A Common Flavour Basis for the coupled QED×QCD Evolution

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Abstract

In this document I will present a suitable flavout basis for the coupled QCD×QED DGLAP evolution of PDFs.

Contents

1 The Structure of the DGLAP Equation

The DGLAP equation that governs the PDF evolution has a general structure that in QCD holds at any perturbative order. Suppose one wants to study the coupled evolution of the gluon distrubution function $g(x,\mu)$, the i-th quark distribution function $q_i(x,\mu)$ and the j-th anti-quark distribution function $\overline{q}_j(x,\mu)$. In this case evolution equation would look like this:

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} q_i \\ g \\ \overline{q}_j \end{pmatrix} = \sum_{k,l} \begin{pmatrix} P_{q_i q_k} & P_{q_i g} & P_{q_i \overline{q}_l} \\ P_{g q_k} & P_{gg} & P_{g\overline{q}_l} \\ P_{\overline{q}_j q_k} & P_{\overline{q}_j g} & P_{\overline{q}_j \overline{q}_l} \end{pmatrix} \begin{pmatrix} q_k \\ g \\ \overline{q}_l \end{pmatrix} \tag{1.1}$$

where we are understanding the convolution and where the sum over k and l runs over all n_f the active flavours. Because of charge conjugation invariance and $SU(n_f)$ flavour symmetry, one can show that:

$$\begin{aligned} &P_{q_i q_j} = P_{\overline{q}_i \overline{q}_j} = \delta_{ij} P^V_{qq} + P^S_{qq} \\ &P_{\overline{q}_i q_j} = P_{q_i \overline{q}_j} = \delta_{ij} P^V_{\overline{q}q} + P^S_{\overline{q}q} \\ &P_{q_i g} = P_{\overline{q}_i g} = P_{qg} \\ &P_{g q_i} = P_{g \overline{q}_i} = P_{gq}\,. \end{aligned} \tag{1.2}$$

Plugging eq. (1.2) into eq. (1.1), one finds:

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} q_i \\ g \\ \overline{q}_j \end{pmatrix} = \begin{pmatrix} P^V_{qq} & P_{qg} & P^V_{q\overline{q}} \\ P_{gq} & P_{gg} & P_{gq} \\ P^V_{q\overline{q}} & P_{qg} & P^V_{qq} \end{pmatrix} \begin{pmatrix} q_i \\ g \\ \overline{q}_j \end{pmatrix} + \begin{pmatrix} P^S_{qq} & 0 & P^S_{q\overline{q}} \\ 0 & 0 & 0 \\ P^S_{q\overline{q}} & 0 & P^S_{qq} \end{pmatrix} \begin{pmatrix} \sum_k q_k \\ g \\ \sum_l \overline{q}_l \end{pmatrix} . \tag{1.3}$$

Setting $i = j$ and summing and subtracting the first and the third row/column, we find:

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} q_i^+ \\ g \\ q_i^- \end{pmatrix} = \begin{pmatrix} (P^V_{qq} + P^V_{q\overline{q}}) & 2P_{qg} & 0 \\ P_{gq} & P_{gg} & 0 \\ 0 & 0 & (P^V_{qq} - P^V_{q\overline{q}}) \end{pmatrix} \begin{pmatrix} q_i^+ \\ g \\ q_i^- \end{pmatrix} + \begin{pmatrix} (P^S_{qq} + P^S_{q\overline{q}}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (P^S_{qq} - P^S_{q\overline{q}}) \end{pmatrix} \begin{pmatrix} \sum_k q_k^+ \\ g \\ \sum_k q_k^- \end{pmatrix} , \tag{1.4}$$

where we have defined:

$$q_i^{\pm} \equiv q_i \pm \overline{q}_i\,. \tag{1.5}$$

It is evident that in this way we have semi diagonalized the initial system because now the third equation is decaupled from the rest of the system. Using the following definitions:

$$\begin{aligned} &\Sigma \equiv \sum_k q_k^+ \\ &V \equiv \sum_k q_k^- \\ &P^{\pm} \equiv P^V_{qq} \pm P^V_{q\overline{q}} \\ &P_{qq} \equiv P^+ + n_f(P^S_{qq} + P^S_{q\overline{q}}) \\ &P^V \equiv P^- + n_f(P^S_{qq} - P^S_{q\overline{q}}) \end{aligned} \quad , \tag{1.6}$$

we have:

$$\begin{cases} \mu^2 \dfrac{\partial}{\partial \mu^2} g = P_{gg} g + P_{gq} \Sigma \\ \mu^2 \dfrac{\partial}{\partial \mu^2} q_i^+ = P^+ q_i^+ + \dfrac{1}{n_f}(P_{qq} - P^+)\Sigma + 2P_{qg} g \\ \mu^2 \dfrac{\partial}{\partial \mu^2} q_i^- = P^- q_i^- + \dfrac{1}{n_f}(P^V - P^-)V \end{cases} . \tag{1.7}$$

At this point we want to generalize this discussion including the QED corrections. There are two main differences. The first is obviously the fact that we need to intruduce in the DGLAP equation the parton distribution associated to the photon $\gamma(x,\mu)$. The second difference is the fact that the all-order splitting functions no longer undergo to the stringent simplications of eq. (1.2). In fact, the QED corrections introduce an asymmetry between down-like quarks (d, s and b) and top-like quarks (u, c, t), due essentially to the different electric charge, that breaks the flavour symmetry for the quark splitting functions. We then must consider the following extended evolution system:

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix}u_j\\ d_i\\ g\\ \gamma\\ \overline{d}_k\\ \overline{u}_h\end{pmatrix}=\sum_{e,l,m,n}\begin{pmatrix}\mathcal{P}_{u_ju_e} & \mathcal{P}_{u_jd_l} & \mathcal{P}_{u_jg} & \mathcal{P}_{u_j\gamma} & \mathcal{P}_{u_j\overline{d}_m} & \mathcal{P}_{u_j\overline{u}_n}\\ \mathcal{P}_{d_iu_e} & \mathcal{P}_{d_id_l} & \mathcal{P}_{d_ig} & \mathcal{P}_{d_i\gamma} & \mathcal{P}_{d_i\overline{d}_m} & \mathcal{P}_{d_i\overline{u}_n}\\ \mathcal{P}_{gu_e} & \mathcal{P}_{gd_l} & \mathcal{P}_{gg} & \mathcal{P}_{g\gamma} & \mathcal{P}_{g\overline{d}_m} & \mathcal{P}_{g\overline{u}_n}\\ \mathcal{P}_{\gamma u_e} & \mathcal{P}_{\gamma d_l} & \mathcal{P}_{\gamma g} & \mathcal{P}_{\gamma\gamma} & \mathcal{P}_{\gamma\overline{d}_m} & \mathcal{P}_{\gamma\overline{u}_n}\\ \mathcal{P}_{\overline{d}_ku_e} & \mathcal{P}_{\overline{d}_kd_l} & \mathcal{P}_{\overline{d}_kg} & \mathcal{P}_{\overline{d}_k\gamma} & \mathcal{P}_{\overline{d}_k\overline{d}_m} & \mathcal{P}_{\overline{d}_k\overline{u}_n}\\ \mathcal{P}_{\overline{u}_hu_e} & \mathcal{P}_{\overline{u}_hd_l} & \mathcal{P}_{\overline{u}_hg} & \mathcal{P}_{\overline{u}_h\gamma} & \mathcal{P}_{\overline{u}_h\overline{d}_m} & \mathcal{P}_{\overline{u}_h\overline{u}_n}\end{pmatrix}\begin{pmatrix}u_e\\ d_l\\ g\\ \gamma\\ \overline{d}_m\\ \overline{u}_n\end{pmatrix} \tag{1.8}$$

where:

$$u_i=\{u,c,t\}\,,\qquad d_i=\{d,s,b\}\,,\qquad \overline{u}_i=\{\overline{u},\overline{c},\overline{t}\}\,,\qquad \overline{d}_i=\{\overline{d},\overline{s},\overline{b}\}\,. \tag{1.9}$$

Each splitting function in eq. (1.8) can be split into two pieces:

$$\mathcal{P}_{ab}=P_{ab}+\widetilde{P}_{ab}\,, \tag{1.10}$$

where P_{ab} is the usual QCD splitting function, *i.e.* which does not contain any power of the fine structure constant α and it only contains powers of the strong coupling α_s and therefore undergoes to the same simplifications we discussed above. As a further consequence if a or b is equal to γ, P_{ab} must vanish. $\widetilde{P}_{ab}$ instead contains at least one power of α. In this way we can rearrange eq. (1.8) as follows:

$$\begin{aligned}\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix}u_j\\ d_i\\ g\\ \gamma\\ \overline{d}_k\\ \overline{u}_h\end{pmatrix} &= \sum_{e,l,m,n}\left[\begin{pmatrix}\delta_{je}P^V_{qq}+P^S_{qq} & P^S_{qq} & P_{qg} & 0 & P^S_{q\overline{q}} & \delta_{jn}P^V_{q\overline{q}}+P^S_{q\overline{q}}\\ P^S_{qq} & \delta_{il}P^V_{qq}+P^S_{qq} & P_{qg} & 0 & \delta_{im}P^V_{q\overline{q}}+P^S_{q\overline{q}} & P^S_{q\overline{q}}\\ P_{gq} & P_{gq} & P_{gg} & 0 & P_{gq} & P_{gq}\\ 0 & 0 & 0 & 0 & 0 & 0\\ P^S_{q\overline{q}} & \delta_{kl}P^V_{q\overline{q}}+P^S_{q\overline{q}} & P_{qg} & 0 & \delta_{km}P^V_{qq}+P^S_{qq} & P^S_{qq}\\ \delta_{he}P^V_{q\overline{q}}+P^S_{q\overline{q}} & P^S_{q\overline{q}} & P_{qg} & 0 & P^S_{qq} & \delta_{hn}P^V_{qq}+P^S_{qq}\end{pmatrix}\right.\\ &\left.+\begin{pmatrix}\widetilde{P}_{u_ju_e} & \widetilde{P}_{u_jd_l} & \widetilde{P}_{u_jg} & \widetilde{P}_{u_j\gamma} & \widetilde{P}_{u_j\overline{d}_m} & \widetilde{P}_{u_j\overline{u}_n}\\ \widetilde{P}_{d_iu_e} & \widetilde{P}_{d_id_l} & \widetilde{P}_{d_ig} & \widetilde{P}_{d_i\gamma} & \widetilde{P}_{d_i\overline{d}_m} & \widetilde{P}_{d_i\overline{u}_n}\\ \widetilde{P}_{gu_e} & \widetilde{P}_{gd_l} & \widetilde{P}_{gg} & \widetilde{P}_{g\gamma} & \widetilde{P}_{g\overline{d}_m} & \widetilde{P}_{g\overline{u}_n}\\ \widetilde{P}_{\gamma u_e} & \widetilde{P}_{\gamma d_l} & \widetilde{P}_{\gamma g} & \widetilde{P}_{\gamma\gamma} & \widetilde{P}_{\gamma\overline{d}_m} & \widetilde{P}_{\gamma\overline{u}_n}\\ \widetilde{P}_{\overline{d}_ku_e} & \widetilde{P}_{\overline{d}_kd_l} & \widetilde{P}_{\overline{d}_kg} & \widetilde{P}_{\overline{d}_k\gamma} & \widetilde{P}_{\overline{d}_k\overline{d}_m} & \widetilde{P}_{\overline{d}_k\overline{u}_n}\\ \widetilde{P}_{\overline{u}_hu_e} & \widetilde{P}_{\overline{u}_hd_l} & \widetilde{P}_{\overline{u}_hg} & \widetilde{P}_{\overline{u}_h\gamma} & \widetilde{P}_{\overline{u}_h\overline{d}_m} & \widetilde{P}_{\overline{u}_h\overline{u}_n}\end{pmatrix}\right]\begin{pmatrix}u_e\\ d_l\\ g\\ \gamma\\ \overline{d}_m\\ \overline{u}_n\end{pmatrix}\end{aligned} \tag{1.11}$$

Now, since α comes always with an electric charge associated, every $\widetilde{P}_{ab}$ can factorize out at least an electric charge $e_u^2=4/9$ or $e_d^2=1/9$. In order to see how this factorization takes place, we should analyze one by one the splitting functions $\widetilde{P}_{ab}$.

Defining:

$$e^2_\Sigma=N_c(e_u^2n_u+e_d^2n_d)\,, \tag{1.12}$$

where n_u and n_d are respectively the number of up- and down-type active quarks such that $n_u+n_d=n_f$ and $N_c=3$ is the number of colors, we have that:

$$\begin{aligned}&\widetilde{P}_{gg}\to e^2_\Sigma\widetilde{P}_{gg}\,,\quad \widetilde{P}_{g\gamma}\to e^2_\Sigma\widetilde{P}_{g\gamma}\,,\\ &\widetilde{P}_{\gamma g}\to e^2_\Sigma\widetilde{P}_{\gamma g}\,,\quad \widetilde{P}_{\gamma\gamma}\to e^2_\Sigma\widetilde{P}_{\gamma\gamma}\,.\end{aligned} \tag{1.13}$$

This is the consequence of the fact that, having only bosons as external particles, the presence of any fermion in the splitting must be summed over all the flavours. This is (should be) true at any perturbative order.

Now we consider the splitting functions involving one boson and one quark. Here the situation is more involved because at higer orders it may happen that the incoming/outcoming quark never couples with a photon and thus, given that there is at least one power aof α, apart from a term proportional to the charge of the incomin/outcoming quark there must also be a term proportional to the charge e_Σ^2. However, such contributions only appear at three loops (NNLO) and since here we are only intersted in the two-loop splitting functions, we have:

$$
\begin{aligned}
&\widetilde{P}_{gu_i} = \widetilde{P}_{g\overline{u}_i} = e_u^2 \widetilde{P}_{gq}\,, \quad \widetilde{P}_{gd_i} = \widetilde{P}_{g\overline{d}_i} = e_d^2 \widetilde{P}_{gq}\,, \\
&\widetilde{P}_{u_i g} = \widetilde{P}_{\overline{u}_i g} = e_u^2 \widetilde{P}_{qg}\,, \quad \widetilde{P}_{d_i g} = \widetilde{P}_{\overline{d}_i g} = e_d^2 \widetilde{P}_{qg}\,, \\
&\widetilde{P}_{\gamma u_i} = \widetilde{P}_{\gamma\overline{u}_i} = e_u^2 \widetilde{P}_{\gamma q}\,, \quad \widetilde{P}_{\gamma d_i} = \widetilde{P}_{\gamma\overline{d}_i} = e_d^2 \widetilde{P}_{\gamma q}\,, \\
&\widetilde{P}_{u_i \gamma} = \widetilde{P}_{\overline{u}_i \gamma} = e_u^2 \widetilde{P}_{q\gamma}\,, \quad \widetilde{P}_{d_i \gamma} = \widetilde{P}_{\overline{d}_i \gamma} = e_d^2 \widetilde{P}_{q\gamma}\,.
\end{aligned}
\tag{1.14}
$$

Finally, we consider the splitting functions involving quarks or anti-quarks in the final and initial states. Again we will limit ourselves to tow loops and under this resctriction we have:

$$
\begin{array}{l}
\widetilde{P}_{u_i u_j} = \widetilde{P}_{\overline{u}_i \overline{u}_j} = e_u^2 \delta_{ij} \widetilde{P}_{qq}^V + e_u^4 \widetilde{P}_{qq}^S \\
\widetilde{P}_{d_i d_j} = \widetilde{P}_{\overline{d}_i \overline{d}_j} = e_d^2 \delta_{ij} \widetilde{P}_{qq}^V + e_d^4 \widetilde{P}_{qq}^S \\
\widetilde{P}_{\overline{u}_i u_j} = \widetilde{P}_{u_i \overline{u}_j} = e_u^4 \widetilde{P}_{qq}^S \\
\widetilde{P}_{\overline{d}_i d_j} = \widetilde{P}_{d_i \overline{d}_j} = e_d^4 \widetilde{P}_{qq}^S \\
\widetilde{P}_{u_i d_j} = \widetilde{P}_{d_i u_j} = \widetilde{P}_{\overline{u}_i d_j} = \widetilde{P}_{d_i \overline{u}_j} = \widetilde{P}_{\overline{d}_i u_j} = \widetilde{P}_{u_i \overline{d}_j} = \widetilde{P}_{\overline{u}_i \overline{d}_j} = \widetilde{P}_{\overline{d}_i \overline{u}_j} = e_u^2 e_d^2 \widetilde{P}_{qq}^S
\end{array}
\,. \tag{1.15}
$$

Using the information above, we can now write the QED correction matrix of the splitting functions up to two loops as follows:

$$
\begin{pmatrix}
e_u^2 \delta_{je} \widetilde{P}_{qq}^V + e_u^4 \widetilde{P}_{qq}^S & e_u^2 e_d^2 \widetilde{P}_{qq}^S & e_u^2 \widetilde{P}_{qg} & e_u^2 \widetilde{P}_{q\gamma} & e_u^2 e_d^2 \widetilde{P}_{qq}^S & e_u^4 \widetilde{P}_{qq}^S \\
e_u^2 e_d^2 \widetilde{P}_{qq}^S & e_d^2 \delta_{il} \widetilde{P}_{qq}^V + e_d^4 \widetilde{P}_{qq}^S & e_d^2 \widetilde{P}_{qg} & e_d^2 \widetilde{P}_{q\gamma} & e_d^4 \widetilde{P}_{qq}^S & e_u^2 e_d^2 \widetilde{P}_{qq}^S \\
e_u^2 \widetilde{P}_{gq} & e_d^2 \widetilde{P}_{gq} & e_\Sigma^2 \widetilde{P}_{gg} & e_\Sigma^2 \widetilde{P}_{g\gamma} & e_d^2 \widetilde{P}_{gq} & e_u^2 \widetilde{P}_{gq} \\
e_u^2 \widetilde{P}_{\gamma q} & e_d^2 \widetilde{P}_{\gamma q} & e_\Sigma^2 \widetilde{P}_{\gamma g} & e_\Sigma^2 \widetilde{P}_{\gamma\gamma} & e_d^2 \widetilde{P}_{\gamma q} & e_u^2 \widetilde{P}_{\gamma q} \\
e_u^2 e_d^2 \widetilde{P}_{qq}^S & e_d^4 \widetilde{P}_{qq}^S & e_d^2 \widetilde{P}_{qg} & e_d^2 \widetilde{P}_{q\gamma} & e_d^2 \delta_{km} \widetilde{P}_{qq}^V + e_d^4 \widetilde{P}_{qq}^S & e_u^2 e_d^2 \widetilde{P}_{qq}^S \\
e_u^4 \widetilde{P}_{qq}^S & e_u^2 e_d^2 \widetilde{P}_{qq}^S & e_u^2 \widetilde{P}_{qg} & e_u^2 \widetilde{P}_{q\gamma} & e_u^2 e_d^2 \widetilde{P}_{qq}^S & e_u^2 \delta_{hn} \widetilde{P}_{qq}^V + e_u^4 \widetilde{P}_{qq}^S
\end{pmatrix} =
$$

$$
\begin{pmatrix}
e_u^2 \delta_{je} \widetilde{P}_{qq} & 0 & e_u^2 \widetilde{P}_{qg} & e_u^2 \widetilde{P}_{q\gamma} & 0 & 0 \\
0 & e_d^2 \delta_{il} \widetilde{P}_{qq}^V & e_d^2 \widetilde{P}_{qg} & e_d^2 \widetilde{P}_{q\gamma} & 0 & 0 \\
e_u^2 \widetilde{P}_{gq} & e_d^2 \widetilde{P}_{gq} & e_\Sigma^2 \widetilde{P}_{gg} & e_\Sigma^2 \widetilde{P}_{g\gamma} & e_d^2 \widetilde{P}_{gq} & e_u^2 \widetilde{P}_{gq} \\
e_u^2 \widetilde{P}_{\gamma q} & e_d^2 \widetilde{P}_{\gamma q} & e_\Sigma^2 \widetilde{P}_{\gamma g} & e_\Sigma^2 \widetilde{P}_{\gamma\gamma} & e_d^2 \widetilde{P}_{\gamma q} & e_u^2 \widetilde{P}_{\gamma q} \\
0 & 0 & e_d^2 \widetilde{P}_{qg} & e_d^2 \widetilde{P}_{q\gamma} & e_d^2 \delta_{km} \widetilde{P}_{qq}^V & 0 \\
0 & 0 & e_u^2 \widetilde{P}_{qg} & e_u^2 \widetilde{P}_{q\gamma} & 0 & e_u^2 \delta_{hn} \widetilde{P}_{qq}^V
\end{pmatrix}
+
\begin{pmatrix}
e_u^4 \widetilde{P}_{qq}^S & e_u^2 e_d^2 \widetilde{P}_{qq}^S & 0 & 0 & e_u^2 e_d^2 \widetilde{P}_{qq}^S & e_u^4 \widetilde{P}_{qq}^S \\
e_u^2 e_d^2 \widetilde{P}_{qq}^S & e_d^4 \widetilde{P}_{qq}^S & 0 & 0 & e_d^4 \widetilde{P}_{qq}^S & e_u^2 e_d^2 \widetilde{P}_{qq}^S \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
e_u^2 e_d^2 \widetilde{P}_{qq}^S & e_d^4 \widetilde{P}_{qq}^S & 0 & 0 & e_d^4 \widetilde{P}_{qq}^S & e_u^2 e_d^2 \widetilde{P}_{qq}^S \\
e_u^4 \widetilde{P}_{qq}^S & e_u^2 e_d^2 \widetilde{P}_{qq}^S & 0 & 0 & e_u^2 e_d^2 \widetilde{P}_{qq}^S & e_u^4 \widetilde{P}_{qq}^S
\end{pmatrix}
\tag{1.16}
$$

We now apply the same decomposition to the purely QCD matrix, obtaining:

$$
\begin{pmatrix}
\delta_{je} P_{qq}^V + P_{qq}^S & P_{qq}^S & P_{qg} & 0 & P_{q\overline{q}}^S & \delta_{jn} P_{q\overline{q}}^V + P_{q\overline{q}}^S \\
P_{qq}^S & \delta_{il} P_{qq}^V + P_{qq}^S & P_{qg} & 0 & \delta_{im} P_{q\overline{q}}^V + P_{q\overline{q}}^S & P_{q\overline{q}}^S \\
P_{gq} & P_{gq} & P_{gg} & 0 & P_{gq} & P_{gq} \\
0 & 0 & 0 & 0 & 0 & 0 \\
P_{q\overline{q}}^S & \delta_{kl} P_{q\overline{q}}^V + P_{q\overline{q}}^S & P_{qg} & 0 & \delta_{km} P_{qq}^V + P_{qq}^S & P_{qq}^S \\
\delta_{he} P_{q\overline{q}}^V + P_{q\overline{q}}^S & P_{q\overline{q}}^S & P_{qg} & 0 & P_{qq}^S & \delta_{hn} P_{qq}^V + P_{qq}^S
\end{pmatrix} =
$$

$$
\begin{pmatrix}
\delta_{je} P_{qq}^V & 0 & P_{qg} & 0 & \delta_{jm} P_{q\overline{q}}^V & 0 \\
0 & \delta_{il} P_{qq}^V & P_{qg} & 0 & 0 & \delta_{in} P_{q\overline{q}}^V \\
P_{gq} & P_{gq} & P_{gg} & 0 & P_{gq} & P_{gq} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \delta_{kl} P_{q\overline{q}}^V & P_{qg} & 0 & \delta_{km} P_{qq}^V & 0 \\
\delta_{he} P_{q\overline{q}}^V & 0 & P_{qg} & 0 & 0 & \delta_{hn} P_{qq}^V
\end{pmatrix}
+
\begin{pmatrix}
P_{qq}^S & P_{qq}^S & 0 & 0 & P_{q\overline{q}}^S & P_{q\overline{q}}^S \\
P_{qq}^S & P_{qq}^S & 0 & 0 & P_{q\overline{q}}^S & P_{q\overline{q}}^S \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
P_{q\overline{q}}^S & P_{q\overline{q}}^S & 0 & 0 & P_{qq}^S & P_{qq}^S \\
P_{q\overline{q}}^S & P_{q\overline{q}}^S & 0 & 0 & P_{qq}^S & P_{qq}^S
\end{pmatrix}
\tag{1.17}
$$

Finally, plugging eqs. (1.16) and (1.17) into eq. (1.11), performing the sum over e, l, m and n and indentifying

$k=i$ and $h=j$, we obtain:

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix}u_j\\ d_i\\ g\\ \gamma\\ \overline{d}_i\\ \overline{u}_j\end{pmatrix} = \left[\begin{pmatrix}P^V_{qq} & 0 & P_{qg} & 0 & 0 & P^V_{q\overline{q}}\\ 0 & P^V_{qq} & P_{qg} & 0 & P^V_{q\overline{q}} & 0\\ P_{gq} & P_{gq} & P_{gg} & 0 & P_{gq} & P_{gq}\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & P^V_{q\overline{q}} & P_{qg} & 0 & P^V_{qq} & 0\\ P^V_{q\overline{q}} & 0 & P_{qg} & 0 & 0 & P^V_{qq}\end{pmatrix} + \begin{pmatrix}e_u^2\widetilde{P}_{qq} & 0 & e_u^2\widetilde{P}_{qg} & e_u^2\widetilde{P}_{q\gamma} & 0 & 0\\ 0 & e_d^2\widetilde{P}_{qq} & e_d^2\widetilde{P}_{qg} & e_d^2\widetilde{P}_{q\gamma} & 0 & 0\\ e_u^2\widetilde{P}_{gq} & e_d^2\widetilde{P}_{gq} & e_\Sigma^2\widetilde{P}_{gg} & e_\Sigma^2\widetilde{P}_{g\gamma} & e_d^2\widetilde{P}_{gq} & e_u^2\widetilde{P}_{gq}\\ e_u^2\widetilde{P}_{\gamma q} & e_d^2\widetilde{P}_{\gamma q} & e_\Sigma^2\widetilde{P}_{\gamma g} & e_\Sigma^2\widetilde{P}_{\gamma\gamma} & e_d^2\widetilde{P}_{\gamma q} & e_u^2\widetilde{P}_{\gamma q}\\ 0 & 0 & e_d^2\widetilde{P}_{qg} & e_d^2\widetilde{P}_{q\gamma} & e_d^2\widetilde{P}^V_{qq} & 0\\ 0 & 0 & e_u^2\widetilde{P}_{qg} & e_u^2\widetilde{P}_{q\gamma} & 0 & e_u^2\widetilde{P}^V_{qq}\end{pmatrix}\right]\begin{pmatrix}u_j\\ d_i\\ g\\ \gamma\\ \overline{d}_i\\ \overline{u}_j\end{pmatrix}$$

$$+\left[\begin{pmatrix}P^S_{qq} & P^S_{qq} & 0 & 0 & P^S_{q\overline{q}} & P^S_{q\overline{q}}\\ P^S_{qq} & P^S_{qq} & 0 & 0 & P^S_{q\overline{q}} & P^S_{q\overline{q}}\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ P^S_{q\overline{q}} & P^S_{q\overline{q}} & 0 & 0 & P^S_{qq} & P^S_{qq}\\ P^S_{q\overline{q}} & P^S_{q\overline{q}} & 0 & 0 & P^S_{qq} & P^S_{qq}\end{pmatrix} + \begin{pmatrix}e_u^4\widetilde{P}^S_{qq} & e_u^2e_d^2\widetilde{P}^S_{qq} & 0 & 0 & e_u^2e_d^2\widetilde{P}^S_{qq} & e_u^4\widetilde{P}^S_{qq}\\ e_u^2e_d^2\widetilde{P}^S_{qq} & e_d^4\widetilde{P}^S_{qq} & 0 & 0 & e_d^4\widetilde{P}^S_{qq} & e_u^2e_d^2\widetilde{P}^S_{qq}\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ e_u^2e_d^2\widetilde{P}^S_{qq} & e_d^4\widetilde{P}^S_{qq} & 0 & 0 & e_d^4\widetilde{P}^S_{qq} & e_u^2e_d^2\widetilde{P}^S_{qq}\\ e_u^4\widetilde{P}^S_{qq} & e_u^2e_d^2\widetilde{P}^S_{qq} & 0 & 0 & e_u^2e_d^2\widetilde{P}^S_{qq} & e_u^4\widetilde{P}^S_{qq}\end{pmatrix}\right]\begin{pmatrix}\sum_e u_e\\ \sum_l d_l\\ g\\ \gamma\\ \sum_m \overline{d}_m\\ \sum_n \overline{u}_n\end{pmatrix} \tag{1.18}$$

In order to have the same evolution system in terms of plus- and minus-distributions, we apply to eq. (1.18) the following transformation:

$$\mathbf{T} = \begin{pmatrix}1 & 0 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 0\\ 1 & 0 & 0 & 0 & 0 & -1\end{pmatrix} \quad\Longrightarrow\quad \mathbf{T}^{-1} = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 2 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 0\\ 1 & 0 & 0 & 0 & 0 & -1\end{pmatrix} \tag{1.19}$$

so that we get:

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix}u_j^+\\ d_i^+\\ g\\ \gamma\\ d_i^-\\ u_j^-\end{pmatrix} = \left[\begin{pmatrix}P^+ & 0 & 2P_{qg} & 0 & 0 & 0\\ 0 & P^+ & 2P_{qg} & 0 & 0 & 0\\ P_{gq} & P_{gq} & P_{gg} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & P^- & 0\\ 0 & 0 & 0 & 0 & 0 & P^-\end{pmatrix} + \begin{pmatrix}e_u^2\widetilde{P}^+ & 0 & 2e_u^2\widetilde{P}_{qg} & 2e_u^2\widetilde{P}_{q\gamma} & 0 & 0\\ 0 & e_d^2\widetilde{P}^+ & 2e_d^2\widetilde{P}_{qg} & 2e_d^2\widetilde{P}_{q\gamma} & 0 & 0\\ e_u^2\widetilde{P}_{gq} & e_d^2\widetilde{P}_{gq} & e_\Sigma^2\widetilde{P}_{gg} & e_\Sigma^2\widetilde{P}_{g\gamma} & 0 & 0\\ e_u^2\widetilde{P}_{\gamma q} & e_d^2\widetilde{P}_{\gamma q} & e_\Sigma^2\widetilde{P}_{\gamma g} & e_\Sigma^2\widetilde{P}_{\gamma\gamma} & 0 & 0\\ 0 & 0 & 0 & 0 & e_d^2\widetilde{P}^- & 0\\ 0 & 0 & 0 & 0 & 0 & e_u^2\widetilde{P}^-\end{pmatrix}\right]\begin{pmatrix}u_j^+\\ d_i^+\\ g\\ \gamma\\ d_i^-\\ u_j^-\end{pmatrix}$$

$$+\left[\begin{pmatrix}P_{qq}-P^+ & P_{qq}-P^+ & 0 & 0 & 0 & 0\\ P_{qq}-P^+ & P_{qq}-P^+ & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & P^V-P^- & P^V-P^-\\ 0 & 0 & 0 & 0 & P^V-P^- & P^V-P^-\end{pmatrix}\right.$$

$$\left.+\begin{pmatrix}e_u^4(\widetilde{P}_{qq}-\widetilde{P}^+) & e_u^2e_d^2(\widetilde{P}_{qq}-\widetilde{P}^+) & 0 & 0 & 0 & 0\\ e_u^2e_d^2(\widetilde{P}_{qq}-\widetilde{P}^+) & e_d^4(\widetilde{P}_{qq}-\widetilde{P}^+) & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}\right]\frac{1}{n_f}\begin{pmatrix}\sum_e u_e^+\\ \sum_l d_l^+\\ g\\ \gamma\\ \sum_m d_m^-\\ \sum_n u_n^-\end{pmatrix} \tag{1.20}$$

Using the following definitions:

$$\Sigma_u = \sum_{k=i}^{n_u} u_k^+ \qquad \Sigma_d = \sum_{k=i}^{n_d} d_k^+$$
$$V_u = \sum_{k=i}^{n_u} u_k^- \qquad V_d = \sum_{k=i}^{n_d} d_k^- \,, \tag{1.21}$$

which are such that:

$$\Sigma = \Sigma_u + \Sigma_d \quad \text{and} \quad V = V_u + V_d\,, \tag{1.22}$$

we can further manipulate eq. (4.8) obtaining the coupled system:

$$\left\{\begin{array}{rcl}
\mu^2 \dfrac{\partial g}{\partial \mu^2} & = & (P_{gq} + e_u^2 \widetilde{P}_{gq})\Sigma_u + (P_{gq} + e_d^2 \widetilde{P}_{gq})\Sigma_d + (P_{gg} + e_\Sigma^2 \widetilde{P}_{gg})g + e_\Sigma^2 \widetilde{P}_{g\gamma}\gamma \\
\mu^2 \dfrac{\partial \gamma}{\partial \mu^2} & = & e_u^2 \widetilde{P}_{\gamma q}\Sigma_u + e_d^2 \widetilde{P}_{\gamma q}\Sigma_d + e_\Sigma^2 \widetilde{P}_{\gamma g} g + e_\Sigma^2 \widetilde{P}_{\gamma\gamma}\gamma \\
\mu^2 \dfrac{\partial d_i^+}{\partial \mu^2} & = & (P^+ + e_d^2 \widetilde{P}^+)d_i^+ + 2(P_{qg} + e_d^2 \widetilde{P}_{qg})g + 2e_d^2 \widetilde{P}_{q\gamma}\gamma \\
& + & \dfrac{1}{n_f}[(P_{qq} - P^+) + e_u^2 e_d^2(\widetilde{P}_{qq} - \widetilde{P}^+)]\Sigma_u + \dfrac{1}{n_f}[(P_{qq} - P^+) + e_d^4(\widetilde{P}_{qq} - \widetilde{P}^+)]\Sigma_d \\
\mu^2 \dfrac{\partial u_j^+}{\partial \mu^2} & = & (P^+ + e_u^2 \widetilde{P}^+)u_j^+ + 2(P_{qg} + e_u^2 \widetilde{P}_{qg})g + 2e_u^2 \widetilde{P}_{q\gamma}\gamma \\
& + & \dfrac{1}{n_f}[(P_{qq} - P^+) + e_u^4(\widetilde{P}_{qq} - \widetilde{P}^+)]\Sigma_u + \dfrac{1}{n_f}[(P_{qq} - P^+) + e_u^2 e_d^2(\widetilde{P}_{qq} - \widetilde{P}^+)]\Sigma_d \\
\mu^2 \dfrac{\partial d_i^-}{\partial \mu^2} & = & (P^- + e_d^2 \widetilde{P}^-)d_i^- + \dfrac{1}{n_f}(P^V - P^-)V_u + \dfrac{1}{n_f}(P^V - P^-)V_d \\
\mu^2 \dfrac{\partial u_j^-}{\partial \mu^2} & = & (P^- + e_u^2 \widetilde{P}^-)u_j^- + \dfrac{1}{n_f}(P^V - P^-)V_u + \dfrac{1}{n_f}(P^V - P^-)V_d
\end{array}\right. . \tag{1.23}$$

2 Evolution Basis

In order to diagonlize as much as possible the evolution matrix in the presence of QED corrections avoiding unnecessery couplings between parton distributions, we propose the following evolution basis:

$$\begin{array}{ll}
1)\ g & \\
2)\ \gamma & \\
3)\ \Sigma = \Sigma_u + \Sigma_d & 9)\ V = V_u + V_d \\
4)\ \Delta_\Sigma = \Sigma_u - \Sigma_d & 10)\ \Delta_V = V_u - V_d \\
5)\ T_1^u = u^+ - c^+ & 11)\ V_1^u = u^- - c^- \\
6)\ T_2^u = u^+ + c^+ - 2t^+ & 12)\ V_2^u = u^- + c^- - 2t^- \\
7)\ T_1^d = d^+ - s^+ & 13)\ V_1^d = d^- - s^- \\
8)\ T_2^d = d^+ + s^+ - 2b^+ & 14)\ V_2^d = d^- + s^- - 2b^-
\end{array} \tag{2.1}$$

In this basis the evolution system becomes:

$$\left\{\begin{array}{rcl}
\mu^2\dfrac{\partial g}{\partial\mu^2} &=& (P_{gq}+\eta^+\widetilde{P}_{gq})\Sigma+\eta^-\widetilde{P}_{gq}\Delta_\Sigma+(P_{gg}+e^2_\Sigma\widetilde{P}_{gg})g+e^2_\Sigma\widetilde{P}_{g\gamma}\gamma \\
\mu^2\dfrac{\partial\gamma}{\partial\mu^2} &=& \eta^+\widetilde{P}_{\gamma q}\Sigma+\eta^-\widetilde{P}_{\gamma q}\Delta_\Sigma+e^2_\Sigma\widetilde{P}_{\gamma g}g+e^2_\Sigma\widetilde{P}_{\gamma\gamma}\gamma \\
\mu^2\dfrac{\partial\Sigma}{\partial\mu^2} &=& \left[P_{qq}+\eta^+\widetilde{P}^++\dfrac{\eta^+e^2_\Sigma}{n_f}(\widetilde{P}_{qq}-\widetilde{P}^+)\right]\Sigma+\left[\eta^-\widetilde{P}^++\dfrac{\eta^-e^2_\Sigma}{n_f}(\widetilde{P}_{qq}-\widetilde{P}^+)\right]\Delta_\Sigma \\
&+& 2(n_fP_{qg}+e^2_\Sigma\widetilde{P}_{qg})g+2e^2_\Sigma\widetilde{P}_{q\gamma}\gamma \\
\mu^2\dfrac{\partial\Delta_\Sigma}{\partial\mu^2} &=& \left[\eta^-\widetilde{P}^++\dfrac{n_u-n_d}{n_f}(P_{qq}-P^+)+\dfrac{\eta^+\delta^2_e}{n_f}(\widetilde{P}_{qq}-\widetilde{P}^+)\right]\Sigma+\left[P^++\eta^+\widetilde{P}^++\dfrac{\eta^-\delta^2_e}{n_f}(\widetilde{P}_{qq}-\widetilde{P}^+)\right]\Delta_\Sigma \\
&+& 2[(n_u-n_d)P_{qg}+\delta^2_e\widetilde{P}_{qg}]g+2\delta^2_e\widetilde{P}_{q\gamma}\gamma
\end{array}\right.$$

$$\begin{array}{rcl}
\mu^2\dfrac{\partial T^u_{1,2}}{\partial\mu^2} &=& (P^++e^2_u\widetilde{P}^+)T^u_{1,2} \\
\mu^2\dfrac{\partial T^d_{1,2}}{\partial\mu^2} &=& (P^++e^2_d\widetilde{P}^+)T^d_{1,2}
\end{array}$$

$$\left\{\begin{array}{rcl}
\mu^2\dfrac{\partial V}{\partial\mu^2} &=& (P^V+\eta^+\widetilde{P}^-)V+\eta^-\widetilde{P}^-\Delta_V \\
\mu^2\dfrac{\partial\Delta_V}{\partial\mu^2} &=& \left[\dfrac{n_u-n_d}{n_f}(P^V-P^-)+\eta^-\widetilde{P}^-\right]V+\left[P^-+\eta^+\widetilde{P}^-\right]\Delta_V
\end{array}\right.$$

$$\begin{array}{rcl}
\mu^2\dfrac{\partial V^u_{1,2}}{\partial\mu^2} &=& (P^-+e^2_u\widetilde{P}^-)V^u_{1,2} \\
\mu^2\dfrac{\partial V^d_{1,2}}{\partial\mu^2} &=& (P^-+e^2_d\widetilde{P}^-)V^d_{1,2}
\end{array} \tag{2.2}$$

with the definition:

$$\delta^2_e=N_c(n_ue^2_u-n_de^2_d) \tag{2.3}$$

and where we have used the curly bracket to denote the coupled equations. The main thing to notice here is the fact that there are two coupled sub-system. This is in contrast with what we had in pure QCD where there was only one coupled system.

Now, let's write the eq. (2.2) in a matricial form, separating the pure QCD splitting functions (those without tilde) from the QED contributions:

$$\begin{array}{rcl}
\mu^2\dfrac{\partial}{\partial\mu^2}\begin{pmatrix} g\\ \gamma\\ \Sigma\\ \Delta_\Sigma\end{pmatrix} &=& \left[\begin{pmatrix} P_{gg} & 0 & P_{gq} & 0\\ 0 & 0 & 0 & 0\\ 2n_fP_{qg} & 0 & P_{qq} & 0\\ \frac{n_u-n_d}{n_f}2n_fP_{qg} & 0 & \frac{n_u-n_d}{n_f}(P_{qq}-P^+) & P^+\end{pmatrix}\right. \\
&+& \left.\begin{pmatrix} e^2_\Sigma\widetilde{P}_{gg} & e^2_\Sigma\widetilde{P}_{g\gamma} & \eta^+\widetilde{P}_{gq} & \eta^-\widetilde{P}_{gq}\\ e^2_\Sigma\widetilde{P}_{\gamma g} & e^2_\Sigma\widetilde{P}_{\gamma\gamma} & \eta^+\widetilde{P}_{\gamma q} & \eta^-\widetilde{P}_{\gamma q}\\ 2e^2_\Sigma\widetilde{P}_{qg} & 2e^2_\Sigma\widetilde{P}_{q\gamma} & \eta^+\widetilde{P}^++\frac{\eta^+e^2_\Sigma}{n_f}(\widetilde{P}_{qq}-\widetilde{P}^+) & \eta^-\widetilde{P}^++\frac{\eta^-e^2_\Sigma}{n_f}(\widetilde{P}_{qq}-\widetilde{P}^+)\\ 2\delta^2_e\widetilde{P}_{qg} & 2\delta^2_e\widetilde{P}_{q\gamma} & \eta^-\widetilde{P}^++\frac{\eta^+\delta^2_e}{n_f}(\widetilde{P}_{qq}-\widetilde{P}^+) & \eta^+\widetilde{P}^++\frac{\eta^-\delta^2_e}{n_f}(\widetilde{P}_{qq}-\widetilde{P}^+)\end{pmatrix}\right]\begin{pmatrix} g\\ \gamma\\ \Sigma\\ \Delta_\Sigma\end{pmatrix}
\end{array} \tag{2.4}$$

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix}V\\ \Delta_V\end{pmatrix}=\left[\begin{pmatrix}P^V & 0\\ \frac{n_u-n_d}{n_f}(P^V-P^-) & P^-\end{pmatrix}+\begin{pmatrix}\eta^+\widetilde{P}^- & \eta^-\widetilde{P}^-\\ \eta^-\widetilde{P}^- & \eta^+\widetilde{P}^-\end{pmatrix}\right]\begin{pmatrix}V\\ \Delta_V\end{pmatrix} \tag{2.5}$$

It should finally be said that, every time one of one quark flavour is not active, the non-singlet distributions $T_{1,2}^{u,d}$ and $V_{1,2}^{u,d}$ involving that quark flavour, starts evolving as a singlet distribution according to the following equations:

$$\begin{aligned}T_{1,2}^u &= \frac{\Sigma+\Delta_\Sigma}{2}\,,\\ T_{1,2}^d &= \frac{\Sigma-\Delta_\Sigma}{2}\,,\\ V_{1,2}^u &= \frac{V+\Delta_V}{2}\,,\\ V_{1,2}^d &= \frac{V-\Delta_V}{2}\,.\end{aligned} \tag{2.6}$$

3 QED corrections at LO

If we consider only LO QED corrections to the PDF evolution equations, there are a few simplifications that make the evolution sistem simpler. In particular we have that:

$$\widetilde{P}_{gg}=\widetilde{P}_{g\gamma}=\widetilde{P}_{\gamma g}=\widetilde{P}_{gq}=\widetilde{P}_{qg}=0\,. \tag{3.1}$$

In addition:

$$\widetilde{P}^+=\widetilde{P}^-=\widetilde{P}_{qq}\,. \tag{3.2}$$

With these simplifications we can rewrite the above evolution systems as follows:

$$\begin{aligned}\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix}g\\ \gamma\\ \Sigma\\ \Delta_\Sigma\end{pmatrix} &= \left[\begin{pmatrix}P_{gg} & 0 & P_{gq} & 0\\ 0 & 0 & 0 & 0\\ 2n_fP_{qg} & 0 & P_{qq} & 0\\ \frac{n_u-n_d}{n_f}2n_fP_{qg} & 0 & \frac{n_u-n_d}{n_f}(P_{qq}-P^+) & P^+\end{pmatrix}\right.\\ &\left.+\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & e_\Sigma^2\widetilde{P}_{\gamma\gamma} & \eta^+\widetilde{P}_{\gamma q} & \eta^-\widetilde{P}_{\gamma q}\\ 0 & 2e_\Sigma^2\widetilde{P}_{q\gamma} & \eta^+\widetilde{P}_{qq} & \eta^-\widetilde{P}_{qq}\\ 0 & 2\delta_e^2\widetilde{P}_{q\gamma} & \eta^-\widetilde{P}_{qq} & \eta^+\widetilde{P}_{qq}\end{pmatrix}\right]\begin{pmatrix}g\\ \gamma\\ \Sigma\\ \Delta_\Sigma\end{pmatrix}\end{aligned} \tag{3.3}$$

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix}V\\ \Delta_V\end{pmatrix}=\left[\begin{pmatrix}P^V & 0\\ \frac{n_u-n_d}{n_f}(P^V-P^-) & P^-\end{pmatrix}+\begin{pmatrix}\eta^+\widetilde{P}_{qq} & \eta^-\widetilde{P}_{qq}\\ \eta^-\widetilde{P}_{qq} & \eta^+\widetilde{P}_{qq}\end{pmatrix}\right]\begin{pmatrix}V\\ \Delta_V\end{pmatrix} \tag{3.4}$$

$$\begin{aligned}\mu^2\frac{\partial T_{1,2}^u}{\partial\mu^2} &= (P^++e_u^2\widetilde{P}_{qq})T_{1,2}^u\\ \mu^2\frac{\partial T_{1,2}^d}{\partial\mu^2} &= (P^++e_d^2\widetilde{P}_{qq})T_{1,2}^d\\ \mu^2\frac{\partial V_{1,2}^u}{\partial\mu^2} &= (P^-+e_u^2\widetilde{P}_{qq})V_{1,2}^u\\ \mu^2\frac{\partial V_{1,2}^d}{\partial\mu^2} &= (P^-+e_d^2\widetilde{P}_{qq})V_{1,2}^d\end{aligned} \tag{3.5}$$

Notice that eq. (3.3), recognizing that $2e_\Sigma^2=\theta^-$ and $2\delta_e^2=\theta^+$, is consistent with eq. (9) of the APFEL paper.

4 Including the Lepton PDFs

In order to include the lepton PDFs in the coupled QCD×QED DGLAP evolution we first need to make some preliminary considerations. The first thing to note is the fact that, considering only QED corrections and not electroweak corrections, we do not need to introduce neutrino PDFs as neutrinos do not couple neither to the gluon nor to the photon. Thefore the only PDFs that need to be introduced are those of the charged leptons $e^{\pm}$, $\mu^{\pm}$ and $\tau^{\pm}$. The second point to consider is that the absolute value of the charge of all leptons is always equal to one and, since charges enter the DGLAP evolution as squares, this allows us to maintain, at least in the leptonic sector, the isospin symmetry $l^+ \leftrightarrow l^-$. As a final remark, we notice that, while the muon and the electron mass, $\simeq 0.5$ MeV and $\simeq 105$ MeV respectively, are below the $\Lambda_{\rm QCD}$ and thus they do not introduce any threshold in the DGLAP evolution, the tauon mass, whose mass is $m_\tau = 1.777$ GeV, is well above $\Lambda_{\rm QCD}$ and above the initial scale at which PDFs are usually parametized ($Q_0 = 1 - 1.4$ GeV). As a consequence, the presence of tauons in the evolution implies the introduction of a new threshold between m_c and m_b at which the τ PDFs are dynamically generated from the photon. On the contrary, e and μ PDFs cannot be dynamically generated by evolution and need to be parametrized at the initial scale. We will see later how the e and μ PDF functional form at the initial scale can be guessed by assuming a dynamical generation by photon splitting at their respective mass thresholds.

In order to write the full DGLAP equations in the presence of quarks, leptons, gluon and photon, we start considering eq. (1.8) where we add the leptons ℓ_α and $\bar{\ell}_\beta$:

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix}\ell_\alpha\\ u_j\\ d_i\\ g\\ \gamma\\ \bar{d}_k\\ \bar{u}_h\\ \bar{\ell}_\beta\end{pmatrix} = \sum_{e,l,m,n,\gamma,\delta}\begin{pmatrix}
\mathcal{P}_{\ell_\alpha\ell_\gamma} & \mathcal{P}_{\ell_\alpha u_e} & \mathcal{P}_{\ell_\alpha d_l} & \mathcal{P}_{\ell_\alpha g} & \mathcal{P}_{\ell_\alpha\gamma} & \mathcal{P}_{\ell_\alpha\bar{d}_m} & \mathcal{P}_{\ell_\alpha\bar{u}_n} & \mathcal{P}_{\ell_\alpha\bar{\ell}_\delta}\\
\mathcal{P}_{u_j\ell_\gamma} & \mathcal{P}_{u_j u_e} & \mathcal{P}_{u_j d_l} & \mathcal{P}_{u_j g} & \mathcal{P}_{u_j\gamma} & \mathcal{P}_{u_j\bar{d}_m} & \mathcal{P}_{u_j\bar{u}_n} & \mathcal{P}_{u_j\bar{\ell}_\delta}\\
\mathcal{P}_{d_i\ell_\gamma} & \mathcal{P}_{d_i u_e} & \mathcal{P}_{d_i d_l} & \mathcal{P}_{d_i g} & \mathcal{P}_{d_i\gamma} & \mathcal{P}_{d_i\bar{d}_m} & \mathcal{P}_{d_i\bar{u}_n} & \mathcal{P}_{d_i\bar{\ell}_\delta}\\
\mathcal{P}_{g\ell_\gamma} & \mathcal{P}_{g u_e} & \mathcal{P}_{g d_l} & \mathcal{P}_{gg} & \mathcal{P}_{g\gamma} & \mathcal{P}_{g\bar{d}_m} & \mathcal{P}_{g\bar{u}_n} & \mathcal{P}_{g\bar{\ell}_\delta}\\
\mathcal{P}_{\gamma\ell_\gamma} & \mathcal{P}_{\gamma u_e} & \mathcal{P}_{\gamma d_l} & \mathcal{P}_{\gamma g} & \mathcal{P}_{\gamma\gamma} & \mathcal{P}_{\gamma\bar{d}_m} & \mathcal{P}_{\gamma\bar{u}_n} & \mathcal{P}_{\gamma\bar{\ell}_\delta}\\
\mathcal{P}_{\bar{d}_k\ell_\gamma} & \mathcal{P}_{\bar{d}_k u_e} & \mathcal{P}_{\bar{d}_k d_l} & \mathcal{P}_{\bar{d}_k g} & \mathcal{P}_{\bar{d}_k\gamma} & \mathcal{P}_{\bar{d}_k\bar{d}_m} & \mathcal{P}_{\bar{d}_k\bar{u}_n} & \mathcal{P}_{\bar{d}_k\bar{\ell}_\delta}\\
\mathcal{P}_{\bar{u}_h\ell_\gamma} & \mathcal{P}_{\bar{u}_h u_e} & \mathcal{P}_{\bar{u}_h d_l} & \mathcal{P}_{\bar{u}_h g} & \mathcal{P}_{\bar{u}_h\gamma} & \mathcal{P}_{\bar{u}_h\bar{d}_m} & \mathcal{P}_{\bar{u}_h\bar{u}_n} & \mathcal{P}_{\bar{u}_h\bar{\ell}_\delta}\\
\mathcal{P}_{\bar{\ell}_\beta\ell_\gamma} & \mathcal{P}_{\bar{\ell}_\beta u_e} & \mathcal{P}_{\bar{\ell}_\beta d_l} & \mathcal{P}_{\bar{\ell}_\beta g} & \mathcal{P}_{\bar{\ell}_\beta\gamma} & \mathcal{P}_{\bar{\ell}_\beta\bar{d}_m} & \mathcal{P}_{\bar{\ell}_\beta\bar{u}_n} & \mathcal{P}_{\bar{\ell}_\beta\bar{\ell}_\delta}
\end{pmatrix}\begin{pmatrix}\ell_\gamma\\ u_e\\ d_l\\ g\\ \gamma\\ \bar{d}_m\\ \bar{u}_n\\ \bar{\ell}_\delta\end{pmatrix} \tag{4.1}$$

where $\ell_\alpha, \ell_\gamma \in \{e^-, \mu^-, \tau^-\}$ and $\bar{\ell}_\beta, \bar{\ell}_\delta \in \{e^+, \mu^+, \tau^+\}$. Now we notice that up to two loops we have that $\mathcal{P}_{\ell_\alpha g} = \mathcal{P}_{\bar{\ell}_\beta g} = \mathcal{P}_{g\ell_\gamma} = \mathcal{P}_{g\bar{\ell}_\delta} = 0$. In additions, all splitting functions connecting quarks and gluons like $\mathcal{P}_{\ell_\alpha u_e}$ start at $\mathcal{O}(\alpha^2)$, that is at two loops and thus do not have any pure-QCD contribution. Finally, the splitting functions connecting leptons with leptons and photon with leptons start at one loop but they are $\mathcal{O}(\alpha)$ and thus do not have any pure-QCD contribution. In additions, it is easy to realize that such splitting functions do not get QCD corrections up to 3 loops. In conclusion, the pure-QCD matrix in the r.h.s. of eq. (1.11) is untouched by the inclusion of the leptons in the evolution and only the QED-correction matrix gets contributions.

Based on these considerations, using the we can the usual charge conjugation invariance and flavour symmetry, up to two loops, one has that:

$$\begin{aligned}
&\mathcal{P}_{\ell_\alpha\ell_\beta} = \mathcal{P}_{\bar{\ell}_\alpha\bar{\ell}_\beta} = \delta_{ij}\mathcal{P}^V_{\ell\ell} + \mathcal{P}^S_{\ell\ell}\\
&\mathcal{P}_{\bar{\ell}_\alpha\ell_\beta} = \mathcal{P}_{\ell_\alpha\bar{\ell}_\beta} = \delta_{ij}\mathcal{P}^V_{\bar{\ell}\ell} + \mathcal{P}^S_{\ell\ell}\\
&\mathcal{P}_{\ell_\alpha\gamma} = \mathcal{P}_{\bar{\ell}_\alpha\gamma} = \mathcal{P}_{\ell\gamma}\\
&\mathcal{P}_{\gamma\ell_\alpha} = \mathcal{P}_{\gamma\bar{\ell}_\alpha} = \mathcal{P}_{\gamma\ell}\,.
\end{aligned} \tag{4.2}$$

In addition, since to this perturbative order they are pure QED splitting functions, they can be derived starting from the pure QCD splitting functions just by adjusting the color factors. Therefore, apart from replacing the strong coupling α_s with the fine-structure running constant α_s and assuming that the lepton charge is one, we can simply write:

$$\begin{aligned}
&\mathcal{P}^{V,S}_{\ell\ell} = P^{V,S}_{qq}(T_R = 1, C_F = 1, C_A = 0)\,,\\
&\mathcal{P}^{V}_{\bar{\ell}\ell} = P^{V}_{\bar{q}q}(T_R = 1, C_F = 1, C_A = 0)\,,\\
&\mathcal{P}_{\ell\gamma} = P_{qg}(T_R = 1, C_F = 1, C_A = 0)\,,\\
&\mathcal{P}_{\gamma\ell} = P_{gq}(T_R = 1, C_F = 1, C_A = 0)\,.
\end{aligned} \tag{4.3}$$

The last cathegory of splitting functions we need to cosider is that connecting quarks and leptons of the kind $\mathcal{P}_{\ell_\alpha q_i}$ or $\mathcal{P}_{q_i\ell_\alpha}$. As already mentioned, they start at two loops $\mathcal{O}(\alpha^2)$ and at this order they can be written as:

$$\mathcal{P}_{\ell_\alpha q_i} = \mathcal{P}_{q_i \ell_\alpha} = \mathcal{P}_{\bar{\ell}_\alpha q_i} = \mathcal{P}_{q_i \bar{\ell}_\alpha} = \mathcal{P}_{\ell_\alpha \bar{q}_i} = \mathcal{P}_{\bar{q}_i \ell_\alpha} = \mathcal{P}_{\bar{\ell}_\alpha \bar{q}_i} = \mathcal{P}_{\bar{q}_i \bar{\ell}_\alpha} = e_{q_i}^2 \mathcal{P}_{q\ell}^S \,. \tag{4.4}$$

My guess is that at $\mathcal{O}(\alpha\alpha_s)$ the $\mathcal{P}_{q\ell}^S$ can be obtained from the $\mathcal{O}(\alpha_s^2)$ contribution of the P_{qq} splitting functions just by adjusting the color factors. However, we will leave these consideration for a later study when the full two-loop splitting functions will be explicitly considered.

We can now generalize eq. (1.18) to include the presence of leptons in the DGLAP evolution into the system:

$$\begin{aligned}
\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} \ell_\alpha \\ u_j \\ d_i \\ g \\ \gamma \\ \bar{d}_i \\ \bar{u}_j \\ \bar{\ell}_\alpha \end{pmatrix} &= \left[\mathrm{QCD}_{\mathrm{NS}} + \mathrm{QED}_{\mathrm{NS}} + \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e_u^2 \widetilde{P}_{qq} & 0 & e_u^2 \widetilde{P}_{qg} & e_u^2 \widetilde{P}_{q\gamma} & 0 & 0 & 0 \\
0 & 0 & e_d^2 \widetilde{P}_{qq} & e_d^2 \widetilde{P}_{qg} & e_d^2 \widetilde{P}_{q\gamma} & 0 & 0 & 0 \\
0 & e_u^2 \widetilde{P}_{gq} & e_d^2 \widetilde{P}_{gq} & e_\Sigma^2 \widetilde{P}_{gg} & e_\Sigma^2 \widetilde{P}_{g\gamma} & e_d^2 \widetilde{P}_{gq} & e_u^2 \widetilde{P}_{gq} & 0 \\
0 & e_u^2 \widetilde{P}_{\gamma q} & e_d^2 \widetilde{P}_{\gamma q} & e_\Sigma^2 \widetilde{P}_{\gamma g} & e_\Sigma^2 \widetilde{P}_{\gamma\gamma} & e_d^2 \widetilde{P}_{\gamma q} & e_u^2 \widetilde{P}_{\gamma q} & 0 \\
0 & 0 & 0 & e_d^2 \widetilde{P}_{qg} & e_d^2 \widetilde{P}_{q\gamma} & e_d^2 \widetilde{P}_{qq}^V & 0 & 0 \\
0 & 0 & 0 & e_u^2 \widetilde{P}_{qg} & e_u^2 \widetilde{P}_{q\gamma} & 0 & e_u^2 \widetilde{P}_{qq}^V & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \right] \begin{pmatrix} \ell_\alpha \\ u_j \\ d_i \\ g \\ \gamma \\ \bar{d}_i \\ \bar{u}_j \\ \bar{\ell}_\alpha \end{pmatrix} \\
&+ \left[\mathrm{QCD}_{\mathrm{SG}} + \mathrm{QED}_{\mathrm{SG}} + \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e_u^4 \widetilde{P}_{qq}^S & e_u^2 e_d^2 \widetilde{P}_{qq}^S & 0 & 0 & e_u^2 e_d^2 \widetilde{P}_{qq}^S & e_u^4 \widetilde{P}_{qq}^S & 0 \\
0 & e_u^2 e_d^2 \widetilde{P}_{qq}^S & e_d^4 \widetilde{P}_{qq}^S & 0 & 0 & e_d^4 \widetilde{P}_{qq}^S & e_u^2 e_d^2 \widetilde{P}_{qq}^S & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e_u^2 e_d^2 \widetilde{P}_{qq}^S & e_d^4 \widetilde{P}_{qq}^S & 0 & 0 & e_d^4 \widetilde{P}_{qq}^S & e_u^2 e_d^2 \widetilde{P}_{qq}^S & 0 \\
0 & e_u^4 \widetilde{P}_{qq}^S & e_u^2 e_d^2 \widetilde{P}_{qq}^S & 0 & 0 & e_u^2 e_d^2 \widetilde{P}_{qq}^S & e_u^4 \widetilde{P}_{qq}^S & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \right] \begin{pmatrix} \sum_\gamma \ell_\gamma \\ \sum_e u_e \\ \sum_l d_l \\ g \\ \gamma \\ \sum_m \bar{d}_m \\ \sum_n \bar{u}_n \\ \sum_\delta \bar{\ell}_\delta \end{pmatrix}
\end{aligned} \tag{4.5}$$

where:

$$\mathrm{QED}_{\mathrm{NS}} = \begin{pmatrix}
\mathcal{P}_{\ell\ell}^V & 0 & 0 & 0 & \mathcal{P}_{\ell\gamma} & 0 & 0 & \mathcal{P}_{\bar{\ell}\ell}^V \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathcal{P}_{\gamma\ell} & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{P}_{\gamma\ell} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathcal{P}_{\bar{\ell}\ell}^V & 0 & 0 & 0 & \mathcal{P}_{\ell\gamma} & 0 & 0 & \mathcal{P}_{\ell\ell}^V
\end{pmatrix} \tag{4.6}$$

and:

$$\mathrm{QED}_{\mathrm{SG}} = \begin{pmatrix}
\mathcal{P}_{\ell\ell}^S & e_u^2 \mathcal{P}_{q\ell}^S & e_d^2 \mathcal{P}_{q\ell}^S & 0 & 0 & e_d^2 \mathcal{P}_{q\ell}^S & e_u^2 \mathcal{P}_{q\ell}^S & \mathcal{P}_{\ell\ell}^S \\
e_u^2 \mathcal{P}_{q\ell}^S & 0 & 0 & 0 & 0 & 0 & 0 & e_u^2 \mathcal{P}_{q\ell}^S \\
e_d^2 \mathcal{P}_{q\ell}^S & 0 & 0 & 0 & 0 & 0 & 0 & e_d^2 \mathcal{P}_{q\ell}^S \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e_d^2 \mathcal{P}_{q\ell}^S & 0 & 0 & 0 & 0 & 0 & 0 & e_d^2 \mathcal{P}_{q\ell}^S \\
e_u^2 \mathcal{P}_{q\ell}^S & 0 & 0 & 0 & 0 & 0 & 0 & e_u^2 \mathcal{P}_{q\ell}^S \\
\mathcal{P}_{\ell\ell}^S & e_u^2 \mathcal{P}_{q\ell}^S & e_d^2 \mathcal{P}_{q\ell}^S & 0 & 0 & e_d^2 \mathcal{P}_{q\ell}^S & e_u^2 \mathcal{P}_{q\ell}^S & \mathcal{P}_{\ell\ell}^S
\end{pmatrix} \tag{4.7}$$

Now, using the tranformation:

$$\mathbf{T} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} \quad \Longrightarrow \quad \mathbf{T}^{-1} = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} \tag{4.8}$$

For the non-leptonic parts, we get again the system in eq. (1.20) while for the leptonic parts, defining:

$$
\begin{aligned}
\mathcal{P}^{\pm} &\equiv \mathcal{P}^{V}_{\ell\ell} \pm \mathcal{P}^{V}_{\ell\bar{\ell}} \\
\mathcal{P}_{\ell\ell} &\equiv P^{+} + 2n_\ell \mathcal{P}^{S}_{\ell\ell} \quad , \\
\mathcal{P}^{V} &\equiv P^{-}
\end{aligned}
\tag{4.9}
$$

being n_ℓ the number of active leptons, we have that:

$$
\mathbf{T}\mathrm{QED}_{\mathrm{NS}}\mathbf{T}^{-1} = \begin{pmatrix}
\mathcal{P}^{+} & 0 & 0 & 0 & 2\mathcal{P}_{\ell\gamma} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathcal{P}_{\gamma\ell} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{P}^{-}
\end{pmatrix}
\tag{4.10}
$$

and:

$$
\mathbf{T}\mathrm{QED}_{\mathrm{SG}}\mathbf{T}^{-}1 = \frac{1}{n_\ell}\begin{pmatrix}
(\mathcal{P}_{\ell\ell}-\mathcal{P}^{+}) & 2n_\ell e_u^2 \mathcal{P}^{S}_{q\ell} & 2n_\ell e_d^2 \mathcal{P}^{S}_{q\ell} & 0 & 0 & 0 & 0 & 0 \\
2n_\ell e_u^2 \mathcal{P}^{S}_{q\ell} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2n_\ell e_d^2 \mathcal{P}^{S}_{q\ell} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\tag{4.11}
$$

In conclusion, the single equations are:

$$
\begin{aligned}
\mu^2 \frac{\partial \ell^{+}_{\alpha}}{\partial \mu^2} &= \mathcal{P}^{+}\ell^{+}_{\alpha} + \frac{1}{n_\ell}(\mathcal{P}_{\ell\ell}-\mathcal{P}^{+})\Sigma_\ell + 2\mathcal{P}^{S}_{q\ell}(\eta^{+}\Sigma + \eta^{-}\Delta_\Sigma) + 2\mathcal{P}_{\ell\gamma}\gamma \\
\mu^2 \frac{\partial \ell^{-}_{\alpha}}{\partial \mu^2} &= \mathcal{P}^{-}\ell^{-}_{\alpha} \\
\mu^2 \frac{\partial u^{+}_{j}}{\partial \mu^2} &= \cdots + 2e_u^2 \mathcal{P}^{S}_{q\ell}\Sigma_\ell \\
\mu^2 \frac{\partial d^{+}_{i}}{\partial \mu^2} &= \cdots + 2e_d^2 \mathcal{P}^{S}_{q\ell}\Sigma_\ell \\
\mu^2 \frac{\partial \gamma}{\partial \mu^2} &= \cdots + \mathcal{P}_{\gamma\ell}\Sigma_\ell
\end{aligned}
\tag{4.12}
$$

where, as usual, we have defined:

$$
\ell^{\pm}_{\alpha} = \ell_\alpha \pm \bar{\ell}_\alpha \, , \tag{4.13}
$$

and:

$$
\eta^{\pm} = \frac{1}{2}(e_u^2 \pm e_d^2) \, . \tag{4.14}
$$

Now, let us consider the following combinations:

$$\begin{aligned}
\Sigma_\ell &= \sum_{\alpha=e,\mu,\tau} \ell_\alpha^+ \\
V_\ell &= \sum_{\alpha=e,\mu,\tau} \ell_\alpha^- \\
T_3^\ell &= \ell_e^+ - \ell_\mu^+ \\
T_8^\ell &= \ell_e^+ + \ell_\mu^+ - 2\ell_\tau^+ \\
V_3^\ell &= \ell_e^- - \ell_\mu^- \\
V_8^\ell &= \ell_e^- + \ell_\mu^- - 2\ell_\tau^-
\end{aligned} \tag{4.15}$$

It is easy to see that above the τ mass threshold, *i.e.* where $n_\ell = 3$, they evolve according to the following equations:

$$\begin{aligned}
\mu^2 \frac{\partial \Sigma_\ell}{\partial \mu^2} &= 2n_\ell \mathcal{P}_{\ell\gamma}\gamma + 2n_\ell \mathcal{P}_{q\ell}^S(\eta^+\Sigma + \eta^-\Delta_\Sigma) + \mathcal{P}_{\ell\ell}\Sigma_\ell \\
\mu^2 \frac{\partial V_\ell}{\partial \mu^2} &= \mathcal{P}^- V_\ell = \mathcal{P}^V V_\ell \\
\mu^2 \frac{\partial T_{3,8}^\ell}{\partial \mu^2} &= \mathcal{P}^+ T_{3,8}^\ell \\
\mu^2 \frac{\partial V_{3,8}^\ell}{\partial \mu^2} &= \mathcal{P}^- V_{3,8}^\ell
\end{aligned} \tag{4.16}$$

In addition, the photon and QCD singlet distributions Σ and Δ_Σ acquire the following terms:

$$\begin{aligned}
\mu^2 \frac{\partial \Sigma}{\partial \mu^2} &= \cdots + 2e_\Sigma^2 \mathcal{P}_{q\ell}^S \Sigma_\ell \\
\mu^2 \frac{\partial \Delta_\Sigma}{\partial \mu^2} &= \cdots + 2\delta_e^2 \mathcal{P}_{q\ell}^S \Sigma_\ell \\
\mu^2 \frac{\partial \gamma}{\partial \mu^2} &= \cdots + \mathcal{P}_{\gamma\ell}\Sigma_\ell
\end{aligned} \tag{4.17}$$

In conclusion, the full system of equations in the evolution basis including leptons is the following:

$$\begin{aligned}
\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} g \\ \gamma \\ \Sigma \\ \Delta_\Sigma \\ \Sigma_\ell \end{pmatrix} = & \left[\begin{pmatrix} P_{gg} & 0 & P_{gq} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2n_f P_{qg} & 0 & P_{qq} & 0 & 0 \\ \frac{n_u - n_d}{n_f} 2n_f P_{qg} & 0 & \frac{n_u-n_d}{n_f}(P_{qq} - P^+) & P^+ & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right. \\
& + \begin{pmatrix} e_\Sigma^2 \widetilde{P}_{gg} & e_\Sigma^2 \widetilde{P}_{g\gamma} & \eta^+ \widetilde{P}_{gq} & \eta^- \widetilde{P}_{gq} & 0 \\ e_\Sigma^2 \widetilde{P}_{\gamma g} & e_\Sigma^2 \widetilde{P}_{\gamma\gamma} & \eta^+ \widetilde{P}_{\gamma q} & \eta^- \widetilde{P}_{\gamma q} & 0 \\ 2e_\Sigma^2 \widetilde{P}_{qg} & 2e_\Sigma^2 \widetilde{P}_{q\gamma} & \eta^+ \widetilde{P}^+ + \frac{\eta^+ e_\Sigma^2}{n_f}(\widetilde{P}_{qq} - \widetilde{P}^+) & \eta^- \widetilde{P}^+ + \frac{\eta^- e_\Sigma^2}{n_f}(\widetilde{P}_{qq} - \widetilde{P}^+) & 0 \\ 2\delta_e^2 \widetilde{P}_{qg} & 2\delta_e^2 \widetilde{P}_{q\gamma} & \eta^- \widetilde{P}^+ + \frac{\eta^+ \delta_e^2}{n_f}(\widetilde{P}_{qq} - \widetilde{P}^+) & \eta^+ \widetilde{P}^+ + \frac{\eta^- \delta_e^2}{n_f}(\widetilde{P}_{qq} - \widetilde{P}^+) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
& + \left. \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{P}_{\gamma\ell} \\ 0 & 0 & 0 & 0 & 2e_\Sigma^2 \mathcal{P}_{q\ell}^S \\ 0 & 0 & 0 & 0 & 2\delta_e^2 \mathcal{P}_{q\ell}^S \Sigma_\ell \\ 0 & 2n_\ell \mathcal{P}_{\ell\gamma} & 2n_\ell \mathcal{P}_{q\ell}^S \eta^+ & 2n_\ell \mathcal{P}_{q\ell}^S \eta^- & \mathcal{P}_{\ell\ell} \end{pmatrix} \right] \begin{pmatrix} g \\ \gamma \\ \Sigma \\ \Delta_\Sigma \\ \Sigma_\ell \end{pmatrix}
\end{aligned} \tag{4.18}$$

$$\mu^2 \frac{\partial}{\partial\mu^2}\begin{pmatrix} V \\ \Delta_V \end{pmatrix} = \left[\begin{pmatrix} P^V & 0 \\ \frac{n_u-n_d}{n_f}(P^V - P^-) & P^- \end{pmatrix} + \begin{pmatrix} \eta^+\widetilde{P}^- & \eta^-\widetilde{P}^- \\ \eta^-\widetilde{P}^- & \eta^+\widetilde{P}^- \end{pmatrix}\right]\begin{pmatrix} V \\ \Delta_V \end{pmatrix} \tag{4.19}$$

$$\mu^2 \frac{\partial V_\ell}{\partial\mu^2} = \mathcal{P}^- V_\ell = \mathcal{P}^V V_\ell \tag{4.20}$$

$$\begin{aligned} \mu^2 \frac{\partial T^u_{1,2}}{\partial\mu^2} &= (P^+ + e_u^2 \widetilde{P}^+) T^u_{1,2} \\ \mu^2 \frac{\partial T^d_{1,2}}{\partial\mu^2} &= (P^+ + e_d^2 \widetilde{P}^+) T^d_{1,2} \end{aligned} \tag{4.21}$$

$$\begin{aligned} \mu^2 \frac{\partial V^u_{1,2}}{\partial\mu^2} &= (P^- + e_u^2 \widetilde{P}^-) V^u_{1,2} \\ \mu^2 \frac{\partial V^d_{1,2}}{\partial\mu^2} &= (P^- + e_d^2 \widetilde{P}^-) V^d_{1,2} \end{aligned} \tag{4.22}$$

$$\begin{aligned} \mu^2 \frac{\partial T^\ell_{3,8}}{\partial\mu^2} &= \mathcal{P}^+ T^\ell_{3,8} \\ \mu^2 \frac{\partial V^\ell_{3,8}}{\partial\mu^2} &= \mathcal{P}^- V^\ell_{3,8} \end{aligned} \tag{4.23}$$

4.1 Evolution Equations at LO

At LO in QED, considering that $\mathcal{P}^S_{q\ell} = 0$ and that $\mathcal{P}_{ij} = \widetilde{P}_{ij}$, the evolution equations above reduce to:

$$\begin{aligned} \mu^2 \frac{\partial}{\partial\mu^2}\begin{pmatrix} g \\ \gamma \\ \Sigma \\ \Delta_\Sigma \\ \Sigma_\ell \end{pmatrix} &= \left[\begin{pmatrix} P_{gg} & 0 & P_{gq} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2n_f P_{qg} & 0 & P_{qq} & 0 & 0 \\ \frac{n_u-n_d}{n_f} 2n_f P_{qg} & 0 & \frac{n_u-n_d}{n_f}(P_{qq} - P^+) & P^+ & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right. \\ &+ \left.\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & e_\Sigma^2 \widetilde{P}_{\gamma\gamma} & \eta^+\widetilde{P}_{\gamma q} & \eta^-\widetilde{P}_{\gamma q} & \widetilde{P}_{\gamma q} \\ 0 & 2e_\Sigma^2 \widetilde{P}_{q\gamma} & \eta^+\widetilde{P}_{qq} & \eta^-\widetilde{P}_{qq} & 0 \\ 0 & 2\delta_e^2 \widetilde{P}_{q\gamma} & \eta^-\widetilde{P}_{qq} & \eta^+\widetilde{P}_{qq} & 0 \\ 0 & 2n_\ell \widetilde{P}_{q\gamma} & 0 & 0 & \widetilde{P}_{qq} \end{pmatrix}\right]\begin{pmatrix} g \\ \gamma \\ \Sigma \\ \Delta_\Sigma \\ \Sigma_\ell \end{pmatrix} \end{aligned} \tag{4.24}$$

$$\mu^2 \frac{\partial}{\partial\mu^2}\begin{pmatrix} V \\ \Delta_V \end{pmatrix} = \left[\begin{pmatrix} P^V & 0 \\ \frac{n_u-n_d}{n_f}(P^V - P^-) & P^- \end{pmatrix} + \begin{pmatrix} \eta^+\widetilde{P}_{qq} & \eta^-\widetilde{P}_{qq} \\ \eta^-\widetilde{P}_{qq} & \eta^+\widetilde{P}_{qq} \end{pmatrix}\right]\begin{pmatrix} V \\ \Delta_V \end{pmatrix} \tag{4.25}$$

$$\mu^2 \frac{\partial V_\ell}{\partial\mu^2} = \widetilde{P}_{qq} V_\ell \tag{4.26}$$

$$\begin{aligned} \mu^2 \frac{\partial T^u_{1,2}}{\partial\mu^2} &= (P^+ + e_u^2 \widetilde{P}_{qq}) T^u_{1,2} \\ \mu^2 \frac{\partial T^d_{1,2}}{\partial\mu^2} &= (P^+ + e_d^2 \widetilde{P}_{qq}) T^d_{1,2} \\ \mu^2 \frac{\partial V^u_{1,2}}{\partial\mu^2} &= (P^- + e_u^2 \widetilde{P}_{qq}) V^u_{1,2} \\ \mu^2 \frac{\partial V^d_{1,2}}{\partial\mu^2} &= (P^- + e_d^2 \widetilde{P}_{qq}) V^d_{1,2} \end{aligned} \tag{4.27}$$

$$\mu^2 \frac{\partial T^\ell_{3,8}}{\partial \mu^2} = \widetilde{P}_{qq} T^\ell_{3,8}$$
$$\mu^2 \frac{\partial V^\ell_{3,8}}{\partial \mu^2} = \widetilde{P}_{qq} V^\ell_{3,8} \tag{4.28}$$

There is one last detail to be discussed. Contraty to electrons and muons, whose masses are well below $\Lambda_{\rm QCD}$, the τ has a mass equal to $m_\tau = 1.777$ GeV which is well above $\Lambda_{\rm QCD}$ and even above the typical initial scale $Q_0 \simeq 1$ GeV from which PDFs are usually evolved. As a consequence, we need to account for the possibility to cross the τ mass threshold. To do so, we just need to realize that below the τ threshold, where $n_\ell = 2$, the T^ℓ_8 and V^ℓ_8 reduce to the lepton singlet Σ_ℓ and total valence V_ℓ distributions and and thus evolve as such.

From the implementation point of view, the main problem is the fact that we need to introduce a new threshold between the charm and the bottom thresholds and this will complicate the structure of the code.