# Definition of CC Light and Heavy Structure Functions

## Contents

# 1 $F_2^{\nu(\overline{\nu})}$ and $F_L^{\nu(\overline{\nu})}$

Writing explicitely the CKM matrix elements, the proton stucture function $F_2^{\nu,p}$ in the Zero-Mass Variable Flavour Number Scheme looks like this:

$$\begin{aligned}
F_2^{\nu,p} &= 2x\Big\{C_{2,q}\otimes\Big[\left(|V_{ud}|^2+|V_{cd}|^2+|V_{td}|^2\right)d \\
&+ \left(|V_{ud}|^2+|V_{us}|^2+|V_{ub}|^2\right)\overline{u} \\
&+ \left(|V_{us}|^2+|V_{cs}|^2+|V_{ts}|^2\right)s \\
&+ \left(|V_{cd}|^2+|V_{cs}|^2+|V_{cb}|^2\right)\overline{c} \\
&+ \left(|V_{ub}|^2+|V_{cb}|^2+|V_{tb}|^2\right)b \\
&+ \left(|V_{td}|^2+|V_{ts}|^2+|V_{tb}|^2\right)\overline{t} \\
&+ c_g^{CC}(N_f)C_{2,q}\otimes g\Big\}
\end{aligned} \tag{1}$$

where:

$$c_g^{CC}(N_f) = 2\sum_{i=u,c,t}\sum_{j=d,s,b}|V_{ij}|^2 \tag{2}$$

Note that everything we will do in this Section applies also to $F_L^{\nu,p}$ just substuting $C_{2,q}$ and $C_{2,g}$ with $C_{L,q}$ and $C_{L,g}$, respectively. Therefore, the stucture of the observables is exactly the same.

Now, we want to split up the above structure function into its light part plus the stucture functions of the single heavy quarks, that is:

$$F_2^{\nu,p} = F_{2,l}^{\nu,p} + F_{2,c}^{\nu,p} + F_{2,b}^{\nu,p} + F_{2,t}^{\nu,p} \tag{3}$$

without having any overlap between the single components.

To this end, we use the CKM matrix elements and we say that:

1. the light part is composed by the terms of eq. (1) which are proportional to those CKM matrix elements containing *only* light flavours (i.e. $u$, $d$ and $s$)

2. the heavy part due to the heavy flavour $h$ is instead given by the terms proportional to $V_{hk}$ or $V_{kh}$, where $m_h^2 > m_k^2$

Actually, this means dividing the CKM matrix in the following way:

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{ub} \end{pmatrix} \tag{4}$$

where the red terms are those which contribute to the light part of the proton structure functions, the blue terms to the charm part, the green terms to the bottom part and the violet terms to the top part.

Explicitely we have that:

$$\begin{aligned}
F_{2,l}^{\nu,p} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{ud}|^2 d + \left(|V_{ud}|^2+|V_{us}|^2\right)\overline{u} + |V_{us}|^2 s\Big] \\
&+ 2\left(|V_{ud}|^2+|V_{us}|^2\right)C_{2,g}\otimes g\Big\} \\
F_{2,c}^{\nu,p} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{cd}|^2(d+\overline{c}) + |V_{cs}|^2(s+\overline{c})\Big] \\
&+ 2\left(|V_{cd}|^2+|V_{cs}|^2\right)C_{2,g}\otimes g\Big\} \\
F_{2,b}^{\nu,p} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{ub}|^2(\overline{u}+b) + |V_{cb}|^2(\overline{c}+b)\Big] \\
&+ 2\left(|V_{ub}|^2+|V_{cb}|^2\right)C_{2,g}\otimes g\Big\} \\
F_{2,t}^{\nu,p} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{td}|^2(d+\overline{t}) + |V_{ts}|^2(s+\overline{t}) + |V_{tb}|^2(b+\overline{t})\Big] \\
&+ 2\left(|V_{td}|^2+|V_{ts}|^2+|V_{tb}|^2\right)C_{2,g}\otimes g\Big\}
\end{aligned} \tag{5}$$

We can obtain $F_2^{\overline{\nu},p}$ directly from $F_2^{\nu,p}$ just by exchanging each quark with the respective antiquark and viceversa. So, we get:

$$
\begin{aligned}
F_{2,l}^{\overline{\nu},p} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{ud}|^2\overline{d}+\left(|V_{ud}|^2+|V_{us}|^2\right)u+|V_{us}|^2\overline{s}\Big] \\
&+ 2\left(|V_{ud}|^2+|V_{us}|^2\right)C_{2,g}\otimes g\Big\} \\
F_{2,c}^{\overline{\nu},p} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{cd}|^2(\overline{d}+c)+|V_{cs}|^2(\overline{s}+c)\Big] \\
&+ 2\left(|V_{cd}|^2+|V_{cs}|^2\right)C_{2,g}\otimes g\Big\} \\
F_{2,b}^{\overline{\nu},p} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{ub}|^2(u+\overline{b})+|V_{cb}|^2(c+\overline{b})\Big] \\
&+ 2\left(|V_{ub}|^2+|V_{cb}|^2\right)C_{2,g}\otimes g\Big\} \\
F_{2,t}^{\overline{\nu},p} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{td}|^2(\overline{d}+t)+|V_{ts}|^2(\overline{s}+t)+|V_{tb}|^2(\overline{b}+t)\Big] \\
&+ 2\left(|V_{td}|^2+|V_{ts}|^2+|V_{tb}|^2\right)C_{2,g}\otimes g\Big\}
\end{aligned}
\tag{6}
$$

And finally the neutron structure functions can be obtained from the proton ones just by exchanging $u(\overline{u})\leftrightarrow d(\overline{d})$. So:

$$
\begin{aligned}
F_{2,l}^{\nu,n} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{ud}|^2u+\left(|V_{ud}|^2+|V_{us}|^2\right)\overline{d}+|V_{us}|^2s\Big] \\
&+ 2\left(|V_{ud}|^2+|V_{us}|^2\right)C_{2,g}\otimes g\Big\} \\
F_{2,c}^{\nu,n} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{cd}|^2(u+\overline{c})+|V_{cs}|^2(s+\overline{c})\Big] \\
&+ 2\left(|V_{cd}|^2+|V_{cs}|^2\right)C_{2,g}\otimes g\Big\} \\
F_{2,b}^{\nu,n} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{ub}|^2(\overline{d}+b)+|V_{cb}|^2(\overline{c}+b)\Big] \\
&+ 2\left(|V_{ub}|^2+|V_{cb}|^2\right)C_{2,g}\otimes g\Big\} \\
F_{2,t}^{\nu,n} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{td}|^2(u+\overline{t})+|V_{ts}|^2(s+\overline{t})+|V_{tb}|^2(b+\overline{t})\Big] \\
&+ 2\left(|V_{td}|^2+|V_{ts}|^2+|V_{tb}|^2\right)C_{2,g}\otimes g\Big\}
\end{aligned}
\tag{7}
$$

and:

$$
\begin{aligned}
F_{2,l}^{\overline{\nu},n} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{ud}|^2\overline{u}+\left(|V_{ud}|^2+|V_{us}|^2\right)d+|V_{us}|^2\overline{s}\Big] \\
&+ 2\left(|V_{ud}|^2+|V_{us}|^2\right)C_{2,g}\otimes g\Big\} \\
F_{2,c}^{\overline{\nu},n} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{cd}|^2(\overline{u}+c)+|V_{cs}|^2(\overline{s}+c)\Big] \\
&+ 2\left(|V_{cd}|^2+|V_{cs}|^2\right)C_{2,g}\otimes g\Big\} \\
F_{2,b}^{\overline{\nu},n} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{ub}|^2(d+\overline{b})+|V_{cb}|^2(c+\overline{b})\Big] \\
&+ 2\left(|V_{ub}|^2+|V_{cb}|^2\right)C_{2,g}\otimes g\Big\} \\
F_{2,t}^{\overline{\nu},n} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{td}|^2(\overline{u}+t)+|V_{ts}|^2(\overline{s}+t)+|V_{tb}|^2(\overline{b}+t)\Big] \\
&+ 2\left(|V_{td}|^2+|V_{ts}|^2+|V_{tb}|^2\right)C_{2,g}\otimes g\Big\}
\end{aligned}
\tag{8}
$$

Now, since the average of a given structure function $F_2^{\nu(\overline{\nu})}$ is given by:

$$F_2^{\nu(\overline{\nu})} = fF_2^{\nu(\overline{\nu}),p} + (1-f)F_2^{\nu(\overline{\nu}),p} \tag{9}$$

where:

$$f = \frac{N_p}{N_p + N_n} \tag{10}$$

we have that:

$$\begin{aligned}
F_{2,l}^{\nu} &= 2x\Big\{C_{2,q} \otimes \Big[|V_{ud}|^2(fd + (1-f)u) + \left(|V_{ud}|^2 + |V_{us}|^2\right)(f\overline{u} + (1-f)\overline{d}) + |V_{us}|^2 s\Big] \\
&+ 2\left(|V_{ud}|^2 + |V_{us}|^2\right) C_{2,g} \otimes g\Big\} \\
F_{2,c}^{\nu} &= 2x\Big\{C_{2,q} \otimes \Big[|V_{cd}|^2(fd + (1-f)u + \overline{c}) + |V_{cs}|^2(s + \overline{c})\Big] \\
&+ 2\left(|V_{cd}|^2 + |V_{cs}|^2\right) C_{2,g} \otimes g\Big\} \\
F_{2,b}^{\nu} &= 2x\Big\{C_{2,q} \otimes \Big[|V_{ub}|^2(f\overline{u} + (1-f)\overline{d} + b) + |V_{cb}|^2(\overline{c} + b)\Big] \\
&+ 2\left(|V_{ub}|^2 + |V_{cb}|^2\right) C_{2,g} \otimes g\Big\} \\
F_{2,t}^{\nu} &= 2x\Big\{C_{2,q} \otimes \Big[|V_{td}|^2(fd + (1-f)u + \overline{t}) + |V_{ts}|^2(s + \overline{t}) + |V_{tb}|^2(b + \overline{t})\Big] \\
&+ 2\left(|V_{td}|^2 + |V_{ts}|^2 + |V_{tb}|^2\right) C_{2,g} \otimes g\Big\}
\end{aligned} \tag{11}$$

and:

$$\begin{aligned}
F_{2,l}^{\overline{\nu}} &= 2x\Big\{C_{2,q} \otimes \Big[|V_{ud}|^2(f\overline{d} + (1-f)\overline{u}) + \left(|V_{ud}|^2 + |V_{us}|^2\right)(fu + (1-f)d) + |V_{us}|^2 \overline{s}\Big] \\
&+ 2\left(|V_{ud}|^2 + |V_{us}|^2\right) C_{2,g} \otimes g\Big\} \\
F_{2,c}^{\overline{\nu}} &= 2x\Big\{C_{2,q} \otimes \Big[|V_{cd}|^2(f\overline{d} + (1-f)\overline{u} + c) + |V_{cs}|^2(\overline{s} + c)\Big] \\
&+ 2\left(|V_{cd}|^2 + |V_{cs}|^2\right) C_{2,g} \otimes g\Big\} \\
F_{2,b}^{\overline{\nu}} &= 2x\Big\{C_{2,q} \otimes \Big[|V_{ub}|^2(fu + (1-f)d + \overline{b}) + |V_{cb}|^2(c + \overline{b})\Big] \\
&+ 2\left(|V_{ub}|^2 + |V_{cb}|^2\right) C_{2,g} \otimes g\Big\} \\
F_{2,t}^{\overline{\nu}} &= 2x\Big\{C_{2,q} \otimes \Big[|V_{td}|^2(f\overline{d} + (1-f)\overline{u} + t) + |V_{ts}|^2(\overline{s} + t) + |V_{tb}|^2(\overline{b} + t)\Big] \\
&+ 2\left(|V_{td}|^2 + |V_{ts}|^2 + |V_{tb}|^2\right) C_{2,g} \otimes g\Big\}
\end{aligned} \tag{12}$$

## 1.1 Light Stucture Function

At first, let's deal with the light structure function $F_{2,l}^{\nu}$. We notice that it can be written as:

$$\begin{aligned}
F_{2,l}^{\nu} &= 2x\Big\{C_{2,q} \otimes \Big[|V_{ud}|^2\left(f(d - u + \overline{u} - \overline{d}) + (u + \overline{d})\right) + |V_{us}|^2\left(f(\overline{u} - \overline{d}) + (\overline{d} + s)\right)\Big] \\
&+ \left(|V_{ud}|^2 + |V_{us}|^2\right) C_{2,g} \otimes g\Big\}
\end{aligned} \tag{13}$$

As usual we have to rotate the basis $\{u, \overline{u}, d, \overline{d}, \dots\}$ into the basis $\{\Sigma, g, V, T_3, V_3, \dots\}$, but knowing that:

$$\begin{pmatrix} u(\overline{u}) \\ d(\overline{d}) \\ s(\overline{s}) \\ c(\overline{c}) \\ b(\overline{b}) \\ t(\overline{t}) \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} u^+ \\ d^+ \\ s^+ \\ c^+ \\ b^+ \\ t^+ \end{pmatrix} \pm \begin{pmatrix} u^- \\ d^- \\ s^- \\ c^- \\ b^- \\ t^- \end{pmatrix}\right] = \frac{1}{120}\begin{pmatrix} 10 & 30 & 10 & 5 & 3 & 2 \\ 10 & -30 & 10 & 5 & 3 & 2 \\ 10 & 0 & -20 & 5 & 3 & 2 \\ 10 & 0 & 0 & -15 & 3 & 2 \\ 10 & 0 & 0 & 0 & -12 & 2 \\ 10 & 0 & 0 & 0 & 0 & -10 \end{pmatrix}\begin{pmatrix} \Sigma \pm V \\ T_3 \pm V_3 \\ T_8 \pm V_8 \\ T_{15} \pm V_{15} \\ T_{24} \pm V_{24} \\ T_{35} \pm V_{35} \end{pmatrix}. \quad (14)$$

we find that:

$$\begin{aligned} d - u &= -\frac{1}{2}(T_3 + V_3) \\ \overline{u} - \overline{d} &= \frac{1}{2}(T_3 - V_3) \end{aligned} \quad (15)$$

so that:

$$d - u + \overline{u} - \overline{d} = -V_3 \quad (16)$$

while:

$$u + \overline{d} = \frac{1}{60}[10\Sigma + 30V_3 + 10T_8 + 5T_{15} + 3T_{24} + 2T_{35}] \quad (17)$$

and:

$$\overline{d} + s = \frac{1}{60}[10\Sigma - 15(T_3 - V_3) - 5(T_8 + 3V_8) + 5T_{15} + 3T_{24} + 2T_{35}] \quad (18)$$

thus:

$$\begin{aligned} F_{2,l}^{\nu} &= 2x\Big\{C_{2,q} \otimes \Big[\frac{1}{60}|V_{ud}|^2\left(10\Sigma + 30(1-2f)V_3 + 10T_8 + 5T_{15} + 3T_{24} + 2T_{35}\right) \\ &+ \frac{1}{60}|V_{us}|^2\left(10\Sigma - 15(1-2f)(T_3 - V_3) - 5(T_8 + 3V_8) + 5T_{15} + 3T_{24} + 2T_{35}\right)\Big] \\ &+ \left(|V_{ud}|^2 + |V_{us}|^2\right)C_{2,g} \otimes g\Big\} \\ &= 2x\Big\{\frac{1}{60}C_{2,q} \otimes \Big[10K_l\Sigma - 15(1-2f)\left[|V_{us}|^2T_3 - (2|V_{ud}|^2 + |V_{us}|^2)V_3\right] \\ &+ 5(2|V_{ud}|^2 - |V_{us}|^2)T_8 - 15|V_{us}|^2V_8 + 5K_lT_{15} + 3K_lT_{24} + 2K_lT_{35}\Big] \\ &+ K_lC_{2,g} \otimes g\Big\} \end{aligned} \quad (19)$$

being:

$$K_l = 2\left(|V_{ud}|^2 + |V_{us}|^2\right) \quad (20)$$

Now let's consider $F_2^{\nu(\overline{\nu})}$. First we notice that it can be written as:

$$\begin{aligned} F_{2,l}^{\overline{\nu}} &= 2x\Big\{C_{2,q} \otimes \Big[|V_{ud}|^2\left(f(\overline{d} - \overline{u} + u - d) + (\overline{u} + d)\right) + |V_{us}|^2\left(f(u - d) + (d + \overline{s})\right)\Big] \\ &+ \left(|V_{ud}|^2 + |V_{us}|^2\right)C_{2,g} \otimes g\Big\} \end{aligned} \quad (21)$$

but, from eq. (14):

$$\begin{aligned} \overline{d} - \overline{u} &= -\frac{1}{2}(T_3 - V_3) \\ u - d &= \frac{1}{2}(T_3 + V_3) \end{aligned} \quad (22)$$

so that:

$$\overline{d} - \overline{u} + u - d = V_3 \tag{23}$$

while:

$$\overline{u} + d = \frac{1}{60}[10\Sigma - 30V_3 + 10T_8 + 5T_{15} + 3T_{24} + 2T_{35}] \tag{24}$$

and:

$$d + \overline{s} = \frac{1}{60}[10\Sigma - 15(T_3 + V_3) - 5(T_8 - 3V_8) + 5T_{15} + 3T_{24} + 2T_{35}] \tag{25}$$

thus:

$$\begin{aligned} F_{2,l}^{\overline{\nu}} &= 2x\Bigg\{\frac{1}{60}C_{2,q} \otimes \Bigg[10K_l\Sigma - 15(1-2f)\left[|V_{us}|^2 T_3 + (2|V_{ud}|^2 + |V_{us}|^2)V_3\right] \\ &+ 5(2|V_{ud}|^2 - |V_{us}|^2)T_8 + 15|V_{us}|^2 V_8 + 5K_l T_{15} + 3K_l T_{24} + 2K_l T_{35}\Bigg] \\ &+ K_l C_{2,g} \otimes g\Bigg\} \end{aligned} \tag{26}$$

## 1.2 Charm Structure Function

Now we consider the charm structure function for the neutrino. We see that it can be written as:

$$\begin{aligned} F_{2,c}^{\nu} &= 2x\Big\{C_{2,q} \otimes \Big[|V_{cd}|^2(f(d-u) + u + \overline{c}) + |V_{cs}|^2(s + \overline{c})\Big] \\ &+ \left(|V_{cd}|^2 + |V_{cs}|^2\right) C_{2,g} \otimes g\Big\} \end{aligned} \tag{27}$$

but:

$$d - u = -\frac{1}{2}(T_3 + V_3) \tag{28}$$

while:

$$u + \overline{c} = \frac{1}{60}[10\Sigma + 15(T_3 + V_3) + 5(T_8 + V_8) - 5(T_{15} - 2V_{15}) + 3T_{24} + 2T_{35}] \tag{29}$$

and:

$$s + \overline{c} = \frac{1}{60}[10\Sigma - 10(T_8 + V_8) - 5(T_{15} - 2V_{15}) + 3T_{24} + 2T_{35}] \tag{30}$$

so that:

$$f(d-u) + u + \overline{c} = \frac{1}{60}[10\Sigma + 15(1-2f)(T_3 + V_3) + 5(T_8 + V_8) - 5(T_{15} - 2V_{15}) + 3T_{24} + 2T_{35}] \tag{31}$$

so we find that:

$$\begin{aligned} F_{2,c}^{\nu} &= 2x\Bigg\{\frac{1}{60}C_{2,q} \otimes \Bigg[10K_c\Sigma + 15(1-2f)|V_{cd}|^2(T_3 + V_3) \\ &+ 5(|V_{cd}|^2 - 2|V_{cs}|^2)(T_8 + V_8) - 5K_c(T_{15} - 2V_{15}) + 3K_c T_{24} + 2K_c T_{35}\Bigg] \\ &+ K_c C_{2,g} \otimes g\Bigg\} \end{aligned} \tag{32}$$

where we have defined:

$$K_c = |V_{cd}|^2 + |V_{cs}|^2 \tag{33}$$

### 1.3 Bottom Structure Function

Now we consider the bottom structure function for the neutrino. We see that it can be written as:

$$\begin{aligned} F_{2,b}^{\nu} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{ub}|^2(f(\overline{u}-\overline{d})+\overline{d}+b)+|V_{cb}|^2(\overline{c}+b)\Big] \\ &+ \big(|V_{ub}|^2+|V_{cb}|^2\big)\,C_{2,g}\otimes g\Big\} \end{aligned} \tag{34}$$

but:

$$\overline{u}-\overline{d}=\frac{1}{2}(T_3-V_3) \tag{35}$$

while:

$$\overline{d}+b=\frac{1}{120}[20\Sigma-30(T_3-V_3)+10(T_8-V_8)+5(T_{15}-V_{15})-3(3T_{24}+5V_{24})+4T_{35}] \tag{36}$$

and:

$$\overline{c}+b=\frac{1}{120}[20\Sigma-15(T_{15}-V_{15})-3(3T_{24}+5V_{24})+4T_{35}] \tag{37}$$

so that:

$$f(\overline{u}-\overline{d})+\overline{d}+b=\frac{1}{120}[20\Sigma-30(1-2f)(T_3-V_3)+10(T_8-V_8)+5(T_{15}-V_{15})-3(3T_{24}+5V_{24})+4T_{35}] \tag{38}$$

so we find that:

$$\begin{aligned} F_{2,b}^{\nu} &= 2x\Bigg\{\frac{1}{120}C_{2,q}\otimes\Bigg[20K_b\Sigma-30(1-2f)|V_{ub}|^2(T_3-V_3)+10|V_{ub}|^2(T_8-V_8) \\ &+ 5(|V_{ub}|^2-3|V_{cb}|^2)(T_{15}-V_{15})-3K_b(3T_{24}+5V_{24})+4K_bT_{35}\Bigg] \\ &+ K_cC_{2,g}\otimes g\Bigg\} \end{aligned} \tag{39}$$

where we have defined:

$$K_b=|V_{ub}|^2+|V_{cb}|^2 \tag{40}$$

### 1.4 Top Structure Function

Now we consider the top structure function for the neutrino. We see that it can be written as:

$$\begin{aligned} F_{2,t}^{\nu} &= 2x\Big\{C_{2,q}\otimes\Big[|V_{td}|^2(f(d-u)+u+\overline{t})+|V_{ts}|^2(s+\overline{t})+|V_{tb}|^2(b+\overline{t})\Big] \\ &+ \big(|V_{td}|^2+|V_{ts}|^2+|V_{tb}|^2\big)\,C_{2,g}\otimes g\Big\} \end{aligned} \tag{41}$$

but:

$$d-u=-\frac{1}{2}(T_3+V_3) \tag{42}$$

while:

$$u+\overline{t}=\frac{1}{120}[20\Sigma+30(T_3+V_3)+10(T_8+V_8)+5(T_{15}+V_{15})+3(T_{24}+V_{24})-4(2T_{35}-3V_{35})] \tag{43}$$

and:

$$\begin{aligned} s+\overline{t} &= \frac{1}{120}[20\Sigma-20(T_8+V_8)+5(T_{15}+V_{15})+3(T_{24}+V_{24})-4(2T_{35}-3V_{35})] \\ b+\overline{t} &= \frac{1}{120}[20\Sigma-12(T_{24}+V_{24})-4(2T_{35}-3V_{35})] \end{aligned} \tag{44}$$

so that:

$$f(d-u)+u+\bar{t} = \frac{1}{120}[20\Sigma+30(1-2f)(T_3+V_3)+10(T_8+V_8)+5(T_{15}+V_{15})+3(T_{24}+V_{24})-4(2T_{35}-3V_{35})] \tag{45}$$

so we find that:

$$\begin{aligned} F^{\nu}_{2,t} &= 2x\Bigg\{\frac{1}{120}C_{2,q}\otimes\Bigg[20\Sigma+30(1-2f)|V_{td}|^2(T_3+V_3)+10(|V_{td}|^2-2|V_{ts}|)(T_8+V_8) \\ &+ 5(|V_{td}|^2+|V_{ts}|^2)(T_{15}+V_{15})+3(|V_{td}|^2+|V_{ts}|^2-4|V_{tb}|^2)(T_{24}+V_{24})-4(2T_{35}-3V_{35})\Bigg] \\ &+ K_c C_{2,g}\otimes g\Bigg\} \end{aligned} \tag{46}$$

where we have used the fact that:

$$|V_{td}|^2+|V_{ts}|^2+|V_{tb}|^2 = 1 \tag{47}$$

# 2 $xF_3^{\nu(\overline{\nu})}$

Now, we consider $xF_3^{\nu,p}$ which is defined as:

$$\begin{aligned} xF_3^{\nu,p} &= 2x\Big\{C_{3,q}\otimes\Big[\left(|V_{ud}|^2+|V_{cd}|^2+|V_{td}|^2\right)d \\ &- \left(|V_{ud}|^2+|V_{us}|^2+|V_{ub}|^2\right)\overline{u} \\ &+ \left(|V_{us}|^2+|V_{cs}|^2+|V_{ts}|^2\right)s \\ &- \left(|V_{cd}|^2+|V_{cs}|^2+|V_{cb}|^2\right)\overline{c} \\ &+ \left(|V_{ub}|^2+|V_{cb}|^2+|V_{tb}|^2\right)b \\ &- \left(|V_{td}|^2+|V_{ts}|^2+|V_{tb}|^2\right)\overline{t} \\ &+ c_g^{CC}(N_f)C_{3,q}\otimes g\Big\} \end{aligned} \tag{48}$$

Note that, we are keeping the gluon term which in the ZM-VFNS would be zero. On the other hand in the massive scheme it's not the case.

Again, we want to split up this structure function into its light part plus the heavy parts, that is:

$$xF_3^{\nu,p} = xF_{3,l}^{\nu,p}+xF_{3,c}^{\nu,p}+xF_{3,b}^{\nu,p}+xF_{3,t}^{\nu,p} \tag{49}$$

Using the same tricks of the previous Section, we get:

$$\begin{aligned} xF_{3,l}^{\nu,p} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{ud}|^2 d-\left(|V_{ud}|^2+|V_{us}|^2\right)\overline{u}+|V_{us}|^2 s\Big] \\ &+ \left(|V_{ud}|^2+|V_{us}|^2\right)C_{3,g}\otimes g\Big\} \\ xF_{3,c}^{\nu,p} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{cd}|^2(d-\overline{c})+|V_{cs}|^2(s-\overline{c})\Big] \\ &+ \left(|V_{cd}|^2+|V_{cs}|^2\right)C_{3,g}\otimes g\Big\} \\ xF_{3,b}^{\nu,p} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{ub}|^2(-\overline{u}+b)+|V_{cb}|^2(-\overline{c}+b)\Big] \\ &+ \left(|V_{ub}|^2+|V_{cb}|^2\right)C_{3,g}\otimes g\Big\} \\ xF_{3,t}^{\nu,p} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{td}|^2(d-\overline{t})+|V_{ts}|^2(s-\overline{t})+|V_{tb}|^2(b-\overline{t})\Big] \\ &+ \left(|V_{td}|^2+|V_{ts}|^2+|V_{tb}|^2\right)C_{3,g}\otimes g\Big\} \end{aligned} \tag{50}$$

and:

$$
\begin{aligned}
xF^{\overline{\nu},p}_{3,l} &= 2x\Big\{C_{3,q}\otimes\Big[-|V_{ud}|^2\overline{d}+\big(|V_{ud}|^2+|V_{us}|^2\big)\,u-|V_{us}|^2\overline{s}\Big] \\
&+ \big(|V_{ud}|^2+|V_{us}|^2\big)\,C_{3,g}\otimes g\Big\} \\
xF^{\overline{\nu},p}_{3,c} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{cd}|^2(-\overline{d}+c)+|V_{cs}|^2(-\overline{s}+c)\Big] \\
&+ \big(|V_{cd}|^2+|V_{cs}|^2\big)\,C_{3,g}\otimes g\Big\} \\
xF^{\overline{\nu},p}_{3,b} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{ub}|^2(u-\overline{b})+|V_{cb}|^2(c-\overline{b})\Big] \\
&+ \big(|V_{ub}|^2+|V_{cb}|^2\big)\,C_{3,g}\otimes g\Big\} \\
xF^{\overline{\nu},p}_{3,t} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{td}|^2(-\overline{d}+t)+|V_{ts}|^2(-\overline{s}+t)+|V_{tb}|^2(\overline{b}+t)\Big] \\
&+ \big(|V_{td}|^2+|V_{ts}|^2+|V_{tb}|^2\big)\,C_{3,g}\otimes g\Big\}
\end{aligned} \tag{51}
$$

While for the neutron we have:

$$
\begin{aligned}
xF^{\nu,n}_{3,l} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{ud}|^2u-\big(|V_{ud}|^2+|V_{us}|^2\big)\,\overline{d}+|V_{us}|^2s\Big] \\
&+ \big(|V_{ud}|^2+|V_{us}|^2\big)\,C_{3,g}\otimes g\Big\} \\
xF^{\nu,n}_{3,c} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{cd}|^2(u-\overline{c})+|V_{cs}|^2(s-\overline{c})\Big] \\
&+ \big(|V_{cd}|^2+|V_{cs}|^2\big)\,C_{3,g}\otimes g\Big\} \\
xF^{\nu,n}_{3,b} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{ub}|^2(-\overline{d}+b)+|V_{cb}|^2(-\overline{c}+b)\Big] \\
&+ \big(|V_{ub}|^2+|V_{cb}|^2\big)\,C_{3,g}\otimes g\Big\} \\
xF^{\nu,n}_{3,t} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{td}|^2(u-\overline{t})+|V_{ts}|^2(s-\overline{t})+|V_{tb}|^2(b-\overline{t})\Big] \\
&+ \big(|V_{td}|^2+|V_{ts}|^2+|V_{tb}|^2\big)\,C_{3,g}\otimes g\Big\}
\end{aligned} \tag{52}
$$

and:

$$
\begin{aligned}
xF^{\overline{\nu},n}_{3,l} &= 2x\Big\{C_{3,q}\otimes\Big[-|V_{ud}|^2\overline{u}+\big(|V_{ud}|^2+|V_{us}|^2\big)\,d-|V_{us}|^2\overline{s}\Big] \\
&+ \big(|V_{ud}|^2+|V_{us}|^2\big)\,C_{3,g}\otimes g\Big\} \\
xF^{\overline{\nu},n}_{3,c} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{cd}|^2(-\overline{u}+c)+|V_{cs}|^2(-\overline{s}+c)\Big] \\
&+ \big(|V_{cd}|^2+|V_{cs}|^2\big)\,C_{3,g}\otimes g\Big\} \\
xF^{\overline{\nu},n}_{3,b} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{ub}|^2(d-\overline{b})+|V_{cb}|^2(c-\overline{b})\Big] \\
&+ \big(|V_{ub}|^2+|V_{cb}|^2\big)\,C_{3,g}\otimes g\Big\}\Big\} \\
xF^{\overline{\nu},n}_{3,t} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{td}|^2(-\overline{u}+t)+|V_{ts}|^2(-\overline{s}+t)+|V_{tb}|^2(\overline{b}+t)\Big] \\
&+ \big(|V_{td}|^2+|V_{ts}|^2+|V_{tb}|^2\big)\,C_{3,g}\otimes g\Big\}
\end{aligned} \tag{53}
$$

So, combining proton and neutron structure functions we get:

$$
\begin{aligned}
xF^{\nu}_{3,l} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{ud}|^2(fd+(1-f)u)-\big(|V_{ud}|^2+|V_{us}|^2\big)\,(f\overline{u}+(1-f)\overline{d})+|V_{us}|^2s\Big]\\
&+ \big(|V_{ud}|^2+|V_{us}|^2\big)\,C_{3,g}\otimes g\Big\}\\
xF^{\nu}_{3,c} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{cd}|^2(fd+(1-f)u-\overline{c})+|V_{cs}|^2(s-\overline{c})\Big]\\
&+ \big(|V_{cd}|^2+|V_{cs}|^2\big)\,C_{3,g}\otimes g\Big\}\\
xF^{\nu}_{3,b} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{ub}|^2(-f\overline{u}-(1-f)\overline{d}+b)+|V_{cb}|^2(-\overline{c}+b)\Big]\\
&+ \big(|V_{ub}|^2+|V_{cb}|^2\big)\,C_{3,g}\otimes g\Big\}\Big\}\\
xF^{\nu}_{3,t} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{td}|^2(fd+(1-f)u-\overline{t})+|V_{ts}|^2(s-\overline{t})+|V_{tb}|^2(b-\overline{t})\Big]\\
&+ \big(|V_{td}|^2+|V_{ts}|^2+|V_{tb}|^2\big)\,C_{3,g}\otimes g\Big\}
\end{aligned}
\tag{54}
$$

and:

$$
\begin{aligned}
xF^{\overline{\nu}}_{3,l} &= 2x\Big\{C_{3,q}\otimes\Big[-|V_{ud}|^2(f\overline{d}+(1-f)\overline{u})+\big(|V_{ud}|^2+|V_{us}|^2\big)\,(fu+(1-f)d)-|V_{us}|^2\overline{s}\Big]\\
&+ \big(|V_{ud}|^2+|V_{us}|^2\big)\,C_{3,g}\otimes g\Big\}\\
xF^{\overline{\nu}}_{3,c} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{cd}|^2(-f\overline{d}-(1-f)\overline{u}+c)+|V_{cs}|^2(-\overline{s}+c)\Big]\\
&+ \big(|V_{cd}|^2+|V_{cs}|^2\big)\,C_{3,g}\otimes g\Big\}\\
xF^{\overline{\nu}}_{3,b} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{ub}|^2(fu+(1-f)d-\overline{b})+|V_{cb}|^2(c-\overline{b})\Big]\\
&+ \big(|V_{ub}|^2+|V_{cb}|^2\big)\,C_{3,g}\otimes g\Big\}\Big\}\\
xF^{\overline{\nu}}_{3,t} &= 2x\Big\{C_{3,q}\otimes\Big[|V_{td}|^2(-f\overline{d}-(1-f)\overline{u}+t)+|V_{ts}|^2(-\overline{s}+t)+|V_{tb}|^2(\overline{b}+t)\Big]\\
&+ \big(|V_{td}|^2+|V_{ts}|^2+|V_{tb}|^2\big)\,C_{3,g}\otimes g\Big\}
\end{aligned}
\tag{55}
$$

At this point we notice that $xF_3^{\nu(\overline{\nu})}$, a part from the fact that there are $C_{3,q}$ and $C_{3,g}$ rather than $C_{2,q}$ and $C_{2,g}$, is almost equal to $F_2^{\nu(\overline{\nu})}$ but with every antiflavour having opposite sign. Now, starting from eq. (14), we can write that:

$$
\begin{pmatrix} u(-\overline{u})\\ d(-\overline{d})\\ s(-\overline{s})\\ c(-\overline{c})\\ b(-\overline{b})\\ t(-\overline{t}) \end{pmatrix} = \frac{1}{120}\begin{pmatrix} 10 & 30 & 10 & 5 & 3 & 2\\ 10 & -30 & 10 & 5 & 3 & 2\\ 10 & 0 & -20 & 5 & 3 & 2\\ 10 & 0 & 0 & -15 & 3 & 2\\ 10 & 0 & 0 & 0 & -12 & 2\\ 10 & 0 & 0 & 0 & 0 & -10 \end{pmatrix}\begin{pmatrix} V\pm\Sigma\\ V_3\pm T_3\\ V_8\pm T_8\\ V_{15}\pm T_{15}\\ V_{24}\pm T_{24}\\ V_{35}\pm T_{35} \end{pmatrix}. \tag{56}
$$

So, we see that replacing $\overline{q}$ with $-\overline{q}$ in the basis $\{u,\overline{u},d,\overline{d},\dots\}$ is equivalent to exchange $\Sigma\leftrightarrow V$, $T_3\leftrightarrow V_3$, $T_8\leftrightarrow V_8$ and so on in the basis $\{\Sigma,g,V,T_3,V_3,\dots\}$. For this reason we can directly write down the expressions for $xF_3^{\nu(\overline{\nu})}$.

# 3 Heavy Quark Structure Functions in the Massive Scheme

In this sectio we will try to understand how the structure of the heavy quark structure functions change in the massive scheme where the heavy quark PDFs are absent. In principle, this step, if computing structure functions in the purely massive scheme, is not needed becuase the PDF evolution would automatically adjust the structure of the observables. However, when considering the FONLL scheme where massive and zero-mass schemes are combined and PDFs evolve in the ZM-VFNS, it is necessary to know how massive structure functions are written in terms of PDFs in the evolution basis to ensure a proper combination.

It is possible to write a set of simple rules that allow us to write the massive structure functions starting from the expressions we found in the previous sections. The principle is simple: in the $N_f$ massive scheme, all PDFs from $N_f + 1$ to 6 are absent. In order to see what happens at the level of PDFs in the evolution basis, we need to consider all the cases from $N_f = 3$ to $N_f = 5$ ($N_f = 6$ is equivalent to the massless scheme).

In the $N_f = 3$ massive scheme we have:

$$\begin{aligned} T_{15} &\to \Sigma \\ T_{24} &\to \Sigma \\ T_{35} &\to \Sigma \\ V_{15} &\to V \\ V_{24} &\to V \\ V_{35} &\to V \end{aligned} \tag{57}$$

In the $N_f = 4$ massive scheme:

$$\begin{aligned} T_{24} &\to \Sigma \\ T_{35} &\to \Sigma \\ V_{24} &\to V \\ V_{35} &\to V \end{aligned} \tag{58}$$

Finally, in the $N_f = 5$ massive scheme:

$$\begin{aligned} T_{35} &\to \Sigma \\ V_{35} &\to V \end{aligned} \tag{59}$$