# Correcting Underrepresentation and Intersectional Bias for Fair Classification

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#### **Abstract**

We consider the problem of learning from data corrupted by underrepresentation bias, where positive examples are filtered from the data at different, unknown rates for a fixed number of sensitive groups. We show that with a small amount of unbiased data, we can efficiently estimate the group-wise drop-out parameters, even in settings where intersectional group membership makes learning each intersectional rate computationally infeasible. Using this estimate for the group-wise drop-out rate, we construct a re-weighting scheme that allows us to approximate the loss of any hypothesis on the true distribution, even if we only observe the empirical error on a biased sample. Finally, we present an algorithm encapsulating this learning and re-weighting process, and we provide strong PAC-style guarantees that, with high probability, our estimate of the risk of the hypothesis over the true distribution will be arbitrarily close to the true risk.

# 1 Introduction

Intersectional bias in machine learning refers to the phenomenon where algorithms exhibit discriminatory outcomes based on the combination of multiple attributes, such as race, gender, or socioe-conomic status, and when the biases associated with each attribute intersect. One example of intersectional bias can be seen in online advertising algorithms. Studies have revealed that certain ads for housing or job opportunities are disproportionately shown to specific demographic groups while excluding others. Ethnic discrimination in the hiring process is another example of intersectional bias. Here, the bias intersects along ethnicity and socioeconomic status, as individuals from certain ethnic backgrounds may face compounded biases based on their race and social class[21, 4].

Moreover, research has shown that black males have been unfairly incarcerated on drug charges at significantly higher rates than white males. This example illustrates intersectional bias based on the intersection of race and gender, as black males experience discrimination that is distinct from both black females and white males[1, 13]. These biased outcomes are often the result of historical data that reflects discriminatory patterns or biased decision-making processes. When these biased data are used to train algorithms, they perpetuate and amplify existing inequalities, further disadvantaging individuals who belong to multiple marginalized groups. Ensuring that training datasets are diverse, representative, and free from biases is essential in addressing and mitigating intersectional bias.

Philosophically, we can think of intersectional biases as representing hidden factors that distort our perception of the true state of affairs. The philosophy of science distinguishes between observables that we can see and the ground truth that we cannot observe [5]. This distinction is essential because it highlights the limitations of empirical observation and the role of theory in science. While we can observe the effects of a phenomenon, we can never directly observe the underlying causes that produce those effects. Instead, we must infer the causal mechanisms that generate the observable data through theoretical models. Causal modeling attempts to reason about and model this explicitly, including unobserved variables, referred to as "latents" [22]. By explicitly modeling the transforma-

tion data undergoes and utilizing a biased observable distribution, we gain insight into the ground truth we cannot directly observe. This allows us to model the bias that generates the observed biased distribution, akin to having a causal model. By recognizing this analogy, we can reason about bias more rigorously. Just as a causal model allows us to reason about the unobserved mechanisms that generate observable data, the bias model enables us to reason about the hidden factors that lead to biased observations. This deeper understanding of bias and its relationship to the ground truth helps us devise strategies to recover the true distribution and train fair classifiers. These strategies allow us to bridge the gap between what we can directly observe and the underlying reality we aim to capture, leading to more robust and equitable machine-learning systems.

Often, the term "intersectionality" is invoked without regard for its origins in Black feminism [12, 11]. It is misinterpreted to refer to any combination of attribute categories without explicitly connecting to social power. Bauer et al. [3] discovered in their research surveying the role of intersectionality in quantitative studies that the engagement with the fundamental principles of intersectionality was frequently superficial. They reported that 26.9% of the papers did not define intersectionality, and 32.0% did not cite the pioneering authors. Moreover, they observed that 17.5% of the papers utilized 'intersectional' categories that were not explicitly linked to social power. In response to the critique of using 'intersectional' categories without clear ties to social power, we propose a quantitative model of intersectionality. This model offers a methodology that quantifies the extent of bias associated with each group. We incorporate bias parameters for each group but do not make assumptions about whether any specific instance or combination of bias equates to power or oppression. This is because power and oppression are substantive ethical & political concepts that require a rigorous ethical & political framework to identify their instances. However, introducing bias parameters for each group allows our model to explicitly incorporate a way to identify social power into analyzing intersectional biases when supplemented with a more robust normative theory. These parameters can provide a quantitative measure of the power dynamics and social hierarchies that intersect within the dataset based on one's ethical & political theory of choice for analyzing what constitutes oppression or power.

In the present work, our contributions are interdisciplinary; we introduce a philosophical model of intersectionality bias as well as an algorithm that mitigates this bias with provable guarantees. In terms of philosophy, we introduce an agnostic approach termed *epistemic intersectionality* and outline the characteristics and strengths of this framing. Leveraging methods typically used for correcting statistical underrepresentation, we present a simple algorithm that accounts for interactions arising from multiple group memberships while maintaining empirical risk minimization. We show that if the base rates of positive labels are not uniform among intersecting groups, but we know the bias parameters, we can still learn near-optimal classifiers. We provide strong theoretical guarantees that show we can learn the bias parameters accurately with a very small sample of unbiased data, so this gives a proof of concept that a small amount of unbiased data can take us from "impossible" to "nearly optimal" at least in our intersectionality bias model. The main result is that the amount of unbiased data we need is much smaller than we would need to learn the model from that alone — we need a large quantity of biased data.

## 2 Background and Related Work

Intersectionality is a concept that was introduced in the 1980s by the legal scholar Crenshaw [12] to describe how multiple forms of oppression, such as racism, sexism, and classism, intersect to create unique experiences of discrimination for individuals who are members of more than one marginalized group. Since then, intersectionality has become a crucial lens through which researchers and activists analyze social inequality. Quantitative intersectionality studies have utilized various methods to examine the complex relationships between multiple social identities and outcomes. Bauer et al. [3] comprehensively surveys the literature on quantitative intersectionality.

Significant research has been conducted on machine learning techniques to predict outcomes in the face of corrupted data [2, 8, 15]. Blum and Stangl [6] explore two ways the training data for learning models can exhibit bias: by under-representing positives or mislabeling examples from certain groups.

Several approaches have been devised to address imbalances in machine learning algorithms. Some of the most common approaches involve incorporating demographic constraints into the learning

process, which define the standards that a fair classifier should adhere to [10]. Commonly, these approaches segment communities based on relevant characteristics, and the classifier is expected to exhibit comparable performance across all protected demographic groups [9], [17], [18]. Ensuring equality in predictive outcomes based on group membership can be pursued through fairness metrics or definitions, which encompass various aspects of parity constraints. Several definitions have been proposed in the literature to capture different facets of fairness [9], [16], [17]. A substantial amount of research has been dedicated to exploring the interrelationships between these various fairness definitions and identifying conflicts among them [20] [9].

A developing area of research within the domain of fair machine learning, which bears significant relevance to our study, revolves around the concept that observed data does not fully capture the underlying unobserved data. This line of inquiry explores how enforcing fairness can aid machine learning models in mitigating biases [19, 6]. Our work draws direct inspiration from the investigation conducted by Blum and Stangl [6], who demonstrate that the equal opportunity criterion enables the recovery of the Bayes classifier, even when faced with challenges such as under-representation and labeling biases. This body of research contributes to the growing body of literature advocating that fairness and accuracy should not be traded against each other. Notably, our research expands the knowledge base concerning equitable classification by proposing an approach to address under-representation bias experienced by multiple groups without making any assumptions about the consistency of base rates across these groups.

# 3 Framework for Epistemic Intersectionality

**Epistemic Intersectionality**: We introduce a philosophical framework we coin *epistemic intersectionality* that incorporates and analyzes intersectional biases in machine learning models while considering the epistemic dimension of intersectionality. It combines empirical verification and normative considerations to comprehensively understand biases arising from the intersection of multiple social identities.

#### **Characterization:**

- Empirical Verification: Epistemic intersectionality recognizes the importance of empirical verification in understanding the existence and extent of intersectional biases. It emphasizes the need to gather empirical evidence to determine the presence of bias in the dataset and the impact of multiple group memberships on these biases.
- 2. **Normative Analysis**: Epistemic intersectionality acknowledges that normative theories influence the interpretation and significance of empirical findings. It incorporates normative considerations to assess the relevance and significance of the detected biases. Normative theories help determine whether the observed biases can be oppressive or contribute to power imbalances.
- 3. **Disambiguation of Epistemic and Normative Dimensions**: Epistemic intersectionality distinguishes between epistemic facts and normative considerations. It recognizes that empirical evidence provides information about the state of the world (epistemic facts), while normative theories inform judgments about what is morally or socially desirable (normative considerations). That is, it helps disambiguate the epistemic dimension (what is) from the normative dimension (what ought to be), leading to a more precise understanding of the role of intersectionality in academic research. This disambiguation allows for a rigorous analysis that ensures empirical findings are appropriately interpreted within a normative framework.
- 4. Comprehensive Analysis of Intersectionality: Epistemic intersectionality promotes a comprehensive analysis of intersectionality by considering the complex interactions between social identities and systems of power. It goes beyond merely recognizing the presence of multiple identities and emphasizes how these identities intersect and interact, which can potentially constitute social power structures. This analysis helps capture the nuanced and context-specific biases that emerge from intersecting identities.
- 5. **Role in Academic Research**: Epistemic intersectionality contributes to academic research by offering a rigorous and evidence-based approach to understanding the impact of intersectional biases in machine learning models. It facilitates the formulation of testable

hypotheses and the generation of empirical content, enhancing the empirical robustness and generalizability of intersectionality theory. This approach fosters a more precise understanding of the complexities of bias and fairness in machine learning systems.

By considering the bias parameters when analyzing the biased dataset, our model ensures that data points are included based on the collective impact of multiple group memberships, which may be explicitly tied to social power given one's moral and political theory. This provides flexibility because assuming power relationships or oppression at the outset is not sufficient, and empirical evidence needs to be gathered to verify the existence and extent of such relationships. Once this empirical evidence is obtained, a normative theory can be applied to determine if the amount of bias detected is significant enough to be considered oppressive.

Bright et al. [7] have pointed out that some versions of the intersectionality framework have faced criticism for their lack of explicit methodology for testing intersectional claims and even the high potential for false predictions. Critics have argued that intersectionality theorists have not provided a straightforward methodological approach, and their predictions about the social consequences of intersecting identities have been questioned. For example, Curry [14] has argued that intersectionality utilizes a form of categorical analysis that imposes generalizations on groups contrary to the empirically observed behaviors and properties assigned to those groups. Curry [14] suggested that disaggregation is better than current intersectional trends.

Our contribution to the literature on intersectionality and bias in machine learning is significant because it is the first study to formally model and provide theoretical guarantees for correcting intersectional bias. We model group-wise drop-out parameters and propose an algorithmic approach to address them. Our work represents a significant step forward in designing fair and equitable machine learning systems that account for the complexity of biases faced by individuals from diverse backgrounds. By explicitly considering the biases that emerge from individuals belonging to multiple groups, our model provides a framework to understand better and mitigate potential false predictions and unintended consequences that may arise from ignoring intersectionality.

Our model addresses the criticism of the intersectionality framework by introducing a specific methodology to incorporate and analyze intersectional biases in machine learning models. First, it recognizes that biases can exist within different groups and aims to capture the complex interactions and collective impact of multiple group memberships on biases in the dataset. Second, by introducing bias parameters for each group, our model quantifies the degree of bias associated with each group. This provides a more explicit approach to measuring and understanding biases in the dataset. It offers a quantitative approach to generate empirical content and facilitate falsifiability, enabling the formulation of testable hypotheses about intersectional effects within specific populations and supporting generalizations that enhance the empirical robustness of intersectionality theory.

**Strengths** The model provides a flexible and generalizable method of estimating bias across multiple intersectional categories without any prior assumptions about the degree of bias in each group. The model allows for a more nuanced understanding of bias by examining the joint effects of multiple categories rather than just one at a time. The model can capture the compounding effects of an individual being part of multiple marginalized groups, such as racial and gender groups. The model also has the strength that once the bias parameters are estimated, they can be used on future draws of biased data to find a hypothesis approximately minimizing prediction risk with respect to the true distribution with high probability.

**Limitations** The model assumes that group membership is independent, which may oversimplify the complexities of social power structures and how they intersect in real-world contexts. In addition, we only consider underrepresentation bias and believe this approach would be valuable in modeling other types of bias.

## 4 Summary of Results

## 4.1 Model Oveview

In our model, we acknowledge that biases can exist within different groups, and we aim to capture the intersectionality of these biases. To achieve this, we introduce bias parameters  $\beta_i$ , which are

associated with each group  $G_i$  and capture the extent to which positive examples from that group are filtered out. To estimate these bias parameters, we use  $\hat{\beta} = (\hat{\beta}_1, \dots, \hat{\beta}_k)$ . Importantly. we assume group membership is *independent*. We discussed the strengths and weaknesses of this assumption in Section 3.

We construct a new biased dataset called  $S_{\beta}$  to incorporate the intersectional nature of biases. This dataset reflects the biases that emerge when individuals belong to multiple groups simultaneously. Consider an individual  ${\bf x}$  who may be a member of various groups, denoted by  $G({\bf x})$ . If  ${\bf x}$  has a positive label, we include it in  $S_{\beta}$  with a probability determined by the product of the bias parameters of all the groups it belongs to, i.e.,  $\prod_{i\in G({\bf x})}\beta_i$ . Formally, the biased dataset  $S_{\beta}$  comprises pairs  $({\bf x}_i,y_i)$ , where  ${\bf x}_i$  represents a data point from the original dataset  $S_{\beta}$ , and  $S_{\beta}$  denotes its associated label. The size of  $S_{\beta}$  is denoted as  $S_{\beta}$ , which is less than or equal to the size of the original dataset  $S_{\beta}$ . We let  $S_{\beta}$  be the distribution over  $S_{\beta}$  is a binary label belonging to the label space  $S_{\beta}$ . We let  $S_{\beta}$  be the distribution over  $S_{\beta}$  is determined by the following probability:

$$\Pr(\mathbf{x} \in S_{\beta} | y = 0) = 1$$

$$\Pr(\mathbf{x} \in S_{\beta} | y = 1, \mathbf{x} \in \bigcap_{i \in G(\mathbf{x})} G_i) = \prod_{i \in G(\mathbf{x})} \beta_i$$

This probability captures the likelihood of including a data point in  $S_{\beta}$  given that it has a positive label and belongs to the intersection of the groups specified by  $\mathcal{X}$ . Finally, we define  $\mathcal{D}_{\beta}$ , the distribution induced on  $S_{\beta}$  by the filtering process as it relates to  $p_{\mathcal{D}}(\mathbf{x}, y)$ 

$$p_{\mathcal{D}}(\mathbf{x}, y) = \frac{w(\mathbf{x}, y)p_{\mathcal{D}_{\beta}}(\mathbf{x}, y)}{\sum_{(\mathbf{x}, y) \sim \mathcal{D}_{\beta}} w(\mathbf{x})p_{\mathcal{D}_{\beta}}(\mathbf{x}, y)} = \frac{w(\mathbf{x}, y)p_{\mathcal{D}_{\beta}}(\mathbf{x}, y)}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(x)]}$$

which implies

$$p_{\mathcal{D}_{\beta}}(\mathbf{x}, y) = \frac{p_{\mathcal{D}}(\mathbf{x}, y) \mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x})]}{w(\mathbf{x}, y)}$$

for 
$$w(\mathbf{x},y) = \frac{1}{\prod_{i \in G(\mathbf{x})} \beta_i} \mathbb{I}(y=1) + \mathbb{I}(y=0)$$

We call w the reweighting factor and use it to construct a reweighting mechanism that approximates the loss of any hypothesis on the ground truth distribution, even when we only have access to the empirical error computed from the biased sample. We further develop an algorithm operationalizing this learning and reweighting process, culminating in a powerful tool for mitigating intersectional bias in machine learning.

By incorporating the bias parameters, we emphasize the intersectionality of biases, accounting for the collective impact of multiple group memberships on the biases in the dataset. The construction of  $S_{\beta}$  enables us to explicitly consider and analyze the intersectional biases in machine learning models. By training on this biased dataset, we can develop models that account for the complex interactions of biases arising from multiple group memberships. This approach provides a more comprehensive understanding of bias and its effects, ultimately leading to more equitable and fair machine learning systems.

## 4.2 Algorithm Overview

Our algorithm, described in detail as Algorithm 1 in Section 6, aims to mitigate the biases present in the dataset and learn a fair and accurate model. We start with an unbiased training set S and a biased training set  $S_{\beta}$ . The biased training set  $S_{\beta}$  is a function of the product of the inverse of the  $\beta$ 's, which we estimate by the product of the inverse of the  $\hat{\beta}$ 's to capture the intersectional biases. Not that by learning each  $\frac{1}{\beta_i}$  individually, we can estimate the product accurately and efficiently without having to estimate a bias parameter for each unique intersection of groups. To ensure fairness, we apply the intersectional bias learning algorithm to obtain a hypothesis h that minimizes the risk of the learned model while considering the biases. The algorithm proceeds as follows:

- 1. We begin by estimating the positive rate  $p_i$  for each group  $G_i$  according to the underlying distribution  $\mathcal{D}$ ,
- 2. Next, we estimate the positive rate  $p_i\beta_i$  for each group  $G_i$  according to the *biased* distribution  $\mathcal{D}_{\beta}$ .
- 3. Using the estimates for  $p_i$  and  $p_i\beta_i$ , we can then estimate  $\frac{1}{\beta_i}$  for each group  $G_i$ .
- 4. We determine the sample sizes required for training our model. The sample sizes are chosen to balance the complexity of the hypothesis class  $\mathcal{H}$  and the desired confidence level in the learned model they depend on the logarithm of the VC dimension of the hypothesis class size, the target error  $\epsilon$ , and the confidence parameter  $\delta$ . We also consider the bias parameter  $\beta$  and the estimated fraction of biased examples  $\widehat{p\beta}$  in the calculation.
- 5. We then create a training set S by randomly selecting m examples from the unbiased distribution set  $\mathcal{D}$  and  $m_{\beta}$  examples from the biased distribution  $\mathcal{D}_{\beta}$ , with an appropriate number of samples per group. This random selection ensures that we have a representative sample from both datasets.
- 6. Finally, we apply the empirical risk minimization algorithm to train a model using the training set S. The algorithm minimizes the risk associated with the learned model while considering the biases captured by the intersectional bias parameters. The algorithm's output is the hypothesis h, representing the learned model.

By incorporating intersectional biases in the learning process, our algorithm aims to develop fair and accurate models that account for the complexities of bias arising from multiple group memberships. It allows us to learn models that address the specific challenges of intersectionality, thereby promoting fairness and equity in machine-learning applications.

#### 4.3 Theorem Overview

The main theorem states that given a hypothesis class  $\mathcal{H}$  and an unknown distribution  $\mathcal{D}$  over feature space  $\mathcal{X} \times \{0,1\}$ , if we have an unbiased sample S of size m and a biased sample  $S_{\beta}$  of size  $m_{\beta}$ , with the marginal probability of a positive example in group i of  $S_{\beta}$  being  $p_i\beta_i$ , then running Algorithm 1 with appropriate sample sizes ensures that, with high probability, the algorithm outputs a hypothesis h that has a low error on the distribution  $\mathcal{D}$ .

The proof of the main theorem consists of three parts:

**Part A**: This part focuses on normalizing the weighted empirical loss on the biased sample. The goal is to bound the difference between the sum of weights in the biased sample and the reciprocal of the expected weight. By achieving this normalization, we can account for the bias introduced by the weights and ensure that the algorithm's performance is not adversely affected. This normalization step is crucial for the subsequent analysis of the proof. See Lemmas A.2 and A.3 for details.

**Part B**: Part B of the proof is concerned with estimating the inverse of the product of the bias parameters. It involves first obtaining reliable estimates of the probabilities of positive examples in each group of the unbiased data and biased data, respectively, which can then be used to estimate  $\frac{1}{\beta}$  and, finally,  $\frac{1}{\prod_{i=1}^k \beta_i}$  The key lemma established in this part specifies the sample size requirements to achieve reliable probability estimation, which is crucial for accurately evaluating the algorithm's performance. It shows that we can estimate the probabilities of positive examples in each group with reasonable accuracy by having only a small number of unbiased samples in addition to our larger set of biased samples. See Lemmas B.1 – B.9 for details.

**Part C**: Part C combines the results from Parts A and B to establish the sample complexity guarantee for Algorithm 1. It utilizes the bounds obtained from Lemmas established in Parts A and B, which cover both the intersectional case where the groups in the biased sample have overlapping instances and the computationally simpler case where they are disjoint. By combining these results, the proof demonstrates that running Algorithm 1 with appropriate sample sizes outputs a hypothesis h with a low error on the true distribution  $\mathcal{D}$  with high probability. See Lemmas C.1 - C.2 for details.

## 5 Preliminaries

In this setting, the probability distribution  $\mathcal{D}$  is defined over the space  $\mathcal{X} \times \mathcal{Y}$ , where  $\mathcal{X}$  is the feature space and  $\mathcal{Y}$  is the label space. Specifically,  $\mathcal{D}$  is the joint distribution of the feature-label pairs  $(\mathbf{x}, y)$ , where  $\mathbf{x}$  is a d-dimensional feature vector and  $y \in \{0, 1\}$  is the binary label. This type of distribution can be interpreted as having two segments: one segment, represented as  $\mathcal{D}_{\mathbf{x}}$ , is over domain points that are not labeled (commonly known as the marginal distribution), and the other is a conditional probability that assigns labels to each point in the domain, denoted as  $\mathcal{D}((\mathbf{x},y)|\mathbf{x})$ . We define  $\mathcal{D}$  as:

$$\mathcal{D} = \{(\mathbf{x}, y) \mid \mathbf{x} \in X, y \in Y, p_D(\mathbf{x}, y) \text{ is the probability of } (\mathbf{x}, y)\}$$

Where  $p_{\mathcal{D}}(\mathbf{x}, y)$  is the joint probability mass function of the feature-label pairs  $(\mathbf{x}, y)$ .  $S = ((\mathbf{x}_1, y_1), \dots, (\mathbf{x}_m, y_m))$  is an unbiased finite sequence of pairs in  $\mathcal{X} \times \mathcal{Y}$ , that is, a sequence of labeled domain points. This is the input that the learner has access to.

We consider a setting where there are k groups  $\{G_i\}_{i=1}^k$  that can intersect so that a person can be a member of any of the  $2^k$  subsets of  $\{G_i\}_{i=1}^k$ . We assume that group membership is independent, that is, for  $X \subseteq 1, \ldots, k$ , the rate at which positive examples appear in  $\bigcap_{i \in G(\mathbf{x})} G_i$  in the unbiased data is:

$$\Pr(y = 1 | \mathbf{x} \in \bigcap_{i \in G(\mathbf{x})} G_i) = \prod_{i \in G(\mathbf{x})} p_i.$$

The data distribution  $\mathcal{D}$  is a mixture of distributions  $\mathcal{D}_{G_i}$ , where  $\mathbf{x}|\mathbf{x} \in G_i \sim \mathcal{D}_{G_i}$ .

## 5.1 Intersectionality Bias Model

Each group  $G_i$  is associated with a bias parameter  $\beta_i > 0$ , and let  $\hat{\beta} = (\hat{\beta}_1, \dots, \hat{\beta}_k)$  be an estimate of the bias parameters. These bias parameters are used procedurally to form a new biased dataset  $S_{\beta}$ . We define the bias parameters as follows: Say that a person  $\mathbf{x}$  is in the set of groups  $G(\mathbf{x})$ ; then, if they are a positive example, they are included in  $S_{\beta}$  with probability  $\prod_{i \in G(\mathbf{x})} \beta_i$ .

Formally,  $S_{\beta} = (\mathbf{x}_1, y_1), \dots, (\mathbf{x}_{m_{\beta}}, y_{m_{\beta}})$  is a biased training dataset of size  $m_{\beta} \leq m$ , where  $\mathbf{x}_i \in D$  and  $y_i \in \mathcal{Y}$ . Recall from Section 4.1 that for each  $(\mathbf{x}_i, y_i) \in S$ ,  $(\mathbf{x}_i, y_i)$  is included in  $S_{\beta}$  with the following probability:

$$\Pr(\mathbf{x} \in S_{\beta} | y = 0) = 1$$

$$\Pr(\mathbf{x} \in S_{\beta} | y = 1, \mathbf{x} \in \bigcap_{i \in G(\mathbf{x})} G_i) = \prod_{i \in G(\mathbf{x})} \beta_i$$

We will also define  $m_i$  as the number of samples in S from group  $G_i$  and  $m_{-i}$  as the number of samples in S that are not from group  $G_i$ . Similarly, we will say that  $m_{\beta_i}$  is the number of samples in  $S_{\beta}$  from group  $G_i$  and  $m_{\beta_{-i}}$  is the number of samples in  $S_{\beta}$  not from group  $G_i$ .

#### 5.2 The Learning Problem

Let S be an unbiased dataset of size m drawn i.i.d. from the distribution  $\mathcal{D}$  over the input space  $\mathcal{X}$  and the label space  $\mathcal{Y}$ . Then, the true error of hypothesis  $h: \mathcal{X} \to \{0,1\}$  is defined as:

$$L_{\mathcal{D}}(h) = P_{(\mathbf{x},y) \sim \mathcal{D}}[h(\mathbf{x}) \neq y]$$

This measures how likely h is to make an error when labeled points are randomly drawn according to  $\mathcal{D}$ . For the empirical risk on the unbiased sample S, we have  $L_S(h) = \frac{1}{m} \sum_{i=1}^m \mathbb{I}(h(\mathbf{x}_i) \neq y_i)$ 

Similarly, consider the biased sample  $S_{\beta} \subset S$ , where  $S_{\beta} = ((\mathbf{x}_1, y_1), \dots, (\mathbf{x}_m, y_m)) \in \mathcal{D}$ . We define the biased empirical risk to be the average loss over  $S_{\beta}$ . Now we will define the reweighted biased empirical risk (RBER), which will be our attempt to approximate the empirical risk on S. The RBER for  $S_{\beta}$  reweighted by the true inverse of  $\beta$  is:

$$L_{S_{\beta}\beta^{-1}}(h) = \mathbb{E}_{(\mathbf{x},y)\sim S_{\beta}}\left[w(\mathbf{x},y)\mathbb{I}(h(\mathbf{x})\neq y)\right] = \frac{1}{m_{\beta}}\sum_{i=1}^{m_{\beta}}w(\mathbf{x}_i,y_i)\mathbb{I}(h(\mathbf{x}_i)\neq y_i)$$

where 
$$w(\mathbf{x},y) = \prod_{i \in G(\mathbf{x})} \mathbb{I}(y=1) \frac{1}{\beta_i} + \mathbb{I}(y=0)$$

The RBER for  $S_{\beta}$  reweighted by our estimate of  $\beta$  is:

$$L_{S_{\beta}\hat{\beta}^{-1}}(h) = \mathbb{E}_{(\mathbf{x},y) \sim S_{\beta}} \left[ \hat{w}(\mathbf{x},y) \mathbb{I}(h(\mathbf{x}) \neq y) \right] = \frac{1}{m_{\beta}} \sum_{i=1}^{m_{\beta}} \hat{w}(\mathbf{x}_i, y_i) \mathbb{I}(h(\mathbf{x}_i) \neq y_i)$$

where  $\hat{w}(\mathbf{x}, y) = \prod_{i \in G(\mathbf{x})} \mathbb{I}(y = 1) \frac{1}{\hat{\beta}_i} + \mathbb{I}(y = 0)$ .

## 6 Theoretical Guarantees

## Algorithm 1: Intersectional Bias Learning Algorithm

**Input:** Unbiased training set S of size m with  $m_i$  samples  $\in G_i$  and  $m_{-i}$  samples  $\notin G_i$  for i=1,...,k, biased training set  $S_\beta$  of size  $m_\beta$  with  $m_{\beta_i}$  samples  $\in G_i$  and  $m_{\beta_{-i}}$  samples  $\notin G_i$  for i=1,...,k, target error  $\epsilon$ , confidence parameter  $\delta$ 

- 1. Estimate  $\hat{p}_i$  for each group i from S
- 2. Estimate  $\widehat{\beta_i p_i}$  for each group i from  $S_\beta$
- 3. Let  $\frac{1}{\hat{\beta}_i} = \frac{\hat{p}_i}{\hat{\beta_i} p_i}$  be the estimated bias for each group *i*.
- 4. Use ERM to return a hypothesis h that minimizes the empirical risk on  $\frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}} \hat{w}(\mathbf{x}_{i}, y_{i})} L_{S_{\beta}\hat{\beta}^{-1}}(h)$

**Theorem 6.1.** Let  $\epsilon, \delta > 0$ ,  $\mathcal{H}$  be a hypothesis class with VC-dimension  $|\mathcal{H}|$ , and let  $\mathcal{D}$  be an unknown distribution over  $\mathcal{X} \times \{0,1\}$ , where  $\mathcal{X}$  is a feature space. Let S be an unbiased sample of m examples drawn i.i.d. from  $\mathcal{D}$ , and let  $S_{\beta}$  be a biased sample of  $m_{\beta}$  examples drawn i.i.d. from  $\mathcal{D}_{\beta}$ . If Algorithm 1 is run with sample sizes  $m_{\beta} \geq \frac{5^2}{2\epsilon^2 \prod_{j=1}^k \beta_j^2} \ln \frac{4|\mathcal{H}|(k+1)}{\delta}$ ,  $m_{\beta_i} \geq \frac{3^3 5^2}{\epsilon^2 \prod_{j=1}^k \beta_j^3} \ln \frac{8(k+1)}{\delta}$ ,  $m_{\beta_i} \geq \frac{3^3 5^2}{\epsilon^2 \prod_{j=1}^k \beta_j^2 \prod_{j\neq i} \beta_j^2 \prod$ 

$$\left| \frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}} \hat{w}(\mathbf{x}_{i}, y_{i})} L_{S_{\beta}\hat{\beta}^{-1}}(h) - L_{\mathcal{D}}(h) \right| \leq \epsilon$$

#### 6.1 Proof Sketch of Theorem 6.1

The proof proceeds by establishing several lemmas.

- 1. **Lemma A.1**: The risk difference between the learned hypothesis h on the biased sample  $S_{\beta}$  and the risk of h on the true distribution  $\mathcal{D}$  is bounded with probability  $1 \delta$ . The proof uses the triangle inequality and the Hoeffding bound.
- 2. **Lemma A.2**: The absolute difference between the estimate of the normalizing factor and the true normalizing factor is bounded with probability  $1 \frac{\delta}{2k+2}$ . The proof uses the Hoeffding bound.
- 3. **Lemma A.3**: The reweighted loss  $L_{S_{\beta}}(h)$  normalized by the empirical estimate is always less than or equal to 1. The proof uses the sum of weighted indicator functions.
- 4. **Lemma B.1**: The ratio of the expected value of y given  $\mathbf{x}$  for samples in group  $G_i$  and samples not in group  $G_i$  is equal to  $p_i$ . The proof uses the definition of expectation and the assumption of independence of the groups.
- 5. **Lemma B.2**: The sample size required to estimate  $p_i$  within a certain level of precision in the disjoint case is given by the multiplicative Chernoff Bound. The proof uses the concept of Bernoulli trials and the multiplicative Chernoff Bound.

- 6. **Lemma B.3**: The sample complexity requirement for estimating  $p_i$  in the intersectional case is given by a multiplicative Chernoff bound. The proof uses a multiplicative Chernoff bound.
- 7. **Lemma B.5**: The ratio of the expected outcome variable y given that  $\mathbf{x} \in \mathcal{D}_{\beta}$  belongs to group  $G_i$  and the expected outcome variable y given that  $\mathbf{x}$  belongs to  $\mathcal{D}_{\beta}$  but not  $G_i$  is equal to  $\beta_i p_i$ . The proof uses the independence of the variables.
- 8. **Lemma B.4**: The sample size required to estimate  $p_i\beta_i$  within a certain level of precision is given by the multiplicative Chernoff Bound. The proof uses the concept of indicator variables and the multiplicative Chernoff bound.
- 9. **Lemma B.6**: If the sizes of each group in the biased dataset and the biased dataset as a whole are large enough, then with high probability, the estimated value of  $\beta_i p_i$  is close to the true value within a given tolerance  $\frac{\epsilon \prod_{i=1}^k \beta_i}{5}$ . The proof uses the bounds from Lemma B5 and applies these bounds to the product of inverse biases.
- 10. **Lemma B.7**: Given the expected values of the outcomes y for samples in group i and samples not in group i of  $\mathcal{D}_{\beta}$ , as well as the expected values of y for samples in group i of  $\mathcal{D}$  and samples not in group i of  $\mathcal{D}$ , we can calculate the value of  $1/\beta_i$ . The proof follows directly from the definitions and properties of the subpopulations of group i in  $\mathcal{D}_{\beta}$  and  $\mathcal{D}$ .
- 11. **Lemma B.8**: If the sizes of group i in  $\mathcal{D}_{\beta}$  and the size of the population not in group i of  $\mathcal{D}_{\beta}$  satisfy certain conditions, then with high probability, our estimated inverse  $1/\beta_i$  will be within an error margin of  $\frac{3\epsilon \prod_{j \neq i} \beta_j}{5}$  from the true inverse  $1/\beta_i$ . The proof uses the ratio inequality between the estimated and true ratios of parameters and applies the Hoeffding bound to the difference between the estimated and true inverses.
- 12. **Lemma B.9**: If the sizes of the samples satisfy certain conditions, then with high probability, the estimated inverse of the product of biases  $\prod_{i=1}^k \widehat{\beta}_i$  is close to the true inverse of the product of biases  $\prod_{i=1}^k \beta_i$  within a margin of  $\frac{3\epsilon}{5}$ . The proof uses the bounds from Lemma B8 and applies these bounds to the product of inverse biases.
- 13. **Lemma C.1**: The expected loss of hypothesis h on the distribution  $\mathcal{D}$  is equal to the expected loss of h on the weighted distribution  $\mathcal{D}_{\beta}$  scaled by the inverse of the expected weight  $\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x},y)]$  of the weighted distribution. The proof follows from the definition of expected value.
- 14. **Lemma C.2**: A sample size of at least  $m_{\beta} \geq \frac{5^2}{2\epsilon^2 \prod_{j=1}^k \beta_j^2} \ln \frac{4|H|(k+1)}{\delta}$  is required to estimate the true expected loss  $L_D(h)$  using a reweighting of the biased sample loss  $L_{S_{\beta}}(h)$ . The proof uses a multiplicative Hoeffding bound.

Part C ties everything together and establishes the algorithm's sample complexity, providing the desired guarantee for its performance. This is novel in that a small amount of unbiased data can take us from "impossible" to "nearly optimal," at least in the context of underrepresentation and intersectional bias. The core observation is only a small fraction of your data needs to be unbiased. The unbiased data can be used to estimate the base rate. We then can use a large quantity of biased data to learn the model. By relying on these lemmas, Theorem 6.1 is proven, demonstrating the reliability and accuracy of the proposed method for empirical risk minimization in biased datasets.

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## A Appendix

#### A.1 Proofs

In Lemma A.1, we want to prove that with probability  $1-\delta$  and a certain number of samples, the difference between the reweighted risk of the learned hypothesis h on the biased sample  $S_{\beta}\hat{\beta}^{-1}$  (normalized by  $\sum_{i=1}^{m_{\beta}} \hat{w}(\mathbf{x}_i, y_i)$  and reweighted with  $\hat{\beta}$ ) and the risk of h on the true distribution  $\mathcal{D}$  is bounded by an epsilon multiplicative factor of the risk on the true distribution.

To prove this, we start by expanding the expression using the triangle inequality. We split it into three terms: A, B, and C. A represents the difference between the normalized risk on the biased sample using  $\hat{w}(\mathbf{x},y)$  and the normalized risk using the expected weights  $\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x},y)]$ . B represents the difference between the normalized risk on the biased sample reweighted with  $\hat{\beta}$  and the normalized risk on the biased sample reweighted with  $\beta$ . C represents the difference between the normalized risk on the biased sample using  $\beta$  and the risk on the true distribution  $\mathcal{D}$ .

Next, we simplify the expression further. We use the absolute value to ensure non-negativity and introduce additional terms to manipulate the expression. We apply the Hoeffding bound to bound the first term, which involves the difference in weights. We also introduce the empirical risk  $L_S(h)$  and manipulate the terms to arrive at the final inequality.

The final inequality shows that the difference between the two risks is bounded by  $\epsilon$ . This inequality holds with probability  $1-\delta$  and depends on the number of samples used.

By proving this lemma, we establish a bound on the difference between the reweighted risk of the learned hypothesis on the biased sample and the risk on the true distribution. This helps us understand the generalization performance of the learned hypothesis in the presence of bias.

$$\begin{array}{c} \text{Lemma A.1. } \text{Let } \epsilon, \delta > 0. \quad \text{If } m_{\beta} \, \geq \, \frac{5^2}{2\epsilon^2 \prod_{j=1}^k \beta_j^2} \ln \frac{4|H|(k+1)}{\delta}, \, m_{\beta_i} \, \geq \, \frac{3^3 5^2}{\epsilon^2 \prod_{j=1}^k \beta_j^3} \ln \frac{8(k+1)}{\delta}, \\ m_{\beta_{-i}} \, \geq \, \frac{3^3 5^2}{\epsilon^2 \prod_{j=1}^k \beta_j^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{8(k+1)}{\delta}, \, m_i \, \geq \, \frac{3^3 5^2}{\epsilon^2 \prod_{j=1}^k \beta_j^2 p_j} \ln \frac{8(k+1)}{\delta} \, \text{ and } \, m_{-i} \, \geq \\ \frac{3^3 5^2}{\epsilon^2 \prod_{j=1}^k \beta_j^2 \prod_{j \neq i} p_j} \ln \frac{8(k+1)}{\delta} \, \text{ for all groups } G_i, \, \text{then with probability } 1 - \delta, \\ |\frac{m_{\beta}}{\sum_{i=1}^k \hat{y}_i^2 \sum_{j=1}^k \hat{y}_i^2 \sum_{j=1}^k \hat{y}_j^2 \prod_{j \neq i} p_j} L_{S_{\beta}\hat{\beta}^{-1}}(h) - L_{\mathcal{D}}(h)| \leq \epsilon \\ \end{array}$$

*Proof.* We can use the triangle inequality to expand this expression into three terms:

$$\begin{split} &|\frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}} \hat{w}(\mathbf{x}_{i}, y_{i})} L_{S_{\beta} \hat{\beta}^{-1}}(h) - L_{\mathcal{D}}(h)| \leq |\frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}} \hat{w}(\mathbf{x}_{i}, y_{i})} L_{S_{\beta} \hat{\beta}^{-1}}(h) - \frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}, y)]} L_{S_{\beta} \hat{\beta}^{-1}}(h)| \\ &+ |\frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}, y)]} L_{S_{\beta} \hat{\beta}^{-1}}(h) - \frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}, y)]} L_{S_{\beta} \beta^{-1}}(h)| + |\frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}_{i}, y_{i})]} L_{S_{\beta} \beta^{-1}}(h) - L_{\mathcal{D}}(h)| \end{split}$$

Simplifying the expression above, we have (with probability  $1 - \delta$ )

$$\begin{split} &|\frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}}\hat{w}(\mathbf{x}_{i},y_{i})}L_{S_{\beta}\hat{\beta}^{-1}}(h)-L_{\mathcal{D}}(h)|\\ &\leq |\frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}}\hat{w}(\mathbf{x}_{i},y_{i})}-\frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x},y)]}|L_{S_{\beta}\hat{\beta}^{-1}}(h)+|L_{S_{\beta}\hat{\beta}^{-1}}(h)-L_{S_{\beta}\beta^{-1}}(h)|\frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x},y)]}\\ &+|\frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x},y)]}L_{S_{\beta}\beta^{-1}}(h)-L_{\mathcal{D}}(h)|\\ &\leq \frac{\epsilon m_{\beta}}{\sum_{i=1}^{m_{\beta}}w(\mathbf{x}_{i},y_{i})}L_{S_{\beta}\hat{\beta}^{-1}}(h)+|\frac{1}{m_{\beta}}\sum_{i=1}^{m_{\beta}}\mathbb{I}[h(\mathbf{x}_{i})\neq y_{i}](\hat{w}(\mathbf{x}_{i},y_{i})-w(\mathbf{x}_{i},y_{i})|\frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x},y)]}\\ &+|\frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x},y)]}L_{S_{\beta}\beta^{-1}}(h)-L_{\mathcal{D}}(h)|\\ &=\frac{\epsilon m_{\beta}}{\sum_{i=1}^{m_{\beta}}w(\mathbf{x}_{i},y_{i})}L_{S_{\beta}\hat{\beta}^{-1}}(h)+\frac{L_{S}(h)}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x},y)]}|\frac{1}{\prod_{i=1}^{k}\hat{\beta}_{i}}-\frac{1}{\prod_{i=1}^{k}\beta_{i}}| \end{split}$$

$$+ \left| \frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}, y)]} L_{S_{\beta}\beta^{-1}}(h) - L_{\mathcal{D}}(h) \right|$$

$$\leq 2\epsilon + \frac{1}{\prod_{i=1}^{k} \beta_{i}} 3\epsilon$$

Since  $\frac{\epsilon \prod_{i=1}^k \beta_i}{5} \leq \frac{\epsilon}{2 + \frac{3}{\prod_{i=1}^k \beta_i}}$ , we can replace  $\epsilon$  with  $\frac{\epsilon \prod_{i=1}^k \beta_i}{5}$  in our sample size bounds from the remaining lemmas, giving us:

$$\left|\frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}} \hat{w}(\mathbf{x}_i, y_i)} L_{S_{\beta}\hat{\beta}^{-1}}(h) - L_{\mathcal{D}}(h)\right| \le \epsilon$$

## A.1.1 Part A

In Lemma A.2, we want to bound the absolute difference between our estimate of the normalizing factor and the true normalizing factor. We aim to find the number of biased samples needed to ensure this difference is within  $\epsilon \frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}} w(\mathbf{x}_{i}, y_{i})}$  with probability  $1 - \frac{\delta}{2k+2}$ .

To prove this, we start by using a Hoeffding bound. Since the weights w lie within the interval  $[1,\frac{1}{\prod_{i=1}^k\beta_i}]$ , we can apply the Hoeffding bound to bound the absolute difference between the sum of weights and  $m_\beta\mathbb{E}_{\mathcal{D}_\beta}[w(\mathbf{x},y)]$ . This step allows us to control the difference between the estimated normalizing factor and the true normalizing factor.

We manipulate the expression and arrive at an inequality involving the absolute difference between  $\frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x},y)]}$  and  $\frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}}w(\mathbf{x}_{i},y_{i})}$ . To satisfy this inequality, we set the probability of the event exceeding the bound to be less than  $\delta$ . Solving for the required number of samples  $m_{\beta}$ , we obtain the condition  $m_{\mathcal{D}_{\beta}} \geq \frac{1}{2\epsilon^{2}} \ln \frac{2(2k+2)}{\delta}$ .

Therefore, the lemma states that with at least  $m_{\beta}$  samples from the biased distribution  $\mathcal{D}_{\beta}$ , the probability that the absolute difference between the estimated normalizing factor and the true normalizing factor exceeds  $\epsilon \frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}} w(\mathbf{x}_i, y_i)}$  is bounded by  $\delta$ . This provides a bound on the number of biased samples needed to ensure an accurate estimation of the normalizing factor.

We begin by bounding the absolute difference between our estimate of the normalizing factor and the true normalizing factor, and we find the number of biased samples needed to keep this difference with  $\epsilon \frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}} w(\mathbf{x}_{i}, y_{i})}$  with probability  $1 - \frac{\delta}{2k+2}$ .

**Lemma A.2.** With 
$$m_{\beta} \geq \frac{1}{2\epsilon^2} \ln \frac{2(2k+2)}{\delta}$$
 samples from  $\mathcal{D}_{\beta}$ : 
$$\Pr\left[\left|\frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}} w(x_i, y_i)} - \frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(x, y)]}\right| > \frac{\epsilon m_{\beta}}{\sum_{i=1}^{k} w(x_i, y_i)}\right] \leq \frac{\delta}{2k+2}$$

*Proof.* We can use a Hoeffding bound, as  $w(\mathbf{x}, y) \in [1, \frac{1}{\prod_{i=1}^k \beta_i}]$ 

$$\begin{split} & \Pr[\sum_{i=1}^{m_{\beta}} w(\mathbf{x}_{i}, y_{i}) - m_{\beta} \mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}, y)]| > \epsilon m_{\beta} \mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}, y)] \leq 2e^{\frac{-2\epsilon^{2} m \mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}, y)]^{2}}{(\frac{1}{\prod_{i=1}^{k} \beta_{i}} - 1)^{2}}} \\ & \Longrightarrow \Pr\left[\left|\frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}, y)]} - \frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}} w(\mathbf{x}_{i}, y_{i})}\right| > \frac{\epsilon m_{\beta}}{\sum_{i=1}^{m_{\beta}} w(\mathbf{x}_{i}, y_{i})}\right] \leq 2e^{\frac{-2\epsilon^{2} m \mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}, y)]^{2}}{(\frac{1}{\prod_{i=1}^{k} \beta_{i}} - 1)^{2}}} \end{split}$$

Setting this to be less than  $\delta$ , we see that this requires  $m \geq \frac{(\prod_{i=1}^k \beta_i^{-1} - 1)^2}{-2\epsilon^2 \mathbb{E}_{\mathcal{D}_\beta}[w(\mathbf{x},y)]^2} \ln \frac{\delta}{2(2k+2)}$  samples from the biased distribution:

$$2e^{\frac{-2\epsilon^2 m \mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x},y)]^2}{(\frac{1}{\prod_{i=1}^k \beta_i} - 1)^2}} \leq \frac{\delta}{2k+2}$$

$$\Rightarrow \frac{-2\epsilon^2 m \mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x},y)]^2}{(\frac{1}{\prod_{i=1}^k \beta_i} - 1)^2} \leq \ln \frac{\delta}{2(2k+2)}$$

$$\Rightarrow -2\epsilon^2 m \mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x},y)]^2 \leq (\frac{1}{\prod_{i=1}^k \beta_i} - 1)^2 \ln \frac{\delta}{2(2k+2)}$$

$$\Rightarrow -2\epsilon^2 m \mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x},y)]^2 \leq (\prod_{i=1}^k \beta_i^{-1} - 1)^2 \ln \frac{\delta}{2(2k+2)}$$

$$\Rightarrow m \geq \frac{(\prod_{i=1}^k \beta_i^{-1} - 1)^2}{-2\epsilon^2 \mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x},y)]^2} \ln \frac{2(2k+2)}{\delta}$$

Finally, we can upper bound the right-hand side to say that if we have at least  $\frac{1}{2\epsilon^2} \ln \frac{2(2k+2)}{\delta}$  samples, we achieve the desired bound.

Lemma A.2 provides a bound on the absolute difference between the estimated normalizing factor and the true normalizing factor. It states that with a sufficient number of samples from the biased distribution  $\mathcal{D}_{\beta}$ , specifically  $m_{\beta} \geq \frac{1}{2\epsilon^2} \ln \frac{2(2k+2)}{\delta}$ , the probability of the absolute difference exceeding  $\frac{\epsilon m_{\beta}}{\sum_{i=1}^{m_{\beta}} w(\mathbf{x}_i, y_i)}$  is bounded by  $\delta$ .

In Lemma A.3, we prove that the reweighted loss  $L_{S_{\beta}\beta^{-1}}$  normalized by the empirical estimate  $\frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}} w(\mathbf{x}_{i}, y_{i})}$  is always less than or equal to 1. This result allows us to upper bound this term in the triangle inequality of Lemma A.1 by 1.

To prove this lemma, we start by expressing the reweighted loss as the sum of weighted indicator functions  $\mathbb{I}(h(\mathbf{x}_i) \neq y_i)$  divided by  $m_\beta$ . We then multiply each term by  $w(\mathbf{x}_i, y_i)$  and divide the whole expression by  $\sum_{i=1}^{m_\beta} w(\mathbf{x}_i, y_i)$ .

Simplifying the expression, we find that the numerator and denominator cancel out, resulting in the bound  $\frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}} w(\mathbf{x}_i, y_i)} L_{S_{\beta}\beta^{-1}} \leq 1$ . This shows that the reweighted loss normalized by  $\frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}} w(\mathbf{x}_i, y_i)}$  is always less than or equal to 1.

Therefore, Lemma A.3 states that the normalized reweighted loss is bounded by 1, which is a crucial result for subsequent analysis and bounding of terms in Lemma A.1.

Lemma A.3. 
$$\frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}} w(\mathbf{x}_i, y_i)} L_{S_{\beta}\beta^{-1}} \leq 1$$

Proof.

$$\frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}} w(\mathbf{x}_{i}, y_{i})} L_{S_{\beta}\beta^{-1}} = \frac{m_{\beta}}{\sum_{i=1}^{m_{\beta}} w(\mathbf{x}_{i}, y_{i})} \frac{1}{m_{\beta}} \sum_{i=1}^{m_{\beta}} w(\mathbf{x}_{i}, y_{i}) \mathbb{I}(h(\mathbf{x}_{i}) \neq y_{i}) \leq \frac{\sum_{i=1}^{m_{\beta}} w(\mathbf{x}_{i}, y_{i})}{\sum_{i=1}^{m_{\beta}} w(\mathbf{x}_{i}, y_{i})} = 1$$

## B Part B

In Lemma B.1, we show that the ratio of the expected values of y given  $\mathbf{x}$  for samples in group  $G_i$  and samples not in group  $G_i$  is equal to  $p_i$ . Here,  $p_i$  represents the marginal probability that an example is positive if it belongs to group i, and  $q_i$  represents the marginal probability that an example falls in group i.

To prove this lemma, we start by considering the numerator and denominator separately. By the definition of expectation, the numerator represents the expected value of y given x for samples in

group  $G_i$ . Similarly, the denominator represents the expected value of y given  $\mathbf{x}$  for samples not in group  $G_i$ .

By assuming independence of the groups, we can express the numerator and denominator in terms of the probabilities of belonging to each group. We multiply the probability of being positive given that a sample is in i,  $p_i$ , with the product of the probabilities of being positive and belonging to the other groups  $(\prod_{j\neq i}q_jp_j)$ . Since the denominator involves the same product of probabilities, the terms cancel out, leaving us with  $p_i$ .

Therefore, Lemma B.1 establishes that the ratio of expected values of y given  $\mathbf{x}$  for samples in group  $G_i$  and samples not in group  $G_i$  is equal to the probability  $p_i$ . This result highlights the relationship between the expected values and the group base rates, which will be useful for further analysis.

**Lemma B.1.** 
$$\frac{\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\in G_i]}{\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\notin G_i']} = p_i$$

Proof. By independence,

$$\frac{\mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \in G_i]}{\mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \notin G_i]} = \frac{p_i \prod_{j \neq i} q_j p_j}{\prod_{j \neq i} q_j p_j} = p_i$$

Lemma B.2 states the sample size required to estimate  $\hat{p}_i$ , the empirical estimate of  $p_i$  (the base positive rate of group i), within a certain level of precision.

The proof begins by considering  $m_i$  independent Bernoulli trials with success probability  $p_i$ . Let S be the number of successes out of these trials, and  $\hat{p}$  be the ratio of successes to the total number of trials, which serves as the empirical estimate of  $p_i$ .

Using the multiplicative Chernoff Bound, the lemma establishes an upper bound on the probability that the absolute difference between the estimated  $\hat{p}_i$  and the true value  $p_i$  exceeds a given threshold  $p_i \epsilon$ . This upper bound is given by  $2e^{-\frac{m_i \epsilon^2 p_i}{3}}$ .

To ensure that the estimated  $\widehat{p}_i$  falls within the desired threshold with a probability of at least  $1-\frac{\delta}{2k+2}$ , we set  $\delta$  to be greater than or equal to  $2e^{-\frac{m_i\epsilon^2p_i}{3}}$ . Solving this inequality for  $m_i$ , we find that  $m_i \geq \frac{3}{\epsilon^2p_i} \ln \frac{2(2k+2)}{\delta}$ .

Thus, Lemma B.2 concludes that a sample size of at least  $\frac{3}{\epsilon^2 p_i} \ln \frac{2(2k+2)}{\delta}$  is required to estimate  $\hat{p}_i$  with a high probability, satisfying the precision constraint given by Equation 1.

**Lemma B.2.** (Disjoint Case) Given  $\epsilon, \delta \in (0,1)$ , we require  $m_i \geq \frac{3}{\epsilon^2 p_i} \ln \frac{2(2k+2)}{\delta}$  samples of unbiased data to estimate  $\hat{p}_i$  such that with probability  $1 - \frac{\delta}{2k+2}$ ,

$$|\widehat{p}_i - p_i| < p_i \epsilon \tag{1}$$

*Proof.* Let  $X_1, X_2, \ldots, X_m$  be m independent Bernoulli trials with success probability  $p_i$ . Let  $S = X_1 + X_2 + \cdots + X_m$  be the number of successes, and  $\widehat{p} = S/m_i$  be the empirical estimate of  $p_i$ .

By the multiplicative Chernoff Bound, for any  $\epsilon > 0$  and  $\delta > 0$ , we have:

$$\Pr\left(|\widehat{p} - p| \ge p\epsilon\right) \le 2e^{-\frac{m\epsilon^2 p_i}{3}}.\tag{2}$$

Thus, by setting 
$$\frac{\delta}{2k+2} \ge 2e^{-\frac{m_i\epsilon^2p_i}{3}}$$
 and solving for  $m_i$ , we get  $m_i \ge \frac{3}{\epsilon^2p_i} \ln \frac{2(2k+2)}{\delta}$ 

Lemma B.3 states the sample complexity requirement for estimating  $\hat{p_i}$ , the probability of positive labels in a subgroup i, in the intersectional case. Given a desired accuracy  $\epsilon$  and confidence level  $\delta$ , the lemma provides lower bounds on the number of samples from  $G_i$  and the number of samples not from  $G_i$  needed to estimate  $\hat{p_i}$  within an additive error of  $\epsilon p_i$  with probability at least  $1 - \frac{\delta}{2k+2}$ .

The lemma states that to estimate  $p_i$  with a relative error of at most  $\epsilon$  and a confidence of at least  $1-\frac{\delta}{2k+2}$ , one needs to collect at least  $m_i$  samples from  $G_i$  and  $m_{-i}$  samples not from  $G_i$ , where the expressions in the lemma give  $m_i$  and  $m_{-i}$ . The sample size guarantees are derived by using a multiplicative Chernoff bound to provide concentration inequalities for the empirical means of y in  $G_i$  and  $y \notin G_i$ , where y is the label of the sample.

In the intersectional case, as our estimate for  $\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\in G_i]$ , we will take  $\frac{1}{m_i}\sum_{j=1}^{m_i}y_j\mathbb{I}[\mathbf{x}_j\in G_i]$ , and as our estimate for  $\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\notin G_i]$ , we will use  $\frac{1}{m_{-i}}\sum_{j=1}^{m_{-i}}y_j\mathbb{I}[\mathbf{x}_j\notin G_i]$ . The estimated values for  $\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\in G_i]$  and  $\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\notin G_i]$  are obtained by averaging the labels within the respective subgroups. The lemma establishes that if the sample sizes satisfy  $m_i\geq \frac{27\ln\frac{4(2k+2)}{\delta}}{\epsilon^2\prod_{j=1}^k p_j}$  and  $m_{-i}\geq \frac{27\ln\frac{4(2k+2)}{\delta}}{\epsilon^2\mathbb{E}_{\mathcal{D}}\prod_{j\neq i} p_j}$ , then with probability at least  $1-\frac{\delta}{2k+2}$ , the estimated  $\hat{p}_i$  will be within an error of  $\epsilon p_i$  of the true  $p_i$ .

This lemma considers the influence of group positive rates and proportions —  $p_i$  and  $q_i$  — on the sample size requirements, with the denominators involving the product of these probabilities. These bounds ensure that the sample sizes account for the impact of group probabilities in the intersectional case

**Lemma B.3.** (Intersectional Case) Given  $\epsilon, \delta \in (0,1)$ , we require  $m_i \geq \frac{27}{\epsilon^2 \prod_{j=1}^k p_j} \ln \frac{4(2k+2)}{\delta}$  and  $m_{-i} \geq \frac{27}{\epsilon^2 \prod_{j\neq i} p_j} \ln \frac{4(2k+2)}{\delta}$  samples of unbiased data to estimate  $\hat{p}_i$  such that with probability  $1 - \frac{\delta}{2k+2}$ 

$$|\hat{p_i} - p_i| < p_i \epsilon$$

*Proof.* We will begin by deriving concentration bounds for the numerators and denominators separately. Using the multiplicative Chernoff bound, for any  $0 \le \epsilon \le 1$ 

$$Pr[|\sum_{i=1}^{m_i} y_j \mathbb{I}[\mathbf{x}_j \in G_i] - m_i \mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \in G_i]| \ge \frac{\epsilon}{3} m_i \mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \in G_i] \le 2e^{-\frac{\epsilon^2 m_i \mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \in G_i]}{27}}$$

Setting this probability to be bounded above by  $\frac{\delta}{2(2k+2)}$ , we see that

$$2e^{-\epsilon^2 m_i \mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\in G_i]/27} \le \frac{\delta}{2(2k+2)}$$

$$e^{-\epsilon^2 m_i \mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\in G_i]/27} \le \frac{\delta}{4(2k+2)}$$

$$-\epsilon^2 m_i \mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\in G_i]/27 \le \ln\frac{\delta}{4(2k+2)}$$

$$m_i \ge \frac{27 \ln\frac{4(2k+2)}{\delta}}{\epsilon^2 \mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\in G_i]}$$

Therefore, if the size of group i is at least  $\frac{27 \ln \frac{4(2k+2)}{\epsilon^2 \mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \in G_i]}}{\epsilon^2 \mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \in G_i]}$ , we have confidence at least  $1 - \frac{\delta}{2(2k+2)}$  that our empirical estimate for  $|m_i|\mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \in G_i]$  will be within a multiplicative factor of  $1 \pm \frac{\epsilon}{3}$  of its expected value.

Similarly,

$$Pr[|\sum_{i=1}^{m_{-i}} y_j \mathbb{I}[\mathbf{x}_j \notin G_i] - m_{-i} \mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \notin G_i]| \ge \frac{\epsilon}{3} m_{-i} \mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \notin G_i] \le 2e^{-\epsilon^2 m_{-i} \mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \notin G_i]/27}$$

If we want this probability to be less than  $\frac{\delta}{2(2k+2)}$ , this implies that we need

$$m_{-i} \ge \frac{27 \ln \frac{4}{\delta}}{\epsilon^2 \mathbb{E}_{\mathcal{D}}[y | \mathbf{x} \notin G_i]}$$

Then, with probability  $1 - \delta$ , and as long as

$$m_{i} \geq \frac{27 \ln \frac{4(2k+2)}{\delta}}{\epsilon^{2} \mathbb{E}_{\mathcal{D}}[y | \mathbf{x} \in G_{i}]}$$
$$m_{-i} \geq \frac{27 \ln \frac{4(2k+2)}{\delta}}{\epsilon^{2} \mathbb{E}_{\mathcal{D}}[y | \mathbf{x} \notin G_{i}]}$$

we have that both

$$(1 - \frac{\epsilon}{3}) m_i \mathbb{E}_{\mathcal{D}}[y | \mathbf{x} \in G_i] \le \sum_{j=1}^{m_i} y_j \mathbb{I}[\mathbf{x}_j \in G_i] \le (1 + \frac{\epsilon}{3}) m_i \mathbb{E}_{\mathcal{D}}[y | \mathbf{x} \in G_i]$$

$$(1 - \frac{\epsilon}{3}) m_{-i} \mathbb{E}_{\mathcal{D}}[y | \mathbf{x} \notin G_i] \le \sum_{j=1}^{m_{-i}} y_j \mathbb{I}[\mathbf{x}_j \notin G_i] \le (1 + \frac{\epsilon}{3}) m_{-i} \mathbb{E}_{\mathcal{D}}[y | \mathbf{x} \notin G_i]$$

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$$\frac{(1 - \frac{\epsilon}{3})m_i \mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \in G_i]}{(1 + \frac{\epsilon}{3})m_{-i}\mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \notin G_i]} \le \frac{\sum_{j=1}^{m_i} y_j \mathbb{I}[\mathbf{x}_j \in G_i]}{\sum_{j=1}^{m_{-i}} y_j \mathbb{I}[\mathbf{x}_j \notin G_i]} \le \frac{(1 + \frac{\epsilon}{3})m_i \mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \in G_i]}{(1 - \frac{\epsilon}{3})m_{-i}\mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \notin G_i]} 
(1 - \epsilon) \frac{m_i \mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \in G_i]}{m_{-i}\mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \notin G_i]} \le \frac{\sum_{j=1}^{m_i} y_j \mathbb{I}[\mathbf{x}_j \in G_i]}{\sum_{j=1}^{m_{-i}} y_j \mathbb{I}[\mathbf{x}_j \notin G_i]} \le (1 + \epsilon) \frac{m_i \mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \in G_i]}{m_{-i}\mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \notin G_i]}$$

Finally, this implies that with the required sample size:

$$\begin{split} ⪻[|\frac{\sum_{j=1}^{m_i}y_j\mathbb{I}[\mathbf{x}_j\in G_i]}{\sum_{j=1}^{m_i}y_j\mathbb{I}[\mathbf{x}_j\notin G_i]} - \frac{m_i\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\in G_i]}{m_{-i}\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\notin G_i]}| > \epsilon\frac{m_i\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\in G_i]}{m_{-i}\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\notin G_i]}| \le \frac{\delta}{2k+2} \\ &\Longrightarrow Pr[|\frac{\frac{1}{m_i}\sum_{j=1}^{m_i}y_j\mathbb{I}[\mathbf{x}_j\in G_i]}{\sum_{j=1}^{m_{-i}}y_j\mathbb{I}[\mathbf{x}_j\notin G_i]} - \frac{\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\in G_i]}{\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\notin G_i]}| > \epsilon\frac{\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\in G_i]}{\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\notin G_i]}| \le \frac{\delta}{2k+2} \\ &\Longrightarrow Pr[|\hat{p_i} - p_i| > \epsilon p_i] \le \frac{\delta}{2k+2} \end{split}$$

Lastly, we can lower bound the expectations  $\mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \in G_i]$  and  $\mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \notin G_i]$  using the probabilities  $p_i$  defined earlier. Specifically, we have:

$$\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\in G_i] \ge \prod_{j=1}^k p_j \tag{3}$$

and similarly,

$$\mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \notin G_i] \ge \prod_{j \ne i} p_j \tag{4}$$

Lemma B.5 states that the ratio of the expected outcome variable y given that  $\mathbf{x} \in \mathcal{D}_{\beta}$  belongs to group  $G_i$  and the expected outcome variable y given that  $\mathbf{x}$  belongs to the complement of  $\mathcal{D}_{\beta}$  but not  $G_i$  is equal to  $\beta_i p_i$ , where  $p_i$  is the probability that an example is positive given that it is in  $G_i$  of the unbiased data, and  $\beta_i$  is the probability of a positive example from group i surviving into  $\mathcal{D}_{\beta}$ .

The proof uses the independence of the variables to show that the ratio simplifies to  $\beta_i p_i$ . Specifically, the numerator in the ratio is the product of  $\beta_i$ , which is the proportion of positive observations in  $G_i$  that are included in  $\mathcal{D}_{\beta}$ , and the product of  $p_j$  over all  $j \neq i$ , which is the probability that an example is positive given that it belongs to each group other than  $G_i$ . The denominator is the same as the numerator but without the  $\beta_i$  term. Canceling out the common terms in the numerator and denominator yields the result  $\beta_i p_i$ .

Lemma B.4. 
$$\frac{\mathbb{E}_{\mathcal{D}_{\beta}}[y|\mathbf{x}\in G_i]}{\mathbb{E}_{\mathcal{D}_{\beta}}[y|\mathbf{x}\notin G_i]}=\beta_i p_i$$

Proof. Again by independence,

$$\frac{\mathbb{E}_{\mathcal{D}_{\beta}}[y|\mathbf{x} \in G_i]}{\mathbb{E}_{\mathcal{D}_{\beta}}[y|\mathbf{x} \notin G_i]} = \frac{\beta_i p_i \prod_{j \neq i} \beta_j q_j p_j}{\prod_{j \neq i} \beta_j q_j p_j} = \beta_i p_i$$

An analogous result follows for the biased sample  $\mathcal{D}_{\beta}$ . Lemma B.4, provides a requirement for estimating the value of  $p_i\beta_i$  using biased data. It states that given parameters  $\epsilon$  and  $\delta$  in the range (0,1), we need a minimum of  $m_{\beta}$  samples of biased data. Here,  $m_{\beta_i}$  represents the number of samples in group i of the biased data.

The lemma guarantees that with probability  $1 - \frac{\delta}{2k+2}$ , the estimated value  $\widehat{p_i\beta_i}$  will be within a certain range of the true value  $p_i\beta_i$ . Specifically, the absolute difference between  $\widehat{p_i\beta_i}$  and  $p_i\beta_i$  will be less than  $\epsilon p_i\beta_i$ .

To prove this, we consider the indicator variable  $Z_j$ , which denotes whether sample j in group i is a positive sample. The sum of these indicators, denoted as R, represents the total number of positive samples in group i. The expected value of  $Z_j$  is  $p_i\beta_i$ , and the expected value of R is  $m_{\beta_i}p_i\beta_i$ , where  $m_{\beta_i}$  is the number of samples in group i.

By applying a multiplicative Chernoff bound, we obtain an upper bound on the probability of the absolute difference between  $\widehat{p_i\beta_i}$  and  $p_i\beta_i$  exceeding  $\epsilon p_i\beta_i$ . This probability is bounded above by  $2\exp\left(-\frac{\epsilon^2 m_{\beta_i}p_i\beta_i}{3}\right)$ .

To ensure that this probability is less than  $\frac{\delta}{2k+2}$ , we set the inequality  $2\exp\left(-\frac{\epsilon^2 m_{\beta_i} p_i \beta_i}{3}\right) \leq \frac{\delta}{2k+2}$ .

Solving this inequality yields the minimum requirement for  $m_{\beta_i}$ , which is  $m_{\beta_i} \geq \frac{3 \ln \frac{2(2k'+2)}{\delta}}{\epsilon^2 p_i \beta_i}$ . Thus, we need at least  $m_{\beta_i}$  biased data samples to estimate  $p_i \beta_i$  within the desired range with a probability of at least  $1 - \frac{\delta}{2k+2}$ .

**Lemma B.5.** (Disjoint case) Given  $\epsilon, \delta \in (0,1)$ , we require  $m_{\beta_i} \geq \frac{3}{p_i \beta_i \epsilon^2} \left( \ln \frac{2(2k+2)}{\delta} \right)$  samples of biased data to estimate  $p_i \beta_i$  such that with probability  $1 - \frac{\delta}{2k+2}$ ,

$$|p_i\beta_i - \widehat{p_i\beta_i}| < p_i\beta_i\epsilon$$

*Proof.* Let  $Z_j$  be the indicator that sample j in group i is a positive sample, and let  $R = \sum_{j=1}^{m_{\beta_i}} Z_j$  be their sum. Then,  $\mathbb{E}[Z] = p_i \beta_i$  and  $\mathbb{E}[S] = m_{\beta_i} p_i \beta_i$ , where  $m_{\beta_i}$  is the number of samples in group i of the biased data. Applying a multiplicative Chernoff bound gives:

$$\mathbb{P}\left(\left|\widehat{p_i\beta_i} - p_i\beta_i\right| \ge \epsilon p_i\beta_i\right) \le 2\exp\left(-\frac{\epsilon^2 m_{\beta_i} p_i\beta_i}{3}\right) \tag{5}$$

Setting this probability to be less than  $\delta$  gives

$$2\exp\left(-\frac{\epsilon^2 m_{\beta_i} p_i \beta_i}{3}\right) \le \frac{\delta}{2k+2}$$

$$\implies -\epsilon^2 m_{\beta_i} p_i \beta_i \le 3\ln\frac{\delta}{2(2k+2)}$$

$$\implies m_{\beta_i} \ge \frac{3\ln\frac{2(2k+2)}{\delta}}{\epsilon^2 n_i \beta_i}$$

Lemma B.6 states that if the sizes of each group in the biased dataset and the biased dataset as a whole are large enough, then with high probability, the estimated value of  $\beta_i p_i$  is close to the true value within a given tolerance  $\epsilon$ , based on the sample mean. The probability of error is upper bounded by  $\frac{\delta}{2k+2}$ . The proof is similar to the previous lemma and relies on the multiplicative Chernoff bound.

**Lemma B.6.** Fix  $\epsilon, \delta > 0$ . Then if  $m_{\beta_i} \geq \frac{27}{\epsilon^2 \prod_{i=1}^k p_i \beta_i} \ln \frac{4(2k+2)}{\delta}$  and  $m_{\beta_{-i}} \geq \frac{27}{\epsilon^2 \prod_{i\neq j}^k p_i \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,

$$Pr[|\widehat{\beta_i p_i} - \beta_i p_i| > p_i \beta_i \epsilon] \le \frac{\delta}{2k+2}$$

*Proof.* The proof follows the same structure as the previous lemma.

Lemmma B.7 states that given the expected values of the outcomes y for samples in group i and samples not in group i of  $\mathcal{D}_{\beta}$ , as well as the expected values of y for samples in group i of  $\mathcal{D}$  and samples not in group i of  $\mathcal{D}$ , we can calculate the value of  $\frac{1}{\beta_i}$ . Specifically, we can calculate it as the product of  $\frac{\mathbb{E}_{\mathcal{D}_{\beta}}[y|\mathbf{x}\notin G_i]}{\mathbb{E}_{\mathcal{D}_{\beta}}[y|\mathbf{x}\in G_i]}$  and  $\frac{\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\notin G_i]}{\mathbb{E}_{\mathcal{D}}[y|\mathbf{x}\notin G_i]}$ , both of which can be estimated from the available data. Finally, we use the fact that  $\frac{p_i}{\beta_i p_i} = \frac{1}{\beta_i}$  to arrive at the desired result. This lemma follows directly from the definitions and properties of the subpopulations of group i in  $\mathcal{D}_{\beta}$  and  $\mathcal{D}$ .

**Lemma B.7.** Using these results, we can then calculate  $\frac{1}{\beta_i}$  as

$$\frac{\mathbb{E}_{\mathcal{D}_{\beta}}[y|\mathbf{x} \notin G_i]}{\mathbb{E}_{\mathcal{D}_{\beta}}[y|\mathbf{x} \in G_i]} \frac{\mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \in G_i]}{\mathbb{E}_{\mathcal{D}}[y|\mathbf{x} \notin G_i]} = \frac{p_i}{\beta_i p_i} = \frac{1}{\beta_i}$$

Lemma B.8, states that if the sizes of group i in  $\mathcal{D}_{\beta}$  and the size of the population not in group i of  $\mathcal{D}_{\beta}$  satisfy certain conditions, then with high probability, our estimated inverse  $\frac{1}{\beta_i}$  will be within an error margin of  $\frac{3\epsilon}{\beta_i}$  from the true inverse  $\frac{1}{\beta_i}$ . By manipulating the ratio inequality between the estimated and true ratios of parameters, we obtain an inequality for the difference between the estimated and true inverses. By setting the probability of violating this inequality to be less than  $\delta$ , we conclude that the estimated inverse is a reliable approximation of the true inverse with a high level of confidence.

**Lemma B.8.** Set 
$$\epsilon, \delta > 0$$
. If  $m_{\beta_i} \geq \frac{3^3}{\epsilon^2 \prod_{i=1}^k p_i \beta_i} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ .

$$Pr\left[\left|\frac{1}{\hat{\beta}_i} - \frac{1}{\beta_i}\right| > \frac{3\epsilon}{\beta_i}\right] \le \frac{\delta}{2k+2}$$

*Proof.* With probability  $1 - \delta$ ,

$$\frac{(1-\epsilon)p_i}{(1+\epsilon)\beta_i p_i} \le \frac{\hat{p_i}}{\hat{\beta_i p_i}} \le \frac{(1+\epsilon)p_i}{(1-\epsilon)\beta_i p_i}$$

For  $\epsilon < \frac{1}{3}$ ,

$$(1-3\epsilon)\frac{p_i}{\beta_i p_i} \le \frac{\hat{p_i}}{\hat{\beta_i p_i}} \le (1+3\epsilon)\frac{p_i}{\beta_i p_i}$$

Then

$$Pr\left[|\frac{\hat{p_i}}{\hat{\beta_i p_i}} - \frac{p_i}{\beta_i p_i}| > 3\epsilon \frac{p_i}{\beta_i p_i}\right] \le \frac{\delta}{2k+2} \implies Pr\left[|\frac{1}{\widehat{\beta_i}} - \frac{1}{\beta_i}| > \frac{3\epsilon}{\beta_i}\right] \le \frac{\delta}{2k+2}$$

Lemma B.9 states given  $\epsilon, \delta > 0$  and sample sizes  $m_{\beta_i}, m_{\beta_{-i}}, m_i$  and  $m_{-i}$ , if the sizes of the samples satisfy certain conditions, then with high probability  $(1 - \frac{\delta}{2k+2})$ , the estimated product of inverse biases  $\frac{1}{\prod_{i=1}^k \hat{\beta}_i}$  is close to the true product of inverse biases  $\frac{1}{\prod_{i=1}^k \beta_i}$  within a margin of  $\frac{3\epsilon}{\prod_{i=1}^k \beta_i}$ . By using the bounds from Lemma B.8, we establish that the estimated inverse biases  $\frac{1}{\hat{\beta}_i}$  deviate from the true inverse biases  $\frac{1}{\beta_i}$  by at most  $\frac{3\epsilon}{\beta_i}$  with probability at least  $1 - \frac{\delta}{2k+2}$ . By applying these bounds to the product of inverse biases, we conclude that the estimated product  $\frac{1}{\prod_{i=1}^k \hat{\beta}_i}$  is within  $\frac{3\epsilon}{\prod_{i=1}^k \beta_i}$  of the true product  $\frac{1}{\prod_{i=1}^k \beta_i}$  with probability at least  $1 - \frac{\delta}{2k+2}$ .

**Lemma B.9.** Set  $\epsilon, \delta > 0$ . If  $m_{\beta_i} \geq \frac{3^3}{\epsilon^2 \prod_{i=1}^k p_i \beta_i} \ln \frac{4(2k+2)}{\delta}$ ,  $m_{\beta_{-i}} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j \beta_j} \ln \frac{4(2k+2)}{\delta}$ ,  $m_i \geq \frac{3^3}{\epsilon^2 \prod_{i=1}^k p_i} \ln \frac{4(2k+2)}{\delta}$  and  $m_{-i} \geq \frac{3^3}{\epsilon^2 \prod_{j \neq i} p_j} \ln \frac{4(2k+2)}{\delta}$  for all groups  $G_i$ 

$$\Pr\left[\left|\frac{1}{\prod_{i=1}^{k} \hat{\beta}_{i}} - \frac{1}{\prod_{i=1}^{k} \beta_{i}}\right| \ge \frac{3\epsilon}{\prod_{i=1}^{k} \beta_{i}}\right] \le \frac{\delta}{2k+2}$$

*Proof.* We can use our bound on the marginal  $\frac{1}{\hat{\beta}_i}$  to bound the difference of this product from its expectation. Lemma B.8 tells us that with probability  $1-\delta$ ,  $|\frac{1}{\hat{\beta}_i}-\frac{1}{\beta_i}|<3\epsilon\frac{1}{\beta}$ . Then, with probability  $1-\delta$  and  $\epsilon<1/3$ 

$$\left| \frac{1}{\prod_{i=1}^{k} \hat{\beta}_{i}} - \frac{1}{\prod_{i=1}^{k} \beta_{i}} \right| \leq \left| \frac{1}{\prod_{i=1}^{k} \beta_{i}} (1 - 3\epsilon) - \frac{1}{\prod_{i=1}^{k} \beta_{i}} \right| \leq \frac{1}{\prod_{i=1}^{k} \beta_{i}} (3\epsilon)^{k} \leq \frac{3\epsilon}{\prod_{i=1}^{k} \beta_{i}}$$

## C Part C

Lemma C.1 states that for any hypothesis h, the expected loss of h on the distribution  $\mathcal{D}$  is equal to the expected loss of h on the weighted distribution  $\mathcal{D}_{\beta}$  scaled by the inverse of the expected weight  $E_{D_{\beta}}[w(\mathbf{x},y)]$  of the weighted distribution.

In other words, if we weight the examples in  $\mathcal{D}$  by the weights given by  $\beta$ , the expected loss of h on the weighted distribution  $\mathcal{D}_{\beta}$  divided by the expected weight of  $\mathcal{D}_{\beta}$  is equal to the expected loss of h on the original distribution  $\mathcal{D}$ .

The proof uses the definition of expected value and the fact that the expected weight of  $\mathcal{D}_{\beta}$  is the denominator in the definition of the weighted empirical risk.

Lemma C.1. 
$$\frac{1}{E_{D_{\beta}}[w(\mathbf{x},y)]}\mathbb{E}_{\mathcal{D}_{\beta}}[L_{S_{\beta}\beta^{-1}}(h)] = L_{\mathcal{D}}(h)$$

Proof.

$$\frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}, y)]} \mathbb{E}_{\mathcal{D}_{\beta}}[L_{S_{\beta}\beta^{-1}}(h)] = \frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}, y)]} \mathbb{E}_{\mathcal{D}_{\beta}}[\frac{1}{m_{\beta}} \sum_{i=1}^{m_{\beta}} w(\mathbf{x}_{i}, y_{i}) \mathbb{I}(h(\mathbf{x}_{i}) \neq y_{i})]$$

$$= \frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}, y)]} \mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}, y) \mathbb{I}(h(\mathbf{x}) \neq y)]$$

$$= \sum_{x, y \sim \mathcal{D}_{\beta}} w(\mathbf{x}, y) \mathbb{I}(h(\mathbf{x}) \neq y) \frac{p_{D_{\beta}}(\mathbf{x}, y)}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}, y)]}$$

$$= \sum_{x, y \sim \mathcal{D}} \mathbb{I}(h(\mathbf{x}_{i}) \neq y_{i}) p_{D}(\mathbf{x}, y)$$

$$= L_{\mathcal{D}}(h)$$

Lemma C.2 states that with a sufficient number of samples, we can estimate the true expected loss  $L_{\mathcal{D}}(h)$  using the normalized and reweighted biased sample loss  $L_{S_{\beta}\beta^{-1}}(h)$ .

By applying a multiplicative Hoeffding bound, we can bound the difference between the normalized and reweighted biased sample loss and the true expected loss. The number of samples required to guarantee this bound is at least  $m_{\beta} \geq \frac{1}{2\epsilon^2} \ln \frac{2|H|(2k+2)}{\delta}$ .

In other words, by collecting sufficient samples, we can estimate the true expected loss within a desired precision, providing a reliable approximation for empirical risk minimization.

**Lemma C.2.** With  $m_{\beta} \geq \frac{1}{2\epsilon^2} \ln \frac{2|H|(2k+2)}{\delta}$  samples,

$$\Pr\left[\left|\frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\boldsymbol{x}_{i},y_{i})]}L_{S_{\beta}\beta^{-1}}(h) - L_{\mathcal{D}}(h)\right| > \epsilon L_{\mathcal{D}}(h)\right] \leq \frac{\delta}{2k+2}$$

*Proof.* From Lemma C.1, we know that  $\frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}_i,y_i)]}L_{S_{\beta}\beta^{-1}}(h)$  is bounded between 0 and  $\frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}_i,y_i)]\prod_{i=1}^k\beta_i}$ . Applying a multiplicative Hoeffding bound gives

$$\Pr\left[\left|\frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}_{i},y_{i})]}L_{S_{\beta}\beta^{-1}}(h)-L_{\mathcal{D}}(h)\right|>\epsilon L_{\mathcal{D}}(h)\right]\leq 2e^{-2\epsilon^{2}(\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}_{i},y_{i})])^{2}\prod_{i=1}^{k}\beta_{i}^{2}m}$$

Applying a union bound over the VC dimension of the class H, |H|, gives:

$$\Pr\left[\left|\frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}_{i},y_{i})]}L_{S_{\beta}\beta^{-1}}(h)-L_{\mathcal{D}}(h)\right|>\epsilon L_{\mathcal{D}}(h)\right]\leq 2|H|e^{-2\epsilon^{2}(\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}_{i},y_{i})])^{2}\prod_{i=1}^{k}\beta_{i}^{2}m}$$

Upper bounding by  $\frac{\delta}{2k+2}$  gives:

$$\begin{aligned} 2|H|e^{-2\epsilon^2(\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}_i,y_i)])^2\prod_{i=1}^k\beta_i^2m} &\leq \frac{\delta}{2k+2} \\ &\Longrightarrow e^{-2\epsilon^2(\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}_i,y_i)])^2\prod_{i=1}^k\beta_i^2m} \leq \frac{\delta}{2|H|(2k+2)} \\ &\Longrightarrow -2\epsilon^2(\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}_i,y_i)])^2\prod_{i=1}^k\beta_i^2m \leq \ln\frac{\delta}{2|H|(2k+2)} \\ &\Longrightarrow (\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}_i,y_i)])^2\prod_{i=1}^k\beta_i^2m \geq \frac{1}{-2\epsilon^2}\ln\frac{\delta}{2|H|(2k+2)} \\ &\Longrightarrow m \geq \frac{1}{2\epsilon^2}\ln\frac{2|H|(2k+2)}{\delta} \frac{1}{\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}_i,y_i)])^2\prod_{i=1}^k\beta_i^2} \end{aligned}$$

Finally, because  $\mathbb{E}_{\mathcal{D}_{\beta}}[w(\mathbf{x}_i, y_i)] \leq \frac{1}{\prod_{i=1}^k \beta_i}$ , choosing  $m \geq \frac{1}{2\epsilon^2} \ln \frac{2|H|(2k+2)}{\delta}$  suffices.