Intrinsic Charm Implementation

Valerio Bertone[a]

[a]**PH Department, TH Unit, CERN, CH-1211 Geneva 23, Switzerland**

Abstract

In these set of notes I will describe the strategy to include the intrinsic charm (IC) contribution to the FONLL structure functions as implemented in APFEL. I will first consider the massive sector (and its massless limit), where the IC implies the presence of the charm in the initial state with the consequence of additional diagrams to be include in the computation. I will the consider the massless sector where the presence of an IC implies a retratment of the PDF matching conditions at the charm threshold.

Contents

1 Intrinsic Charm Contribution to the Massive Structure Functions

Assuming the presence of IC in the proton, the massive structure functions with $N_f = 3$ light flavours acquire a further contribution coming from the presence of a massive charm in the initial state. As a consequence, the massive structure functions get a term that is proportional to a *static* charm PDF, *i.e.* a PDF that, being massive, does not evolve according to the DGLAP equantion. Such a contribution starts already at order α_s^0 and has the novel effect to "allign" the massive scheme to the massless scheme in terms of power counting because, contrary to what happens without IC, the two sectors start at α_s^0.

1.1 Order α_s^0 Contributions

In order to write explicitly the form of such LO contributions to the DIS structure functions, I consider eq. (2) of Ref. [1] where the function Q_1 schould be indentified with the charm PDF. It should be noticed that in the $N_f = 3$ scheme, such PDF does not obey the DGLAP equation because, due to the presence of the mass of the charm m_c, no large collinear logarithms appear in the calculation and thus there is no need to resum them.

From eq. (2) of [1] one reads that the $\mathcal{O}(\alpha_s^0)$ IC contributions to the massive structrure functions are given by:

$$F_1^{\rm FF,IC}(x,Q^2) = \frac{S_+\Sigma_{++} - 2m_1m_2S_-}{2\Delta}c(\chi) \tag{1.1a}$$

$$F_2^{\rm FF,IC}(x,Q^2) = \frac{S_+\Delta}{2Q^2}2xc(\chi) \tag{1.1b}$$

$$xF_3^{\rm FF,IC}(x,Q^2) = 2R_+xc(\chi) \tag{1.1c}$$

where m_1 and m_2 are the masses of the incoming and outcoming quarks, respectively, while $\Delta \equiv \Delta(m_1^2, m_2^2, -Q^2)$ with the function Δ defined as:

$$\Delta(a,b,c) = \sqrt{a^2 + b^2 + c^2 - 2(ab + ac + bc)} \tag{1.2}$$

and:

$$\Sigma_{\pm\pm} = Q^2 \pm m_2^2 \pm m_1^2 \tag{1.3}$$

$$\chi = \frac{x}{2Q^2}(\Sigma_{+-} + \Delta) \tag{1.4}$$

The quantities $S_\pm$ and $R_\pm$, instead, are linked to the EW couplings and depend on the vector boson that strikes the heavy quark with mass m_1 in the initial state. Notice that in eq. (1.1) the PDF c does not depend on any factorization scale and, as mentioned before, the reason is that it is a static distribution of non-perturbative origin that does not evolve according to the DGLAP equation.

In practice, assuming the presence of IC in the proton, the massive (FF) structure functions become:

$$F_i^{\mathrm{FF}}(x,Q^2) \longrightarrow F_i^{\mathrm{FF}}(x,Q^2) + F_i^{\mathrm{FF,IC}}(x,Q^2) \quad \text{with} \quad i = 1,2,3 \tag{1.5}$$

Now, for a purely *electromagnetic* process, where only a γ strikes the charm, one has:

$$S_+ = S_- = e_c^2 \quad \text{and} \quad R_+ = 0 \tag{1.6}$$

Moreover, in this case both the incoming and the outcoming quarks are of the same flavour (charm) therefore we have $m_1 = m_2 = m_c$. Under this conditions one finds:

$$F_1^{\mathrm{FF,IC}}(x,Q^2) = \frac{1}{2\sqrt{1+4\lambda}} e_c^2 c(\chi) \tag{1.7a}$$

$$F_2^{\mathrm{FF,IC}}(x,Q^2) = \left(\sqrt{1+4\lambda}\right) e_c^2 x c(\chi) \tag{1.7b}$$

$$xF_3^{\mathrm{FF,IC}}(x,Q^2) = 0 \tag{1.7c}$$

with:

$$\chi = \frac{x}{2}\left(1+\sqrt{1+4\lambda}\right) = \frac{x}{\eta}\,, \tag{1.8}$$

where I have defined:

$$\eta = \frac{2Q^2}{\Sigma_{+-}+\Delta} = 2\left(1+\sqrt{1+4\lambda}\right)^{-1} = 2\left(1+\sqrt{1+4\lambda}\right)^{-1}\,, \tag{1.9}$$

with $\lambda = \frac{m_c^2}{Q^2}$.

For a *neutral current* process, where all the γ, the Z and the interference γZ contributions are considered, one has:

$$S_\pm = B_c(\tilde{B}_c) = e_c^2 - 2e_c V_e V_c P_Z + (V_e^2 + A_e^2)(V_c^2 \pm A_c^2)P_Z^2 \quad \text{and} \quad R_+ = D_c = -2e_c A_c A_e P_Z + 4V_c A_c V_e A_e P_Z^2 \tag{1.10}$$

with:

$$V_c = \frac{1}{2} - \frac{4}{3}\sin^2\theta_W \quad \text{and} \quad A_c = \frac{1}{2} \tag{1.11}$$

and

$$V_e = -\frac{1}{2} + 2\sin^2\theta_W \quad \text{and} \quad A_e = -\frac{1}{2} \tag{1.12}$$

the vector and the axial coupling of charm and electron to the Z and where:

$$P_Z = \frac{1}{4\sin^2\theta_W(1-\sin^2\theta_W)}\frac{Q^2}{Q^2+M_Z^2} \tag{1.13}$$

Here, exactly as in the electromagnetic case, $m_1 = m_2 = m_c$ so that one ends up with:

$$F_1^{\mathrm{FF,IC}}(x,Q^2) = \frac{B_c + 2\lambda(B_c - \tilde{B}_c)}{2\sqrt{1+4\lambda}} c(\chi)\,, \tag{1.14a}$$

$$F_2^{\mathrm{FF,IC}}(x,Q^2) = \left(\sqrt{1+4\lambda}\right) B_c x c(\chi)\,, \tag{1.14b}$$

$$xF_3^{\mathrm{FF,IC}}(x,Q^2) = 2D_c x c(\chi)\,. \tag{1.14c}$$

Finally, for a *charged current* process, where a charged boson $W^\pm$ strikes the charm, one has:

$$S_+ = 2|V_{cs}|^2\,, \quad S_- = 0\,, \quad \text{and} \quad R_+ = |V_{cs}|^2 \tag{1.15}$$

if the outcoming quark is a strange or an anti-strange, and:

$$S_+ = 2|V_{cd}|^2\,, \quad S_- = 0\,, \quad \text{and} \quad R_+ = |V_{cd}|^2 \tag{1.16}$$

if the outcoming quark is a down or an anti-down.

In this case $m_1 = m_c$ but $m_2 = 0$ with the consequence that:

$$F_1^{\mathrm{FF,IC}}(x,Q^2) = |V_{cj}|^2 c(x) \tag{1.17a}$$

$$F_2^{\mathrm{FF,IC}}(x,Q^2) = 2\,(1+\lambda)\,|V_{cj}|^2 x c(x) \tag{1.17b}$$

$$xF_3^{\mathrm{FF,IC}}(x,Q^2) = 2|V_{cj}|^2 x c(x) \tag{1.17c}$$

with $j = d, s$. Note that in this case $\eta = 1$ and thus $\chi = x$.

In order to take into account the possible contributions due to intrinsic charm, one has to consider all diagrams contributing to a given process. As far as the neutral current (electromagnetic) case is concerned, one has to consider also the presence of $\overline{c}$ in the proton which, summed to the contribution of the c, gives:

$$F_1^{\mathrm{FF,IC}}(x,Q^2) = \frac{1}{2\sqrt{1+4\lambda}} B_c c^+(\chi) = \frac{1}{2x} B_c \frac{\eta^2}{2-\eta}\left[1 + \frac{2(1-\eta)}{\eta^2}\left(1 - \frac{\tilde{B}_c}{B_c}\right)\right]\chi c^+(\chi) \tag{1.18a}$$

$$F_2^{\mathrm{FF,IC}}(x,Q^2) = \frac{2\sqrt{1+4\lambda}}{1+\sqrt{1+4\lambda}} B_c \chi c^+(\chi) = (2-\eta) B_c \chi c^+(\chi) \tag{1.18b}$$

$$xF_3^{\mathrm{FF,IC}}(x,Q^2) = \frac{4}{1+\sqrt{1+4\lambda}} D_c \chi c^-(\chi) = 2\eta D_c \chi c^-(\chi) \tag{1.18c}$$

where:

$$c^\pm = c \pm \overline{c} \tag{1.19}$$

therefore:

$$F_L^{\mathrm{FF,IC}}(x,Q^2) = F_2^{\mathrm{FF,IC}}(x,Q^2) - 2xF_1^{\mathrm{FF,IC}}(x,Q^2) = \frac{1}{2}\left(1 + \frac{\tilde{B}_c}{B_c}\right) 4\frac{1-\eta}{2-\eta} B_c \chi c^+(\chi) \tag{1.20}$$

In the charged current case, instead, one has to distinguish between neutrino and anti-neutrino scattering. The neutrino scattering gives as a result the following structure functions:

$$F_1^{\nu,\mathrm{FF,IC}}(x,Q^2) = (|V_{cd}|^2 + |V_{cs}|^2)\overline{c}(x) \tag{1.21a}$$

$$F_2^{\nu,\mathrm{FF,IC}}(x,Q^2) = 2\,(1+\lambda)\,(|V_{cd}|^2 + |V_{cs}|^2) x\overline{c}(x) \tag{1.21b}$$

$$xF_3^{\nu,\mathrm{FF,IC}}(x,Q^2) = 2(|V_{cd}|^2 + |V_{cs}|^2) x\overline{c}(x) \tag{1.21c}$$

$$F_L^{\nu,\mathrm{FF,IC}}(x,Q^2) = 2\lambda(|V_{cd}|^2 + |V_{cs}|^2) x\overline{c}(x) \tag{1.21d}$$

The anti-neutrino scattering instead gives as a result the following structure functions:

$$F_1^{\overline{\nu},\mathrm{FF,IC}}(x,Q^2) = (|V_{cd}|^2 + |V_{cs}|^2) c(x) \tag{1.22a}$$

$$F_2^{\overline{\nu},\mathrm{FF,IC}}(x,Q^2) = 2\,(1+\lambda)\,(|V_{cd}|^2 + |V_{cs}|^2) x c(x) \tag{1.22b}$$

$$xF_3^{\overline{\nu},\mathrm{FF,IC}}(x,Q^2) = 2(|V_{cd}|^2 + |V_{cs}|^2) x c(x) \tag{1.22c}$$

$$F_L^{\overline{\nu},\mathrm{FF,IC}}(x,Q^2) = 2\lambda(|V_{cd}|^2 + |V_{cs}|^2) x c(x) \tag{1.22d}$$

It should be pointed out that since the charm quark belongs to the sea it is symmetric under isospin symmetry and thus all the above structure functions are the same for proton and neutron.

1.2 Massless Limit

When implementing the FONLL scheme, one also needs to consider the massless limit of the massive structuture functions (FF0). To this end, we just need to take the limit for $m_c \to 0$ of eqs. (1.24), (1.25) and (3.5). Considering that:

$$\eta \underset{m_c\to 0}{\longrightarrow} 1 \quad \Rightarrow \quad \chi \underset{m_c\to 0}{\longrightarrow} x\,, \tag{1.23}$$

one finds :

$$F_1^{\mathrm{FF0,IC}}(x,Q^2) = \frac{1}{2}B_c c^+(x) \tag{1.24a}$$

$$F_2^{\mathrm{FF0,IC}}(x,Q^2) = B_c x c^+(x) \tag{1.24b}$$

$$xF_3^{\mathrm{FF0,IC}}(x,Q^2) = 2D_c x c^-(x) \tag{1.24c}$$

$$F_L^{\mathrm{FF0,IC}}(x,Q^2) = 0 \tag{1.24d}$$

and:

$$F_1^{\nu,\mathrm{FF0,IC}}(x,Q^2) = (|V_{cd}|^2 + |V_{cs}|^2)\bar{c}(x) \tag{1.25a}$$

$$F_2^{\nu,\mathrm{FF0,IC}}(x,Q^2) = 2(|V_{cd}|^2 + |V_{cs}|^2)x\bar{c}(x) \tag{1.25b}$$

$$xF_3^{\nu,\mathrm{FF0,IC}}(x,Q^2) = 2(|V_{cd}|^2 + |V_{cs}|^2)x\bar{c}(x) \tag{1.25c}$$

$$F_L^{\nu,\mathrm{FF0,IC}}(x,Q^2) = 0 \tag{1.25d}$$

and:

$$F_1^{\overline{\nu},\mathrm{FF0,IC}}(x,Q^2) = (|V_{cd}|^2 + |V_{cs}|^2)c(x) \tag{1.26a}$$

$$F_2^{\overline{\nu},\mathrm{FF0,IC}}(x,Q^2) = 2(|V_{cd}|^2 + |V_{cs}|^2)xc(x) \tag{1.26b}$$

$$xF_3^{\overline{\nu},\mathrm{FF0,IC}}(x,Q^2) = 2(|V_{cd}|^2 + |V_{cs}|^2)xc(x) \tag{1.26c}$$

$$F_L^{\overline{\nu},\mathrm{FF0,IC}}(x,Q^2) = 0 \tag{1.26d}$$

1.3 Order α_s Contributions

We can now turn to describe the NLO contributions to the IC component of the DIS structure functions. The explicit expressions can be found in Appendix C of Ref. [1]. The main difficulty here is the fact that are particularly involved and it is not possible to identify the singular terms from the regular ones. As a consequence, in this case we have to adopt a different strategy.

As we know, when convoluting a coefficient function with a PDF, say, $f(y)$ interpolated over an x-space grid to obtaing the structure function $F(x,m_1,m_2)$ the resulting expression is([1]):

$$F(x_\beta,m_1,m_2) = x_\beta \int_{x_\beta}^1 \frac{dy}{y}\widetilde{C}\left(\frac{x_\beta}{y},m_1,m_2\right) f(y) = \sum_{\alpha=0}^{N_x} \Gamma_{\beta\alpha}(m_1,m_2)x_\alpha f(x_\alpha,t)\,, \tag{1.27}$$

with:

$$\Gamma_{\beta\alpha}(m_1,m_2) = \int_c^d dy\,\widetilde{C}\,(y,m_1,m_2)\,w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right)\,, \tag{1.28}$$

where $w_\alpha^{(k)}$ is the order-k interpolation functions on the grid node α and:

$$c = \max(x_\beta, x_\beta/x_{\alpha+1}) \quad \text{and} \quad d = \min(1, x_\beta/x_{\alpha-k})\,. \tag{1.29}$$

On the other hand, as it usually happens when mass effects are taken into account, the phase-space available to the process gets reduced and this is reflected by the fact that the convolution between coeffient functions and PDFs needed to obtain the structure functions takes the following form:

$$F(x) = x\int_\chi^1 \frac{dz}{z}C(z)f\left(\frac{\chi}{z}\right) = x\int_\chi^1 \frac{dz}{z}C\left(\frac{\chi}{z}\right)f(z)\,, \tag{1.30}$$

[1] Note that the factor x_β in front of the intergral in eq. (1.27) is not always included in the definition of the structure functions. In particular, while F_2 and F_L include it, F_1 and and F_3 do not so we need to keep it in mind in what follows.

where $\chi = x/\eta$ with $\eta \leq 1$ given in eq. (1.9) and where we have dropped all the unnecessary mass dependencies. Usually, expressions for the coefficient functions are given in this form. However, in order to write this integral in the form given in eq. (1.27), that is in such a way that lower bound of the integral is not the rescaled variable χ but the physical Bjorken x, one needs to perform the change of variable $y = \eta z$, so that:

$$F(x) = x \int_x^\eta \frac{dy}{y} C\left(\frac{y}{\eta}\right) f\left(\frac{x}{y}\right) = x \int_x^1 \frac{dy}{y} \widetilde{C}(y) f\left(\frac{x}{y}\right) , \tag{1.31}$$

with:

$$\widetilde{C}(y) \equiv \theta\left(1 - \frac{y}{\eta}\right) C\left(\frac{y}{\eta}\right) . \tag{1.32}$$

The kind of expressions we have to deal with have this apparently simple form:

$$C(z) = \frac{R(z)}{(1-z)_+} + L\delta(1-z) , \tag{1.33}$$

where R is a regular function in $z = 1$, that is:

$$R(1) = \lim_{z\to 1} R(z) = K , \tag{1.34}$$

K being a finite function of the masses, and L is also constant in z and depends only on the masses.

Now, plugging eq. (1.32) into eq. (1.28) and taking into account eq. (1.33), we have that:

$$\begin{aligned}
\Gamma_{\beta\alpha} &= \int_c^d dy\, \theta\left(1 - \frac{y}{\eta}\right) C\left(\frac{y}{\eta}\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) = \int_c^{\bar{d}} dy\, C\left(\frac{y}{\eta}\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) \\
&= \int_c^{\bar{d}} dy \left[\frac{1}{(1-y/\eta)_+} R\left(\frac{y}{\eta}\right) + \delta\left(1 - \frac{y}{\eta}\right) L\right] w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) \\
&= \int_c^{\bar{d}} dy \frac{1}{1-y/\eta} \left[R\left(\frac{y}{\eta}\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) - R(1) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right) \theta(\bar{d} - \eta)\right] \\
&+ \left[R(1) \ln\left(1 - \frac{c}{\eta}\right) \theta(\bar{d} - \eta) + L\right] \eta w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right) .
\end{aligned} \tag{1.35}$$

where we have redefined:

$$\bar{d} = \min(\eta, x_\beta / x_{\alpha-k}) . \tag{1.36}$$

Finally, changing the integration variable in $z = y/\eta$, we have:

$$\begin{aligned}
\Gamma_{\beta\alpha} &= \eta \int_{c/\eta}^{\bar{d}/\eta} dz \frac{1}{1-z} \left[R(z)\, w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right) - R(1) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right) \theta(\bar{d} - \eta)\right] \\
&+ \eta \left[R(1) \ln\left(1 - \frac{c}{\eta}\right) \theta(\bar{d} - \eta) + L\right] w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right) .
\end{aligned} \tag{1.37}$$

As a remark, it should be noted that the Heaviside function $\theta(\bar{d} - \eta)$ is indeed superflous because the same condition is guaranteed by the form of the interpolation functions. As a consequence, we can simply write:

$$\begin{aligned}
\Gamma_{\beta\alpha} &= \eta \int_{c/\eta}^{\bar{d}/\eta} dz \frac{1}{1-z} \left[R(z)\, w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right) - R(1) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right)\right] \\
&+ \eta \left[R(1) \ln\left(1 - \frac{c}{\eta}\right) + L\right] w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right) .
\end{aligned} \tag{1.38}$$

Explicit expressions for the functions R and L for the structure functions xF_1, F_2 and xF_3 can be extracted Appendix C of Ref. [1]. The relative expressions for F_L can be constructed using the usual relation:

$$F_L(x,Q) = F_2(x,Q) - 2xF_1(x,Q) . \tag{1.39}$$

Using the same notation of Ref. [1] and dropping an overall factor $\alpha_s/4\pi$, we can write the $\mathcal{O}(\alpha_s)$ contribution to the "reduced" structure functions as:

$$x\hat{\mathcal{F}}_i^{QS^{(1)}} = x\int_\chi^1 \frac{dz}{z} 2\hat{H}_i^q(z) c\left(\frac{\chi}{z}\right) \quad \text{with} \quad i = 1,2,3\,, \tag{1.40}$$

and:

$$2\hat{H}_i^q(z) = 2C_F\left[(S_i+V_i)\delta(1-z) + \frac{(1-z)}{(1-z)_+}\frac{\hat{s}-m_2^2}{8\hat{s}} N_i^{-1}\hat{f}_i^Q(z)\right]. \tag{1.41}$$

The standard structrure functions can be recovered just by applying the correct kinematic factors. We will do this in the next section where we will consider the NC and the CC sectors separately.

It is now easy to identify the function $C(z)$ in eq. (1.30) with $2\hat{H}_i^q(z)$. As a consequence, we also have that the function $\widetilde{C}(y)$ in eq. (1.31) is equal to:

$$\widetilde{C}(y) = \theta\left(1-\frac{y}{\eta}\right) 2\hat{H}_i^q\left(\frac{y}{\eta}\right). \tag{1.42}$$

Finally, comparing eq. (1.41) to eq. (1.33), we can easily make the following identifications:

$$\begin{aligned} R(z) &= 2C_F(1-z)\frac{\hat{s}-m_2^2}{8\hat{s}} N_i^{-1}\hat{f}_i^Q(z)\,, \\ L &= 2C_F(S_i+V_i)\,. \end{aligned} \tag{1.43}$$

In the following, we will treat the NC and the CC cases separately, showing how to implement the $\mathcal{O}(\alpha_s)$ corrections to the relative structure functions keeping into account all the relevant kinematic factors. To do so, using eq (7) of Ref. [1], we write here the general correspondence between the standard and the reduced structure functions that holds at all orders in perturbation theory:

$$\begin{aligned} 2xF_1 &= S_+\left[\frac{\Sigma_{++} - 2m_1m_2\frac{S_-}{S_+}}{\Delta}\right] x\hat{\mathcal{F}}_1^{QS}\,, \\ F_2 &= S_+\left[\frac{\Delta}{Q^2}\right] x\hat{\mathcal{F}}_2^{QS}\,, \\ xF_3 &= R_+\,[2]\; x\hat{\mathcal{F}}_3^{QS}\,, \\ F_L &= \left[\frac{2S_+}{S_+ + S_-}\right]_{\text{NC only}} (F_2 - 2xF_1) \end{aligned} \tag{1.44}$$

Notice that we have factorized a factor which in APFEL is included *a posteriori* when constructing the structure functions and that thus does not go into the coefficient functions. The factors in the square bracket goes into the coefficient functions with the exception of the factor in front of F_L that must be included only the NC case to balance the factor in the r.h.s. of eq. (1.20)

In the NC case, we have:

$$\begin{aligned} &m_1 = m_2 = m_c \\ &S_+ = B_q \\ &S_- = \widetilde{B}_q \\ &R_+ = D_q \end{aligned} \tag{1.45}$$

In addition, the parameter η to be used in eq. (1.38) takes the form:

$$\eta = \frac{2}{1+\sqrt{1+4\lambda}}\,. \tag{1.46}$$

Finally, to obtain the coefficient functions to implement, one just needs to multiply the functions R and L given in eq. (1.43) by the kinematical factor that multiplies the respective $x\hat{\mathcal{F}}_i^{QS^{(1)}}$ in the equation above.

In the CC case, where the outcoming parton is massless (either a strange or a down quark), the kinematics is simpler than in the NC case. In addition, no mass singularies are caused by the masslessness of the outcoming

parton. Therefore the limit $m_2 \to 0$ can safely be taken. In fact, in the CC case for a strange+down quark in the final state,we have:

$$\begin{aligned} &m_1 = m_c \\ &m_2 = 0 \\ &S_+ = 2(|V_{cs}|^2 + |V_{cd}|^2) \\ &S_- = 0 \\ &R_+ = |V_{cs}|^2 + |V_{cd}|^2 \end{aligned} \tag{1.47}$$

However, for numerical reasons we cannot really set $m_2 = 0$. We set it instead to a small number. As a further simplification:

$$\eta = 1 \tag{1.48}$$

As a consequence, eq. (1.38) reduces to:

$$\Gamma_{\beta\alpha} = \int_c^d dz \frac{1}{1-z}\left[R\left(z\right) w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) - R(1)\delta_{\alpha\beta}\right] + \left[R(1)\ln\left(1-c\right) + L\right]\delta_{\alpha\beta}\,. \tag{1.49}$$

1.4 Massless Limit

What is left to be done to allow us to implement the FONLL scheme with IC to $\mathcal{O}(\alpha_s)$ is to compute and implement the massless limit, *i.e.* $m_1 = m_c \to 0$, of the $\mathcal{O}(\alpha_s)$ coeffient functions. We do not really need to compute these coeffient functions because in Ref. [1] the authors show that the logarithmically divergent part of all coefficient functions is equal to:

$$\begin{aligned} D(x, m_1^2) &= 2C_F\left[\frac{1+x^2}{1-x}\left(\ln\frac{\mu_F^2}{m_1^2} - 1 - 2\ln(1-x)\right)\right]_+ \\ &= 2C_F\left\{\left(\ln\frac{\mu_F^2}{m_1^2} - 1\right)\left[(1+x^2)\frac{1}{1-x}\right]_+ - 2\left[(1+x^2)\frac{\ln(1-x)}{1-x}\right]_+\right\}. \end{aligned} \tag{1.50}$$

We can now manipulate this expression using the following property of the plus-prescripted distributions:

$$\left[g(x)f(x)\right]_+ = \left[g(x)\right]_+ f(x) - \delta(1-x)\int_0^1 dy\left[g(y)\right]_+ f(y)\,, \tag{1.51}$$

so that:

$$\begin{aligned} D(x, m_1^2) &= 2C_F\Bigg\{\ln\frac{\mu_F^2}{m_1^2}\left(2\left[\frac{1}{1-x}\right]_+ - (1+x) + \frac{3}{2}\delta(1-x)\right) \\ &\quad -4\left[\frac{\ln(1-x)}{1-x}\right]_+ - 2\left[\frac{1}{1-x}\right]_+ + 2(1+x)\ln(1-x) + (1+x) + 2\delta(1-x)\Bigg\}. \end{aligned} \tag{1.52}$$

Finally, what we need to do to obtain the massless limit is to add to the $\overline{\text{MS}}$ zero-mass coeffient functions the term in eq. (1.52).

2 The FONLL Structure Functions

Once the inclusion of the IC into the massive sectors has been established, one can construct the FONLL structure functions using the usual recipe but now including the additional contributions. Calling F_i^{FONLL} the usual FONLL structure functions without IC and $F_i^{\text{FONLL,IC}}$ the structure function with IC, the relation is:

$$\begin{aligned} F_i^{\text{FONLL,IC}} &= F_i^{\text{FF}} + F_i^{\text{FF,IC}} + D(Q^2)\left[F_i^{\text{ZM}} - F_i^{\text{FF0}} - F_i^{\text{FF0,IC}}\right] \\ &= F_i^{\text{FONLL}} + \left[F_i^{\text{FF,IC}} - D(Q^2)F_i^{\text{FF0,IC}}\right] = F_i^{\text{FONLL}} + \Delta F_i^{\text{FONLL,IC}} \end{aligned} \tag{2.1}$$

where $D(Q^2)$ is a damping factor needed to quench undesired possibly large subleading terms at small energies. In the rest of theses notes I will concentrate on the implementation of the $\Delta F_i^{\text{FONLL,IC}}$ in APFEL.

3 The Implementation

At $\mathcal{O}(\alpha_s^0)$ there is no convolution between PDFs and coefficient functions and the charm PDFs appear directly in the expressions. According to whether one considers CC or NC heavy-quark-initiated processes, PDFs enter either as $xc(x)$, where x is the measured Bjorken variable, or as $\chi c(\chi)$, where χ is the rescaled variable defined in eq. (1.8). Now, in order to achive a proper implementation of the FONLL scheme in APFEL, I need to know all the component of the struncture functions (massive and massless) on the same x-space interpolation grid, defined as $\{x_\alpha\}$, , $\alpha \in= 0, \dots, N_x$. At LO, this essentially means knowing both $xc(x)$ and $\chi c(\chi)$ on the same grid. But choosing to tabulate $xc(x)$, such that in the CC case:

$$F^{\rm CC}(x_\alpha) \propto x_\alpha c(x_\alpha) = \tilde{c}(x_\alpha) = \sum_{\beta=0}^{N_x} \delta_{\alpha\beta} \tilde{c}(x_\beta)\,, \tag{3.1}$$

in the NC case, using the usual interpolation formula, the structure function can be expanded as:

$$F^{\rm NC}(x_\alpha) \propto \chi(x_\alpha) c(\chi(x_\alpha)) = \tilde{c}(\chi(x_\alpha)) = \sum_{\beta=0}^{N_x} w_\beta^{(k)}(\chi(x_\alpha)) \tilde{c}(x_\beta) = \sum_{\beta=0}^{N_x} w_\beta^{(k)}\left(\frac{x_\alpha}{\eta}\right) \tilde{c}(x_\beta)\,. \tag{3.2}$$

As a consequence, in the APFEL framework, quantities to store are $\delta_{\alpha\beta}$ and $w_\beta^{(k)}(x_\alpha/\eta)$ to be combined in a proper way to the other coefficient functions. First of all, let us compute case by case the quantity $\Delta F_i^{\rm FONLL,IC}$. In the NC case one has:

$$\Delta F_2^{\rm FONLL,IC}(x_\alpha) = \sum_{\beta=0}^{N_x} B_c \left[(2-\eta) w_\beta^{(k)}\left(\frac{x_\alpha}{\eta}\right) - D(Q^2)\delta_{\alpha\beta}\right] \tilde{c}^+(x_\beta) \tag{3.3a}$$

$$x_\alpha \Delta F_3^{\rm FONLL,IC}(x_\alpha) = \sum_{\beta=0}^{N_x} D_c \left[2\eta w_\beta^{(k)}\left(\frac{x_\alpha}{\eta}\right) - D(Q^2) 2\delta_{\alpha\beta}\right] \tilde{c}^-(x_\beta) \tag{3.3b}$$

$$\Delta F_L^{\rm FONLL,IC}(x_\alpha) = \sum_{\beta=0}^{N_x} B_c \left[1 - \frac{1}{2}\left(1 - \frac{\tilde{B}_c}{B_c}\right)\right] 4\frac{1-\eta}{2-\eta} w_\beta^{(k)}\left(\frac{x_\alpha}{\eta}\right) \tilde{c}^+(x_\beta) \tag{3.3c}$$

Finally, the CC case is slightly simpler:

$$\Delta F_2^{\nu,\rm FONLL,IC}(x_\alpha) = \sum_{\beta=0}^{N_x} 2(|V_{cd}|^2 + |V_{cs}|^2)\left[(1+\lambda) - D(Q^2)\right] \delta_{\alpha\beta} \tilde{\bar{c}}(x_\beta) \tag{3.4a}$$

$$x_\alpha \Delta F_3^{\nu,\rm FONLL,IC}(x_\alpha) = \sum_{\beta=0}^{N_x} 2(|V_{cd}|^2 + |V_{cs}|^2)\left[1 - D(Q^2)\right] \delta_{\alpha\beta} \tilde{\bar{c}}(x_\beta) \tag{3.4b}$$

$$\Delta F_L^{\nu,\rm FONLL,IC}(x_\alpha) = \sum_{\beta=0}^{N_x} 2(|V_{cd}|^2 + |V_{cs}|^2)\lambda \delta_{\alpha\beta} \tilde{\bar{c}}(x_\beta) \tag{3.4c}$$

and:

$$\Delta F_2^{\bar{\nu},\rm FONLL,IC}(x_\alpha) = \sum_{\beta=0}^{N_x} 2(|V_{cd}|^2 + |V_{cs}|^2)\left[(1+\lambda) - D(Q^2)\right] \delta_{\alpha\beta} \tilde{c}(x_\beta) \tag{3.5a}$$

$$x_\alpha \Delta F_3^{\bar{\nu},\rm FONLL,IC}(x_\alpha) = \sum_{\beta=0}^{N_x} 2(|V_{cd}|^2 + |V_{cs}|^2)\left[1 - D(Q^2)\right] \delta_{\alpha\beta} \tilde{c}(x_\beta) \tag{3.5b}$$

$$\Delta F_L^{\bar{\nu},\rm FONLL,IC}(x_\alpha) = \sum_{\beta=0}^{N_x} 2(|V_{cd}|^2 + |V_{cs}|^2)\lambda \delta_{\alpha\beta} \tilde{c}(x_\beta) \tag{3.5c}$$

Now, since structure functions in APFEL are expressed in the so-called evolution basis $\{\Sigma, g, V, V_3, \dots\}$, we only need to re-express the charm PDFs in terms of the distributions in the evolution basis. In particular, it is easy to show that:

$$c^+ = \frac{1}{6}\Sigma - \frac{1}{4}T_{15} + \frac{1}{20}T_{24} + \frac{1}{30}T_{35}\,, \tag{3.6}$$

and:

$$c^- = \frac{1}{6}V - \frac{1}{4}V_{15} + \frac{1}{20}V_{24} + \frac{1}{30}V_{35}\,. \tag{3.7}$$

In addition:

$$c = \frac{1}{2}(c^+ + c^-) \quad \text{and} \quad \bar{c} = \frac{1}{2}(c^+ - c^-)\,. \tag{3.8}$$

It should finally be noticed that the LO coefficient functions on the grid as written in eqs. (3.3) and (3.4) are non-singlet like and as such should be treated.

In the NC sector, the IC contributions to the massive stucture functions represent the only possible non-singlet contribution([2]). As a consequence, the IC fills the non-singlet "slot" and does not interfere with the non-IC part making the implementation in APFEL easier. Unfortunately this is not the case in the CC sector where instead the IC contribution overlaps with the non-IC one in a non-trivial way. In fact, even at LO the CC IC diagrams have a different kinematics as compared to the non-IC ones and thus their combination on the interpolation grid is not strightforward. What we need to do in APFEL is creating a new *ad hoc* slot for the IC contrbutions in such a way that it does not interfere with the other contributions([3]).

To conclude the section on LO implementation, we mention that, since the contributions reported above are to be included to the $N_F = 3$ massive scheme (be it full or asymptotic), heavy quark PDFs different from charm, *i.e.* bottom and top, do not contribute. As a consequence, it turns out that:

$$\begin{aligned} T_{24} &= T_{35} = \Sigma\,, \\ V_{24} &= V_{35} = V\,, \end{aligned} \tag{3.9}$$

and thus eq. (3.8) can be written as:

$$c = \frac{1}{8}(\Sigma - T_{15} + V - V_{15}) \quad \text{and} \quad \bar{c} = \frac{1}{8}(\Sigma - T_{15} - V + V_{15})\,. \tag{3.10}$$

References

[1] S. Kretzer and I. Schienbein, Phys. Rev. D **58** (1998) 094035 [hep-ph/9805233].

[2] This is essentially due to the fact that, requiring that the incoming photon (or Z) only couples to the charm, the photon vertex is never directly connected with the light initial state.

[3] In order not to burden APFEL with any additional big array, we exploit the fact that the massive CC coefficient functions are presently known up to $\mathcal{O}(\alpha_s)$ and up to this order no pure-singlet contribution is present. We then "artificially" place the IC contribution in the pure-singlet slot bearing in mind that if ever the $\mathcal{O}(\alpha_s^2)$ corrections to this process, which contain a pure-singlet contribution, will be computed, we will need to move the IC contribution in a real new slot.