Notes on the Parton Luminosities

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Abstract

In this document I describe the definition and the implementation of the differential parton-luminosity functions.

1 Definition

The definition of the parton-luminosity functions is given by the expression of the total hadronic cross section σ in terms of the parton ditribution functions $f_{i(j)}(x_{1,(2)},\mu)$ and the partonic cross sections $\hat{\sigma}_{ij}$:

$$\sigma = \sum_{ij} \int_0^1 dx_1 \int_0^1 dx_2 f_i(x_1,\mu) f_j(x_2,\mu) \hat{\sigma}_{ij} \,. \tag{1}$$

Now, defining the kinematic variable y (rapidity) and M_X (invariant mass of the partonic final state, usually normalized to the total center of mass squared energy energy s) as:

$$\begin{aligned} \tau &= \frac{M_X^2}{s} = x_1 x_2 \\ y &= \frac{1}{2}\ln\left(\frac{x_1}{x_2}\right) \end{aligned} \,, \tag{2}$$

one can express the integral in eq. (1) in terms of y and τ. In fact, by means of a change of variables and taking into account the fact that the Jacobian $\partial(y,\tau)/\partial(x_1,x_2) = 1$, we find that:

$$\sigma = \sum_{ij}\left(\int_{-\infty}^0 dy \int_0^{e^{2y}} d\tau + \int_0^{+\infty} dy \int_0^{e^{-2y}} d\tau\right)\frac{d^2\mathcal{L}_{ij}}{dyd\tau}\hat{\sigma}_{ij} = \sum_{ij}\int_0^1 d\tau\left(\int_0^{-\frac{1}{2}\ln\tau} dy + \int_{\frac{1}{2}\ln\tau}^0 dy\right)\frac{d^2\mathcal{L}_{ij}}{dyd\tau}\hat{\sigma}_{ij} \,, \tag{3}$$

where we have defined fully differential parton-luminosity functions as:

$$\frac{d^2\mathcal{L}_{ij}}{dyd\tau} = f_i\left(\sqrt{\tau}e^{y},\sqrt{s\tau}\right) f_j\left(\sqrt{\tau}e^{-y},\sqrt{s\tau}\right) \,, \tag{4}$$

and where we have also set $\mu = M_X = \sqrt{s\tau}$. Considering that the the differential parton-luminosity functions and the partonic cross sections must be symmetric under $y \leftrightarrow -y$, eq. (3) can be written as:

$$\sigma = \sum_{ij}\int_{-\infty}^{+\infty} dy \int_0^{e^{-2|y|}} d\tau \frac{d^2\mathcal{L}_{ij}}{dyd\tau}\hat{\sigma}_{ij} = \sum_{ij}\int_0^1 d\tau \int_{\frac{1}{2}\ln\tau}^{-\frac{1}{2}\ln\tau} dy \frac{d^2\mathcal{L}_{ij}}{dyd\tau}\hat{\sigma}_{ij} \,, \tag{5}$$

that can be written in a unique way as:

$$\sigma = \sum_{ij}\int_{-\infty}^{+\infty} dy \int_0^1 d\tau\, \theta\left(-y-\frac{1}{2}\ln\tau\right)\theta\left(y-\frac{1}{2}\ln\tau\right)\frac{d^2\mathcal{L}_{ij}}{dyd\tau}\hat{\sigma}_{ij} \,. \tag{6}$$

More in general, we can write the following equivalence at the level of phase space as:

$$\int_0^1 dx_1 \int_0^1 dx_2 = \int_{-\infty}^{+\infty} dy \int_0^1 d\tau\, \theta\left(-y-\frac{1}{2}\ln\tau\right)\theta\left(y-\frac{1}{2}\ln\tau\right) = \int_0^1 d\tau \int_{-\infty}^{+\infty} dy\, \theta\left(e^{-2|y|}-\tau\right) \,. \tag{7}$$

Now, let us compute the single differential parton-luminosity functions integrating out one of the kinematic variables y and τ. Let us start with y. In this case we have:

$$\begin{aligned}\frac{d\mathcal{L}_{ij}}{d\tau} &= \sum_{ij}\int_{-\infty}^{+\infty} dy \int_0^1 d\tau\, \theta\left(-y-\frac{1}{2}\ln\tau\right)\theta\left(y-\frac{1}{2}\ln\tau\right)\frac{d^2\mathcal{L}_{ij}}{dyd\tau}\delta(\tau-\tau') \\ &= \int_{\frac{1}{2}\ln\tau}^{-\frac{1}{2}\ln\tau} dy f_i\left(\sqrt{\tau'}e^{y},\sqrt{s\tau'}\right) f_j\left(\sqrt{\tau'}e^{-y},\sqrt{s\tau'}\right)\,,\end{aligned} \tag{8}$$

that, performing the change of variable $x=\sqrt{\tau'}e^{y}$ and taking into account that $s\tau' = M_X^2$, becomes:

$$\Phi_{ij} = \frac{d\mathcal{L}_{ij}}{dM_X^2} = \frac{1}{s}\int_{\tau'}^1 \frac{dx}{x} f_i(x, M_X) f_j\left(\frac{\tau'}{x}, M_X\right)\,. \tag{9}$$

Now we integrate the same for τ:

$$\begin{aligned}\Psi_{ij} = \frac{d\mathcal{L}_{ij}}{dy} &= \sum_{ij}\int_0^1 d\tau \int_{-\infty}^{+\infty} dy\, \theta\left(e^{-2|y|}-\tau\right)\frac{d^2\mathcal{L}_{ij}}{dyd\tau}\delta(y-y') \\ &= \int_0^{e^{-2|y'|}} d\tau f_i\left(\sqrt{\tau}e^{y'},\sqrt{s\tau}\right) f_j\left(\sqrt{\tau}e^{-y'},\sqrt{s\tau}\right)\,,\end{aligned} \tag{10}$$

that, defining $x=\sqrt{\tau}e^{y'}$ and assuming $y' \geq 0$, becomes:

$$\Psi_{ij} = \frac{d\mathcal{L}_{ij}}{dy} = 2e^{-2y'}\int_0^{e^{-y'}} dx\, x f_i(x, \sqrt{s}xe^{-y'}) f_j(xe^{-2y'}, \sqrt{s}xe^{-y'})\,. \tag{11}$$

Eq. (11) has the drawback that the integration range in x goes down to zero. In addition, also the scale in which PDFs are evaluate depends on x linarly. This implies that performing the integral requires the evaluation of PDFs in $x=\mu=0$ and this is clearly impossible. However, for converge reasons, the invariant mass of the final state is alway taken to be bigger than a give threshold. At the LHC 14 TeV this threshold is typically of the order of a few tens of GeV. Here, for explicative purposes, we take $M_{X,\mathrm{cut}} = 10$ GeV and this results in the cutoff $\tau_{\mathrm{cut}} \simeq 6\cdot 10^{-7}$, so that eq. (10) becomes:

$$\Psi_{ij} = \frac{d\mathcal{L}_{ij}}{dy} = \int_{\tau_{\mathrm{cut}}}^{e^{-2y'}} d\tau f_i\left(\sqrt{\tau}e^{y'},\sqrt{s\tau}\right) f_j\left(\sqrt{\tau}e^{-y'},\sqrt{s\tau}\right)\,, \tag{12}$$

and thus:

$$\Psi_{ij} = \frac{d\mathcal{L}_{ij}}{dy} = 2e^{-2y'}\int_{\sqrt{\tau_{\mathrm{cut}}}e^{y'}}^{e^{-y'}} dx\, x f_i(x, \sqrt{s}xe^{-y'}) f_j(xe^{-2y'}, \sqrt{s}xe^{-y'})\,. \tag{13}$$

Eq. (13) should allow for a practical implementation as the minimum of the integration range is now of the order of $7\cdot 10^{-4}$ and the minimum energy in which PDFs are avaluated is equal to $M_{X,\mathrm{cut}}$. In addition, in order for the upper limit of the integral in eq. (13) to be smaller than the lower one, one should also require that:

$$y' \leq -\frac{1}{2}\ln\frac{M_{X,\mathrm{cut}}}{\sqrt{s}} \simeq 3.6\,. \tag{14}$$