Quantum-Inspired Support Vector Machine

Chen Ding,[1, *] Tian-Yi Bao,[2, *] and He-Liang Huang[3, 4, †]

[1]*The School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China*
[2]*Department of Computer Science, College of Literature, Science, and the Arts, University of Michigan, Ann Arbor, MI 48109-2121, USA*
[3]*Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*
[4]*CAS Centre for Excellence and Synergetic Innovation Centre in Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*

Support vector machine (SVM) is a particularly powerful and flexible supervised learning model that analyze data for both classification and regression, whose usual complexity scales polynomially with the dimension and number of data points. Inspired by the quantum SVM, we present a quantum-inspired classical algorithm for SVM using fast sampling techniques. In our approach, we develop a general method to approximately calculate the kernel function and make classification via carefully sampling the data matrix, thus our approach can be applied to various types of SVM, such as linear SVM, poly-kernel SVM and soft SVM. Theoretical analysis shows one can find the supported hyperplanes on a data set which we have sampling access, and thus make classification with arbitrary success probability in logarithmic runtime, matching the runtime of the quantum SVM.

Key Words: quantum-inspired algorithm; SVM; exponential speedup

I. INTRODUCTION

Since 1980s, quantum computing has attracted wide attention due to its enormous advantages in solving some problems, such as integer factorization [1], database search [2], machine learning [3] and so on. In 1997, Daniel R. Simon offered compelling evidence that the quantum model may have significantly more complexity theoretic power than the probabilistic Turing machine [4]. However, it remains an interesting question where is the boundary of the classical computing and quantum computing. Although many proposed quantum algorithms can achieve exponential speedup over the existing classical algorithms, is there really no classical algorithms that can achieve the same computational complexity?

In 2018, inspired by the quantum recommendation system algorithm proposed by Kerenidis and Prakash [5], Ewin Tang designed a classical algorithm to produce a recommendation that can achieve an exponential improvement on previous algorithms [6], which is a breakthrough that show how to apply the subsampling strategy based on FKV algorithm [7] to find a low-rank approximation of a matrix. Subsequently, Tang continued to use same techniques to dequantize two quantum machine learning algorithms, quantum principal component analysis [8] and quantum supervised clustering [9], and showed classical algorithms could also match the bounds and runtime of the corresponding quantum algorithms, with only polynomial slowdown [10]. András Gilyén *et. al.* [11] and Nai-Hui Chia *et. al.* [12] independently and simultaneously proposed a quantum-inspired matrix inverse algorithm with logarithmic computational complexity of matrix size, which eliminates the speedup advantage of the famous HHL algorithm [13] in some certain conditions. Recently, Juan Miguel Arrazola *et. al.* studied the practical performance of quantum-inspired algorithms and found that quantum-inspired algorithms can perform well in practice under given conditions. However, the conditions should be further reduced if we want to apply the algorithms to practical datasets[14]. All of these works give a very bright future for applying the quantum-inspired algorithm into the machine learning area.

In this paper, we want to bring the "magical power" of quantum-inspired methods to the support vector machine (SVM), a data classification algorithm which is commonly used in machine learning area [15, 16]. However, when facing with the big data, a phenomenon called curse of dimensionality describes the complexity and overfitting problem in high dimensional data spaces [17]. In 2014, Patrick Rebentrost *et. al.* proposed a quantum SVM [18], which can achieve an exponential speedup over the classical algorithms. Inspired by the quantum SVM algorithm and Tang's methods [6, 11], we propose a quantum-inspired classical SVM algorithm. The main idea is first transforming the problem of SVM to a linear equations solving problem $X^TX\alpha = y$, where $K = X^TX$ is the kernel matrix, and X

[*]These two authors contributed equally.
[†]Electronic address: quanhhl@ustc.edu.cn

is the data matrix. We note that the quantum-inspired matrix inverse algorithm [11] can't be invoked directly to solve the equations here, since we only have sampling access to the data matrix instead of kernel matrix. Then, we find the approximate singular value decomposition of the kernel matrix K via sampling the data matrix X. Finally we make classification by approximately computing the classification expression, which consists of the solution, the data matrix and the querying point. To avoid a polynomial complexity overhead, we employ methods of sampling dot computation and rejection sampling. In the whole process, we need to avoid the direct operation on vectors or matrices with size as the kernel, in case losing the exponential speedup. Analysis shows that our algorithm can make accurate classification with an appropriate success probability by controlling the computation error, within only logarithmic runtime of dimension and number of data points.

II. SVM

We show the simplest case that the data points are linear separable and leave the other cases to further discussion.

Suppose we have m data points $\{(\vec{x_j}, y_j) : \vec{x_j} \in \mathbb{R}^n, y_j = \pm 1\}_{j=1,...,m}$, where $y_j = \pm 1$ depending on the class to which $\vec{x_j}$ belongs. A SVM finds a pair of parallel hyperplanes $\vec{x} \cdot \vec{w} + b = \pm 1$ that strictly divides the points in two classes depends on the given data. Then for any new input points, it can make classification by its relative position with the hyperplanes. Here $\vec{w}, b$ are parameter of hyperplanes, given by the following optimization problem

$$\begin{gathered} \min \frac{1}{2}\|w\| \\ \text{s.t. } y_i(w^T x_i + b) \geq 1, i = 1, ..., n \end{gathered}$$

for the data is linear separable, as in [18], by taking duel problem we have

$$\begin{gathered} \max \sum_{j=1}^{m} y_i \alpha_i - \frac{1}{2} \sum_{j,k=1}^{m} \alpha_j A_{jk} \alpha_k \\ \text{s.t. } \sum \alpha = 0 \ \& \ y_j \alpha_j \geq 0, i = 1, ..., n \end{gathered}$$

in which $A_{jk} = (X^T X)_{jk} = x_j \cdot x_k$, $X = (x_1, ..., x_m)$. Taking derivation of the objective function, we have

$$X^T X \alpha = y \tag{1}$$

A solution to the optimization problem must be the solution of equation 1. Thus once we find α, we can make classification for any x by $\text{sgn}(x^T X \alpha + b)$, in which $b = y_j - x_j^T X \alpha$ for any $\alpha_j \neq 0$.

III. QUANTUM-INSPIRED SVM ALGORITHM

We use the following notation (see Table I).

TABLE I: Notation

Symbol	Meaning
A	matrix A
y	vector y or matrix y with only one column
A^{-1}	pseudo inverse of A
A^T	adjoint of A
$A_{i\cdot}$	i-th row of A
$A_{\cdot j}$	j-th column of A
$\|A\|$	2-norm of A
$\|A\|_F$	Frobenius norm of A
$O(\cdot)$	time complexity for computing $\cdot$
$Q(\cdot)$	time complexity for querying $\cdot$

There are a main algorithm (algorithm 1) and two sub-algorithms: dot product estimation (algorithm 2) and rejection sampling (algorithm 3) in this paper. Here we discuss the main algorithm that solves the problem and treat the sub-algorithms as oracles that outputs certain outcomes and errors. We first show the idea of sampling, which is the key technique used in our algorithm, as well as in [6, 7, 11].

Definition 1. *Suppose $v \in \mathbb{C}^n$, define $q^{(v)}$ as a probability distribution that:*

$$x \sim q^{(v)}: \quad \mathbb{P}[x = i] = \frac{|v_i|^2}{\|v\|^2}$$

A sampling on $q^{(v)}$ is here called a sampling on v.

Definition 2. *Suppose $A \in \mathbb{C}^{m\times n}$, $a \in \mathbb{R}^m$ a list vector of norm of A's rows, i.e. $a_i := \|A_{i.}\|$. A sample on a getting index i and a sample on $A_{i.}$ is called a row sample on A. Row samples and column samples on A is called samples on A.*

Algorithm 1 Quantum-inspired SVM.

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Input: $m$ training data points of form $\{(\vec{x_j}, y_j) : \vec{x_j} \in \mathbb{R}^n, y_j = \pm 1\}_{j=1,...,m}$, where $y_j = \pm 1$ depending on the class to which $\vec{x_j}$ belongs. Error bound $\epsilon$ and success probability bound $1-\eta$.
Goal 1: Find $\tilde{\alpha}$ that $\|\tilde{\alpha} - \alpha\| \le \epsilon\|\alpha\|$ with success probability at least $1-\eta$, in which $\alpha = (X^TX)^{-1}\vec{y}$.
Goal 2: For any given $\vec{x} \in \mathbb{R}^n$, find its class.
1: Init: Set $r, c$ as described in (3) and (4).
2: Sample columns: Sample $r$ column indices $i_1, i_2, ..., i_r$ according to the column norm squares $\frac{\|X_{\cdot i}\|^2}{\|X\|_F^2}$. Define $\hat{X}$ to be the matrix whose $s$-th column is $\frac{\|X\|_F}{\sqrt{r}}\frac{X_{\cdot i_s}}{\|X_{\cdot i_s}\|}$. Define $\hat{A} = \hat{X}^T\hat{X}$.
3: Sample rows: Sample $s \in [r]$ uniformly, then sample a row index $j$ distributed as $\frac{|\hat{X}_{js}|^2}{\|\hat{X}_{\cdot s}\|^2}$. Sample a total number of $c$ row indices $j_1, j_2, ..., j_c$ this way. Define $\tilde{X}$ whose $t$-th row is $\frac{\|X\|_F}{\sqrt{c}}\frac{\hat{X}_{j_t\cdot}}{\|\hat{X}_{j_t\cdot}\|}$. Define $\tilde{A} = \tilde{X}^T\tilde{X}$.
4: Spectral decomposition: Compute the eigenvalues and eigenvectors of $\tilde{A}$. Denote here as $\tilde{A} = V'\Sigma^2V'^T$, s.t. $V'$ is orthogonal matrix while $\Sigma$ is diagonal matrix with only first $k$ diagonal elements non-zero.
5: Approximate eigenvectors: Let $R = \hat{X}^TX$. Define $V_l = \frac{R^TV_l'}{\sigma_l^2}$ for $l = 1, ..., k$.
6: Estimate matrix elements: Compute $\tilde{\lambda}_l = V_l^T\vec{y}$ to precision $\frac{3\epsilon\sigma_l^2}{8\sqrt{k}}\|y\|$ by algorithm 2, each with success probability $\frac{\eta}{4k}$. Let $u = \sum_{l=1}^k \frac{\tilde{\lambda}_l}{\sigma_l^4}V_l'$.
7: Find sign: Define $\tilde{\alpha} = R^Tu$. Compute $y_j - (\vec{x} - \vec{x_j})^TX\tilde{\alpha}$ to precision $\epsilon\|\alpha\|\|\vec{x} - \vec{x_j}\|$ with success probability $1 - \frac{\eta}{4}$. Tell its sign.
Output: The answer class depends on the sign. Postive corresponds to 1 while negative for $-1$.
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We say we have sample access of X if we can do sample on X. The following theorem is to be proved in section IV and section V.

Theorem 1. *If data matrix $X \in \mathbb{C}^{n\times m}$ satisfies rank$(X) \le k, \|X\| \le 1, \|X^{-1}\| \le \kappa$, and we have sample access of X in logarithmic time on n and m, then algorithm 1 can classify any point x in logarithmic time on n and m with probability at least $1-\eta$.*

IV. ACCURACY

Let $\alpha = (X^TX)^{-1}\vec{y}$, $\alpha' = \sum_{l=1}^k \frac{\lambda_l}{\sigma_l^2}V_l = V\Sigma^{-2}V^T\vec{y}$, in which $\lambda_l = V_l^T\vec{y}$. We show that $\|\tilde{\alpha} - \alpha\| \le \epsilon\|\alpha\|$. In the following two subsections, we show $\|V\Sigma^{-2}V^TA - I_m\| \le \frac{\epsilon}{2}$ and $\|\tilde{\alpha} - \alpha'\| \le \frac{\epsilon}{2}\|\alpha\|$. Then

$$\|\tilde{\alpha} - \alpha\| \le \|V\Sigma^{-2}V^TA\alpha - \alpha\| + \|\tilde{\alpha} - \alpha'\| \le \epsilon\|\alpha\|$$

We compute $(\vec{x} - \vec{x_j})^TX\tilde{\alpha}$ with error $\epsilon\|\alpha\|\|\vec{x} - \vec{x_j}\|$ and success probability $1 - \frac{\eta}{4}$ as mentioned in step 7 of algorithm 1, then the total error is

$$\begin{aligned} E &\le \epsilon\|\alpha\|\|\vec{x} - \vec{x_j}\| + \|(\vec{x} - \vec{x_j})^TX(\tilde{\alpha} - \alpha)\| \\ &\le 2\epsilon\|\alpha\|\|\vec{x} - \vec{x_j}\| \\ &\le 2\epsilon\kappa^2\sqrt{m}\|\vec{x} - \vec{x_j}\| \end{aligned}$$

with success probability $1-\eta$.

For accurate classification, we only need a relative error less than 1. Thus by lessen ϵ, we can achieve this goal in any given probability range.

A. Proof of $\|V\Sigma^{-2}V^TA - I_m\| \leq \frac{\epsilon}{2}$

Here we put 5 theorems(from 2 to 6) for $\|V\Sigma^{-2}V^TA - I_m\| \leq \frac{\epsilon}{2}$, in which theorem 2 and 5 are invoked from [11].

Theorem 2. *Let $\hat{X} \in \mathbb{C}^{n\times r}$ be a matrix and let $\tilde{X} \in \mathbb{C}^{c\times r}$ be the sample matrix that $\mathbb{E}[\tilde{X}^T\tilde{X}] = \hat{X}^T\hat{X}$, then $\forall\epsilon \in [0, \frac{\|\hat{X}\|}{\|\hat{X}\|_F}]$, we have*

$$\mathbb{P}\left[\|\hat{X}^T\hat{X} - \tilde{X}^T\tilde{X}\| \geq \epsilon\|\hat{X}\|\|\hat{X}\|_F\right] \leq 2re^{-\frac{\epsilon^2 c}{4}}$$

Hence, for $c \geq \frac{4\ln(\frac{2r}{\eta})}{\epsilon^2}$, with probability at least $(1-\eta)$ we have

$$\|\hat{X}^T\hat{X} - \tilde{X}^T\tilde{X}\| \leq \epsilon\|\hat{X}\|\|\hat{X}\|_F$$

Theorem 3. *Suppose V_l' is a system of orthogonal vectors while*

$$\tilde{A} = \sum_{l=1}^{k} \sigma_l^2 V_l' V_l'^T.$$

Suppose $\|\hat{A} - \tilde{A}\| \leq \beta$. Then

$$|V_i'^T\hat{A}V_j' - \delta_{ij}\sigma_i^2| \leq \beta$$

Proof.

$$\begin{aligned}|V_i'^T\hat{A}V_j' - \delta_{ij}\sigma_i^2| &\leq |V_i'^T(\hat{A}-\tilde{A})V_j'| + |V_i'^T\tilde{A}V_j' - \delta_{ij}\sigma_i^2| \\ &\leq \|V_i'^T\| \cdot \|(\hat{A}-\tilde{A})V_j'\| \\ &\leq \beta\end{aligned}$$

□

Theorem 4. *Suppose that V_l' is a system of orthogonal vectors that*

$$|V_i'^T\hat{A}V_j' - \delta_{ij}\sigma_i^2| \leq \beta$$

Suppose $\|XX^T - \hat{X}\hat{X}^T\| \leq \epsilon'$, rank$(\hat{X}) = k$, $\frac{1}{\kappa} \leq \sigma_i^2 \leq 1$. Let $V_l = \frac{R^TV_l'}{\sigma_l^2}$, then

$$|V_i^TV_j - \delta_{ij}| \leq \kappa^2((k+1)\beta + \epsilon'),$$

and

$$|V_i^TAV_j - \delta_{ij}\sigma_i^2| \leq \kappa^4(2\epsilon'\kappa^2 + (k^2-2k+2)\beta^3 + (3k-4)\beta^2 + 3\beta).$$

In which $\hat{A} = \hat{X}^T\hat{X}$, $A = X^TX$.

Proof.

$$\begin{aligned}|V_i^TV_j - \delta_{ij}| &= |\frac{V_i'^TRR^TV_j' - \delta_{ij}\sigma_i^4}{\sigma_i^2\sigma_j^2}| \\ &\leq \frac{1}{\sigma_i^2\sigma_j^2}\Big(|V_i'^T\hat{A}\hat{A}V_j' - \delta_{ij}\sigma_i^4| + |V_i'^T(RR^T - \hat{A}\hat{A})V_j'|\Big) \\ &\leq (\sigma_i^2 + \sigma_j^2 + k - 1)\beta + \|\hat{X}^T(XX^T - \hat{X}\hat{X}^T)\hat{X}\| \\ &\leq \frac{1}{\sigma_i^2\sigma_j^2}((\sigma_i^2 + \sigma_j^2 + k - 1)\beta + \epsilon') \\ &\leq \kappa^2((k+1)\beta + \epsilon')\end{aligned}$$

$$\begin{aligned}
|V_i^T A V_j - \delta_{ij}\sigma_i^2| &= \frac{1}{\sigma_i^2\sigma_j^2}|V_i'^T RAR^T V_j' - \delta_{ij}\sigma_i^6| \\
&\le \frac{1}{\sigma_i^2\sigma_j^2}(|V_i'^T(RAR - \hat{A}\hat{A}\hat{A})^T V_j'| + |V_i'^T\hat{A}\hat{A}\hat{A}^T V_j' - \delta_{ij}\sigma_i^6|) \\
&\le \frac{1}{\sigma_i^2\sigma_j^2}(\kappa^2\|XX^TXX^T - \hat{X}\hat{X}^T\hat{X}\hat{X}^T\| + |V_i'^T\hat{A}(\sum_{l_1=1}^{k} V_{l_1}'V_{l_1}'^T)\hat{A}(\sum_{l_2=1}^{k} V_{l_2}'V_{l_2}'^T)\hat{A}V_j' - \delta_{ij}\sigma_i^6|)
\end{aligned}$$

For

$$\begin{aligned}
\|XX^TXX^T - \hat{X}\hat{X}^T\hat{X}\hat{X}^T\| &\le \|XX^T(XX^T - \hat{X}\hat{X}^T)\| + \|(XX^T - \hat{X}\hat{X}^T)\hat{X}\hat{X}^T\| \\
&\le 2\epsilon'
\end{aligned}$$

and

$$\begin{aligned}
|V_i'^T\hat{A}(\sum_{l_1=1}^{k} V_{l_1}'V_{l_1}'^T)\hat{A}(\sum_{l_2=1}^{k} V_{l_2}'V_{l_2}'^T)\hat{A}V_j' - \delta_{ij}\sigma_i^6| &\le (k-1)^2\beta^3 + \sigma_i^2\sigma_j^2\beta + \sigma_i^2((k-1)\beta^2 + \sigma_i^2\beta) \\
&\quad + \sigma_j^2((k-1)\beta^2 + \sigma_j^2\beta) + \beta^2((k^2-3k+3)\beta + k - 2) \\
&\le (k^2-2k+2)\beta^3 + (3k-4)\beta^2 + 3\beta
\end{aligned}$$

Thus

$$|V_i^T A V_j - \delta_{ij}\sigma_i^2| \le \kappa^4(2\epsilon'\kappa^2 + (k^2-2k+2)\beta^3 + (3k-4)\beta^2 + 3\beta)$$

□

Theorem 5. *If rank$(B) \le k$, V has k columns that spans the row and column space of B, then*

$$\|B\| \le \|(V^TV)^{-1}\|\|V^TBV\|.$$

Theorem 6. *Suppose that V_l is a system of approximated orthogonal vectors that*

$$|V_i^T V_j - \delta_{ij}| \le \gamma_1 \le \frac{1}{4k} \tag{2}$$

$$|V_i^T A V_j - \delta_{ij}\sigma_i^2| \le \gamma_2$$

In which $A = X^TX$, rank$(X) = k$, $\|X\| \le 1$, $\|X^{-1}\| \le \kappa$. Then

$$\|V\Sigma^{-2}V^TA - I_m\| \le \epsilon$$

Proof. Let $V = (V_l)_{l=1,\dots,k}$, by (2) we have $\|V^TV - I\| \le k\gamma_1 \le \frac{1}{4}$, thus $\|(V^TV)^{-1}\| \le \frac{4}{3}$.

$$\begin{aligned}
|V_i^T B V_j| &= |\sum_{l=1}^{k} \frac{V_i^TV_l \cdot V_l^TAV_j}{\sigma_l^2} - V_i^TV_j| \\
&\le |\sum_{l=1}^{k} \frac{V_i^TV_l}{\sigma_l^2}(V_l^TAV_j - \delta_{lj}\sigma_l^2)| + |\sum_{l=1}^{k} V_i^TV_l\delta_{lj} - V_i^TV_j| \\
&\le \gamma_2((k-1)\gamma_1\kappa + (\gamma_1+1)\kappa) \\
&\le \frac{5}{4}\gamma_2\kappa
\end{aligned}$$

Let $B = V\Sigma^{-2}V^TA - I_m$ then by theorem 5

$$\|B\| \le \frac{5}{3}\kappa k\gamma_2$$

□

To conclude, for $\mathbb{P}[\|\tilde{\alpha}-\alpha\| > \frac{\epsilon}{2}\|\alpha\|] \le \frac{\eta}{2}$, we need to pick $\eta_1 = \eta_2 = \frac{\eta}{4}$, ϵ' and β such that

$$5k\kappa^5(2\epsilon'\kappa^2 + 4\beta) \le 3\frac{\epsilon}{2}$$

$$4k\kappa^2((k+1)\beta + \epsilon') \le 1$$

$$(k^2 - 2k + 2)\beta^2 + (3k-4)\beta \le 1$$

and decide the sampling parameter as

$$r = \lceil \frac{4\ln(\frac{2n}{\eta_2})}{\epsilon'^2} \rceil \tag{3}$$

$$c = \lceil \frac{4\kappa^2 \ln(\frac{2r}{\eta_1})}{\beta^2} \rceil \tag{4}$$

B. Proof of $\|\tilde{\alpha} - \alpha'\| \le \frac{\epsilon}{2}\|\alpha\|$

For $y = X^T X\alpha$ and $\alpha = X^{-1}X^{-T}y$, we have $\|y\| \le \|\alpha\| \le \kappa^2\|y\|$.

For $\|\tilde{\alpha} - \alpha'\|$, let z be the vector that $z_l = \frac{\lambda_l - \tilde{\lambda}_l}{\sigma_l^2}$, we have

$$\begin{aligned}
\|\tilde{\alpha} - \alpha'\| &= \|\sum_{l=1}^{k} \frac{\lambda_l - \tilde{\lambda}_l}{\sigma_l^2} V_l\| \\
&= \|Vz\| \\
&\le \sqrt{\|V^T V\|}\|z\| \\
&\le \frac{4}{3}\frac{3\epsilon\sigma_l^2}{8\sqrt{k}}\|y\|\frac{1}{\sigma_l^2}\sqrt{k} \\
&\le \frac{1}{2}\epsilon\|\alpha\|
\end{aligned}$$

In which $\|V^T V\| \le \frac{4}{3}$ as shown in proof of theorem 6.

V. COMPLEXITY

A. The spectral decomposition

For $r \times r$ symmetric matrix A, the fastest classical spectral decomposition is through classical spectral symmetric QR method, of which the complexity is $O(r^3)$.

B. Computation of $\tilde{\lambda}_l$

Here we invoke algorithm 2 from [11].

Algorithm 2 Trace inner product estimation.

```
Input: $A \in \mathbb{C}^{m\times n}$ that we have all access in complexity $L(A)$ and $B \in \mathbb{C}^{m\times n}$ that we have query access in complexity $Q(B)$.
    Error bound $\epsilon$ and success probability bound $1-\eta$.
Goal: Estimate $\mathrm{Tr}[A^T B]$ to precision $\xi\|A\|_F\|B\|_F$ with probability at least $1-\eta$.
1: Sample $i$ from row norms of $A$, sample $j$ from $A_i$, let $X = \frac{\|A\|_F^2}{A_{ij}}B_{ij}$.
2: Repeat step 1 $\lceil \frac{9}{\xi^2} \rceil$ times and compute the mean of $X$, note as $Y$.
3: Repeat step 2 $\lceil 6\ln(\frac{2}{\eta}) \rceil$ times and take the median of $Y$, note as $Z$.
Output: $Z$.
```

Easy to find it's complexity is

$$\frac{1}{\xi^2}\ln(\eta)(L(A)+Q(B))$$

For computation of $\tilde{\lambda_l}$ by algorithm 2, we have

$$\lambda_l = \frac{1}{\sigma_l^2} V_l'^T R\vec{y} = \frac{1}{\sigma_l^2}\mathrm{Tr}[V_l'^T \hat{X}^T X\vec{y}] = \frac{1}{\sigma_l^2}\mathrm{Tr}[X\vec{y}V_l'^T \hat{X}^T]$$

Observe that $\|\vec{y}V_l'^T\hat{X}^T\|_F = \|\vec{y}\|\|V_l'^T\hat{X}^T\| \leq \|\vec{y}\|$, and we can query the (i,j) matrix element of $\vec{y}V_l'^T\hat{X}^T$ in cost $O(r)$. Thus the complexity is

$$O((\frac{\|X\|_F\|\vec{y}\|}{\epsilon\|\vec{y}\|/\kappa^4\sqrt{k}})^2\ln(\frac{4k}{\eta})(L(X)+Q(\vec{y}V_l'^T\hat{X}^T))) = O(\frac{\kappa^8 k^2\|X\|_F^2 r}{\epsilon^2}\ln(\frac{4k}{\eta}))$$

C. Computation of $y_j - (\vec{x}-\vec{x_j})^T X\tilde{\alpha}$

1. *Query of* $\tilde{\alpha}$

for any $i=1,...,m$, we have $\tilde{\alpha_i} = \sum_{s=1}^r R_{sj}u_s$. To estimate R_{sj}. We take $R_{sj} = e_s^T\hat{X^T}Xe_j = \mathrm{Tr}[Xe_je_s^T\hat{X^T}]$ and using algorithm 2 computing it to ϵ_1 with success probability η_1 that

$$\epsilon_1 = \frac{\epsilon^3}{4\|X\|_F\ln(\frac{8}{\eta})C}\|\alpha\|\|x-x_j\|$$

$$\eta_1 = \frac{\eta\epsilon^2}{32r\|X\|_F\ln(\frac{8}{\eta})}$$

in which

$$C = \sqrt{r}(\kappa^4\sqrt{k+\kappa^2(k+1)\beta+\kappa\epsilon'} + \frac{3}{8}\kappa^2\epsilon\sqrt{m})$$

ϵ' and β are given in subsection IV A.

Thus for a query of $\tilde{\alpha}$, the error ϵ_2:

$$\begin{aligned}
\epsilon_2 &\leq \epsilon_1(\sum_{s=1}^r u_s)\\
&\leq \epsilon_1\sqrt{r}\|u\|\\
&\leq \epsilon_1\sqrt{r}(\sqrt{\sum_{l=1}^k\frac{\lambda_l^2}{\sigma_l^8}} + \sqrt{\sum_{l=1}^k\frac{|\tilde{\lambda_l}-\lambda_l|^2}{\sigma_l^8}})\\
&\leq \epsilon_1\sqrt{r}(\kappa^4\sqrt{\sum_{l=1}^k\lambda_l^2} + \sqrt{\sum_{l=1}^k\frac{9\epsilon^2\|y\|^2}{64\sigma_l^4 k}})\\
&\leq \epsilon_1\sqrt{r}(\kappa^4\sqrt{\sum_{l=1}^k y^TV_lV_l^Ty} + \frac{3}{8}\kappa^2\epsilon\|y\|)\\
&\leq \epsilon_1\sqrt{r}(\kappa^4\sqrt{\mathrm{Tr}[V_lV_l^T]} + \frac{3}{8}\kappa^2\epsilon\|y\|)\\
&\leq \epsilon_1\sqrt{r}(\kappa^4\sqrt{k+\kappa^2(k+1)\beta+\kappa\epsilon'} + \frac{3}{8}\kappa^2\epsilon\sqrt{m})\\
&\leq \frac{\epsilon^3\|\alpha\|\|x-x_j\|}{4\|X\|_F\ln(\frac{8}{\eta})}
\end{aligned}$$

and the success probability is $1-\eta_2 = 1 - r\eta_1$. The complexity is

$$\begin{aligned} Q(\tilde{\alpha}) &= O(r\frac{\|X\|_F}{\epsilon_1^2\|\alpha\|^2\|x-x_j\|^2}\ln(\frac{1}{\eta_1})) \\ &= O(\frac{16r\|X\|_F^3\ln^3(\frac{8}{\eta})C}{\epsilon^6\|\alpha\|^2\|x-x_j\|^2}\ln(\frac{32r\|X\|_F\ln(\frac{8}{\eta})}{\eta\epsilon^2})) \end{aligned}$$

2. *Computation of j*

We compute j by a sample of $\tilde{\alpha}$ by the following algorithm 3. We don't care about error of j or success probability here because if $|\tilde{\alpha}_j| > \epsilon\kappa^2\sqrt{m} > \epsilon\|\alpha\|$, it suffice to sample another index of $\tilde{\alpha}$. Plus we know we won't always run into small $|\tilde{\alpha}_j|$. The sampling algorithm for $\tilde{\alpha}$ is as follow:

Algorithm 3 Rejection sampling.

```
Input: $A \in \mathbb{C}^{m\times n}$ that we have length-square access and $b \in \mathbb{C}^n$ that we have norm access and $y = Ab$ that we have query access.
Goal: Sample from length-square distribution of $y = Ab$.
1: Take $D \geq \|b\|^2$.
2: Sample a row index $i$ by row norm square of $A$.
3: Query $|y_i|^2 = |A_{i.}b|^2$ and compute $\frac{|A_{i.}b|^2}{D\|A_{i.}\|^2}$.
4: Sample a real number $x$ uniformly distributed in $[0,1]$. If $x < \frac{|A_{i.}b|^2}{D\|A_{i.}\|^2}$, output $i$, else, go to step 2.
Output: The row index $i$.
```

Here we can take D as $\kappa^4\sqrt{k+\kappa^2(k+1)\beta+\kappa\epsilon'} + \frac{3}{8}\kappa^2\epsilon\sqrt{m} \geq \|u\| \geq \|\hat{X}u\|$ and control it within logarithmic range of m by reducing ϵ. Or we can simply compute $\|\hat{X}u\|$ using algorithm 2 and take it as D.

For $\tilde{\alpha} = R^Tu = X^T\hat{X}u$ we need to query $\tilde{\alpha}_i = X_{i.}^T\hat{X}u$ for $\frac{\|X\|_F\|\hat{X}u\|}{\|X^T\hat{X}u\|}$ times on average. If we get D in the first way, the total sampling complexity is

$$\frac{\|X\|_F D}{\|X^T\hat{X}u\|}Q(\tilde{\alpha})$$

3. *Computation of $y_j - (\vec{x}-\vec{x_j})^TX\tilde{\alpha}$*

Once we have index j and query access to $\tilde{\alpha}$, by algorithm 2, we can compute $y_j - (\vec{x}-\vec{x_j})^TX\tilde{\alpha}$ to the assumption in step 7 of algorithm 1

$$\frac{4\|X\|_F}{\epsilon^2}\ln(\frac{8}{\eta})Q(\tilde{\alpha}) + \frac{\|X\|_F D}{\|X^T\hat{X}u\|}Q(\tilde{\alpha}) = O(\frac{16r\|X\|_F^3\ln^3(\frac{8}{\eta})C}{\epsilon^6\|\alpha\|^2\|x-x_j\|^2}\ln(\frac{32r\|X\|_F\ln(\frac{8}{\eta})}{\eta\epsilon^2}))\|X\|_F(\frac{4}{\epsilon^2}\ln(\frac{8}{\eta}) + \frac{D}{\|X^T\hat{X}u\|})$$

which is within the logarithmic range of m and n.

Considering the error and success probability in query process, the total error is

$$\frac{4\|X\|_F}{\epsilon^2}\ln(\frac{8}{\eta})(\sum u_s)\epsilon_1\|\alpha\|\|x-x_j\| + \frac{\epsilon}{2}\|\alpha\|\|x-x_j\| \leq \epsilon\|\alpha\|\|x-x_j\|$$

while the success probability is greater than

$$1 - \frac{\eta}{8} - \frac{4\|X\|_F}{\epsilon^2}\ln(\frac{8}{\eta})r\eta_1 = 1 - \frac{\eta}{4}$$

VI. DISCUSSION

The other way to solve this problem is to solve $X\alpha$ from $X^TX\alpha = y$ by just employing algorithm in [11] and use it in the estimation of $y_j - (\vec{x}-\vec{x_j})^TX\tilde{\alpha}$. Or solve α twice by employing algorithm in [11] twice. Though it works here, it can't deal with further problem like soft SVM because it simply depends on the ability to solve linear equations with sample access on coefficient matrix.

We here give some discussion about the improvements to be made on our algorithm in the future:

A. Sampling for dot product

Remember in algorithm 2 we can estimate dot products for two vectors. However, it doesn't work well for all the conditions, like when $\|\vec{x}\|$ and $\|\vec{y}\|$ are donminated by a coordinate. So for randomness, [19] implies that we can apply a spherically random rotation R to all $\vec{x}$, which doesn't change the kernel matrix K, but will make all the coordinates random variables distributed evenly.

B. Non-linear SVM

When the training data is not linear separable, there is not a pair of parallel hyperplanes that strictly divides the points in two classes depends on the given data. Hence the solution to the original optimization problem does not exist. There're two kinds of improving methods here.

A non-linear SVM improves the fitting ability by changing the kernel function, thus make it possible to classify strictly.

Take poly-kernel SVM for example, we have

$$K(x_i, x_j) = (x_j x_i^T)^d$$

The equation 1 becomes

$$K\vec{\alpha} = \vec{y}$$
$$K = ((x_j x_i^T)^d)_{i,j=1,\dots,m}$$

If we take

$$X = [x_1 \otimes x_1 \otimes \cdots \otimes x_1, x_2 \otimes x_2 \otimes \cdots \otimes x_2, ..., x_m \otimes x_m \otimes \cdots \otimes x_m]$$

Note that X's size is $n^d \times m$, $X_{ij} = x_{j,i/n+1} x_{j,i(mod n)}$, the column norms of X are

$$\|x_k \otimes x_k \otimes \cdots \otimes x_k\|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(x_{ki}x_{kj})^2 = (\sum_{l=1}^{n} x_{kl}^2) = \|x_k\|^4$$

Thus we can sample on X, and for $K = X^T X$, algorithm 1 is still suitable here.

C. Soft SVM

A non-linear SVM may bring overfitting problem while achieving strict classification. Another improving methods is soft SVM, which allows for wrong classification on training data and minimize the offsets.

By introducing a soft variable γ here the equation to solve becomes

$$\begin{pmatrix} 0 & \vec{1}^T \\ \vec{1} & X^T X + \gamma^{-1} I_m \end{pmatrix} \begin{pmatrix} b \\ \vec{\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ \vec{y} \end{pmatrix}$$

We only consider its sub-equation

$$(X^T X + \frac{1}{\gamma} I_m)\vec{\alpha} = \vec{y} \tag{5}$$

For $A = X^T X + \frac{1}{\gamma} I_m$, we have

$$A^{-1} \approx \sum_{l=1}^{k} \frac{1}{\sigma_l^2 + \frac{1}{\gamma}} V_l' V_l'^T$$

Thus to find solution of (5), we only need to add $\frac{1}{\gamma}$ to all the eigenvalues of $\tilde{A}$ in step 4 of algorithm 1 and continue.

VII. CONCLUSION

We have proposed a quantum-inspired SVM algorithm that achieves exponential speedup over the previous classical algorithms. We hope that the techniques developed in our work can promote the emergence of more efficient classical algorithms, such as applying our method to more complex support vector machines [16, 20], and using the method of sampling dot computation as the subroutine for designing a quantum-inspired algorithm according the swap-test related quantum algorithms [21]. Some improvements on our work would be made in the future, such as reducing the conditions on the data matrix and further reducing the computational complexity, which can be achieved through a more subtle investigation on the algorithm and the error transmition process.

We note that our work, as well as the previous quantum-inspired algorithms, are not intended to demonstrate that quantum computing is uncompetitive. We want to find out where the boundaries of classical and quantum computing are, and we expect new quantum algorithms are developed to beat our algorithm.

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[1] P. W. Shor, in *Proceedings 35th Annual Symposium on Foundations of Computer Science* (IEEE, Santa Fe, 1994) pp. 124–134.
[2] L. K. Grover, in *Proceedings of the Twenty-eighth Annual ACM Symposium on Theory of Computing* (ACM, New York, 1996) pp. 212–219.
[3] J. Biamonte, P. Wittek, N. Pancotti, P. Rebentrost, N. Wiebe, and S. Lloyd, Nature **549**, 195 (2017).
[4] D. R. Simon, SIAM Journal on Computing **26**, 1474 (1997).
[5] I. Kerenidis and A. Prakash, in *8th Innovations in Theoretical Computer Science Conference (ITCS 2017)*, Leibniz International Proceedings in Informatics (LIPIcs), Vol. 67 (2017) pp. 49:1–49:21.
[6] E. Tang, Electronic Colloquium on Computational Complexity (ECCC) **25**, 128 (2018).
[7] A. Frieze, R. Kannan, and S. Vempala, J. ACM **51**, 1025 (2004).
[8] S. Lloyd, M. Mohseni, and P. Rebentrost, Nature Physics **10**, 631 (2014).
[9] S. Lloyd, M. Mohseni, and P. Rebentrost, arXiv:1307.0411 (2013).
[10] E. Tang, arXiv:1811.00414 (2018).
[11] A. Gilyén, S. Lloyd, and E. Tang, arXiv:1811.04909 (2018).
[12] N.-H. Chia, T. Li, H.-H. Lin, and C. Wang, arXiv:1901.03254 (2019).
[13] A. W. Harrow, A. Hassidim, and S. Lloyd, Phys. Rev. Lett. **103**, 150502 (2009).
[14] J. M. Arrazola, A. Delgado, B. R. Bardhan, and S. Lloyd, arXiv:1905.10415 (2019).
[15] P. J. Phillips, in *Advances in Neural Information Processing Systems* (1999) pp. 803–809.
[16] J. A. K. Suykens and J. Vandewalle, Neural Processing Letters **9**, 293 (1999).
[17] R. Bellman, Science **153**, 34 (1966).
[18] P. Rebentrost, M. Mohseni, and S. Lloyd, Phys. Rev. Lett. **113**, 130503 (2014).
[19] D. Achlioptas, F. McSherry, and B. Schölkopf, in *Advances in Neural Information Processing Systems* (2002) pp. 335–342.
[20] L. Wang, *Support vector machines: theory and applications*, Vol. 177 (Springer Science & Business Media, 2005).
[21] N. Wiebe, D. Braun, and S. Lloyd, Phys. Rev. Lett. **109**, 050505 (2012).