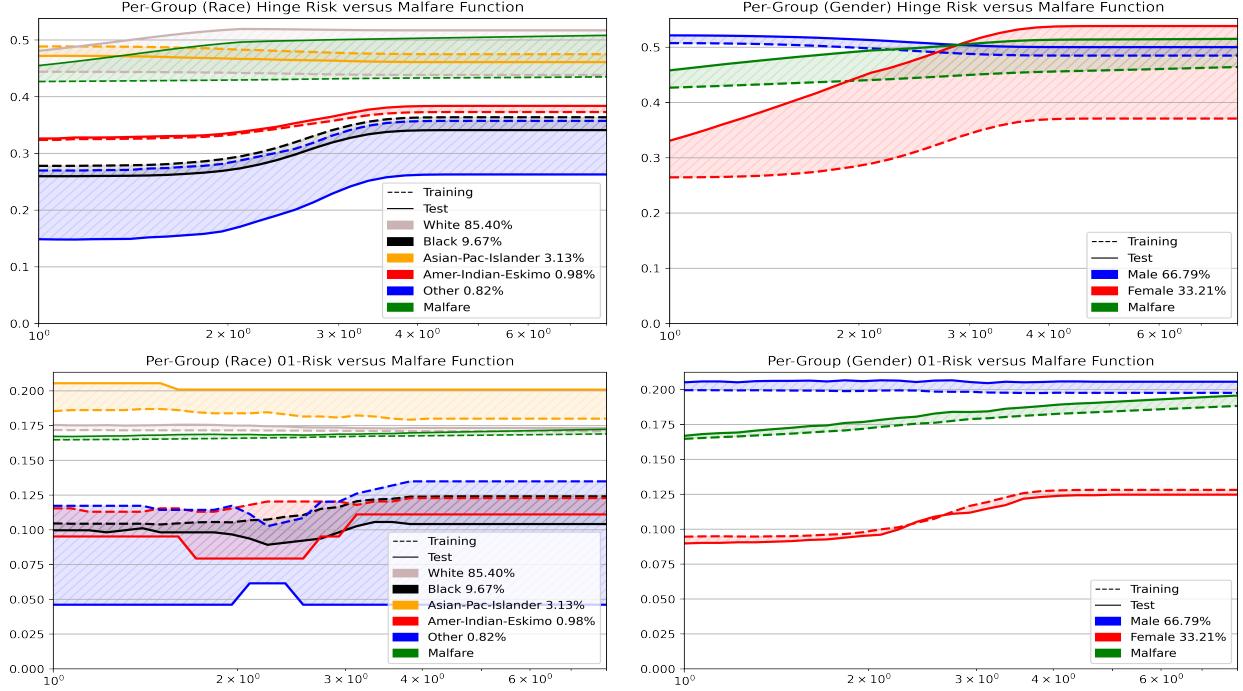


Figure 9: Unweighted linear SVM experiments on `adult` dataset, with groups split by race (left) and gender (right), malfare and risk plotted against p . The upper row depicts hinge-risks and malfare on hinge-risks, and the lower row depicts the 0-1 risks and malfare on 0-1 risks (of the models trained on hinge-risk). All plots show training and test per-group risk and malfare, as a function of p , with shaded regions depicting train-test gaps.

the hinge risk is a *convex proxy* for the 0-1 risk, as it would be computationally intractable to optimize the 0-1 risk directly. Because we optimize hinge risk, but report 0-1 risk, we don't expect to see monotonicity in malfare, and the discontinuity of the 0-1 risk is manifest as noise in risk values. Nevertheless, if hinge risk is a good proxy for 0-1 risk, we should still see a *general trend* of the classifier becoming fairer (improving high-risk group performance) w.r.t. 0-1 risk as it becomes fairer w.r.t. hinge risk, and we do in fact observe this with increasing p .

Unweighted SVM These experiments are quite similar to those of figure 3 and figure 8, except here we optimize the malfare of, and report the values of, the *unweighted* hinge risk. In these experiments, we also take regularity constraint $\|\vec{\theta}\|_2 \leq \lambda = 10$, and report the hinge and 0-1 risks and malfares, using *race* and *gender* groups. As such, the objective is to minimize the $\text{M}_p(\cdot; \mathbf{w})$ malfare of per-group hinge-loss, using per-group-frequencies as malfare weights, i.e.,

$$\hat{h} \doteq \operatorname{argmin}_{h \in \mathcal{H}} \text{M}_p \left(i \mapsto \hat{R}(h; \ell_{\text{hinge}}, \mathbf{z}_i); \mathbf{w} \right).$$

With both gender and race, we see significantly variations in model performance between groups. We stress that group size and affluence are not directly correlated with model accuracy; for instance, here we see that model performance on the (generally affluent) *Male*, *white*, and *Asian* populations is relatively poor, due to greater income homogeneity within these groups (in direct contrast to the *weighted* experiments).

In all cases, we see that increasing p improves the *training set performance* of the model on the high-risk (inaccurate) groups (male, white, and Asian), at the cost of significant performance degradation for the more accurate groups. However, the trend does not always hold in *test set* performance, since raising p increases the relative importance of *high-risk subpopulations* in training, which leads to increased overfitting. This highlights the phenomenon of *overfitting to fairness*, as we see that improved training set malfare does not necessarily translate to the test set.

Logistic Regression Experiments Figure 10 complements the previous experiments, where now we optimize malfare of (weighted) *cross entropy risk* of logit predictors, where weights are chosen as in figure 3, i.e., we optimize

$$\hat{h} \doteq \operatorname{argmin}_{h \in \mathcal{H}} \text{M}_p \left(i \mapsto \frac{1}{b_i} \hat{R}(h; \ell_{\text{LRCE}}, \mathbf{z}_i); \mathbf{w} \right).$$

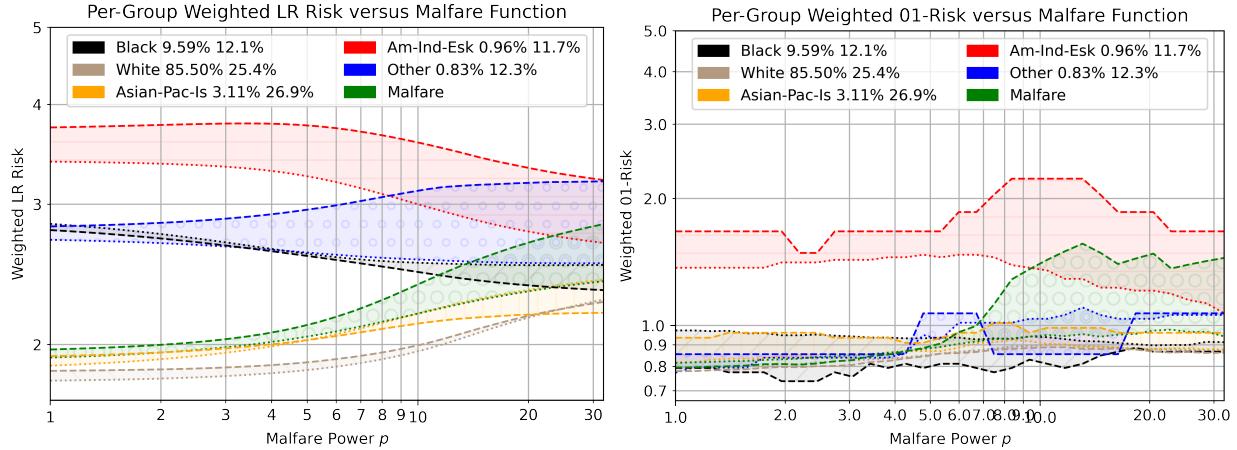


Figure 10: Experiments on `adult` dataset on race groups, with *weighted logistic regression* malfare objective.

We draw essentially the same conclusions as with the hinge risk: malfare minimization yields to better training performance of the model for high-risk (Black, native American, and other) groups, and better test-performance, except in the *other* group, which is tiny and badly overfit.