Algorithms analysis

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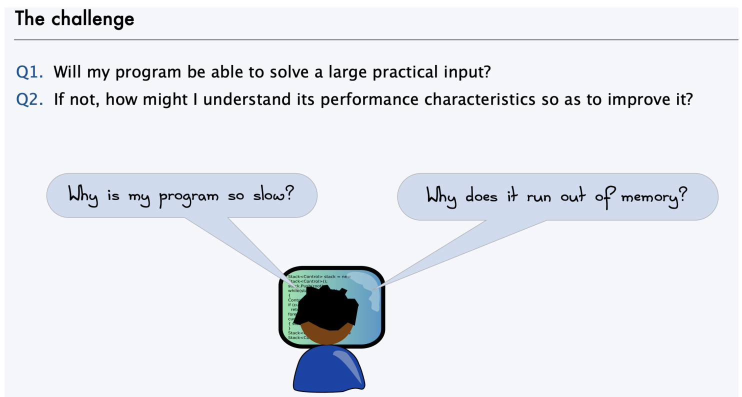


An algorithm's execution time can be expressed as the number of steps required to solve the problem. This abstraction is exactly what we need: it characterizes an algorithm's efficiency in execution time while remaining independent of any particular program or computer.

Understanding this is like getting brand new glasses - you start seeing the world in a way you didn't even know existed before - it's a completely new perspective.

Let's take a closer look at some examples. The most expensive computation unit will be an *apicall()*. So, our goal is to find the number of times *apicall()* is called as a function of the input size n.

For loop

```
for (i = 4; i < n + 42; i += 2)
    apicall()
```

apicall() is called exactly

$$(n + 42 - 4)/2$$

The time complexity is linear; it means that if we increase the input size *10, our program will make 10 times more *apicall()*

```
for (i = 10; i < n + 5; i *= 3)
    apicall()
```

The loop will multiply i by 3 until

$$10 * 3^i >= n + 5$$

$$3^i >= (n + 5)/10$$

$$i = log_3(n + 5)/10$$

Therefore, *apicall()* is called roughly logn times

Double loop

```
for (i = 0; i < n; ++i)
    for (j = 0; j < n; j += 2)
            apicall()
```

```
i=0 => n/2
i=1 => n/2
i=2 => n/2
...
i=n => 0
```

The total number of calls is

$$\frac{n}{2} + \frac{n}{2} + ... + \frac{n}{2} = \frac{n^2}{2}$$

.

Triple loop

```
for (i = 1; i <= n; ++i)
    for (j = 1; j <= i; ++j)
        for (k = 1; k <= i; ++k)
            apicall()
```

```
i=1 => 1*1
i=2 => 2*2
i=3 => 3*3
...
i=n => n*n
```

The total number of times *apicall()* is performed is

$$1^2 + 2^2 + ... + n^2 = n^3$$

To understand why, consider the following observations:

1. It's not geometric or arithmetic progression
2. Upped bound:

$$1^2 + 2^2 + ... + n^2 \leq n^2 + n^2 + ... + n^2 \leq n * n^2 = n^3$$

   Therefore, the order of growth is not more than n^3
3. Lower bound:

$$1^2+2^2+...+n^2 \geq (\frac{n}{2})^2+(\frac{n}{2}+1)^2+...+n^2 \geq (\frac{n}{2})^2+(\frac{n}{2})^2+...+(\frac{n}{2})^2 \geq n/2*(n/2)^2 = \frac{n^3}{8}$$

   Therefore, the order of growth is not less than $\frac{n^3}{8}$

Since we drop constant coefficients, the upper and lower bounds imply that the order of growth is n^3

Recursion

```
def f(n: int) -> None:
    if n == 0: return
    for (i = 0; i < n; ++i)
        apicall()
    f(n-1)
```

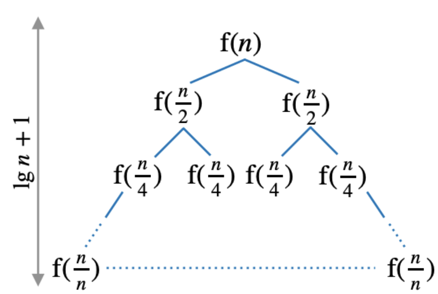
apicall() is executed n times for $f(n)$ and $n-1$ times for $f(n-1)$, etc. This means that *apicall()* is executed

$$n + (n-1) + (n-2) + ... + 1 = n * (n+1)/2$$

The amount of space required to execute our code is linear.

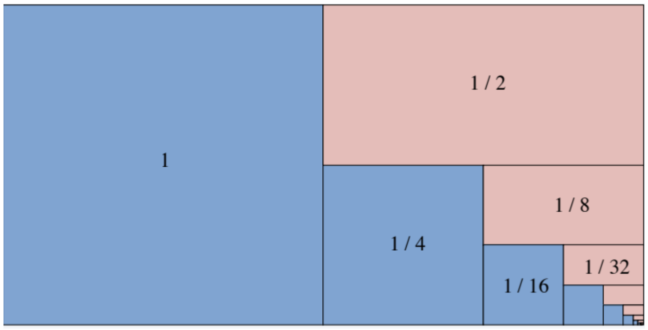
```
def f(n: int) -> None {
    if n == 0: return
    apicall()
    f(n/2)
    f(n/2)
}
```

Consider the following visualization:



apicall() is performed once in each recursive call:

$$1 + 2 + 4 + ... + n/4 + n/2 + n = 2n - 1$$




```python
def fib(n: int) -> int:
    if n < 2:
        return n
    return fib(n-1) + fib(n-2)
```

Here is the animation of the code execution (zoom it if it is small).
The number of additions is:

$$2^0 + 2^1 + 2^2 + ... + 2^n$$

Using the formula for the sum of the first $n+1$ terms of a geometric series with the first term $a = 1$ and the common ratio $r = 2$ we got $2^{n+1} - 1$.