The Vasicek PD Model and Transition Matrices

Optimization of the Systemic Factor Z

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The Vasicek PD Model in IFRS9 Modeling

- The Vasicek Probability of Default (PD) model is one of the most commonly used approaches for modeling PD for IFRS9 purposes. Essentially, it relies on the assumption that observed default rates follow the Vasicek distribution.
- Vasicek distribution is a two-parameter ($0 < p < 1$ and $0 < \rho < 1$) continuous distribution on the range 0 to 1. If a variable x has a Vasicek distribution, then x (observed default rate) can be represented as:

$$x = \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right)$$

  where:

  - p and ρ are the parameters of the distribution, commonly referred to as the average default rate and asset correlation, respectively;
  - Z represents the systemic factor drawn from the standard normal distribution;
  - Φ and Φ^{-1} denote the distribution and quantile function of the standard normal distribution, respectively.
- Given the parameters p and ρ, which are commonly estimated at the portfolio level, the systemic factor Z is derived by manipulating the equation of the Vasicek distribution.
- But what if we shift the focus to transition matrices instead of deriving Z from the portfolio level?
- The following slides present an approach to deriving the systemic factor Z based on grade transition matrices. This approach is particularly attractive as it aligns with the common practice of applying the Vasicek model's results to grade transition rates in IFRS9 PD modeling.

The Transition Rates and Z Factor Optimization

- Rather than deriving Z at the portfolio level, practitioners can instead derive the Z factor at the transition matrix level by utilizing ρ, as well as Through-the-Cycle (TtC) and Point-in-Time (PiT) Grade-to-grade (G-to-g) transition rates.

- The usual approach is optimization by solving a least-squares problem. One way to define this optimization problem is as follows:

$$\min_{Z_t} \sum_{G} \sum_{g} \frac{n_{t,g} \left[P_t(G,g) - \Delta(x_{g+1}^G, x_g^G, Z_t) \right]^2}{\Delta(x_{g+1}^G, x_g^G, Z_t)[1 - \Delta(x_{g+1}^G, x_g^G, Z_t)]}$$

  where:

  - $P_t(G,g)$ is the PiT probability of moving from grade G to grade g between two periods;
  - $\Delta(x_{g+1}^G, x_g^G, Z_t)$ is defined as $\Phi\left(\frac{x_{g+1}^G - \sqrt{\rho}Z_t}{\sqrt{1-\rho}}\right) - \Phi\left(\frac{x_g^G - \sqrt{\rho}Z_t}{\sqrt{1-\rho}}\right)$;
  - G and g are the indices of the grades in the transition matrices at the beginning and end of the period, respectively;
  - x_g^G is the inverse standard normal transformation of the cumulative transition rate probabilities for grades G and g;
  - Φ denotes the standard normal distribution;
  - $n_{t,g}$ is the number of obligors in grade g at the time t;
  - Z_t is the systemic factor being optimized.

- The optimization procedure finds the optimal Z_t factor that best approximates the PiT transition matrix, given the TtC matrix and the parameter ρ.

- More details on this approach can be found here.

Simulation Setup

For an asset correlation value of $\rho = 0.0163$ and the TtC and PiT transition matrices provided below, this simulation aims to find the Z_t factor that best approximates the PiT matrix. The number of obligors (`Obs`) from the PiT matrix is used as an additional input in the optimization process described in the previous slide.

TtC transition matrix:

```
## G-to-g    AAA     AA      A    BBB     BB      B    CCC      D
##    AAA 91.13%  8.00%  0.70%  0.10%  0.05%  0.01%  0.01%  0.01%
##     AA  0.70% 91.03%  7.47%  0.60%  0.10%  0.07%  0.02%  0.01%
##      A  0.10%  2.34% 91.54%  5.08%  0.61%  0.26%  0.01%  0.05%
##    BBB  0.02%  0.30%  5.65% 87.98%  4.75%  1.05%  0.10%  0.15%
##     BB  0.01%  0.11%  0.55%  7.77% 81.77%  7.95%  0.85%  1.00%
##      B  0.00%  0.05%  0.25%  0.45%  7.00% 83.50%  3.75%  5.00%
##    CCC  0.00%  0.01%  0.10%  0.30%  2.59% 12.00% 65.00% 20.00%
```

PiT transition matrix:

```
## G-to-g Obs    AAA     AA      A    BBB     BB      B    CCC      D
##    AAA  85 92.94%  4.71%  2.35%  0.00%  0.00%  0.00%  0.00%  0.00%
##     AA 220  0.46% 92.52%  6.08%  0.47%  0.47%  0.00%  0.00%  0.00%
##      A 480  0.00%  4.45% 84.95%  9.54%  0.64%  0.00%  0.00%  0.42%
##    BBB 298  0.37%  0.37%  3.26% 85.52%  9.78%  0.37%  0.00%  0.34%
##     BB 168  0.00%  0.68%  0.00%  2.68% 82.42% 10.05%  0.00%  4.17%
##      B 161  0.00%  0.00%  0.72%  0.72%  2.89% 87.50%  5.06%  3.11%
##    CCC  16  0.00%  0.00%  0.00%  0.00%  0.00%  7.39% 73.86% 18.75%
```

Simulation Results

- Given the simulation inputs, the optimal value of Z_t is `0.87`.
- The matrix below presents the resulting transition rates that best approximate the observed PiT matrix under the described optimization process.
- The following slide visualizes the TtC to fitted PiT transformation procedure using the optimized Z_t value for the grade `A`.

Fitted Matrix

```
## G-to-g Obs    AAA    AA      A    BBB     BB      B    CCC      D
##    AAA  85 89.40%  9.49%  0.89%  0.13%  0.07%  0.01%  0.01%  0.00%
##     AA 220  0.48% 89.61%  8.88%  0.76%  0.13%  0.09%  0.03%  0.01%
##      A 480  0.06%  1.73% 90.91%  6.10%  0.77%  0.34%  0.01%  0.07%
##    BBB 298  0.01%  0.20%  4.42% 88.04%  5.68%  1.32%  0.13%  0.20%
##     BB 168  0.01%  0.07%  0.38%  6.23% 81.66%  9.34%  1.05%  1.26%
##      B 161  0.00%  0.03%  0.17%  0.32%  5.59% 83.43%  4.36%  6.10%
##    CCC  16  0.00%  0.01%  0.06%  0.21%  1.96% 10.14% 64.57% 23.06%
```

Simulation Results cont.