A Game-theoretic Understanding of Repeated Explanations in ML Models

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Abstract

This paper formally models the strategic repeated interactions between a system, comprising of a machine learning (ML) model and associated explanation method, and an end-user who is seeking a prediction/label and its explanation for a query/input, by means of game theory. In this game, a malicious end-user must strategically decide when to stop querying and attempt to compromise the system, while the system must strategically decide how much information (in the form of noisy explanations) it should share with the end-user and when to stop sharing, all without knowing the type (honest/malicious) of the This paper formally models this end-user. trade-off using a continuous-time stochastic Signaling game framework and characterizes the Markov perfect equilibrium state within such a framework.

1 Introduction

Today, complex machine learning (ML) models are employed by almost every major sector of our economy for making critical data-based decisions [Bhavsar et al., 2017], [Ahmad et al., 2018], [Ballestar et al., 2019], [Dixon et al., 2020]. One challenge while using complex ML models, such as, Deep Neural Networks (DNN), is that model description is hidden behind millions of parameters, making it impossible to comprehend model decisions. Consequently, there have been significant research efforts to design explanation techniques that provide a better understanding of the behavior and predictions by complex, pre-trained ML models, effectively improving their transparency and reliability [Shrikumar et al., 2017] [Lundberg and Lee, 2017] [Ribeiro et al., 2016] [Arras et al., 2017] [Jacovi et al., 2018] [Ancona et al., 2018].

One drawback of providing additional information as explanations is that it exposes an attack surface that can be exploited by an adversary to infer private model information [Shokri *et al.*, 2021] or launch adversarial attacks against the model [Ignatiev *et al.*, 2019][Slack *et al.*, 2020]. It is intuitive that the length and pattern of repeated interactions with a ML model, and the accompanying explanations, could have an effect on the amount of private information that can be leaked from the model. Currently, there is no formal understanding

of this phenomenon, especially in the presence of a strategic adversary who wants to minimize the attack path (and cost) towards compromising model privacy and a strategic ML system who wants to maximize information in the released explanations to non-adversaries. We address this research gap by formally modeling the strategic interactions between an adversary who wishes to exploit an explanation technique (to leak sensitive information) and a ML system that wishes to prevent such attacks while still providing an acceptable quality of explanations to all users (including honest ones).

We employ game theory to model the dynamics of explanations generated by a system (comprising of a ML model and the corresponding explanation technique) for predictions/labels related to queries sent by end-users (comprising of both honest and malicious users). Specifically, we model the interactions between an end-user and the system, where explanations generated by the system evolve according to a stochastic differential equation (SDE), as a two-player continuous-time Signaling Game. We then characterize the Markov Perfect Equilibrium (or MPE) of the above stochastic game as a pair of two optimal functions:- $U(\pi)$ and $L(\pi)$, where $U(\pi)$ represents the optimal explanation path for the system, $L(\pi)$ represents the optimal explanation path for the end-user and π represents the belief of the system about the type of end-user. We then prove conditions for the existence and uniqueness of a MPE in the above game model. Following that, we numerically analyze the existence of MPE and the optimal play (actions) of both the system and the end-user for four popular gradient-based explanation methods: Integrated Gradients [Sundararajan et al., 2017], Saliency Maps [Simonyan et al., 2014], Gradient*Input [Shrikumar et al., 2016] and ϵ -LRP [Bach et al., 2015].

2 Background and Preliminaries

2.1 Machine Learning using Gradient Descent

A ML algorithm (or model) is typically used to find underlying patterns within (vast amounts of) data, which enables systems that employ them learn and improve from experience. In this paper, we assume that a ML model performs a classification task, i.e., mapping a given input vector \overrightarrow{x} to a predicted label $y = F(\overrightarrow{x})$. Any ML model, denoted by a function $F: \mathbb{R}^N \to \mathbb{R}$, is defined by a set of parameters θ , and

the model "learns" by calculating the optimal set of these parameters using some optimization algorithm. To be explicit, we may write $F_{\theta}(\overrightarrow{x})$ to refer to the model $F(\overrightarrow{x})$ with parameters θ . One popular approach to train ML models is the *Stochastic Gradient Descent (SGD)* approach [Bottou, 2010]. Given a set of labeled training examples \mathcal{X}_{tr} with examples \overrightarrow{x} and corresponding labels y, SGD attempts to solve:

$$\underset{\theta}{\arg\min} \underset{(\overrightarrow{x},y) \in \mathcal{X}_{tr}}{\mathbb{E}} l(F_{\theta}(\overrightarrow{x}), y)$$

where $l(\cdot)$ is the loss function which intuitively measures how "wrong" the prediction $F_{\theta}(\overrightarrow{x'})$ is compared to the true label y. SGD solves the above problem by iteratively updating the parameters:

$$\theta \leftarrow \theta - \alpha \cdot \nabla_{\theta} (\sum_{(\overrightarrow{x},y) \in \mathcal{X}_{tr}} (F_{\theta}(\overrightarrow{x}), y))$$

where ∇_{θ} is the gradient of the loss with respect to the weights θ and α is the learning rate which controls by how much the weights θ should be changed.

2.2 Gradient based Explanations

In general, an explanation method \mathcal{H} corresponding to a learning-based classification model F_{θ} outputs some justification of why the model returns a particular label $y = F_{\theta}(\overrightarrow{x})$ for an input \overrightarrow{x} . Several explanation techniques have been proposed in the research literature [Molnar, 2020], however for concreteness we outline four popular gradient based explanation methods: Integrated Gradients, Saliency maps, Gradient*Input and ϵ -LRP. Let's consider an input $\overrightarrow{x} = (f_1, f_2, ..., f_n)$, where f_i is the i^{th} feature value and n is the number of features. Intuitively, in gradient-based methods the explanation $\mathcal{H}(\overrightarrow{x}) \in \mathbb{R}^n$ of an input \overrightarrow{x} is $\mathcal{H}(\overrightarrow{x}) = \{e_1, e_2, ..., e_n\}$, where e_i is the contribution of f_i in the prediction of $F(\overrightarrow{x})$. Below is the brief introduction of all four gradient based explanation methods.

- Saliency maps: This method allows to compute the relevance of the class score of interest by taking the gradient of that class score with respect to the input features while setting the other classes to zero, i.e., $Saliency(\overrightarrow{x}) = \frac{\partial S_c}{\partial \overrightarrow{x}}$, where S_c is the class score of interest.
- ϵ -LRP: In this method, attributions are obtained by doing a backward pass on the model network. The algorithm defines the relevance in the last layer L as the output of that layer itself and for the previous layers, it redistributes the layer's relevance according to the weighted contribution of the neurons of the previous layer to the current layer's neurons. One rule for the redistribution of a layer's relevance to the previous layer is the ϵ -rule [Ancona et al., 2018].
- Integrated Gradients: Integrated gradients are obtained by accumulating gradients, computed at all points along the linear path from a baseline \overrightarrow{x}' to the actual \overrightarrow{x} [Sundararajan *et al.*, 2017]. In other words, integrated gradients are the path integral of the gradients along the straight-line path from the baseline \overrightarrow{x}' to the input \overrightarrow{x} . The integrated gradient along the *i*-th feature for an input \overrightarrow{x} and baseline \overrightarrow{x}' is defined as:

$$IntegratedGrads_{i}(\overrightarrow{x}) = (f_{i} - f'_{i}) \times \int_{\alpha=0}^{1} \frac{\partial F(\overrightarrow{x}' + \alpha \times (\overrightarrow{x} - \overrightarrow{x}'))}{\partial f_{i}} d\alpha$$

• Gradient*Input: In this method, attributions are computed by taking the signed partial derivative of the model function with respect to the input query and then multiplying the obtained derivative with the input itself, i.e., $Gradient * Input(\overrightarrow{x}) = \overrightarrow{x} \times \frac{\partial S_c}{\partial \overrightarrow{x}}$

2.3 Geometric Brownian Motion

We model the explanation process (EX) in this work as a *Geometric Brownian Motion (GBM)* process. GBM is a continuous-time stochastic process that takes non-negative values. The system uses an explanation method \mathcal{H} to generate explanations for labels predicted by the machine learning model $F_{\theta}(\overrightarrow{x})$. We assume that there are random fluctuations in the generated explanations, because they vary from label to label for different queries. In summary, we assume that the explanation generation (EX) is a stochastic process that follows a GBM, and which can experience shocks both upward and downward. Given a generic state process s_t , it follows a GBM if it satisfies the following stochastic differential equation (SDE):

$$ds_t = a(s, u(s,t), t)s_t dt + b(s, u(s,t), t)s_t dW_t$$

where a(s, u(s,t), t) and b(s, u(s,t), t) are the drift and volatility parameters of the state process s_t , respectively, W_t is a standard Brownian motion with mean = 0 and variance = t, and u(s,t) is the control.

2.4 Continuous-time Bellman Equation

Problems involving optimal control are usually modeled with Bellman's equation and solved using optimization techniques such as dynamic programming. In our paper, we have assumed $s=(EX_t,\pi_t)$ as a state pair and control u(s,t) is the decision of the end-user and the system to either continue or stop the process of sending queries and (their) explanations, respectively. The value function H(s,t) represents the optimal payoff/reward of the system or the end-user over the interval t=[0,T] when started at the time t=0 having state variable $s_0=(ex_0,\pi_0)$. Now, given that s_t follows a GBM, the value function $(H(s_t,t))$ can be written as:

$$H(s_t, t) = \max_{u} \int_0^T f(s_t, u(s_t, t), t) dt$$

where, $f(s_t, u(s_t, t), t)$ is the instantaneous payoff/reward a player can get given the state (s_t) and the control used (u) at time t. For continuous-time optimization problems, the Bellman equation is a partial differential equation or PDE, referred to as the Hamilton Jacobi Bellman (HJB) equation, and can be written as:

$$rH(s_t, t) = f(s_t, u^*, t) + \frac{\partial H}{\partial t} + \frac{\partial H}{\partial s_t} a(s_t, u^*, t)$$
$$+ \frac{1}{2} \frac{\partial^2 H}{\partial s_t^2} b(s_t, u^*, t)^2$$

 $u^* = u(s_t, t)$ = optimal value of control variable

We represent the value functions of both the system and the end-user in this work using the above equation. The optimal control u for an entity or agent is a binary decision, i.e, u=1 to stop the task being done in the previous time instant, while u=0 to continue.

• Stopping Problem and Terminal Condition: A stopping problem models a situation where an agent must decide whether to continue the activity he/she is involved in (in the current time instant) or cease it, based on the payoff he/she is expected to receive in the next instant. In the case of the two agents in this paper, i.e., the system and the end-user, they need to make an optimal decision on whether to continue their respective activities and get an instantaneous flow payoff, $f(s_t, u(s_t, t), t)$ or stop the activity they were involved in (previous time instant) and get the termination payoff, $\lambda(s_t, T)$. If s_t^* is the state boundary value at which an agent decides to stop and get the termination payoff, then the solution to the stopping problem is a stopping rule: -

$$u(s,t) = \begin{cases} \text{stop,} & s_t >= s_t^* \\ \text{continue,} & s_t < s_t^* \end{cases}$$

In other words, when the agent decides to stop, he/she gets:

$$H(s_t, T) = \lambda(s_t, T) \quad \forall s_t \ge s_t^*$$

• Value Matching and Smooth Pasting Condition: In order to solve the HJB equation outlined above, two boundary conditions are required which we describe next. The first condition, called the *value matching condition*, defines a constraint at the boundary which tells an agent that if they decide to stop (at that defined boundary), then the payoff it would get is at least the same as the previous time instant. The value matching condition determines whether the value function is continuous at the boundary or not:

$$H(s_t^*, t) = \lambda(s_t^*, t) \quad \forall t$$

However, as the boundary, s_t^* is also an unknown variable, we need another condition which will help in finding s_t^* along with $H(s_t,t)$. The *smooth pasting condition* helps in pinning the optimal decision boundary, s_t^* . Intuitively, it also helps to formulate an agent's indifference between continuation and stopping.

$$H_{s_t}(s_t^*, t) = \lambda_{s_t}(s_t^*, t) \quad \forall t$$

where $H_{s_t}(s_t^*,t)$ is the derivative of $H(s_t^*,t)$ with respect to the state s_t . If one or both the above conditions are not satisfied, then stopping at the boundary s_t^* can't be optimal. Therefore, an agent should continue and again decide at the next time instant.

3 Game Model

Next, we present an intuitive description of the problem followed by its formal setup as a Signaling game. Further, we also characterize the equilibrium concept in this setup.

3.1 Intuition

We consider a platform, referred to as the system, that makes available an ML model and an explanation method as a (black box) service. Customers of this platform, referred to as end-users, seek a label and it's explanation for their query (sent to the system), but cannot download the model itself from the platform. A query is classified as suspicious if the explanation generated for its label lies near some boundary of maximum relevant explanations which can be given by the system to the end-user, otherwise, it is non-suspicious. The system can serve two types of end-users: honest and malicious. Honest end-users have no malicious goal (of compromising the system) and always send non-suspicious queries, referred to as "signals" (or "signaling"). Malicious end-users' primary goal is to carry out privacy or other attacks against the ML model by employing query labels and explanations received from the system. To accomplish this objective without getting detected (or blocked) by the system, a malicious end-user will attempt to frame queries by mimicking honest end-users as much as possible to confuse the system. Eventually when the moment (state) is right, the malicious end-user must strategically deviate from this signaling strategy (i.e., mimicking the honest end-user) to accomplish the attack objective. This deviation is accomplished by the malicious end-user selecting suspicious queries from a different, much larger, query space than that of the honest end-user. In summary, as the game evolves, an honest end-user is always expected to signal, while a malicious end-user can choose to signal by sending queries similar to that of honest end-user or deviate from signaling by reverting to its own (malicious/suspicious) query space and eventually carry out the attack against the system. We assume that the malicious end-user has enough resources to compromise the system. Thus, he must decide to either continue getting explanations through honest queries (signaling) from the system or deviate from honest behavior based on an optimal explanation path $L(\pi)$, as described later in Section 3.3. This is important as the malicious end-user wants to avoid being detected by the system prior to compromising it. The system on receiving requests from the end-users should decide whether to continue giving explanations and how much noise/perturbation to add to it, or to just block the end-user based on an optimal explanation path $U(\pi)$, as outlined later in Section 3.3. It should be noted that the system has imperfect information about the type of the end-user sending the query, i.e., it does not know whether the query is from an honest or malicious end-user. Thus, the added noise/perturbation to the generated explanation is based on the system's belief about the type of the end-user. In this preliminary effort, we formally model the above interactions between a single end-user (type determined by nature) and the system within a stochastic game-theoretic framework, and further analyze it to answer the following two high-level questions: When does a malicious end-user decide to deviate from honest behavior (i.e., stop signaling) and attack the system? How does the system make the strategic decision to block a potential malicious end-user while continuing to give relevant expla-

3.2 Setup and Assumptions

We model the above scenario as a two-player continuous-time imperfect information game with repeated play. The game has two players: Player 1 is the end-user, of privately known type $\Theta \to \{h, m\}$ (i.e., honest or malicious), who wants to convince Player 2 (i.e., system) that he is honest. The game begins with nature picking an end-user of a particular type, and we analyze repeated play between this end-user (selected by nature) and the system which occurs in each continuous-time, $t \in R$. As the system has imperfect information about the type of end-user, it assigns an initial belief $\pi_0 = Pr(\Theta = h)$. We assume both players are risk neutral, i.e., indifferent to taking risk, and each player discounts payoffs at a constant rate r. Explanations (EX_t) generated by an explanation method of the system follows a Geometric Brownian Motion (GBM), and is given by (details in Section 2.3):

$$dEX_t = \mu EX_t dt + \sigma EX_t dW_t$$

where, μ is the constant drift and $\sigma>0$ is the constant volatility of the explanation process EX_t , and $EX_0=ex_0>0$. W_t is a standard Brownian motion with mean = 0 and variance = t. To ensure finite payoffs at each continuous time t, we assume $\mu< r$. The state of the game is represented by the pair (ex_t,π_t) , where ex_t is the realization of the process EX_t and π_t is the belief of the system about the type of the end-user at time t.

The system wants to give informative or relevant explanations to the honest end-user, but noisy explanations to the malicious end-user. Hence, depending on the system's belief about the type of the end-user, it will decide how much noise/perturbation to add to each released explanation. Let $U(\pi_t)$ denote the optimal explanation path (or functional path) for the system - an increasing cut-off function which tells the system the maximum relevant explanation generated by an explanation method and $L(\pi_t)$ denote the optimal explanation path for the end-user - an increasing cut-off function which tells the end-user the maximum relevant explanation he/she can expect from the system at given belief π_t . To simplify the resulting analysis, we assume that explanations generated and explanations given are just different realizations of the explanation process EX_t . Moreover, as the system would add some calculated noise to the generated explanation based on it's belief about the type of the end-user (discussed in detail later), we assume that $U(\pi_t) \geq L(\pi_t), \ \forall \pi_t$. Also, both explanation generated by an explanation method and explanation given to the end-user follow a GBM. Finally, as discussed earlier, at each continuous time $t \in R$, a malicious end-user can decide either to signal or not, while a honest end-user always signals. As we are only interested in modeling interactions between a malicious end-user and the system, any reference to an end-user from this point on implies a malicious end-user, unless explicitly stated otherwise. Next, we outline a few other relevant model parameters before characterizing the concept of equilibrium in the proposed game model. Information Environment: Let $\mathcal{F}_t = \sigma(\{EX_s\}: 0 \leq s \leq t)$ be the end-user's information environment, which is the sigma-algebra generated by the explanation process EX. In other words, \mathcal{F}_t represents the information contained in the public history of the explanation process. The system's information environment is denoted by $\mathcal{F}_t^+ = \sigma(\{EX_s, \phi_s\}: 0 \leq s \leq t)$, where EX_s is the explanation process representing the history of explanations and ϕ_s is the stochastic process representing the signaling activity (history) of the end-user. If ρ is the time that end-user reveals or deviates, then $\phi_t = \rho$ if $\rho \leq t$ and ∞ otherwise.

Strategies:

- end-user: We only define strategies for the malicious end-user type (m) as the honest type (h) has no incentive to stop signaling. We assume that the (malicious) end-user plays a randomized strategy, i.e., at each time t he either continues to signal or deviates from honest behavior (stop signaling) and attack the system. The end-user's strategy space is dependent on the history of the explanations given by the system, hence it is a collection of \mathcal{F}_t adapted stopping times $\{\phi_t\}$ such that $\phi_t = \rho$ if $\rho \leq t$ and ∞ otherwise.
- system: We assume system plays a pure stopping time (τ^t) strategy, i.e., the time at which it blocks the end-user (if deemed malicious). The system's strategy is dependent on the evolution of the explanation process EX and record of end-user's signaling activity. Hence, the strategy space of the system is a collection of \mathcal{F}_t^+ -adapted stopping times $\{\tau^t\}$.

We use a path-wise Cumulative Distribution Function (CDF), represented as $R_t^{t_0}$, to characterize how fast the explanation generated at a given time t is reaching some maximum relevance explanation threshold (defined later). We will compute this CDF from the probability density function $(f_t(ex))$ of the GBM, given by:

$$f_t(ex) = \frac{1}{\sqrt{2\pi t}\sigma ex} \exp\left(-\frac{[ln(ex) - (\mu - \sigma^2)t]^2}{2\sigma^2 t}\right),$$

where, $ex \in (0, \infty)$. In other words, the CDF $(R_t^{t_0})$ will give the probability of how close the generated explanation is to the maximum relevant explanation at time t starting from the explanation generated at time t_0 , i.e., ex_0 .

Beliefs: Given information \mathcal{F}_t^+ , the system updates it's beliefs at time t from time $t_0 < t$ using *Bayes' rule*, as shown below:

$$\pi_t = \begin{cases} \frac{1}{1 + (1 - \pi_{t_0})R_t^{t_0}}, & \text{if} \quad \pi_{t_0} > 0 \quad \text{and} \quad \rho > t. & \textbf{(i)} \\ 0, & \text{if} \quad \rho \leq t \quad \text{or} \quad \pi_{t_0} = 0. & \textbf{(ii)} \end{cases}$$

Bayes' rule (i) is used when the <code>end-user</code> has not stopped signaling ($\rho \geq t$) and the initial belief of the <code>system</code> about the <code>end-user</code>'s type is also not zero. However, if the <code>system</code> has already identified the <code>end-user</code>'s type as m or the <code>end-user</code> has already stopped signaling, then <code>system</code>'s belief π_t will be zero, as indicated in (ii).

Payoffs: The system earns a reward of $D_B^{\Theta=m}=kEX_t$ for detecting and blocking the malicious end-user. The

Table 1: Flow Payoff Coefficients before and after Detection by the system.

	Before Detection		After Detection	
	Signaling (Pooling)	No Signaling (Separation Starts)	Detection and Block (Game Ends)	
end-user, type m	P	M_{NS}^m	-k	
end-user, $type\ h$	P	P	P	
system	r_e	D_{NS}^{Θ}	D_B^Θ	

end-user's type (malicious) is immediately revealed at this time, thus a cost of -k is incurred by the end-user. In case of interaction with an honest end-user, the system will always earn a payoff of $r_e E X_t$, i.e., $D_{NS}^{\Theta=h} = r_e E X_t$ and $D_{R}^{\Theta=h} = r_{e}EX_{t}$, while the honest end-user always earns a reward of PEX_t in each stage of the game. In case of a malicious end-user who signals, he receives a payoff (relevant explanation) of PEX_t . Prior to detection, if the malicious end-user deviates (i.e., stop signaling) and is able to compromise the system, then the system will have to pay a lump-sum cost of $D_{NS}^{\Theta\equiv m}=-d^{'}$ and the malicious end-user will earn $M_{NS}^{m}=(M^{m}+d^{'})EX_{t}-d$, where $M^{m}EX_{t}$ is the gain which relates to the relevant explanation gained from the system, $d'EX_t$ is the benefit (can be monetary) achieved after attacking the system and d is the cost of deviation. We make the following assumptions about the payoff coefficients (outlined in Table 1): We assume that $D_{R}^{\Theta=m}=kEX_{t}>r_{e}EX_{t}$, as the system will gain more in successfully preventing the attack from the malicious end-user. When the malicious end-user deviates to attack the system and is not successful in compromising the system, then $PEX_t > M^m EX_t - d (d' = 0)$ as the system has not yet blocked the malicious end-user and because of the cost of deviation.

3.3 Equilibrium Description

A MPE consists of a strategy profile and a state process (EX, π) such that the malicious end-user and the system are acting optimally, and π_t is consistent with Bayes' rule whenever possible (in addition to the requirement that strategies be Markovian). A unique MPE occurs when the two types of end-users display pooling behavior, i.e., malicious end-user imitates or mimics the honest end-user. This behavior is referred as the on-equilibrium path behavior. Alternatively, if the malicious end-user deviates from the honest signaling behavior, it is referred to as the separating or off-equilibrium path behavior. To characterize the MPE, we only consider the payoff relevant variables at any time, i.e., the current explanation value EX_t and the current belief held by the system about the end-user's type π_t . We impose the following restrictions on belief-updating after off-path events: After any decision by the end-user to stop signaling, the system believes that end-user is of type m if the explanation generated for that end-user has reached the maximum relevant explanation value (defined later). This includes the case where the end-user has earned a reputation of $\pi_t = 1$, but is actually type m and chooses to stop signaling. Moreover, if the end-user is ever believed to be malicious with certainty ($\pi_t = 0$), he

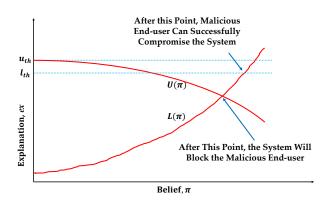


Figure 1: Illustration of a continuous path analysis of $U(\pi)$ and $L(\pi)$ in Markov Perfect Equilibrium

can never recover; reputation of that end-user remains at $\pi_t=0$.

Given this equilibrium concept, our main result is the characterization of signaling by the malicious end-user and detection (stopping) strategies by the system in a unique equilibrium. We assume that a decision to stop signaling (i.e., deviating from honest behavior) is the last action in the game taken by the (malicious) end-user. This decision allows the end-user to either achieve the target of compromising the system and then getting blocked by it or getting blocked without reaching this target at all. In either case, the system's belief about this end-user will jump to $\pi_t=0$. The end-user has no further action, and the game reduces to a straightforward stopping problem (Section 2) for the system. In that case, the continuation payoffs from that point on can be interpreted as the termination payoffs of the original signaling game.

Next, consider the state of the game before the end-user deviates/reveals and before the system's block action. Since malicious end-user plays a mixed strategy that occurs onequilibrium path, the system's belief about end-user evolves over time. Thus, a unique MPE consists of a state variable process (EX_t,π_t) and two cutoff functions, an increasing function $U(\pi_t)$ for the system and an increasing function $L(\pi_t)$ for the end-user, where

- The system immediately blocks the end-user if $ex_t \ge U(\pi_t)$, i.e., $\tau = \inf\{t >= 0 : ex_t \ge U(\pi_t)\}$. Here ex_t is the realization of EX_t at time t.
- The malicious end-user signals whenever $ex_t < L(\pi_t)$ and mixes between revealing and not revealing whenever $ex_t \ge L(\pi_t)$, so that the curve $\{(L(\pi), \pi) : \pi \in [0, 1]\}$ serves as a reflecting boundary for the process (ex_t, π_t) .

We call such a unique MPE equilibrium a (U, L) equilibrium. The first condition defines an upper boundary which tells the system that if an explanation value ex_t at time t (corresponding to a query by the end-user) is greater than or equal to this boundary $(U(\pi_t))$, then the end-user is trying to compromise the system. In this case, the system should block the end-user. The second condition above guides the behavior of the malicious end-user. Function $L(\pi_t)$ represents the upper-bound of the target explanation value the malicious end-user wants to achieve given a certain belief π_t at time t. When the explanation value corresponding to a query by an end-user is less than this boundary function, i.e., $ex_t < L(\pi_t)$, then it's strategically better for the malicious end-user to keep signaling (i.e., imitating the honest end-user). However, if $ex_t \geq L(\pi_t)$ then the malicious end-user has an incentive to deviate from the honest behavior. For the malicious end-user, this condition also represents that it is near the desired (or target) explanation value - one more step by the malicious end-user can either lead to success (compromise of the system) or failure (getting blocked by the system before achieving its goal).

For intuition on the structure of a MPE, suppose that the current belief is π_t . If the explanation generated by an explanation method is sufficiently large, i.e., close to maximum relevant explanation value, then system should block the end-user who is suspiciously trying to move in the direction of the boundary explanation value. This cutoff for explanation value is a non-increasing function of π because, by definition, the end-user is less likely to be honest when the explanation value is sufficiently close to the maximum relevant explanation value and π_t is large. Thus, when explanation value becomes greater than or equal to the optimal function $U(\pi_t)$ at any time t, system will block the end-user. This is intuitively shown in Figure 1, where u_{th} represents the explanation with the maximum information (or relevance) generated by the system and l_{th} represents explanation with the maximum information after system adds noise based on its belief, which can be given to the end-user. Next, we argue that end-user types must separate above some upper cutoff of $L(\pi_t)$. Suppose that initially both types pool and in particular, type m always signals (equilibrium condition). Then starting from the state $(ex_t, \pi_t) = (ex_0, \pi)$, as no information arrives about the end-user's type, the belief π_t holds constant at π . For sufficiently large ex_t , if the malicious end-user is very close to the desired target, then she/he would strictly prefer to deviate in order to compromise the system now, contradicting equilibrium. Thus, deviation must be in the support of type m's strategy above some cutoff $L(\pi_t)$.

4 Equilibrium Analysis

Next, we try to analyze conditions under which a unique MPE exists in the game described above, i.e., we try to construct a (U,L) equilibrium by finding conditions under which optimal functions $L(\pi_t)$ and $U(\pi_t)$ exist. First, we try to prove the continuity and differentiability of $L(\pi_t)$ and $U(\pi_t)$ in the belief domain $(\pi_t \in [0,1])$. We assume that $U(\pi_t)$ (for the system) is optimal, and show that it exists and is contin-

uous. However for $L(\pi_t)$, we first construct two bounding functions $L^+(\pi_t)$ (upper) and $L^-(\pi_t)$ (lower) and show that such functions exists. We then show that as π_t increases, $L^+(\pi_t)$ and and $L^-(\pi_t)$ will converge, and the first intersection point (root) of $L^+(\pi_t)$ and $L^-(\pi_t)$ is a unique MPE. For both players, we express boundary conditions (as outlined in Section 2) at the decision boundaries, i.e., Signaling, No signaling, and Detection and Block (outlined in table 1). At these decision boundaries, both players decide to either continue or stop. Therefore, the optimal choice (control u: in Section 2.4) for both the system and the end-user is a binary variable (0 to stop and 1 to continue), and we have written down value matching, smooth pasting and reversed explanation condition, i.e., for example, if end-user was expecting $L^+(\pi_t)$, but he/she received $L^-(\pi_t)$, at the possible decision boundaries and decided if a player wants to continue or stop for optimality.

4.1 Value Functions

As mentioned earlier, we are considering an equilibrium that occurs when both types of end-user pool, i.e., ex_t is strictly below $L(\pi_t)$. In this pooling scenario, no information becomes available to the system about the type of the end-user; thus, belief (π_t) remains constant. Hence, we write the value functions for both the players conditioned on no deviation by the end-user. As we consider an infinite horizon in our game, there is no known terminal value function (Section 2.4). Moreover, these value functions are independent w.r.t. $t \in R$, as t singularly has no effect on them. The end-user's value function (F) should solve the following equation representing his risk-less return (as described in Section 2.4):

$$rF(ex,\pi) = \mu exF_{ex}^{'}(ex,\pi) + \frac{1}{2}\sigma^{2}ex^{2}F_{ex}^{''}(ex,\pi) + \psi ex$$

where $F_{ex}^{'}$ and $F_{ex}^{''}$ are the first and second order partial derivative of the value function $F(ex,\pi)$ w.r.t. ex, respectively, and, μ and σ are the drift/mean and the variance/volatility of the explanation process EX_t , respectively. ψ is the payoff coefficient which depends on the stage payoffs of the end-user as represented in Table 1. The solution to the above equation can be represented as:

$$F(ex, \pi) = A_1(\pi)ex^{\beta_1} + A_2(\pi)ex^{\beta_2} + \frac{\psi ex}{r - \mu}$$

for some constant $A_1(\pi)$ and $A_2(\pi)$ and where $\beta_1>1$ and $\beta_2<0$ are the roots of the characteristic equation [Dixit and Pindyck, 2012]. Similarly, the system's value function $V(ex,\pi)$ should satisfy the following equation, conditioned on $ex< L(\pi)$ and π staying constant:

$$rV(ex, \pi) = \mu exV'_{ex}(ex, \pi) + \frac{1}{2}\sigma^{2}ex^{2}V''_{ex}(ex, \pi) + \psi ex$$

where $V_{ex}^{'}$ and $V_{ex}^{''}$ are the first and second order partial derivative of the value function $V(ex,\pi)$ w.r.t. ex, respectively. As before, ψ is the payoff coefficient which depends on the stage payoffs of the system as shown in Table 1. The solution to the above equation can be represented as:

$$V(ex, \pi) = B_1(\pi)ex^{\beta_1} + B_2(\pi)ex^{\beta_2} + \frac{\psi ex}{r - \mu}$$

for some constant $B_1(\pi)$ and $B_2(\pi)$. We will use different boundary conditions to determine $A_1(\pi)$, $A_2(\pi)$, $B_1(\pi)$ and $B_2(\pi)$. Then, we will use these conditions to determine $U(\pi_t)$ and $L(\pi_t)$.

4.2 Analytical Results

As mentioned in Section 3.3, we have assumed that the equilibrium occurs only when both types of end-user's pool. Moreover, malicious end-user will pool (or imitate) with the honest end-user with high probability (there is a chance, end-user may deviate earlier because he/she is playing a mixed strategy) only if the maximum relevance explanation threshold is not reached. Hence, our first aim is to compute the system's threshold u_{th} (Lemma 1) and the corresponding end-user's threshold l_{th} (Lemma 2). u_{th} represents the explanation with the maximum information (or relevance) generated by the system and l_{th} represents explanation with the maximum information after system adds noise based on its belief, which can be given to the end-user. These thresholds define the region in which an MPE may occur in the game, if the necessary conditions are satisfied. Therefore, these two lemmas will be used to determine if a unique MPE exists in the game or not.

Lemma 1. There exists a positive upper bound u_{th} on the explanations generated by an explanation method representing maximum relevant information.

Proof. If the explanation generated by the system reaches u_{th} , then system's belief about the type of end-user becomes zero, i.e., $\pi_t=0$, as the end-user's type is revealed to be malicious. Let's assume that the system is perfectly able to block the end-user if he/she tries to compromise the system. Hence, using the system's value function, we get

$$V(u_{th}) = \frac{Ku_{th}}{r - \mu} - c_d \tag{1}$$

$$V'(u_{th}) = \frac{K}{r - \mu} \tag{2}$$

Equation (1) is the value matching condition, which tells that a system will gain ku_{th} while incurring a cost of $-c_d$ (detection cost) if it's able to block the end-user before he/she attacks. Equation (1) is a smooth pasting condition. In equation (1) and (2), we replace $V(u_{th})$ and it's derivative $V'(u_{th})$ with the general solution of the value function of the system and it's derivative, respectively. As mentioned earlier, at $\pi=0$ the explanation value given by the system to the end-user becomes zero. As a result, since root $\beta_2<0$, second term in the value function can go to $-\infty$. To eliminate this case we make the constant $B_2=0$ and we only calculate the constant B_1 . Therefore, equations (1) and (2) are resolved to.

$$B_1 u_{th}^{\beta_1} + \frac{r_e u_{th}}{r - \mu} = \frac{k u_{th}}{r - \mu} - c_d$$

$$B_1 \beta_1 u_{th}^{\beta_1 - 1} + \frac{r_e}{r - \mu} = \frac{k}{r - \mu}$$

We first compute the constant B_1 , then find the threshold u_{th} . By solving the above two equations using the elimination and substitution method, we get

$$u_{th} = \frac{\beta_1 c_d(r-\mu)}{(\beta_1 - 1)(k - r_e)} > 0$$

As outlined in the payoff assumption, $k>r_e$, which implies that $u_{th}>0$.

Lemma 2. There exists a positive upper bound l_{th} on the explanations given by the system to the end-user representing maximum relevant information required by the end-user to compromise the system.

Proof. It is assumed that if an opportunity arrives for the end-user to compromise the system, then he/she will take one more step after signaling. This step can either make end-user to accomplish his/her's goal of compromising the system or get's detected and blocked by it. If end-user gets blocked by the system, the belief will jump to $\pi=0$. Thus, to eliminate the possibility of the second term, $A_2 l_{th}^{\beta_2} \to -\infty$ (as $\beta_2 < 0$) in the general solution of the end-user's value function, we assume $A_2=0$.

Let's assume that end-user is successful in compromising the system, then the payoff gained by the end-user will be greater than the signaling stage payoff as there will be some monetary benefit to the end-user because of the privacy leak. Using the value matching and the smooth pasting conditions for end-user between signaling and not-signaling stages and accordingly assuming the payoffs received by the end-user, we get

$$A_1 l_{th}^{\beta_1} + \frac{P l_{th}}{r - \mu} < \frac{M_{NS}^m l_{th}}{r - \mu} - d \tag{3}$$

$$A_1 \beta_1 l_{th}^{\beta_1 - 1} + \frac{P}{r - \mu} < \frac{M_{NS}^m}{r - \mu} \tag{4}$$

Solving the above two equations we get,

$$l_{th} > \frac{\beta_1 d(r-\mu)}{(\beta_1 - 1)(M_{NS}^m - P)} > 0$$

 $M_{NS}^m=M^m+d^{'},$ if end-user is successful in compromising the system. Thus, $M_{NS}^m>P,$ which implies $l_{th}>0.$

Our next aim is to characterize the optimal cutoff functions: system's $U(\pi_t)$ and end-user's $L(\pi_t)$ as these two functions represent the MPE in the game. These cut-off functions help the system and the end-user to play optimally in each stage of the game. For example, if the system doesn't have any knowledge of $U(\pi_t)$, then it won't know the range of the explanation values being generated and thus, given to the end-user. Hence, an adversary will be able to easily compromise the system. In contrast, $L(\pi_t)$ function knowledge will guide an adversary on how to optimally compromise the system. For that reason, we first prove that $U(\pi_t)$ exists and is non-increasing and continuously differentiable

(Theorem 1). To prove $L(\pi_t)$ (Theorem 2) exists and is increasing and continuously differentiable, we first charcterize an explanation path, $L^+(\pi_t)$ (Lemmas 3), which represents the maximum relevant explanation given by the system and an explanation path, $L^-(\pi_t)$ (Lemma 4), which represents the minimum relevant explanation given by the system to the end-user. Then, we demonstrate that both these functions are increasing and continuously differentiable. The purpose for doing this is to use these lemmas to show that as $\pi_t \to 1$, both $L^+(\pi_t)$ and $L^-(\pi_t)$ starts to converge and becomes equal to $L(\pi_t)$ after some point. Hence, our final aim is to show that such a point exists and thus, a unique MPE (Theorem 3) exists in the game.

Theorem 1. $U(\pi_t)$ is non-increasing and continuously differentiable function in domain [0,1] if and only if either $\beta_2\beta_1J^{\cdot}(\pi,t)^{\beta_2-1} \leq \beta_1\beta_2J^{\cdot}(\pi,t)^{\beta_1-1}$ or $\beta_2(\beta_1-1)J^{\cdot}(\pi,t)^{\beta_2-1} \leq \beta_1(\beta_2-1)J^{\cdot}(\pi,t)^{\beta_1-1}$, where $J(\pi,t) = \frac{L(\pi)}{U(\pi)}$.

Proof. We are assuming optimal cutoff function, $U(\pi_t)$, of the system is optimal as $U(\pi_t)$ is computed using the belief it has about the type of the end-user. Using three boundary conditions:- value matching, smooth pasting ,and instead of $U(\pi_t)$, the system computes $L(\pi_t)$ given belief π_t , we calculate the optimal $U(\pi_t)$. These conditions are assumed at the boundary when the end-user has deviated, and the system has successfully detected the attack. In consequence, account of that end-user is suspended. When system computes $L(\pi_t)$, which is below $U(\pi_t)$, it will get a payoff of $r_eL(\pi_t)$. The reason being, since system computes $L(\pi_t)$, end-user has no incentive to deviate, thus will keep imitating the honest user. As a result, system will get a signaling stage payoff. The three boundary conditions are:

$$V(U(\pi_t), z) = \frac{kU(\pi_t)}{r - \mu} - c_d$$
$$V'_U(U(\pi_t), \pi) = \frac{k}{r - \mu}$$
$$V(L(\pi_t), \pi) = \frac{r_e L(\pi_t)}{r - \mu}$$

By replacing the value of $V(U(\pi_t),\pi), \ V(L(\pi_t),\pi)$ and $V_U^{'}(U(\pi_t),\pi)$ with the general solution of the value function of the system and it's derivative, we get the following equations:-

$$B_1(\pi)U(\pi_t)^{\beta_1} + B_2(z)U(\pi_t)^{\beta_2} + \frac{r_eU(\pi_t)}{r - \mu} = \frac{kU(\pi_t)}{r - \mu} - c_d$$
 (5)

$$B_1(\pi)\beta U(\pi_t)^{\beta_1-1} + B_2(\pi)\beta U(\pi_t)^{\beta_2-1} + \frac{r_e}{r-\mu} = \frac{k}{r-\mu} \eqno(6)$$

$$B_1(z)L(\pi_t)^{\beta_1} + B_2(\pi)L(\pi_t)^{\beta_2} + \frac{r_eL(\pi_t)}{r - \mu} = \frac{r_eL(\pi_t)}{r - \mu}$$
(7)

For simplicity, we will be replacing $L(\pi_t)$ with $L(\pi)$ and $U(\pi_t)$ with $U(\pi)$. Solving equations (5) and (6) using substitution and elimination, we get the coefficients B_1 and B_2 .

$$B_1 = \frac{(\beta_2 - 1)(k - r_e)}{(r - \mu)(\beta_2 - \beta_1)U(\pi)^{\beta_1 - 1}} - \frac{c_d \beta_2}{(\beta_2 - \beta_1)U(\pi)^{\beta_1}}$$

$$B_2 = \frac{(\beta_1 - 1)(k - r_e)}{(r - \mu)(\beta_1 - \beta_2)U(\pi)^{\beta_2 - 1}} - \frac{c_d\beta_1}{(\beta_1 - \beta_2)U(\pi)^{\beta_2}}$$

Now, in order to compute the optimal $U(\pi)$, we substitute the values of B_1 and B_2 computed above in equation (7):

$$(\frac{(\beta_2 - 1)(k - r_e)}{(r - \mu)(\beta_2 - \beta_1)U(\pi)^{\beta_1 - 1}} - \frac{c_d \beta_2}{(\beta_2 - \beta_1)U(\pi)^{\beta_1}})L(\pi)^{\beta_1}$$

$$(\frac{(\beta_1 - 1)(k - r_e)}{(r - \mu)(\beta_1 - \beta_2)U(\pi)^{\beta_2 - 1}} - \frac{c_d \beta_1}{(\beta_1 - \beta_2)U(\pi)^{\beta_2}})L(\pi)^{\beta_2} = 0$$

By simplifying the above equation, we get

$$\begin{split} \frac{(k-r_e)U(\pi)}{(r-\mu)(\beta_2-\beta_1)} [(\beta_2-1)J(\pi,t)^{\beta_1} - (\beta_1-1)J(\pi,t)^{\beta_2}] \\ - \frac{c_d}{\beta_2-\beta_1} [\beta_2J(\pi,t)^{\beta_1} - \beta_1J(\pi,t)^{\beta_2}] = 0 \end{split}$$

$$U(\pi) = \frac{\left[\beta_2 J(\pi, t)^{\beta_1} - \beta_1 J(\pi, t)^{\beta_2}\right] (r - \mu)}{\left[(\beta_2 - 1)J(\pi, t)^{\beta_1} - (\beta_1 - 1)J(\pi, t)^{\beta_2}\right] (k - r_e)} \tag{8}$$

Equation (8) allows us to define a "best response" threshold curve $U(\pi)$ for the system. We use $J(\pi,t)$ to denote the value of $\frac{L(\pi)}{U(\pi)}$. Now, we will calculate the derivative $\frac{\partial U(\pi)}{\partial \pi}$ to check whether $U(\pi)$ is non-increasing or not. The derivative of $U(\pi)$ depends on it's numerator (represented as $n(U(\pi))$) and denominator (represented as $d(U(\pi))$) and are given as:

$$n(U(\pi)) = \left[\beta_2 J(\pi, t)^{\beta_1} - \beta_1 J(\pi, t)^{\beta_2}\right] (r - \mu)$$

$$d(U(\pi)) = \left[(\beta_2 - 1)J(\pi, t)^{\beta_1} - (\beta_1 - 1)J(\pi, t)^{\beta_2} \right] (k - r_e)$$

The derivatives of $n(U(\pi))$ and $d(U(\pi))$ are given as:

$$n_{U}(U(\pi)) = [\beta_{1}\beta_{2}J^{'}(\pi,t)^{\beta_{1}-1} - \beta_{2}\beta_{1}J^{'}(\pi,t)^{\beta_{2}-1}] \times (r-\mu)$$
 (9)

$$d_{U}(U(\pi)) = [\beta_{1}(\beta_{2}-1)\dot{J}^{`}(\pi,t)^{\beta_{1}-1} - \beta_{2}(\beta_{1}-1)\dot{J}^{`}(\pi,t)^{\beta_{2}-1}] \times (k-r_{e}) \tag{10}$$

Thus, $U(\pi)$ will be non-increasing, if either $n_U(U(\pi)) < 0$ or $d_U(U(\pi)) < 0$.

Checking when $n_U(U(\pi)) > 0$: As we know $\beta_1 > 1$, $\beta_2 < 0$ and $r - \mu > 0$. Thus, $n_U(U(\pi)) > 0$ if and only if $\beta_2\beta_1J^{\cdot}(\pi,t)^{\beta_2-1} > \beta_1\beta_2J^{\cdot}(\pi,t)^{\beta_1-1}$ and $n_U(U(\pi)) < 0$ otherwise.

Checking when $d_U(U(\pi)>0$:) We know $k-r_e>0$. Thus, $d_U(U(\pi))>0$ if and only if $\beta_2(\beta_1-1)J^{`}(\pi,t)^{\beta_2-1}>\beta_1(\beta_2-1)J^{`}(\pi,t)^{\beta_1-1}$ and $n_U(U(\pi))<0$ otherwise.

So, $U(\pi)$ will be a non-increasing function if and only if the above conditions are satisfied pertaining to either $n_U(U(\pi)) < 0$ or $d_U(U(\pi)) < 0$.

Now, we will define the conditions under which optimal explanation path $L(\pi)$ (given optimal $U(\pi)$) exists for the end-user. As mentioned earlier, we are trying to find an MPE in a pooling situation, which will occur when end-user's explanation lies in the range, $(0, l_{th})$. l_{th} is the explanation threshold, which an end-user is trying to achieve to accomplish his/her aim of compromising the system. To find optimal $L(\pi)$, we define two increasing functions, $L^+(\pi)$ and $L^-(\pi)$. $L^+(\pi)$ is the upper threshold function, which states the explanation with maximum relevance given by the system to the end-user with the least amount of noise added by it. $L^{-}(\pi)$ is the lower threshold function, which states the explanation with minimum relevance given by the system to the end-user with the maximum amount of noise added by it. First, we will prove that these two functions exist and then show these two converge by finding out the first intersection point of these two curves as the root of a polynomial (defined below) and finally show that a unique root exists of that polynomial. We will write 3 equations each for $L^+(\pi)$ and $L^-(z)$ according to the value matching, smooth pasting, and the condition in which the explanation received is opposite of what end-user expected. That is, if instead of $L^+(\pi)$, an end-user gets $L^-(\pi)$ or if instead of $L^-(\pi)$, an end-user gets $L^+(\pi)$). Three conditions for $L^+(\pi)$ are: -

$$F(L^+(\pi), \pi) = \lambda(L^+(\pi), \pi)$$

Replacing $F(L^+(\pi), \pi)$ with it's solution, we get

$$A_1^+(\pi)L^+(\pi)^{\beta_1} + A_2^+(\pi)L^+(\pi)^{\beta_2}$$

$$+\frac{PL^{+}(\pi)}{r-\mu} = \lambda(L^{+}(\pi), \pi)$$
 (11)

$$F'_{L^{+}}(L^{+}(\pi), \pi) = \lambda'(L^{+}(\pi), \pi)$$

Replacing $F_{L^+}^{'}(L^+(\pi),\pi)$ with it's solution, we get

$$A_1^+(\pi)\beta_1 L^+(\pi)^{\beta_1-1} + A_2^+(\pi)\beta_2 L^+(\pi)^{\beta_2-1}$$

$$+\frac{P}{r-\mu} = \lambda'(L^{+}(\pi),\pi)$$
 (12)

$$F(L^-(\pi),\pi) = \frac{P \cdot L^-(\pi)}{r - \mu}$$

Replacing $F(L^-(\pi), \pi)$ with it's solution, we get

$$A_1^+(\pi)L^-(\pi)^{\beta_1} + A_2^+(\pi)L^-(\pi)^{\beta_2} = 0$$
 (13)

The above conditions are at the boundary between signaling and not-signaling. Equation (11) states that whether successful in compromising the system or not, end-user will get $\lambda(L^+(\pi),\pi)$ given the state of the game (ex,π) . Equation (12) is the smooth pasting condition and equation (13) is when instead of getting $L^+(\pi)$, end-user gets $L^-(\pi)$. Thus, will receive $(P\cdot L^-(\pi))$ from the system.

Three conditions for $L^-(\pi)$ are: -

$$F(L^{-}(\pi),\pi) = \frac{PL^{-}(\pi)}{r-\mu}$$

Replacing $F(L^-(\pi), \pi)$ with it's solution, we get

$$A_1^-(\pi)L^-(\pi)^{\beta_1} + A_2^-(\pi)L^-(\pi)^{\beta_2} = 0 \tag{14}$$

$$F'_{L^{-}}(L^{-}(\pi),\pi)=0$$

Replacing $F'_{L^{-}}(L^{-}(\pi),\pi)$ with it's solution, we get

$$A_1^-(\pi)\beta_1 L^-(\pi)^{\beta_1 - 1} + A_2^-(\pi)\beta_2 L^-(\pi)^{\beta_2 - 1} = 0$$
 (15)

$$F(L^+(\pi), \pi) = \lambda(L^+(\pi), \pi)$$

Replacing $F(L^+(\pi), \pi)$ with it's solution, we get

$$A_1^{-}(\pi)L^{+}(\pi)^{\beta_1} + A_2^{-}(\pi)L^{+}(\pi)^{\beta_2} + \frac{PL^{+}(\pi)}{r - \mu} = \lambda(L^{+}(\pi), \pi)$$
(16)

Equation (14) is the value matching condition which states that if the end-user takes one more step, and gets at least the payoff one was getting in the previous stage, then it will stop playing the game. The reason is that $L^-(\pi)$ is a value with the highest amount of noise being added to it, hence end-user is already not getting anything. Thus, will want to stop attacking the system. Equation (15) is a smooth pasting condition and equation (16) is when end-user gets $L^+(\pi)$ instead of $L^-(\pi)$. Thus, there is a chance that end-user can succeed in compromising the system. Hence, getting $\lambda(L^+(\pi),\pi)$. In lemma's 3 and 4, we have shown the continuity and differentiability of $L^+(\pi)$ and $L^-(\pi)$ respectively. The result from these two lemmas will be used in proving the existence and uniqueness of MPE in the game.

Lemma 3. $L^+(\pi)$ is a well-defined, increasing, continuous and differentiable function in domain [0,1] if and only if $\lambda'(L^+(\pi),\pi)>0$ and P>0, where $\lambda()$ is the termination payoff if the end-user decides to deviate and attack the system.

Proof. To compute optimal $L^+(\pi)$, we assume π is set to 1. The reason being, since it's an explanation with maximum relevance, the <code>system</code>'s belief about the type of <code>end-user</code> is set to 1. Given belief $\pi=1$, we solve equation (14) and (15) by substitution and elimination approach to get constants $A_1^-(\pi)$ and $A_2^-(\pi)$.

We get
$$A_1^-(\pi)=0$$
 and $A_2^-(\pi)=0$.

Next, using boundary condition (16) and substituting the values of the constants $A_1^-(\pi)$ and $A_1^-(\pi)$ in this equation, we get an implicit expression for $L^+(\pi)$,

$$\frac{PL^{+}(\pi)}{r-\mu} = \lambda(L^{+}(\pi), \pi)$$

By simplifying the above equation, we get

$$L^{+}(\pi) = \frac{\lambda(L^{+}(\pi), \pi)(r - \mu)}{P}$$
 (17)

Taking the derivative of equation (17) we get,

$$\frac{\partial L^{+}(\pi)}{\partial z} = \frac{\lambda'(L^{+}(\pi), \pi)(r - \mu)}{P}$$
 (18)

Since $\lambda'(L^+(\pi),\pi)>0$, $L^+(\pi)$ is a well-defined, increasing, continuous and differentiable function. Hence proved.

Lemma 4. $L^-(\pi)$) is a well-defined, increasing, continuous and differentiable function domain [0,1] if and only if either $(\frac{\partial A_1^+(z)}{\partial \pi}L^-(\pi)^{\beta_1}+\frac{\partial A_2^+(\pi)}{\partial \pi}L^-(\pi)^{\beta_2})<0$ or $(A_1^+(\pi)\beta_1L^-(\pi)^{\beta_1-1}+A_2^+(\pi)\beta_2L^-(\pi)^{\beta_2-1})<0$.

Proof. Given belief π , we solve equations (11) and (12) by substitution and elimination approach to get $A_1^+(\pi)$ and $A_2^+(\pi)$,

$$A_{1}^{+}(\pi) = \frac{\beta_{2}\lambda(L^{+}(\pi),\pi) - \lambda'(L^{+}(\pi),\pi)L^{+}(\pi)}{(\beta_{2} - \beta_{1})L^{+}(\pi)^{\beta_{1}}} + \frac{P(1 - \beta_{2})}{(r - \mu)(\beta_{2} - \beta_{1})L^{+}(\pi)^{\beta_{1} - 1}}$$
(19)
$$A_{2}^{+}(\pi) = \frac{\beta_{1}\lambda(L^{+}(\pi),\pi) - \lambda'(L^{+}(\pi),\pi)L^{+}(\pi)}{(\beta_{1} - \beta_{2})L^{+}(\pi)^{\beta_{2}}} + \frac{P(1 - \beta_{1})}{(r - \mu)(\beta_{1} - \beta_{2})L^{+}(\pi)^{\beta_{2} - 1}}$$
(20)

Now, we will calculate $\frac{\partial A_1^+(\pi)}{\partial z}$ and $\frac{\partial A_2^+(\pi)}{\partial z}$ to check for the continuity $A_1^+(\pi)$ and $A_2^+(\pi)$ which we will use in checking the continuity of $L^-(\pi)$.

$$\frac{\partial A_{1}^{+}(\pi)}{\partial \pi} = \frac{1}{\beta_{2} - \beta_{1}} \times \frac{\partial L^{+}(\pi)}{\partial z} \left(\frac{\beta_{2} \lambda'(L^{+}(\pi), \pi)}{L^{+}(\pi)^{\beta_{1}}} \right) - \frac{\lambda''(L^{+}(\pi), \pi)}{L^{+}(\pi)^{\beta_{1} - 1}} + \frac{\partial L^{+}(\pi)}{\partial \pi} \times \frac{1}{L^{+}(\pi)} (A_{1}^{+}(\pi)(1 - \beta_{1})) - \frac{\beta_{2} \lambda(L^{+}(\pi), \pi)}{(\beta_{2} - \beta_{1})L^{+}(\pi)^{\beta_{1}}} \right) \tag{21}$$

$$\frac{\partial A_{2}^{+}(\pi)}{\partial z} = \frac{1}{\beta_{1} - \beta_{2}} \times \frac{\partial L^{+}(\pi)}{\partial z} \left(\frac{\beta_{1} \lambda'(L^{+}(\pi), \pi)}{L^{+}(\pi)^{\beta_{2}}} \right) - \frac{\lambda''(L^{+}(\pi), \pi)}{L^{+}(\pi)^{\beta_{2} - 1}} + \frac{\partial L^{+}(\pi)}{\partial \pi} \times \frac{1}{L^{+}(\pi)} (A_{1}^{+}(\pi)(1 - \beta_{2}) - \frac{\beta_{1} \lambda(L^{+}(\pi), \pi)}{(\beta_{1} - \beta_{2})L^{+}(\pi)^{\beta_{2}}} \right) \tag{22}$$

Using boundary condition (13) and taking the derivative of it, gives an implicit expression for $\frac{\partial L^-(\pi)}{\partial \pi}$,

$$\frac{\partial A_1^+(\pi)}{\partial z} L^-(\pi)^{\beta_1} + \frac{\partial A_2^+(\pi)}{\partial z} L^-(\pi)^{\beta_2}$$

$$+A_{1}^{+}(\pi)\beta_{1}L^{-}(\pi)^{\beta_{1}-1}\frac{\partial L^{-}(\pi)}{\partial z}$$

$$+A_{2}^{+}(\pi)\beta_{2}L^{-}(\pi)^{\beta_{2}-1}\frac{\partial L^{-}(\pi)}{\partial \pi}$$
(23)

By simplifying the above equation, we get

$$\frac{\partial L^{-}(\pi)}{\partial \pi} = -\frac{\frac{\partial A_{1}^{+}(\pi)}{\partial \pi} L^{-}(\pi)^{\beta_{1}} + \frac{\partial A_{2}^{+}(\pi)}{\partial \pi} L^{-}(\pi)^{\beta_{2}}}{A_{1}^{+}(\pi)\beta_{1}L^{-}(\pi)^{\beta_{1}-1} + A_{2}^{+}(\pi)\beta_{2}L^{-}(\pi)^{\beta_{2}-1}} \tag{24}$$

For equation (24) to be greater than zero, either the numerator (represented as $n(L^-(\pi))$) or the denominator (represented as $d(L^-(\pi))$) has to be negative.

$$n(L^{-}(\pi)) = \frac{\partial A_{1}^{+}(z)}{\partial \pi} L^{-}(\pi)^{\beta_{1}} + \frac{\partial A_{2}^{+}(\pi)}{\partial \pi} L^{-}(\pi)^{\beta_{2}}$$
(25)

$$d(L^{-}(\pi)) = A_1^{+}(\pi)\beta_1 L^{-}(\pi)^{\beta_1 - 1} + A_2^{+}(\pi)\beta_2 L^{-}(\pi)^{\beta_2 - 1}$$
(26)

Thus, $\frac{\partial L^-(\pi)}{\partial \pi}>0$ or $L^-(\pi)>0$ if and only if either $n(L^-(\pi))<0$ or $d(L^-(\pi))<0$.

Theorem 2. $L(\pi)$ is a well-defined, increasing, continuous and differentiable function domain [0,1] if and only if either $\lambda'(L(\pi),\pi) > 0$ and P > 0.

Proof. Computation of $L(\pi)$ follows the same approach as the computation of $L^+(\pi)$, except the belief is not set to 1, i.e., $\pi \neq 1$. The reason being it's an optimal functional path for the end-user computed at each belief (π) of the system about the type of the end-user. So, $L(\pi)$ is formulated as:

$$L(\pi) = \frac{\lambda(L(\pi), \pi)(r - \mu)}{P}$$

Taking the derivative of the above equation we get,

$$\frac{\partial L(\pi)}{\partial z} = \frac{\lambda'(L(\pi), \pi)(r - \mu)}{P}$$

Since $\lambda^{'}(L(\pi),\pi)>0$ and P>0, $L(\pi)$ is a well-defined, increasing, continuous and differentiable function. Hence proved.

5 Characterization and Uniqueness:

In the previous section we defined the upper bound, $L^+(\pi)$ and the lower bound, $L^-(\pi)$ of the best response function of the end-user. In this section, we will show there exist a best response function, $L(\pi)$, and it is a point where $L^+(\pi)$ and $L^-(\pi)$ starts to converge.

Theorem 3. A unique MPE or a point, $\varsigma = \frac{\lambda(L^+(\pi),\pi)\times(r-\mu)}{P\times L^-(\pi)}$, exists in the game where the two curves $L^+(\pi)$ and $L^-(\pi)$ starts to converge, if and only if $\frac{\beta_2}{\varsigma^{\beta_2+1}}\left[L^+(\lambda^{'}-\frac{P}{r-\mu})-\beta_1(\lambda-\frac{PL^+}{r-\mu})\right]$ $\geq \frac{\beta_1}{\varsigma^{\beta_1+1}}\left[\beta_2(\lambda-\frac{PL^+}{r-\mu})-L^+(\lambda^{'}-\frac{P}{r-\mu})\right]$.

Proof. In this proof, first, we will convert the system of equations ((11), (12), (13), (14), (15), (16)) into a system of two equations (shown below). Then, we will try to prove that a unique root exists for this pair of equations. Hence, any solution to this pair of equations will also be a root of the system of equations ((11), (12), (13), (14), (15), (16)). If this root exists and is unique, we say a unique MPE exists in the game. The pair of equations are derived as follows:

First, we replace the constants in equations (13) and (16), then after rearranging the terms, we get the following two equations:

$$\left[\frac{\beta_2 \lambda(L^+(\pi), \pi) - \lambda^{'}(L^+(\pi), \pi)L^+(\pi)}{(\beta_2 - \beta_1)} + \frac{P(1 - \beta_2)L^+(\pi)}{(r - \mu)(\beta_2 - \beta_1)}\right] \times (\frac{1}{\varsigma})^{\beta_1}$$

$$+ \left[\frac{\beta_1 \lambda(L^+(\pi), \pi) - \lambda'(L^+(\pi), \pi)L^+(\pi)}{(\beta_1 - \beta_2)} + \frac{P(1 - \beta_1)L^+(\pi)}{(r - \mu)(\beta_1 - \beta_2)} \right] \times (\frac{1}{\varsigma})^{\beta_2}$$
(27)

$$\frac{P\varsigma}{r-\mu} - \frac{\lambda(L^+(\pi),\pi)}{L^-(\pi)} = 0 \tag{28}$$

In the above equations, we have introduced a term ς , which is equal to $\frac{L^+(\pi)}{L^-(\pi)}$ and is a new dependent variable. Thus, any possible root of the above pair of equations will be a value of $\frac{L^+(\pi)}{L^-(\pi)}$ or of ς .

Proof of the existence of ς : We will be using concepts from the calculus to prove that a root exists for equations (27) and (28).

From equation (28), we get one of the value of ς ,

$$\varsigma = \frac{\lambda(L^{+}(\pi), \pi) \times (r - \mu)}{P \times L^{-}(\pi)}$$
 (29)

To show that this is the root for the system of equations (27) and (28), we take the curve represented by equation (27) and denote it by Λ . Then, we will show that only one real root of Λ exists in the range $ex \in [0, l_{th}]$, which is the solution to the six boundary conditions.

$$\Lambda(\varsigma) = \left[\frac{\beta_2 \lambda(L^+(z), z) - \lambda'(L^+(z), z) L^+(z)}{(\beta_2 - \beta_1)} + \frac{P(1 - \beta_2) L^+(z)}{(r - \mu)(\beta_2 - \beta_1)} \right] \\
+ \left[\frac{\lambda(L^+(z), z) - \lambda'(L^+(z), z) L^+(z)}{(\beta_1 - \beta_2)} + \frac{P(1 - \beta_1) L^+(z)}{(r - \mu)(\beta_1 - \beta_2)} \right] \\
\times \left(\frac{1}{\varsigma} \right)^{\beta_2} \\
(30)$$

The minimum explanation which can be given by the system to the end-user is zero which is at $\pi=0$, thus at this belief, both $L^-(\pi)\to 0$ and $L^+(\pi)\to 0$. As a result of which, $\varsigma=\frac{0}{0}$. To remove this case, we will assume that $L^-(\pi)$ is close to zero, not exactly 0. For the sake of simplicity, let's denote the coefficient of $(\frac{1}{\varsigma})^{\beta_1}$ as A and the coefficient of $(\frac{1}{\varsigma})^{\beta_2}$ as B in equation (30). Now, we will check if there exists a root in $ex\in[0,l_{th}]$ or not.

- Case 1: As $ex \to 0$, then $L^+(\pi) \to 0$ and $\varsigma \to 0$. As we know $\beta_1 > 1$ and $\beta_2 < 0$, thus $(\frac{1}{\varsigma})^{\beta_2} \to 0$ and $(\frac{1}{\varsigma})^{\beta_1} \to \infty$. Since we are considering the case when $\pi \to 0$, hence the payoff an end-user gets from the system will be approximately equal to zero. Thus, in the first part of A, $\lambda(L^+(\pi),\pi) \to 0$ and hence $\lambda'(L^+(\pi) \to 0$. Therefore, $\frac{\beta_2\lambda(L^+(\pi),\pi)-\lambda'(L^+(\pi),\pi)L^+(\pi)}{(\beta_2-\beta_1)} = 0$. For the second term of A i.e. $+\frac{P(1-\beta_2)L^+(\pi)}{(r-\mu)(\beta_2-\beta_1)}$, the numerator > 0 and denominator < 0. As a result, in this case, $\Lambda(\varsigma) \to -\infty$.
- Case 2: As $ex \to u_{th}$, then $L^+(\pi) \to l_{th}$ and $\varsigma \to 1$. If you put these values in equation (24), we will see that $\Lambda(\varsigma) > 0$.

Hence, $\Lambda(\varsigma)$ has at-least one root which lies in range $ex \in [0, u_{th}]$.

Proof of uniqueness: Now, we will show that only one root exists for the pair of equations defined before by proving that $\partial \Lambda(\varsigma)$ or $\Lambda_{L^+}(\varsigma)$ is increasing in the interval $L^+(\pi) \in [0, l_{th}]$ (we could have also taken $L^-(\pi)$). Using chain of rule: $-\Lambda_{L^+}(\varsigma) = \Lambda_{L^+}(\varsigma) + \Lambda_{\varsigma}(\varsigma) \times \varsigma'$. Hence, $\Lambda(\varsigma)$ will be increasing in the interval or will have a unique root in $[0, l_{th}]$ if $\Lambda_{L^+}(\varsigma) > 0$ and $\Lambda_{\varsigma}(\varsigma) \times \varsigma' > 0$.

Claim i: For all ex, π , we have $\Lambda_{L^+}(\varsigma) > 0$. We have,

$$\Lambda(\varsigma) = \left[\frac{\beta_2 \lambda(L^+(\pi), \pi) - \lambda'(L^+(\pi), \pi)L^+(\pi)}{(\beta_2 - \beta_1)} + \frac{P(1 - \beta_2)L^+(\pi)}{(r - \mu)(\beta_2 - \beta_1)} \right] \times \left(\frac{1}{\varsigma}\right)^{\beta_1} + \left[\frac{\beta_1 \lambda(L^+(\pi), \pi) - \lambda'(L^+(\pi), \pi)L^+(\pi)}{(\beta_1 - \beta_2)} + \frac{P(1 - \beta_1)L^+(\pi)}{(r - \mu)(\beta_1 - \beta_2)} \right] \times \left(\frac{1}{\varsigma}\right)^{\beta_2}$$
(31)

Rearranging the terms will give us:

$$\Lambda(\varsigma) = \frac{1}{\beta_1 - \beta_2} \\
\left[\beta_1 \lambda(L^+(\pi), \pi) - \lambda'(L^+(\pi), \pi) L^+(\pi) + \frac{P(1 - \beta_1) L^+(\pi)}{(r - \mu)} \right] \\
\times \frac{1}{\varsigma^{\beta_2}} - \frac{1}{\beta_1 - \beta_2} \tag{32}$$

$$\left[\beta_2 \lambda(L^+(\pi), \pi) - \lambda'(L^+(\pi), \pi) L^+(\pi) + \frac{P(1 - \beta_2) L^+(\pi)}{(r - \mu)} \right] \\
\times \frac{1}{\varsigma^{\beta_1}}$$

Now taking derivative of equation (32) with respect to $L^+(\pi)$. Taking $L^+(\pi) = L^+$ and $\lambda(L^+(\pi),\pi) = \lambda$ for simplicity.

$$\begin{split} & \Lambda_{L^{+}}(\varsigma) = \frac{1}{\beta_{1} - \beta_{2}} \left[\beta_{1} \lambda^{'} - \lambda^{''} L^{+} - \lambda^{'} + \frac{P(1 - \beta_{1}}{r - \mu} \right] \times \frac{1}{\varsigma^{\beta_{2}}} \\ & \frac{1}{\beta_{1} - \beta_{2}} \left[-\beta_{2} \lambda^{'} + \lambda^{''} L^{+} + \lambda^{'} - \frac{P(1 - \beta_{2}}{r - \mu} \right] \times \frac{1}{\varsigma^{\beta_{1}}} \end{split}$$

By rearranging the terms, we get

$$\Lambda_{L+}(\varsigma) = \frac{1}{\beta_1 - \beta_2} \left[(\beta_1 - 1)(\lambda' - \frac{P}{r - \mu}) - \lambda'' L^+ \right] \times \frac{1}{\varsigma^{\beta_2}}$$

$$\frac{1}{\beta_1 - \beta_2} \left[\lambda'' L^+ + (1 - \beta_2)(\lambda' - \frac{P}{r - \mu}) \right] \times \frac{1}{\varsigma^{\beta_1}}$$
(33)

As one can notice, $\lambda^{'} - \frac{sig}{r-\mu}$ is part of equation (11) or (16), hence positive. Thus, $\Lambda_{L^{+}}(\varsigma) > 0$.

Claim ii: For all ex, π , we have $\Lambda_{\varsigma}(\varsigma) \times \varsigma' > 0$. We know that $\varsigma = L^+(z)/L^-(z)$, thus $\varsigma_{L^+} > 0$.

$$\Lambda_{\varsigma}(\varsigma) = \frac{-\beta_{2}}{\beta_{1} - \beta_{2}} \left[\beta_{1} \lambda - \lambda' L^{+} + \frac{P(1 - \beta_{1})L^{+}}{(r - \mu)} \right] \times \frac{1}{\varsigma^{\beta_{2} + 1}} - \frac{\beta_{1}}{\beta_{1} - \beta_{2}} \left[\beta_{2} \lambda - \lambda' L^{+} + \frac{P(1 - \beta_{2})L^{+}}{(r - \mu)} \right] \times \frac{1}{\varsigma^{\beta_{1} + 1}}$$
(34)

Rearranging equation (34) gives us,

$$\Lambda_{\varsigma}(\varsigma) = \frac{-\beta_2}{\beta_1 - \beta_2} \left[\beta_1 \left(\lambda - \frac{PL^+}{r - \mu}\right) - L^+ \left(\lambda' - \frac{sig}{r - \mu}\right) \right] \times \frac{1}{\varsigma^{\beta_2 + 1}}$$
$$-\frac{\beta_1}{\beta_1 - \beta_2} \left[L^+ \left(\lambda' - \frac{sig}{r - \mu}\right) + \beta_2 \left(\frac{PL^+}{r - \mu} - \lambda\right) \right] \times \frac{1}{\varsigma^{\beta_1 + 1}}$$
(35)

For $\Lambda_{\varsigma}(\varsigma) > 0$, we must have,

$$\frac{\beta_2}{\varsigma^{\beta_2+1}} \left[L^+(\lambda' - \frac{P}{r-\mu}) - \beta_1(\lambda - \frac{PL^+}{r-\mu}) \right]$$

$$\geq \frac{\beta_1}{\varsigma^{\beta_1+1}} \left[\beta_2(\lambda - \frac{PL^+}{r-\mu}) - L^+(\lambda' - \frac{P}{r-\mu}) \right]$$

Hence, if the above condition is true, we will have a unique root of $\Lambda(\varsigma)$ which exists in the range $[0,l_{th}]$. Hence proved.

6 Experiments

Setup: We employ *MobileNet* - a CNN pre-trained on the ImageNet database - to evaluate our game model, and use the DeepExplain [Marco, 2020] framework to generate four different types of explanations (namely, Integrated Gradients [Sundararajan et al., 2017], Saliency maps [Simonyan et al., 2014], Gradient*Input [Shrikumar et al., 2016] and ϵ -LRP [Bach et al., 2015]). As all these methods generate explanations in the form of a vector, representing contribution of each feature (in the input) to the output label, our first task is to map these vectors into a single real value representing explanation relevance. This is required as we assume that the domain of the explanation process random variable is real. We accomplish this by computing the absolute normalized means of the values in the vector and use that as a measure of relevance. Then, we estimate the mean and standard deviation of explanations generated for the output labels by the *MobileNet* model on 500 random flower images, and use that to sample a series of (future) explanations by employing GBM. Noise to be added to the explanations provided by the system to the end-user is sampled from a Normal distribution with the same mean and standard deviation of the generated explanations. The sampled noise is added to the generated explanations by the system based on the computed belief at that time (using Bayes' rule) such that larger the belief that the end-user is honest, the smaller is the noise value added to the explanation, and vice-versa markers¹. Game model parameters that we chose in our numerical simulations for all explanation techniques are outlined in Table 2.

Parameters	ϵ -LRP	IG	G*I	SMaps
P	0.005	0.005	0.005	0.005
k	5.38	5.4	10	6.227
r_e	5	5	5	5
d	19.5	35	4.62	13
$d^{'}$	7.615	20	30	20
c_d	1.001	1.001	1.001	1.0009
a	0.002	0.002	0.002	0.002
b	227	226	135	237.5

Table 2: Game parameters assumed for each of the explanation methods. ϵ -LRP: ϵ Layer-wise Relevance Propagation, IG: Integrated Gradeints, G*I: Gradient*Input and SMaps: Saliency Maps.

Equilibrium Characterization: We compute $U(\pi_t)$, $L^+(\pi_t)$, $L^-(\pi_t)$ and $L(\pi_t)$ using the closed form representation presented in Section 4 for these functions. The termination payoff, $\lambda(ex_g,\pi_t)$ (defined in Section 2.4), which is used to write the boundary conditions and in the computation of $L^+(\pi_t)$, $L^-(\pi_t)$ and $L(\pi_t)$ is assumed to be:

$$\lambda(ex_g, \pi_t) = \frac{0.8 \times ex_g \times e^{\pi_t \times a} + \pi_t \times ex_g}{b}$$

where, ex_g is the value of any end-user's functional path (considered for the specific computation) at time t, and a and b (for each explanation method) are model parameters as outlined in Table 2. The parameters to compute $\lambda(ex_g,\pi_t)$ are empirically chosen based on their suitability to each of the four explanation methods considered in the paper. Based on our numerical simulations, below are some of the observations we make.

- From the plots of the optimal functional path $U(\pi_t)$ of the system for each of the explanation method, as shown in Figure 2, we can observe that as $\pi_t \to 1$, $U(\pi_t)$ starts decreasing. This is because, as the system's belief about the type of end-user approaches 1, both the explanation generated by the system and the noisy explanation given to the end-user approach u_{th} and l_{th} , respectively. After a certain point, $U(\pi_t)$ starts to decrease, indicating that when $ex_{ge}(\pi_t) > U(\pi_t)$, the system will block the end-user, which confirms to our intuition.
- From the optimal functional paths $L^+(\pi_t)$, $L^-(\pi_t)$ and $L(\pi_t)$ of the end-user for each of the explanation

¹Code is available at https://github.com/kkmidnyt/XAI_gAME

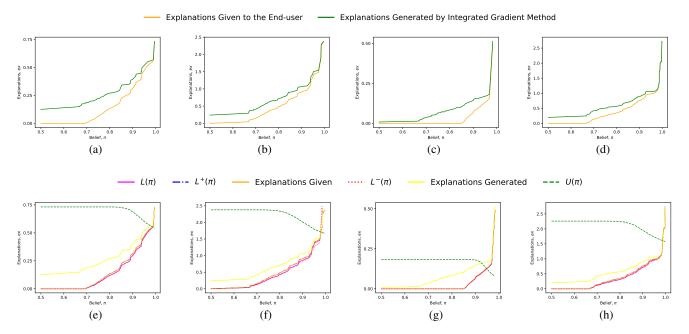


Figure 2: Functional paths for the end-user and the system. (a), (b), (c) and (d) represents the explanations generated and the explanations given to the end-user in Saliency Maps, ϵ -LRP, Integrated Gradients and Gradeint*Input respectively. (e), (f), (g) and (h) shows the optimal functional path $U(\pi_t)$ for the system and optimal functional paths $L(\pi_t)$, $L^+(\pi_t)$ and $L^-(\pi_t)$ for the end-user in Saliency Maps, ϵ -LRP, Integrated Gradients and Gradeint*Input, respectively.

method, as shown in Figure 2, we can observe that as $\pi_t \to 1$, $L^+(\pi_t)$, $L^-(\pi_t)$ and $L(\pi_t)$ approach the threshold l_{th} . Thus, if the system doesn't block the end-user at the right time (or doesn't have knowledge about optimal $U(\pi)$), then the end-user can easily compromise the system.

• Earlier we showed that a unique MPE exists when $L^+(\pi_t)$ and $L^-(\pi_t)$ begin to converge as $\pi_t \to 1$. This is also visible from our results (Figure 2), where we can observe that as $\pi_t \to 1$, $L^+(\pi_t)$ and $L^-(\pi_t)$ starts converging after a certain point. After the first intersection of curves $L^+(\pi_t)$ and $L^-(\pi_t)$, $L(\pi_t) = L^+(\pi_t) = L^-(\pi_t)$. Thus, if the system doesn't block the end-user before that point, we say a unique MPE exists in the game.

Optimal Communications: Next, we describe the optimal play of the end-user, the optimal play of the system and the optimal communication path between them as shown in Figure 3. Optimal play is a player's action without considering the play/action of the other player. Optimal communication occurs when both players (end-user and system) engage in optimal plays/actions. As the end-user plays a mixed strategy, we can see from these figures that before compromising the system or getting getting blocked, the end-user can continue to signal or can deviate and fail (if the threshold l_{th} is not reached). We also observe that for the saliency maps, ϵ -LRP and integrated gradient explanation methods, with the current set of assumed game parameters (Table 2), the system was able to block the end-user before he/she was able to reach the threshold l_{th} . However for the Gradient*Input method, the malicious end-user achieved success in his/her aim of compromising the system because explanation generated didn't cross the optimal cut-off function, $U(\pi_t)$, of the system.

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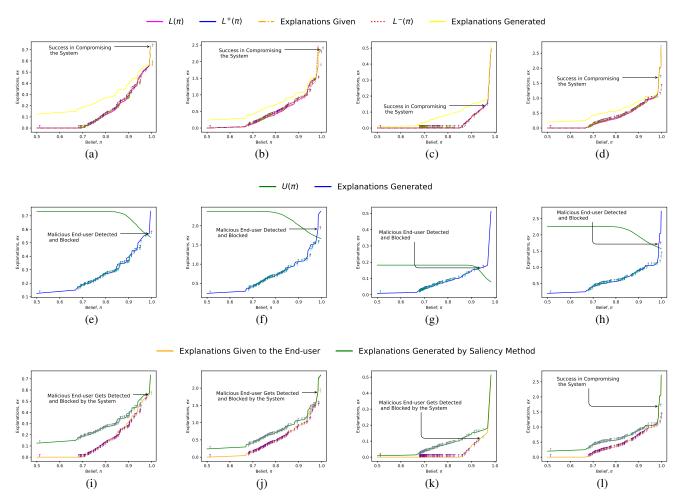


Figure 3: Optimal communication paths. (a), (e) and (i) represents the plots of optimal play of the end-user (without the interence from the system), optimal play of the system and the optimal communication between the system and the end-user for Saliency maps, respectively. (b), (f) and (j) represents the same for ε-LRPmethod. (c), (g) and (k) represents the same for Integrated Gradients method. (d), (h) and (l) represents the same for Gradient*Input method. In (a), (b), (c) and (d), † represents the successful compromise of the system by the end-user after deviation, ↑ represents the signaling activity of the end-user and ↓ represents the separation and failed to compromise the system case of end-user. In (e), (f), (g) and (h), ↑ represents the signaling activity of the system and † represents the successful detection and block of the end-user by the system. In (i), (j), (k) and (l), † represents the separation and failed case of end-user, ↑ represents the signaling activity of the system, † represents the detection and block of the end-user by the system from the perspective of end-user and † represents the detection and block of the end-user for the system.

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