Tensor Gluons in APFEL

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Abstract

In this document I will present the strategy to implement the DGLAP evolution in APFEL in the presence of tensor gluons.

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1 Tensor Gluon Splitting Functions

When considering the PDF evolution in the presence of tensor gluons we need consider a generalized prescription to subtract the soft difergence in in $x = 1$ from the plitting functions. In particular, one of the new typologies of integral we need to consider is:

$$J_{n,m}[f(z)] = \int_x^1 dz \frac{z^n f(z)}{(1-z)_+^m}\,, \qquad n \geq 0\,. \tag{1.1}$$

But using the binomial expansion we can write:

$$z^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (1-z)^i\,. \tag{1.2}$$

Therefore:

$$\begin{aligned} J_{n,m}[f(z)] &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \int_x^1 dz \frac{f(z)}{(1-z)_+^{m-i}} \\ &= \sum_{i=0}^{m-1} (-1)^i \binom{n}{i} \int_x^1 dz \frac{f(z)}{(1-z)_+^{m-i}} + \sum_{j=0}^{n-m} (-1)^{j+m} \binom{n}{j+m} \int_x^1 dz f(z)(1-z)^j\,. \end{aligned} \tag{1.3}$$

So, it's clear that, if $n \geq m$, the second sum in the equation above does not need any +-prescription because it is always convergent, while the first term is relatively easy to treat, provided that we know how to compute the derivatives of $f(z)$. We will deat with this kind of integrals later. Now, to complete the picture, the second new typoligy of integrals to consider in the presence of tensor gluons is:

$$J_{-1,m}[f(z)] = \int_x^1 dz \frac{f(z)}{z(1-z)_+^m}\,. \tag{1.4}$$

Now, using the geometrical series, we can write that:

$$\frac{1}{z} = \sum_{i=0}^{\infty} (1-z)^i\,. \tag{1.5}$$

Therefore:

$$
\begin{aligned}
J_{-1,m}[f(z)] &= \sum_{i=0}^{\infty}\int_x^1 dz\frac{f(z)}{(1-z)_+^{m-i}} = \sum_{i=0}^{m-1}\int_x^1 dz\frac{f(z)}{(1-z)_+^{m-i}} + \sum_{j=0}^{\infty}\int_x^1 dz f(z)(1-z)^j \\
&= \sum_{i=0}^{m-1}\int_x^1 dz\frac{f(z)}{(1-z)_+^{m-i}} + \int_x^1 dz\frac{f(z)}{z}\,.
\end{aligned}
\tag{1.6}
$$

where again the second integral in the equation above is convergent without the need of any prescriton.

Now, looking at eqs. (1.3) and (1.6), we notice that both contain the same typology of integral which need to be regularized according to the prescription given in Ref. [1]. In particular we need to consider the following integral:

$$
\begin{aligned}
L_j[f(z)] &= \int_x^1 dz\frac{f(z)}{(1-z)_+^j} = \int_0^1 dz\frac{f(z)}{(1-z)_+^j} - \int_0^x dz\frac{f(z)}{(1-z)^j} \\
&= \int_0^1 dz\frac{1}{(1-z)_+^j}\left\{f(z) - \sum_{l=0}^{j-1}\frac{(-1)^l}{l!}f^{(l)}(1)(1-z)^l\right\} - \int_0^x dz\frac{f(z)}{(1-z)^j} \\
&= \int_x^1 dz\frac{1}{(1-z)_+^j}\left\{f(z) - \sum_{l=0}^{j-1}\frac{(-1)^l}{l!}f^{(l)}(1)(1-z)^l\right\} \\
&- \sum_{l=0}^{j-1}\frac{(-1)^l}{l!}f^{(l)}(1)\int_0^x\frac{dz}{(1-z)^{j-l}}\,.
\end{aligned}
\tag{1.7}
$$

In addition we have that:

$$
\int_0^x \frac{dz}{(1-z)^s} = \begin{cases} -\ln(1-x) & s=1 \\ -\dfrac{1}{s-1}\left[\dfrac{1}{(1-x)^{s-1}} - 1\right] & s>1 \end{cases}\,,
\tag{1.8}
$$

so that:

$$
\begin{aligned}
L_j[f(z)] &= \int_x^1 dz\frac{1}{(1-z)_+^j}\left\{f(z) - \sum_{l=0}^{j-1}\frac{(-1)^l}{l!}f^{(l)}(1)(1-z)^l\right\} \\
&+ \sum_{l=0}^{j-2}\frac{(-1)^l}{l!}\frac{f^{(l)}(1)}{j-l-1}\left[\frac{1}{(1-x)^{j-l-1}} - 1\right] \\
&+ \frac{(-1)^{j-1}}{(j-1)!}f^{(j-1)}(1)\ln(1-x)
\end{aligned}
\tag{1.9}
$$

Now, if $f(z) = w_\alpha^{(k)}(x_\beta/z)$, that is a Lagrange interpolation polynomial, the difficult part here is computing the derivatives $f^{(l)}(1)$. In order to do that, we try with an indirect strategy: we try to compute the expansion of the function $w_\alpha^{(k)}(x_\beta/z)$ around $z=1$ and then we put this expansion in relation with the Taylor's expansion to retrieve the derivatives.

By means of the variable change $x_\beta/z \to y$, expanding the function $w_\alpha^{(k)}(x_\beta/z)$ around $z=1$ is equivalent to expand the function $w_\alpha^{(k)}(y)$ around $y = x_\beta$ and such an expansion takes the usual form:

$$
w_\alpha^{(k)}(y) = \sum_{t=0}^{\infty}\frac{1}{t!}\left[\frac{d^t w_\alpha^{(k)}(x_\beta)}{dy^t}\right](y-x_\beta)^t\,.
\tag{1.10}
$$

Now, restoring the old variable we find:

$$
w_\alpha^{(k)}(x_\beta/z) = \delta_{\alpha\beta} + \sum_{t=1}^{\infty}\frac{x_\beta^t}{t!}\left[\frac{d^t w_\alpha^{(k)}(x_\beta)}{dy^t}\right]\frac{(1-z)^t}{z^t}\,.
\tag{1.11}
$$

where we have used the fact that $w_\alpha^{(k)}(x_\beta) = \delta_{\alpha\beta}$. This is not all because now we need to expand the function z^{-t} $(t \geq 1)$ around $z = 1$. But one can sho that:

$$\frac{1}{z^t} = \sum_{s=0}^{\infty} \frac{(s+t-1)!}{s!(t-1)!}(1-z)^s \,, \tag{1.12}$$

which can be plugged into eq. (1.11), obtaining:

$$w_\alpha^{(k)}(x_\beta/z) = \delta_{\alpha\beta} + \sum_{s=0}^{\infty}\sum_{t=1}^{\infty} \frac{(s+t-1)!}{s!(t-1)!}\frac{x_\beta^t}{t!}\left[\frac{d^t w_\alpha^{(k)}(x_\beta)}{dy^t}\right](1-z)^{t+s} \,. \tag{1.13}$$

Now, defining $s + t = l$, we can rearrange the series above as follows:

$$w_\alpha^{(k)}(x_\beta/z) = \delta_{\alpha\beta} + \sum_{l=1}^{\infty}\frac{1}{l!}\left\{ l! \sum_{t=1}^{l} \frac{(l-1)!}{(l-t)!(t-1)!}\frac{x_\beta^t}{t!}\left[\frac{d^t w_\alpha^{(k)}(x_\beta)}{dy^t}\right]\right\}(1-z)^l \tag{1.14}$$

Now, since the Taylor's expansion of $w_\alpha^{(k)}(x_\beta/z)$ around $z = 1$ is:

$$w_\alpha^{(k)}(x_\beta/z) = \delta_{\alpha\beta} + \sum_{l=1}^{\infty}\frac{1}{l!}\left[\frac{d^l w_\alpha^{(k)}(x_\beta/z)}{dz^l}\right]_{z=1}(1-z)^l \,, \tag{1.15}$$

we immediately read that, for $l \geq 1$:

$$(f^{(l)}(1) =)\left[\frac{d^l w_\alpha^{(k)}(x_\beta/z)}{dz^l}\right]_{z=1} = l! \sum_{t=1}^{l} \frac{(l-1)!}{(l-t)!(t-1)!}\frac{x_\beta^t}{t!}\left[\frac{d^t w_\alpha^{(k)}(x_\beta)}{dy^t}\right] \tag{1.16}$$

Plugging eq. (1.16) into eq. (1.17), we have:

$$\begin{aligned} L_j[w_\alpha^{(k)}(x_\beta/z)] \;&= \int_x^1 dz \frac{1}{(1-z)_+^j}\left\{ w_\alpha^{(k)}(x_\beta/z) - \sum_{l=0}^{j-1}(-1)^l(1-z)^l \sum_{t=1}^{l}\frac{(l-1)!}{(l-t)!(t-1)!}\frac{x_\beta^t}{t!}\left[\frac{d^t w_\alpha^{(k)}(x_\beta)}{dy^t}\right]\right\} \\ &+ \sum_{l=0}^{j-2}\frac{(-1)^l}{j-l-1}\left[\frac{1}{(1-x)^{j-l-1}} - 1\right]\sum_{t=1}^{l}\frac{(l-1)!}{(l-t)!(t-1)!}\frac{x_\beta^t}{t!}\left[\frac{d^t w_\alpha^{(k)}(x_\beta)}{dy^t}\right] \\ &+ (-1)^{j-1}\ln(1-x)\sum_{t=1}^{j-1}\frac{(j-2)!}{(j-t-1)!(t-1)!}\frac{x_\beta^t}{t!}\left[\frac{d^t w_\alpha^{(k)}(x_\beta)}{dy^t}\right] \end{aligned} \tag{1.17}$$

Finally, we can write that:

$$\begin{aligned} J_{n,m}[w_\alpha^{(k)}(x_\beta/z)] \;&= \sum_{i=0}^{m-1}(-1)^i\binom{n}{i}L_{m-i}[w_\alpha^{(k)}(x_\beta/z)] \\ &+ \sum_{j=0}^{n-m}(-1)^{j+m}\binom{n}{j+m}\int_x^1 dz\, w_\alpha^{(k)}(x_\beta/z)(1-z)^j \,. \end{aligned} \tag{1.18}$$

and:

$$J_{-1,m}[w_\alpha^{(k)}(x_\beta/z)] = \sum_{i=0}^{m-1} L_{m-i}[w_\alpha^{(k)}(x_\beta/z)] + \int_x^1 dz \frac{w_\alpha^{(k)}(x_\beta/z)}{z} \,. \tag{1.19}$$

2 Derivatives of the Lagrange Polynomials

Now we need to compute the derivative of the Lagrange polynomial $w_\alpha^{(k)}(x)$ of degree k on the grid nodes appearing in eq. (1.16). A Lagrange polynomial solves the following differential system:

$$\begin{cases} \dfrac{dw_\alpha^{(k)}(x)}{dx} = \left(\displaystyle\sum_{\substack{\beta=0 \\ \beta\neq\alpha}}^{k} \frac{1}{x - x_\beta} \right) w_\alpha^{(k)}(x) \\ \\ w_\alpha^{(k)}(x_\alpha) = 1 \end{cases} \tag{2.1}$$

whose solution is:

$$w_\alpha^{(k)}(x) = \prod_{\substack{\beta=0 \\ \beta\neq\alpha}}^{k} \frac{x - x_\beta}{x_\alpha - x_\beta} \tag{2.2}$$

Using eq. (2.1), one can show that the n-th derivative of $w_\alpha^{(k)}(x)$ is equal to zero for $n > k$, while for $n \leq k$:

$$\frac{d^n w_\alpha^{(k)}(x)}{dx^n} = \left[\underbrace{\sum_{\substack{\beta=0 \\ \beta\neq\alpha}}^{k} \frac{1}{x - x_\beta} \sum_{\substack{\gamma=0 \\ \gamma\neq\alpha,\beta}}^{k} \frac{1}{x - x_\gamma} \sum_{\substack{\delta=0 \\ \delta\neq\alpha,\beta,\gamma}}^{k} \frac{1}{x - x_\delta} \cdots}_{n \text{ factors}} \right] w_\alpha^{(k)}(x) \tag{2.3}$$

2.1 Value of the Derivatives on the Nodes

The value of the Lagrange interpolation function $w_\alpha^{(k)}(x)$ in correspondence of the nodes is:

$$w_\alpha^{(k)}(x_\rho) = \delta_{\alpha\rho} \,. \tag{2.4}$$

As a consequence, it looks like we could use this equation in eq. (2.3) to get the value of any derivative on the nodes. Unfortunately this is not so straightforward because, except for the case $\alpha = \rho$, this expression for the derivatives is defined only in the limit $x \to x_\rho$. In particular:

$$\frac{d^n w_\alpha^{(k)}(x_\alpha)}{dx^n} = \underbrace{\sum_{\substack{\beta=0 \\ \beta\neq\alpha}}^{k} \frac{1}{x_\alpha - x_\beta} \sum_{\substack{\gamma=0 \\ \gamma\neq\alpha,\beta}}^{k} \frac{1}{x_\alpha - x_\gamma} \sum_{\substack{\delta=0 \\ \delta\neq\alpha,\beta,\gamma}}^{k} \frac{1}{x_\alpha - x_\delta} \cdots}_{n \text{ factors}} \tag{2.5}$$

while for $\rho \neq \alpha$ we write eq. (2.3) in a different form, that is:

$$\frac{d^n w_\alpha^{(k)}(x)}{dx^n} = \underbrace{\sum_{\substack{\beta=0 \\ \beta\neq\alpha}}^{k} \frac{1}{x_\alpha - x_\beta} \sum_{\substack{\gamma=0 \\ \gamma\neq\alpha,\beta}}^{k} \frac{1}{x_\alpha - x_\gamma} \sum_{\substack{\delta=0 \\ \delta\neq\alpha,\beta,\gamma}}^{k} \frac{1}{x_\alpha - x_\delta} \cdots}_{n \text{ factors}} \prod_{\substack{\sigma=0 \\ \sigma\neq\alpha,\beta,\gamma,\delta,\ldots}}^{k} \frac{x - x_\sigma}{x_\alpha - x_\sigma} \tag{2.6}$$

which can now be evaluate on any node without the need of the limit, yielding:

$$\frac{d^n w_\alpha^{(k)}(x_\rho)}{dx^n} = \underbrace{\sum_{\substack{\beta=0 \\ \beta\neq\alpha}}^{k} \frac{1}{x_\alpha - x_\beta} \sum_{\substack{\gamma=0 \\ \gamma\neq\alpha,\beta}}^{k} \frac{1}{x_\alpha - x_\gamma} \sum_{\substack{\delta=0 \\ \delta\neq\alpha,\beta,\gamma}}^{k} \frac{1}{x_\alpha - x_\delta} \cdots}_{n \text{ factors}} \prod_{\substack{\sigma=0 \\ \sigma\neq\alpha,\beta,\gamma,\delta,\ldots}}^{k} \frac{x_\rho - x_\sigma}{x_\alpha - x_\sigma} \tag{2.7}$$

which in turn obviously reduces to eq. (2.5) for $\rho = \alpha$. But for $\rho \neq \alpha$ eq. (2.7) can be further simplified. In fact, the product in eq. (2.7) is different from only if $\rho = \beta, \gamma, \delta, \ldots$ (here by definition we are assuming $\rho \neq \alpha$). Therefore we could write:

$$\frac{d^n w_\alpha^{(k)}(x_\rho)}{dx^n} = \frac{1}{(x_\alpha - x_\rho)^n} \prod_{\substack{\sigma=0 \\ \sigma\neq\alpha,\rho}}^{k} \frac{x_\rho - x_\sigma}{x_\alpha - x_\sigma} \tag{2.8}$$

Eqs. (2.5) and (2.7) could be plugged into eq. (1.16) but, before doing it we can use a simplification. Suppose that the interpolation grid is such that:

$$x_\alpha - x_\beta = \delta x(\alpha - \beta) \tag{2.9}$$

where δx is a constant. Under this assumption, we have that:

$$\frac{d^n w_\alpha^{(k)}(x_\rho)}{dx^n} = \begin{cases} \frac{1}{(\delta x)^n} \underbrace{\sum_{\substack{\beta=0\\ \beta\neq\alpha}}^{k} \frac{1}{\alpha-\beta} \sum_{\substack{\gamma=0\\ \gamma\neq\alpha,\beta}}^{k} \frac{1}{\alpha-\gamma} \sum_{\substack{\delta=0\\ \delta\neq\alpha,\beta,\gamma}}^{k} \frac{1}{\alpha-\delta} \cdots}_{n \text{ factors}} & \rho = \alpha \\ \frac{1}{(\delta x)^n} \prod_{\substack{\sigma=0\\ \sigma\neq\alpha,\rho}}^{k} \frac{\rho-\sigma}{\alpha-\sigma} & \rho \neq \alpha \end{cases} \tag{2.10}$$

3 Implementation in APFEL

In APFEL we express PDFs by means of a moving interpolation over an x-space grid, according to the formula:

$$q(x,\mu^2) = \sum_{\alpha=0}^{N_x} w_\alpha^{(k)}(x) q(x_\alpha,\mu^2)\,, \tag{3.1}$$

where $w_\alpha^{(k)}$ are the "moving" Lagrange interpolation functions of degree k which take the following form:

$$w_\alpha^{(k)}(x) = \sum_{j=0, j\leq\alpha}^{k} \theta(x - x_{\alpha-j})\theta(x_{\alpha-j+1} - x) \prod_{\delta=0,\delta\neq j}^{k} \frac{x - x_{\alpha-j+\delta}}{x_\alpha - x_{\alpha-j+\delta}}\,. \tag{3.2}$$

References

[1] G. Savvidy, J. Phys. A **47**, 055204 (2014) [arXiv:1308.2695 [hep-th]].