

From Fair Decision Making To Social Equality

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Abstract

The study of fairness in intelligent decision systems has mostly ignored long-term influence on the underlying population. Yet fairness considerations (e.g. affirmative action) have often the implicit goal of achieving balance among groups within the population. The most basic notion of balance is eventual equality between the qualifications of the groups. How can we incorporate influence dynamics in decision making? How well do dynamics-oblivious fairness policies fare in terms of reaching equality? In this paper, we propose a simple yet revealing model that encompasses (1) a selection process where an institution chooses from multiple groups according to their qualifications so as to maximize an institutional utility and (2) dynamics that govern the evolution of the groups' qualifications according to the imposed policies. We focus on demographic parity as the formalism of affirmative action.

We then give conditions under which an unconstrained policy reaches equality on its own. In this case, surprisingly, imposing demographic parity may break equality. When it doesn't, one would expect the additional constraint to reduce utility, however, we show that utility may in fact increase. In more realistic scenarios, unconstrained policies do not lead to equality. In such cases, we show that although imposing demographic parity may remedy it, there is a danger that groups settle at a worse set of qualifications. As a silver lining, we also identify when the constraint not only leads to equality, but also improves all groups. This gives quantifiable insight into both sides of the mismatch hypothesis. These cases and trade-offs are instrumental in determining when and how imposing demographic parity can be beneficial in selection processes, both for the institution and for society on the long run.

1 Introduction

As many aspects of human society become increasingly automated, questions of ethical nature migrate into the technological sphere. Admittedly, no amount of formalization can capture all the complexity and subtlety of these issues. Yet, one is compelled to mathematize them and integrate them with familiar frameworks. The result of this process is the design of purportedly ethically-minded technology. An important component of this process, then, is to understand just how well such technology interacts with society and whether its use does indeed result in the intended effect.

Automated decision and policy making, particularly the data-driven case embodied by machine learning, is one such technology that has recently allotted considerable attention to these ethical questions. In particular, guaranteeing non-discriminatory behavior has been at the forefront of

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research. Various formalizations of non-discrimination have been proposed, relationships between these notions have been studied [DHP⁺12, HPS⁺16], and fundamental trade-offs have been identified [KMR16]. Yet the influence of the resulting decision systems on society has only garnered modest treatment [LDR⁺18]. One place where this question has been thoroughly considered is in studying the evolution of negative stereotypes in labor markets, in economics [CL93, HC18]. The dynamics there, however, are very carefully crafted to mimic the labor market. To relate the formalism of the current paper with these prior work, see Section 2.

This paper starts by proposing a selection process where an institution chooses from multiple groups according to their qualifications so as to maximize an institutional utility. In this context, non-discrimination often takes the form of constraining the type of selection policies. Demographic parity, for example, enforces groups to have equal selection rates. This is a simple yet rich model that has been well studied [HPS⁺16] [KMR16], and as such this paper chooses it as the archetype of non-discrimination, referring to it with the more colloquial name of affirmative action (AA). But non-discrimination is rarely a goal in and of itself. Ultimately, one tacitly expects a benefit to society. What kind of expected benefit is implicit in AA? The fact that a fully transformed society does not require non-discrimination hints at a possible answer. This is in particular true if the groups themselves are indistinguishable. If one accepts that such social equality is the intended effect of non-discrimination, then the question becomes: does AA lead to it? To answer this, modeling how the populations change in response to policies is not simply a luxury, but a necessity. This paper starts with some natural tenets, and derives from them a model of dynamics that govern the evolution of the groups' qualifications according to the imposed policies. These dynamics are general enough to model behavior beyond the specific ones in prior work.

The paper then proceeds by first giving conditions under which an unconstrained policy reaches equality on its own. In this case, surprisingly, imposing AA may break equality. When it doesn't, one would expect the additional constraint to reduce utility, however, it can be shown that utility may in fact increase. In practice, AA acts in two possible ways. Either selection rates of the privileged are reduced or those of the underprivileged are increased [FK13]. This dichotomy is at the heart of the starkly different ways in which non-discrimination manifests itself. In real world scenarios, unconstrained policies do not lead to equality. In such cases, it can be shown that although imposing AA may remedy it, there is a danger that groups settle at a worse set of qualifications. As a silver lining, one can identify exactly when the constraint not only leads to equality, but also improves all groups.

To summarize, the contributions of this paper are as follows:

- A simple yet flexible model for both a selection process by an institution and dynamics describing the change in the population due to selection policies
- A characterization of the policies resulting from institutional utility maximization without and with imposing affirmative action as non-discrimination
- Conditions under which society equalizes for each type of policy, along with the impact on the long-term institutional utility

These cases and trade-offs are instrumental in determining when and how imposing demographic parity can be beneficial in selection processes, both for the institution and for society on the long run.

The rest of the paper is organized as follows. Section 2 covers the most relevant related work. Section 3 formulates the problem and establishes the models. Section 4 derives the policies that

arise from the models and characterizes the conditions under which they lead to social equality. Sections 6 and 7 study the contrast between imposing affirmative action or not, in the case when not imposing it naturally leads to equality or not, respectively. Section 8 concludes.

2 Related Work

On the economics side, a large literature exists examining statistical discrimination, dating back to Arrow [A⁺73]. Prior work has attempted to study the effect of affirmative action in the labor market to remove stereotypes [CL93, HC18, MN04, ASW06, FV92], where stereotypes are defined as misrepresenting the quality of a worker from a disadvantaged group, when groups are assumed to be ex-ante equal. The models of [CL93] and [HC18] are the most related to the present work: they both set up a game in a market between workers and employers, where workers have the choice to invest in themselves to become qualified, and emit a test signal that employers threshold as their policy. The dynamics that this game generates on the distribution of qualified individuals per group is very specific. In particular, it is restricted to be unimodal. Both papers look at sufficient conditions for equality and propose interventions to actively reach it. The current paper trades off the concreteness of this line of work for more domain generality, through less specific dynamics. This helps garner more insight about the fundamental interplay between the constraints and changes in society. Moreover, examining dynamics and tracking the populations through time instead of only looking at equilibria as in [MN04, FV92], allows comparing cumulative utility which plays a role in short term considerations.

A recent work [LDR⁺18] raised very similar questions to the ones considered here. The main difference in that setup is that individuals are characterized by a potentially non-binary score and are studied over only a single step of the selection process. The influence of the selection is modeled by a social utility function Δ that can be interpreted as the expected change in score for an individual due to selection. The authors do not suggest that this change is due to population shifts, though it could be. Without tailoring Δ specifically to have such an interpretation, it is not possible to track distributions of scores over long time horizons. The dynamics proposed here implements precisely that aspect and enables the study of both institutional utility and social equality that emerge from fairness constraints. Another somewhat related work is that of [HSNL18], which models the retention rate of users of a particular service as a function of that service’s quality. There, this is thought of as the error of a learning algorithm and the authors propose a robust learning approach to avoid amplifying the difference in retention rate by group.

The present formulation also bears some similarity to reinforcement learning due to the dynamics being essentially Markovian. As such, it can be thought of as an optimal control problem. The goal here however is to analyze policies that are unaware of the dynamics, and do not make an explicit effort to learn it. Also, the focus is not simply on utility, but the secondary objective of social equality too. Previous work [JJK⁺17] has explored fairness with learning, guaranteeing that actions are preferentially selected only if they effectively generate more reward. However, there are many key differences, such as the state and action space in the current model being continuous as opposed to the discrete versions treated in [JJK⁺17] and having deterministic transitions. Work in online decision making [JKMR16] has also explored fairness constraints, but in a bandit setting that lacks the dynamical aspect.

3 Problem Formulation

3.1 Society: groups, qualification profiles

Throughout this paper, the total population is divided into two groups indexed by the variable G : group $G = A$ which constitutes the g_A fraction and group $G = B$ the $g_B = 1 - g_A$ fraction. The limitation to only two groups simplifies the exposition yet maintains the essence of the problem.

One can conceptually think of each individual in either group as having some attribute $\theta \in \Theta$ that bears information about qualification. For example in a college admission scenario the attribute θ can be: [GPA, SAT, Letters of recommendation]. This attribute is thought to provide a complete description in this context. In what follows, θ is implicitly mapped through an estimator $F : \theta \rightarrow \{0, 1\}$ to a crisp evaluation of qualification: say $v = 1$ if qualified and $v = 0$ otherwise. The assumption is that this binary classification is sufficient to characterize the utility that the individual would bring. This is compatible with the work on statistical discrimination [CL93, HC18]. An alternative way to reach this model is to first consider a more general multiclass characterization and then show that a threshold policy is optimal with respect to utility and societal improvement, as was done in [LDR⁺18]. This again indicates that a binary characterization, i.e. above and below the threshold, is often sufficient. For the example of admissions, one can think of $v = 1$ as indicating an individual likely to be successful in college.

Let the distribution of evaluations $V = v$ in group $G = j$ be denoted by $\pi(\cdot|G = j)$. This probability distribution is the *qualification profile* of group $G = j$. Instead of tracking individuals, it is more natural to study changes in the population in terms of the qualification profiles of groups. For example, in college admissions it is not the same students that would change their qualifications in response to admission policies, but rather those of the next application cycle.

3.2 Institution: policy, utility, selection rates

The population undergoes a selection process. An *institution*, referred to also by *policy maker*, designs a *policy* $\tau(V = v; G = j) : \{0, 1\} \times \{A, B\} \rightarrow [0, 1]$ that maps each individual to a probability of selection, possibly depending on the group identity. The institution also places a *utility* on each possible evaluation v , through a map $u : \{0, 1\} \rightarrow \mathbb{R}$, $v \mapsto u(v)$. Since $v = 1$ and $v = 0$ are assumed to be beneficial and detrimental respectively, and to avoid trivial policies where none/all are selected in what follows, assume $u(0) \leq 0 \leq u(1)$. The (average) *institutional utility* of a policy can then be defined as follows:

$$U(\tau) = \sum_{j \in \{A, B\}} g_j \sum_{v \in \{0, 1\}} u(v) \cdot \tau(V = v; G = j) \cdot \pi(v|G = j). \quad (1)$$

Any given policy also defines (per group) *selection rates*: $\beta(G = j) = \sum_{v \in \{0, 1\}} \tau(V = v; G = j) \cdot \pi(V = v|G = j)$. For college admission, this corresponds to acceptance rates per group. Note that each term in the sum is the rate corresponding to a particular evaluation. Denote these by $\beta(V = v; G = j) = \tau(V = v; G = j) \cdot \pi(V = v|G = j)$. In what follows, for notational convenience, the explicit ' $V =$ ' and ' $G =$ ' are dropped from the expressions of the qualification profiles, policies, and selection rates, since the context of the argument is clear.

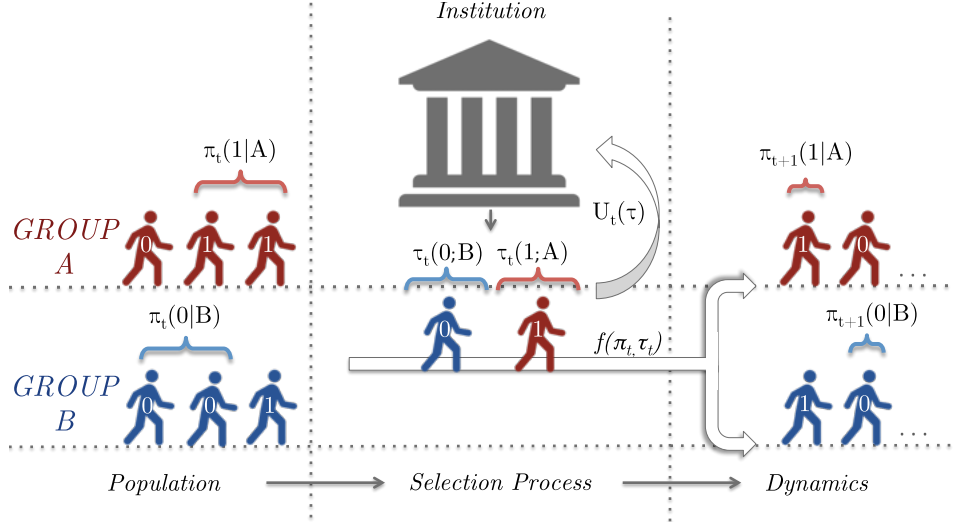


Figure 1: Selection process and influence dynamics – Two subgroups of the population have each publicly known qualification profiles π . An institution performs a selection by maximizing utility U , possibly subject to a non-discrimination constraint. The selection policies τ influence the profiles at the next round, through dynamics f .

3.3 Influence Dynamics

The execution of a selection policy can be thought of as demarcating time t . The main premise of this paper is that the selection process affects the population, namely by changing the qualification profiles of either group at time $t + 1$. This change can be interpreted in different ways. It could be due to genuine change within society: the institutions' policies can then be thought of as having a secondary effect as social incentive or deterrent. But change could also be due to self-selection in the pool of individuals available to the institution on the next time step: policies then have the effect of filtering through time, possibly without society itself changing. Although the model is not specific to an interpretation, when the perspective moves to social equality in Section 4, the tacit assumption is that dynamics are due to genuine change.

Dynamics are in general very complicated. They may vary in time, be stochastic, and depend on many parameters of the problem. Some of these aspects, however, are more relevant than others for capturing the fundamental effects. In what follows, dynamics are taken to be time-invariant and deterministic. Time-invariance is a convenient assumption, which can be supported by the slow change in the way in which society responds to stimuli. Determinism does not mean that there is no stochasticity within society, but that any such change is summarized through the evolution of the qualifications. Moreover, one can interpret these deterministic dynamics as the expectation of a more general stochastic dynamics. As such it is a Markovian assumption, with the state consisting of the qualification profiles.

Finally, what parameters of the problem should govern change and in what manner should they do so? There are three tenets that lead to this paper's proposal:

- (a) Within each group, there is an (a priori) distribution of *potential* qualifications, identical to the qualification profiles at the previous time step.

- (b) Upon observing the previous time step's policy, individuals respond by instantiating with some probability as (a posteriori) qualified or unqualified.
- (c) The response probabilities, which summarize how policies influence individuals, depend only on the selection rates within the group of the individual and their potential qualification.

Tenet (a) describes society's inertia in the absence of influencing policies. Tenet (b) could be thought of as a result of adequate or lacking individual effort in response to policies. In tenet (c), the fact that potentially qualified and potentially unqualified individuals may be influenced differently is necessary for the history of qualifications to have relevance. The main restriction thus is that the only influence of policies is through selection rates within a group. Lack of influence across groups is an idealization of such influence arguably being weaker than within a group. As for selection rates being the key variable rather than the selection policy itself, one can motivate it by considering extremes: indeed, if no qualified (or unqualified) individuals exist then selection policy for qualified (or unqualified) individuals is never observed and cannot have influence.

With these tenets, dynamics induced by the influence of policies can be defined as follows.

Assumption 3.1 (Dynamics). *For a given group j , let $\pi_t(1|j) =: \pi_t(1)$ (for clarity, the group index is dropped here and later) denote the qualification profile of group j for $v = 1$ at time t and let the policies τ_t at that time step induce the selection rates β_t . Then the qualification profiles at time $t + 1$ are given by*

$$\pi_{t+1}(1) = \pi_t(1) f_1(\beta_t(0), \beta_t(1)) + \pi_t(0) f_0(\beta_t(0), \beta_t(1)) \quad (2)$$

where f_0 and f_1 are two arbitrary continuously differentiable functions $[0, 1] \times [0, 1] \rightarrow [0, 1]$. The pair (f_0, f_1) is referred to as the dynamics.

These dynamics make the tenets concrete. For each group $G = j$, $\pi_t(\cdot|j)$ describes the potential qualification profile. The function f_1 represents the *retention at the top*: the rate of retention of the sub-population with potential $v = 1$ due to the current policy. The function f_0 represents *change for the better*: the fraction of the potential $v = 0$ sub-population that progresses to have evaluation 1. Equivalently, one can pair these with their counterparts to describe full conditional distributions. The pair $1 - f_1$ (*change for the worse*) and f_1 are the conditional distribution describing the response of an individual with potential $v = 1$ into actual evaluations 0 or 1 respectively, while the pair $1 - f_0$ (*retention at the bottom*) and f_0 are the same respectively for an individual with potential $v = 0$.

It is worth making a couple of remarks about these dynamics. First, the two groups' responses are identical since the choice of f_1 and f_0 does not depend on the group. To justify this, note that in fairness considerations such as demographic parity one makes the inherent assumption that the groups are ex-ante equal in all respect except for their qualification profiles, as otherwise there may be other reasons to treat them differently based only on their identity. Second, the groups do not interact through the functional form of the dynamics. Differences between groups and any potential coupling between groups can only happen through the different and interacting selection rates induced by the policies.

Finally, leaving f_0 and f_1 entirely arbitrary is good for generality but may lead to esoteric behavior. It may be more reasonable to restrict them in certain natural ways. One assumption to make, for example, is that change is harder than retention.

Assumption 3.2 (Status quo bias). *For all $x, y \in [0, 1]$:*

$$f_1(x, y) \geq f_0(x, y).$$

3.4 Utility Maximization and Non-discrimination

The primary tendency of the institution is to maximize its average utility. Rarely, if ever, are institutions aware of the underlying dynamics. Thus *dynamics-oblivious* policies are the appropriate ones, where the utility is maximized over a single time step. Next, two approaches taken in practice are modeled: unconstrained maximization without any non-discrimination considerations and affirmative action via demographic parity.

3.4.1 Unconstrained

In the simplest of settings, institutional utility maximization amounts to solving the following linear program (LP) at every time step t :

$$\max_{\tau} U_t(\tau), \tag{3}$$

where U_t is as in (1), with all quantities defined at time step t .

3.4.2 Affirmative Action

If the evaluation v is perfect, then the unconstrained policy is trivially *identity blind*, thus one could argue that it is fair in the sense of equalized opportunity [HPS⁺16]. However, as distributions of qualified individuals differ per group, the policy will have different selection rates per group. *Affirmative action* (AA), at an intuitive level, attempts to bridge such inequalities between different groups. More precisely, AA forces equal aggregate selection rates between groups [CL93, HC18], and it is more formally known as demographic parity [BHN18, LDR⁺18]. This paper adheres to this as the archetype of non-discrimination.

Definition 3.3 (Affirmative Action). The *affirmative action* constraint forces the policy to select at an equal rate between both groups:

$$\beta(A) = \beta(B) . \tag{4}$$

Thus in this case, the policy maker solves (3) with the additional constraint (4). Since the constraint is linear in τ , the optimal policy is also found via a LP. This is elaborated in Section 4.

4 Road to Equality

With the models of selection process, dynamics, and affirmative action in place, one can now ask the main question: when and how can resulting policies lead to social equality? Perhaps the simplest notion of equality within the population can be defined as the groups becoming indistinguishable in their qualification profiles. This is formalized as follows.

Definition 4.1 (Social Equality). Under given dynamics (f_0, f_1) , a policy is said to be *equalizing* if for all starting $\pi_0(1|A)$ and $\pi_0(1|B)$

$$\lim_{t \rightarrow \infty} |\pi_t(1|A) - \pi_t(1|B)| = 0$$

When this happens, say that the policy/population *reaches equality*, which is understood to be in an asymptotic sense.

This section characterizes the policies that result in both the unconstrained and affirmative action settings, and studies the relationships between these policies and the conditions under which they become equalizing.

4.1 Unconstrained

4.1.1 Resulting policy

First consider the case when the institution does not impose non-discrimination, that is the utility is maximized as in (3) without further constraints. Denote this case by UN. It is then straightforward to see that the optimal policies are $\tau_t(1; \cdot) = 1$ and $\tau_t(0; \cdot) = 0$ for all time t .

Note that the policy only depends on the distribution at time t . Any dynamics for group j is thus simply of the form:

$$\pi_{t+1}(1|j) = f(\pi_t(1|j)),$$

where f is defined in terms of (f_0, f_1) according to Equation (2) as:

$$f(\pi) = \pi f_1(0, \pi) + (1 - \pi) f_0(0, \pi).$$

For convenience denote the t^{th} iterate of f by f^t .

4.1.2 Condition for equality

Can an unconstrained policy be equalizing? The following result from classical stability analysis, proved for completeness in Appendix A, characterizes exactly when this is possible.

Proposition 4.2. *Unconstrained dynamics, of the form $\pi_{t+1} = f(\pi_t)$ within each group, are equalizing if and only if f has a unique globally attracting equilibrium point π^e .*

It is worth noting that f may have a unique attracting point, in addition to a negligible number of points that do not lead to this attracting point (say finitely many or a set of measure zero). Since in real dynamics have some level of noise, these points won't be influential. Without dwelling on this further, this suggests that one could alternatively assume that f has a unique attracting point, globally except for such a negligible set.

If f has a unique globally attracting equilibrium point then f^t converges to π^e as $t \rightarrow \infty$, a constant function which is trivially a contractive mapping. If this convergence is uniform, then there is a large enough T such that f^T is contractive. One could instead consider the following stronger condition in the form of f itself being a contractive mapping.

Definition 4.3 (Contractivity). $f : [0, 1] \rightarrow [0, 1]$ is a contractive mapping if $\exists L \in [0, 1)$ such that f is L -Lipschitz: $\forall x, y \in [0, 1]$,

$$|f(x) - f(y)| \leq L|x - y|.$$

If f is differentiable, then this condition is equivalent to having the derivative of f bounded by L on $[0, 1]$. By an application of the Banach fixed-point theorem [Lat14], this shows that the condition of Definition 4.3 is indeed sufficient for f to have a globally attracting equilibrium point. Recall the following.

Theorem 4.4. (*Banach fixed-point theorem*) Let (X, d) be a non-empty complete metric space and let $f : X \rightarrow X$ be a contractive mapping. Then f admits a unique fixed-point $x^e \in X$ where x^e can be found by choosing any arbitrary element $x_0 \in X$ and iterating the function f , i.e. define $x_t = f^t(x_0)$, then $x_t \rightarrow x^e$.

To stress again, contractivity is not a necessary condition to reach equality, but it is a simple condition to place on the dynamics that captures the intuition of reaching equality. More importantly, since the dynamics are defined in terms of the functions f_0 and f_1 , it is more useful to give conditions on the latter to obtain a contractive f . For π and π' such that $\pi - \pi' = \Delta > 0$, rewrite the requirement that $|f(\pi) - f(\pi')| \leq L \cdot \Delta$ for $L \in [0, 1)$ as:

$$\begin{aligned} |f(\pi) - f(\pi')| &= | \pi f_1(0, \pi) + f_0(0, \pi) - \pi f_0(0, \pi) - (\pi' f_1(0, \pi') + f_0(0, \pi') - \pi' f_0(0, \pi')) | \\ &= | \pi(f_1(0, \pi) - f_1(0, \pi - \Delta)) + (1 - \pi)(f_0(0, \pi) - f_0(0, \pi - \Delta)) \\ &\quad + \Delta(f_1(0, \pi - \Delta) - f_0(0, \pi - \Delta)) | < L \end{aligned} \quad (5)$$

A first step is to take both f_1 and f_0 to be Lipschitz.

Assumption 4.5. f_1 and f_0 are ℓ_1 Lipschitz continuous with constants L_1 and L_0 respectively. That is $\forall x_1, x_2, y_1, y_2 \in [0, 1]$:

$$|f_1(x_1, x_2) - f_1(y_1, y_2)| \leq L_1(|x_1 - y_1| + |x_2 - y_2|)$$

and

$$|f_0(x_1, x_2) - f_0(y_1, y_2)| \leq L_0(|x_1 - y_1| + |x_2 - y_2|).$$

This assumption is reasonable as one would expect the jump in the rate of retention (f_1) or rate of change (f_0) to be bounded as a function of the jump in the selection rates. But looking at Equation (5), it is not sufficient. How large should L_0 and L_1 be?

$$\begin{aligned} |f(\pi) - f(\pi')| &\leq |\pi(f_1(0, \pi) - f_1(0, \pi - \Delta))| + |(1 - \pi)(f_0(0, \pi) - f_0(0, \pi - \Delta))| \\ &\quad + |\Delta(f_1(0, \pi - \Delta) - f_0(0, \pi - \Delta))| \quad (\text{triangle inequality}) \\ &< \Delta(\pi L_1 + (1 - \pi)L_0 + |f_1(0, \pi - \Delta) - f_0(0, \pi - \Delta)|) \end{aligned} \quad (6)$$

Thus a sufficient condition for f to be a contractive mapping is:

$$L_{\text{UN}} := \max_{\pi \in [0, 1], \Delta < \pi} \pi L_1 + (1 - \pi)L_0 + |f_1(0, \pi - \Delta) - f_0(0, \pi - \Delta)| < 1 \quad (7)$$

4.1.3 Speed of convergence

Contractivity is a restrictive condition but it has the obvious benefit of easily describing the rate at which the population reaches equality. In particular, if

$$\max_{j \in \{A, B\}} |\pi_0(1|j) - \pi^e| \leq \Delta,$$

then

$$|\pi_t(1|A) - \pi_t(1|B)| \leq 2\Delta L^t.$$

Equality is reached at a linear rate (in the exponent): to equalize to within an ϵ difference, about $\frac{\log \epsilon / \Delta}{\log L}$ time steps are sufficient.

4.2 Affirmative Action

4.2.1 Two resulting policies

The following characterizes the solution of maximizing the utility as in (3) under the additional constraint (4). The proof can be found in Appendix A.

Proposition 4.6. *Assume that at time t group j is advantaged, defined as $\pi_t(1|j) \geq \pi_t(1|\neg j)$. Then the optimal policy is one of two cases, depending only on g_j , $u(0)$, and $u(1)$:*

AA⁻ *If $g_j u(1) + (1 - g_j)u(0) \leq 0$, then*
 $\tau_t(1; j) = \frac{\pi_t(1|\neg j)}{\pi_t(1|j)}$, $\tau_t(0; j) = 0$ (*under-acceptance*),
 $\tau_t(1; \neg j) = 1$, $\tau_t(0; \neg j) = 0$.

AA⁺ *If $g_j u(1) + (1 - g_j)u(0) \geq 0$, then*
 $\tau_t(1; j) = 1$, $\tau_t(0; j) = 0$,
 $\tau_t(1; \neg j) = 1$, $\tau_t(0; \neg j) = \frac{\pi_t(1|j) - \pi_t(1|\neg j)}{1 - \pi_t(1|\neg j)}$ (*over-acceptance*).

There are two possible cases through which affirmative action impacts the policy, denoted by AA⁻ and AA⁺. These represent two drastically different approaches to fairness. AA⁻ (*under-acceptance*) accepts fewer qualified individuals from the advantaged group so as to equalize the selection rates for qualified individuals between both groups. On the other hand, AA⁺ (*over-acceptance*) accepts unqualified individuals from the disadvantaged group in order to equalize the aggregate selection rates. One could think of AA⁻ as increasing the standard for the advantaged group and as such reducing total selection rates. As a peek at the intuition to develop, this could possibly reduce the motivation for individuals to become qualified. As for AA⁺, one could think of it as reducing the standard for the disadvantaged group. It is not as evident whether this could potentially increase or decrease the motivation to become qualified. On one hand, it does present the possibility of leading to a more qualified overall society by increasing the total selection rates and training more unqualified people. On the other hand, selecting unqualified corresponds to the condition of the so-called *mismatch hypothesis* [San04], which claims that this could actually reduce the number of qualified individuals over time. This mismatch effect has been amply debated in discussions on public policy with arguments on both sides [Ho04, AAFS11, San04]. One perspective of the current investigation is to understand how the dynamics tip the scales in this debate.

4.2.2 Conditions for equality

Under AA, the analysis of equality gets complicated. This is mainly due to the fact that in Proposition 4.6 there is a specific advantaged group. If from one time step to the next the advantage changes, the identities of the groups swap. This in itself is not an issue, unless the swap also causes a switch between the cases AA⁻ and AA⁺. This last situation will not be considered for two reasons: first, because it is too unwieldy, and second, because it does not arise if the dynamics are slow enough, as formalized in Section 5. The conditions for equality given here are thus provisional on remaining in AA⁻ or AA⁺, with swaps or without. Here is a characterization to remain in one case at all times, given with no proof as it follows from Proposition 4.6.

Proposition 4.7. *Let $\mathcal{C}_{j,t} := [\pi_t(1|j) \geq \pi_t(1|\neg j)]$ be the j -advantage condition. Resolving equality either way, note that $\mathcal{C}_{\neg j,t} = \neg \mathcal{C}_{j,t}$. Let $\mathcal{C}_{-|j} := [g_j u(1) + (1 - g_j)u(0) \leq 0]$ be the AA^- condition under j -advantage and let $\mathcal{C}_{+|j} := [g_j u(1) + (1 - g_j)u(0) \geq 0] = \neg \mathcal{C}_{-|j}$ be the AA^+ condition under j -advantage.*

Then to remain in AA^- at all times, it is necessary that for all t :

$$(\mathcal{C}_{A,t} \wedge \mathcal{C}_{-|A}) \vee (\mathcal{C}_{B,t} \wedge \mathcal{C}_{-|B}),$$

and in particular it suffices that $\mathcal{C}_{-|A} \wedge \mathcal{C}_{-|B}$. Similarly, to remain in AA^+ at all times, it is necessary that for all t :

$$(\mathcal{C}_{A,t} \wedge \mathcal{C}_{+|A}) \vee (\mathcal{C}_{B,t} \wedge \mathcal{C}_{+|B}),$$

and in particular it suffices that $\mathcal{C}_{+|A} \wedge \mathcal{C}_{+|B}$.

Another reason for the complication is because, under AA , the dynamics for one group depends on the other group: in Proposition 4.6 there always exists one j for which the policy depends on the qualification profiles of $\neg j$. The dynamics therefore have to be tracked jointly, unlike in UN . Write

$$\begin{aligned}\pi_{t+1}(1|A) &= f_A(\pi_t(1|A), \pi_t(1|B)) \\ \pi_{t+1}(1|B) &= f_B(\pi_t(1|A), \pi_t(1|B)) ,\end{aligned}$$

where f_A and f_B are two components of a joint dynamics function $f : [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1]$. These components can be written in terms of f_0 and f_1 according to Equation (2) and Proposition 4.6, but have a complicated expression that accounts for all four cases (who is advantaged and which AA case it is). Nevertheless, one could say that given dynamics of this form, equality will be reached if as $t \rightarrow \infty$ $|f_A^t(\pi_0(1|B), \pi_0(1|A)) - f_B^t(\pi_0(1|B), \pi_0(1|A))| \rightarrow 0$, for all $\pi_0(1|A)$ and $\pi_0(1|B)$.

Similarly to the unconstrained case, since this would otherwise give no guarantees on the speed of convergence, one can require f to satisfy the following contraction condition. There exists $L \in [0, 1)$ such that f satisfies:

$$|f_A(\pi, \pi') - f_B(\pi, \pi')| \leq L|\pi - \pi'| \quad \forall \pi, \pi' \in [0, 1] \quad (8)$$

This does not quite fit the classical Banach fixed-point theorem, but it is easy to show sufficiency. That is, if the dynamics satisfy this condition then the population will reach equality. Indeed, let $\pi_0, \pi'_0 \in [0, 1]$:

$$\begin{aligned}|f_A(\pi_0, \pi'_0) - f_B(\pi_0, \pi'_0)| &\leq L|\pi_0 - \pi'_0| \\ |f_A(\pi_1, \pi'_1) - f_B(\pi_1, \pi'_1)| &\leq L|\pi_1 - \pi'_1| \leq L^2|\pi_0 - \pi'_0| \\ &\vdots \\ |f_A(\pi_{t-1}, \pi'_{t-1}) - f_B(\pi_{t-1}, \pi'_{t-1})| &\leq L^t|\pi_0 - \pi'_0|\end{aligned}$$

Since $L < 1$ and $|\pi_0 - \pi'_0| \leq 1$ then $\lim_{t \rightarrow \infty} L^t = 0$ and $|f_A(\pi_{t-1}, \pi'_{t-1}) - f_B(\pi_{t-1}, \pi'_{t-1})| \geq 0 \quad \forall t$ therefore $\lim_{t \rightarrow \infty} |f_A(\pi_t, \pi'_t) - f_B(\pi_t, \pi'_t)| = 0$ and the population will reach equality.

As in the unconstrained case, this contractivity requirement on f can be reduced to sufficient conditions on f_0 and f_1 .

Proposition 4.8. Assume that AA^- holds at all times, by the necessary or sufficient conditions of Proposition 4.7. Then if

$$L_{AA^-} := \max_{\pi \in [0,1]} |f_1(0, \pi) - f_0(0, \pi)| < 1 \quad (9)$$

holds, then society equalizes.

Proposition 4.9. Assume that AA^+ holds at all times, by the necessary or sufficient conditions of Proposition 4.7. Then if

$$L_{AA^+} := \max_{0 \leq \Delta \leq \pi \leq 1} 2[\pi L_1 + (1 - \pi)L_0] + |f_1(\Delta, \pi - \Delta) - f_0(\Delta, \pi - \Delta)| < 1 \quad (10)$$

holds, then society equalizes.

The proofs can be found in Appendix A. Note that while social equality conditions for UN and AA^- can be related, the condition for AA^+ is not comparable to either of them in the sense that there exist dynamics where one holds but the other does not. Observe however that the same two components of condition (7) appear in (10). First, responses generally need to be *stable*: L_1 and L_0 have to be small. In AA^+ , this condition is more stringent than UN, due to the additional 2 factor. Second, responses of both the qualified and unqualified have to be *comparable*: f_1 and f_0 need to be close. In AA^+ , the main difference is that the comparison becomes sensitive to β_0 . This condition may be indeed more lenient, if more unqualified selection makes the responses not necessarily higher or lower, but *more similar*. To appreciate the kind of insight this provides, assume the mismatch hypothesis holds, in the sense that f_0 drops with larger β_0 . Proponents of the hypothesis assume that this is a failure of affirmative action. Proposition 4.9 says that if f_1 drops enough to meet f_0 , AA^+ could still lead to equality. And, as discussed in what follows (cf. Theorems 6.2 and 7.3), this can be beneficial to society on the long run.

4.3 Effect of Stereotypes

Everywhere in this paper, except in this subsection, the policy maker is assumed to know exactly the distribution π of qualifications of both groups. But the institution relies on its estimator F to identify qualified individuals, errors in F translate to errors ϵ in estimates of group qualifications and thus $\hat{\pi}$. As a result of these errors, it becomes hard to properly impose fairness constraints, as even selections rates per true groups are unknown. But the framework allows one to investigate the effect of certain types of errors on the dynamics.

At each time step t , define the errors in the qualification's estimates as ϵ_t^A and ϵ_t^B such that $-\pi_t(1|A) \leq \epsilon_t^A \leq 1 - \pi_t(1|A)$ and $-\pi_t(1|B) \leq \epsilon_t^B \leq 1 - \pi_t(1|B)$. Then the institution has the following qualification estimates:

$$\begin{aligned} \hat{\pi}_t(1|A) &= \pi_t(1|A) + \epsilon_t^A \\ \hat{\pi}_t(1|B) &= \pi_t(1|B) + \epsilon_t^B \end{aligned}$$

Assume the errors are small enough that one can still identify which group truly is more qualified. This translates to having $\epsilon_t^B - \epsilon_t^A \leq \pi_t(1|A) - \pi_t(1|B)$. Furthermore, restrict errors to be of the form of positive or negative stereotypes. Concretely, if $\epsilon_t^B \leq 0$, call this negative stereotype, the institution identifies only $\frac{\pi_t(1|B) + \epsilon_t^B}{\pi_t(1|B)}$ of qualified individuals and considers the rest unqualified. If $\epsilon_t^A \geq 0$, call this positive stereotype, then it identifies all qualified individuals correctly and mistakes $\frac{\epsilon_t^A}{\pi_t(0|A)}$ of unqualified individuals as qualified.

The institution bases its policy on these estimates, however in terms of the actual distribution the implemented policy actually differs.

If the policy acted upon is UN, meaning $\tau(1) = 1$ and $\tau(0) = 0$, with a positive stereotype $\hat{\pi} = \pi + \epsilon \geq \pi$, the actual policy is:

$$\tau(1) = 1, \tau(0) = \frac{\epsilon}{1 - \pi}$$

And with a negative stereotype $\hat{\pi} = \pi + \epsilon \leq \pi$:

$$\tau(1) = \frac{\pi + \epsilon}{\pi}, \tau(0) = 0$$

These policies are those that influence the next step distribution. Note that in both cases the unconstrained condition for equality might fail to hold as the difference in selection rates between the two groups can be exaggerated.

Now consider that the institution applies affirmative action. In the case of AA^- , surprisingly, given negative stereotype against group B, no matter if group A's qualification is overestimated or underestimated, equality is reached under the same cases as if the true distribution been known (no stereotypes). However, given a positive stereotype for group B, then the conditions for equality under AA^- derived previously are no longer sufficient. Thus the AA^- policy is robust to negative stereotypes against disadvantaged groups.

On the other hand with AA^+ , with a negative bias against group A and a positive bias for group B one can guarantee equality under the same cases as without stereotypes. A further treatment of the actual policies and analysis of the above conclusions is left to Appendix B.

5 Dynamics in Continuous Time

While discrete time (DT) describes successive selection-response steps naturally, the arbitrary nature of the dynamics can lead to sudden jumps and oscillatory behavior. These do not quite correspond to how populations evolve in the real world, where one expects change to happen gradually and slowly through time. Figure 2 shows the evolution of the percentage of bachelor degree holders by sex in the United States of America. In 1967, 13% of men 25 years and older held a bachelor's degree or higher while 8% of women did [RB16]; this gap has decreased through time as degree attainment increased for both sexes. In 2015, 33% of women held a college degree compared to 32% of men, thus the proportion of degree holders equalized between both sexes in 47 years time. One of the reported reasons behind this increase has been attributed to Title IX [oJ], a civil rights law passed in 1972 aiming to eliminate discrimination in educational programs and the removal of quotas against disadvantaged groups.

A change that is smooth and gradual can be modeled by transforming the discrete time dynamics into continuous time (CT). The transformation is standard, starting from dynamics (2) and dropping group indices, one can rewrite it as a difference equation:

$$\pi_{t+1}(1) - \pi_t(1) = \pi_t(1)(f_1(\beta_t(0), \beta_t(1)) - 1) + \pi_t(0)f_0(\beta_t(0), \beta_t(1))$$

Now assume that the difference from t to $t + \Delta t$ for all $\Delta t > 0$ is proportional to that from t to

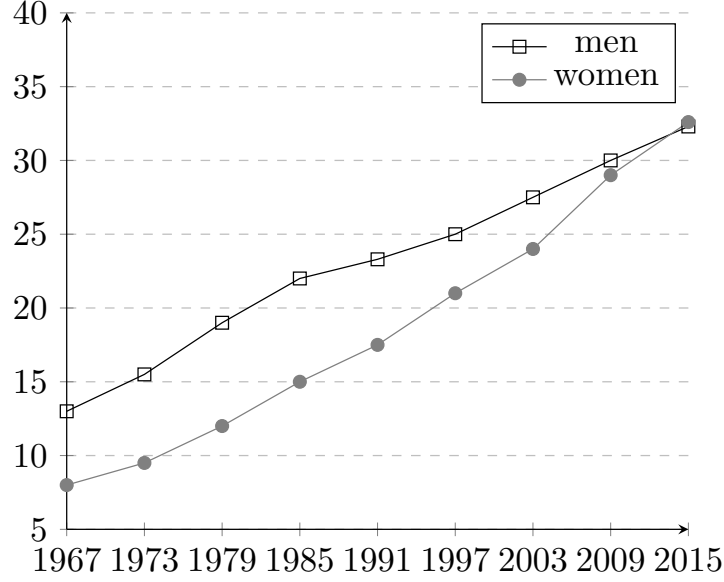


Figure 2: Percentage of the US population 25 years and older holding a bachelor degree by sex from 1967 till 2015, adapted from [RB16].

$t + 1$ (linear interpolation):

$$\pi_{t+\Delta t}(1) - \pi_t(1) = \Delta t \cdot (\pi_t(1)(f_1(\beta_t(0), \beta_t(1)) - 1) + \pi_t(0)f_0(\beta_t(0), \beta_t(1)))$$

dividing by Δt :

$$\frac{\pi_{t+\Delta t}(1) - \pi_t(1)}{\Delta t} = \pi_t(1)(f_1(\beta_t(0), \beta_t(1)) - 1) + \pi_t(0)f_0(\beta_t(0), \beta_t(1))$$

Now taking the limit as $\Delta t \rightarrow 0$:

$$\frac{d\pi_t}{dt} = \pi_t(f_1(\beta_t(0), \beta_t(1)) - 1) + (1 - \pi_t)f_0(\beta_t(0), \beta_t(1)) \quad (11)$$

The CT dynamics thus derived (11) can be motivated in two ways. As is shown next, it maintains the properties of the DT dynamics regarding conditions leading to social equality. Moreover, it possesses additional properties that greatly facilitate the analysis.

The proof of this and all the following lemmas can be found in Appendix A.

Lemma 5.1. *Assume the unconstrained policy is implemented in both discrete time (DT) and continuous time (CT). If the sufficient condition of Definition 4.3 for social equality holds for the DT dynamics with equilibrium point π^e , then the CT dynamics also converges to π^e starting from any initial condition.*

The following shows that populations in CT, unlike in DT, maintain their order.

Lemma 5.2. *Following either an unconstrained or affirmative action policy in CT will preserve the initial advantage. For example, if at $t = 0$, $\pi_0(1|B) \leq \pi_0(1|A)$ then for all $t \geq 0$ $\pi_t(1|A) \leq \pi_t(1|B)$.*

With Lemma 5.2, reaching equality with AA is straightforward in CT if it does in DT.

Lemma 5.3. *If an AA policy in DT reaches equality by virtue of condition (8), implied in particular by the conditions of Propositions 4.8 and 4.9, then in CT equality is likewise reached.*

Thus to summarize, transitioning to CT does not break the conditions previously derived for DT but in fact facilitates them. One can additionally give a bound on the difference of the distributions between groups at each time step:

Lemma 5.4. *If the DT dynamics is contractive with constant $L \in [0, 1)$, then under CT the difference between the distributions of both groups $\Delta_t = \pi_t(1|A) - \pi_t(1|B)$ at any time t obeys:*

$$\Delta_0 e^{-t(1+L)} \leq \Delta_t \leq \Delta_0 e^{-t(1-L)} \quad (12)$$

Motivated by these properties, in what follows the remainder of the analysis of the behavior of the selection process on the groups' distributions due to the different policies is performed under CT dynamics. The treatment could have initially started with CT dynamics, however the setup is more intuitive in DT and motivates the transition better.

6 Affirmative Action under Natural Equality

This section is concerned with the following scenario: assume that using the *unconstrained* policy equality is reached, what can happen if one instead follows an affirmative action policy? Even though affirmative action may seem unnecessary, as without requiring further intervention equality is reached, but can there be any additional benefit to applying fairness constraints and is one guaranteed that in this setting equality will also be reached? This is relevant since the policy maker may not be aware of operating in this regime.

Formally, make the assumption that the dynamics under the unconstrained policy (UN) is contractive with constant $L_{UN} < 1$, implying that there exists a globally attracting point π^* . The precise questions are then: are the conditions for equality with affirmative action also met? And how does the cumulative utility obtained under AA policies compare to that under the UN policy?

Because in CT advantage is maintained (Lemma 5.2) and Proposition 4.8 applies (Lemma 5.3), if AA^- holds initially it continues to hold and the only remaining condition, which is therefore sufficient, is Equation (9).

The following then holds.

Theorem 6.1. *If equality is reached with an unconstrained (UN) policy then it is necessarily reached by an AA^- policy, however with no more, and possibly less, utility at each step.*

Proof. Since equality is reached under the UN dynamics then $\forall \pi \in [0, 1]$:

$$|f'(\pi)| \leq L_{UN}$$

where f' is the derivative of the dynamics under the UN policy. Expanding in terms of the proposed dynamics:

$$|f_1(0, \pi) - f_0(0, \pi) + \pi f'_1(0, \pi) + (1 - \pi) f'_0(0, \pi)| \leq L_{UN} \quad (13)$$

where f'_1 and f'_0 are the partial derivatives of $f_1(0, \pi)$ and $f_0(0, \pi)$ with respect to π .

Suppose for the sake of contradiction that $|f_1(0, \pi^*) - f_0(0, \pi^*)| = 1$ for some $\pi^* \in [0, 1]$. Note that since both f_1 and f_0 take values in $[0, 1]$, this implies that one of $f_1(0, \pi^*)$ or $f_0(0, \pi^*)$ is 0 and the other is 1. Suppose $\pi^* = 1$. If $f_1(0, \pi^*) = 1$ it follows that $f'_1(0, \pi^*) \geq 0$ and thus $f'(\pi) = 1 + f'_1(0, \pi^*) \geq 1$. If instead $f_1(0, \pi^*) = 0$ then $f'_1(0, \pi^*) \leq 0$ and thus $f'(\pi) = -1 + f'_1(0, \pi^*) \leq -1$. Both cases contradict the condition that $|f'(\pi)| \leq L_{\text{UN}} < 1$. The argument is similar for ruling out $\pi^* = 0$. Finally, suppose $\pi^* \in (0, 1)$. Then one of f_1 and f_0 attains its maximum at π^* and the other its minimum, thus $f'_1(0, \pi^*) = 0$ and $f'_0(0, \pi^*) = 0$, implying that $|f'(\pi)| = 1$. This also contradicts condition (13). It follows that $|f_1(0, \pi^*) - f_0(0, \pi^*)| \neq 1$ and thus AA^- holds.

As the sufficient conditions for both AA^- and UN to reach equality are satisfied, in CT the equilibrium point π^e will be reached by both policies, however their trajectories toward π^e are different and thus each achieves a different utility.

At each time-step t , to differentiate between the distributions due to different policies denote by $\pi_t(1|A, \text{UN})$ the distribution of $\pi(1|A)$ at time t due to the UN policy starting from $t = 0$, and similarly for AA^- . The utility of the UN policy at time t is:

$$U_t(\text{UN}) = g_A \cdot u(1) \cdot \pi_t(1|A, \text{UN}) + (1 - g_A) \cdot u(1) \cdot \pi_t(1|B, \text{UN})$$

and that of AA^- at time t is:

$$U_t(\text{AA}^-) = g_A \cdot u(1) \cdot \pi_t(1|B, \text{AA}^-) + (1 - g_A) \cdot u(1) \cdot \pi_t(1|B, \text{AA}^-)$$

Consider the difference of the utilities:

$$\begin{aligned} U_t(\text{UN}) - U_t(\text{AA}^-) &= g_A \cdot u(1) \cdot \pi_t(1|A, \text{UN}) + (1 - g_A) \cdot u(1) \cdot \pi_t(1|B, \text{UN}) \\ &\quad - (g_A \cdot u(1) \cdot \pi_t(1|B, \text{AA}^-) + (1 - g_A) \cdot u(1) \cdot \pi_t(1|B, \text{AA}^-)) \end{aligned}$$

Since under AA^- and UN the trajectory of $\pi(1|B)$ is identical then $\pi_t(1|B, \text{AA}^-) = \pi_t(1|B, \text{UN})$, thus:

$$U_t(\text{UN}) - U_t(\text{AA}^-) = g_A \cdot u(1) \cdot (\pi_t(1|A, \text{UN}) - \pi_t(1|B, \text{UN}))$$

Following Proposition 5.2, one has that $\forall t \geq 0$, $\pi_t(1|A, \text{UN}) - \pi_t(1|B, \text{UN}) \geq 0$, and thus the utility of the UN policy is always no less than that of AA^- . \square

In light of this, the policy maker has no gain in enforcing affirmative action in this case.

Consider next how AA^+ acts on the population in this setting. The sufficient condition (10) for equality with the AA^+ policy cannot be directly implied from the UN equality. However, under AA^+ group A's distribution trajectory will reach the equilibrium point π^e as the group's policy is identical to the UN policy in CT. On the other hand, group B's distribution evolution through time is unclear. The dynamics governing it is, with $\Delta(t) = \pi_t(1|A, \text{AA}^+) - \pi_t(1|B, \text{AA}^+)$:

$$\begin{aligned} \frac{d\pi_t(1|B, \text{AA}^+)}{dt} &= \pi_t(1|B, \text{AA}^+)(f_1(\Delta(t), \pi_t(1|B, \text{AA}^+) - \Delta(t)) \\ &\quad + (1 - \pi_t(1|B, \text{AA}^+))(f_0(\Delta(t), \pi_t(1|B, \text{AA}^+) - \Delta(t)) - \pi_t(1|B, \text{AA}^+)) \end{aligned}$$

Since $\pi_t(1|A, \text{AA}^+)$ converges to π^e as $t \rightarrow \infty$, then $\Delta(t)$ will only become a function of $\pi_t(1|B, \text{AA}^+)$ and π^e . Therefore, the dynamics will approximately become an ODE with one variable, and oscillatory behavior cannot exist in first order one dimensional ODEs [Str18], then $\pi_t(1|B, \text{AA}^+)$ will converge to some fixed point of the dynamics necessarily below π^e . Looking closely, if π^e was

initially below $\pi_0(1|B, AA^+)$, then the trajectory of $\pi_t(1|A, AA^+)$ will force group B to converge toward the equilibrium point. On the other hand, if $\pi^e \geq \pi_0(1|B, AA^+)$ then nothing can be said, thus unfortunately only worse off equilibria are guaranteed.

Now let's assume the conditions for reaching equality with the AA^+ policy are met to see what could be the consequence. Specifically there exists $L_{AA^+} \in [0, 1)$ such that for all π, π' in $[0, 1]$ one has:

$$|f_A(\pi, \pi') - f_B(\pi, \pi')| \leq L_{AA^+} |\pi - \pi'|$$

Similarly to the discussion in the AA^- section, by comparing the cumulative utility with UN one surprisingly finds a lifeline for enforcing affirmative action.

Theorem 6.2. *Let $\alpha = (1 - g_A)u(1)/((1 - g_A)u(1) + |u(0)|)$. Assume UN and AA^+ both satisfy sufficient conditions (7) and (10) to reach equality in DT, assuming remaining in the same case. Then if L_{UN} and L_{AA^+} satisfy the following:*

$$L_{UN} \geq 1 - \alpha$$

and

$$L_{AA^+} \leq 1 + (L_{UN} - 1)/\alpha,$$

Then AA^+ provides more utility in CT over an infinite time horizon.

Proof. Let us try to look at the difference between the utility at any time step t for the UN policy and AA^+ policy:

$$\begin{aligned} U_t(UN) - U_t(AA^+) &= g_A \cdot u(1) \cdot \pi_t(1|A, UN) + (1 - g_A) \cdot u(1) \cdot \pi_t(1|B, UN) \\ &\quad - (g_A \cdot u(1) \cdot \pi_t(1|A, AA^+) + (1 - g_A) \cdot (u(1) \cdot \pi_t(1|B, AA^+) \\ &\quad + u(0) \cdot (\pi_t(1|A, AA^+) - \pi_t(1|B, AA^+))) \end{aligned}$$

Since under AA^+ and UN the trajectory of $\pi(1|A)$ is identical then $\pi_t(1|A, AA^+) = \pi_t(1|A, UN)$, simplifying things:

$$U_t(UN) - U_t(AA^+) = (1 - g_A) \cdot (u(1) \cdot (\pi_t(1|B, UN) - \pi_t(1|B, AA^+)) - u(0) \cdot (\pi_t(1|A, AA^+) - \pi_t(1|B, AA^+)))$$

using the fact that $u(0) < 0$ and $u(1) > 0$:

$$\begin{aligned} U_t(UN) - U_t(AA^+) &= (1 - g_A)(u(1) + |u(0)|) \cdot (\pi_t(1|A, AA^+) - \pi_t(1|B, AA^+)) \\ &\quad - (1 - g_A) \cdot u(1) \cdot (\pi_t(1|A, UN) - \pi_t(1|B, UN)) \end{aligned}$$

In DT the cumulative utility was the sum over all time steps, similarly in CT the utility will be the integral of the utility function at each step evaluated from 0 till the final time step.

Now let us look at the total difference in cumulative utility over an infinite time horizon:

$$\begin{aligned} U(UN) - U(AA^+) &= (1 - g_A) \cdot (u(1) + |u(0)|) \cdot \int_0^\infty (\pi_t(1|A, AA^+) - \pi_t(1|B, AA^+)) dt \\ &\quad - (1 - g_A) \cdot u(1) \cdot \int_0^\infty (\pi_t(1|A, UN) - \pi_t(1|B, UN)) dt \end{aligned}$$

Using the two-sided bound on the difference between group distributions of Lemma 5.4, we can

lower bound the utility difference:

$$\begin{aligned}
&\leq (1 - g_A) \cdot (u(1) + |u(0)|) \cdot \int_0^\infty \exp(-t(1 - L_{AA^+}) + \ln(\Delta_0)) dt \\
&- (1 - g_A) \cdot u(1) \cdot \int_0^\infty \exp(-t(1 + L_{UN}) + \ln(\Delta_0)) dt \\
&= (1 - g_A) \cdot (u(1) + |u(0)|) \frac{\Delta_0}{1 - L_{AA^+}} \\
&- (1 - g_A) \cdot u(1) \frac{\Delta_0}{1 - L_{UN}}
\end{aligned}$$

We ask under what conditions of L_{AA^+} and L_{UN} can the above difference be negative:

$$\begin{aligned}
&(1 - g_A) \cdot (u(1) + |u(0)|) \frac{\Delta_0}{1 - L_{AA^+}} - (1 - g_A) \cdot u(1) \frac{\Delta_0}{1 - L_{UN}} \leq 0 \\
&\Leftrightarrow (1 - g_A) \cdot (u(1) + |u(0)|) \frac{\Delta_0}{1 - L_{AA^+}} \leq (1 - g_A) \cdot u(1) \frac{\Delta_0}{1 - L_{UN}} \\
&\Leftrightarrow \frac{1 - L_{UN}}{1 - L_{AA^+}} \leq \frac{(1 - g_A) \cdot u(1)}{(1 - g_A) \cdot u(1) + |u(0)|} \\
&\Leftrightarrow L_{AA^+} \leq L_{UN} \frac{(1 - g_A) \cdot u(1) + |u(0)|}{(1 - g_A) \cdot u(1)} + 1 - \frac{(1 - g_A) \cdot u(1) + |u(0)|}{(1 - g_A) \cdot u(1)} \\
&\Leftrightarrow L_{AA^+} \leq 1 + (L_{UN} - 1) \cdot \frac{(1 - g_A) \cdot u(1) + |u(0)|}{(1 - g_A) \cdot u(1)}
\end{aligned}$$

Since we must have $L_{AA^+} \geq 0$, then we require $L_{UN} \leq 1 - \frac{(1 - g_A) \cdot u(1)}{(1 - g_A) \cdot u(1) + |u(0)|}$ as a necessary additional condition. \square

It is straightforward to see that $L_{UN} \leq 1 + (L_{UN} - 1)/\alpha$. A weaker version of this theorem can thus be informally stated as follows: if under the unconstrained policy equality is reached slowly enough yet under AA^+ equality is reached faster, then AA^+ results in a utility gain. This implies that, rather than trading off, speed of convergence and utility go hand in hand.

7 Affirmative Action under Disparate Equilibria

7.1 Multiple Equilibria

It is more realistic to expect that myopically maximizing utility is unlikely to equalize the qualifications of groups. That is, one expects following an unconstrained policy to not lead to equality. To model this scenario, the dynamics should not have a unique attracting equilibrium point. In general one could have arbitrarily many, even infinitely many, fixed points for the dynamics. In this section this is simplified to assuming that the dynamics under UN has a finite number of attracting equilibrium points, each with its own basin of attraction. Thus the UN policy would not equalize the groups unless they are initially close enough to fall in the same basin. The notion of such dynamics can be formalized as follows.

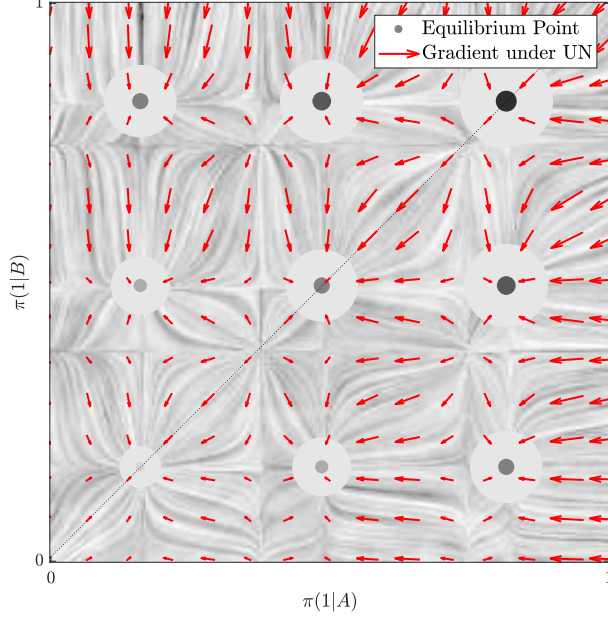


Figure 3: 3-equilibrium mapping under UN policy

Definition 7.1 (*k*-Equilibrium Dynamics). The function f obtained from (f_0, f_1) under the UN policy is called a *k*-equilibrium (DT or CT) dynamics, if it is continuously differentiable and has k fixed points $\pi_1^\epsilon \leq \dots \leq \pi_k^\epsilon$ that are locally attracting (DT or CT) equilibrium points. Namely, there exist $k - 1$ delimiting fixed points $\delta_1 \in (\pi_1^\epsilon, \pi_2^\epsilon), \dots, \delta_{k-1} \in (\pi_{k-1}^\epsilon, \pi_k^\epsilon)$, such that for all $i \in [k]$ and all $\pi_0 \in (\delta_{i-1}, \delta_i)$, using the convention $\delta_0 = 0, \delta_k = 1$, and semi-closed basins $[0, \delta_1)$ and $(\delta_k, 1]$:

$$\pi_t \rightarrow \pi_i^\epsilon \quad \text{as } t \rightarrow \infty,$$

where $\pi_t = f^t(\pi_0)$ (for DT) or π_t is the solution at t of $\frac{d\pi}{dt} = f(\pi) - \pi$ initialized at π_0 (for CT).

Remark. Note that it follows that the $\delta_i, i \in [k - 1]$ are unstable equilibria, under both DT and CT dynamics, by virtue of being fixed points. The extremes $\pi = 0$ or $\pi = 1$ could also be fixed points. If one or the other is an attracting equilibrium, then the notation implies $\pi_1^\epsilon = 0$ or $\pi_k^\epsilon = 1$ respectively. If one or the other or both is an unstable equilibrium, then δ_0 or δ_k or both join $\{\delta_i : i = 1, \dots, k - 1\}$ as unstable equilibria.

Figure 3 illustrates a 3-equilibrium CT dynamics under an UN dynamics with the direction of the gradient illustrated at each point. Observe the 9 possible joint equilibria that result by following the UN policy, depending on where each group is initialized. See Appendix C for the specific dynamics (f_0, f_1) for this illustration.

In the CT case, the necessary and sufficient conditions for f to be a *k*-equilibrium dynamics is that $\forall i \in [k]$:

$$\begin{aligned} f(\pi_i^\epsilon) &= \pi_i^\epsilon \\ \forall \pi \in (\delta_{i-1}, \pi_i^\epsilon) : f(\pi) &> \pi \\ \forall \pi \in (\pi_i^\epsilon, \delta_i) : f(\pi) &< \pi \end{aligned} \tag{14}$$

Once again, sufficient conditions for CT are not so for DT. For example, one may have oscillations (think of the single equilibrium at 0.5 when $f(\pi) = 1 - \pi$.) One way to adapt (14) is to modify it as follows:

$$\begin{aligned} f(\pi_i^\epsilon) &= \pi_i^\epsilon \\ \forall \pi \in (\delta_{i-1}, \pi_i^\epsilon) : \pi_i^\epsilon &> f(\pi) > \pi \\ \forall \pi \in (\pi_i^\epsilon, \delta_i) : \pi_i^\epsilon &< f(\pi) < \pi \end{aligned} \tag{15}$$

Condition (15) allows dynamics that flatten out, giving an entire interval of equilibrium points. Intuitively, this corresponds to slowdown. To address it, one can require that there exists a neighborhood of radius $r^i > 0$ around every equilibrium point π_i^ϵ such that:

$$\frac{df}{d\pi} \leq L_i \tag{16}$$

for some $L_i \in [0, 1)$ for all $\pi \in [\pi_i^\epsilon - r^i, \pi_i^\epsilon + r^i]$. Conditions (15) and (16) guarantee that f is a k -equilibrium DT dynamics. However, if f is already known to be a k -equilibrium dynamics, one can still require (16) as local Lipschitz conditions that control the rate of convergence in both DT and CT dynamics, just like was done for the single equilibrium case.

7.2 Eliminating Disparate Equilibria

The following shows that similarly to the single equilibrium UN, the AA^- policy is equalizing even under a k -equilibrium UN.

Theorem 7.2. *Assume status quo bias, i.e. Assumption 3.2. Let f be a k -equilibrium dynamics as in Definition 7.1, that further satisfies conditions (16). Let j be the initially advantaged group. If the disadvantaged group starts at $\pi_0(1|\neg j) \neq \delta_i$ for any $i \in \{1, \dots, k-1\}$, following AA^- reaches equality in both DT and CT.*

Proof. Consider first DT dynamics. By examining the proof of Proposition 4.8, it suffices to show that $|f_1(0, \pi) - f_0(0, \pi)| \neq 1$, not for all π , but only for $\pi \in [0, 1]$ reachable by the disadvantaged group $\neg j$. In AA^- , recall that the dynamics of group $\neg j$ are the same as in the unconstrained case. Since we assume that initially $\pi_0(1|\neg j) \neq \delta_i$ for any $i \in \{1, \dots, k-1\}$, it follows that the same holds for all time t and none of these δ_i are reachable, because they are unstable equilibria.

Assume then, for the sake of contradiction, that there exists a $\pi^* \in [0, 1]$ such that $|f_1(0, \pi^*) - f_0(0, \pi^*)| = 1$, from the status-quo bias Assumption 3.2, the condition reduces to $f_1(0, \pi^*) - f_0(0, \pi^*) = 1$.

This implies that $f_1(0, \pi^*) = 1$ and $f_0(0, \pi^*) = 0$, thus $f(\pi^*) = \pi^*$, or equivalently π^* is an equilibrium of the dynamics f . Additionally, by the same arguments as in the proof of Theorem 6.1, we can deduce that if $\pi^* \in \{0, 1\}$ then $f'(\pi^*) \geq 1$ and if $\pi^* \in (0, 1)$ then $f'(\pi^*) = 1$.

It is not possible for π^* to be a stable equilibrium, i.e. for all $i \in [k]$, $\pi^* \neq \pi_i^\epsilon$. Otherwise, condition (16), which requires that for all $i \in [k]$, $f'(\pi_i^\epsilon) \leq L_i < 1$, is contradicted. The only remaining possibility is for π^* to be an unstable equilibrium. But since these are not reachable, we conclude by contradiction that we cannot have $|f_1(0, \pi) - f_0(0, \pi)| = 1$.

The CT case follows from Lemma 5.3. □

The above theorem states that following AA^- is a guaranteed way to reach equality, however equality comes at a price. Even more generally than Theorem 7.2, whenever AA^- equalizes a k -equilibrium CT dynamics (14), it always leads to worse long-term utility than under UN by leading to a population with lower qualification. On the other hand when following AA^+ , equality is always beneficial. The caveat is that, as in the single equilibrium case, under the conditions of a k -equilibrium dynamics, one cannot deduce conditions for AA^+ that lead to equality. Assuming that these conditions do hold, however, guarantee a utility no worse than under UN by leading to a population with higher qualification. The following Theorem characterizes this.

Theorem 7.3. *Let f be a k -equilibrium CT dynamics. If the policy is AA^- , and the equalizing condition of Proposition 4.8 holds, then the equalized population generates long-term utility no higher (and possibly lower) than the limiting population under UN. If the policy is AA^+ , and the equalizing condition of Proposition 4.9 holds, then the equalized population generates long-term utility no lower (and possibly higher) than the limiting population under UN.*

Proof. Without loss of generality, let A be the advantaged group. Let $i, j \in [k]$ with $i \leq j$, let $\pi_0(1|B) \in (\delta_{i-1}, \delta_i)$ and $\pi_0(1|A) \in (\delta_{j-1}, \delta_j)$.

Consider first the AA^- case. Then the policy for group B is identical to the UN policy as long as $\pi_t(1|B) \leq \pi_t(1|A)$ and from Lemma 5.2 this is assured. Therefore as $t \rightarrow \infty$, $\pi_t(1|B) \rightarrow \pi_i^\epsilon$. Since the Proposition 4.8 condition holds, Lemma 5.3 implies that $\pi_t(1|A) \rightarrow \pi_t(1|B)$ and consequently $\pi_t(1|A) \rightarrow \pi_i^\epsilon$. The equilibrium state is thus $\{\pi_i^\epsilon, \pi_i^\epsilon\}$.

Under the UN policy, one ends up with distributions $\{\pi_i^\epsilon, \pi_j^\epsilon\}$ for groups B and A respectively. While this state is possibly unequal however it possesses a higher utility value per time step. Therefore, by reducing selection rates the population is forced to an overall less qualified state.

Consider next the AA^+ case. Then the policy for group A is identical to the UN policy. Therefore as $t \rightarrow \infty$ we have $\pi_t(1|A) \rightarrow \pi_j^\epsilon$. If the condition for equality under AA^+ is satisfied then $\pi_t(1|B) \rightarrow \pi_t(1|A) \rightarrow \pi_j^\epsilon$. Thus the population converges to the equilibrium state $\{\pi_j^\epsilon, \pi_j^\epsilon\}$. Compared again to the UN policy, this is possibly of higher utility value as $i \leq j$. Therefore, by increasing selection rates the population is pushed to be more qualified, but only if AA^+ indeed reaches equality. \square

Figure 4 shows the difference between the gradient of π_t under either AA^- and AA^+ and UN, for the dynamics which resulted in Figure 3 in a 3-equilibrium UN. This can be shown to satisfy the conditions to make the AA^- and AA^+ policies equalizing. Note how the gradient difference for AA^- points generally downward and leftward toward states with lower utility as opposed to that of AA^+ which generally points upward and rightward toward equilibrium points with higher utility. These illustrate qualitatively the kind of behavior characterized in Theorem 7.3.

8 Conclusions

Imposing fairness considerations on decision making systems requires understanding the influence that they will have on the population at hand. Without knowledge of this influence, seemingly fair constraints might in fact exacerbate existing differences between groups of the population.

This paper proposed a simple but expressive model of a selection process to concretely consider these concerns. Namely, affirmative action was studied in terms of its ability to equalize the qualifications of different groups. Imposing one case of affirmative action (under-acceptance of qualified individuals) was shown to guarantee equality at the cost of worse institutional utility and

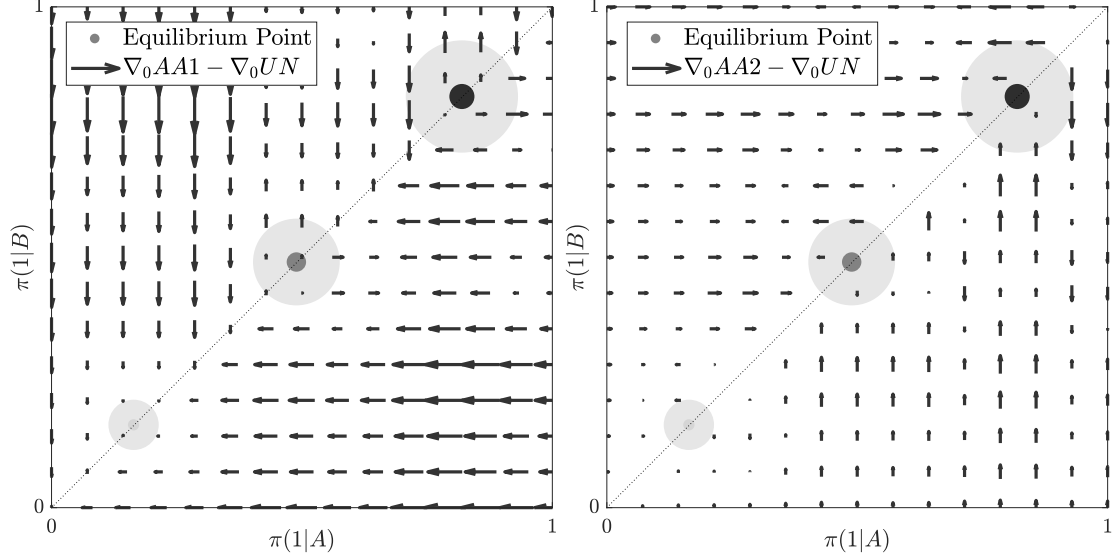


Figure 4: Illustration of Theorem 7.3 for the AA^- and AA^+ policies respectively under the same dynamics as that of Figure 3

possibly decreasing the population's overall qualification level. In another case (over-acceptance of unqualified individuals), however, affirmative action was shown to lead to a policy with different characteristics: equality cannot be directly guaranteed to hold but when it does, it results, in equilibria where the population becomes more qualified. Interestingly, this last case corresponds to the debate on the mismatch hypothesis. The analysis here quantifies how both sides of the debate can be correct, depending on whether society equalizes or not. This invites devoting attention to the conditions under which over-acceptance tips society toward equality.

This paper is certainly not the last word on the role of dynamics in non-discrimination. The hope is to spur many new lines of investigation. The proposed model and analysis could serve as a framework to evaluate other non-discrimination constraints. More ambitiously, it could be used to motivate and derive new constraints with explicit awareness of their long term impact.

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A Deferred proofs

A.1 Section 4

A.1.1 Proposition 4.2

Proof. If f has a unique globally attracting equilibrium point, it trivially holds that the dynamics are equalizing. Next, assume the dynamics are equalizing. We prove by contradiction that there can only be a unique globally attracting equilibrium point. Let π_1, π_2 and π_3 be three distinct points in $[0, 1]$. Think of π_1 as the initial qualified fraction of group A . Then, if π_2 is the initial qualified fraction of group B , and if starting at π_1 and π_2 respectively the trajectories of A and B were to reach equality, then they must converge over time to a point $\pi_1^* \in [0, 1]$. Next, think of π_3 as the initial qualified fraction of group B . By the same argument, now the trajectories must converge over time to a (possibly distinct) point $\pi_2^* \in [0, 1]$, since the definition of equalization doesn't assume a unique limit. However, if π_1^* and π_2^* are different, it implies that the trajectory of A from π_1 under f can differ, which is a contradiction since f is deterministic. \square

A.1.2 Proposition 4.6

Proof. By assumption at time t we have $\pi_t(1|A) \geq \pi_t(1|B)$, from this point on we drop the time subscript for clarity. Let us now express the total utility (3) when the demographic parity constraint (4) is satisfied:

$$U(\tau) = g_A \cdot (u(1) + |u(0)| \cdot (\pi(1|A)\tau(1; A) - \pi(1|B)\tau(1; B)) \\ + u(1) \cdot \pi(1|B)\tau(1; B) - |u(0)| \cdot \pi(0|B)\tau(0; B)$$

First of all we claim that $\pi(1|A)\tau(1; A) \geq \pi(1|B)\tau(1; B)$ for the optimal policy. If this is not the case then it is possible to increase $\tau(1; A)$ and decrease either $\tau(0; A)$ or $\tau(0; B)$ while still guaranteeing constraint (4) as $\pi(1|A) \geq \pi(1|B)$ all the while obtaining higher utility. Moreover, the optimal policy will have $\tau(1; B) = 1$ as increasing $\tau(1; B)$ will increase utility as $\tau(1; A)$ can only increase and $\tau(0; \cdot)$ decrease in response. Similarly $\tau(0; A) = 0$ as increasing it will only decrease utility and force $\tau(0; B)$ to increase as a consequence, to meet the demographic parity constraint.

This in turn induces a lower bound on $\tau(1; A)$, where $\frac{\pi(1|B)}{\pi(1|A)} \leq \tau(1; A)$, next we show that $\tau(1; A)$ can take only the two possible values: $\tau(1; A) = \frac{\pi(1|B)}{\pi(1|A)}$ or $\tau(1; A) = 1$.

Suppose $\tau(1; A) = c$, where c is a value between its two possible bounds, then to satisfy the demographic parity constraint (4):

$$\pi(1|A)\tau(1; A) = \pi(0|B)\tau(0; B) + \pi(1|B)$$

we need:

$$\tau(0; B) = \frac{c\pi(1|A) - \pi(1|B)}{\pi(0|B)}$$

Let us compare the utility between the policy with $\tau(1; A) = c$ and that with $\tau(1; A) = \frac{\pi(1|B)}{\pi(1|A)}$. Since by assumption $\tau(1; A) = c$ is utility maximizing, the difference is positive:

$$0 \leq g_A \cdot u(1) \cdot (c - \pi(1|B) \cdot \pi(1|A)) \cdot \pi(1|A) + (1 - g_A) \cdot u(0) \cdot \left(\frac{c\pi(1|A) - \pi(1|B)}{\pi(0|B)} \right) \cdot \pi(0|B)$$

$$\begin{aligned}
&= g_A \cdot u(1) \cdot (c\pi(1|A) - \pi(1|B)) - g_A \cdot u(0) \cdot (c\pi(1|A) - \pi(1|B)) + u(0) \cdot (c\pi(1|A) - \pi(1|B)) \\
&= (g_A u(1) + g_A |u(0)| + u(0)) \cdot (c\pi(1|A) - \pi(1|B))
\end{aligned}$$

We already have that $c\pi(1|A) - \pi(1|B) \geq 0$ from our first claim, and thus for the overall difference of utilities to be positive, then $g_A u(1) + g_A |u(0)| + u(0) \geq 0$. However increasing c to $c = 1$ will result in higher utility, therefore any interpolation is not optimal.

To summarize if $g_A u(1) + g_A |u(0)| + u(0) \geq 0$, we have $\tau(1; A) = 1$, otherwise $\tau(1; A) = \frac{\pi(1|B)}{\pi(1|A)}$. As a final note, if instead $\pi_t(1|A) \leq \pi_t(1|B)$, then it suffices to switch A by B in the conditions and policies. \square

A.1.3 Proposition 4.8

Proof. Without loss of generality, assume that A is advantaged. Let π be shorthand for $\pi(1|A)$ and π' for $\pi(1|B)$. Due to advantage, we can write $\pi' = \pi - \Delta$ for some $\Delta > 0$:

$$\begin{aligned}
|f_A(\pi, \pi') - f_B(\pi, \pi')| &= |\pi f_1(0, \pi - \Delta) + f_0(0, \pi - \Delta) - \pi f_0(0, \pi - \Delta) \\
&\quad - (\pi - \Delta) f_1(0, \pi - \Delta) - f_0(0, \pi - \Delta) + (\pi - \Delta) f_0(0, \pi - \Delta)| \\
&= \Delta |f_1(0, \pi - \Delta) - f_0(0, \pi - \Delta)|.
\end{aligned}$$

Since the latter is bounded by L_{AA^-} , it follows that the condition of the proposition guarantees the contraction expressed in Equation (8). \square

A.1.4 Proposition 4.9

Proof. Without loss of generality, assume that A is advantaged. Let π be shorthand for $\pi(1|A)$ and π' for $\pi(1|B)$. Due to advantage, we can write $\pi' = \pi - \Delta$ for some $\Delta > 0$:

$$\begin{aligned}
|f_A(\pi, \pi') - f_B(\pi, \pi')| &= |\pi(f_1(0, \pi) - f_1(\Delta, \pi - \Delta)) \\
&\quad + (1 - \pi)(f_0(0, \pi) - f_0(\Delta, \pi - \Delta)) + \Delta(f_1(\Delta, \pi - \Delta) - f_0(\Delta, \pi - \Delta))| \\
&\leq \Delta \cdot (2\pi L_1 + 2(1 - \pi)L_0 + |f_1(\Delta, \pi - \Delta) - f_0(\Delta, \pi - \Delta)|) \\
&\quad (\text{triangle inequality} + \text{Assumption 4.5}).
\end{aligned}$$

Since the latter is bounded by L_{AA^+} , it follows that the condition of the proposition guarantees the contraction expressed in Equation (8). \square

A.2 Section 5

A.2.1 Lemma 5.1

Proof. Under the unconstrained policy, the dynamics are an autonomous ordinary differential equation:

$$\frac{d\pi}{dt} = \pi(t) \cdot (f_1(0, \pi(t)) - 1) + (1 - \pi(t)) \cdot f_0(0, \pi(t))$$

At any time step t , the following will be shown. If $\pi_t < \pi^\epsilon$, $\frac{d\pi}{dt}$ is strictly positive. Otherwise if $\pi_t > \pi^\epsilon$, $\frac{d\pi}{dt}$ is strictly negative and finally if $\pi_t = \pi^\epsilon$ then $\frac{d\pi}{dt} = 0$. This implies that the dynamics converges to π^ϵ no matter the starting point.

Indeed, if $\pi_t < \pi^\epsilon$, then $\frac{d\pi}{dt} = \pi_t \cdot f_1(0, \pi_t) + (1 - \pi_t) \cdot f_0(0, \pi_t) - \pi_t = f(\pi_t) - \pi_t$, where $f(\pi)$ is the DT dynamics. Since the DT dynamics satisfies the following:

$$0 \leq |f(\pi^\epsilon) - f(\pi_t)| \leq L|\pi^\epsilon - \pi_t|$$

and using the fact that $f(\pi^\epsilon) = \pi^\epsilon$:

$$0 \leq |\pi^\epsilon - f(\pi_t)| \leq L(\pi^\epsilon - \pi_t) \quad (17)$$

From Equation (17) we can see that we must have $f(\pi_t) > \pi_t$ making the derivative positive.

If $\pi_t = \pi^\epsilon$, then $\frac{d\pi}{dt} = f(\pi_t) - \pi_t = \pi^\epsilon - \pi^\epsilon = 0$, and finally if $\pi_t > \pi^\epsilon$ the proof is similar to the case where $\pi_t < \pi^\epsilon$. \square

A.2.2 Lemma 5.2

Proof. The conservation of advantage will be shown when following each of the noted policies. In all of the following, at time $t = 0$: $\pi_0(1|B) \leq \pi_0(1|A)$.

UN: we remind that the UN policy is $\tau(1; \cdot) = 1$ and $\tau(0; \cdot) = 0$. As we are in CT, to have at some t , $\pi_t(1|G = B)' > \pi_t(1|G = A)$ at any time t then from a simple continuity argument of the trajectory of π over time, there must exist a time point $t' < t$ such that $\pi_{t'}(1|G = A) = \pi_{t'}(1|G = B)$. However, then we must have $\forall t > t'$ that $\pi(t, G = A)' = \pi(t, G = B)$ so that order is conserved.

AA⁻: we remind that initially the AA⁻ policy is $\tau_0(1; A) = \frac{\pi_0(1|B)}{\pi_0(1|A)}$, $\tau_0(1; B) = 1$ and $\tau(0; \cdot) = 0$. Following the same argument of the UN case proof, for the groups to change order then there must exist a time-point t' such that $\pi_{t'}(1|G = A) = \pi_{t'}(1|G = B)$, at that point the policy then becomes $\tau_{t'}(1; \cdot) = 1$ and $\tau_{t'}(0; \cdot) = 0$ and thus the argument is exactly as in the UN case.

AA⁺: finally the AA⁺ policy is $\tau_0(1; A) = 1$, $\tau_0(0; A) = 0$, $\tau_0(1; B) = 1$ and $\tau_0(0; B) = \frac{\pi_0(1|A) - \pi_0(1|B)}{\pi_0(0|B)}$. Following the same argument for AA⁻, at the time t' the policy becomes: $\tau_{t'}(1; \cdot) = 1$ and $\tau_{t'}(0; \cdot) = 0$, therefore the argument is exactly as in the UN case. \square

A.2.3 Lemma 5.3

Proof. Any solution π_t to the CT dynamics will satisfy the following integral equation [GCF⁺08]:

$$\pi_t = \pi_0 + \int_0^t \frac{d\pi}{ds} ds$$

Thus let us write the difference between the solutions of the two groups denoted by π and π' under AA and note that the CT dynamics (11) can be written as:

$$\begin{aligned} \frac{d\pi}{dt} &= f_A(\pi_t, \pi'_t) - \pi_t \\ \frac{d\pi'}{dt} &= f_B(\pi_t, \pi'_t) - \pi'_t \end{aligned}$$

where $f_A(\pi, \pi')$ and $f_B(\pi, \pi')$ specify the DT dynamics under affirmative action. Now let us track the difference at any time $t \geq 0$ of the distribution between both groups, denote $\Delta_t := \pi_t - \pi'_t$ and assume that $\pi_0 \geq \pi'_0$:

$$\pi_t - \pi'_t = \pi_0 + \int_0^t (f_A(\pi_s, \pi'_s) - \pi_s) ds - \pi'_0 + \int_0^t (-f_B(\pi_s, \pi'_s) + \pi'_s) ds$$

taking the derivative with respect to t :

$$\Delta_t = \Delta_0 + \int_0^t (f_A(\pi_s, \pi'_s) - f_B(\pi_s, \pi'_s)) ds - \int_0^t \Delta_s ds$$

Our assumption that the AA DT policy reaches equality implies that there exists $L \in [0, 1)$ such that inequality (8) holds, i.e. $|f_A(\pi_s, \pi'_s) - f_B(\pi_s, \pi'_s)| \leq L\Delta_s$, and since from proposition 5.2 order is preserved:

$$\begin{aligned} \Delta_t &\leq \Delta_0 + \int_0^t L\Delta_s ds - \int_0^t \Delta_s ds \\ \frac{d\Delta}{dt} &\leq (L - 1)\Delta_t \end{aligned} \tag{18}$$

Equation (18) is obtained by differentiating with respect to t . Similarly since $f_A(\pi_s, \pi'_s) - f_B(\pi_s, \pi'_s) \geq -L\Delta_s$:

$$\frac{d\Delta}{dt} \geq -(L + 1)\Delta_t \tag{19}$$

From our assumption we know that $L - 1 < 0$ and $\Delta_t \geq 0$ from proposition 5.2 so that $\forall t \geq 0$, $\frac{d\Delta}{dt} \leq 0$. Thus Δ_t is strictly decreasing over time and is lower bounded by 0 and when for some T , Δ_T becomes 0 then this will be the case for all time $t' > T$ by combining the lower bound (19) and upper bound (18). \square

A.2.4 Lemma 5.4

Proof. Equation (18) gives us an upper bound on the derivative of Δ_t given the DT dynamics are contractive with Lipschitz constant $L \in [0, 1)$:

$$\frac{d\Delta}{dt} \leq (L - 1)\Delta_t$$

assume that $\Delta_t \neq 0 \forall t \geq 0$:

$$\frac{d\Delta}{dt} \frac{1}{\Delta_t} \leq (L - 1)$$

integrating from 0 to t :

$$\int_0^t \frac{d\Delta}{ds} \frac{1}{\Delta_s} ds \leq \int_0^t (L - 1) ds$$

we get:

$$\ln(\Delta_t) - \ln(\Delta_0) \leq t(L - 1)$$

arranging the terms:

$$\Delta_t \leq e^{-t(1-L)+\ln(\Delta_0)} \tag{20}$$

If for some $t' \geq 0$ we have $\Delta_{t'} = 0$, then the conclusion of our theorem will hold for all $t < t'$ with the same proof and for all $t \geq t'$ it trivially holds. Similarly using the lower bound in (19) we can obtain the lower bound on Δ_t . \square

B Effects of Stereotypes

Table 1: True policies under \mathbf{AA}^- with stereotypes

Stereotypes — Policy	$\tau(1; A)$	$\tau(0; A)$	$\tau(1; B)$	$\tau(0; B)$
$\epsilon_t^A \geq 0, \epsilon_t^B \geq 0$	$\min(1, \frac{\pi(1 B)+\epsilon^B}{\pi(1 A)})$	0	1	$\frac{\epsilon^B}{1-\pi(1 B)}$
$\epsilon_t^A \leq 0, \epsilon_t^B \geq 0$	$\frac{\pi(1 B)+\epsilon^B}{\pi(1 A)}$	0	1	$\frac{\epsilon^B}{1-\pi(1 B)}$
$\epsilon_t^A \geq 0, \epsilon_t^B \leq 0$	$\frac{\pi(1 B)+\epsilon^B}{\pi(1 A)}$	0	$\frac{\pi(1 B)+\epsilon^B}{\pi(1 B)}$	0
$\epsilon_t^A \leq 0, \epsilon_t^B \leq 0$	$\frac{\pi(1 B)+\epsilon^B}{\pi(1 A)}$	0	$\frac{\pi(1 B)+\epsilon^B}{\pi(1 B)}$	0

Table 2: True policies under \mathbf{AA}^+ with stereotypes

Stereotype — Policy	$\tau(1; A)$	$\tau(0; A)$	$\tau(1; B)$	$\tau(0; B)$
$\epsilon_t^A \geq 0, \epsilon_t^B \geq 0$	1	$\frac{\epsilon^A}{1-\pi(1 A)}$	1	$\frac{\pi(1 A)-\pi(1 B)+\epsilon^A-\epsilon^B}{1-\pi(1 B)}$
$\epsilon_t^A \leq 0, \epsilon_t^B \geq 0$	$\frac{\pi(1 A)+\epsilon^A}{\pi(1 A)}$	0	1	$\frac{\pi(1 A)-\pi(1 B)+\epsilon^A-\epsilon^B}{1-\pi(1 B)}$
$\epsilon_t^A \geq 0, \epsilon_t^B \leq 0$	1	$\frac{\epsilon^A}{1-\pi(1 A)}$	1	$\frac{\pi(1 A)-\pi(1 B)+\epsilon^A-\epsilon^B}{1-\pi(1 B)}$
$\epsilon_t^A \leq 0, \epsilon_t^B \leq 0$	$\frac{\pi(1 A)+\epsilon^A}{\pi(1 A)}$	0	1	$\frac{\pi(1 A)-\pi(1 B)+\epsilon^A-\epsilon^B}{1-\pi(1 B)}$

Whenever $\epsilon_t^B \leq 0$, we can see that the selection rates for value $v = 1$ are equal across both groups. Therefore with the same analysis as the case without stereotype, the necessary condition to reach equality still holds as long as for all $t \geq 0$ $\epsilon_t^B \leq 0$. However, this does not imply that the trajectory of the distributions is the same but only that equality will be achieved.

On the other side, when applying \mathbf{AA}^+ the resulting policies are shown in Table 2. We never recover in any of the cases the exact selection rates as those of \mathbf{AA}^+ without stereotype. However, with the sufficient condition stated in (10), \mathbf{AA}^+ equality can be maintained:

When $\epsilon_t^A \leq 0, \epsilon_t^B \geq 0$, let $\Delta = \pi(1|A) - \pi(1|B) \geq 0$, the resulting selection rates are as follows: $\beta(1; A) = \pi(1|A) + \epsilon^A \geq \pi(1|A) - \Delta$, $\beta(0; A) = 0$, $\beta(1; B) = \pi(1|A) - \Delta$ and $\beta(0; B) = \Delta + \epsilon^A - \epsilon^B \leq \Delta$. Now we check that condition (10) is met:

$$\begin{aligned}
|f_A(\pi(A), \pi(B)) - f_B(\pi(A), \pi(B))| &= |\pi(A)(f_1(0, \pi(A) + \epsilon^A) - f_1(\Delta + \epsilon^A - \epsilon^B, \pi(A) - \Delta)) \\
&\quad + (1 - \pi(A))(f_0(0, \pi(A) + \epsilon^A) - f_0(\Delta + \epsilon^A - \epsilon^B, \pi(A) - \Delta)) \\
&\quad + \Delta(f_1(\Delta + \epsilon^A - \epsilon^B, \pi(A) - \Delta) - f_0(\Delta + \epsilon^A - \epsilon^B, \pi(A) - \Delta))| \\
&\leq (2\delta - \epsilon^B)\pi(A)L_1 + (2\delta - \epsilon^B)(1 - \pi(A))L_0 \\
&\quad + |f_1(\Delta + \epsilon^A - \epsilon^B, \pi(A) - \Delta) - f_0(\Delta + \epsilon^A - \epsilon^B, \pi(A) - \Delta)|
\end{aligned} \tag{21}$$

Finally, since (21) is less than L_{AA+} then the condition is satisfied.

C Figure 3

The 3-equilibrium dynamics in figure 3 are generated with the following f_1 and f_0 functions:

$$\begin{aligned} f_1(\beta(0), \beta(1)) &= 0.5(\beta(1) + \frac{\beta(1)}{5})/1.4 + e^{-10^{-9} \cdot (\beta(0) + \beta(1))} \cdot \sin(18(\beta(0) + \beta(1))) + 0.1 \\ f_0(\beta(0), \beta(1)) &= (\beta(1) + \frac{\beta(1)}{5})/1.2 + 0.01 \end{aligned}$$

The resulting function f is shown in figure 5.

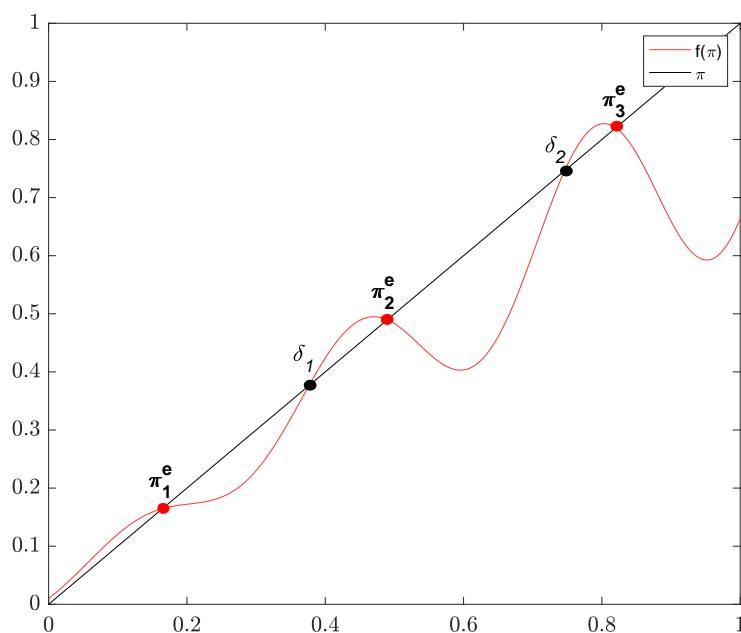


Figure 5: 3-equilibrium dynamic function