Individual Fairness in Sponsored Search Auctions

Shuchi Chawla *[1], Christina Ilvento †[2], and Meena Jagadeesan ‡[2]

[1]Department of Computer Science, University of Wisconsin-Madison
[2]Harvard John A. Paulson School of Engineering and Applied Sciences

Abstract

Fairness in advertising is a topic of particular interest in both the computer science and economics literatures, supported by theoretical and empirical observations. We initiate the study of tradeoffs between individual fairness and performance in online advertising, where advertisers place bids on ad slots for each user and the platform must determine which ads to display. Our main focus is to investigate the "cost of fairness": more specifically, whether a fair allocation mechanism can achieve utility close to that of a utility-optimal unfair mechanism. Motivated by practice, we consider both the case of many advertisers in a single category, e.g. sponsored results on a job search website, and ads spanning multiple categories, e.g. personalized display advertising on a social networking site, and show the tradeoffs are inherently different in these settings.

We prove lower and upper bounds on the cost of fairness for each of these settings. For the single category setting, we show constraints on the "fairness" of advertisers' bids are necessary to achieve good utility. Moreover, with bid fairness constraints, we construct a mechanism that simultaneously achieves a high utility and a strengthening of typical fairness constraints that we call **total variation fairness**. For the multiple category setting, we show that fairness relaxations are necessary to achieve good utility. We consider a relaxed definition based on user-specified category preferences that we call **user-directed fairness**, and we show that with this fairness notion a high utility is achievable. Finally, we show that our mechanisms in the single and multiple category settings compose well, yielding a high utility combined mechanism that satisfies user-directed fairness across categories and conditional total variation fairness within categories.

1 Introduction

In the ongoing discussion of what it means for automated decision-making systems to be fair, the topic of online advertising has merited particular interest. In the United States, segregated employment ads for men and women proved to be a flashpoint in the 1960s, and the introduction of ever-more finely-tuned advertising online has renewed concerns about discrimination in ads for critical categories such as employment, housing and credit. In fact, recent news articles and empirical studies have demonstrated troubling patterns of systematically different online advertising experiences for different gender or racial groups [2, 4, 3, 9, 1]. Although individual advertisers certainly have opportunities to use fine-grained targeting (in some cases, targeting individual users) to implement biased advertising strategies, there is both empirical evidence and theoretical support for the idea that these trends do not occur solely because of bad actors among advertisers. In fact, these trends can arise simply from competition between advertisers and other platform implementation decisions [9, 5, 4].

Dwork and Ilvento [5] initiated a theoretical study of fairness under composition in several different contexts including advertising. They observed that auctions between advertisers in different categories (e.g. employment and home goods) can result in unfair outcomes even if every advertiser believes they are behaving fairly. In particular, individuals who are equally qualified for employment may see different

*shuchi@cs.wisc.edu
†cilvento@g.harvard.edu, this work was funded in part by the Sloan Foundation.
‡mjagadeesan@college.harvard.edu

numbers of employment ads depending on differing levels of competition for their attention from home goods advertisers. Furthermore they argued that competition between advertisers targeting different demographics can exacerbate the effects of intersectionality[1] without obvious indications in larger group statistics. They presented mechanisms to ensure fairness in competitive settings (such as auctions), but noted that these mechanisms do not optimize for platform objectives such as revenue or maximal advertisement allocation.

In this paper we initiate a study of the tradeoffs between fairness and performance or utility in online advertising in order to quantify the "cost of fairness." We consider a simple stylized setting where k advertisers place bids on n users that arrive in sequence. Each user is shown a single ad. As users arrive, the platform must determine in an online fashion which ad to display to the user, potentially using randomness in its choice. We call this sequence of choices the allocation mechanism. The platform has two competing goals: (1) **Maximizing utility:** ads displayed must be maximally relevant or useful in that the sum of the bids corresponding to the displayed ads is (approximately) maximized; (2) **Ensuring fairness:** the allocation of ads should be fair across users. Our goal is to understand the conditions under which fairness and high utility can be achieved simultaneously. In particular, we ask: can a fair allocation mechanism achieve utility close to that of a utility-optimal unfair mechanism? We use the term "fair value" to describe the ratio between the utility of a fair mechanism and the optimal unfair utility[2].

Online advertising platforms are generally interested in optimizing a combination of different objectives (e.g. short term revenue obtained from the advertisers, total budget spend of the advertisers, social welfare or aggregate value of the matches as a proxy for long term market share, etc). We capture these considerations through a single simple objective: maximizing the utility of the allocation defined as the sum over the users of the expected bid of the ad displayed. The utility of the allocation is equal to the revenue obtained by the platform when the allocation mechanism is accompanied by a first-price payment rule. On the other hand, if advertisers bid their true values, the utility is equal to the social welfare or economic efficiency of the allocation. We ignore incentive issues in this paper and assume that advertisers' bids are not affected by the choices of the allocation mechanism.[3]

To formalize the notion of a fair allocation we use the concept of Individual Fairness [6], which requires *similar* individuals to be treated *similarly*. (A formal definition is provided in Section 2.) In the context of online advertising, we use the "multiple-task fairness" formalization proposed in [5], which requires that for any advertiser i and any two users u and v that are similarly qualified with respect to the category of advertiser i, the probabilities of displaying i's ad to u and v should be close, regardless of the behavior of other advertisers[4]. We start with this basic definition of fairness and consider both a strengthening (total variation fairness) and a weakening (user-directed fairness) of it in appropriate contexts. These variations are discussed below and formally defined in Section 2. Observe that multiple-task fairness as defined above requires specifying a similarity metric over users with respect to every advertiser i: this similarity metric can depend on the suitability and qualifications of the users for the particular category of products that the ad belongs to. As a result, ads for products or services that belong to very different categories may impose very different similarity metrics. For example, two users, one who works in healthcare and one who works in technology, may be equally qualified for a home loan, but have very different levels of qualification for a software engineering job. We assume that these metrics are fixed and functions of the attributes of the users and product categories. In particular, the metrics do not depend on preferences of the advertisers.

If advertisers are permitted to place significantly different bids on similar individuals, it turns out to be impossible to achieve a high utility fair allocation as we show in Section 2.2. Since we focus on the effects of composition (rather than incentivizing fair behavior in advertisers) in this work, we assume that the bids of every advertiser satisfy an appropriate notion of individual fairness. The notion of bid fairness that we consider is that advertiser i places similar bids on any two users u and v that are close with respect to the similarity metric for i's product. More specifically, we impose a *bid ratio condition* requiring that each advertiser's bids on any pair of users are within a certain multiplicative factor that depends on the distance between the users in the similarity metric. The mechanisms that we propose are guaranteed to be fair as long as each advertiser individually places fair bids.[5] This property is highly desirable from an implementation

[1]e.g. protecting "women" and "parents" separately, but failing to protect "women who are also parents."

[2]In particular, fair value is a quantity in $[0, 1]$. Mechanisms with higher fair value have higher utility.

[3]Moreover, we ignore advertiser budget issues.

[4]While the intention of [5] was to use multiple-task fairness for different metrics representing different categories, we also consider multiple-task fairness for advertisers with identical metrics (within a single category).

[5]We do not tackle the issue of how to enforce fair bidding. One way to enforce fair bidding is for the platform to set up

perspective, as each advertiser is only responsible for enforcing fairness on their own bids, and neither the advertisers nor the platform need to explicitly reason about or account for interference between advertisers beyond correctly executing the mechanism.

As we will show, the tradeoffs achievable between fairness and utility depend heavily on whether the competing advertisers are subject to similar or different metrics. Consider the following motivating example. Suppose there are $n+1$ users and n advertisers, each advertising a unique product and therefore subject to a different metric. In the similarity metric for product i, the users i and $n+1$ are considered "maximally qualified" for advertiser i, and are placed at a maximal distance from the rest of the users (i.e. users i and $n+1$ are essentially placed in a separate class). This means that user i is a "specialist" who is maximally qualified for product i, but maximally unqualified for any other product. On the other hand, user $n+1$ is a "jack of all trades": maximally qualified for each of the n products. Suppose that each advertiser i places a bid of 1 on users i and $n+1$ and a bid of 0 on everyone else. Each advertiser's bids are "fair" with respect to their respective metrics. Consider any allocation mechanism and suppose that this mechanism chooses an ad from distribution $(q_1, \cdots, q_n)$ to display for user $n+1$. Then, in order to respect multiple task fairness as defined above, the mechanism cannot allocate ad i to user i with probability much greater than q_i. As a result, the utility of any fair allocation is bounded by some constant, whereas an unfair allocation can achieve utility $n+1$. Roughly speaking, the platform is forced to choose between a high utility unfair allocation (i.e. allowing the "jack of all trades" user to see fewer ads in each category than their equally qualified "specialist" counterpart) and a poor utility fair allocation (i.e. limiting each "specialist" user to see only as many ads in their category as the "jack of all trades"). A main driver of the gap between fair and unfair allocations in this example is competition between product categories that impose very different similarity metrics.

For this reason, we distinguish between and focus most of our attention on two extreme cases: a single category with many advertisers and different categories each with a single advertiser. We also combine these results to consider the case of multiple categories with multiple advertisers. In Section 1.1, we summarize our results for the case where all advertisers are within a single category (and are subject to identical metrics over users). In Section 1.2, we summarize our results for the case where each advertiser is in a different category, each of which is subject to a different metric over users. In Section 1.3, we summarize our "intermediate" result that uses the results from the two extreme cases to study the case of multiple categories with multiple advertisers. In Section 1.4, we compare our work to concurrent work of Kim, Korolova, Rothblum, and Yona [8]. In Section 1.5, we outline the rest of the paper.

1.1 Fairness within a single category

The first setting we study is motivated by the example of sponsored results on a job search platform. In the context of a job search site, we expect that a particular job-seeker is searching within a narrow enough field (e.g. software engineering, small firm human resources, retail marketing) that each advertiser relevant to the user's search is subject to the same similarity metric.

As noted in [5], multiple-task fairness guarantees over advertisers in a single category will not be sufficient to guarantee fairness as it is intuitively understood.[6] Consider two users that are similar to each other but not identical, so that multiple task fairness requires their allocation probabilities for each ad to differ by at most ϵ additively. Suppose that the second user is consistently shown high paying job ads at a probability ϵ smaller than that of the first user, and is shown low paying jobs at a probability ϵ larger than that of the first user. For any individual ad, the probabilities that the two users see this ad are similar. However, if we consider the set of all high paying jobs, then the first user has a significant advantage over the second user.

In addition to original fairness problem, we remark that the ϵ-blowup can actually be exploited by dishonest advertisers to their advantage. In particular, advertisers have the opportunity to submit multiple different ads to the platform targeting certain individuals at slightly higher bids. In doing so, they may gain a slight advantage over the targeted users on each ad, adding up to a significant advantage across all of the ads taken together. Such a "splitting into subadvertisers" approach subverts the desired fairness guarantees.

a separate auditing mechanism that periodically audits bids placed in the recent past and penalizes advertisers that violate fairness requirements.

[6] As discussed before, we interpret multiple-task fairness over advertisers in a single category to mean each advertiser satisfies fairness with respect to its metric.

We present a much stronger definition for fairness for the single-category setting that addresses both the fairness and subversion issues raised above. We formally define, **total variation fairness**, in Section 2. Total variation fairness requires that the probability distributions over ads received by two similar users should be close to each other in total variation (or ℓ_1) distance. This means that for any subset of advertisers, the probability that a user observes an ad in this set is close to the probabilities that a similar user observes an ad in this set. As a result, any group of sub-advertisers is subject to the same fairness constraints as would have been on the original advertiser, thus addressing the subversion issue. An additional consequence is that if a user u regards some subset S as substitutes (e.g. high-paying jobs), then the definition gives fairness guarantees over the probability of viewing an ad in S. Note that our definition provides much stronger guarantees than the OR-fairness definition proposed in [5] for the single-category setting[7].

This setting gives rise to several questions. Is it easy to achieve a high utility fair allocation if the competition is among similar products? What if, in particular, the allocation mechanism simply assigns the ad with the highest bid to every user? Such a mechanism may not satisfy fairness, because small changes in bids can cause large jumps in allocation probabilities. In order to simultaneously achieve fairness and good utility, a mechanism must "smooth out" allocation probabilities across several top bids. How much information does the mechanism need to perform such a smoothening? Is it necessary to know the entire metric over users beforehand? Does the allocation need to be a carefully designed function of bids and allocations assigned to previously seen users?

Our contributions. We begin by establishing upper bounds on the utility achieved by any fair online allocation mechanism as a fraction of the utility of the optimal-in-hindsight (unfair) allocation. First, we show that a mechanism that determines its allocation purely on the ordering of the bids and that achieves multiple-task fairness must perform poorly in terms of utility.[8] For this reason, we permit the platform to use the *values* of the bids and identities of the advertisers in determining the allocation. Under this stronger model, we show upper bounds by constructing an instance with a uniform metric (i.e. a metric under which all users are the same distance d apart). We first show that without a condition on the bids, the mechanism can achieve at most a $\frac{1}{k}$ fraction of the unfair optimal utility in some cases, where k is the number of advertisers. Thus, we specify a bid ratio condition that requires that advertiser bids on any pair of users are within a multiplicative factor α. The factor α captures the strength of the bid fairness constraint. We show upper bounds on utility in terms on α: we prove that the fair value (i.e. the ratio between the utility of a fair mechanism and the optimal unfair utility) is bounded above by approximately $\alpha^{-1} + d$ for offline mechanisms satisfying multiple-task fairness and by approximately $\alpha^{-2} + d$ for online mechanisms satisfying multiple-task fairness, which demonstrates the need for a reasonably strong bid ratio condition to achieve good utility.

We complement these upper bounds with a positive result for the case of a general (not necessarily uniform) similarity metric. We construct an allocation mechanism that always achieves a good utility allocation and is guaranteed to satisfy total variation fairness as long as bids satisfy an appropriate bid ratio condition. Furthermore, this mechanism is *metric-oblivious* (the probability assignment does not explicitly use knowledge of the metric[9]), *history-oblivious* (the probability assignment does not depend on bids on previous users, which allows use in settings where users may appear in adversarial or unknown ordering), and *identity-oblivious* (the probability assignment does not depend on the identities of the advertisers). This allows us to consider a setting in which the platform is entirely metric-blind and a separate central body is responsible for auditing fair behavior of the advertisers.

We emphasize that our upper bounds (negative results) apply to mechanisms that satisfy the weaker multiple task fairness and are allowed to use all of the knowledge they possess—the metric over the entire universe and the bids and allocations of previous users. On the other hand, our lower bound (positive result) is achieved via a mechanism that satisfies the stronger total variation fairness and allows the platform to be metric blind.

[7]In particular, we provide OR-style guarantees on *all* subsets of *all* sizes, while the notion in [5] restricts to a single predetermined set.

[8]In position auctions, introduced by Edelman, Ostrovsky, and Schwarz [7] and Varian [10], the platform bases the allocation solely based on the ordering of the bids. In Appendix F, we show that mechanisms that only have access to the ordering of the bids (and do not have information about the identities of the advertisers or the bid values) and achieves multiple-task fairness cannot achieve a competitive revenue.

[9]The mechanism instead relies on the fairness of the bids in order to guarantee the fairness of the allocation.

1.2 Fairness across different categories

There are, of course, real world settings where products from very different categories compete for the attention of each user, for example, personalized display advertising. Let's revisit the "jack of all trades" example discussed above. In this example, the fairness constraint translates into ensuring that for all i, user $n+1$ sees ad i just as frequently as user i. Since user $n+1$ is limited to seeing at most one ad in expectation, this single user's fairness requirement hurts the allocation of all other users. Within this context, we view the multiple task fairness objective to be unduly skewed in favor of the individual over the collective good. In fact although user $n+1$ is qualified in all categories, they may not explicitly care about fairness guarantees in all of those categories equally. Is it possible to achieve a better balance? It is indeed if we slightly shift our viewpoint. Consider a mechanism that forces user $n+1$ to pick a (small) subset of ads over which to guarantee fairness. Now, we can allocate ad i to user i for users outside of the set specified by user $n+1$ without imposing fairness conditions on user $n+1$. This simultaneously achieves good utility and gives users control over how they want to spend their "fairness budget".

Formally, we propose the notion of **User-Directed Fairness**. Each user is allowed to pick a subset of categories over which they would like to receive a fair allocation of ads. The platform then ensures that the probability that each user is allocated an ad within their favorite subset is at least as large as the probability that any other user is allocated an ad within the same subset, though we do not provide guarantees on how the ads are distributed between categories. Observe that this is a directional notion of fairness. Furthermore, it is agnostic to similarity metrics over the users.

Our contributions. We first consider the setting where all users specify subsets with C categories for their user-directed fairness requirement. For this setting, we show matching upper and lower bounds on the fair value. More specifically, we show that the fair value[10] is inversely proportional to C. Our lower bound follows from a simple mechanism that achieves the optimal balance between allocating to the highest bidding category and allocating to user-specified categories. Our upper bound is constructed through considering the extremal case where user categories do not align with advertiser bids in the sense that advertisers bid low on categories specified by users.

We then consider a relaxed benchmark, with respect to which higher fair values are possible. In many advertising platform scenarios, users are unlikely to click or meaningfully engage with ads in which they have not expressed an interest, and platform revenue is actually based on a click-through rate. Viewing categories specified by the user as an indication of interest, this motivates considering a relaxed revenue model where the platform only obtains revenue on allocations on categories that the user specified. With respect to this relaxed benchmark, we show a simple mechanism that achieves the same utility as a utility-optimal unfair mechanism.

1.3 An intermediate setting

Finally we consider a model that combines the two extremes described above. In particular, we consider a setting where the advertised products fall into multiple different categories, with individual categories potentially containing multiple products. For example, the categories may correspond to household products, jobs, and cars. Each of these categories may contain several products advertised by different advertisers. Each category imposes its own distance metric over advertisers and these metrics can be arbitrarily different from each other. Within each category, however, every advertiser is subject to the same metric. We then ask: is it possible to simultaneously achieve user-directed fairness across categories as well as a form of total variation fairness within each category individually?

Our contributions. We show that our mechanisms developed for the two extreme settings discussed above are composable in a manner that simultaneously achieves user-directed fairness across categories and a *conditional* version of total variation fairness within each category. The composed mechanism obtains a fair value that is the product of the fair values of the two mechanisms individually.

[10]For the strongest form of user-directed fairness, the fair value when $C = 1$ takes the value of $1/2$, but this increases with the multiplicative relaxation coefficient.

1.4 An alternative approach: Preference-Informed Fairness

Concurrent with this work, Kim, Korolova, Rothblum and Yona also considered the problem of multiple task composition in the context of online advertising [8]. However, Kim et al. focus primarily on optimizing user utility via offline optimization, whereas our work is primarily focused on quantifying the "cost of fairness" (i.e. showing tradeoffs between platform utility and fairness) in the online setting. Furthermore, while their user preference based relaxation of multiple task fairness, Preference Informed Multiple Task Fairness, is motivated by similar observation, their proposed relaxation is different from our notion of user-directed fairness. Preference Informed Multiple Task Fairness still requires strong guarantees in *each* category, while our notion relaxes the fairness guarantees *across* categories. We view our work as complementary to [8], and we anticipate that combining the insights from their perspective and ours will be useful for proposing alternative mechanisms and fairness relaxations.

1.5 Outline for the rest of the paper

The rest of this work is organized as follows: In Section 2, we introduce the relevant definitions. In Section 3, we consider the single category (identical metrics) case. In Section 4, we consider the different categories (different metrics) case. In Section 5, we study a combination of the two settings. Directions for future work are discussed in Section 6. All proofs are deferred to the appendix.

2 Definitions

We model the online advertising problem as follows. We have a universe U of users that arrive in an online fashion. There are k advertisers indexed by $i \in [k]$. When a user u arrives, each advertiser i places a bid b_u^i on the user. The allocation algorithm then assigns allocation probabilities $p_u^i \in [0, 1]$ to the advertisers with $\sum_{i=1}^k p_u^i \leq 1$. The allocation mechanism is an online algorithm. We assume that this algorithm has access to the universe U (as well as any relevant underlying metrics on this set, as we will discuss later). However, it assigns allocation probabilities without observing bids on users that arrive in the future or the ordering of future user arrivals. We assume for simplicity that each user arrives at most once. We use $\mathbf{p}$ to denote the allocation rule output by the allocation algorithm.

The goal of the allocation mechanism is to maximize the sum of the bids of the ads displayed. Formally, this is given by:

$$\text{Utility}(\mathbf{p}) = \sum_{u \in U} \sum_{i \in [k]} p_u^i b_u^i.$$

The utility is easy to maximize in the absence of any constraints on how probabilities vary across users: the mechanism can simply assign a probability mass of 1 to the highest bidder for every user. We call the corresponding utility the unfair optimum:

$$\text{Unfair-OPT} = \sum_{u \in U} \max_{i \in [k]} b_u^i.$$

The **Fair Value** of an allocation mechanism is the ratio of its utility to the Unfair-OPT.[11] Note that this ratio is always less than 1; the larger the fair value the better the utility of the allocation is.

2.1 Fairness of allocation

Our fairness guarantees are based on the concept of individual fairness defined by [6]. At a high level, individual fairness guarantees that similar individuals are treated similarly. Similarity between individuals is captured through a fairness metric $\mathbf{d}$ over U and similarity between outcomes is captured by defining a metric D over distributions over outcomes.

[11]In other places in this paper, we use the phrase "cost of fairness" to denote the gap between the fair utility and unfair optimum. The cost of fairness can be viewed as the reciprocal of the fair value as defined here. Our bounds are easiest to express and understand as ratios of the fair utility to the unfair optimum and are therefore described as quantities smaller than 1 with larger ratios being better than smaller ratios.

Definition 1 ([6]). A function $f : U \to \Delta(O)$ assigning users to distributions over outcomes is said to be **individually fair** with respect to distance metrics $\mathbf{d}$ over U and D over $\Delta(O)$, if for all $u, v \in U$ we have $D(f(u), f(v)) \leq \mathbf{d}(u, v)$.

Dwork and Ilvento [5] proposed extending the notion of individual fairness to settings involving multi-dimensional allocations by ensuring fairness separately within each dimension or "task". This gives rise to the notion of multiple-task fairness, which we define next in the context of online advertising. Let $\mathbf{d}^i$ denote a pseudometric over the users relevant to task (advertiser) i; $\mathbf{d}^i : U \times U \to [0, 1]$. In the context of our setting, the outcome assigned to each user u corresponds to the advertiser who is assigned to the slot for user u. Our mechanism maps users to distributions over outcomes, i.e. to the allocation probabilities $\{p_u^i\}_{1\leq i\leq k}$. We use the absolute difference between these probabilities (i.e. the ℓ_1 metric) to capture the similarity of allocations.[12]

Definition 2 (Multiple-Task Fairness [5]). An allocation function $\mathbf{p}$ satisfies **multiple-task fairness** with respect to distance metrics $\{\mathbf{d}^i\}_{i\in[k]}$ if for all $u, v \in U$ and $i \in [k]$ we have $|p_u^i - p_v^i| \leq \mathbf{d}^i(u, v)$.

We now describe two specializations of the above definition, **total variation fairness** and **user-directed fairness**.

For the single-category setting, where all of the metrics $\mathbf{d}^i$ are identical, we propose **total variation fairness** which requires that the allocation vectors p_u and p_v are not only close component-wise, but are also close in terms of ℓ_1 distance or total variation distance. This stronger definition has two nice consequences. First, consider a user u who regards some arbitrary subset S of advertisers to be substitutes. In that case, the probability that the user observes an ad from this subset is $\sum_{i\in S} p_u^i$, and we total variation fairness ensures that this sum is close to that of similar users. Second, consider an advertiser that submits multiple different ads (pretending to be distinct advertisers) for the same product and bids separately on each user for each of those ads. Enforcing multiple-task fairness for each ad individually may allow the advertiser to subvert the fairness constraint, whereas total variation fairness enforces the constraint across any such subset of ads. We now present the formal definition.

Definition 3 (Total Variation Fairness). An allocation function $\mathbf{p}$ satisfies **total variation fairness** with respect to a metric $\mathbf{d}$ if for all $u, v \in U$ and all $S \subseteq [k]$, we have $|\sum_{i\in S} p_u^i - \sum_{i\in S} p_v^i| \leq \mathbf{d}(u, v)$. Equivalently, for all $u, v \in U$, $\|p_u - p_v\|_1 \leq \mathbf{d}(u, v)$.

Our second specialization ignores distance metrics entirely and applies to settings where multiple-task fairness is unachievable (see Section 4). In this context, we enforce fairness along dimensions that are preferred by users. We require every user u to pick a preferred set $S_u \subseteq [k]$ of advertisers. We then ensure that user u is satisfied with her own allocation of ads within this set relative to the allocation any other user v receives from this set, regardless of how similar or dissimilar u and v are. In particular, we require that $\sum_{i\in S_u} p_u^i \geq \sum_{i\in S_u} p_v^i$ for all $v \in U$. Observe that this is a directional notion: we do not require u's allocation probability on her preferred set to be close to that of v; we just require it to be weakly greater.

Definition 4 (User-Directed Fairness). An allocation function $\mathbf{p}$ satisfies **user-directed fairness** with respect to preferred sets $\{S_u\}_{u\in U}$ if for all $u, v \in U$, we have $\sum_{i\in S_u} p_v^i \leq \sum_{i\in S_u} p_u^i$.

We can further refine each of the above notions by defining multiplicative relaxations parameterized by $\beta \in [1, \infty)$:

- β **multi task fairness:** for all $u, v \in U$ and $i \in [k]$ we have $|p_u^i - p_v^i| \leq \beta\mathbf{d}^i(u, v)$.
- β **total variation fairness:** for all $u, v \in U$ we have $\|p_u - p_v\|_1 \leq \beta\mathbf{d}(u, v)$.
- β **user-directed fairness:** for all $u, v \in U$ we have $\sum_{i\in S_u} p_v^i \leq \beta \sum_{i\in S_u} p_u^i$.

[12]We can also view the assignment $\{p_u^i\}_{1\leq i\leq k}$ as a fractional allocation. In this case, the distance corresponds to the difference in the portion of allocation for each advertiser.

2.2 Fairness in bids

Our goal is to develop an allocation mechanism that satisfies fairness according to the definitions described above which simultaneously achieves large fair value with respect to the Unfair-OPT. As one might expect, it is impossible to achieve this if advertisers are allowed to place arbitrary bids on users without regard to the relevant similarity metric over users. The question thus becomes: what kind of fairness constraint on bids enables a desirable fair value? The following example illustrates the need for a fairness constraint that requires an advertiser's bids on pairs of similar users to be close in ratio. In particular, being close in terms of their absolute difference is not sufficient to achieve a desirable fair value.

Example 1. Suppose that there are k advertisers and the universe has k users. Suppose that the metric is uniform: for some parameter $d \in [0,1]$, every pair of users is a distance of d apart. For each $i \in [k]$, advertiser i bids b^{high} on user i and $b^{\text{low}}(< b^{\text{high}})$ on all other users $j \neq i$. Observe that Unfair-OPT $= kb^{\text{high}}$. On the other hand, due the symmetry of this instance and the fact that a fair allocation requires that $|p_i^i - p_i^j| \leq d$, for each user, it turns out the optimal fair allocation assigns an allocation probability of $(1-d)/k$ to all advertisers with the low bid and a probability of $(1-d)/k + d$ to the advertiser with the high bid. The fair value of this allocation turns out to be[13]

$$d + \frac{1-d}{k} + (1-d)\frac{(k-1)}{k}\frac{b^{\text{low}}}{b^{\text{high}}}.$$

Observe that a fair value of $d + (1-d)/k$ is trivial to achieve via a fair allocation.[14] If $b^{\text{high}} \gg b^{\text{low}}$, then, in this example, no fair allocation can perform much better than the trivial algorithm. In order to be able to do better, $b^{\text{high}}/b^{\text{low}}$ must be bounded.

Motivated by the example above, we require that for every advertiser and every pair of users, the ratio of the bids the advertiser places on the users is bounded by a function of the distance between the users. The closer the two users, the closer this ratio should be to 1. On the other hand, if the users are maximally distant, i.e. $d(u,v) = 1$, then we place no constraint on how the bids on the two users are related. We formally define this constraint as follows.

Definition 5. A **bid ratio constraint** is a function $f : [0,1] \to [1,\infty]$. We say that the bid function b^i of advertiser i satisfies the bid ratio constraint f with respect to metric $\mathbf{d}$ if we have for all $u,v \in U$: $\frac{1}{f(\mathbf{d}(u,v))} \leq \frac{b_u^i}{b_v^i} \leq f(\mathbf{d}(u,v))$.

In order to be reasonable, the bid ratio constraint should satisfy some conditions. On the one extreme, if there are two users that are identical from the viewpoint of an advertiser, in particular, $\mathbf{d}(u,v) = 0$, then the advertiser should be forced to place the same bid on these users. In other words, we require $f(0) = 1$. At the other extreme, when two users are maximally apart, that is, $\mathbf{d}(u,v) = 1$, then the bid ratio function should place no constraint on how different the bids for the two users can be. In other words, $f(1) = \infty$. At intermediate distances, we require $f > 0$ and f should be a weakly increasing function. For ease of analysis, it also makes sense to require that f is continuous and differentiable.

In fact any differentiable increasing function composed with $1/(1-d)$ satisfies all of the above properties. In this work, we consider a specific parameterized class of polynomial bid ratio constraints that satisfy the above properties. The family is parameterized by $l \geq 1$, and is defined as:

$$f_l(d) = \left(\frac{1+d}{1-d}\right)^l.$$

Figure 1 displays some functions in this family. Note that as the parameter l increases, the bid ratio condition becomes more and more strict. We emphasize that bid ratios are not required to satisfy the constraint exactly, but rather should lie *at or below* the imposed curve. From an algorithmic viewpoint, this means that if we design an algorithm based on the polynomial family described above but the actual

[13]We prove this explicitly in Lemma 13 in Appendix A.1.1 as a corollary of a more general result about the optimal utility achieved by a fair offline mechanism on a uniform metric.

[14]In particular, for each user, assigning an allocation probability of $(1-d)/k + d$ to the highest bidder and $(1-d)/k$ to all other advertisers achieves this bound.

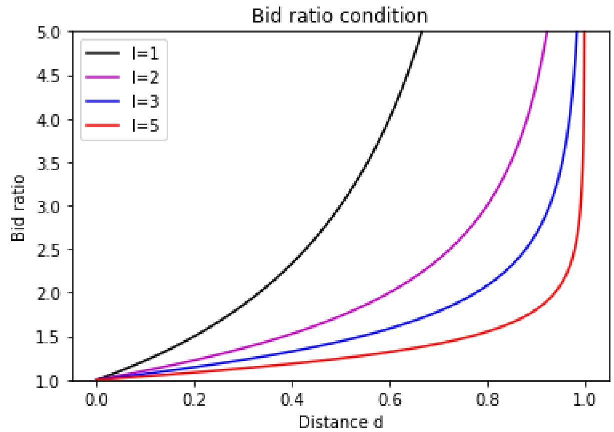

Figure 1: Bid ratio condition $f_l(d) = \left(\frac{1+d}{1-d}\right)^{1/l}$

bid ratio constraint imposed on bids, say g, does not belong to this family, it nevertheless suffices for the algorithm to find a value of the parameter l for which $f_l(d) \geq g(d)$ for all $d \in [0, 1]$ and use the function f_l in making allocations.

Convexity of the bid ratio constraint.

The family of constraints described above has the following super-multiplicative-like property at all points: if $0 < d_1, d_2 < 1$, then $f(d_1 + d_2) > f(d_1)f(d_2)$. This has the following peculiar consequence. Suppose that u_1 and u_2 satisfy that $\mathbf{d}(u_1, u_2) = d$ where $d > 0$. In this case, the bid ratio condition is $\frac{1}{f(d)} \leq \frac{b_{u_1}}{b_{u_2}} \leq f(d)$. Suppose that u_3 is on a line in between u_1 and u_2 (i.e. that $\mathbf{d}(u_1, u_3) + \mathbf{d}(u_3, u_2) = \mathbf{d}(u_1, u_2)$, where u_3 is not at either endpoint (i.e. that $\mathbf{d}(u_3, u_1), \mathbf{d}(u_3, u_2) \neq 0$. Including the user u_3 in between u_1 and u_2 necessarily *strengthens* the bid ratio condition between u_1 and u_2 since $\frac{b_{u_1}}{b_{u_2}} = \frac{b_{u_1}}{b_{u_3}} \cdot \frac{b_{u_3}}{b_{u_2}}$ which is upper bounded by $f(\mathbf{d}(u_3, u_1))f(\mathbf{d}(u_3, u_2)) < f(\mathbf{d}(u_1, u_2))$ and lower bounded by $\frac{1}{f(\mathbf{d}(u_3,u_1))f(\mathbf{d}(u_3,u_2))} > \frac{1}{f(\mathbf{d}(u_1,u_2))}$. This means that the users cannot be specified in an online manner to the advertisers: the advertiser may bid on users in a way that satisfies the bid ratio conditions on u and v at the current time step but violates it on a future time step.

If a bid ratio condition f does not have this super-multiplicative-like property at any points (i.e. that $f(d_1 + d_2) \leq f(d_1)f(d_2)$ for all $d_1, d_2 \geq 0$), then this means that the bid ratio condition is concave or linear. We briefly consider concave bid ratio constraints with a finite bound on $f(1)$ in Appendix G and place a restrictive upper bound on the fair value.

3 Identical metrics

In this section, we focus on the case where every advertiser faces the same metric over users, in particular, $\mathbf{d} = \mathbf{d}^1 = \mathbf{d}^2 = \ldots = \mathbf{d}^k$. We begin by investigating the special case of uniform metrics and establishing impossibility results in Section 3.1, i.e. upper bounds on the fair value of any allocation mechanism that satisfies multiple-task fairness. In Section 3.2, we consider settings with arbitrary distance metrics and a class of convex bid ratio constraints, and establish our main positive result for this setting. We exhibit an allocation mechanism that is metric-oblivious, history-oblivious, and achieves total variation fairness with respect to the given metric. We then bound the fair value achieved by this mechanism as a function of k and a parameter defining the bid ratio constraint. In Section 3.3, we show that the fair value achieved by this mechanism is close to the upper bound established in Section 3.1 for the special case of uniform metrics. We emphasize that while our negative result applies to mechanisms which satisfy multiple-task fairness with access to the underlying metric, our positive result applies to a mechanism that is metric-oblivious and satisfies the stronger notion of total variation fairness.

3.1 Uniform metric: upper and lower bounds on fair value

We prove upper bounds on the fair value using uniform metrics, i.e. metrics of the form $\mathbf{d}(u, v) = d$ for all $u \neq v$. We use Example 1 to show an upper bound on the fair value as a function of the bid ratio constraint.[15]

Lemma 1. *Given $d \in [0, 1]$ and $\alpha = f(d)$, there exists an instance of the online advertising problem for which the fair value of every offline allocation mechanism satisfying multiple-task fairness with respect to the uniform metric $\mathbf{d}(u, v) = d$ is at most $d + \frac{1-d}{k} + \frac{(1-d)(k-1)}{k\alpha} \leq \frac{1}{k} + \frac{1}{\alpha} + d$.*

Observe that as d increases, the upper bound on the fair value increases, due to a weaker fairness constraint on setting allocation probabilities. On the other hand, for any fixed d and k, the upper bound decreases as a function of α: as α increases, weakening the fairness constraint on bids, the algorithm's performance becomes worse.

In the online setting it is possible to prove prove even stronger bounds on the fair value.[16] More specifically, we can tighten the $1/\alpha$ term in the fair value to $1/\alpha^2$. In Appendix C, we develop online allocation mechanisms that are tailored to uniform metrics and achieve a fair value that nearly matches these lower bounds, demonstrating that stronger lower bounds cannot be obtained through uniform metrics.

Lemma 2. *Given $d \in [0, 1 - 1/k]$ and $\alpha = f(d)$, there exists an instance of the online advertising problem for which no online allocation mechanism satisfying multiple-task fairness with respect to the uniform metric $\mathbf{d}(u, v) = d$ can obtain fair value better than $\alpha^{-2}\left(1 - \frac{1}{k} - d\right) + \frac{1}{k} + d \leq \frac{1}{k} + \frac{1}{\alpha^2} + d$.*

The main idea of the proof of Lemma 2 is to make the bids on the first user equal, and then for the next user, the adversary maximally increases and decreases bids so that the advertiser receiving the lowest probability on the first user has the highest bid. The distance metric limits the extent to which the mechanism can increase the probability placed on this advertiser on this user due to the low probability placed on the first user.

3.2 The general metric case

Here, we present a mechanism that satisfies the strongest notion of total variation fairness and that achieves a fair value which can be set arbitrarily close to 1 with the appropriate choice of parameters. The key intuition is to convert the bids on a user into probabilities using a (deterministic) function that places higher probabilities on higher bids. We then normalize the allocations so that they sum up to 1. We observe that the overall optimal solution (Unfair-OPT) can be placed in this framework: the mechanism that distributes the probability mass equally among the highest bidders for each user corresponds to the function that places the full mass on the highest bids. This function can be viewed as assigning allocation probabilities in proportion to their contribution to the ℓ_∞-norm over the bids. More generally, we consider a mechanism that assigns allocations in proportion to each bid's contribution to the ℓ_l-norm of the bid vector. We call this mechanism the **proportional allocation mechanism with exponent** l.

Mechanism 1. Given a parameter $l \geq 1$, **the proportional allocation mechanism with exponent** l assigns $p_u^i = \frac{(b_u^i)^l}{\sum_{j=1}^k (b_u^j)^l}$ for every user $u \in U$ and advertiser $i \in [k]$.

Observe that although the proportional allocation mechanism uses the parameter l, it is otherwise completely oblivious of the underlying metric $\mathbf{d}$. We will show that if the bids satisfy the bid ratio constraint f_l, then regardless of the underlying metric, the mechanism achieves total variation fairness with respect to the metric $\mathbf{d}$. We then analyze the utility achieved by the mechanism, thereby bounding the fair value achieved, in terms of k and l. We remark that fair value tends to 1 as l tends to infinity.

We remark that in addition to the metric obliviousness, Mechanism 1 satisfies several other nice properties. First, the mechanism treats advertisers in a symmetric manner in each individual iteration — the allocation to

[15]In Appendix C, we give an explicit formula for the optimal revenue that can be achieved in the offline setting for a uniform metrics as a function of the bids and d. This formula matches the bound in Lemma 1 on Example 1.

[16]To be explicit, the online lower bound applies in the limit as $|U| \to \infty$, where the adversary is given access to the probabilities output by the mechanism when designing the next set of bids.

an advertiser does not depend on their identity and how they bid on other users. In particular, although this mechanism is not a position auction since it heavily relies of the values of the bids, it is "close" to a position auction in that the highest bidder is always assigned the highest probability regardless of their identity. Second, the mechanism protects against an advertiser splitting up into sub-advertisers (or submitting multiple ads) in an attempt to obtain a higher probability allocation. It is straightforward to see that the net probability of assignment for an advertiser bidding $b_1 + b_2 + \ldots + b_s$ is no smaller than the sum of probabilities of s advertisers placing bids b_1, b_2, $\ldots$, b_s in an attempt to exploit the mechanism.[17] Third, allocation probabilities increase monotonically as functions of the advertisers' bids, which means that we can make this mechanism truthful by setting the payoffs appropriately by Myerson's lemma. Finally, the mechanism is history-oblivious, which means that the memory required by the mechanism is independent of the number of users and the solution is *independent* of the ordering of the users so we don't need to worry about impact of the order in which the users are given on advertiser strategies and/or utility.

We show that Mechanism 1 satisfies total variation fairness with the bid ratio condition discussed in Section 2.2.

Proposition 3. *For any distance metric* $\mathbf{d}$ *over the users, if bids satisfy the bid ratio constraint* $f_l(d) = \left(\frac{1+d}{1-d}\right)^{1/l}$ *with respect to* $\mathbf{d}$ *for some parameter* $l \geq 1$*, the proportional allocation mechanism with exponent* l *(Mechanism 1) satisfies total variation fairness with respect to metric* $\mathbf{d}$.

We next analyze the utility achieved by the mechanism. We emphasize that our bound on the utility does not require bids to satisfy the bid ratio constraint.

Lemma 4. *If* $k \geq 9$ *and* $l \geq 1$*, then the fair value achieved by the proportional allocation mechanism with exponent* l *(Mechanism 1) is at least* $(k-1)^{-1/l} \frac{k-1}{k} + \frac{1}{k}$.

Observe that fair value is an increasing function of l, and as $l \to \infty$ (where Mechanism 1 places the entire mass on the highest bid), the bound equals 1 as expected. This means that fair value can be made arbitrarily close to 1 by sufficiently strengthening the bid ratio constraint.

In Appendix D we discuss relaxations of the fairness constraint and analyze the performance of Mechanism 1 under those relaxations.

3.3 Discussion

It is a little tricky to directly compare the upper and lower bounds established on fair value in the previous sections because the bounds depend on different parameters. In particular, our lower bounds (positive results) hold for arbitrary metrics whereas our upper bounds (negative results) are designed only for the uniform metric. To perform an apples-to-apples comparison, we fix parameters k, l, and some number $d \in (0,1)$, and set $\alpha = f_l(d)$. Figures 2, 4, 3, and 5 display the ratio of the upper bound and lower bound for various parameter settings. Observe that the ratio is bounded by a reasonably small constant except when l is very small. This indicates that Mechanism 1 in general obtains a large fraction of the utility that can be obtained by any mechanism satisfying multiple-task fairness, despite satisfying a stronger form of fairness and a number of other nice properties. Closing the gap between the upper bound and lower bound for small values of l is an important direction for future work.

4 Different metrics

In this section, we consider the setting where different product categories correspond to very different metrics. We first show it is not possible to achieve the same tradeoff between multiple-task fairness and utility as in the case of identical metrics. We then show that with the user-directed fairness, significantly better tradeoffs are possible.

In Section 4.1, we focus on the Unfair-OPT benchmark considered in the previous section. Using the "jack-of-all-trades" example, we show that when different advertisers are subject to different metrics, it is

[17] We show that the probability of assignment placed on the other advertisers decreases (which suffices since the sum is 1 for this mechanism). The denominator is no larger for the case of b_1, b_2, $\ldots$, b_s than the case of $b_1 + b_2 + \ldots + b_s$ since $\sum_{l=1}^{s} (b_l)^p \leq \left(\sum_{l=1}^{s} b_l\right)^p$, so the probability is no smaller for the other advertisers.

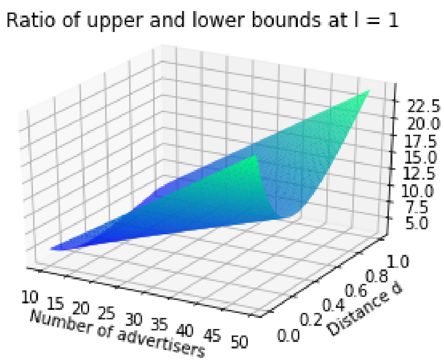

Figure 2: Ratio of fair value bounds at $l = 1$

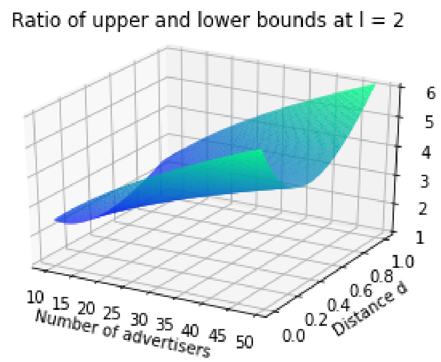

Figure 4: Ratio of fair value bounds at $l = 2$

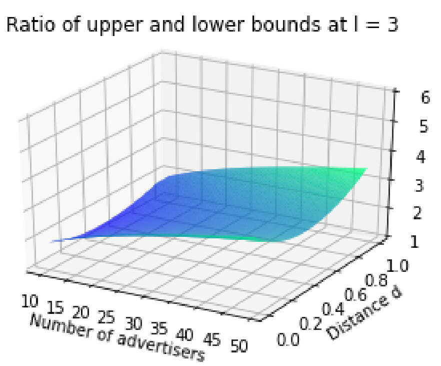

Figure 3: Ratio of fair value bounds at $l = 3$

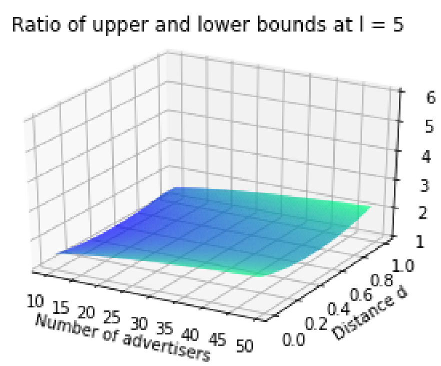

Figure 5: Ratio of fair value bounds at $l = 5$

Figure 6: These figures show the ratio of upper and lower bounds on fair values at different l values. Notice that the z-axis scales for $l = 1$ is different than for $l = 2, 3, 5$.

not possible to achieve fair value better than $1/k$ with multiple-task fairness no matter how strong the bid ratio constraint is. We thus consider the weaker notion of user-directed fairness, where each user is allowed to specify a preferred set of categories over which they desire a fair allocation. We present tight upper and lower bounds on the fair value achievable as a function of the sizes of the preferred sets. In Section 4.2, we argue in favor of a weaker benchmark to evaluate the performance of fair allocation mechanisms, and show that this benchmark can be exactly met by mechanisms that satisfy user-directed fairness.

4.1 User-directed fairness and fair value relative to Unfair-OPT

First, we show a upper bound that demonstrates that multiple-task fairness is in conflict with utility, even in the offline setting, if the metrics are permitted to be different. The key ingredient of this bound is the following "jack-of-all-trades" example, where each user is qualified for exactly one category except for one user who is equally qualified in all of the categories.

Example 2 (Jack-of-all-trades)**.** Suppose that the universe has $k+1$ users $u_1, \ldots, u_{k+1}$ and there are k advertisers. The metric $\mathbf{d}^i$ is defined so that $\mathbf{d}^i(u_{k+1}, u_i) = d$ and all other distances are 1. Suppose that advertiser i bids b^{high} on u_i and u_{k+1}, and b^{low} on everybody else, with $b^{\text{high}}/b^{\text{low}} = f(1)$.

In this example, the Unfair-OPT would assign an allocation of 1 to advertiser i for each user i. However, a mechanism satisfying multiple-task fairness cannot do so: all advertisers together receive a total allocation of 1 for user $k+1$, and multiple-task fairness dictates that advertiser i cannot get a allocation for user i of d more than what he gets for user $k+1$. This severely limits the mass that can be assigned to the users $u_1, \ldots, u_k$ on the category on which they are specialized. We obtain the following bound:

Proposition 5. *Suppose that bids satisfy the bid ratio constraint f, then no offline mechanism that satisfies multiple-task fairness can obtain a fair value more than* $\frac{2}{k+1} + \frac{1}{f(1)} + d$.

We now consider the weaker fairness notion of user-directed fairness defined in Section 2. We assume that the platform obtains from each user $u \in U$ a preferred set $S_u \subseteq [k]$ of advertisers as the user arrives (i.e. the preferences are not known to the allocation mechanism in advance). user-directed fairness requires that the total allocation of ads in S_u to the user u should be at least as large as the total allocation of ads in S_u to any other user. Observe that our definition of fairness does not involve a metric over the users; indeed the category-specific metrics may be worst case, with each user u being identical in every dimension $i \in S_u$ to the user that previously obtained the maximum allocation over the set S_u and being maximally distant in all other dimensions. For this reason, no reasonable/useful fairness constraint can be imposed on advertisers' bids in this setting. We therefore assume that bids are arbitrary. Obtaining good utility requires that we place a reasonable amount of mass on the highest bid(s) for the user u. The two objectives of fairness and utility are therefore once again at odds with each other.

We develop an allocation mechanism that reserves some allocation mass for the highest bid on each user u, and distributes the remaining allocation probability across advertisers in S_u. In doing so, we must ensure that no subset of advertisers gets too much mass in total. This is because if such a set S exists, and a future user v sets $S_v = S$, then the mechanism is forced to give a high allocation to this set for user v. This may then leave little probability mass for the highest bidder for v. One consequence of this observation is that users' preferred sets should be similar in size, and the sizes of the preferred sets have a bearing on the utility achieved by the mechanism.

We now formally define our mechanism and bound the fair value achieved by it.

Mechanism 2. The **equal-spread mechanism with parameters** β **and** C is defined as follows. Suppose that user u specifies a subset $S_u \subset [k]$ and let $A_u = \operatorname{argmax}\{b_u^i\}$. We assume that $|S_u| = C$. The mechanism assigns an allocation probability of $\frac{1+(\beta-1)C}{\beta C+1}$ to the highest bidder A_u, a $\frac{1}{\beta C+1}$ probability to advertisers in $S_u \setminus \{A_u\}$, and any remaining probability mass equally among advertisers outside of $S_u \cup \{A_u\}$.

The parameter $\beta \geq 1$ allows the mechanism to trade-off between fairness and utility. By allowing the mechanism to achieve β-user-directed fairness for larger β values, we obtain a better approximation to the Unfair-OPT.

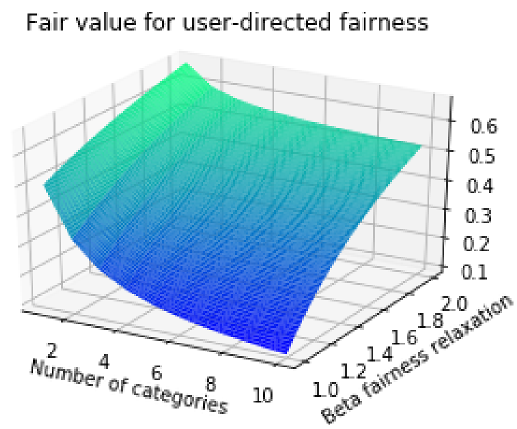

Figure 7: Fair value for user-directed fairness

Proposition 6. *If every user's preferred set contains exactly C categories, then for any $\beta \geq 1$, the equal-spread mechanism with parameters β and C (Mechanism 2) satisfies β-user-directed fairness and achieves a fair value of $\geq \frac{1+(\beta-1)C}{\beta C+1}$.*

We show a matching upper bound on the fair value, thus showing that Mechanism 2 is optimal. Our proof boils down to showing that it is not possible to place more than $\frac{1+(\beta-1)C}{\beta C+1}$ mass on the highest bid. We construct a sequence of users with the property that bids in S_u are always 0 and each user has an advertiser $A_u \notin S_u$ that bids 1. We adaptively construct the sets S_u and A_u to minimize the fair value.

Proposition 7. *Suppose that every user's preferred set contains exactly C categories, with $C < k$. Then, regardless of the bid ratio constraint f imposed on the advertisers (but assuming $f(1) = \infty$), any online mechanism that satisfies β-user-directed fairness obtains a fair value of at most $\frac{1+(\beta-1)C}{\beta C+1}$.*

In the above construction, we consider a uniform metric with distance 1 (so that bid ratio conditions do not provide any guarantees). Nonetheless, even though the users are maximally distant, the user-directed fairness guarantees provide the maximal strength of fairness even when two users are maximally distant. One may think that this issue can be rectified by considering a weaker notion of fairness relaxed by the metric constraints. However, we show in Appendix E that even with this weaker notion of fairness, it is difficult to achieve high fair values without extremely restrictive conditions on the fairness metric or bids, or significantly relaxing fairness.

We plot the tight bound on fair value obtained for user-directed fairness as a function of β and C in Figure 7. For the dependence on β, observe that as β increases, the weakened fairness guarantees cause the fair value to increase. For the dependence on C, observe that the mechanism must balance between allocating to the highest bidding advertiser to achieve the most revenue, and allocating to advertisers in S_u to achieve user-directed fairness. As C increases, the mechanism has a greater number of categories to consider for each user (and the highest bid can still be outside of S_u), thus causing the optimal fair value to decrease. Let's now consider the strongest setting of $\beta = 1$, so the fair value becomes $\frac{1}{C+1}$. The fair value thus becomes small when users are permitted to specify a large number of categories, and when $C = 1$ (and $\beta = 1$), the fair value is 1/2.

4.2 Relaxing the utility benchmark

The results in the previous section place an upper bound of 1/2 (or stronger) on the fair value that can achieved by user-directed fairness. In this section, we consider the setting where we restrict to mechanisms that receive utility only for allocations in S_u: that is, the utility is $\sum_{u \in U} \sum_{i \in S_u} p_i^u b_u^i$. In this case, we assume that user-specified categories are aligned with interest, and in a click-through-rate based revenue/utility model, the platform only obtains benefit from allocations within the user-specified sets.

It is straightforward to see that the best possible utility achieved by any (potentially unfair) mechanism in this restricted class is $\sum_{u \in U} \max_{i \in S_u} b_u^i$, which may be much smaller than unrestricted optimal utility of $\sum_{u \in U} \max_{i \in [k]} b_u^i$. Thus, when considering the cost of fairness for this setting, we compare against $\sum_{u \in U} \max_{i \in S_u} b_u^i$, the best possible user-preference-compatible utility for a mechanism in this setting. This motivates considering the relaxed fair value given by:

$$\frac{\sum_{u \in U} \sum_{i \in S_u} p_u^i b_u^i}{\sum_{u \in U} \max_{i \in S_u} b_u^i}.$$

We show a strong positive result in this setting. More specifically, we remark that a simple highest-bidder-wins mechanism achieves user-directed fairness and a relaxed fair value of 1. This mechanisms does not assume that all users specify the same number of categories and permits users to specify any number of categories.

Mechanism 3. For each $u \in U$, the mechanism allocates a probability of 1 to the highest bidder in S_u. If there are multiple advertisers tied for the highest bid, then the mechanism splits the probability equally between the highest bidding advertisers.

Proposition 8. *Mechanism 3 achieves user-directed fairness with $\beta = 1$ and achieves a relaxed fair value of 1.*

Given the simplicity of Mechanism 3, a natural question is to ask is whether we can obtain better fairness guarantees while still maintaining a high relaxed fair value in this setting. However, in Appendix E, we show upper bounds that demonstrate that it is not possible to recover multiple-task fairness-style guarantees in this setting even against the relaxed benchmark.

5 A combination of the two settings

We now consider a compositional-form of fairness in the more realistic setting in which there are multiple advertisers represented in each category. Let $\{C_1, \ldots, C_c\}$ denote a partition of the set $[k]$ of advertisers into c categories.

We have two goals. First, we want to achieve user-directed fairness across different categories: every user should be able to specify a subset of categories $S_u \subset [c]$ such that the total allocation received by the user within these categories is at least as large as that of any other user. Second, we simultaneously want to achieve some form of fairness *within* each category, where stronger fairness guarantees are possible since advertisers within the same category face the same metric over users.

We demonstrate that within a category, we cannot hope to achieve total variation fairness (or even multiple-task) fairness guarantees in the composed setting. This is because even if two users u and v are similar to each with respect to a category C_j's distance metric, they may nevertheless obtain different total allocations within this category if their preferred sets contain other categories. This follows from the fact that user-directed fairness only considers the total allocation probability over the subset of categories specified by the user. For example, suppose that we have two categories, job ads and household product ads, and two users that are identical to each other for the job ads category. The first user includes only job ads in their preferred set and gets to see a job ad with probability 1. The second user includes both job ads and household product ads and obtains the same bids from both categories, and thus gets to see each with probability 1/2. On a category by category level, this allocation satisfies user-directed fairness. However, there is no way to allocate the respective probabilities within the job ads category across individual advertisers that would satisfy multiple-task fairness simply because of the fact that we have different total probabilities to distribute.

Nonetheless, it is possible to guarantee that each of the two users to see a similar *mix* of job ads, (e.g. the first user shouldn't see a different proportion of high-paying job ads compared with the second). For this reason, we require that the *conditional distribution* of probability within each category satisfy total variation fairness. Formally, we define compositional fairness as follows. We use $\mathbf{d}^j$ to denote the metric specific to category C_j and $q_u^j = \sum_{i \in C_j} p_u^i$ to denote the total allocation within category C_j for user u.

Definition 6 (Compositional Fairness)**.** An allocation function $\mathbf{p}$ satisfies **compositional fairness** with respect to distance metrics $\{\mathbf{d}^j\}_{j\in[c]}$ if the probabilities assignments $\left\{q_u^j\right\}_{u\in U, j\in[c]}$ satisfy user-directed fairness and for each $j \in [c]$, the conditional probabilities $\left\{\frac{p_u^i}{q_u^j}\right\}_{i\in C_j}$ satisfy total variation fairness with respect to $\mathbf{d}^i$.

We can compose Mechanism 3 with Mechanism 1 to achieve compositional fairness guarantees.[18] The idea is that we run Mechanism 3 to identify the category in S_u with the highest bid, and then we run Mechanism 1 to divide the probability mass between advertisers in this category.

Mechanism 4. For each user u, the mechanism runs Mechanism 3 (with parameters β and C) to allocate probabilities between categories. The mechanism then determines the conditional probabilities for advertisers within a category using Mechanism 1 (with parameter l).

It follows from the analysis of Mechanism 1 in Section 3 that Mechanism 4 achieves compositional fairness and a competitive fair value.

Proposition 9. *Let $k' = \max_{1\le j\le c} |C_j|$ be the maximum number of advertisers in any category. If all advertisers $1 \le i \le k$ satisfy the bid ratio condition $f^i(d) \le \left(\frac{1+d}{1-d}\right)^{1/l}$, then Mechanism 4 with a parameter l achieves compositional fairness and a relaxed fair value of at least $(k'-1)^{-1/l}\frac{k'-1}{k'} + \frac{1}{k'}$.*

It is likewise possible to combine mechanisms 1 and 2 to obtain bounds on the fair value against Unfair-OPT.

Mechanism 5. For each user u, the mechanism runs Mechanism 2 (with parameters β and C) to allocate probabilities between categories. The mechanism then determines the conditional probabilities for advertisers within a category using Mechanism 1 (with parameter l).

It follows from the analysis of Mechanism 1 in Section 3 and Mechanism 2 in Section 4 that Mechanism 5 achieves compositional fairness and a competitive fair value.

Proposition 10. *Suppose that every user's preferred set contains exactly C categories. Let $k' = \max_{1\le j\le c} |C_j|$ be the maximum number of advertisers in any category. If all advertisers $1 \le i \le k$ satisfy the bid ratio condition $f^i(d) \le \left(\frac{1+d}{1-d}\right)^{1/l}$, then Mechanism 5 with parameters β, C, and l achieves compositional fairness and a fair value of at least $\left(\frac{1+(\beta-1)C}{\beta C+1}\right)\left((k'-1)^{-1/l}\frac{k'-1}{k'} + \frac{1}{k'}\right)$.*

6 Future work

In this work, we have initiated a study of sponsored search auctions that are simultaneously individually fair and achieve good utility, and have uncovered a number of interesting areas for future exploration. We now briefly summarize several directions for future work.

Incentivizing and auditing fair bidding. A significant benefit of the online mechanism presented for the similar metrics case is that it is *metric oblivious* which frees the platform from explicitly checking whether bids placed are in accordance with the relevant fairness metric. However, this does leave open the question of incentivizing, auditing, and enforcement of fair bids. Although the total variation fair mechanism presented in this work has the nice property of not incentivizing advertisers to split into smaller "sub-advertisers," there are many other stronger incentive properties we might like to have. We anticipate that if there is a mechanism which can impose penalties (either monetary penalties or reduced platform access) on advertisers who do not behave fairly that this may be sufficient incentive to follow the rules.

[18] In order to conclude that $q_u^j = \sum_{i\in C_j} p_u^i$ is actually the probability that the first mechanism assigned to category C_l, we require that the second mechanism assigns the full 1 probability mass to each user. This is true of Mechanism 1.

Closing the gaps and tighter lower bounds. The lower bounds presented in this work can potentially be improved. Although uniform metrics provide a good starting point for lower bounds, we anticipate that tighter lower bounds can be achieved using nonuniform metrics. Additionally, there is room for exploring how tight our lower bounds are in certain oblivious settings, i.e., requiring metric or identity obliviousness.

Generic mechanisms for alternative bid conditions. The bid ratio conditions presented in this work are a critical ingredient for simultaneously achieving fairness and quality. However there may be more generic mechanisms and alternative constraints on bidding which could also yield fair outcomes with good revenue guarantees, perhaps with alternative relaxations in the fairness definitions.

Multiple slot auctions. In this work, we have exclusively considered the case of single slot auctions. However, in practice multiple slot auctions, or auctions where each user appears at several times, are also common. In addition to technical implementation questions, there are several important considerations in defining fairness for the setting. For instance, the ordering of advertisements may be important when there is significant visual distinction (e.g. the user must scroll to see ads lower in the slate). The number of ads shown from each category may also be important, even if we deem it acceptable to "backfill" with additional ads from a previously shown category.

References

[1] Muhammad Ali et al. *Discrimination through optimization: How Facebook's ad delivery can lead to skewed outcomes*. 2019. URL: `https://arxiv.org/abs/1904.02095`.

[2] Julia Angwin and Terry Perris. *Facebook Lets Advertisers Exclude Users by Race*. 2016. URL: `https://www.propublica.org/article/facebook-lets-advertisers-exclude-users-by-race`.

[3] Julia Angwin, Noam Scheiber, and Ariana Tobin. *Facebook Job Ads Raise Concerns About Age Discrimination. The New York Times* in collaboration with *ProPublica*, December 20, 2017. 2017. URL: `https://www.nytimes.com/2017/12/20/business/facebook-job-ads.html`.

[4] Katie Benner, Glenn Thrush, and Mike Isaac. *Facebook Engages in Housing Discrimination With Its Ad Practices, U.S. Says. The New York Times*, March 28, 2019. 2019. URL: `https://www.nytimes.com/2019/03/28/us/politics/facebook-housing-discrimination.html`.

[5] Cynthia Dwork and Christina Ilvento. "Fairness Under Composition". In: *10th Innovations in Theoretical Computer Science Conference, ITCS 2019, January 10-12, 2019, San Diego, California, USA*. 2019, 33:1–33:20.

[6] Cynthia Dwork et al. "Fairness Through Awareness". In: *Proceedings of the 3rd Innovations in Theoretical Computer Science Conference*. ITCS '12. Cambridge, Massachusetts, 2012, pp. 214–226. ISBN: 978-1-4503-1115-1.

[7] Benjamin Edelman, Michael Ostrovsky, and Michael Schwarz. "Internet Advertising and the Generalized Second-Price Auction: Selling Billions of Dollars Worth of Keywords". In: *American Economic Review* 97.1 (Mar. 2007), pp. 242–259.

[8] Michael Kim et al. "Preference-Informed Fairness". In: *arXiv preprint arXiv:1904.01793* (2019).

[9] Anja Lambrecht and Catherine E. Tucker. "Algorithmic Bias? An Empirical Study into Apparent Gender-Based Discrimination in the Display of STEM Career Ads". In: *SSRN Electronic Journal* (Jan. 2016).

[10] Hal Varian. "Position auctions". In: *International Journal of Industrial Organization* 25.6 (2007), pp. 1163–1178.

A Proofs for Section 3

A.1 Proofs for Section 3.1

In Section A.1.1, we prove bounds for offline mechanisms. In Section A.1.2, we prove bounds for online mechanisms.

A.1.1 Offline mechanisms

First, we compute the optimal offline mechanism revenue for uniform metrics.

Lemma 11. *Suppose that A_x be an advertiser $\sum_{u \in U} b_u^x = \max_{1 \leq i \leq k} \left(\sum_{u \in U} b_u^i\right)$. The optimal solution to the multiple-task fairness LP achieves a revenue of exactly:*

$$\max(0, 1 - md) \sum_{u \in U} + d \sum_{i=1}^{m-1} ith\ price\ auction\ revenue$$

$$+ (d - \max(0, md - 1)) mth\ price\ auction\ revenue.$$

where m is the maximum integer in $[1, \min(k-1, \frac{1}{d} + 1)]$ such that the mth price auction revenue is bigger than $\sum_{u \in U} b_u^x$ (where $m = 0$ if $\sum_{u \in U} b_u^x$ is the first price auction revenue).

The proof for this lemma boils down to solving the following linear program. In the offline setting for multiple-task fairness, an optimal-revenue mechanism that achieves multiple-task fairness is solving the following linear program, where the solutions are $\left\{p_u^i\right\}_{1 \leq i \leq k, u \in U}$, and the rest of the variables are inputs.

$$\max \left(\sum_{i=1}^{k} \sum_{u \in U} b_u^i p_u^i\right) \text{ such that:}$$

$$\sum_{i=1}^{k} p_u^i \leq 1 \text{ for all } u \in U$$

$$p_u^i - p_v^i \leq d^i(u, v) \text{ for all } u \neq v \in U$$

$$p_u^i \geq 0$$

We first show the following property of the LP. In order to better study this LP, we also consider the dual with variables $\{z_u\}_{i \in U}$ and $\left\{w_{u,v}^i\right\}_{1 \leq i \leq k, u \neq v \in U}$:

$$\min ||z||_1 + \sum_{i=1}^{k} \sum_{u \neq v} d^i(u, v) w_{u,v}^i \text{ such that:}$$

$$z_u \geq b_u^i + \sum_{v \neq u} w_{v,u}^i - \sum_{v \neq u} w_{u,v}^i \text{ for all } u \in U, 1 \leq i \leq k$$

$$z_u, w_{u,v}^i \geq 0.$$

The variable z_u represents the condition $\sum_{i=1}^{k} p_u^i$. The variable $w_{u,v}^i$ represents the condition $p_u^i - p_v^i \leq d^i(u, v)$.

Proposition 12. *At an optimal solution, if $\sum_{u \in U} b_u^i \neq \max_{j \in [k]} \left(\sum_{u \in U} b_u^j\right)$, then there exists some $u_i \in U$ such that $p_{u_i}^i = 0$.*

Proof. We prove the contrapositive. We use complementary slackness to analyze the structure of this LP. Suppose that we are at an optimal solution and $p_u^i > 0$ for all u for some i. The corresponding condition in the dual is that $z_u = b_u^i + \sum_{v \neq u} w_{v,u}^i - \sum_{v \neq u} w_{u,v}^i$ for all $u \in U$. This means that

$$\sum_{u \in U} z_u = \sum_{u \in U} b_u^i + \sum_{u \in U} \sum_{v \neq u} w_{v,u}^i - \sum_{u \in U} \sum_{v \neq u} w_{u,v}^i = \sum_{u \in U} b_u^i.$$

Moreover, we know that for any $j \in [k]$, it holds that $z_u \geq b_u^j + \sum_{v \neq u} w_{v,u}^j - \sum_{v \neq u} w_{u,v}^j$ for all $u \in U$. This means that

$$\sum_{u \in U} z_u \geq \sum_{u \in U} b_u^j + \sum_{u \in U} \sum_{v \neq u} w_{v,u}^j - \sum_{u \in U} \sum_{v \neq u} w_{u,v}^j = \sum_{u \in U} b_u^j.$$

Thus, we know that $\sum_{u \in U} b_u^i \geq \sum_{u \in U} b_u^j$ for any $1 \leq j \leq k$. □

Now, we prove Lemma 11.

Proof of Lemma 11. We first show this result when there is a unique advertiser A_x that achieves the top bid sum. For $i \neq x$, we know that $p_{u_i}^i = 0$ for some $u_i \in U$ by Proposition 12. Thus, $0 \leq p_v^i \leq d$ for all $v \in U$ by the fairness constraint. Now, we know that the bids of advertiser A_x are in the interval $[p, p+d]$ for some p by the fairness constraints. Any set of bids such that $0 \leq p_u^i \leq d$ for $i \neq x$ and $p_u^x \in [p, p+d]$ satisfies the fairness constraints. Let's examine an individual item $u \in U$. We see that the optimal strategy is to assign p mass to A^x and 0 mass to all other advertisers to start, and then distribute the remaining $1-p$ mass by assigning d mass to the top bidder, then d mass to the next highest bidder, until all of the mass runs out (assigning potentially $< d$ mass to the last bidder considered).

Let's suppose that $p \in [1-md, 1-(m-1)d]$. Let $p = q+1-md$ where $0 \leq q \leq d$. We see that the revenue achieved at p is $p \sum_{u \in U} b_u^x + d \sum_{i=1}^{m-1} \sum_{u \in U}$ *ith* highest bidder on u $+ q \sum_{u \in U}$ *mth* highest bidder on u. This is equal to

$$(1-md) \sum_{u \in U} b_u^x + d \sum_{i=1}^{m-1} ith \text{ price auction revenue} + q \left(\sum_{u \in U} b_u^x - mth \text{ price auction revenue} \right).$$

Thus, the maximum is achieved (not necessarily uniquely) at $q = d$ if $\sum_{u \in U} b_u^x \geq mth$ price auction revenue and achieved at $q = \max(0, md-1)$ if mth price auction revenue $> \sum_{u \in U} b_u^x$.

Now, let m be the maximum value such that mth price auction revenue $> \sum_{u \in U} b_u^x$ and $1-(m-1)d \geq 0$. The above argument shows that the optimal revenue can be achieved by setting $p = \max(0, 1-md)$. This achieves the revenue in the lemma statement.

We can obtain the general case result via a limiting argument in the general case. When bids are modified by $\pm\epsilon$, the primal LP constraints remain the same, and the objective is changed by at most $\epsilon|U|$. Thus, the optimal revenue changes by at most $\epsilon|U|$. As $\epsilon \to 0$, this goes to 0, so we know that the optimal revenue is a continuous function of the bids.

Now, we just need to show that the revenue expression in the lemma statement is the limit of a sequence of bids that converges to the desired bids. Consider all advertisers who achieve the top bid sum. If there is more than one, let's pick one to be the top. For the other such advertisers, we add an ϵ dampening on some term where an advertiser is not the *unique* top bid (this exists for every such advertiser – otherwise this advertiser would be the unique advertiser with the top bid sum), and keep all of the other bids the same. Thus, the only term in the above expression that changes is the ith price revenue. We see that the ith price revenue will be dampened by at most $\epsilon \cdot |U|$, leading to at most a $\epsilon|U|$ reduction in the expression. Moreover, if ϵ is sufficiently small, then m will not be affected (since we have a *strict* inequality in its definition). Thus, the revenue expression converges to its limit as $\epsilon \to 0$ for this sequence of bids. □

We use this lemma to compute the revenue ratio in the offline case on Example 1.

Lemma 13. *The fair value of the offline revenue in Example 1 is*

$$d + \frac{1-d}{k} + (1-d)\frac{k-1}{k}\frac{b^{low}}{b^{high}}.$$

Proof of Lemma 13. We see by Lemma 11 that $m = 1$ since $\sum_{u \in U} b_u^i = b^{high} + (k-1)b^{low}$ while the 1st price auction revenue is $b^{high}k$ and the ith price auction revenue for $i \neq 1$ is kb^{low}. □

We see that Lemma 1 essentially follows from Lemma 13.

Proof of Lemma 1. We use that $\frac{b^{low}}{b^{high}} \geq \alpha^{-1}$ to obtain the desired result. □

A.1.2 Online mechanism

We now consider the online setting.

Proof of Lemma 2. Suppose that all advertisers bid 1 on the first user u. Suppose that the mechanism assigns the minimum probability m (or tied for minimum probability) on u to some advertiser A. Now, suppose that A bids α on every subsequent user, and every other advertiser bids α^{-1} on every subsequent user. Then the first-price auction revenue is $1+\alpha(|U|-1)$. This mechanism's revenue is $\leq 1+(\alpha^{-1}(1-m-d)+(m+d)\alpha)(|U|-1)$. The ratio is

$$\leq \frac{1+(\alpha^{-1}(1-m)+m\alpha)(|U|-1)}{1+\alpha(|U|-1)}.$$

Since there are k advertisers, we know that $m \leq \frac{1}{k}$, so this is:

$$\leq \frac{1+(\alpha^{-1}(1-\frac{1}{k}-d)+(\frac{1}{k}+d)\alpha)(|U|-1)}{1+\alpha(|U|-1)}.$$

Let $\beta = \alpha^{-2}(1-\frac{1}{k}-d)+(\frac{1}{k}+d)$. This is equal to

$$\beta + \frac{1-\beta}{1+\alpha(|U|-1)}$$

As $|U| \to \infty$, this approaches β. $\square$

A.2 Proofs for Section 3.2

First, we show fairness.

Proof of Proposition 3. Let's consider the difference

$$E := \frac{\sum_{i\in S} C_v^i}{\sum_{j=1}^k C_v^j} - \frac{\sum_{i\in S} C_u^i}{\sum_{j=1}^k C_u^j},$$

where $C_u^i = (b_u^i)^l$ and $C_v^i = (b_v^i)^l$. Let's let $\alpha_u = \sum_{i\in S} C_u^i$, $\beta_u = \sum_{i\notin S} C_u^i$ and $\alpha_v = \sum_{i\in S} C_v^i$, $\beta_v = \sum_{i\notin S} C_v^i$. WLOG, assume that $E \geq 0$. Let $R_\alpha = \alpha_v/\alpha_u$ and let $R_\beta = \beta_u/\beta_v$. Bid fairness tells us that $C_u^i \leq RC_v^i$ and $C_v^i \leq RC_u^i$ for all i. This implies that $\frac{1}{R} \leq R_\alpha, R_\beta \leq R$, where $R = f(d)^l$. We have that

$$E = 1 - \frac{\beta_v}{R_\alpha \cdot \alpha_u + \beta_v} - \frac{\alpha_u}{\alpha_u + R_\beta \cdot \beta_v}.$$

Observe that this expression can be upper bounded by the case where R_β is maximized (i.e. where $R_\beta = R$) and R_α is maximized (i.e. where $R_\alpha = R$). Our expression becomes:

$$\begin{aligned} E &\leq \frac{\alpha_u \cdot R}{\alpha_u \cdot R + \beta_v} - \frac{\alpha_u}{\alpha_u + \beta_v \cdot R} \\ &= \frac{\alpha_u\beta_v(R^2-1)}{(\alpha_u + \beta_v \cdot R)(R\cdot\alpha_u + \beta_v)} \\ &= \frac{\alpha_u\beta_v(R^2-1)}{R\alpha_u^2 + R\beta_v^2 + \alpha_u\beta_v(R^2+1)} \\ &\leq \frac{\alpha_u\beta_v(R^2-1)}{2R\alpha_u\beta_v + \alpha_u\beta_v(R^2+1)} \\ &= \frac{R^2-1}{2R+R^2+1} \\ &= \frac{R-1}{R+1} \end{aligned}$$

It suffices to show that
$$\frac{R-1}{R+1} \leq d(u,v).$$
This can be solved to
$$R \leq \frac{1+d(u,v)}{1-d(u,v)}.$$
Thus, we can set the following condition:
$$f(d(u,v)) \leq \left(\frac{1+d(u,v)}{1-d(u,v)}\right)^{1/l}.$$
$\square$

The revenue for item $u \in U$ in Mechanism 1 is:
$$\sum_{i=1}^{k} b_u^i p_u^i = \sum_{i=1}^{k} b_u^i \frac{(b_u^i)^l}{\sum_{j=1}^{k}(b_u^j)^l} = \frac{\sum_{i=1}^{k}(b_u^i)^{l+1}}{\sum_{i=1}^{k}(b_u^i)^l}.$$
Let $\vec{b}_u = [b_u^1, \dots, b_u^k]$. Then this can be written as
$$\frac{||\vec{b}_u||_{l+1}^{l+1}}{||\vec{b}_u||_l^l}.$$

Let's consider how this compares to a first price auction revenue for item i, which can be written as $||\vec{b}_u||_\infty$. The fair value is:
$$\frac{||\vec{b}_u||_{l+1}^{l+1}}{||\vec{b}_u||_l^l ||\vec{b}_u||_\infty}.$$
Now, we compute the revenue. We begin with an analysis of l_p-norm relevant to the calculation.

Proposition 14. *Consider $\vec{x} \in (\mathbb{R}^{\geq 0})^n$ such that $||x||_l^l = C$. Then $||x||_{l+1}^{l+1} \geq n\left(\frac{C}{n}\right)^{\frac{l+1}{l}}$.*

Proof. We prove this by induction. The base case is $n = 1$, where the expression is $C^{\frac{l+1}{l}}$ as desired. Now, we do the Lagrange multipliers for $n = m$. The boundary condition is $x_i = 0$ for some number of i, but this just reduces to the case of a smaller n. We compute the minimum for an interior point. The relevant expression is:
$$x_1^{l+1} + \dots x_m^{l+1} - \lambda(x_1^l + \dots x_m^l).$$
Taking a derivative of x_i, we obtain:
$$(l+1)x_i^l - \lambda l x_i^{l-1} = 0.$$
This can be reduced to:
$$x_l = \frac{\lambda l}{l+1}.$$
This means that all of the x_l are equal, so $x_l = \left(\frac{C}{m}\right)^l$. Plugging this in, we obtain:
$$m\left(\frac{C}{m}\right)^{\frac{l+1}{l}}.$$
This is an increasing function of m, so the boundary cases will not win. $\square$

We prove Lemma 4.

Proof of Lemma 4. We just need to analyze $\frac{||\vec{b}_u||_{l+1}^{l+1}}{||\vec{b}_u||_l^l ||\vec{b}_u||_\infty}$. Multiplicatively scaling by $||\vec{b}_u||_\infty$ leaves the expression unchanged, so we can assume WLOG that $||\vec{b}_u||_\infty = 1$. WLOG, let $b_u^k = 1$. Let $\vec{b'}_u = [b_1^u, \ldots, b_u^{k-1}]$. Now, the expression can be written as:

$$\frac{1 + ||\vec{b'}_u||_{l+1}^{l+1}}{1 + ||\vec{b'}_u||_l^l}.$$

Now, let $C = ||\vec{b'}_u||_l^l$. For any given C, minimizing the expression is equivalent to minimizing $||\vec{b'}_u||_{l+1}$. By Proposition 14, we claim that $||\vec{b'}_u||_{l+1}^{l+1} \geq (k-1)\left(\frac{C}{k-1}\right)^{\frac{l+1}{l}}$. Let $c = \left(\frac{C}{k-1}\right)^{\frac{1}{l}}$. Observe that $0 \leq C \leq k-1$. Then we have that our ratio is lower bounded by:

$$\frac{1 + (k-1)\cdot c^{l+1}}{1 + (k-1)\cdot c^l} = c + \frac{1-c}{1 + (k-1)\cdot c^l}.$$

Now, we need to minimize this expression for $0 \leq c \leq 1$. The derivative of this expression is equal to:

$$\begin{aligned} D &= \frac{(k-1)(l+1)c^l}{(k-1)c^l + 1} - \frac{(k-1)lc^{l-1}((k-1)c^{l+1} + 1)}{((k-1)c^l + 1)^2} \\ &= \frac{(k-1)c^{l-1}}{((k-1)c^l + 1)^2}\left((l+1)c((k-1)c^l + 1) - l((k-1)c^{l+1} + 1)\right) \\ &= \frac{(k-1)c^{l-1}}{((k-1)c^l + 1)^2}\left((k-1)c^{l+1} + c(l+1) - l\right). \end{aligned}$$

Thus, the sign of this expression is the sign of $P(c) = \left((k-1)c^{l+1} + c(l+1) - l\right)$. This expression is increasing as a function of c. Moreover, this expression is $k - 1 + 1 = k > 0$ at $c = 1$ and $-l$ at $c = 0$. Thus, there's exactly one root, and it occurs in the interval $c \in (0, 1)$. Let's suppose that the root of this is c^*. If $P(c) < 0$, then $c < c^*$, and if $P(c) > 0$, then $c > c^*$. Then $c < c^*$, then the ratio is decreasing, and if $c > c^*$, then the ratio is increasing.

Now, consider $c' = (k-1)^{-1/l}$. At this value, $P(c') = (k-1)(k-1)^{-(l+1/l)} + (k-1)^{-1/l}(l+1) - l = (k-1)^{-1/l}(l+2) - l$. Now, notice that $(k-1) \geq e^2 \geq \left(1 + \frac{2}{l}\right)^l$. Thus $(k-1)^{-1/l} \leq \frac{1}{1+\frac{2}{l}} = \frac{l}{l+2}$. This implies that $P(c') < 0$. Thus, we have that $c' < c^*$.

It suffices to show that for all $c \geq c'$, the expression $c + \frac{1-c}{1+(k-1)\cdot c^l}$ is lower bounded by $\frac{(k-1)^{-1/l}(k-1)}{k} + \frac{1}{k}$. We observe that $c + \frac{1-c}{1+(k-1)\cdot c^l} \geq c + \frac{1-c}{1+k-1} = \frac{c(k-1)}{k} + \frac{1}{k}$. This lower bound is an increasing function of c, and so we can plug in $c = c' = (k-1)^{-1/l}$ to obtain a lower bound. □

B Proofs for Section 4

B.1 Proofs for Section 4.1

We now prove Proposition 5.

Proof of Proposition 5. We use Example 2. Observe that the first-price auction revenue is at least $b^{high}(k+1)$.

If A^i receives the protected individual with probability $p^i_{u_{k+1}}$, then we know that that $\sum_{i=1}^k p^i_{u_{k+1}} = 1$. Moreover, the total revenue becomes bounded by $b^{high}(\min(1, p^1_{u_{k+1}} + d)) + b^{high}(\min(1, p^2_{u_{k+1}} + d)) + \ldots + b^{high}(\min(1, p^k_{u_{k+1}} + d)) + b^{high}\cdot(p^1_{u_{k+1}} + \ldots + p^n_{u_{k+1}}) + b^{low}(\text{ leftovers }) \leq 2b^{high} + b^{high}dk + b^{low}\cdot k$.

Thus, the fair value is at most $\frac{2}{k+1} + d + \frac{b^{low}}{b^{high}} = \frac{2}{k+1} + d + \frac{1}{f(1)}$ □

We prove Proposition 6.

Proof of Proposition 6. The total sum of probabilities is ≤ 1. The sum of probabilities on S_u on u is at least $\frac{C}{\beta C+1}$. The sum of the probabilities on S_u on $v \neq u$ is at most $\frac{\beta C}{\beta C+1}$. □

We prove Proposition 7.

Proof of Proposition 7. Suppose that an online allocation mechanism achieves an fair value of $> R$ and β user-directed fairness.

We will specify a sequence of user types $t_0, t_1, \ldots, t_n, \ldots$. A user of type t_n will specify a set S_{t_n} of C advertisers. For users of type t_n, all of the advertisers, except for a special advertiser $A_n \notin S_{t_n}$, bid 0 on the user, while A_n bids 1 on the user. We will adaptively construct the sets S_{t_n}. The advertiser A_n will be selected from the set of advertisers not in S_{t_n}. (Recall that C is less than the total number of categories, so there is always at least one such advertiser.)

The adversary will perform the following high-level strategy. It will iterate through users of types t_0, then type t_1, etc. It will continue to specify users of type t_n while the probability placed on A_n is less than R. If the probability is at least R, then the adversary switches to t_{n+1}. If this process gets stuck at some t_n, then the fair value will necessarily be at most R. Thus, since the fair value is greater than R, we know that users of every type will be seen at some finite time step.

Now, we show how to construct the sets S_{t_n} so that for $n \geq 1$, there is at least a probability of $R\beta^{-1} + R\beta^{-2}\frac{C-1}{C} + \ldots + R\beta^{-n}\left(\frac{C-1}{C}\right)^{n-1}$ must be placed on S_{t_n} on for users of type t_n. We construct the sets inductively. The first set S_{t_0} can be any set of C advertisers. For S_{t_1}, we pick any subset that contains A_0. Since we switched to type t_1 users, there was at least a mass of R placed on A_0 in the last type t_0 user. Thus, there must be at least a $R\beta^{-1}$ mass on S_{t_1} by users of type t_1 as desired. For $n \geq 1$, for set $S_{t_{n+1}}$, we use the fact that there is at least a $R\beta^{-1} + R\beta^{-2}\frac{C-1}{C} + \ldots + R\beta^{-n}\left(\frac{C-1}{C}\right)^{n-1}$ mass on S_{t_n}. Thus, there's some subset of size $C-1$ of S_{t_n} that has at least a $\frac{C-1}{C}\left(R\beta^{-1} + R\beta^{-2}\frac{C-1}{C} + \ldots + R\beta^{-n}\left(\frac{C-1}{C}\right)^{n-1}\right)$ mass. We let $S_{t_{n+1}}$ be this set coupled with A_n. The mass on this set on the last user of type t_n, since we switched to t_{n+1}, is $R + \frac{C-1}{C}\left(R\beta^{-1} + R\beta^{-2}\frac{C-1}{C} + \ldots + R\beta^{-n}\left(\frac{C-1}{C}\right)^{n-1}\right)$. Thus, the mass on the users of type t_{n+1} must be at least $\beta^{-1}\left(R + \frac{C-1}{C}\left(R\beta^{-1} + R\beta^{-2}\frac{C-1}{C} + \ldots + R\beta^{-n}\left(\frac{C-1}{C}\right)^{n-1}\right)\right)$ as desired.

Therefore the last user of type t_n must have a mass of $R + R\beta^{-1} + R\beta^{-2}\frac{C-1}{C} + \ldots + R\beta^{-n}\left(\frac{C-1}{C}\right)^{n-1}$. Since every type is seen at a finite time step, we know that $R + R\beta^{-1}\sum_{i=0}^{\infty}\left(\frac{C-1}{\beta C}\right)^i \leq 1$. This means that $R(1 + \frac{\beta^{-1}}{1-\frac{C-1}{\beta C}}) \leq 1$. This can be written as $R(1 + \frac{\beta^{-1}}{\frac{(\beta-1)C+1}{\beta C}}) = R(1 + \frac{C}{(\beta-1)C+1}) = R(\frac{1+\beta C}{(\beta-1)C+1})$. Thus we have that $R \leq \frac{(\beta-1)C+1}{1+\beta C}$. □

We prove Proposition 8.

Proof of Proposition 8. This follows from the fact that the revenue in Mechanism 3 is $\sum_{u \in U} \max_{i \in S_u} b_u^i$ and the probability of selecting a category in S_u on u is 1. □

Proof of Proposition 9, Proof of Proposition 10. Fairness follows from the fairness of Mechanism 2 and Mechanism 3, as well as the fairness of Mechanism 1 in Proposition 3 and since Mechanism 1 uses the full probability mass on each user. The fair value follows from the fair value of Mechanism 1 in Lemma 4 and the fair value of Mechanism 3 in Proposition 8. □

and Mechanism 2 in Proposition 7.

C Mechanisms for uniform metrics

We consider the following mechanism for the uniform metric setting and show upper bound nearly matches the lower bounds in Section 3.1.

Mechanism 6 (Shifted Mechanism)**.** The mechanism assigns a top bidder for the first user 1 and all other bidders 0. On future users, if the advertiser does not have the top bid, then the mechanism assigns $1-d$ to that advertiser and d to the advertiser with the top bid, and 0 to other advertisers.

Observe that the fair value achieved by this mechanism is at least $(1-d)\alpha^{-2}+d$. This is very close to (in fact within an additive error of $\frac{1}{k}$ of the bound of) the upper bound in Lemma 2. This demonstrates that the upper bound in Lemma 2 essentially cannot be tightened using uniform metrics.

Since this mechanism requires use of the metric and identities, we also consider other mechanisms, though the fair value of these mechanisms is worse. The next mechanism is metric-oblivious and is fair for non-uniform metrics but is highly asymmetric based on bids on the first user. It does not provide good revenue guarantees when $d \to 1$ where the bid ratio goes to ∞.

Mechanism 7 (Top Bidder Mechanism)**.** For item $u \in U$ and $1 \le i \le k$, the mechanism assigns a top bidder on the first user $p_u^i = 1$ for all bids.

It is straightforward to see that the fair value is α^{-2}.

The next mechanism is a modification of the previous mechanism that achieves good revenue guarantees, but that depends on d. It continues to have a strong asymmetry based on bids on the first user. Now, we consider a symmetric mechanism that also depends on d. However, it does not provide reasonable revenue guarantees.

Mechanism 8 (Even Mechanism)**.** For item $u \in U$ and $1 \le i \le k$, the mechanism assigns a top bidder $p_u^i = \frac{1}{k} + \frac{d(k-1)}{k}$ and all other bidders $\frac{1}{k} - \frac{d}{k}$.

It is straightforward to see that the fair value is $\frac{1}{k} + \frac{d(k-1)}{k}$.

The next mechanism is a modification of the previous mechanism that breaks some of the symmetry of the previous mechanism by performing weeding out of advertisers based on bids on the first user. it provides much better revenue guarantees.

Mechanism 9 (Improved Even Mechanism)**.** For item $u \in U$ and $1 \le i \le k$, the mechanism assigns a probably of 0 to all advertisers that aren't within α^{-2} of the top advertiser on the first user 0. The mechanism assigns a top bidder $p_u^i = \frac{1}{k} + \frac{d(l-1)}{l}$ and all other bidders $\frac{1}{l} - \frac{d}{l}$ to the l remaining advertisers.

It is straightforward to see that the fair value is at least $\frac{1}{k} + \frac{d(k-1)}{k} + \alpha^{-4}\left(\frac{k-1}{k} - \frac{d(k-1)}{k}\right)$.

D Relaxations for the identical metric case

One possible relaxation is to consider $(1-\delta)$-relaxed total variation fairness, a Lipschitz relaxation of total variation fairness. This definition essentially disregards distances $> 1-\delta$ and scales the other distances out.

Definition 7 (Relaxed Total Variation Fairness)**.** A central body mechanism satisfies $(1-\delta)$-relaxed-**total variation fairness** for a metric d and constant $\delta > 0$ if $|\sum_{i\in S} p_u^i - \sum_{i \in S} p_v^i| \le (1-\delta)^{-1} d(u,v)$ for every $1 \le i \le k$, $u, v \in U$, and *all* subsets S of advertisers.

We can easily obtain the following by applying the mechanism to $d(u,v)\cdot(1-\delta)$ and ignoring distances bigger than $1-\delta$.

Proposition 15. *With the bid ratio condition* $f(d(u,v)) = \left(\frac{1+(1-\delta)^{-1}d(u,v)}{1-(1-\delta)^{-1}d(u,v)}\right)^{1/l}$ *for* $d(u,v) \le 1-\delta$ *for some parameter* $l \ge 1$*, Mechanism 1 satisfies* $(1-\delta)$*-relaxed total variation fairness.*

This bid ratio has the nice property that as $d \to 1-\delta$, the bid ratio condition goes to ∞. Moreover, the bid ratio is weaker than the bid ratio condition in Section 3.2.

E Additional lower bounds for different metrics

We consider a weaker notion of fairness based on additively relaxing based on the sum of fairness metrics in the fairness constraints.

Definition 8 (Metric-User-Specified Fairness)**.** A central body mechanism satisfies β-**metric-user-specified fairness** if for all $u \in U$, it is true that: $\sum_{i \in S_u} p_v^i \le \beta\left(\sum_{i\in S_u} p_u^i + \sum_{i \in S_u} d^i(u,v)\right)$.

We prove lower bounds by considering the following example:

Example 3. Suppose for all $u, v \in U$, we have that $d^k(u, v) \geq d_{big}$ for $j+1 \leq k \leq c$ and $d^k(u, v) \leq d_{small}$ for $1 \leq k \leq j$. On all users, the bids of $I_1, \ldots, I_j$ are b, for some constant $b > 1$. Suppose that we can partition U into $U_1 \cup U_2$ where users in U_1 are type 1 and users in U_2 are type 2. For all $1 \leq j \leq k$, for $u \in U_1$ we have that $b_u^k = M$ for $M > b$ and for $u \in U_2$, we have that $b_u^k = 1$. Let's suppose that all type 1 users specify $I_1, \ldots, I_j$ for its selected categories[19].

The idea if $b >> 1$, then on type 1 users, the bids from $I_1, \ldots, I_C$ are significantly worse than the bids from $I_{C+1}, \ldots, I_c$, while on type 2 users, the bids from $I_1, \ldots, I_C$ are significantly better than the bids from $I_{C+1}, \ldots, I_c$. Since the type 1 users requested fairness on $I_1, \ldots, I_C$, it is not possible to simultaneously place a high mass on $I_{C+1}, \ldots, I_c$ for type 1 users and $I_1, \ldots, I_C$ for type 2 users. We specifically show the following lower bounds:

Proposition 16. *Consider online mechanisms for different metrics that achieve β metric-user-specified fairness.*

1. *If the bid ratio condition is ∞ at 1 and distances of 0 are permitted, then any such mechanism has a fair value of $R \leq \frac{\beta}{\beta+1}$.*
2. *If the bid ratio condition is ∞ at 1, users are only permitted to specify $\leq C$ advertisers, and $d(u, v) \geq d_{small}$ for $u \neq v$, then any such mechanism has a fair value of $R \leq \frac{\beta}{\beta+1}(1 + C \cdot d_{small})$.*
3. *Suppose that $f(1) < \infty$. Then any such mechanism has a fair value of $R \leq \frac{\beta}{\beta+1} + \frac{1}{\sqrt{f(1)(\beta+1)}}$.*

Proof of Proposition 16. It suffices to show the following: if the metric has minimum distance d_{small}, each category has $\leq C$ advertisers, and m is the minimum-to-maximum bid ratio condition, then any online mechanism that achieves β-metric-user-specified fairness online mechanism has a fair value $R \leq \frac{\beta}{\beta+1}(1 + C \cdot d_{small}) + \frac{(1 - C \cdot d_{small} \beta)\sqrt{m}}{\beta+1}$.

The proof relies on Example 3. Let δ be some value and suppose there are $\geq b/\delta$ users.

Let's consider a type 1 user u and type 2 user v. Let $p_1 := \sum_{i=1}^{j} p_u^i$ and $p_2 := \sum_{i=1}^{j} p_v^i$. Suppose that we have a fairness condition that says that $p_2 \leq \beta(p_1 + D)$.

On u, the fair value is $1 - p_1 + p_1 \frac{b}{M} = 1 - p_1\left(1 - \frac{b}{M}\right)$. On v, the fair value is $p_2 + \frac{1}{b}(1 - p_2) = \frac{1}{b} + p_2\left(1 - \frac{1}{b}\right)$. Now, our fairness condition says that $p_2 \leq \beta(p_1 + D)$. Thus we know that $\frac{1}{b} + p_2\left(1 - \frac{1}{b}\right) \leq \frac{1}{b} + \beta p_1\left(1 - \frac{1}{b}\right) + \beta D\left(1 - \frac{1}{b}\right)$.

We have that $1 - p_1\left(1 - \frac{b}{M}\right) = \frac{1}{b} + \beta p_1\left(1 - \frac{1}{b}\right) + \beta D\left(1 - \frac{1}{b}\right)$ when:

$$p_1 = \frac{(1 - \frac{1}{b})(1 - D\beta)}{\beta(1 - \frac{1}{b}) + 1 - \frac{b}{M}}.$$

Now we use the fact that $M = b^2$ to obtain:

$$p_1 = \frac{1 - D\beta}{\beta + 1}.$$

This implies that the fair value at this value is at least:

$$1 - p_1\left(1 - \frac{b}{M}\right) = 1 - \left(1 - \frac{1}{b}\right)\frac{1 - D\beta}{\beta + 1} = \frac{\beta}{\beta + 1}(1 + D) + \frac{1 - D\beta}{(\beta + 1)b}.$$

Now, we first send in a type 1 user. We continue to send in type 1 users while the total probability assigned to $I_1, \ldots, I_j$ is $\leq \frac{1 - D\beta}{\beta+1}$. As shown above, the fair value is at least $\frac{\beta}{\beta+1}(1 + D) + \frac{1 - D\beta}{(\beta+1)b}$. If the total probability exceeds $\frac{1 - D\beta}{\beta+1}$ at any point, then we switch to sending in type 2 users. Now, the fair value on type 2 users is at least $\leq \frac{1 - D\beta}{\beta+1}$ as well.

[19] We don't even need type 2 users to specify fairness constraints.

Thus, the fair value is at least $r = \frac{1-D\beta}{\beta+1}$ except on potentially one user where the switch to type 2 users occurs. The total fair value can be upper bounded by $\frac{M+rQ}{m+Q}$. As long as $Q \geq M/\delta$, we can bound this by $r + \delta$. We can phrase this as long as there are $\geq b/\delta$ users.

Let's set $\delta \to 0$ and observe that $b \leq \frac{1}{\sqrt{m}}$ to obtain the desired expression. □

We now show that with the relaxed fair value based on user preferences, it is still not possible to recover multiple-task fairness-style guarantees. The fairness notion that we consider is a user-specified variant of version of multiple-task fairness.

Definition 9 (User-Specified Multiple-Task Fairness)**.** A mechanism satisfies **β-user-specified multiple-task fairness** with respect to some function f if $p_v^i \leq \beta(p_u^i + d^i(u,v))$ for all $i \in S_u$ and $u \in U$.

First, we show that if we don't place restrictions on the subsets S_u, then we can't achieve fairness w.r.t any reasonable combination of the metrics. (Here, observe that the guarantees don't necessarily get stronger as C increases because the fair value definition also changes. In fact in the examples we consider, the guarantees get weaker in a sense.) The bound makes it so that some users specify a set of advertisers that all bid low, while other users swap out of one of the advertisers for an advertiser that bids high.

Proposition 17. *Even with a maximum-to-minimum bid ratio condition of* 1 *on the advertisers at all distances (i.e. each advertiser always bids the same on all users), a mechanism that receives sets S_u of size C and satisfies user-specified multiple-task fairness has a fair value of at most:*

$$R \leq \frac{\beta}{\beta + \frac{C-1}{C}} + \frac{d_{small}(C-1)}{\beta + \frac{C-1}{C}},$$

if the metric satisfies $d \geq d_{small}$.

Proof of Proposition 17. Suppose that we have advertisers $A_1, \ldots, A_k$. Let's suppose the distance metric is d everywhere. Let's suppose that $A_1, \ldots, A_j$ bid 1 on all users and $A_{j+1}, \ldots, A_k$ bid $B > 1$ on all users. Let's give the mechanism a user u that specifies $A_1, \ldots, A_j$. Suppose the mechanism places a probability of p on u. If $p < R$, we repeat copies of this user until the mechanism assigns more mass on this user. Now, suppose $p > R$. Otherwise, there is some set of $C-1$ advertisers that have a total of $\geq \frac{p(C-1)}{C}$ mass. Now, the mechanism has to continue to place a mass of $\frac{1}{\beta}\left(\frac{p(C-1)}{C} - d(C-1)\right)$ on these advertisers. Now, the mechanism can only put $1 - \frac{1}{\beta}\left(\frac{p(C-1)}{C} - d(C-1)\right) < 1 - \frac{1}{\beta}\left(\frac{R(C-1)}{C} - d(C-1)\right)$ mass on B. Let's solve to set

$$R = 1 - \frac{1}{\beta}\left(\frac{R(C-1)}{C} - d(C-1)\right).$$

This gives us:

$$\beta - \beta R = \frac{R(C-1)}{C} - d(C-1).$$

We can further solve to obtain:

$$R = \frac{\beta}{\beta + \frac{C-1}{C}}.$$

□

Now, let's try to lower our expectations and allow the user to choose from a collection of prescribed sets $C_1, \ldots, C_l$ that partition the categories so that these sets are non-intersecting. Now, if advertiser bid the same on all users, the problem is trivial (run a first-price auction in each C_i). Without these extremely restrictive conditions, it turns out to still be impossible to achieve user-specified multiple-task fairness with a good fair value. These bounds use Example 3.

Proposition 18. *Consider online mechanisms for different metrics that achieve β-user-specified multiple-fairness. Let's suppose that users can choose from prescribed category choices $C_1, \ldots, C_l$,.*

1. *If the bid ratio condition is ∞ at* 1 *and distances of* 0 *are permitted, then any such mechanism has a fair value of $R \leq \frac{\beta}{\beta+1}$.*

2. *If $f(1) < \infty$, then any such mechanism has a fair value of $R \leq \frac{\beta}{\beta+1} + \frac{1}{\sqrt{f(1)(\beta+1)}}$.*

Proof of Proposition 18. We use Example 3. Let $C_1 = \{I_1, I_{j+1}, I_{j+2}, \ldots, I_{j+C-1}\}$ and suppose all users pick C_1. Since $p_u^i = 0$ on $i \notin 1, [j+1, j+C-1]$, we can cut the other categories from the auction. We see that in this example, the revenue on $\sum_{u \in U} \max_{i \in C_1} b_u^i$ matches the first-price revenue. Now, we use the fact that if a mechanism satisfies β-user-specified multiple-task fairness, then it satisfies β-metric user-specified fairness on I_1, and the relaxed fair value is the same as the regular fair value. Thus, the lower bounds from the proof of Proposition 16 on this example apply in this setting. □

F Position Auctions

The first-price auction, which produces the optimal revenue in the absence of fairness, is a position auction that selects the highest bidder for the first slot with probability 1 produced the optimal revenue, since the mechanism only uses the ordering of the bids, without any information about the identities of the advertisers or the bid values. We show that with multiple-task fairness constraints, it is impossible for a mechanism that only uses the ordering of the bids to achieve a competitive revenue. This example demonstrates why we must give the platform greater information about the bids in order to simultaneously achieve fairness and a competitive revenue approximation.

In our lower bound, we specifically consider the setting where the platform is only given access to the ordering of current bids (and not the identities of advertisers or the values of the current bids) on the current user. *After* selecting a fractional allocation for this user, the platform is giving access to the values of the bids and the identities of the advertisers on those bids. Intuitively, not knowing the identity of the highest bidder or values of the other bids makes it difficult for the central body to place a high probability on the highest bid while satisfying fairness constraints. We show that with *any* non-trivial bid ratio condition, it is impossible for a position auction with fractional allocations to simultaneously achieve multiple-task fairness and a competitive revenue approximation.

Lemma 19. *If the metrics are identical and uniform (i.e. $d^i(u,v) = d$ for all $u, v \in U$) and with* ***any*** *bid ratio condition > 1 for $d > 0$, an online mechanism that only sees the ordering of the bids and achieves multiple-task fairness has a fair value of $\leq \frac{1}{k} + 2d$.*

Proof of Lemma 19. First, we show an online mechanism that only sees the order of the bids must place at most a $\frac{1}{k} + d$ probability on the top bidder, if there is a unique top bid. We consider bid sequences with a unique top bid and all other bids equal.

Let p_1 be the probability on the top bid on the first user. First, we show that fairness alone tells us that $p_1 \leq \frac{1}{k} + d$. Let's say that the first user has bids $1, 1-\delta, 1-\delta, \ldots, 1-\delta$. Now, the second user can (approximately) have bids $\alpha, \alpha^{-1}, \ldots, \alpha^{-1}$. Assuming that $\alpha > 1$, this means that any permutation is possible and we can't differentiate between permutations. Let's say that we put a probability of S on the top advertiser on the next user. We can choose an assignment where the top advertiser from the first user is the top advertiser from the second user, so $S \geq p_1 - d$. We can also choose an assignment where the top advertiser on the first user is assigned to the minimum probability on the second user. This means that $\frac{1-S}{k-1} \geq p_1 - d$. (Solving, we obtain $S \leq 1 - (k-1)(p_1 - d)$.) This implies that $kp_1 - dk \leq S + (k-1)\frac{1-S}{k-1} = 1$, so $p_1 \leq \frac{1}{k} + d$.

Let's suppose the first user actually has bids $1, 0, 0, \ldots, 0$. Now, the top bidder can only receive $\frac{1}{k} + 2d$ on any subsequent bid, so the fair value is at most $\frac{1}{k} + 2d$. □

We see that in the limit as $k \to \infty$ and $\epsilon \to 0$, this condition disallows $d(u,v) \leq \frac{1}{2}$. This condition is far too strong to permit the necessary level of expression for the fairness metric.

G Upper bounds for convex bid ratio constraints

If $f(1)$ is permitted to be a large finite number, it is not possible to have a concave bid ratio condition along with competitive fair value in an offline mechanism that achieves multiple-task fairness.

Lemma 20. *Suppose that the bid ratio condition satisfies $f(0) = 1$ and $f(1) = h$ and is concave (or linear). Then, any offline mechanism that satisfies multiple-task fairness on a general metric has a fair value $\leq \frac{1}{k} + \frac{2}{\sqrt{h-1}}$.*

Proof. Any concave function f will satisfy $f(d) \geq 1 + d(h-1)$. Moreover, by Lemma 1, we know that the fair value is at most $R \leq \frac{1}{k} + \frac{1}{1+d(h-1)} + d$. At $d = \frac{1}{\sqrt{h-1}}$, we see that the fair value is at most $\frac{1}{k} + \frac{1}{\sqrt{h-1}} + \frac{1}{1+\sqrt{h-1}} \leq \frac{1}{k} + \frac{2}{\sqrt{h-1}}$ as desired. $\square$

When h is large, this fair value is small. This demonstrates that to achieve a competitive revenue and multiple-task fairness, we cannot restrict to concave (or linear) bid ratio functions.