Time Series Analysis in Credit Risk Modeling

OLS vs Yule-Walker Estimator for Autoregressive Coefficients

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Time Series Modeling in Credit Risk

- Certain areas of credit risk modeling involve working with and modeling time series.
- In Probability of Default (PD) modeling, time series play an important role in at least three regulatory areas:
  1. IFRS9 forward-looking modeling;
  2. quantifying central tendency uncertainty;
  3. producing simulation-based PD estimates for low-default portfolios.
- A crucial distinction between time series and independently collected data is that the order of observations significantly affects the modeling exercises.
- One of the most challenging aspects of time series modeling is appropriately addressing the autocorrelation structure.
- In simple terms, autocorrelation refers to the relationship between a variable's current and past values.
- Two commonly used methods for estimating autocorrelation coefficients in credit risk modeling are Ordinary Least Squares (OLS) and Yule-Walker (YW).
- The following slides present the basic formulas for OLS and YW estimators and compare their performance through Monte Carlo simulations.

OLS Estimator

Parameter estimation:

$$\hat{\beta} = (X^{'}X)^{-1}X^{'}Y$$

Standard error of parameters:

$$se(\hat{\beta}) = \sqrt{\hat{\sigma}^2(X'X)^{-1}}$$

with $\hat{\sigma}^2$ being:

$$\hat{\sigma}^2 = (n - k - 1)^{-1}\hat{\varepsilon}^{'}\hat{\varepsilon}$$

where:

- X is the design matrix;
- Y denotes the vector of observed values (dependent variable);
- $\hat{\varepsilon}$ represents the regression residuals.

Yule-Walker Estimator and Ljung-Box Test

The Yule-Walker equations relate the autocorrelation function of a time series to the parameters of the autoregressive (AR) model. By solving the following system of equations, the model parameters are obtained:

$$\begin{bmatrix} \rho(0) & \rho(1) & \cdots & \rho(p-1) \\ \rho(1) & \rho(0) & \cdots & \rho(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(p-1) & \rho(p-2) & \cdots & \rho(0) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{bmatrix} = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(p) \end{bmatrix}$$

where:

- $\rho(k)$ is the autocorrelation function at lag k;
- ϕ_i represents the parameters of the AR model;
- p is the order of the autoregressive model.

The Ljung-Box test statistic is given by the following expression:

$$Q = n(n+2) \sum_{k=1}^{h} \frac{\hat{\rho}_k^2}{n-k}$$

where:

- n is the sample size;
- h is the number of lags tested;
- $\hat{\rho}_k$ is the sample autocorrelation at lag k.

The test statistic Q follows a chi-square distribution with h degrees of freedom under the null hypothesis that the data are independently distributed.

Simulation Setup

The following steps detail the simulation design used to compare the two autoregressive coefficient estimators' bias and examine the statistical power of the OLS and Ljung-Box tests. For simplicity, only an autoregressive process of order one is considered.

1. Select the order of the autoregressive process ($\phi = 0.5$).
2. Select sample size n (10, 15, 20, 25, 30, 100).
3. Simulate an autoregressive process of order 1 for the chosen values of ϕ and n:

   $$x_{t,t\leq n} = \phi x_{t-1} + \sqrt{1-\phi^2}\varepsilon_t$$

   where ε_t is drawn from the standard normal distribution.
4. Using the simulated values of x from step 3, estimate the autoregressive coefficient using the OLS and Yule-Walker methods. Calculate the standard error and p-value of the OLS estimate and obtain the p-value from the Ljung-Box test.
5. Repeat steps 3 and 4 for $N = 10,000$ simulations, storing the results of the estimations.
6. Calculate the bias of the estimators as the difference between the average value of the simulated estimates and the true autoregressive coefficient ($\phi = 0.5$).
7. Calculate the statistical power of the OLS and Ljung-Box tests at a significance level of 0.05.

Simulation Results

```
##          type   T mean.estimator    bias stat.power
##           OLS  10         0.2440 -0.2560     0.0712
##   Yule-Walker  10         0.2043 -0.2957     0.1018
##           OLS  15         0.3263 -0.1737     0.2129
##   Yule-Walker  15         0.2990 -0.2010     0.2613
##           OLS  20         0.3719 -0.1281     0.3775
##   Yule-Walker  20         0.3494 -0.1506     0.4167
##           OLS  25         0.3987 -0.1013     0.5219
##   Yule-Walker  25         0.3803 -0.1197     0.5550
##           OLS  30         0.4170 -0.0830     0.6415
##   Yule-Walker  30         0.4014 -0.0986     0.6662
##           OLS 100         0.4745 -0.0255     0.9968
##   Yule-Walker 100         0.4696 -0.0304     0.9969
```

Simulation Results cont.