DGLAP Evolution Equation on the Complex Plane

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August 27, 2015

Abstract

In this document I describe the extension of the DGLAP equation for complex values of the factorization scale μ and the relative implementation in `APFEL`.

1 DGLAP on the Complex Plane

Let us start from the usual DGLAP evolution equation for the distribution f([1]):

$$\mu^2 \frac{\partial f}{\partial \mu^2} = P\left(x, \alpha_s(\mu)\right) \otimes f(x, \mu) \tag{1}$$

where $\otimes$ represents the usual Mellin convolution such that:

$$A(x) \otimes B(x) \equiv \int_0^1 dz \int_0^1 dy A(y) B(z) \delta(x - yz) = \int_x^1 \frac{dy}{y} A(x) B\left(\frac{y}{x}\right) = \int_x^1 \frac{dy}{y} A\left(\frac{y}{x}\right) B(x)\,. \tag{2}$$

Now we consider the RGE for the strong coupling α_s, that reads:

$$\mu^2 \frac{\partial \alpha_s}{\partial \mu^2} = \beta(\alpha_s)\,, \tag{3}$$

and combine it with the DGLAP equation in eq. (1), obtaining the DGLAP equation differential in α_s:

$$\frac{\partial f}{\partial \alpha_s} = R\left(x, \alpha_s\right) \otimes f(x, \alpha_s) \tag{4}$$

where:

$$R\left(x, \alpha_s\right) = \frac{P\left(x, \alpha_s\right)}{\beta(\alpha_s)}\,. \tag{5}$$

The next fundamental step is the promotion of the factorization scale μ from a real to a complex variable:

$$\mu \rightarrow \eta = \mu + i\nu\,. \tag{6}$$

As a consequence, we need to promote also the strong the PDF f and the strong coupling α_s to being complex functions, that is:

$$\begin{array}{ccc} f & \rightarrow & F = f + ig\,, \\ \alpha_s & \rightarrow & \zeta_s = \alpha_s + i\xi_s\,. \end{array} \tag{7}$$

This has as a further consequence that the DGLAP and the α_s evolution equations in eqs. (1) and (3) become complex differential equations:

$$\eta^2 \frac{\partial F}{\partial \eta^2} = P\left(x, \zeta_s(\eta)\right) \otimes F(x, \eta)\,, \tag{8}$$

and:

$$\eta^2 \frac{\partial \zeta_s}{\partial \eta^2} = \beta(\zeta_s)\,, \tag{9}$$

[1] At this stage it is not necessary to distinguish between singlet or non-singlet distributions. We will consider these cases separately later once the formalism has been settled.

that can be again combined in:

$$\frac{\partial F}{\partial \zeta_s} = R\left(x, \zeta_s\right) \otimes F(x, \zeta_s)\,. \tag{10}$$

The main goal is the solution of eq. (10). The starting observation is the fact that the complex function F must be an analytical function of the complex variable ζ_s. This implies that the real and the complex parts of F must obey the Cauchy-Riemann equations, that is:

$$\begin{array}{rcl} \dfrac{\partial f}{\partial \alpha_s} & = & \dfrac{\partial g}{\partial \xi_s} \\ & & \\ \dfrac{\partial f}{\partial \xi_s} & = & -\dfrac{\partial g}{\partial \alpha_s} \end{array}, \tag{11}$$

so that the derivative of F with respect to η_s can be expanded as:

$$\frac{\partial F}{\partial \zeta_s} = \frac{\partial f}{\partial \alpha_s} + i\frac{\partial g}{\partial \alpha_s} = \frac{\partial g}{\partial \xi_s} - i\frac{\partial f}{\partial \xi_s}\,. \tag{12}$$

Now let us consider the function R. Being it a complex function, it can be split into a real and a complex part:

$$R = S + iT\,, \tag{13}$$

and thus:

$$R \otimes F = (S + iT) \otimes (f + ig) = (S \otimes f - T \otimes g) + i(T \otimes f + S \otimes g)\,. \tag{14}$$

We can now combine eqs. (12) and (14) into eq. (10). This allows us to obtain two sets of coupled real differential equations that can be written in the following matricial form:

$$\frac{\partial}{\partial \alpha_s}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} S & -T \\ T & S \end{pmatrix} \otimes \begin{pmatrix} f \\ g \end{pmatrix}, \tag{15}$$

and:

$$\frac{\partial}{\partial \xi_s}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -T & -S \\ S & -T \end{pmatrix} \otimes \begin{pmatrix} f \\ g \end{pmatrix}. \tag{16}$$

The solution of eqs. (15) and (16) allows one to obtain the dependence of the real functions f and g (*i.e.* the real and the complex part of the "complex" PDF F) on the real variables α_s and ξ_s (*i.e.* the real and the complex part of the "complex" strong coupling ζ_s) which in turn are functions of the complex factorization scale η. It should be noticed that while solving eq. (15) the value of ξ_s should be kept constant and conversely while solving eq. (16) the value of α_s should be kept constant. Geometrically, this means that eq. (15) allows one to compute the PDF evolution along the real axis in the complex plane of ζ_s while eq. (16) allows one to compute the PDF evolution along the immaginary axis. Of course, a suitable combination of these evolution allows to reach any point of the complex plane of ζ_s starting from any other point. This is strictly true only if no branch cut is crossed during the evolution. Finally, it is interesting to notice that the splitting function matices in the r.h.s. of eqs. (15) and (16) commute. This has the consequence that the order in which the derivarives with respect of α_s and ξ_s does not affect the result. Of course, this feature must be reflected in the solutions of eqs. (15) and (16). In oder words, this means that, aslo on the complex plane, the evolution factor to be applied to the initial state PDF only depends on the initial and the final point and not on the path followed to connect the two points.

Now we need to extract the functions S and T from R and in the next section we will show their form at leading order (LO) in QCD.

1.1 Solution at LO

At LO in QCD we have that:

$$R\left(x, \zeta_s\right) = \frac{P\left(x, \zeta_s\right)}{\beta(\zeta_s)} = -\frac{P^{(0)}(x)}{\beta_0}\frac{1}{\zeta_s} = -\frac{P^{(0)}(x)}{\beta_0}\frac{\alpha_s - i\xi_s}{\alpha_s^2 + \xi_s^2} \tag{17}$$

and thus:

$$S(x, \alpha_s, \xi_s) = -\frac{P^{(0)}(x)}{\beta_0}\frac{\alpha_s}{\alpha_s^2 + \xi_s^2} \quad \text{and} \quad T(x, \alpha_s, \xi_s) = -\frac{P^{(0)}(x)}{\beta_0}\frac{-\xi_s}{\alpha_s^2 + \xi_s^2}\,. \tag{18}$$

Using eq. (18), eqs. (15) and (16) become:

$$\frac{\partial}{\partial \alpha_s}\begin{pmatrix} f \\ g \end{pmatrix} = -\frac{P^{(0)}(x)}{\beta_0}\frac{1}{\alpha_s^2+\xi_s^2}\begin{pmatrix} \alpha_s & \xi_s \\ -\xi_s & \alpha_s \end{pmatrix} \otimes \begin{pmatrix} f \\ g \end{pmatrix}, \tag{19}$$

and:

$$\frac{\partial}{\partial \xi_s}\begin{pmatrix} f \\ g \end{pmatrix} = -\frac{P^{(0)}(x)}{\beta_0}\frac{1}{\alpha_s^2+\xi_s^2}\begin{pmatrix} \xi_s & -\alpha_s \\ \alpha_s & \xi_s \end{pmatrix} \otimes \begin{pmatrix} f \\ g \end{pmatrix}. \tag{20}$$

In Mellin space, the Mellin convolution of the equations above becomes a simple product and can then be solved more easily. In fact, the equations above in Mellin space become:

$$\frac{\partial}{\partial \alpha_s}\begin{pmatrix} f \\ g \end{pmatrix} = -\frac{\gamma^{(0)}(N)}{\beta_0}\frac{1}{\alpha_s^2+\xi_s^2}\begin{pmatrix} \alpha_s & \xi_s \\ -\xi_s & \alpha_s \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \tag{21}$$

and:

$$\frac{\partial}{\partial \xi_s}\begin{pmatrix} f \\ g \end{pmatrix} = -\frac{\gamma^{(0)}(N)}{\beta_0}\frac{1}{\alpha_s^2+\xi_s^2}\begin{pmatrix} \xi_s & -\alpha_s \\ \alpha_s & \xi_s \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}. \tag{22}$$

Defining:

$$\mathbf{F} \equiv \begin{pmatrix} f \\ g \end{pmatrix} \quad \text{and} \quad R_0 = \frac{\gamma^{(0)}(N)}{\beta_0}, \tag{23}$$

and considering that:

$$\int dx \frac{x}{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2) \quad \text{and} \quad \int dx \frac{y}{x^2+y^2} = \operatorname{atan}\left(\frac{x}{y}\right) = \frac{\pi}{2} - \operatorname{atan}\left(\frac{y}{x}\right), \tag{24}$$

the solution of eqs. (21) and (22) is:

$$\mathbf{F}(N,\alpha_s,\xi_s) = \mathbf{F}(N,\alpha_{s,0},\xi_s)\exp\left[-R_0\Gamma_\alpha\right], \tag{25}$$

and:

$$\mathbf{F}(N,\alpha_s,\xi_s) = \mathbf{F}(N,\alpha_s,\xi_{s,0})\exp\left[-R_0\Gamma_\xi\right], \tag{26}$$

with:

$$\Gamma_\alpha = \begin{pmatrix} \frac{1}{2}\ln\left(\frac{\alpha_s^2+\xi_s^2}{\alpha_{s,0}^2+\xi_s^2}\right) & -\operatorname{atan}\left(\frac{\xi_s}{\alpha_s}\right) + \operatorname{atan}\left(\frac{\xi_s}{\alpha_{s,0}}\right) \\ \operatorname{atan}\left(\frac{\xi_s}{\alpha_s}\right) - \operatorname{atan}\left(\frac{\xi_s}{\alpha_{s,0}}\right) & \frac{1}{2}\ln\left(\frac{\alpha_s^2+\xi_s^2}{\alpha_{s,0}^2+\xi_s^2}\right) \end{pmatrix}, \tag{27}$$

and:

$$\Gamma_\xi = \begin{pmatrix} \frac{1}{2}\ln\left(\frac{\alpha_s^2+\xi_s^2}{\alpha_s^2+\xi_{s,0}^2}\right) & -\operatorname{atan}\left(\frac{\xi_s}{\alpha_s}\right) + \operatorname{atan}\left(\frac{\xi_{s,0}}{\alpha_s}\right) \\ \operatorname{atan}\left(\frac{\xi_s}{\alpha_s}\right) - \operatorname{atan}\left(\frac{\xi_{s,0}}{\alpha_s}\right) & \frac{1}{2}\ln\left(\frac{\alpha_s^2+\xi_s^2}{\alpha_s^2+\xi_{s,0}^2}\right) \end{pmatrix}. \tag{28}$$

It is interesting to observe that, if the complex strong coupling ζ_s becomes real, *i.e.* $\xi_s \to 0$, the matrices above become:

$$\lim_{\xi_s\to 0}\Gamma_\alpha = \ln\left(\frac{\alpha_s}{\alpha_{s,0}}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{29}$$

and:

$$\lim_{\xi_s\to 0}\Gamma_\xi = 0, \tag{30}$$

and thus the evolution in α_s of the real part of the PDF f reduces to the expected one while no evolution in ξ_s is left. As for the evolution in α_s of the imaginary part of the PDF g, there is an evolution factor but it is decoupled from the real part and thus it has an effect only if the imaginary part of the initial scale PDF is different from zero.

The solutions in eqs. (25) and (26) are written in a pretty formal way because they imply the exponential of matrices. However, since Γ_α and Γ_ξ are 2 by 2 matrices, their exponential in known a simple closed form. In particular:

$$\exp\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{\exp\left[(a+d)/2\right]}{\Delta}\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \tag{31}$$

where:

$$\Delta = \sqrt{(a-d)^2+4bc} \tag{32}$$

and:

$$\begin{aligned} m_{11} &= \Delta\text{cosh}\left(\frac{\Delta}{2}\right) + (a-d)\text{sinh}\left(\frac{\Delta}{2}\right) \\ m_{12} &= 2b\,\text{sinh}\left(\frac{\Delta}{2}\right) \\ m_{21} &= 2c\,\text{sinh}\left(\frac{\Delta}{2}\right) \\ m_{22} &= \Delta\text{cosh}\left(\frac{\Delta}{2}\right) - (a-d)\text{sinh}\left(\frac{\Delta}{2}\right) \end{aligned} \tag{33}$$

Given the structure of Γ_α and Γ_ξ, we can simplify the formulas above. In the case of Γ_α the structure is:

$$\Gamma_\alpha = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{34}$$

with:

$$\begin{aligned} a &= \frac{1}{2}\ln\left(\frac{\alpha_s^2+\xi_s^2}{\alpha_{s,0}^2+\xi_s^2}\right) = \ln\left(\frac{|\zeta_s|}{|\zeta_{s,0}|}\right) \\ b &= \text{atan}\left(\frac{\xi_s}{\alpha_s}\right) - \text{atan}\left(\frac{\xi_s}{\alpha_{s,0}}\right) = \theta - \theta_0 = \Delta\theta\,, \end{aligned} \tag{35}$$

and thus, after some simplifications, we find:([2]):

$$\exp\left[-R_0\Gamma_\alpha\right] = \exp[-R_0 a]\begin{pmatrix} \cos(-R_0 b) & -\sin(-R_0 b) \\ \sin(-R_0 b) & \cos(-R_0 b) \end{pmatrix} = \left(\frac{|\zeta_s|}{|\zeta_{s,0}|}\right)^{-R_0}\begin{pmatrix} \cos(-R_0\Delta\theta) & -\sin(-R_0\Delta\theta) \\ \sin(-R_0\Delta\theta) & \cos(-R_0\Delta\theta) \end{pmatrix}. \tag{36}$$

For Γ_ξ,the structure is exactly the same:

$$\Gamma_\alpha = \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \tag{37}$$

with:

$$\begin{aligned} c &= \frac{1}{2}\ln\left(\frac{\alpha_s^2+\xi_s^2}{\alpha_s^2+\xi_{s,0}^2}\right) = \ln\left(\frac{|\zeta_s|}{|\zeta_{s,0}|}\right) \\ d &= \text{atan}\left(\frac{\xi_s}{\alpha_s}\right) - \text{atan}\left(\frac{\xi_{s,0}}{\alpha_s}\right) = \theta - \theta_0 = \Delta\theta\,, \end{aligned} \tag{38}$$

and thus also here we find:

$$\exp\left[-R_0\Gamma_\xi\right] = \exp[-R_0 c]\begin{pmatrix} \cos(-R_0 d) & -\sin(-R_0 d) \\ \sin(-R_0 d) & \cos(-R_0 d) \end{pmatrix} = \left(\frac{|\zeta_s|}{|\zeta_{s,0}|}\right)^{-R_0}\begin{pmatrix} \cos(-R_0\Delta\theta) & -\sin(-R_0\Delta\theta) \\ \sin(-R_0\Delta\theta) & \cos(-R_0\Delta\theta) \end{pmatrix}. \tag{39}$$

Eqs. (36) and (39) are the main result of this section because they are the evolution factors in the real and imaginary direction to be applied to the initial scale PDF. Considering that eqs. (36) and (39) have exactly the same form they can combined in one single evolution factor with the following compact result:

$$\mathbf{F}(N,\zeta_s) = \mathbf{F}(N,\zeta_{s,0})\left(\frac{|\zeta_s|}{|\zeta_{s,0}|}\right)^{-R_0}\begin{pmatrix} \cos\left[-R_0(\theta-\theta_0)\right] & -\sin\left[-R_0(\theta-\theta_0)\right] \\ \sin\left[-R_0(\theta-\theta_0)\right] & \cos\left[-R_0(\theta-\theta_0)\right] \end{pmatrix}, \tag{40}$$

where $|\zeta_s|$ and $|\zeta_{s,0}|$ are the absolute value of the initial and final values of the complex coupling while θ and θ_0 are the respective phases such that:

$$\zeta_s = |\zeta_s|\exp(i\theta) \quad \text{and} \quad \zeta_{s,0} = |\zeta_{s,0}|\exp(i\theta_0)\,, \tag{41}$$

[2]Notice that:

$$a + ib = \ln\left(\frac{\zeta_s}{\zeta_{s,0}}\right).$$

which is equivalent to:

$$F(N,\zeta_s) = F(N,\zeta_{s,0})\left(\frac{|\zeta_s|}{|\zeta_{s,0}|}\right)^{-R_0} e^{-iR_0(\theta-\theta_0)} = F(N,\zeta_{s,0})\left[-R_0\ln\left(\frac{\zeta_s}{\zeta_{s,0}}\right)\right] = \left(\frac{\zeta_s}{\zeta_{s,0}}\right)^{-R_0}, \tag{42}$$

which is the straight LO solution of the Mellin version of eq. (10). We can thus deduce that, also beyond LO, eq. (10) can be solved using the standard techniques.

Since the best way to solve eq. (10) is in N (Mellin) space, under the condition that the x-space PDF is also complex the numerical inversion algorithm from N to x space needs to be adapted.

The inverse Mellin transformationis defined as:

$$F(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dN\, x^{-N} F(N)\,, \tag{43}$$

where the real number c has to be such that the integral $\int_0^1 dx\, x^{c-1}F(x)$ is absolutely convergent. Hence c has to lie to the right of the rightmost singularity of $F(N)$ in the complex plane.

Under the assumption that $F(N)$ is an analytical (or holomorphic) function, the Cauchy theorem states that one can deform the integration path in a continuous way without changing the result of the integral, provided that no pole of the function $F(N)$ is crossed during the deformation. This allows us to cleverly choose a different path that makes the solution of the integral in eq. (43) easy to implement in a numerical code. A possible choice is the so-called Talbot path $\mathcal{C}_T$, such that eq. (43) is equivalent to:

$$F(x) = \frac{1}{2\pi i}\int_{\mathcal{C}_T} dN\, x^{-N} F(N)\,, \tag{44}$$

where:

$$\mathcal{C}_T : \{N(\theta) = r\theta(\cot\theta + i);\ \theta \in (-\pi, +\pi)\} \tag{45}$$

being r a parameter possibly depending only on x. Performing a variable substitution in eq. (44) using eq. (45) and defining $t \equiv -\ln x$, one gets:

$$\begin{aligned} F(x) &= \frac{1}{2\pi i}\int_{-\pi}^{+\pi} d\theta\, \frac{dN(\theta)}{d\theta}\, e^{t\,N(\theta)} F(N(\theta)) \\ &= \frac{r}{2\pi}\int_{-\pi}^{+\pi} d\theta\, [1 + i\sigma(\theta)]\, e^{t\,N(\theta)} F(N(\theta))\,, \end{aligned} \tag{46}$$

with:

$$\sigma(\theta) = \theta + \cot\theta(\theta\cot\theta - 1)\,. \tag{47}$$

Usually the computation of the integral in the r.h.s. of eq. (46) is usually performed numerically using the fact that $F(N)$ is the Mellin tranform of a real function and thus, under this assumption, $F(N^*) = F^*(N)$. In the case we are considering here, *i.e.* also the x-space PDF $F(x)$ is a complex function, this assumption can no longer be used. Defining:

$$G(\theta) = \frac{r}{2\pi}[1 + i\sigma(\theta)]\, e^{t\,N(\theta)} F(N(\theta))\,, \tag{48}$$

and using the trapezoidal method to solve the integral in eq. (46) one gets:

$$F(x) = \delta\left[\sum_{k=1}^{M-1} G(-\pi + k\delta) + \frac{G(-\pi) + G(\pi)}{2}\right], \tag{49}$$

where M is the number of equal intervals in which the integration range is divided and:

$$\delta = \frac{2\pi}{M} \tag{50}$$

is essentially the width of the single interval. One can show that the boundary terms $G(-\pi)$ and $G(\pi)$ in eq. (49) vanish and thus, defining $\theta_k = -\pi + k\delta$, it becomes:

$$F(x) = \frac{r}{M}\sum_{k=1}^{M-1} [1 + i\sigma(\theta_k)]\, e^{t\,N(\theta_k)} F(N(\theta_k))\,. \tag{51}$$

From eq. (45) one can easily see that $N(-\theta) = N^*(\theta)$, while from eq. (47) it is evident that $\sigma(\theta)$ is an odd function of θ. Moreover, $F(N)$ being the Mellin transform of a real function, one automatically has that $F(N^*) = F^*(N)$. Using this information, one gets:

$$F(x) \quad = \quad \frac{r}{\pi}\int_0^{+\pi} d\theta \, \mathrm{Re}\left\{[1 + i\sigma(\theta)]\, e^{t\,N(\theta)} F(N(\theta))\right\} \,, \tag{52}$$

where Re$\{\dots\}$ is the real part of its argument. Finally, eq. (52) can be solved numerically using, for instance, the trapezoidal approximation.

At this point, in order to be able to implement in numerical code, it is necessary to distinguish between non-slinglet and singlet distributions. In the case of the non-singlet distributions eqs. (36) and (39) can be implemented exactly as they are written because the anomalous dimension $\gamma^{(0)}$ appearing in eqs. (21) and (22) is a singled-valued function. In the singlet case instead $\gamma^{(0)}$ (and thus R_0) is actually a 2 by 2 matrix of functions and thus, since it appears in the exponential and in the trigonometric functions, eqs. (36) and (39) need to be treated in a suitable way.