Optimizing Long-term Social Welfare in Recommender Systems: A Constrained Matching Approach

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Abstract

Most recommender systems (RS) research assumes that a user's utility can be maximized independently of the utility of the other agents (e.g., other users, content providers). In realistic settings, this is often not true—the dynamics of an RS ecosystem couple the long-term utility of all agents. In this work, we explore settings in which content providers cannot remain viable unless they receive a certain level of user engagement. We formulate the recommendation problem in this setting as one of *equilibrium selection* in the induced dynamical system, and show that it can be solved as an *optimal constrained matching* problem. Our model ensures the system reaches an equilibrium with maximal social welfare supported by a sufficiently diverse set of viable providers. We demonstrate that even in a simple, stylized dynamical RS model, the standard *myopic* approach to recommendation—always matching a user to the best provider—performs poorly. We develop several scalable techniques to solve the matching problem, and also draw connections to various notions of user regret and fairness, arguing that these outcomes are fairer in a utilitarian sense.

1. Introduction

Investigations of various notions of *fairness* in machine learning (ML) have shown that, without due care, applying ML in many domains can result in biased outcomes that disadvantage specific individuals or groups (Dwork et al., 2012; Barocas & Selbst, 2016). Content *recommender systems (RSs)*, which match users to content (e.g., news, music,

video), typically rely on ML to predict a user's interests to recommend "good" content (Konstan et al., 1997; Jacobson et al., 2016; Covington et al., 2016). Since these predictions are learned from past behavior, many issues of ML fairness arise in RS settings (Beutel et al., 2019).

One aspect of "fairness" that has received little attention emerges when one considers the *dynamics of the RS ecosystem*. Both users and content providers have particular incentives for engaging with an RS platform—incentives which interact, via the RS *matching policy*, to couple the long-term utility of agents on both sides of this content "marketplace." Some work has looked at the impact of RS policies on provider welfare (Singh & Joachims, 2018), on long-term user and provider metrics (e.g., using RL (Chen et al., 2018; Ie et al., 2019) or assessing various other phenomena (Ribeiro et al., 2020; Celma, 2010)). However, little work has looked at the interaction of the two on the ecosystem dynamics induced by the RS policy (though there are some exceptions, e.g., (Ben-Porat & Tennenholtz, 2018), which are discussed below).

In this work, we focus on *provider* behavior using a stylized model of a content RS ecosystem in which providers require a certain degree of user engagement (e.g., views, time spent, satisfaction) to remain *viable*. This required degree (or "threshold") of engagement reflects their incentives (social, economic, or otherwise) to participate in the RS; and if this threshold is not met, a provider will withdraw from the platform (i.e., their content is no longer accessible). If this occurs, user segments for whom that provider's content is ideal may be disadvantaged, leading such users to derive less utility from the RS. This may often arise, say, for users and providers of *niche* content.

Typical RS policies are *myopic*: given a user request or query, it returns the provider that is (predicted to be) best-aligned with that query. In our model, myopic policies often drive the dynamical system to a poor equilibrium, with low *user social welfare* and poor provider diversity. By contrast, a more *holistic* approach to matching requests to providers can derive much greater user welfare. We formulate policy optimization in our model as a constrained matching problem, which *optimizes for the socially optimal equilibrium of the ecosystem*. We develop scalable techniques for

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computing these policies, and show empirically that they produce much higher social welfare than myopic policies, *even in this relatively stylized model*. Such policies also lead to greater provider diversity—even though our objective only involves *user* utility—by (implicitly) "subsidizing" some providers that would not remain viable under the myopic policy. We examine tradeoffs between user regret and social welfare in this model, showing that user maximum regret tends to be quite low, especially with utility functions exhibiting diminishing returns. Finally, we draw connections to notions of ML fairness and argue that the outcomes induced by our matching-based policies are fairer in a utilitarian sense.

2. Challenges for Myopic Content Matching

We first introduce an abstract, but general, formalization of dynamic content RSs. Two key elements drive ecosystem utility and dynamics. First, users derive (possibly non-linear) utility from the *collection* or *sequence* of content recommended to them, not just from the individual items. Second, providers require minimum levels of user engagement to remain *viable*, i.e., incentivized to engage with the RS. Within this model, we show how *myopic* RS policies often poorly serve both users and providers, and discuss policy types that overcome this.

2.1. A Formalization of Dynamic Recommendations

We assume an RS (or platform) that matches *users* $\mathcal{U}$ to *content providers* $\mathcal{C}$. Users issue *queries* for desired content, drawn from some space $\mathcal{Q}$. We assume $\mathcal{U}, \mathcal{C}, \mathcal{Q}$ are finite. Given user u's query q_u, the RS returns a provider c from which u derives *immediate reward* $r(q_u, c)$ reflecting match quality. We often assume the existence of some *latent space* $X \subseteq \mathbb{R}^d$ that is used to represent both content and queries. This is common, say, in collaborative filtering (CF), where X is an embedding space constructed by matrix factorization (Salakhutdinov & Mnih, 2007) or neural CF (He et al., 2017; Beutel et al., 2018)). For $c, q_u \in X$ we let $r(q_u, c) = q_u^T c$ be immediate reward.

User queries are received asynchronously and immediately matched to a provider, giving a discrete-event dynamical system over time periods $1, 2, \ldots, t, \ldots$ At each time t, a user $u[t]$, drawn from some distribution $\rho(\mathcal{U})$, issues a query $q_u[t] \in X$, itself drawn from distribution $P_{u[t]}(X)$ reflecting that user's interests. The RS *matches* $q_u[t]$ to some provider $c[t] \in \mathcal{C}$, and $u[t]$ derives reward $r(q_u[t], c[t])$.[1] We assume the RS has *complete knowledge* of the location in X of c and q_u, and that these embeddings do not change

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[1]For ease of exposition, we do not distinguish the different content *items* offered by provider c. Nothing fundamental changes in our approach if we match queries to specific items if each item is associated with a provider.

over time. While unrealistic in practice—most RSs continually update user and content representations, and may engage in active exploration to help assess user interests—the problems we address under this assumption are further exacerbated by incomplete information. We discuss this further below.

Let $h_t = ((u[i], q_u[i], c[i]))_{i \leq t}$ be a length t history of past queries and recommendations, $H[t]$ be the set of all such histories, and $H[*] = \cup_{t < \infty} H[t]$. An RS *policy* $\pi : H[*] \times (\mathcal{Q}, \mathcal{U}) \to \Delta(\mathcal{C})$ maps a history and a query to a distribution $\pi(h_t, q_u[t])$ over providers.

Generally, an RS aims to maximize user engagement. We assume that over long horizons, user engagement is optimized by maximizing *user utility*, which in turn drives sustained user satisfaction with the platform. We assume that u's utility is a (possibly non-linear) function f of the reward sequence $\mathbf{r}_u$ obtained over some horizon. For instance, this might be the cumulative sum of rewards; recency-weighted cumulative reward; or a function (e.g., sigmoid) that captures various behavioral phenomena, such as decreasing marginal returns. We discuss several such functions in Sec. 3. However, even with this user focus, the RS must also address the provider incentives—in our case, viability—since providers offer the quality content needed for a thriving ecosystem. Each provider c has a *viability threshold* ν_c over the amount of (possibly recency-weighted) user engagement (e.g., visits, time spent) generated for it by the RS. Periodically, c compares its overall engagement against ν_c and abandons the platform—perhaps stochastically–if this threshold is not reached (we detail specific forms of this function below). If a provider becomes unviable, it can no longer be matched by the RS to any user query.

2.2. Suboptimality of Myopic Policies

Before developing our methods, we use two simple examples to illustrate why typical *myopic* recommendation policies may be suboptimal.

Typical RSs behave *myopically*: when query q_u is received, it is matched to the provider giving maximum user reward $c_{q_u}^* = \arg \max_{c \in \mathcal{C}} r(q_u, c)$. In the example in Fig. 1(a), each provider has a viability threshold of 2 and is viable in the initial period. The myopic policy first matches each of u_1, u_2, u_3 to c_1; u_4, u_5 to c_2; and u_6 to c_3. This gives total immediate reward $10 + 2\varepsilon$ according to the reward function $r(u, c) = 2 - |u - c|$ for each recommendation. However, since c_3 receives only one user, it is no longer viable at the next period. Hence, at all subsequent periods, the RS must match u_6 to c_2 (with reward 0), attaining a long-run per-period average reward of $8 + 2\varepsilon$.

A *non-myopic policy* can obtain a long-run average reward of $10 - 2\varepsilon$ by matching u_3 to c_2 and u_5 to c_3 at each period.
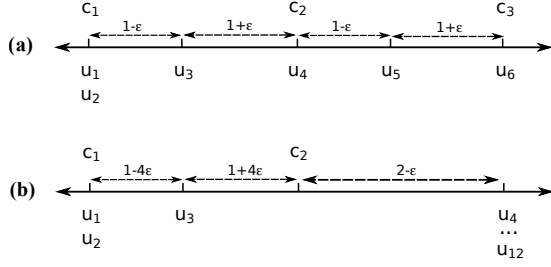
Figure 1. Two 1D examples: (a) User u_i's reward for being matched to provider c_j is 2 less the distance between them (e.g., u_1 has reward 2 for c_1; u_3 has reward $1 + \varepsilon$ (resp., $1 - \varepsilon$) for c_1 (resp., c_2). We equate reward and utility, and a user with her query. assume that each user issues a single query per period, and that each provider requires 2 user impressions in each period to remain viable at the next period. (b) Similar to (a) except that c_1 requires 2 impressions and c_2 requires 10.

Under this policy, c_3 remains viable, allowing u_6 to receive reward 2 (rather than 0) in perpetuity. This comes at a small price to u_3 and u_5, each of whom receive 2ε less per period. This matching *subsidizes* c_3 by matching its content to u_5 (who would slightly prefer provider c_2). This subsidy leaves c_2 vulnerable, so it too is subsidized by the match with u_3. Indeed, this matching is *optimal* for any horizon of at least two periods—its average-per-period *user social welfare* (or total reward) is maximized. The maximum loss of utility experienced by any user at any period w.r.t. the myopic policy is quite small, only 2ε (by both u_3, u_5)—this is the *maximum (user) regret* of the policy. Finally, this policy keeps all providers viable in perpetuity; the set of viable providers $V = \mathcal{C}$ is an *equilibrium* of the dynamical system induced by the policy. By contrast, the myopic policy reaches an equilibrium $V' = \{c_1, c_2\}$ that has fewer viable providers.

Consider now a policy that matches u_3 to c_3 at each period, but otherwise behaves myopically. This induces the same equilibrium $V = \mathcal{C}$ as the optimal policy by subsidizing c_3 with u_3. However, this policy—though improving u_5's utility by 2ε (and her regret to 0)—gives a reward of 0 to u_3 (whose regret is $1 + \varepsilon$). This policy not only has higher max regret, it also has significantly lower welfare of $9 + \varepsilon$.

While not the case in this example, the policy that optimizes social welfare need not minimize max regret. Fig. 1(b) considers a case where the viability threshold differs from each of the two providers. The myopic policy initially matches $\{u_1, \ldots, u_3\}$ to c_1 and $\{u_4, \ldots, u_{12}\}$ to c_2, after which c_2 is no longer viable (and $\{u_4, \ldots, u_{12}\}$ receive no further reward). Thus per-period reward is $5 + 4\varepsilon$ (and max regret is ε.) The welfare-optimal policy subsidizes c_2 by matching u_3, increasing welfare marginally by ε to $5 + 5\varepsilon$, but also increasing max regret (see u_3) to 8ε. This illustrates the trade-off between social welfare maximization and max-

regret minimization.

These examples show that maximizing user social welfare often requires that the RS take action to *ensure the long-run viability of providers*. The example from Fig 1(a) shows that such considerations need not be explicit, but simply emerge as a *by-product of maximizing user welfare alone*. This also promotes diversity among viable providers that can in some sense be interpreted as being "more fair" to the user population. In particular, it creates a smaller gap between the (long-term) utility values attained by different users across the spectrum of possible topic interests. However, as with provider diversity, this type of fairness is not part of the *explicit* objective that drives the RS policy—rather it is implicit, with fairness emerging as a consequence of trying to maximize *overall* user welfare. We discuss connections to work on ML fairness further below. For a richer illustration of this, see Fig. 2.

2.3. Matching Optimization for Recommendation

We now formalize our objectives and optimization approach. For ease of exposition, we assume an *epoch-based* decision problem: time steps are grouped into epochs of fixed length T, with user utility and provider viability both determined at the end of each epoch. Let Q denote the induced distribution over queries during any epoch (since user behavior is stationary, so is Q). Other forms of user/provider evaluation do not impact the qualitative nature of our results—some require different forms of analysis and optimization, while others carry through easily. For example, if providers use recency-weighted engagement, no substantial changes are needed; but if their evaluation occurs on a continual (not epoch-based) basis, more intricate equilibrium analysis is required and optimization becomes more online in nature.

A policy π induces a stochastic dynamical system over a state space, where the state encodes user utility and provider viability at the end of each epoch. Let random variable (RV) $e_t^{\pi}(c)$ be provider c's engagement at time t under π, and $E_k^{\pi}(c)$ its cumulative engagement during epoch $k \geq 1$. If $E_k^{\pi}(c) \geq \nu_c$, c remains viable at epoch $k + 1$, otherwise it abandons the platform. Let V_k^{π} be the set of providers that are viable at the end of epoch k, We assume $V_0^{\pi} = \mathcal{C}$.

Let (RV) $\mathbf{r}_k^{\pi}(u)$ be user u's reward sequence in epoch k under π, and $U_k^{\pi}(u) = f(\mathbf{r}_k^{\pi}(u))$ be u's utility. *Social welfare* generated by π at epoch k is $SW_k^{\pi} = \sum_{u \in \mathcal{U}} U_k^{\pi}(u)$. (Long-run) *average social welfare* is $SW_\infty^{\pi} = \lim_{k \to \infty} [\sum_k SW_k^{\pi}]/k$. The average utility $U_\infty^{\pi}(u)$ of u under π is defined analogously. If π_u^* is the policy that maximizes u's average utility, then u's *regret* under π is $Rgrt^{\pi}(u) = U_\infty^{\pi_u^*}(u) - U_\infty^{\pi}(u)$. The *maximum regret* of π is $MR(\pi) = \max_{u \in \mathcal{U}} Rgrt^{\pi}(u)$. Let $MC(u) \subseteq \mathcal{C}$ be those providers matched by the myopic policy to queries $\mathcal{Q}_u$ with positive support in P_u when all providers are vi-
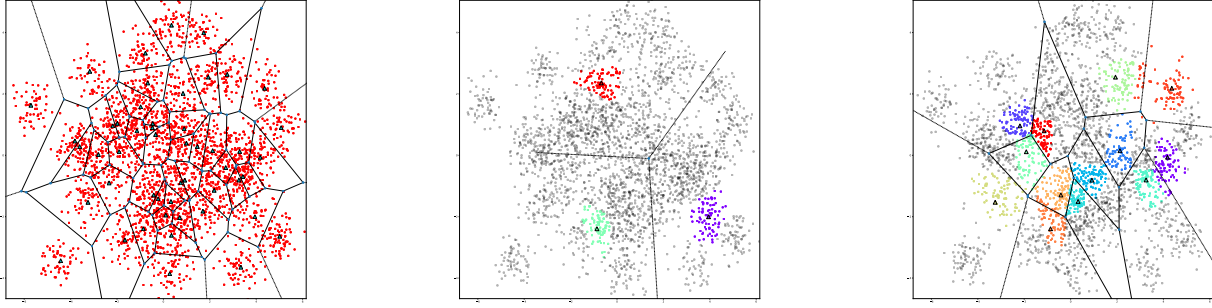
Figure 2. Recommendation using dynamics-informed constraints keeps more content providers viable in the long run, improving social welfare amongst users. The first panel shows providers (blue triangles) and users (red dots) embedded in a 2D topic space. The second and third panel respectively show equilibrium under Myopic and proposed linear program (LP) recommendation; here users whose most-preferred provider has dropped out of the ecosystem are shaded in grey, while the remaining users are colored according to their most-preferred provider.

able. Under the mild assumption that, for any u, there is a policy that keeps $MC(u)$ viable, $U_\infty^{\pi_u^*}(u) = \mathbb{E}[f(\mathbf{r}^*(u))]$ is a *constant* where $\mathbf{r}^*(u)$ is the realized reward sequence when any q_u generates reward $\arg\max_c r(q_u, c)$.

Our interest in long-run performance leads to a focus on policy behavior in equilibrium w.r.t. provider viability. We say $V \subseteq \mathcal{C}$ is an *equilibrium* of π if, for some $k \geq 0$, $V_{k+j}^\pi = V$ for all $j \geq 0$ (i.e., the providers V_k^π that are viable after epoch k remain viable in perpetuity). Since most large-scale RSs have massive numbers of users, we assume that the number of queries received at each point in X (or within a suitable set of small subregions of X) during each epoch is exactly its expectation. This can be justified by appeal to the law of large numbers for epochs of sufficient duration.[2]

Under these assumptions, we show that the recommendation policy that maximizes average social welfare has an especially simple form. Specifically, there is an optimal *stationary policy* (w.r.t. epochs) which can be formulated as an optimal matching problem under viability constraints. Consider the following *single-epoch decision problem*:

$$\max_\pi \sum_{u \in \mathcal{U}} \mathbb{E}[f(\mathbf{r}^\pi(u))|\pi] \qquad (1)$$

$$\text{s.t. } \pi_{q,c} > 0 \text{ only if } \sum_{u, q_u} \overline{Q}(q_u)\pi_{q_u,c} \geq \nu_c, \ \forall q, c \qquad (2)$$

Here π is a vector of matching variables $\pi_{q_u,c}$, denoting the proportion of queries q_u to match to c.[3] $\overline{Q}(q_u)$ is the

expected number of queries of the type q_u, but note that the method holds for any non-negative function of q_u. The expectation of u's utility f is taken w.r.t. user activation at each t in the epoch, the user query distribution and π; i.e., the t-th component of $\mathbf{r}^\pi(u)$ is distributed as

$$r_t^\pi \sim \rho(u) \sum_{q_u \in \mathcal{Q}} P_u(q_u) \sum_c \pi_{q_u,c} r(q_u, c). \qquad (3)$$

Objective (1) optimizes social welfare over the epoch, while constraint (2) ensures that any *matched* provider remains viable at the end of the epoch.

We show that applying this single-epoch policy π across all epochs is guaranteed to optimize average user social welfare. Since the expectation $\overline{Q}(q_u)$ is realized exactly, π induces an equilibrium during the first epoch. Moreover, the optimization ensures it has maximum welfare, i.e., is the optimal stationary policy. Finally, while a non-stationary π' may initially improve welfare relative to π, its equilibrium welfare cannot exceed that of π; so any welfare improvement is transient and cannot increase (long-run) average welfare. The stationary π given by (1) *anticipates the equilibrium it induces* and only matches to providers that are viable in that equilibrium. This obviates the need to consider more complex policies based on the underlying Markov decision process (we discuss richer policy classes below).

3. Solving the Matching Optimization

We now develop several practical methods for solving the optimization (1) to generate optimal policies. We consider formulations that accommodate various forms of user utility: simple linear, cumulative reward; a discounted model that reflects decreasing marginal returns; and non-linear models (of which sigmoidal utility is a motivating example). We

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[2]We discuss relaxations of this assumption below.

[3]This can also be interpreted as a stochastic policy, which is the natural practical implementation. When user utility is time-dependent, we sometimes allow the policy to be non-stationary *within* the epoch, writing $\pi_{q_u,c,t}$ for $t \leq T$.

then show how social welfare and regret can be traded off, and briefly describe how the models can be made more robust to uncertainty in the user query stream.

3.1. Additive Utility: A Linear Programming Model

We first develop a linear programming (LP) model that applies when user utility is suitably additive, capturing both cumulative reward and a *discounted engagement* model that reflects a natural form of decreasing marginal return.

Let $\alpha \in \mathbb{R}_+^T$ be a vector of non-negative weights. We assume linear user utility $f_\alpha : \mathbf{r} \mapsto \sum_{t=1}^T \alpha_t \mathbf{r}_t$, where user utility in an epoch is the α-weighted sum of immediate rewards (cumulative reward is a special case with all $\alpha_t = 1$). Moreover, we consider a class of non-stationary policies that take time itself as their only history feature, $\pi_{h_t, q_u, c} = \pi_{t, q_u, c}$ (if we allow π to depend on arbitrary statistics, the problem is a full POMDP).

The expected welfare of π over the epoch is

$$\sum_u \mathbb{E}[f_\alpha(\mathbf{r}^\pi(u))|\pi] = \sum_{t=1}^T \sum_{u \in \mathcal{U}} \sum_{q_u \in \mathcal{Q}} \sum_{c \in \mathcal{C}} \alpha_t \pi_{t, q_u, c} \bar{r}(q_u, c),$$

where $\bar{r}(q_u, c) = \rho(u) P_u(q_u) r(q_u, c)$. Since (per-epoch) social welfare SW^π is a linear function of π, Eq. (1) can be reformulated as a mixed-integer linear program (MILP):

$$
\begin{aligned}
\max_{\pi, y} \quad & \sum_{t=1}^T \sum_{u \in \mathcal{U}} \sum_{q_u \in \mathcal{Q}} \sum_{c \in \mathcal{C}} \alpha_t \pi_{t, q_u, c} \bar{r}(q_u, c) \\
\text{s.t.} \quad & \sum_c \pi_{q_u, c, t} = 1 \qquad \forall t \in [1:T], q_u \in \mathcal{Q}, \\
& \pi_{q_u, c, t} \leq y_c \qquad \forall t \in [1:T], q_u \in Q, c \in C \\
& \sum_{u, q_u, t} \overline{Q}(q_u) \pi_{q_u, c, t} \geq \nu_c y_c \quad \forall c \in \mathcal{C}
\end{aligned}
$$

(4)

where matching variables $\pi_{t, q_u, c}$ represent the stochastic policy, and provider-viability variables y_c are in $\{0, 1\}$ (for cumulative reward, dependence of π on t can be removed). Problem (4) is akin to *max-utility constrained facility location*, where user-query-time tuples act as customers and providers act as facilities. Related problems have been investigated in various forms (An et al., 2017; Li, 2019; Cornuejols et al., 1977). All variants (including ours) have basic facility location as a special case (where the constraints are trivial (i.e., $\nu_c = 0$) and are thus NP-hard. Even though their formulations are similar, each has different approximation properties. The combination of objective and constraints we consider has not, we believe, been studied in the literature.

Our problem can be approximated in polynomial time up to a constant factor:

Theorem 1. *Problem* (4) *can be approximated up to factor* $\frac{1}{e}$ *in polynomial time.*

The core of the proof (see Appendix A.1) is to consider the problem of computing *maximum constrained welfare*, $cSW(C)$, given a *fixed* set $C \subseteq \mathcal{C}$ of viable providers. $cSW(C)$ is the maximum of Eq. (4) when the provider variables are "set" as $y_c = 1$ iff $c \in C$; i.e., we find the best stochastic matching given that all and only providers in C remain viable. With the integer variables removed, the problem becomes a polynomially sized LP. We show that cSW is submodular in the provider set C, i.e., $cSW(C \cup \{c', c\}) - cSW(C \cup \{c'\}) \leq cSW(C \cup \{c\}) - cSW(C)$ for any $c, c' \in \mathcal{C}, c \subset \mathcal{C}$. This means that social welfare can be approximately optimized by greedily adding providers until no further viable providers can be added.

A more efficient alternative to greedy provider selection is to directly round the results of the LP relaxation of Eq. (4). This is known to yield good results for constrained facility location; we refer the interested reader to Jones (2015) for a discussion of possible rounding strategies and the implications on computational complexity. Beyond the time savings, we find in preliminary experiments that the LP-rounding heuristic performs indistinguishably from the greedy method in terms of social welfare. Given its superior computational performance, we only evaluate the LP rounding approach in our experiments (denoted LP-RS) in Sec. 4.

The weighted linear utility model provides us with a mechanism for modeling *decreasing marginal returns* in user utility, specifically, by discounting user rewards by setting $\alpha_t = \gamma^{t-1}$ for some $\gamma \in (0, 1]$. Such discounting is one simple way to model the realistic assumption of decreasing marginal utility with increased content consumption (e.g., a good match for a second user query has less impact on her utility than it does for the first query). This model makes it easier to maintain provider viability and improve social welfare with lower individual regret (see Sec. 4).

3.2. Non-linear Utility: A Column Generation Model

While additive utility provides us with a useful model class that can be solved approximately in polynomial time, it cannot express important structures such as sigmoidal utility (Kahneman & Tversky, 1979). Optimal matching with such non-linearities can be challenging.

Suppose now that f is nonlinear, e.g., $f(\mathbf{r}^\pi(u)) = \sigma(\mathbf{r}^\pi(u) + \beta)$. The facility location literature provides little guidance for such problems. While approximation results are known for concave problems (Hajiaghayi et al., 2003), it is unclear if these apply with constraints, a setting that, we believe, has not been investigated.

Our approach linearizes the problem to form a MILP that

can subsequently be relaxed to an LP as follows. Let $C \in \mathcal{C}^k$ be a k-tuple of providers. A pair

$(q_u, C) \in \mathcal{Q} \times \mathcal{C}^k$ represents a possible answer to user u's k queries identical to q_u by the provider tuple C. We call such a tuple a *star* $q_u C$. For each star, we use a variable $\pi_{q_u C}$ to represent the policy's match to q_u. The linearized objective is then:

$$\text{maximize}_{\pi, y} \sum_{u \in \mathcal{U}} \sum_{q_u \in \mathcal{Q}} \sum_{C \in \mathcal{C}^k} \pi_{q_u C} \bar{\sigma}(q_u, C) , \quad (5)$$

where $\bar{\sigma}(q_u, C) = \rho(u) P_u(q_u) \sigma(q_u, C)$. The linear constraints from Eq. (4) are adapted to these stars. See Appendix A.2.1 for the complete optimization formulation and our use of column generation to solve this much larger, more complex problem.

3.3. Incorporating Regret Trade-offs

As discussed in Sec. 2.2, pure social welfare maximization can sometimes induce large maximum regret (i.e., high regret for some users). Fortunately, max regret can be traded off against social welfare directly in the optimization. Let μ_u be a constant denoting u's maximum utility $U_\infty^{\pi_u^*}(u)$ (see Sec. 2.3). Since the term $U^\pi(u) = \sum_{t=1}^T \sum_{q_u \in \mathcal{Q}} \sum_{c \in \mathcal{C}} \pi_{q_u, c, t} \bar{r}(q_u, c, t)$ denotes u's expected utility in LP (4), we can express u's realized regret as a variable $Rgrt_u \mu_u - U^\pi(u)$ for all $u \in \mathcal{U}$. Letting variable MR represent the max regret induced by the policy, we can add the term $-\lambda MR$ to the objective in MILPs (4) and (5) where constant λ controls the desired welfare-regret trade-off. Constraining $MR \geq Rgrt_u, \forall u \in \mathcal{U}$ ensures MR takes on the actual max regret induced by π.

3.4. Extensions of the Model

Our matching optimization relies on some restrictive, unrealistic assumptions about real-world RSs. However, our stylized model directly informs approaches to more realistic models (and can sometimes be adapted directly).

Robustness and RL: Strong assumptions about query distribution variance motivated our equilibrium arguments. Even if these assumptions do not hold, our LP formulation can be extended to generate "high-probability" optimal equilibria. For example, lower-confidence bounds on the traffic expected for each provider under the induced policy can be constructed, and viability thresholds inflated to allow for a margin of safety. This can be encoded formally or heuristically within our LP model. A full RL approach offers more flexibility by dynamically adjusting the queries matched to a provider as a function of the state of all providers (e.g., exploiting query variance to opportunistically make additional providers viable, or "salvage" important providers

given, say, an unanticipated drop in traffic). The combinatorial nature of the state space (state of engagement of each provider) complicates any RL model. Adaptive online methods (e.g., as used in ad auctions (Mehta, 2013)) can exploit optimization techniques like ours to dynamically adjust the matching without the full complexity of RL.

Incomplete Information: Our model assumes complete knowledge of both user utility and "interests" (via the query distribution, the reward function and the utility function), and of a provider's position in "topic space" and its utility (via its viability). In practice, these quantities are learned from data and constantly updated, are generally never known with full precision, and are often very uncertain. Indeed, one reason "unfair" recommendations arise is when the RS does not undertake sufficient exploration to discover diversity in user interests (especially for niche users). Likewise an RS is usually somewhat (though perhaps less) uncertain of a provider's content distribution. Incorporating this uncertainty into our model can (a) generate more robust matchings, and (b) drive exploration strategies that uncover *relevant* user interests and utility given ecosystem viability constraints. These extensions are an important direction for future research.

Richer Dynamics: More complex dynamics exist in real RS ecosystems than are captured by our simple model, including: arrival/departure of providers/users; evolving user interests and provider topics/quality; richer provider responses (e.g., not just abandonment, but quality reduction, content throttling, topic shifts); and strategic behavior by providers. Our model serves only as a starting point for richer explorations of these phenomena.

4. Experiments

We evaluate the effectiveness of our LP-rounding method (Sec. 3.1), dubbed LP-RS, for additive utility models (we consider both cumulative and discounted reward). Preliminary experiments show that LP-RS runs faster and performs better than the greedy/submodular approach, so we do not present results for the latter here. We provide a detailed evaluation of column generation for nonlinear models (Sec. 3.2) in Appendix A.2.3. We compare the policy π_{LP} based on LP-RS to a myopic baseline policy π_{My} w.r.t. social welfare, max regret and provider diversity/viability, assessing both on several domains using a RS ecosystem simulator that captures provider viability dynamics. We outline the simulator, describe our datasets and trained embeddings, then discuss our findings.

4.1. Ecosystem

The ecosystem simulator captures the provider viability dynamics described in Sec. 2.1. At each epoch, the RS

observes a single query per user, and must serve a "slate" of s providers to each user. A provider c can only occur *once* in each slate (e.g., the myopic policy rank-orders the s best-fit providers). The user query at each epoch is sampled as $q_u[t] \sim P_{u[t]}(X)$; hence, the realized preference changes at each epoch, but the distribution parameters are fixed throughout the simulation. Providers are static and all viability thresholds have the same value ν.

4.2. Datasets

Synthetic data   To examine the emergent properties of the ecosystem in a controlled setting, we generate synthetic data in a two-dimensional topic space, like those in Fig. 2. The query distribution is a mixture of Gaussians, with one component for each provider centered at that provider's location in topic space X, and its weight reflecting the user's affinity for that provider's topic area. We consider two variants of the mixture model: (a) a *uniform* variant where providers/topics are distributed uniformly, and all users have the same variance; and (b) a *skewed* variant, where (w.l.o.g.) topics near the origin are considered *popular* and receive relatively more users, while topics far from the origin are *niche* with fewer users, but whose users are more *loyal* and exhibit lower variance. User-provider reward is given by $f(a, b) = -||a - b||_2$ (with max value 0).

Movie ratings   We train an embedding on the Movielens dataset (Harper & Konstan, 2015) using non-negative matrix factorization on a sparse matrix of user-movie engagements (see Appendix B for details). The pair of matrix factors are then used as embedding vectors for users and providers for the simulator (here each provider is a movie). User-provider rewards are computed as $r(q_u, c) = q_u^T c$.

Social network   We train user and provider embeddings from the SNAP 2010 Twitter follower dataset (Kwak et al., 2010). This dataset consists of a large list of (followee, follower) pairs, each with a unique account ID. We designate popular accounts as "providers," and other accounts as "users," then learn a low-dimensional embedding that captures affinity between users and providers via a link prediction task (see Appendix B for details). User-provider rewards are computed as $r(q_u, c) = q_u^T c$.

4.3. Results

Exploring Embedding Type   We begin by using synthetically generated embeddings to study how properties of the *embeddings* affect long-term social welfare under π_{My}. We evaluate π_{LP} and π_{My} using the uniform and skewed synthetic embeddings. The results (Table 1) show that when user/provider embeddings are *skewed*, myopic recommendation yields suboptimal user welfare due to less popular providers abandoning the platform. LP-RS improves wel-

Data Type	Method	Avg. Welfare	Viable Providers
Uniform	Myopic	-2.41 ± 0.59	43.80 ± 1.94
	LP-RS	**-1.23 ± 0.57**	**48.00 ± 2.10**
Skewed	Myopic	-5.68 ± 0.61	34.40 ± 1.96
	LP-RS	**-3.40 ± 1.10**	**42.00 ± 3.35**

Table 1. (Synthetic data) Myopic recommendation performs poorly when observing *skewed* user and provider embeddings (some topic space areas are more popular). LP-RS improves social welfare *and* number of viable providers for both data types. Note that zero is the maximum possible welfare in this setting.

fare and increases the number of viable providers in both cases. For the remainder of the paper we use only the skewed variant of the synthetic data.

Tradeoffs in Regret and Welfare   Next we investigate the trade-off between social welfare and individual regret induced by LP-RS at various levels of diminishing returns for user utility, introduced by discounting immediate user rewards as discussed in Sec 3.1. Recall that $Rgrt^\pi(u)$ is defined w.r.t. a policy π_u^* that is "tailor made" for user u, one that keeps all providers with closest affinity to u viable without regard to other users (and will generally serve most users poorly). Under this definition, every policy will generally have very high max regret. But this serves as a useful reference point for understanding how the preferences of any single user trade off with long-term social welfare under a realistic policy $\pi \in \{\pi_{\text{My}}, \pi_{\text{LP}}\}$. We discuss the results for π_{LP} in the following. Results for π_{My} can be found in Appendix E.

We expect that steeper rates of diminishing returns (lower γ) should lead to lower costs—that is, lower individual regret, or sacrifice of individual utility—to generate the optimal provider "subsidies" and generally induce more diverse provider sets in equilibrium, which in turn leads to lower max

γ	Avg. Welfare	Max Regret	Regret-Welfare Ratio	Surviving Providers
0.1	18.02 ± 1.05	7.24 ± 0.77	0.40 ± 0.04	47.20 ± 1.72
0.18	19.79 ± 1.16	7.97 ± 0.84	0.40 ± 0.04	47.20 ± 1.72
0.26	21.87 ± 1.28	8.84 ± 0.91	0.41 ± 0.04	46.60 ± 2.15
0.35	24.30 ± 1.43	10.14 ± 1.18	0.42 ± 0.05	45.20 ± 3.06
0.43	27.17 ± 1.58	11.65 ± 0.91	0.43 ± 0.03	44.00 ± 3.16
0.51	30.44 ± 1.79	14.19 ± 1.46	0.47 ± 0.06	44.80 ± 0.98
0.59	34.30 ± 1.95	16.50 ± 1.73	0.48 ± 0.06	43.00 ± 1.90
0.67	38.67 ± 2.29	20.53 ± 1.46	0.53 ± 0.06	42.60 ± 1.50
0.75	43.69 ± 2.61	24.61 ± 1.31	0.57 ± 0.05	41.80 ± 1.72
0.84	49.51 ± 2.90	28.95 ± 1.11	0.59 ± 0.04	39.80 ± 3.06
0.92	56.15 ± 3.26	33.05 ± 1.16	0.59 ± 0.04	36.80 ± 3.97
1.0	63.62 ± 3.58	37.89 ± 1.39	0.60 ± 0.04	34.20 ± 5.08

Table 2. The trade-off between average user welfare and max user regret depends on the discounting factor γ. The RS makes multiple recommendations to each user during each epoch, and a lower γ indicates more steeply discounted returns for recommendations beyond the first one.

(a) Synthetic data embeddings.　　　　(b) Movielens data embeddings.　　　　(c) SNAP data embeddings.
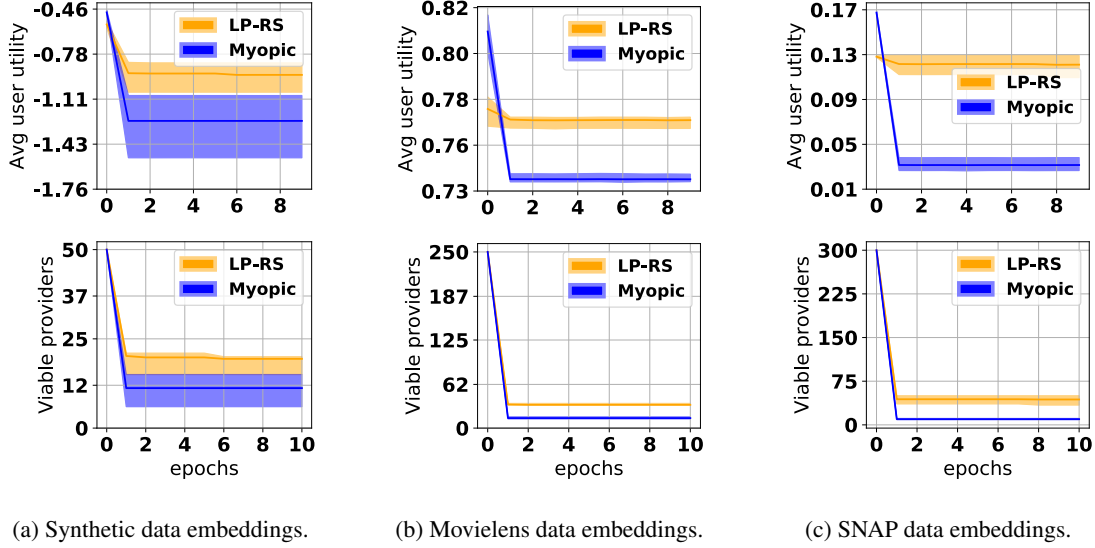
Figure 3. Average social welfare and number of viable providers per epoch of simulation. Bold lines are means over 5 seeds, while shaded regions show 25-th to 75-th percentiles across runs.

regret. This makes our social welfare objective more aligned with both provider and user fairness. Table 2 corroborates this intuition, showing that better max-regret-to-average-welfare ratios are generally achieved at low levels of γ, as are a greater number of viable providers.

Large-scale Simulations　We carry out large-scale simulations using the dataset embeddings described in Section 4.2 (see Appendix C for details). Fig. 3 shows results from the simulations, tracking both the number of viable providers and average user utility (i.e., social welfare divided by number of users). We find that π_{LP} quickly converges to an equilibrium that sustains more providers than π_{My}. Average user utility is also improved under π_{LP}. Fig. 4 shows an example user utility histogram (aggregated over the entire simulation). LP-RS has an overall positive impact on the
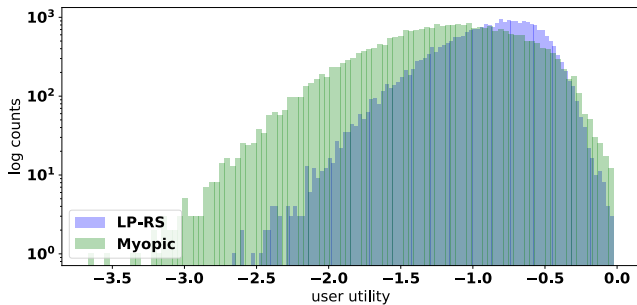

Figure 4. User utility histogram (Synthetic embeddings). The LP recommender improves average social welfare at the expense of some maximum user regret.

distribution of user utility, relative to the myopic baseline. However the increase in social welfare comes at the cost of decreased utility for some of the most well-off users. See Appendix D for utility histograms for all datasets.

5. Related Work

"Fairness" in outcomes for users and providers could be described in a variety of (possibly conflicting) ways, and any computational measurement of fairness in this context will surely reflect normative principles underpinning the RS design (Binns, 2018; Leben, 2020). We have presented a utilitarian view whereby fair outcomes are realized when the average welfare of users is maximized in equilibrium, without taking other factors such as user/provider demographic group membership or diversity of content into account explicitly. This is consistent with some recent approaches that model long-term fairness by constraining the exploration strategy of the decision maker (Joseph et al., 2016; Jabbari et al., 2017) in the sense that fairness and optimality (here w.r.t. expected user rewards) are aligned. However this approach may not always be appropriate, so we note that the general matching strategy we employ captures the *dynamics* of the RS and could in principle be adapted to encourage other types of "fair" outcomes in accordance with different normative principles.

Studies of fairness in ranking typically consider notions of *group fairness*, for example, by regularizing standard ranking metrics to encourage parity of the ranks across demographic groups (Yang & Stoyanovich, 2016; Zehlike et al., 2017). Celis et al. (2017) propose a matching algo-

rithm for ranking a set of items efficiently under a fairness constraint that encourages demographic diversity of items in the top-ranked position.

Research that considers fairness w.r.t. *content providers* (rather than users), fairly allocating the exposure of providers to users (Singh & Joachims, 2018; Biega et al., 2018), is closely related to our work, especially in some aspects of its motivation. These models impose fairness constraints that address "provider" concerns, and maximize user utility subject to these constraints. In this view, the policy makes commitments to every provider in the ecosystem, while user welfare is secondary to satisfying the fairness constraints. This requires that fairness constraints be crafted very carefully (which is a non-trivial problem (Asudeh et al., 2019)) so as to not have an undue impact on user welfare. In our utilitarian framework, provider fairness is justified by user welfare, implicitly incorporating fairness constraints.

Also very related to our work is the work of Ben-Porat & Tennenholtz (2018), who develop a game-theoretic model of RSs whose providers act strategically—by making available or withholding content—to maximize the user engagement derived from an RS platform. They analyze the policies of the RS: using an axiomatic approach, they prove no RS policy can satisfy certain properties jointly (including a form of fairness); and in a cooperative game model, they show the uniqueness and tractability of a simple allocation policy. While related, their models are not dynamic and do not (directly) assess complex user utility models; but they examine more complex, *strategic* behavior of providers in a way we do not. Ben-Porat et al. (2019) draw a connection between (strategic) facility location games and RSs with strategic providers. Our models relate to non-strategic facility location, with an emphasis on scalable optimization methods.

Individual fairness provides an important alternative perspective, requiring that two individuals similar w.r.t. a task should be classified (or otherwise treated) similarly by the ML system (Dwork et al., 2012). Our utilitarian approach guarantees that providers with sufficient impact on user welfare remain viable, regardless of whether their audience lies at the tail or head of content space. As shown in Sec. 4, this provides a dramatic improvement over myopic policies which tend to serve "head" users and providers disproportionately well. While maximizing welfare does not guarantee high individual utility, we generally expect high individual utility to emerge across the user population if utility functions exhibit diminishing returns. If the form of user utility precludes this, the objective can be augmented with a maximum individual regret term as discussed in Sec 3.3.

Also relevant is recent research extending algorithmic fairness to dynamical systems. Several methods for improved fairness in sequential decision-making have been proposed, including work on bandits (Joseph et al., 2016), RL (Jabbari et al., 2017), and importance sampling (Doroudi et al., 2017). Fairness in dynamical systems has been explored in specific domains: predictive policing (Lum & Isaac, 2016; Ensign et al., 2018), hiring (Hu & Chen, 2018; Hu et al., 2019), lending (Mouzannar et al., 2019), and RSs (Chaney et al., 2018; Bountouridis et al., 2019). In general, this work has focused on the role of algorithms in shaping environments over time (Hashimoto et al., 2018; Kannan et al., 2019), observing that the repeated application of algorithms in a changing environment impacts fairness in the long-term differently from short-term effects.

6. Conclusion

We have developed a stylized model of the ecosystem dynamics of a content recommender system. We have used it to study the effects of typical myopic RS policies on content providers whose viability depends on attaining a certain level of user engagement. We showed that myopic policies can serve users poorly by driving the system to an equilibrium in which many providers fail to remain viable, inducing poor long-term (user) social welfare. By formulating the recommendation problem holistically as an optimal constrained matching, these deficiencies can be overcome: we optimize long-term social welfare, while at the same time increasing *provider* viability despite the fact that our objective is to increase *user* welfare. We developed several algorithmic approaches to the matching problem and experiments with our LP-based approach showed significant improvements in user welfare over myopic policies. While our model is stylized, we believe it offers insights into more general, realistic RS model as outlined in Sec. 3.4. It provides a rich framework for studying tradeoffs between individual utility and (utilitarian) social welfare. Extensions to account for group fairness, strategic behavior and exploration policies are critical areas of future research as are new algorithmic techniques (e.g., reinforcement learning, online matching) as discussed in Sec.3.4.

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A. Algorithms and Proofs

A.1. Greedy Optimization and Theorem 2

A.1.1. PRELIMINARIES

Before we begin the proof, we make a few notational modifications to significantly simplify it. Let $\mathcal{C}$ be a set of content providers (hereinafter providers), $\mathcal{U}$ a set of users, and $A \in \mathbb{R}^{|\mathcal{U}| \times |\mathcal{C}|}$ a utility matrix. Furthermore, let $D : \mathcal{U} \to \mathbb{N}$ be a user demand function (specifying how many queries a user u submits to the system) and ν_c for $c \in \mathcal{C}$ be the provider survival threshold, indicating how many queries the provider needs to receive in order to be viable. For this section, we will make the following simplifying assumptions:

- every user has exactly one unique query during the epoch, and

- every user's view contributes exactly one unit towards a provider's viability.

Under these assumptions, the set of queries becomes identical to the set of users ($\mathcal{U} = \mathcal{Q}$), and $\bar{Q}(q_u) = 1$. We proceed to prove the submodularity of user welfare as a function of the provider set subject to the above restrictions. After that, we discuss the reduction of Problem 4 to this restricted case.

The fairness dynamics problem is then to find a matching such that

$$X^* = \arg\max_{X,Y} \sum_{u \in \mathcal{U}} \left(\sum_{t=1}^{D(u)} \sum_{c \in \mathcal{C}} A_{uc} X_{uct} \right)$$

$$\text{subject to } \sum_{c \in \mathcal{C}} X_{uct} = 1 \quad \forall u \in \mathcal{U}, t \in \{1, \dots, D(u)\}$$

$$X_{uct} \leq Y_c \quad \forall u \in \mathcal{U}, c \in \mathcal{C}$$

$$\sum_{u \in \mathcal{U}} \sum_{t=1}^{D(u)} X_{uct} \geq \nu_c Y_c, \quad \forall c \in \mathcal{C}$$

$$X_{uct}, Y_c \in \{0, 1\}, \quad \forall u \in \mathcal{U}, c \in \mathcal{C}, t \in [1, \dots, D(u)], \tag{6}$$

The problem (6) is a hard combinatorial problem, so the question is if we can derive good heuristics for solving it. Of particular interest is the following greedy heuristic: let $C \subseteq \mathcal{C}$ and define $g : 2^{\mathcal{C}} \to \mathbb{R}$ as

$$g(C) \mapsto \max_{X} \sum_{u \in \mathcal{U}} \left(\sum_{t=1}^{D(u)} \sum_{c \in C} A_{uc} X_{uct} \right)$$

$$\text{subject to } \sum_{c \in C} X_{uct} = 1 \quad \forall u \in \mathcal{U}, t \in [1, \dots, D(u)]$$

$$\sum_{u \in \mathcal{U}} \sum_{t=1}^{D(u)} X_{uct} \geq \nu_c \quad \forall c \in C$$

$$X_{uct} \in \{0, 1\}, \quad \forall u \in \mathcal{U}, c \in \mathcal{C}, t \in [1, \dots, D(u)]. \tag{7}$$

That is, $g(C)$ is the best matching if the provider set $C \subset \mathcal{C}$ is fixed externally. Despite that (7) has binary constraints on X_{uct}, its constraint matrix is Totally Unimodular; hence, we are guaranteed that (7) is integral. The goal is then to start with $C = \emptyset$ and greedily add providers while $g(C)$ keeps improving. In order for this to work well, g would need to be sub-modular, which is precisely what we prove next.

Theorem 2. *For every two providers $c_0, c_1 \in \mathcal{C}$ and $C \subseteq \mathcal{C} \setminus \{c_0, c_1\}$, it holds that*

$$g(C \cup \{c_0, c_1\}) - g(C \cup \{c_1\}) \leq g(C \cup \{c_0\}) - g(C). \tag{8}$$

A.1.2. PROOF OF THEOREM 2 IN THE UNIT CASE

Let us make a simplification: a user with $D(u)$ queries is equivalent to $D(u)$ independent users; thus, we will just work with an extended user set. We now present the terminology used in this proof. A matching $X : \mathcal{U} \to \mathcal{C}$, is a function from users to providers. We denote by $\mathcal{C}(X)$ the serving providers under X, i.e., $\mathcal{C}(X) = \{c \mid \exists u \in \mathcal{U}, X(u) = c\}$. Further, we say that a matching X is *feasible* if every provider in $\mathcal{C}(X)$ meets her threshold under X, namely, if for every $c \in \mathcal{C}(X)$ it holds that $|\{u \in \mathcal{U} \mid X(u) = c\}| \geq \nu_c$. We denote by $F(X)$ the value obtained for a feasible matching X in Problem (7) (note that X may not be optimal w.r.t.. $\mathcal{C}(X)$). In the rest of the proof, we rely on optimal matchings for $C, C \cup \{c_1\}$ and $C \cup \{c_0, c_1\}$ to construct a new matching, X^0. The active providers under X^0 are $C \cup \{c_0\}$ and, as we shall show, X^0 satisfies

$$g(C \cup \{c_0, c_1\}) - g(C \cup \{c_1\}) \leq F(X^0) - g(C). \tag{9}$$

The latter immediately implies Inequality (8), since by definition of g,

$$F(C \cup \{c_0, c_1\}) \leq \max_{X : \mathcal{C}(X) = C \cup \{c_0\}} F(X) = g(C \cup \{c_0\}).$$

We are now ready to develop the tools required for the proof. The next notion assists to succinctly quantify the difference in user utility between two matchings.

Definition 1. *Let X be and Y be two feasible matchings. We call a triplet (c, c', u) a relocation triplet w.r.t. X, Y if $X(u) = c$, $Y(u) = c'$ and $c \neq c'$.*

Importantly, two matchings define a unique set of (ordered) relocation triplets. conversely, a source matching and relocation triplets uniquely define the target matching.

Let X and X^1 denote (any) optimal matching induced by $g(C)$ and $g(C \cup \{c_1\})$ in Problem (7), respectively. We now construct a graph whose nodes are the providers and its edges correspond to relocation triplets w.r.t. X, X^1. Formally, let $G^1 = (\mathcal{C}, E^1, w)$ denote a directed multi-graph, where the set of nodes is $\mathcal{C}$; E^1 is the set of all relocation triplets w.r.t. X, X^1, where every triplet (c, c', u) forms a directed edge from $c = X(u)$ to $c' = X^1(u)$ with an ID of u; and the weight function w is defined by $w(c, c', u) = A_{uc'} - A_{uc}$. Observe that the number of users each provider c (a node in the graph) obtains under X^1 equals

$$|\{u \in \mathcal{U} \mid X(u) = c\}| + \deg^+(c) - \deg^-(c), \tag{10}$$

where $\deg^+(c)$ denotes the indegree of c and its outdegree is denoted by $\deg^-(c)$. Moreover, the sum of weights is precisely the difference in utility between X and X^1, i.e.,

$$F(X^1) - F(X) = g(C \cup \{c_1\}) - g(C) = \sum_{e \in E^1} w(e).$$

In the next proposition, we use the fact that X, X^1 are optimal w.r.t. their provider sets to characterize properties of G^1.

Proposition 1. *It holds that:*

(1) G^1 does not contain directed cycles.

(2) The only sink in G^1 is c_1.

The proof of Proposition 1 appears below. Proposition 1 suggests that G^1 is a DAG with flow conservation, so we can decompose its edges into a set of independent paths (for any arbitrary partition into paths) between a source, i.e. a provider with an excess of users under the matching X, and the sink c_1.

Next, we introduce a second graph, $G^{0,1}$, with relocation triplets from X^1 to $X^{0,1}$, the optimal matching for $g(C \cup \{c_0, c_1\})$. Formally, $G^{0,1} = (\mathcal{C}, E^{0,1}, w)$ is a directed multi-graph, with the same set of nodes and the same weight function w. $E^{0,1}$ is composed of all relocation triplets from X^1 to $X^{0,1}$. By mirroring the proof of Proposition 1, we conclude that $G^{0,1}$ contains no cycles and that c_0 is the unique sink of every directed path in it. This graph is of special interest because its sum of weights is the left hand side of Inequality (8). Namely, $\sum_{e \in E^{0,1}} w(e) = F(X^{0,1}) - F(X^1) = g(C \cup \{c_0, c_1\}) - g(C \cup \{c_1\})$. It also describes how to optimally relocate users from $C \cup \{c_1\}$ to $C \cup \{c_0, c_1\}$.

After understanding the structural properties of G^1 and $G^{0,1}$, we are ready to construct the promised matching X^0 (recall Inequality (9)). Let $G = (\mathcal{C}, E^1 \cup E^{0,1}, w)$ be the graph on the same set of nodes $\mathcal{C}$, with all the edges from both E^1 and $E^{0,1}$ (notice that the same edge cannot appear in both). For simplicity, we refer to paths in E^1 as *blue* and to paths in $E^{0,1}$ as *red*, for some arbitrary partition into paths. Our goal is to select a subset E of edges from $E^1 \cup E^{0,1}$, which, when applied to X, will induce the matching X^0. To that end, we devise an iterative process to construct the set E, by adding one path at the time. The key property of this process, which we formalize via Algorithm 1, is that there exists a mapping from every red path to a new path, composed of red and (potentially) blue edges, with a less or equal weight than that red path.

To illustrate why this process is necessary, observe that not every subset of $E^1 \cup E^{0,1}$ can be applied to X in order to obtain a new valid matching. In particular, recall that $E^{0,1}$ is the difference between X^1 and $X^{0,1}$; thus, a red path may involve the relocation (c, c', u), where u might have been matched to c' due some blue relocation (c'', c, u). To ensure that the subset we pick will result in a valid matching, we make the following distinction: a subset E such that $E \subseteq E^1 \cup E^{0,1}$ is called *consistent* if for any relocation triplet $(c, c', u) \in E$ either $X(u) = c$ or there exists another relocation triplet $(c'', c, u) \in E$. Informally, E is consistent if every user u that was relocated to c' from c was either matched to c in X, or was relocated to c from another provider. Consistency of the relocation triplets is a necessary, but not a sufficient condition for the resulting matching to be feasible.

Another useful notion is that of a *junction node*. We say that a node $c \in \mathcal{C}$ is a junction w.r.t. $E^1, E^{0,1}$ if there exists a blue edge $(c'', c, u) \in E^1$ and a red edge $(c, c', u) \in E^{0,1}$ for some $c', c'' \in \mathcal{C}$ and $u \in \mathcal{U}$. See Fig. 5 for illustration.
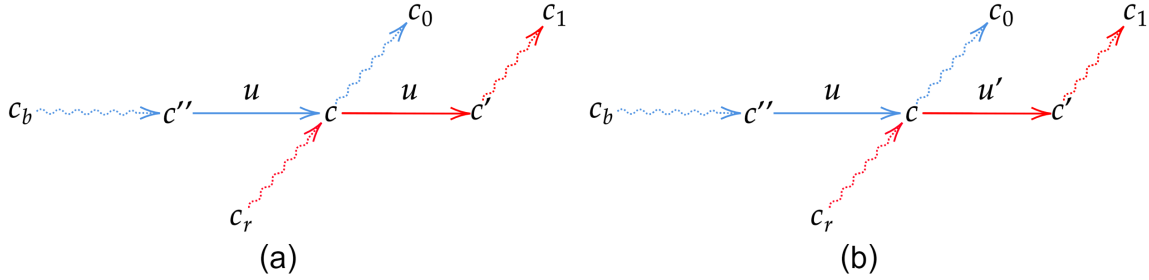

Figure 5. Exemplifying the definition of a junction node. In (a), the node c is a junction between the red path that starts at c_r (and ends at c_0) and the blue path that starts at c_b (and ends at c_1). The reason is that it receives the user u from c'' along a blue edge, and passes u along a red edge. In (b), however, c is not a junction, since the user it passes onward along the red path is u', which is not the user c receives along the blue path.

Next, we employ Algorithm 1 on the blue and red paths in $E^1 \cup E^{0,1}$. We show that

Lemma 1. *The output E of Algorithm 1 satisfies the following:*

1. *E is consistent.*

2. *When applied to X, the resulting matching X^0 is feasible.*

3. *$\sum_{e \in E} w(e) \geq \sum_{e \in E^{0,1}} w(e)$.*

The proof of Lemma 1 appears below. As elaborated above, the matching X and the relocation triplets in E uniquely define the matching X^0. By the second part of Lemma 1, X^0 is feasible. Moreover, by the third part of Lemma 1 and the definition of relocation triplets, we have

$$g(C \cup \{c_0, c_1\}) - g(C \cup \{c_1\}) = \sum_{e \in E^{0,1}} w(e) \leq \sum_{e \in E} w(e) = F(X^0) - g(X).$$

This completes the proof of the theorem.

A.1.3. PROOFS OF PROPOSITION 1 AND LEMMA 1

Proof of Proposition 1. For (1), assume by contradiction that a simple cycle $e_1, e_2, \ldots e_k$ exists for some $k \in \mathbb{N}$. Since $c_1 \notin \mathcal{C}(X)$, there is no relocation triplet with c_1 in the first entry, and hence c_1 does not participate in the cycle. We proceed

---

Algorithm 1 Flow Construction for X^0

1: let B be the set of blue paths and R be the set of red paths.
2: let $E \leftarrow \emptyset$ be the set of new paths.
3: **while** $R \neq \emptyset$ **do**
4:     **if** there is a junction node w.r.t. B, R **then**
5:         let c be a junction node, and b, r denote the paths whose edges $(c'', c, u) \in b$, $(c, c', u) \in r$ form the junction such that c is the closest junction node to the sink of b.
6:         add to E the edge (c'', c, u) and all the directed edges that precede it in b, and (c, c', u) and all subsequent directed edges in r.
7:         remove r from R, b from B.
8:         **continue**
9:     **else**
10:        add all the edges in R to E, set $R \leftarrow \emptyset$.
11:     **end if**
12: **end while**
13: **return** E

---

by analyzing the weight of the cycle, $\sum_{i=1}^{k} w(e_i)$. [4]

- If $\sum_{i=1}^{k} w(e^i) = 0$, we can remove the cycle from the graph and obtain a new graph $\tilde{G}^1$ and a corresponding matching $\tilde{X}^1$. Observe that the number of users every provider gets is the same as in X (see Equation (10)), and hence not only $\mathcal{C}(\tilde{X}^1) = \mathcal{C}(X^1) = C \cup \{c_1\}$ but also every provider in that set meets her threshold. Further, we did not change the sum of weights, and $F(X^1) - F(X) = F(\tilde{X}^1) - F(X)$ implies $F(X^1) = F(\tilde{X}^1)$; hence, $\tilde{X}^1$ is also optimal and we can assume w.l.o.g. that X^1 does not contain such cycles.

- If $\sum_{i=1}^{k} w(e^i) > 0$, we denote by $\tilde{X}$ a matching such that

$$\tilde{X}(u) = \begin{cases} c' & \text{if the edge } (c, c', u) \text{ belongs to the cycle} \\ X(u) & \text{otherwise} \end{cases}.$$

Since the number of users each provider in $C = \mathcal{C}(X) = \mathcal{C}(\tilde{X})$ gets under $\tilde{X}$ is the same as under X, $\tilde{X}$ is feasible. Moreover, $F(\tilde{X}) - F(X) > 0$; hence, we obtain a contradiction to the optimality of X.

- If $\sum_{i=1}^{k} w(e^i) < 0$, we can use an argument similar to the previous case to claim sub-optimality of X^1.

For (2), assume by contradiction that a node $v \in \mathcal{C}, v \neq c_1$ is a sink, and observe that we must have $v \in C$ since $\mathcal{C}(X^1) = C \cup \{c_1\}$. Let $v_1, \ldots, v_k, v$ denote the shortest path ending at v. Because $c_1 \notin \mathcal{C}(X)$, we know that c_1 cannot participate in this path. Further, X is feasible and hence v gets at least ν_v users under X^1. The analysis identically to the first part of the proposition, arguing that the contradiction assumption entails the existence of a path with positive/negative weights, in contrast to the optimality of X and X^1. $\qquad\square$

Proof of Lemma 1. Assume by contradiction that the output E is not consistent. By definition of consistency, there exists an edge $e = (c, c', u)$ such that

1. $X(u) \neq c$, and

2. $(c'', c', u) \notin E$ for every $c \in \mathcal{C}$.

Notice that $e \in E \subseteq E^1 \cup E^{0,1}$; hence, e is either blue or red. If e is blue, let $b(e)$ denote the path e is part of. Since $e \in E$ and is blue, the only way it could have been added to E is via Line 1. This means that either e is the first edge in $b(e)$, in

---

[4]In general, a set of relocation triplets can contain cycles with positive/negative weights, if providers pass different users along the cycle. However, as we prove, this cannot happen in G^1 due to the optimality of X^1.

which case $X(u) = c$ since E^1 is consistent; or e is an intermediate edge in $b(e)$, in which case there exists another edge $e' = (c'', c', u) \in b(e)$ that precedes it, again because E^1 is consistent. In both cases, we obtain contradiction.

Otherwise e is red. Let r denote the path that contains e. We have two cases:

- If c is a junction w.r.t. the initial B, R. In this case, there exists a blue path $b \in B$ that contains an edge (c'', c, u), by the definition of a junction node. Moreover, at some point in the execution e was added, so b must have been identified as a path containing an edge that forms a junction node v (not necessarily c) in Line 1. Recall that in Line 1 we assume that v is the closest junction node to the sink of b, which is c_1; hence, all edges of b that precedes the outgoing edge from v are added to E too, including (c'', c, u). This implies a contradiction.

- Else, c is not a junction. If e is the first edge in the red path r, then $X^1(u) = c$, and since c is not a junction, $X(u) = c$ as well. This holds because both X, X^1 are feasible. Otherwise, if e is an intermediate edge in r, then there must exists a red edge (c'', c, u) for some $c'' \in C$, because $E^1 \cup E^{0,1}$ is consistent. Since red edges like e are inserted to E in Lines 1 and 1, the preceding edges in their red path, including (c'', c, u), are added as well. In both cases, we reach a contradiction.

Second part Denote the matching obtained by applying the relocation triplets of E to X by X^0. To show that X^0 is feasible, we need to show that for every $c \in C \cup \{c_0\}$, it holds that $\left|\{u : X^0(u) = c\}\right| \geq \nu_c$. To do so, we rely on the feasibility of X^1 and $X^{1,0}$, whose relocation edges were used to construct X^0. We divide the analysis into three parts:

- If $c = c_0$. Since $X^{1,0}$ is feasible, we know that the $\deg^+(c_0)$ in $G^{0,1}$ is at least ν_{c_0} (Recall the quantification of the number of matched users in Equation (10)). Since E contains the final edge of every red path, the indegree of c_0 in $G^0 = (C, E, w)$ is the same as in $G^{0,1}$.

- Else, if c is the source of at least one path in E. In this case, it must have been the source of some paths in E^1 (blue) and $E^0, 1$ (red). Recall that the sink of every blue path is c_1, and the sink of every red path is c_0. Moreover, if c participates in other red/blue paths, it must be an intermediate node; thus, we can analyze its loss of users due to the paths in which c is the source solely. Since $E \subseteq E^1 \cup E^{0,1}$, c is the source of less paths in $G^0 = (C, E, w)$ than in $G^{0,1}$; therefore, its indegree in G^0 is greater or equal to its indegree in $G^{0,1}$, which implies that X^0 matched c with at least as many users as $X^{0,1}$.

- Finally, for any other c, X^0 matches c with the same number of users as $X^{0,1}$, since its difference between the indegree and the outdegree in $G^0 = (C, E, w)$ remains as in $G^{0,1}$.

Third part The proof of this part is based on the following observation:

Observation 1. *Let b be a blue path with source c_b and sink c_1, r be a red path with source c_r and sink c_0, and let c be a junction w.r.t. b and r, with edges $(c'', c, u) \in b$ and $(c, c', u) \in r$. Denote by p the path that starts from c_b, takes the edge (c'', c, u) and the edges that precedes in b, and then takes (c, c', u) and its subsequent edges in r, ending at c_0. Then, $\sum_{e \in p} w(e) \leq \sum_{e \in E^{0,1}} w(e)$.*

To see why Observation 1 holds, recall that the prefix of p from its source to (c'', c, u) inclusive, all blue edges, must have higher weight than the prefix of r from its source to (c, c', u), exclusive. This is true since otherwise we could find a heavier blue path to replace b in E^0. However, this cannot be true as X^1, which accounts for the blue edges in G^1, is an optimal matching for $g(C \cup \{c_1\})$. Finally, Algorithm 1 adds red paths either in their entirety (Line 1) or by modifying them to be heavier according to Observation 1; hence, $\sum_{e \in E} w(e) \geq \sum_{e \in E^{0,1}} w(e)$.

To complete the picture, it remains to argue that problems in which user queries contribute non-unit amounts to provider viability can be reduced to the unit case analyzed in the previous sections. That is, constraints of the type $\sum_{u \in \mathcal{U}} w_{uc} X_{uc} \geq \nu_c Y_c$, where the weights w are non-negative rational numbers, can be converted into constraints of the type $\sum_{u \in \mathcal{U}} X_{uc} \geq \nu_c Y_c$. Assuming that the weights are rational numbers, the linear program can be replaced with an equivalent one with integer coefficients, that is, w_{uc} are integer. Each of these integer counts then can be replaced by a fictional user that contributes a unitary count towards the provider's viability. □

A.2. Non-linear Optimization via Column Generation

A.2.1. FORMULATION

As discussed in Sec. 3.2, it is desirable to have a procedure that can optimize social welfare under non-linear utility models. To this end, we extend the mixed-integer linear program in Problem (4) to handle non-linear utilities. Let $C \in \mathcal{C}^k$ be a k-tuple of providers. A pair $(q_u, C) \in \mathcal{Q} \times \mathcal{C}^k$ represents a possible answer to user u's k queries identical to q_u by the provider tuple C. We call such a tuple a *star* $q_u C$. For each star, we use a variable $\pi_{q_u C}$ to represent the policy's match to q_u.

$$
\max_{\pi, y} \quad \sum_{u \in \mathcal{U}} \sum_{q_u \in \mathcal{Q}} \sum_{C \in \mathcal{C}^k} \pi_{q_u, C} \bar{\sigma}(q_u, C)
$$

$$
\text{s.t.} \qquad \sum_{C \in \mathcal{C}^k} \pi_{q_u, C} \leq 1 \qquad u \in \mathcal{U},
$$

$$
\sum_{\{C \in \mathcal{C}^k | c \in C\}} \pi_{q_u, C} \leq y_c \qquad u \in \mathcal{U}, c \in \mathcal{C},
$$

$$
\sum_{u \in \mathcal{U}} \sum_{C \in \mathcal{C}^k} \#[q_u C, c] \overline{Q}(q_u) \pi_{q_u, C} \geq \nu_c y_c, \quad c \in \mathcal{C},
$$

(11)

where $\#[uC, c]$ is the number of times provider c appears in star $q_u C$, and $\bar{\sigma}(q_u, C) = \rho(u) P_u(q_u) \sigma(q_u, C)$. We rely on the linear relaxation of the integrality constraints to approximate the solution of (11) efficiently. It is not obvious if and how this problem can be approximated via discrete algorithmic techniques, so we resort to relaxing the integrality constraints and solving the problem as a linear program. Even under the linear relaxation, the problem size still grows proportionally to $\mathcal{C}^k$ due to the number of variables introduced by linearization. The redeeming property of this problem, however, is that the number of constraints grows proportionally to $\mathcal{U} \times \mathcal{Q} \times \mathcal{C}$ and not $\mathcal{C}^k$. Hence, it is feasible to approach the problem from a column generation perspective.

A.2.2. COLUMN GENERATION

A standard column generation approach for solving a large linear program is a two-step iterative algorithm in which the LP is initially constructed using a small subset of its variables to obtained a reduced-size (master) problem. The dual of the master problem is solved to obtain a dual optimal solution, which is then used to find a (as of yet not generated) variable with maximal reduced cost. That variable is added to the master problem. The method iterates until no variable with positive reduced cost can be found, or some convergence tolerance is reached.

When the set of primal variables is large, the problem of finding a variable with maximal reduced cost (also called a column generation oracle) is still a hard combinatorial optimization problem (typically some flavor of knapsack). However, these problems tend to be massively decomposable so the running time does not scale exponentially in practice.

We now proceed to derive a column generation oracle for Problem (11). Let $\mathcal{A} = (A, b, c)$ denote an LP in inequality form, denoting the optimization problem $x^* = \arg\max_{x : Ax \leq b, x \geq 0} c^T x$. Let y^* be an optimal dual solution to $\mathcal{A}$. The reduced cost problem is thus $\hat{c} = c - A^T y^*$. The column generation oracle thus solves the problem $i^* = \arg\max_i \hat{c}$, which corresponds to the index of the primal variable with highest reduced cost. We now discuss solving the column generation problem given the specific form of (11).

We adopt the following convention for naming the dual variables corresponding to constraints in (11):

$$
\beta_u : \sum_{C \in \mathcal{C}^k} \pi_{q_u, C} \leq 1 \qquad\qquad\qquad u \in \mathcal{U},
$$

$$
\gamma_{uc} : \sum_{\{C \in \mathcal{C}^k | c \in C\}} \pi_{q_u, C} \leq y_c \qquad\qquad\qquad u \in \mathcal{U}, c \in \mathcal{C},
$$

$$
\alpha_c : \sum_{u \in \mathcal{U}} \sum_{C \in \mathcal{C}^k} \#[q_u C, c] \overline{Q}(q_u) \pi_{q_u, C} \geq \nu_c y_c \qquad\qquad c \in \mathcal{C}.
$$

The column generation problem (derived by computing the dual and maximizing the reduced cost) then becomes:

(a) Synthetic embeddings.
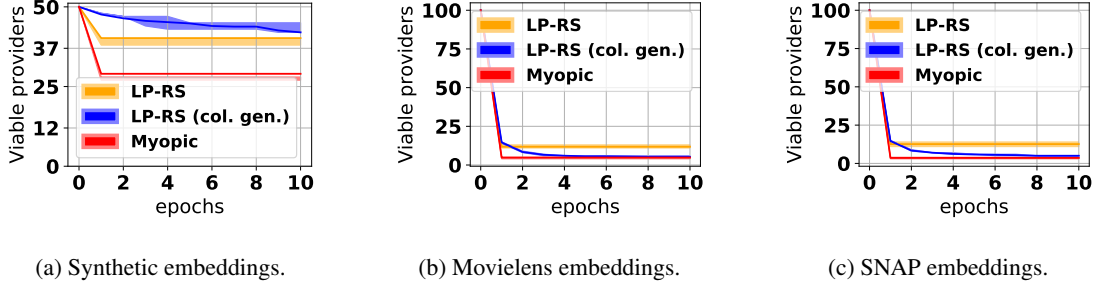
(b) Movielens embeddings.

(c) SNAP embeddings.

Figure 6. Simulations that evaluate the column generation matching strategy (LP-RS col. gen.) on smaller problems. While LP-RS col. gen. is capable of finding a good matching at a given step of simulation, its solutions are relatively inconsistent compared with LP-RS in the limit of several time steps.

$$uC^* = \arg \max_{u \in \mathcal{U}, C \in \mathcal{C}^k} \quad \bar{\sigma}(q_u, C) - \left(\beta_u + \sum_{c \in C} \gamma_{uc} - \sum_{c \in C} \#[q_u C, c] \overline{Q}(q_u) \alpha_c \right).$$

Let us now discuss how the above maximization can be solved. First, observe that the problem decomposes in the user variable u. That is for each $u \in \mathcal{U}$, we can independently solve the maximization over C. This can be done in parallel for each user and the maximum over u can be computed by enumeration. Supposing u is fixed, we still have to solve a series of non-linear integer optimization problems due to the non-linear nature of $\bar{\sigma}$. We can covert the non-linear problems to linear in two steps. First, we convert the tuple maximization problem to a binary-variable one as by introducing slot indicator variables for each of the elements of the tuple C. That is:

$$\max_x \quad \bar{\sigma} \left(\sum_{t \in 1:k} \sum_c x_{ct} A_{uc} \right) - \left(\beta_u + \sum_{t \in 1:k} \sum_{c \in \mathcal{C}} x_{ct} \gamma_{uc} - \sum_{c \in \mathcal{C}} \left(\sum_t x_{ct} \right) \alpha_c \right) \quad \text{s.t.} \quad \sum_c x_{ct} = 1 \quad \forall t \in 1:k .$$

$$\max_x \quad \bar{\sigma}'(m_i) \cdot \left(\sum_{t \in 1:k} \sum_c x_{ct} A_{uc} \right) - \left(\beta_u + \sum_{t \in 1:k} \sum_{c \in \mathcal{C}} x_{ct} \gamma_{uc} - \sum_{c \in \mathcal{C}} \left(\sum_t x_{ct} \right) \alpha_c \right)$$

$$\text{s.t.} \quad \sum_c x_{ct} = 1 \quad \forall t \in 1:k , \quad l_i \le \sum_{t \in 1:k} \sum_c x_{ct} A_{uc} \le u_i,$$

where $\bar{\sigma}'(m_i)$ is the derivative of $\bar{\sigma}$ at m_i. Under smoothness assumptions on $\bar{\sigma}$, this linearization provides a bounded approximation to the original problem. Again, these interval sub-problems can be solved in parallel.

A.2.3. ILLUSTRATIVE EVALUATION OF COLUMN GENERATION

Here we describe some preliminary experiments using the column generation strategy described above. We find that column generation is capable of keeping more providers viable than the myopic baseline at early steps. However the performance was less reliable than the LP-RS approach that was evaluated in Section 4. In some settings, the column generation approach fails to maintain a consistent matching in successive iterations, resulting in a slowly declining number of viable providers over time. We hypothesize that this is due to rounding errors in the procedure, or early stopping before convergence (our implementation used 300 iterations of column generation rather than running exhaustively until convergence). Therefore we expect that improvements can be made by fine-tuning this approach, but leave this to future work.

Figure 6 shows the results of the experiments. While we use the same embeddings data as in the main body of the paper, but we scale down the problem size to compensate for the slower runtime of the column generation approach; this explains the differing number of viable providers at equilibrium compared with the $LP - RS$ approach presented in Section 4. In the synthetic setting we used 50 providers, about 260 users and viability threshold of $\nu = 5$. In the other two datasets we used a competitive (from the provider perspective) setting of 100 providers, 100 users, and viability threshold of $\nu = 8$. We used slate size of 1 for all datasets.

B. Training Details

Here we provide details of training for the embeddings described in Section 4.2.

Movielens We trained a non-negative matrix factorization embedding space using the Movielens dataset (Harper & Konstan, 2015). We use the distribution of this dataset containing about $100,000$ ratings of about $9,000$ movies by about 600 users. The dataset comprises a sparse ratings matrix Given the sparse ratings matrix $R \in \mathbb{R}_{\geq 0}^{N_{\text{users}} \times N_{\text{movies}}}$. We use the binarized engagement matrix $E \in \{0,1\}^{N_{\text{users}} \times N_{\text{providers}}}$ with $E_{i,j} = \mathbb{1}(R_{i,j})$. The embeddings are produced by finding low-rank non-negative factors of the engagement matrix $E \in \{0,1\}^{N_{\text{users}} \times N_{\text{providers}}}$, by solving the optimization problem

$$\min_{U,V} ||(E - UV^T)||_F^2 + \lambda_U ||U||_F^2 + \lambda_V ||V||_F^2 \qquad (12)$$

which yields factors $U \in \mathbb{R}_{\geq 0}^{N_{\text{users}} \times N_{\text{topic}}}$ and $U \in \mathbb{R}_{\geq 0}^{N_{\text{providers}} \times N_{\text{topic}}}$.

The factors $U \in \mathbb{R}_{\geq 0}^{N_{\text{users}} \times N_{\text{topic}}}$ and $C \in \mathbb{R}_{\geq 0}^{N_{\text{providers}} \times N_{\text{topic}}}$ yield row and column vectors that are treated as the embedding vectors; in this case, a single content provider is equivalent to a single movie from the dataset.

The rows of these factor matrices were used to sample user and provider vectors in the RS ecosystem. Note that the value of the ratings were not used, so the "affinity" between user and movie in this embedding space is a measure of how likely the user is to *watch* the movie, rather than rate it highly. The randomly initialized factors U, V are alternatively updated via Weighted alternating least squares (Hu et al., 2008) for 100 iterations. We used embedding rank $N_{\text{topic}} = 20$, and set $\lambda_U = 1$ and $\lambda_v = 1..$

SNAP The dataset consists of a large list of (followee, follower) pairs, where each user is given a unique node ID label. We turn this dataset into a set of providers and users as follows. First, we randomly subsample 100k of the 41 million users. We designate followees as providers. For every provider, we then remove their follow edges, so that they do not follow anyone else. This makes the graph bipartite, where users follow providers. We then choose the top 500 providers in terms of follower count, and remove any users that do not follow at least one of them. This leaves a total of 500 providers, and 59,394 users.

for each user i, we learn a 24-dimensional vector $\mathbf{u}_i$, and for each provider j, a 24-dimensional vector $\mathbf{v}_j$. We train these embeddings by cross-entropy to predict whether there is an edge A_{ij} between user i and provider j, where the probability is given by,

$$P(A_{ij} = 1) = \sigma(\mathbf{u}_i^\top \mathbf{v}_j) \qquad (13)$$

Where $\sigma(\cdot)$ is the sigmoid function. A_{ij} is 1 if user i follows provider j. We add a small amount of weight decay to ensure that the embeddings are well behaved.

C. Simulation Details

This section contains details to reproduce the simulations described in Section 4.3.

Exploring Embedding Type We generate 50 provider vectors and a varying number of user vectors (between 4412 and 4672 per run). Provider and user vectors are sampled in a 10-dimensional topic space, with provider vectors sampled normally with variance 50. These provider vectors serve as cluster means for the mixture-of-Gaussians that generates user vectors. The prior over cluster assignments depends on the variant (*uniform* vs. *skewed* described in the text). User variance was set to 0.1 in the *uniform* variant, and user variance scaled inversely with popularity in the *skewed* variant. The slate size was $s = 1$, with viability threshold set to $\nu = 80$. We run for 10 epochs using 5 seeds for each method/data type pair, and report average values plus or minus one standard deviation in Table 1.

Tradeoffs in regret and welfare We generate synthetic embeddings of the *skewed* variant, with 50 providers and around 900 users. We use slate size $s = 4$ with viability threshold $\nu = 9$. Other settings are carried over from the previous experiment. We run steps of simulation until the policies converge then measure the welfare and max regret metrics.
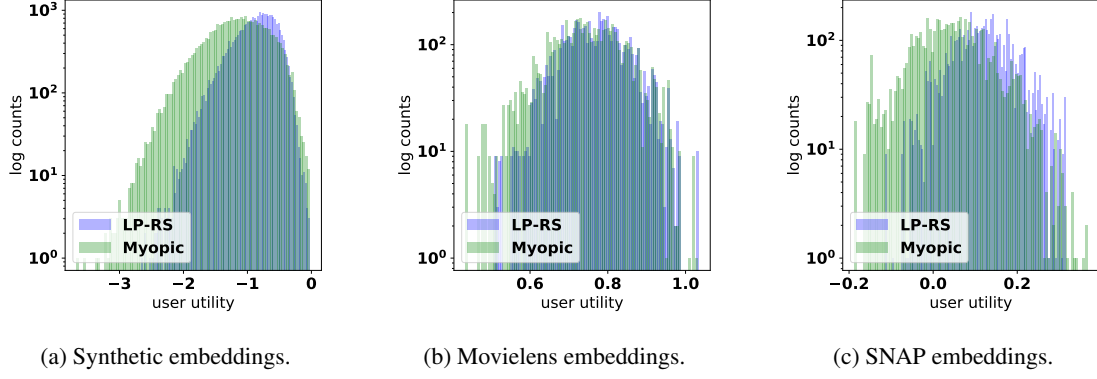
(a) Synthetic embeddings.

(b) Movielens embeddings.

(c) SNAP embeddings.

Figure 7. Histogram of user utility.

Synthetic simulation We generate synthetic embeddings of the *skewed*. The parameters are similar to those described above, with 50 providers (distributed normally with $\sigma^2 = 5$) and about $10,000$ users. We simulate the RS for ten epochs with slate size $s = 1$ and viability threshold $\nu = 78.5$.

Movielens simulation Starting with the learned low-rank factors, we subsample 250 movie column (which serve as providers) and $1,000$ user columns. We simulate the RS for ten epochs with slate size $s = 1$ and viability threshold $\nu = 10$.

SNAP simulation Starting with the learned embedding, we subsample 300 providers and 566 users. We simulate the RS for ten epochs with slate size $s = 1$ and viability threshold $\nu = 10$.

D. Additional Histograms

Figure 7 shows user utility histograms for all simulations.

E. Extended Welfare-Regret Tradeoff Results

Table 3 extends the result from Table 2 by including the Myopic baseline recommender.

		Avg. Welfare	Max Regret	Viable Providers
0.1	LP-RS	18.02 ± 1.05	7.24 ± 0.77	47.20 ± 1.72
	Myopic	13.49 ± 1.26	10.17 ± 0.20	11.80 ± 1.47
0.18	LP-RS	19.79 ± 1.16	7.97 ± 0.84	47.20 ± 1.72
	Myopic	14.83 ± 1.39	11.18 ± 0.22	11.80 ± 1.47
0.26	LP-RS	21.87 ± 1.28	8.84 ± 0.91	46.60 ± 2.15
	Myopic	16.42 ± 1.54	12.37 ± 0.25	11.80 ± 1.47
0.35	LP-RS	24.30 ± 1.43	10.14 ± 1.18	45.20 ± 3.06
	Myopic	18.29 ± 1.71	13.79 ± 0.27	11.80 ± 1.47
0.43	LP-RS	27.17 ± 1.58	11.65 ± 0.91	44.00 ± 3.16
	Myopic	20.50 ± 1.92	15.45 ± 0.31	11.80 ± 1.47
0.51	LP-RS	30.44 ± 1.79	14.19 ± 1.46	44.80 ± 0.98
	Myopic	23.08 ± 2.16	17.40 ± 0.35	11.80 ± 1.47
0.59	LP-RS	34.30 ± 1.95	16.50 ± 1.73	43.00 ± 1.90
	Myopic	26.07 ± 2.44	19.65 ± 0.39	11.80 ± 1.47
0.67	LP-RS	38.67 ± 2.29	20.53 ± 1.46	42.60 ± 1.50
	Myopic	29.51 ± 2.76	22.24 ± 0.44	11.80 ± 1.47
0.75	LP-RS	43.69 ± 2.61	24.61 ± 1.31	41.80 ± 1.72
	Myopic	33.44 ± 3.13	25.21 ± 0.50	11.80 ± 1.47
0.84	LP-RS	49.51 ± 2.90	28.95 ± 1.11	39.80 ± 3.06
	Myopic	37.91 ± 3.55	28.57 ± 0.57	11.80 ± 1.47
0.92	LP-RS	56.15 ± 3.26	33.05 ± 1.16	36.80 ± 3.97
	Myopic	42.94 ± 4.02	32.36 ± 0.64	11.80 ± 1.47
1.0	LP-RS	63.62 ± 3.58	37.89 ± 1.39	34.20 ± 5.08
	Myopic	48.59 ± 4.55	36.62 ± 0.73	11.80 ± 1.47

Table 3. The discounting factor γ allows LP-RS to trade off between average user welfare and max user regret.