Fair and Useful Cohort Selection

Niklas Smedemark-Margulies*
Northeastern University
niklas@ccs.neu.edu

Paul Langton*
Northeastern University
langtonp@ccs.neu.edu

Huy Lê Nguyễn
Northeastern University
hu.nguyen@northeastern.edu

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Abstract

As important decisions about the distribution of society's resources become increasingly automated, it is essential to consider the measurement and enforcement of fairness in these decisions. In this work we build on the results of Dwork and Ilvento in [1], which laid the foundations for the study of fair algorithms under composition. In particular, we study the cohort selection problem, where we wish to use a fair classifier to select k candidates from an arbitrarily ordered set of size $n > k$, while preserving individual fairness and maximizing utility. We define a linear utility function to measure performance relative to the behavior of the original classifier. We develop a fair, utility-optimal $O(n)$-time cohort selection algorithm for the offline setting, and our primary result, a solution to the problem in the streaming setting that keeps no more than $O(k)$ pending candidates at all time.

1 Introduction

Many aspects of life are now determined by automated systems, leading to increasing concerns about the fairness of such systems [2] [3]. Following the pioneering work of Dwork *et al.* [4], a large body of work has been developed simultaneously in the machine learning, theoretical computer science and economics communities. There are two common families of notions regarding fairness: individual fairness and group fairness. In this work, we focus on the setting of individual fairness. Intuitively, individual fairness requires the algorithm under consideration to treat similarly-qualified individuals (using a task-specific metric) similarly. Under this notion, algorithms have been developed for many tasks that are individually fair and efficient. However, in practice, most systems are complex and consist of many separate building blocks. A natural question is whether it is possible to combine fair building blocks into a fair composition. Dwork and Ilvento initiated a rigorous study of this topic in [1]. Interestingly, even for simple compositions, the competition among different tasks for the same individual and the competition among individuals for the same task already leads to *dependent* outcomes, affecting fairness. This situation arises naturally, for example, when there is a limited resource to distribute and individuals must be considered in order of appearance (see Example 1).

Example 1 (Cohort Selection)**.** *Suppose we are building an automated selection system that will select k candidates by the end of an application period. To ensure our system is useful, we wish to maximize the benefit of the cohort, but also ensure each applicant is evaluated solely on their relevant qualifications for the task.*

More precisely, our goal is to select k individuals from a universe U of n candidates. To do so, we are given access to a fair classifier C to evaluate each candidate. It is important to note that the fair classifier is oblivious to the limit k and thus, if more than k candidates are qualified, we must to enforce the limit in a fair way. This problem was previously studied in [1] as an archetype of fairness under composition. They developed several algorithms for the offline setting, i.e. where all individuals are available for consideration at the same time, and the online setting, i.e. when the individuals appear one by one and the system needs to make an irrevocable decision before seeing the next person.

When considering fairness, we can always compare to a simple baseline: ignore the classifier C and select k random individuals. In some settings, this baseline shows that a fair algorithm trivially exists, and in some settings, showing that this baseline cannot be implemented can suggest that the problem has no fair solution.

In this work, our contribution is twofold:

*Equal contribution

- **The offline setting.** When multiple algorithms exist, to differentiate them from the baseline, we consider a natural notion of utility and develop a new fair algorithm achieving the optimal utility. Interestingly a simple example shows that even in the offline setting, the algorithms from [1] might lose a constant factor compared with the optimal utility.
- **The streaming setting.** The work [1] shows that no fair online algorithm exists (even the baseline of uniform random selection is not implementable). We develop a relaxed model where decisions need not be immediate, but at any given point in time, the number of pending decisions must be as small as possible, thus retaining the efficiency of online algorithms. We give an algorithm for fair cohort selection which leaves no more than $O(k)$ candidates pending at any given step and achieves the optimal utility.

1.1 Related Work

The baseline algorithm mentioned above, when implemented in the online setting, is precisely the canonical reservoir sampling algorithm. When also taking the classifier results into account, the problem is related to weighted reservoir sampling (see for example [5] and references therein). Interestingly, the special case $k = 1$ is exactly weighted reservoir sampling, but even for $k = 2$, the problem is already very different from weighted reservoir sampling with replacement.

In Dwork and Ilvento's [1] discussion on equal degrees of composition, it is apparent that theoretically fair systems can lose a great deal of their usefulness in practice. The authors propose utility as an important metric for understanding the loss incurred due to enforcing fairness. [6] discusses the implications of composing relaxed-fair systems. It identifies scenarios where the composed system can be fair, but does not focus heavily on the tradeoffs made to achieve fairness. This makes it difficult for potential implementors of fair algorithms to assess the costs and risks. Armed with the tools for utility described in this paper, [6] could provide critical understanding for the reader about the practical implications of enforcing the fairness they describe.

We make heavy use of the individual fairness definition in [4] (refined in [1]), who source their inspiration for fairness formalisms directly from [7].

Relevance, impacts, and importance of algorithmic fairness in the current cultural context are discussed throughout our works cited, but especially in [3], [8]. [9], for example, underscores the importance of algorithmic fairness in analyzing recidivism.

2 Background

In this work we consider a set of n individuals from a universe U, where we need to select exactly k individuals fairly based on the output of an individually fair classifier C. In this section we review the definition of fairness we are considering, build a measure of utility that expresses the discriminative power of C, and develop a fundamental method for later use, which we call integer exact marginal rounding (EMR).

2.1 Notation

We will use Y to denote a set or list, y to denote an element of that list, and y_i for the ith element of Y. A call to a function named "example" will appear in monospaced font. We include a short index of commonly used symbols and their meanings for quick reference at the end of the paper.

2.2 Fairness

We follow the definition of individual fairness proposed by Dwork *et al* in [4].

Definition 1 (Individual Fairness)**.** *For a given metric D over a universe of individuals to be classified U, and a random classifier $C : U \to \{0, 1\}$. We say that the classifier is **individually fair** with respect to that metric if and only if, $\forall u, v \in U$,*

$$D(u, v) \geq |Pr[C(u) = 1] - Pr[C(v) = 1]|$$

2.3 Utility

We begin by establishing a basic notion of utility for any group of candidates.

Even as we try to measure and enforce fair outcomes from some classification procedure, we must also define and optimize a utility objective to maintain the ability to distinguish between well-qualified and poorly-qualified candidates.

Without a utility objective, we can almost always resort to trivially "fair" solutions, such as treating all individuals the same and using no information about their qualifications. This approach is intuitively not useful, because there are valid reasons for distinguishing between individuals, and a fair allocation of resources need not be uniform (discussed in Related Work and further in [1]).

Instead, we consider a simple linear utility function.

Definition 2 (Utility)**.** *Consider a set of candidates W whose individual probabilities of selection by C are $\vec{s} = s_1, \ldots, s_n$, and with utilities $\vec{u} \in [0,1]^n$ and $\sum_i u_i > 0$. We define the utility of W to be*

$$\sum_{i=0}^{n} s_i u_i \tag{1}$$

Even though we would like to optimize the utility, following [1], we consider only composition algorithms that do not have access to the utility values $\vec{u}$. In many cases, utility may be derived from metrics not relevant to the task itself. For example, utility may be gained from team synergy, role model status, company culture fit, diversity, etc. Thus this is a natural setting to consider. Intuitively, in this setting, the algorithm needs to optimize for the worst case and tries to match the probabilities of C as closely as possible. We formalize this intuition in the following simple definition and lemma.

Definition 3 (Optimal Utility Cohort Selection)**.** *An algorithm A with selection probabilities $p_1, \ldots, p_n$ achieves optimal utility for cohort selection on $\vec{s}$ if it is a solution to*

$$\max_{alg} \min_{\vec{u}} \frac{\sum_i p_i u_i}{\sum_i s_i u_i} \tag{2}$$

The following lemma shows that we can optimize the worst case utility, even when the composition algorithm only has access to the probabilities s_i from classifier C, and does not have access to the utilities $\vec{u}$.

Lemma 1.

$$\min_{\{\vec{u}\}} \frac{\sum_{i=0}^{n} p_i u_i}{\sum_{i=0}^{n} s_i u_i} \leq \min_i \frac{p_i}{s_i} \tag{3}$$

Proof. Let j be the coordinate with the smallest ratio between p_j and s_j: $p_j/s_j \leq p_i/s_i, \forall i \neq j$. Consider the $\vec{u}^*$ vector with $u_j^* = 1$, and $u_i^* = 0, \forall i \neq j$. Then we have:

$$\begin{aligned} \min_{\vec{u}} \frac{\sum_{i=0}^{n} p_i u_i}{\sum_{i=0}^{n} s_i u_i} &\leq \frac{\sum_{i=0}^{n} p_i u_i^*}{\sum_{i=0}^{n} s_i u_i^*} \\ &= \frac{p_j u_j^*}{s_j u_j^*} = \frac{p_j}{s_j} \\ &= \min_i \frac{p_i}{s_i} \end{aligned}$$

□

Thus we see that the ratio of probabilities at the most extreme coordinate gives an upper bound on the utility that any algorithm can achieve, relative to the original utility. In order to improve the worst case utility, an algorithm must improve its most extreme coordinate.

2.4 Dependent Rounding

In this section we develop a solution for Fair Useful Cohort Selection in the special case where the probabilities from C sum up exactly to the desired number of selection k. The solutions for the general case of both the offline and streaming settings will build on this subroutine. The solution to this special case is a simple dependent rounding, but it is worth noting that several common strategies fail to achieve the optimal utility in it.

Example 2. *Consider the example with 3 candidates* $s_1 = 0.5, s_2 = 0.5, s_3 = 1$ *and we would like to select* $k = 2$ *individuals. The optimal solution is to pick* $\{1, 3\}$ *and* $\{2, 3\}$ *each with probability* 0.5*. The WeightedSampling algorithm of [1] selects each subset with probability proportional to its weight, and would select* $\{1, 2\}$ *with probability* $1/4$*,* $\{1, 3\}$ *with probability* $3/8$*, and* $\{2, 3\}$ *with probability* $3/8$*, losing a factor of* $3/4$ *in utility in the worst case. Weighted reservoir sampling with replacement selects* $\{1, 2\}$ *with probability* $1/6$*,* $\{1, 3\}$ *with probability* $5/12$*, and* $\{2, 3\}$ *with probability* $5/12$*, losing a factor of* $5/6$ *in utility in the worst case.*

Our procedure, called Exact Marginal Rounding (EMR), randomly rounds a list of probabilities that sum to $k \in \mathbb{Z}$ into a list of integers so that the expected value of each element after rounding is equal to its original marginal probability. The algorithm iteratively rounds two fractional entries at a time until there is no fractional entry left. Suppose the two fractional values are a and b. If $a + b \leq 1$, the algorithm randomly rounds one of them to $a + b$ and the other one to 0 with appropriate probabilities to preserve the marginal probabilities. The case $a + b > 1$ is analogous.

Notationally, when we use this tool later, we will either use "EMR($list$)" or "EMR($list$, 1)", referring to using Integer Exact Marginal Rounding in Lemma 2 on $list$, or we will use "EMR($list$, a)", referring to using the Non-Integer Exact Marginal Rounding in Corollary 2, up to a value of $a < 1$.

Algorithm 1: Integer Exact Marginal Rounding

input : Array S of length N s.t. $S[i] \in [0, 1]$, $\sum S = K$
output: $\tilde{S} \in \{0, 1\}^N$

set $pendingIndex$ to 1
for i *from* 0 *to* $N - 1$ **do**
  if $i == pendingIndex$ **then**
    continue to next i
  set a to $S[i]$
  set b to $S[pendingIndex]$
  set u to `Unif(0,1)`
  if $a + b \leq 1$ **then**
    `// One gets all, other gets 0`
    if $u < \frac{a}{a+b}$ **then**
      set $\tilde{S}[i]$ to $a + b$
      set pendingIndex to i
      set $\tilde{S}[i + 1]$ to 0
    else
      set $\tilde{S}[i]$ to 0
      set pendingIndex to $i + 1$
      set $\tilde{S}[i + 1]$ to $a + b$
  else
    `// One gets 1, other gets remainder`
    if $u < \frac{1-b}{2-a-b}$ **then**
      set $\tilde{S}[i]$ to 1
      set pendingIndex to $i + 1$
      set $\tilde{S}[i + 1]$ to $a + b - 1$
    else
      set $\tilde{S}[i]$ to $a + b - 1$
      set pendingIndex to i
      set $\tilde{S}[i + 1]$ to 1
return S

Lemma 2 (Integer Exact Marginal Rounding)**.** *Let $s_1 \dots s_n \in [0,1]$ be a list of probabilities with $\sum s_i = k \in \mathbb{N}$. Algorithm 1 rounds each s_i such that k elements will be equal to 1, $n-k$ elements will be equal to 0, and $\mathbb{E}[EMR(s_i)] = s_i$.*

Proof. We first show that $\mathbb{E}[EMR(s_i)] = s_i, \forall i$.

At the current step i, let the two candidates under consideration be A and B, and let their initial probability of selection be a and b, and their adjusted value after the current step be $\tilde{a}$ and $\tilde{b}$. We must consider two cases:

Case 1: $a + b \leq 1$. With probability $\frac{a}{a+b}$, we set $\tilde{a} = a + b$, and with probability $1 - \frac{a}{a+b}$, we set $\tilde{a} = 0$. Thus, we have for individual A:

$$\begin{aligned}\mathbb{E}[\tilde{a}] &= \left(\frac{a}{a+b}\right)(a+b) + \left(1 - \frac{a}{a+b}\right)(0) \\ &= a\end{aligned}$$

The same holds analogously for B.

Case 2: $1 < a + b \leq 2$. With probability $\frac{1-b}{2-a-b}$, we set $\tilde{a} = 1$, and with probability $1 - \frac{1-b}{2-a-b} = \frac{1-a}{2-a-b}$, we set $\tilde{a} = a + b - 1$. Thus we have for individual A:

$$\begin{aligned}\mathbb{E}[\tilde{a}] &= \left(\frac{1-b}{2-a-b}\right)(1) + \left(\frac{1-a}{2-a-b}\right)(a+b-1) \\ &= \frac{(1-b) + (a+b-1) - a(a+b-1)}{2-a-b} \\ &= a\end{aligned}$$

Again, this holds analogously for B.

Next, we show that there are exactly k items equal to 1 and $n - k$ items equal to 0. Note that at every step of the process, either one of the elements was already integer, or the number of integer elements increases by at least 1 (in a rare case, $a + b = 1$ and both elements become integer in a single step). We take n steps, and therefore the final list $\tilde{S}$ contains n integer elements. Furthermore, notice that probability mass is conserved during this process, so that the final list $\tilde{S}$ still sums to k. Since $\tilde{S}$ contains all integer elements and sums to k, then we know that at the end there are exactly k individuals with 1 and the other $n - k$ have 0. □

Corollary 1. *Alg 1 is individually fair.*

Proof. By Lemma 2 each individual is rounded with exactly their original marginal probability. □

Corollary 2 (Non-Integer Exact Marginal Rounding)**.** *Let $s_1 \dots, s_n \in [0, \beta], \beta < 1$ be a list of probabilities whose sum is equal to βK. We can use Alg 1 to round each s_i such that K elements will be equal to β and $n - k$ elements will be equal to 0, and $\mathbb{E}[EMR(s_i)] = s_i$.*

Proof. Note that there is a 1-to-1 mapping to an input for Alg 1; we simply scale all entries s_i by $1/\beta$. We know that Algorithm 1 results in a rounded list $\tilde{S}$ containing exactly k elements equal to 1, so we can then map back to a rounded list containing exactly k elements equal to β. □

3 Offline Cohort Selection

We begin with the scenario of selecting k of n individuals when all candidates are known in advance. Considering Example 1, this corresponds to the idealized case in which all candidates simultaneously submit their applications and are evaluated in a single batch. Dwork and Ilvento give a solution to this scenario in [1] (with a constant factor lost in utility as shown above); we present an alternate one that achieves the optimal utility.

In contrast with the above special case, the sum of probabilities given by $\sum_i s_i$ need not add up to k. If the sum exceeds k, the algorithm simply scales down all probabilities so that the new sum is k and it is not hard to show that this operation preserves fairness. The harder case is when the sum is smaller than k. Scaling up the probabilities is not possible since it increases the gap among candidates and can be unfair. Intuitively, the solution is to additively increase all probabilities by the same amount, thus preserving the gap and the fairness. However, some care is needed, as no probability can exceed 1.

Algorithm 2: Useful Offline Cohort Selection

input : S: a list of $Pr[C(w_i) = 1]$ for candidates $w_1, \ldots, w_n$
k: number of individuals that must be selected
output: cohort: list of k indices corresponding to chosen candidates

$sum \leftarrow \sum_i s_i$
set P to S
if $sum < k$ **then**
  set c to $\frac{k-sum}{n}$
  set p_i to $p_i + c, \forall i \in \{1, \ldots, n\}$
  while $\exists p_i > 1$ *in* P **do**
    distribute $(p_i - 1)$ uniformly to all $p_j < 1$ in P
    set p_i to 1
  // Now the list sums to exactly k and each item is in $[0, 1]$
  set *rounded* to `EMR(`P`)`
  set *cohort* to all $x \in$ *rounded* with $x = 1$
else
  set P to $P \cdot \frac{k}{sum}$
  set *rounded* to `EMR(`P`)`
  set *cohort* to all $x \in$ *rounded* with $x = 1$
return cohort

Table 1: Behavior of Alg 2

n = 5, k = 2, $\sum_i s_i < k$, no candidates adjusted to 1

s_i value	0.1	0.1	0.2	0.2	0.9
$A(s_i)$ output	0.2	0.2	0.3	0.3	1.0

n = 5, k = 2, $\sum_i s_i < k$, some candidates adjusted to 1

s_i value	0.1	0.1	0.2	0.3	0.8
$A(s_i)$ output	0.2	0.2	0.3	0.4	0.9

n = 5, k = 1, $\sum_i s_i > k$

s_i value	0.3	0.2	0.5	0.4	0.6
$A(s_i)$ output	0.15	0.1	0.25	0.2	0.3

Problem 1 (Fair Offline Cohort Selection)**.** *Given a set of candidates W, $|W| = n$, with utilities $\vec{u}$ and an individually fair classifier C, where $Pr[C(w_i) = 1] \equiv s_i$, select exactly k candidates from W while maintaining fairness with respect to C.*

Lemma 3. *If $sum < k$, Algorithm 2 increases each probability s_i to $p_i = s_i + \alpha_i \geq s_i$ and all of the candidates j with $p_j < 1$ receive the same adjustment value $\alpha_j = v$.*

Proof. First we observe that $p_i \geq s_i, \forall i$. At each step of the algorithm the value of p_i either increases, or is clipped at 1. Since $s_i \in [0, 1], \forall i$, we therefore know that $p_i \geq s_i$.

Algorithm 2 initially adds a constant amount c to every candidate, then repeatedly redistributes overflow evenly across all candidates with $s_i + \alpha_i < 1$. We prove by induction that for all i with $p_i < 1$, their increment α_i are all the same. Consider two candidates i, j such that their final $p_i, p_j < 1$. As argued above, both p_i, p_j are smaller than 1 throughout the execution of the algorithm. At the beginning, $\alpha_i = \alpha_j = c$ so the claim holds.

For the inductive step, assume that after step d of redistribution, we have $\alpha_i = p_i - s_i = \alpha_j = p_j - s_j$. We will argue that this equality still holds after step $d+1$. Suppose that in step $d+1$, the algorithm distributes an overflow value x to m candidates. As noted before, i, j are among these m candidates. Thus, both p_i and p_j are increased by x/m and $\alpha_i = \alpha_j$ after step $d + 1$. □

Theorem 1. *Alg 2 is individually fair.*

Proof. By assumption, the s_i we are given are from an individually fair C, thus $\mathcal{D}(u,v) \geq |\mathbb{E}[C(u)] - \mathbb{E}[C(v)]|, \forall u, v \in U$. To show Alg. 2, is individually fair, we must either preserve or decrease the difference between any pair of individuals. Let $A(u)$ for $u \in U$ denote the probability that u is selected by Alg. 2. Thus we must show, for all (u, v):

$$|\mathbb{E}[C(u)] - \mathbb{E}[C(v)]| \geq |A(u) - A(v)|$$

We apply C to a set of candidates from U and obtain two disjoint sets of output candidates: let $S_{(=1)}$ be the set of candidates whose output $A(s_i) = 1$, and let $S_{(<1)}$ be the set of candidates whose output $A(s_i) < 1$. Let S_{all} be the union of these two disjoint sets.

1. First, consider when $\sum_i s_i < k$. Here, each element $s_i \in S_{all}$ is increased by an amount α_i s.t. $\sum_i (s_i + \alpha_i) = k$. Then we apply EMR, where we know by Lemma 2 that Pr[EMR selects $s_i + \alpha_i$] $= s_i + \alpha_i$. We thus need to cover three cases.

   (a) First, we consider two elements $s_i, s_j \in S_{(=1)}$

   $$\begin{aligned} |A(s_i) - A(s_j)| &= |s_i + \alpha_i - (s_j + \alpha_j)| \\ &= |1 - 1| \leq |s_i - s_j| \end{aligned}$$

   (b) Next, we have $s_i, s_j \in S_{(<1)}$ By lemma 3, these candidates all receive the same adjustment. Let this adjustment be called b.

   $$\begin{aligned} |A(s_i) - A(s_j)| &= |s_i + b - (s_j + b)| \\ &= |s_i - s_j| \end{aligned}$$

   (c) Finally we have $s_i \in S_{(=1)}, s_j \in S_{(<1)}$. Note that $\alpha_i - \alpha_j \leq 0$, and $s_i - s_j > 0$.

   $$\begin{aligned} |A(s_i) - A(s_j)| &= |s_i + \alpha_i - (s_j + \alpha_j)| \\ &= |s_i - s_j + \alpha_i - \alpha_j| \\ &\leq |s_i - s_j| \end{aligned}$$

2. Next, consider $\sum_i s_i \geq k$. Here we know $\frac{k}{\sum_i s_i} \leq 1$. By Lemma 2, $A(s_i) = \frac{ks_i}{\sum_i s_i}$. Then, for candidates i and j, we have:

   $$\begin{aligned} |A(s_i) - A(s_j)| &= |\frac{ks_i}{\sum_i s_i} - \frac{ks_j}{\sum_i s_i}| \\ &= |\frac{k}{\sum_i s_i}(s_i - s_j)| \\ &\leq |s_i - s_j| \end{aligned}$$

Thus A is individually fair. □

Theorem 2. *Alg 2 achieves optimal utility for Fair Useful Offline Cohort Selection.*

Proof. We must show that Alg. 2 is a solution to

$$\max_{alg} \min_i \frac{p_i}{s_i} \tag{4}$$

Let $S_{(<1)}$ and $S_{(=1)}$ be defined as in Theorem 1, and note that $S_{(=1)}$ may be empty. Consider for contradiction an arbitrary fair algorithm A' with probabilities of selection p'_i s.t.

$$\min_i \frac{p'_i}{s_i} > \min_i \frac{p_i}{s_i}$$

1. First, consider the case where $\sum_i s_i < k$ and $S_{(=1)}$ is nonempty. Let individual j have the most extreme ratio: $\frac{p_j}{s_j} = \min_i \frac{p_i}{s_i}$. In order for this ratio to be minimal, we know that individual j must have $s_j > s_i \forall i$ and be a member of $S_{(=1)}$. Therefore $s_j + \alpha_j = 1$, and we have that for Algorithm A, $\min_i \frac{p_i}{s_i} = \frac{s_j + \alpha_j}{s_j} = \frac{1}{s_j}$.

   By our definition of A', there is some potentially different most extreme element l s.t. $\frac{p'_l}{s_l} = \min_i \frac{p'_i}{s_i} > \frac{p_j}{s_j}$.

- If $s_l = s_j$, the largest minimum ratio $\frac{p'_l}{s_l}$ that A' can achieve is $\frac{1}{s_l} = \frac{1}{s_j}$, which is the same ratio as Algorithm A and contradicts our assumption.
- If $s_l \neq s_j$, we know $s_l < s_j$, since we already know that s_j is at least as big as all other elements. Then, by fairness, $\alpha'_j \leq \alpha'_l$. Then we can construct the following:

$$\begin{aligned} \alpha'_l/s_l &> \alpha'_j/s_j \\ 1 + \alpha'_l/s_l &> 1 + \alpha'_j/s_j \\ (s_l + \alpha'_l)/s_l &> (s_j + a'_j)/s_j \end{aligned} \tag{5}$$

  This contradicts our assumption that $\frac{p'_l}{s_l}$ is the minimum ratio for A'.

2. Next, consider the case where $\sum_i s_i < k$ and $S_{(=1)}$ group is empty. Again, let element j be the most extreme element for algorithm A, i.e. $\min_i \frac{p_i}{s_i} = \frac{p_j}{s_j}$.

   Consider A' on its most extreme element l constructed as in Item 1.

   (a) This time, if $s_l = s_j$, we may have $p'_l = 1 > p_j$, achieving greater utility. As before, $s_j \geq s_i, \forall i \neq j$. By fairness, if A' adjusts element l by a certain amount, it must adjust all smaller elements by at least the same amount. However, we know that $\sum_i p_i = k$, so this required extra adjustment will mean that $\sum_i p'_i > k$, and A' will not be selecting exactly k candidates.

   (b) If $s_l < s_j$, then we still know that $a'_j \leq a'_l$, and we again have the contradiction in (5)

3. Finally, consider the case where $\sum_i s_i \geq k$.

   In alg 2, all elements will be multiplied by a factor of $\frac{k}{\sum_i s_i}$, and then by Lemma 2, any element will satisfy $p_i = \frac{ks_i}{\sum_i s_i}$, so we have:

$$\min_i p_i/s_i = \min_i \left(\frac{ks_i}{\sum_i s_i}\right)/s_i = \frac{k}{\sum_i s_i}$$

   Notice that in this case, all elements are adjusted by the same constant ratio. Using extreme element l for algorithm A' as constructed previously, and again supposing that this element is more extreme than any element in algorithm A, we derive the following.

$$\begin{aligned} \min_i \frac{p'_i}{s_i} = \frac{p'_l}{s_l} &> \frac{k}{\sum_i s_i} \\ p'_l &> s_l \frac{k}{\sum_i s_i} \end{aligned}$$

   Since element l was the minimum ratio for algorithm A', it follows that for all i,

$$\begin{aligned} p'_i/s_i &> \left(\frac{k}{\sum_i s_i}\right) \\ p'_i &> s_i \left(\frac{k}{\sum_i s_i}\right) \end{aligned}$$

   Taking the sum on both sides we see

$$\sum_i p'_i > \sum_i s_i \left(\frac{k}{\sum_i s_i}\right) = k$$

   As before, the number of elements chosen by algorithm A' will be more than k, which is a contradiction.

Thus we conclude A' cannot exist and Alg. 2 achieves optimal utility for Fair Offline Cohort Selection in all cases. □

4 Streaming Cohort Selection

We now consider the streaming scenario, in which decisions are made before all candidates have been seen. Dwork *et al.* [1] show that in the true online setting, where decisions must be made before next candidate is observed, a fair selection is impossible. Specifically, consider an individually fair algorithm that receives a stream of exactly k individuals; it must clearly select all k individuals in order to complete the cohort selection task. Now, let the algorithm receive an identical stream, followed by one additional candidate. We know it must select the first k candidates, based on its behavior on the previous stream; however this means that the $k+1$st candidate has no chance to be chosen, violating individual fairness. Therefore, we know the requirement for strict online decision making must be relaxed to make progress on this problem.

Motivated in part by the deferred-acceptance algorithm in the context of matching [10], we consider the setting where candidates are tentatively accepted but might be rejected later on as more candidates come. To maintain the efficiency of online algorithms, our goal is to minimize the number of candidates with a pending decision at all points in time. It is clear that the number of candidates on hold must be at least k since we need to accept k candidates at the end.

Problem 2 (Fair Useful Streaming Cohort Selection)**.** *Consider the fair classifier C and candidates $W = \{w_1 \dots w_n\}$, with utilities x_i. Given π, a stream of classifications from C on $w_i \in W$, select exactly k candidates from π while maximizing utility, respecting individual fairness, and with as few candidates pending as possible.*

Below, we provide an algorithm that achieves optimal utility as defined in 3 for the Streaming Cohort Selection problem, while leaving only $O(k)$ candidates pending. The algorithm uses a parameter $\alpha \in [0, 1/2]$. The number of candidates pending will depend on α and we can set α to minimize this number; we will pick $\alpha = 1/2$ by default. The algorithm contains three update rules which are used depending on the sum of candidates seen so far.

4.1 Algorithm Overview

The algorithm maintains a set of k/α candidates with highest probabilities called `top` and a set `rest` of the remaining candidates. As a new candidate s appears, s is added to either `top` (and thus bumps some other candidate from `top` to `rest`), or they are added to `rest`. The algorithm needs to make sure the list `rest` is not too big by rounding and eliminating candidates. It is tempting to use the idea from the offline case: take two candidates from `rest` with probabilities a and b and round them to $a+b$ and 0. However, if at the end of the stream, the total probabilities is less than k and we need to increase all probabilities, then some probability might exceed 1. The key idea (and motivation for having `top`) is that if the total is less than k, the probabilities for candidates not in the top k/α is at most α. Furthermore, since the adjustment to make the total probability equal to k preserves the ordering of the probabilities, their increments are also at most α. Thus, it is safe to round probabilities in `rest` to 0 or $1-\alpha < 1$.

At the end of the stream, if the sum of probabilities is less than k then we need to increase all probabilities. Observe that all candidates in `rest` have the same increment (due to the additive adjustment procedure, and the fact that all their probabilities are less than 1 after the increment). We can compute exactly the increment for the probabilities in `top` and the increment for all probabilities outside of `top`. Before the increment, we store all candidates with non-zero probabilities in `top` $\cup$ `rest`; refer to this set as A. After the increment, we still have the probabilities of everyone in A and we know everyone in $W \setminus A$ has the same probability (the value of their increment). The second key idea is that we can round the probabilities in $W \setminus A$ by maintaining a set called `randoms` consisting of k uniformly random samples from $W \setminus A$. Each member of `randoms` receives probability equal to $1/k$ times the total probability of $W \setminus A$. Since everyone in $S \setminus A$ has the same probability, this rounding clearly preserves the marginal probabilities. Finally, we can use the offline algorithm to select the cohort from the union of `randoms` and A.

The case where the sum of probabilities exceeds k is simpler. The algorithm always scales down all probabilities so that they add up to exactly k. When a new candidate appears, they receive a probability equal to what C gives them, times the current scaling factor, and get added to `top` $\cup$ `rest`. The algorithm then scales down all probabilities so that the sum is exactly k and reduces the size of `top` $\cup$ `rest` by repeatedly rounding pairs of candidate as in the offline setting. In this case, the list `randoms` is not used.

We present the algorithm as a series of update rules to be applied when a new stream element s is encountered. The algorithm keeps track of the sum of the probabilities encountered so far (including the new element s) and performs different updates depending on the comparison between the sum and k. At the end of the stream, depending on the comparison between the sum and k, the algorithm also selects the cohort among the pending candidates differently.

4.2 Before k

Algorithm 3: Streaming Update for new element s with $sum + s < k$

set sum to $sum + s$
if *`top` has fewer than* $\lceil k/\alpha \rceil$ *elements* **then**
  add s to `top`
else if *minimum element in `top`* $< s$ **then**
  remove minimum element from `top`, add it to `rest`
  add s to `top`
else
  add s to `rest`
set *rounded* to `EMR(rest,` $1 - \alpha$`)`
for x *with probability* 0 *in rounded* **do**
  feed x to uniform random reservoir sampling of size k
  set `randoms` to this sample
set `rest` to all x with probability > 0 in rounded
if π *ends* **then**
  go to Alg 4

Algorithm 4: Stream End Adjustment, $sum < k$

set P to `top` $\bigcup$ `rest`
set c to $\frac{k - sum}{|\pi|}$
set p_i to $p_i + c, \forall p_i \in P$
for $e \in$ *`randoms`* **do**
  set e to $\left(k - \sum_{p_i \in P} p_i\right)$ /| `randoms` |
while $\exists p_i > 1$ *in* P **do**
  set c to $(p_i - 1)/(|\pi| - 1)$
  set p_j to $p_j + c,\ \forall p_j \in P \setminus \{p_i\}$
  set e to $e + c \cdot (|\pi| - |P|)$ /| `randoms` |
  set p_i to 1
output `EMR(`$P \bigcup$ *`randoms`*`, 1)`

4.3 Transition

Algorithm 5: Transition Step Streaming Update, $sum + s \geq k$ and $sum < k$

set sum to $sum + s$
add `top`, `rest` to `pending`
set `top`, `rest`, `randoms` to empty
set $scale$ to $\frac{k}{sum}$
set `pending` to (`pending` $\bigcup s$) $\cdot scale$
set `pending` to all $x \in$ `EMR(`*`pending`*`, 1)` with $x > 0$
if π *ends* **then**
  output `pending`

4.4 After k

Algorithm 6: Streaming Update when $sum \geq k$

```
// scale is the product of previous incremental weights, and renormalizes the incoming s
set sum to sum + s
set incr to (sum − s)/sum
set s to s · scale
set pending to (pending ⋃ s) · incr
set scale to scale · incr
set pending to all x ∈ EMR(pending, 1) with x > 0
if π ends then
    output pending
```

Lemma 4. *Given a list L of candidates, let p_i denote the probabilities that the i'th candidate is selected by the Streaming Cohort Selection Algorithm acting on L as a stream, and let q_i denote the same for Alg 2 acting on L as an offline array. Then,*

$$p_i = q_i \; \forall i \tag{6}$$

Proof. Let the process outlined in Algs 3, 4, 5, 6 be denoted A and let A have probability p_i of selecting candidate i. Let Alg. 2 be denoted B with probability q_i selecting candidate i. A never accepts candidates until the end of π, thus there are two major cases.

1. π ends with $sum \geq k$. Thus we finished after using Alg 6; this procedure selects all candidates who were in `pending` at the end of the stream. A candidate i is added to `pending` in one of three ways:

   (a) Candidate i was in `top` when the sum reached k. At each round, they must survive EMR, which by Lemma 2 happens with exactly their marginal probability. Therefore we only need to consider the effect of the incremental scaling adjustments.

   First, let $scale_i$, sum_i, $incr_i$ represent the value of these variables at some step i.

   Assume we entered Alg 5 and initialized $scale$ at some step j:

$$\begin{aligned} scale_j &= \frac{k}{sum_j} \\ scale_{j+1} &= scale_j \cdot \frac{sum_{j+1} - s_{j+1}}{sum_{j+1}} \\ &= \frac{k}{sum_j} \cdot \frac{sum_j}{sum_{j+1}} \\ &= \frac{k}{sum_{j+1}} \end{aligned}$$

   Thus we can see by induction that $scale_i = \frac{k}{sum_i}, \forall i$. Furthermore, we can express the final value of a single element s_i, which by definition is at position i in the stream, as:

$$\begin{aligned} p_i &= s_i \cdot scale_i \cdot \prod_{t=i+1}^{|\pi|} incr_t \\ &= s_i \cdot \frac{k}{sum_i} \cdot \prod_{t=i+1}^{|\pi|} \frac{sum_t - s_t}{sum_t} \\ &= s_i \cdot \frac{k}{sum_i} \cdot \prod_{t=i+1}^{|\pi|} \frac{sum_{t-1}}{sum_t} \\ &= s_i \cdot \frac{k}{sum_{|\pi|}} \end{aligned}$$

   Thus we see that the final p_i is adjusted exactly the same as it would be in the offline case.

(b) Candidate i was added to `rest` when the sum reached k. This means there was less than k total probability in `top`, excluding i. First, we know $s_i \leq 1-\alpha$, as follows: every candidate in `top` is at least as large as s_i (who is in `rest`). There are $\lceil k/\alpha \rceil$ `top` candidates. If $s_i > 1-\alpha$, then the total probability in `top` is $\lceil k/\alpha \rceil \cdot (1-\alpha) > k$, which is a contradiction.

Candidate i joined `rest` using EMR up to $1-\alpha$, which respects its marginal. Then it was added to `pending` via EMR up to 1, which again respects the original marginal. As in Case 1a, their chance of being selected is then

$$p_i = k\frac{s_i}{sum}$$

which is the same as Alg 2.

(c) Candidate i was encountered when $sum \geq k$. This is the same as 1a.

2. The stream π ends with $sum < k$. A has three arrays at this point: `top` (the greatest $\lceil k/\alpha \rceil$ elements in π), `rest` (at most $\lceil k/\alpha \rceil$ elements rounded to $1-\alpha$) and `randoms` (a randomly selected group of k candidates, disjoint from `top` and `rest`).

No acceptances are made in (Alg 3) until π ends. The top candidates are placed into `top`, and elements are added to `rest` via EMR, which preserves original marginal probabilities exactly by Lemma 2. The only change in probabilities then comes from Alg 4.

For the purpose of proof, consider a conceptual algorithm A' that behaves exactly the same, except instead of `randoms` it has *zeros*, a list of all candidates outside of `top` and `rest`. This conceptual algorithm ends by collecting all of the mass in *zeros* into a set of k random candidates, which becomes the equivalent of the list `randoms` in A, and performing a final EMR step to select the outputs.

Each member of `top`, `rest`, and *zeros* is given $\frac{k-sum}{|\pi|}$, and any overflow is redistributed evenly. Any candidate i now has the same probability of selection by both A' and Alg 2: $s_i + \alpha_i$, and thus Alg A', `top`, `rest`, and *zeros* are given the exact same treatment they would receive in the offline algorithm.

The only difference between A and A' is that, before the final step of selecting candidates by EMR, A' selects k random elements from *zeros*, and evenly distributes the mass of *zeros* across these k elements, thereby constructing a list equivalent to `randoms`. This step does not change the marginal probabilities of any element in *zeros*, because they have a uniform probability before and afterwards and we did not add or subtract mass. A' then selects candidates using EMR on the entire list `top` $\bigcup$ `rest` $\bigcup$ `randoms`, which does not affect the adjusted marginals $s_i + \alpha_i$. Thus $p_i = s_i + \alpha_i = q_i \forall i$.

□

Theorem 3. *The Streaming Cohort Selection Algorithm A denoted by Algs 3, 4, 5, 6 is individually fair.*

Proof. We have shown Alg 2 to be individually fair. By Lemma 4, A has the same outcome as Alg 2, thus A must also be individually fair. □

Theorem 4. *The Streaming Cohort Selection Algorithm A denoted by Algs 3, 4, 5, 6 achieves optimal utility for Fair Useful Cohort Selection.*

Proof. We have shown Alg 2 achieves optimal utility for Fair Cohort Selection. By lemma 4, A has the same outcome as Alg 2, thus A must also achieve optimal utility. □

Theorem 5. *The Streaming Cohort Selection Algorithm A denoted by Algs 3, 4, 5, 6 keeps no more than $O(k)$ candidates pending.*

Proof. Any candidate not in one of: `top`, `rest`, `randoms`, `pending` is considered rejected. The sizes of `top` and `randoms` are explicitly bounded by $\lceil k/\alpha \rceil$. `rest` is not bounded in size explicitly, but note that the array is maintained by constantly applying EMR(`rest`, $1-\alpha$). If we ever have more than $\lceil k/(1-\alpha) \rceil$ elements, we have

$$sum \geq \left\lceil \frac{k}{1-\alpha} \right\rceil (1-\alpha) > k$$

This goes to the $sum \geq k$ case, which no longer adds elements to `rest`, thus it is bounded in length by $\lceil k/(1-\alpha) \rceil$. `pending` is created initially by adding `top` and `rest` ($\lceil k/(1-\alpha) \rceil + \lceil k/\alpha \rceil$ pending), but for subsequent steps is never longer than k by Lemma 2. Thus the worst we can do is remain under a sum of k for the entire stream. `top`, `rest`, and `randoms` will all be kept pending until the end of the stream, but still we have at most $\lceil k/\alpha \rceil + \lceil k/(1-\alpha) \rceil + \lceil k/\alpha \rceil = O(k)$ candidates pending. □

5 Conclusion

Quantifying and optimizing the utility of fair algorithms is an important step for increasing adoption of fair algorithms requires considering the utility of the algorithm. Our key contribution is to solve the streaming cohort selection problem both with respect to fairness and with respect to utility, while allowing only $O(k)$ candidates to remain pending.

5.1 Future Work

An immediate open question following our work is to develop an algorithm with exactly k pending candidates. In a broader perspective, as demonstrated in this problem, there is a strong connection between fair composition algorithms and dependent rounding algorithms in approximation algorithms. Perhaps this connection can be used for other composition settings.

This work focuses only on Individual Fairness, but other definitions of fairness that could be considered. For example, using Group Fairness (Conditional Parity), discussed in [1], would change the nature of this problem by removing the strict linear constraint imposed by Individual Fairness. With this definition decisions could be made sooner in the online case and potentially with better utility.

5.2 Index

Symbol	Semantics
k	number of candidates to select
n	length of input list of candidates
U	universe of possible candidates
C	a classifier $C : U \to \{0, 1\}$
W, w_i	the ith candidate from candidate set W
$\vec{x}$	utilities for candidate set W
$\vec{s}$	$s_i = Pr[C(w_i) = 1]$
p_i	algorithm's probability of selecting candidate w_i
α	for $\sum S < k$, max constant increase any candidate can receive
α_i	the actual constant increase w_i receives in a candidate pool, $\alpha_i \leq \alpha$

Table 2: Notation and symbols

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