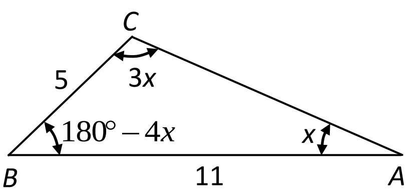
SOLUSI
PART 4
Creative Problem Solving in School Matematics

1. Nilai dari $\sin 18^\circ$ adalah ….
   Solusi:
   $x = 18^\circ$
   $5x = 90^\circ$
   $2x = 90^\circ - 3x$
   $\sin 2x = \sin(90^\circ - 3x)$
   $\sin 2x = \cos 3x$
   $2\sin x\cos x = 4\cos^3 x - 3\cos x$, dengan $\cos x > 0$
   $2\sin x = 4\cos^2 x - 3$
   $2\sin x = 4(1 - \sin^2 x) - 3$
   $4\sin^2 x + 2\sin x - 1 = 0$
   $\sin x = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-1 \pm \sqrt{5}}{4}$
   Karena $\sin x > 0$, maka $\sin x = \frac{-1+\sqrt{5}}{4}$
   $\therefore \sin 18^\circ = \frac{1}{4}\left(\sqrt{5} - 1\right)$

2. Jika $0^\circ \le x \le 90^\circ$ dan $\cos\theta - \sin\theta = \frac{\sqrt{5}}{3}$, maka nilai dari $\cos\theta + \sin\theta$ adalah ….
   Solusi:
   $\cos\theta - \sin\theta = \frac{\sqrt{5}}{3}$ (kedua ruas dikuadratkan)
   $\cos 2\theta + \sin^2\theta - 2\cos\theta\sin\theta = \frac{5}{9}$
   $1 - \sin 2\theta = \frac{5}{9}$
   $\sin 2\theta = \frac{4}{9}$
   $\cos 2\theta = \sqrt{1 - \sin^2 2\theta} = \sqrt{1 - \left(\frac{4}{9}\right)^2} = \frac{1}{9}\sqrt{65}$
   $\cos\theta + \sin\theta = \frac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta} = \frac{\cos 2\theta}{\cos\theta - \sin\theta} = \frac{\frac{1}{9}\sqrt{65}}{\frac{\sqrt{5}}{3}} = \frac{1}{3}\sqrt{13}$

3. Dalam segitiga *ABC*, $\angle ACB = 3\angle BAC$, $BC = 5$, $AB = 11$. Panjang *AC* adalah ….
   Solusi:
   Alternatif 1:
   Menurut aturan Sinus: $\frac{5}{\sin x} = \frac{11}{\sin 3x} = \frac{AC}{\sin(180^\circ - 4x)}$
   $\frac{5}{\sin x} = \frac{11}{\sin 3x} = \frac{AC}{\sin 4x}$



$$\frac{5}{\sin x}=\frac{11}{\sin 3x}$$
$$5\sin 3x=11\sin x$$
$$5\left(3\sin x-4\sin^3 x\right)=11\sin x \text{ (x sudut lancip dan} \sin x\neq 0\text{)}$$
$$15-20\sin^2 x=11$$
$$20\sin^2 x=4$$
$$\sin^2 x=\frac{1}{5}$$
$$\sin x=\frac{1}{\sqrt{5}}$$
$$\cos x=\frac{2}{\sqrt{5}}$$
$$\frac{5}{\sin x}=\frac{AC}{\sin 4x}$$
$$AC=\frac{5\sin 4x}{\sin x}=\frac{10\sin 2x\cos 2x}{\sin x}=\frac{20\sin x\cos x\left(1-2\sin^2 x\right)}{\sin x}=20\cos x\left(1-2\sin^2 x\right)$$
$$=20\times\frac{2}{\sqrt{5}}\left\{1-2\left(\frac{1}{\sqrt{5}}\right)^2\right\}=\frac{40}{\sqrt{5}}\left(1-\frac{2}{5}\right)=\frac{24}{5}\sqrt{5}$$

Alternatif 2:

Tarik garis CD, dengan titik D pada AB, sehingga $\angle ACD=\angle BAC=x$.

Jadi, segitiga ACD sama kaki.

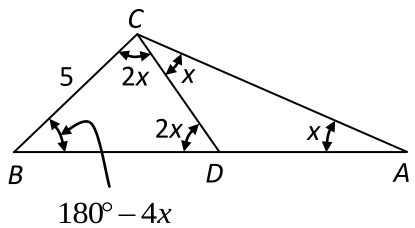
$\angle CDB=\angle DAC+\angle ACD=x+x=2x$.

Jadi, segitiga BDC sama kaki.

$BC=BD=5$

$AD=11-5=6$

$AD=CD=6$



Perhatikan segitiga DBC.

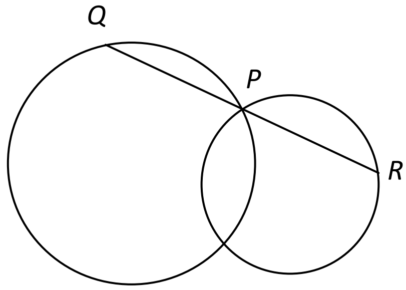
$$\cos\angle DBC=\frac{5^2+5^2-6^2}{2\cdot 5\cdot 5}=\frac{25+25-36}{2\cdot 5\cdot 5}=\frac{7}{25}$$
$$AC^2=11^2+5^2-2\cdot 11\cdot 5\cdot\cos\angle DBC=121+25-110\times\frac{7}{25}=\frac{576}{5}$$
$$AC=\sqrt{\frac{576}{5}}=\frac{24}{5}\sqrt{5}$$

4. Jika $\cos\phi=\frac{1}{7}$ dan $\sin\varphi=\frac{3}{14}\sqrt{3}$, maka nilai $\phi-\varphi=\ldots.$

Solusi:

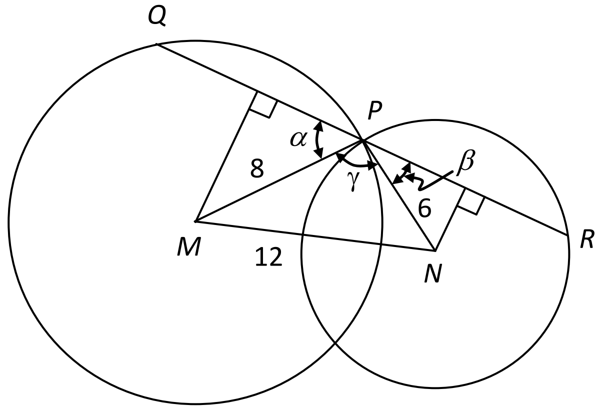
$$\sin\phi=\sqrt{1-\cos^2\phi}=\sqrt{1-\left(\frac{1}{7}\right)^2}=\frac{4}{7}\sqrt{3}$$
$$\cos\varphi=\sqrt{1-\sin^2\varphi}=\sqrt{1-\left(\frac{3}{14}\sqrt{3}\right)^2}=\frac{13}{14}$$
$$\cos(\phi-\varphi)=\cos\phi\cos\varphi+\sin\phi\sin\varphi$$
$$\cos(\phi-\varphi)=\frac{1}{7}\times\frac{13}{14}+\frac{4}{7}\sqrt{3}\times\frac{3}{14}\sqrt{3}=\frac{1}{98}(13+36)=\frac{1}{2}$$
$$\phi-\varphi=60^\circ$$

5. Pada gambar dua lingkaran masing-masing berjari-jari 8 cm dan 6 cm dan pusatnya berjarak 12 cm. Melalui sebuah titik perpotongan P, sebuah garis digambarkan sehingga tali busur QP dan PR sama panjang. Panjang tali busur PQ adalah ….



Solusi:

Karena $PQ=PR$, maka $8\cos\alpha=6\cos\beta$.



$$\cos\gamma=\frac{8^2+6^2-12^2}{2\cdot 8\cdot 6}=-\frac{11}{24}$$

$$\cos\gamma=\cos\{180^\circ-(\alpha+\beta)\}=-\frac{11}{24}$$

$$\cos\gamma=-\cos(\alpha+\beta)=-\frac{11}{24}$$

$$\cos(\alpha+\beta)=\frac{11}{24}$$

$$\cos\alpha\cos\beta-\sin\alpha\sin\beta=\frac{11}{24}$$

$$\frac{6}{8}\cos\beta\times\cos\beta-\sqrt{1-\cos^2\alpha}\times\sqrt{1-\cos^2\beta}=\frac{11}{24}$$

$$\frac{3}{4}\cos^2\beta-\sqrt{1-\left(\frac{6}{8}\cos\beta\right)^2}\times\sqrt{1-\cos^2\beta}=\frac{11}{24}$$

$$\frac{3}{4}\cos^2\beta-\sqrt{1-\frac{9}{16}\cos^2\beta}\times\sqrt{1-\cos^2\beta}=\frac{11}{24}$$

$$\frac{3}{4}\cos^2\beta-\frac{1}{4}\sqrt{16-9\cos^2\beta}\times\sqrt{1-\cos^2\beta}=\frac{11}{24}$$

$$18\cos^2\beta-6\sqrt{16-25\cos^2\beta+9\cos^4\beta}=11$$

$$18\cos^2\beta-11=6\sqrt{16-25\cos^2\beta+9\cos^4\beta}$$

$$324\cos^4\beta-396\cos^2\beta+121=36(16-25\cos^2\beta+9\cos^4\beta)$$

$$324\cos^4\beta-396\cos^2\beta+121=576-900\cos^2\beta+324\cos^4\beta$$

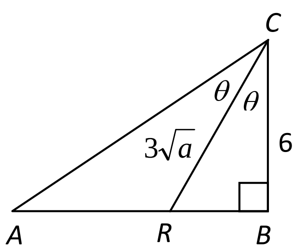
$$504\cos^2\beta=455$$

$$\cos\beta=\sqrt{\frac{455}{504}}=\sqrt{\frac{65}{72}}$$

$$PQ=2\times 6\cos\beta=2\times 6\sqrt{\frac{65}{72}}=\sqrt{130}\text{ cm}$$

6. Dalam segitiga *ABC*, $AB=8\,\text{cm}$, $BC=6\,\text{cm}$, dan $\angle ABC=90^\circ$. Jika garis bagi $\angle ACB$ memotong *AB* di *R* dan $CR=3\sqrt{a}$ cm, maka nilai dari a adalah ….

Solusi:



$$\tan 2\theta=\frac{8}{6}$$

$$\frac{2\tan\theta}{1-\tan^2\theta}=\frac{4}{3}$$

$$3\tan\theta=2-2\tan^2\theta$$

$2\tan^2\theta + 3\tan\theta - 2 = 0$

$(2\tan\theta - 1)(\tan\theta + 2) = 0$

$\tan\theta = \frac{1}{2}$ (diterima) atau $\tan\theta = -2$ (ditolak)

$\tan\theta = \frac{1}{2} = \frac{RB}{6}$

$RB = 3$ cm

$CR^2 = RB^2 + BC^2$

$\left(3\sqrt{a}\right)^2 = 3^2 + 6^2$

$9a = 9 + 36$

$a = 5$

7. Jika nilai minimum fungsi $f(x) = 4\cos x + p\sin x + 8$ adalah 2, maka nilai maksimum fungsi tersebut adalah ….

Solusi:

$f(x) = a\cos x + b\sin x + c$

$\frac{f(x)}{a} = \cos x + \frac{b}{a}\sin x + \frac{c}{a}$ (ambillah $\tan\theta = \frac{b}{a}$, maka $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$)

$\frac{f(x)}{a} = \cos x + \tan\theta\sin x + \frac{c}{a}$

$\frac{f(x)}{a} = \cos x + \frac{\sin\theta}{\cos\theta}\sin x + \frac{c}{a}$

$\frac{f(x)}{a} \times \cos\theta = \cos x\cos\theta + \sin x\sin\theta + \frac{c}{a} \times \cos\theta$

$\frac{f(x)}{a} \times \cos\theta = \cos(x-\theta) + \frac{c}{a} \times \cos\theta$

$f(x) = \frac{a}{\cos\theta} \times \cos(x-\theta) + c$

$f(x) = \frac{a}{\frac{a}{\sqrt{a^2+b^2}}} \times \cos(x-\theta) + c$

$f(x) = \sqrt{a^2+b^2}\cos(x-\theta) + c$

Nilai minimum fungsi f adalah $\sqrt{a^2+b^2} + c$ yang dicapai untuk $\cos(x-\theta) = -1$ dan nilai maksimum fungsi f adalah $-\sqrt{a^2+b^2} + c$ yang dicapai untuk $\cos(x-\theta) = 1$.

Nilai minimum fungsi $f(x) = 4\cos x + p\sin x + 8$ adalah 2.

$-\sqrt{4^2+p^2} + 8 = 2$

$\sqrt{16+p^2} = 6$

$16 + p^2 = 36$

$p^2 = 20$

Nilai maksimum fungsi $f(x) = 4\cos x + p\sin x + 8$ adalah

$\sqrt{4^2+p^2} + 8 = \sqrt{16+20} + 8 = 6 + 8 = 14$

8. Jika nilai maksimum $\frac{m}{15\sin x - 8\cos x + 25}$ adalah 2, maka nilai m adalah ….

Solusi:

$$\frac{m}{-\sqrt{15^2+(-8)^2}+25}=2$$
$$\frac{m}{-17+25}=2$$
$m=16$

9. Diberikan $5\cos x+12\cos y=13$, nilai maksimum dari $5\sin x+12\sin y$ adalah ….

Solusi:

$(5\cos x+12\cos y)^2=13^2$

$(5\cos x+12\cos y)^2+(5\sin x+12\sin y)^2=13^2+(5\sin x+12\sin y)^2$

$25(\cos^2 x+\sin^2 x)+144(\cos^2 y+\sin^2 y)+120(\cos x\cos y+\sin x\sin y)$

$=13^2+(5\sin x+12\sin y)^2$

$25+144+120\cos(x-y)=169+(5\sin x+12\sin y)^2$

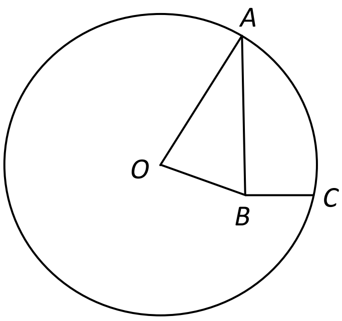
$120\cos(x-y)=(5\sin x+12\sin y)^2$

$5\sin x+12\sin y=\sqrt{120\cos(x-y)}$

Persamaan terakhir akan menjadikan $5\sin x+12\sin y$ mempunyai nilai maksimum, jika $\cos(x-y)=1$.

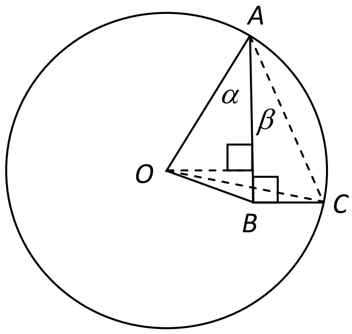
Jadi, nilai maksimum $5\sin x+12\sin y$ adalah $\sqrt{120\cdot 1}=2\sqrt{30}$.

10. Jari-jari lingkaran adalah $\sqrt{50}$ satuan, panjang AB adalah 6 satuan, dan panjang BC adalah 2 satuan. Sudut ABC adalah siku-siku. Kuadrat jarak dari B ke pusat lingkaran adalah ….



Solusi:

Alternatif 1:



$AC=\sqrt{BC^2+AB^2}=\sqrt{6^2+2^2}=\sqrt{40}$

$\sin\beta=\frac{2}{\sqrt{40}}$

$\cos\beta=\frac{6}{\sqrt{40}}$

$\cos(\alpha+\beta)=\frac{OA^2+AC^2-OC^2}{2\times OA\times AC}$

$\cos\alpha\cos\beta-\sin\alpha\sin\beta=\frac{(\sqrt{50})^2+(\sqrt{40})^2-(\sqrt{50})^2}{2\times\sqrt{50}\times\sqrt{40}}$

$\frac{6}{\sqrt{40}}\cos\alpha-\frac{2}{\sqrt{40}}\sin\alpha=\frac{40}{40\sqrt{5}}$

$6\cos\alpha-2\sin\alpha=\frac{1}{\sqrt{5}}\times\sqrt{40}$

$6\cos\alpha-2\sin\alpha=\sqrt{8}$

$3\cos\alpha-\sin\alpha=\sqrt{2}$

$3\cos\alpha-\sqrt{1-\cos^2\alpha}=\sqrt{2}$

$3\cos\alpha-\sqrt{2}=\sqrt{1-\cos^2\alpha}$

$9\cos^2\alpha-6\sqrt{2}\cos\alpha+2=1-\cos^2\alpha$

$10\cos^2\alpha-6\sqrt{2}\cos\alpha+1=0$

$\cos\alpha=\dfrac{6\sqrt{2}\pm\sqrt{72-40}}{20}=\dfrac{6\sqrt{2}\pm4\sqrt{2}}{20}$

$\cos\alpha=\dfrac{\sqrt{2}}{2}$ (diterima) atau $\cos\alpha=\dfrac{\sqrt{2}}{10}$ (ditolak)

Perhatikan segitiga *AOB*:

$OB^2=OA^2+AB^2-2\times OA\times AB\times\cos\alpha$

$OB^2=\left(\sqrt{50}\right)^2+6^2-2\times\sqrt{50}\times6\times\dfrac{\sqrt{2}}{2}$

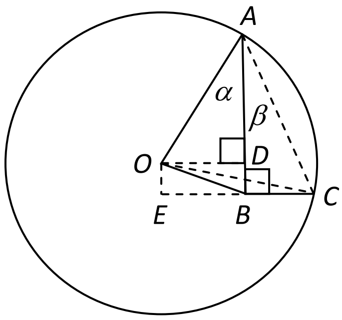
$OB^2=50+36-60$

$OB^2=26$

Jadi, kuadrat jarak dari *B* ke pusat lingkaran adalah 26 satuan.

Alternatif 2:

Ambillah $OD=y$ dan $BD=x$, maka $AD=6-x$.



$OE=BD=x$ dan $BE=y$

$OD^2=OA^2-AD^2$

$y^2=50-(6-x)^2$

$y^2=50-36+12x-x^2$

$x^2+y^2=12x+14$ …….. (1)

$OC^2=OE^2+EC^2$

$50=x^2+(2+y)^2$

$50=x^2+4+4y+y^2$

$x^2+y^2=46-4y$ ………. (2)

Dari persamaan (1) dan (2) diperoleh:

$12x+14=46-4y$

$4y=32-12x$

$y=8-3x$

$y=8-3x\rightarrow x^2+y^2=12x+14$

$x^2+(8-3x)^2=12x+14$

$x^2+64-48x+9x^2=12x+14$

$10x^2-60x+50=0$

$x^2-6x+5=0$

$(x-5)(x-1)=0$

$x=5$ (ditolak) atau $x=1$ (diterima)

$x=1\rightarrow y=8-3x=8-3\times1=5$

$OB^2=x^2+y^2=1^2+5^2=26$

Jadi, kuadrat jarak dari *B* ke pusat lingkaran adalah 26 satuan.

Alternatif 3:

$OB^2-BD^2=OD^2$ …. (1)

$OA^2-AD^2=OD^2$ …. (2)

Dari persamaan (1) dan (2) diperoleh:

$OB^2-BD^2=OA^2-AD^2$

$OB^2 - x^2 = \left(\sqrt{50}\right)^2 - (6-x)^2$

$OB^2 - x^2 = 50 - 36 + 12x - x^2$

$OB^2 = 14 + 12x$

Karena OB < jari-jari OA, maka diambil x = 1 (?), sehingga diperoleh

$OB^2 = 14 + 12 \cdot 1 = 26$.

Jadi, kuadrat jarak dari B ke pusat lingkaran adalah 26 satuan.

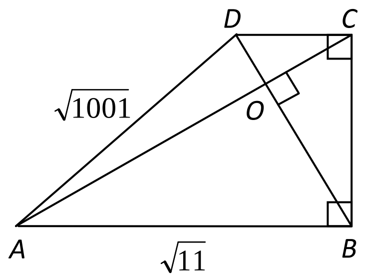
11. Diketahui trapesium $ABCD$, dengan AB // CD, $AB \perp CB$, $BC \perp DC$, $AD = \sqrt{1001}$ cm, dan $AB = \sqrt{11}$ cm. Panjang BC adalah ….

Solusi:

Alternatif 1:

Ambillah $OA = a$ dan $OB = ar$

Perhatikan $\Delta AOB \sim \Delta BOC$



$\frac{OA}{OB} = \frac{OB}{OC}$

$\frac{a}{ar} = \frac{ar}{OC}$

$OC = ar^2$

Perhatikan $\Delta BOC \sim \Delta COD$

$\frac{OB}{OC} = \frac{OC}{OD}$

$\frac{ar}{ar^2} = \frac{ar^2}{OD}$

$OD = ar^3$

Perhatikan ΔAOB:

$AB = \sqrt{OA^2 + OB^2} = \sqrt{(a)^2 + (ar)^2} = a\sqrt{1 + r^2}$

Perhatikan ΔAOD:

$AD = \sqrt{OA^2 + OD^2} = \sqrt{(a)^2 + \left(ar^3\right)^2} = a\sqrt{1 + r^6}$

$\frac{AD^2}{AB^2} = \frac{\left(\sqrt{1001}\right)^2}{\left(\sqrt{11}\right)^2}$

$\frac{a^2 + a^2r^6}{a^2 + a^2r^2} = \frac{1001}{11}$

$\frac{1 + r^6}{1 + r^2} = 91$

$\frac{\left(1 + r^2\right)\left(1 - r^2 + r^4\right)}{1 + r^2} = 91$

$1 - r^2 + r^4 = 91$

$r^4 - r^2 - 90 = 0$

$\left(r^2 - 10\right)\left(r^2 + 9\right) = 0$

$r^2 = 10$ atau $r^2 = -9$ (ditolak)

$r^2 = 10 \rightarrow AB = a\sqrt{1 + r^2} = \sqrt{11}$

$a\sqrt{1 + 10} = \sqrt{11}$

$a = 1$

Perhatikan ΔBOC:

$BC = \sqrt{OB^2 + OC^2} = \sqrt{(ar)^2 + \left(ar^2\right)^2} = \sqrt{\left(1 \cdot \sqrt{10}\right)^2 + (1 \cdot 10)^2} = \sqrt{110}$ cm.

Alternatif 2:

Gradien $m_{BD} = \dfrac{d}{c}$

Gradien $m_{AC} = \dfrac{\sqrt{11}}{-c}$

$AC \perp BD$

$m_{AC} \times m_{AC} = -1$

$\dfrac{\sqrt{11}}{-c} \times \dfrac{d}{c} = -1$

$c^2 = d\sqrt{11}$

$AD^2 = c^2 + \left(\sqrt{11} - d\right)^2$

$\left(\sqrt{1001}\right)^2 = c^2 + \left(\sqrt{11} - \dfrac{c^2}{\sqrt{11}}\right)^2$

$1001 = c^2 + 11 - 2c^2 + \dfrac{c^4}{11}$

$0 = -990 - c^2 + \dfrac{c^4}{11}$

$c^4 - 11c^2 - 10890 = 0$

$\left(c^2 - 110\right)\left(c^2 + 99\right) = 0$

$c^2 = 110$ (diterima) atau $c^2 = -99$ (ditolak)

$c = \sqrt{110}$

Jadi, panjang BC adalah $\sqrt{110}$ cm.