IMAGING A BLACK HOLE

GENERAL RELATIVITY

RESEARCH PROJECT

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1 INTRODUCTION

2 BACKGROUND

2.1 SCHWARZSCHILD SPACETIME

We begin our analysis in the Schwarzschild spacetime where the metric for a static vacuum black hole can be written as,

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

Where r, θ and φ are the spherical co-ordinates in the geometrized unit system with $G = c = 1$ [1]. M is the relativistic mass of the black hole whose surface is given by the Schwarzschild radius $r_s = 2M$.

2.2 GEODESIC EQUATIONS

In order to understand the trajectories of photons around a black hole, we need to solve for the geodesic equations in the Schwarzschild spacetime. The geodesic equation for a co-ordinate x^μ is (where λ is an affine parameter along the geodesic path) [**aw-gr1**],

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{2}$$

Therefore, we need to first find the nonzero Christoffels for this geometry are given by [**aw-gr1**],

$$\Gamma^t_{tr} = \frac{r_s}{2r(r - r_s)} \tag{3}$$

$$\Gamma^r_{tr} = -\Gamma^t_{rr},\ \Gamma^r_{tt} = \frac{r_s}{2r^3}(r - r_s),\ \Gamma^r_{\theta\theta} = -(r - r_s),\ \Gamma^r_{\varphi\varphi} = \sin^2\theta\Gamma^r_{\theta\theta} \tag{4}$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r},\ \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta \tag{5}$$

$$\Gamma^\varphi_{r\varphi} == \frac{1}{r},\ \Gamma^\varphi_{\theta\varphi} = \cot\theta \tag{6}$$

Using these Christoffels for each spherical co-ordinate's geodesic equation we get the following equations [**aw-gr1**],

$$\frac{d^2t}{d\lambda^2} + \frac{r_s}{r(r-r_s)}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0 \tag{7}$$

$$\frac{d^2r}{d\lambda^2} + \frac{r_s}{2r^3}(r-r_s)\left(\frac{dt}{d\lambda}\right)^2 - \frac{r_s}{2r(r-r_s)}\left(\frac{dr}{d\lambda}\right)^2 \tag{8}$$

$$-(r-r_s)\left\{\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2\right\} = 0 \tag{9}$$

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\varphi}{d\lambda}\right)^2 = 0 \tag{10}$$

$$\frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} + 2\cot\theta\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda} = 0 \tag{11}$$

However, we can use Killing vectors to find two integrals of motion. The first is the energy that is conserved,

$$\partial_t g_{\mu\nu} = 0 \implies E \equiv p_t = \left(1 - \frac{r_s}{r}\right)\frac{dt}{d\lambda} \tag{12}$$

and the second integral of motion is the conserved angular momentum,

$$\partial_\varphi g_{\mu\nu} = 0 \implies L \equiv p_\varphi = r^2\sin\theta\frac{d\varphi}{d\lambda} \tag{13}$$

Using the equation,

$$g_{\mu\nu}U^\mu U^\mu = 0 \tag{14}$$

for photon geodesics we get the equation,

$$0 = \left(1 - \frac{r_s}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2\right] \tag{15}$$

Spherical symmetry implies that the motion of a test particle must be fully contained on a plane. Therefore, to simplify the equations we can consider $\theta = \frac{\pi}{2}$ and $\frac{d\theta}{d\lambda} = 0$ [1]. This reduces the four geodesic equations we had before to a set of three simple equations,

$$\left(1 - \frac{r_s}{r}\right)\frac{dt}{d\lambda} = E \tag{16}$$

$$r^2\frac{d\varphi}{d\lambda} = L \tag{17}$$

$$\left(1 - \frac{r_s}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\varphi}{d\lambda}\right)^2 = 0 \tag{18}$$

Using this and our conserved quantities Equation 18 simplifies to,

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1 - \frac{r_s}{r}\right)\left(\varepsilon + \frac{L^2}{r^2}\right) \tag{19}$$

Since we are interested in the geodesics of photons,

$$\left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{r_s}{r}\right)\frac{L^2}{r^2} = E^2 \tag{20}$$

It is convenient to define the shape equation as follows,

$$\left(\frac{dr}{d\varphi}\frac{d\varphi}{d\lambda}\right)^2 + \left(1 - \frac{r_s}{r}\right)\frac{L^2}{r^2} = E^2 \tag{21}$$

$$\left(\frac{dr}{d\varphi}\right)^2\left(\frac{L}{r^2}\right)^2 + \left(1 - \frac{r_s}{r}\right)\frac{L^2}{r^2} = E^2 \tag{22}$$

$$\left\{\frac{1}{r^2}\left(\frac{dr}{d\varphi}\right)^2\right\} + \left(1 - \frac{r_s}{r}\right)\frac{1}{r^2} = \left(\frac{E}{L}\right)^2 \tag{23}$$

Defining $b \equiv \frac{L}{E}$ as the impact parameter at infinity and $V(r)$ as the effective potential we finally have the shape equation for a photon orbiting a black hole,

$$\boxed{\left\{\frac{1}{r^2}\left(\frac{dr}{d\varphi}\right)^2\right\} + V(r) = \frac{1}{b^2},\ V(r) = \frac{1}{r^2}\left(1 - \frac{r_s}{r}\right)} \tag{24}$$

2.3 PHOTON TRAJECTORIES

Figure 1 shows the behaviour of the effective potential found in Equation 24.

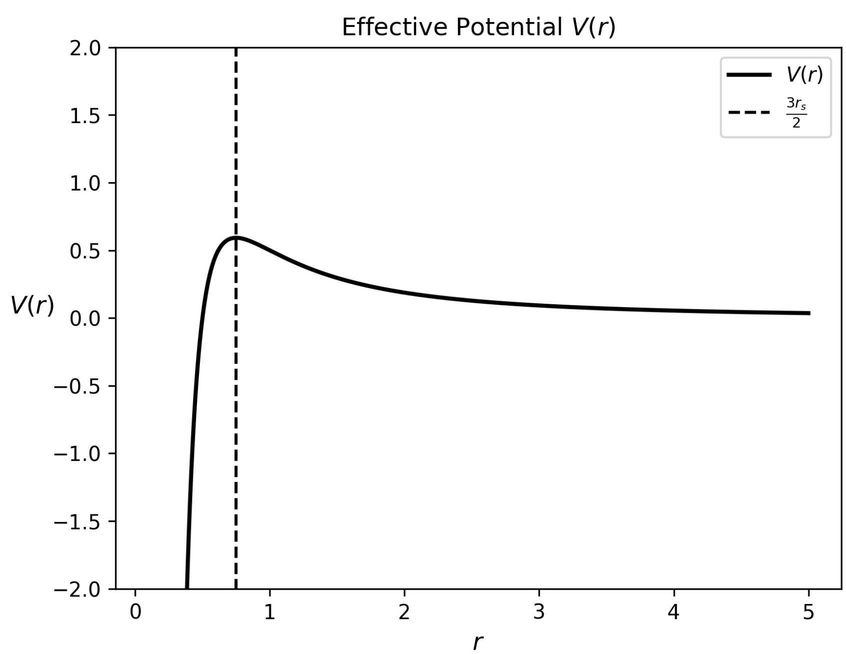

Figure 1: Behaviour of effective potential in Equation 24.

We can see that an unstable minimum occurs at the radius $r_\star = \frac{3}{2}r_s = 3M$ where the potential takes on the value $V(r_\star) = \frac{1}{27M^2}$. This results in a critical impact parameter,

$$b_\star = \sqrt{27}M = 3\sqrt{3}M \tag{25}$$

Any photon with impact parameters $b > b_\star$ will be deflected from the black hole and $b < b_\star$ will be captured by the black hole. Interestingly, this results in an apparent black hole diameter of,

$$d_{BH} = 2b_\star = 6\sqrt{3}M \tag{26}$$

Figure 2 shows the behaviour of photons with various impact parameters b on the $\theta = \frac{\pi}{2}$ plane with initial distance $r_0 = 10M$.

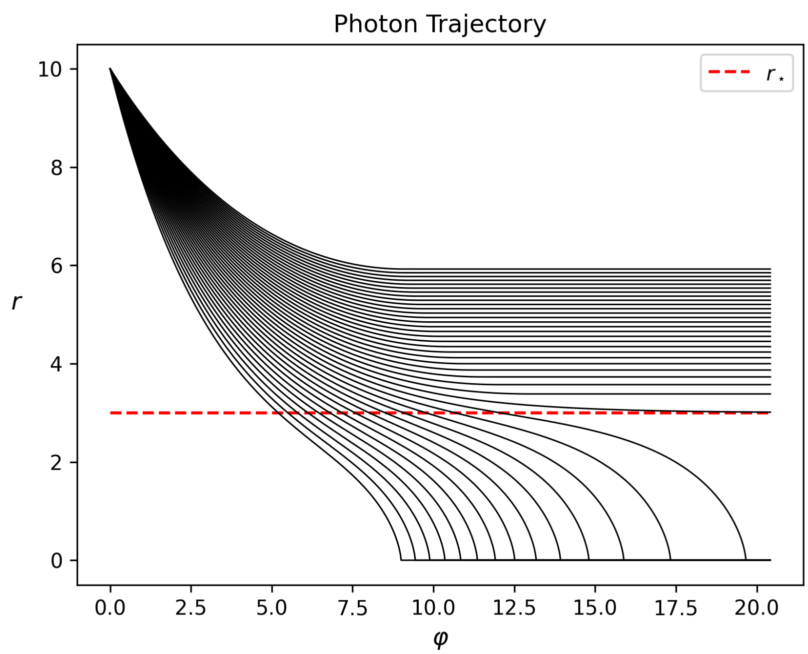

Figure 2: Trajectories of photons with various impact parameters. Only those with impact parameters less than the critical value will fall below the surface into the singularity.

3 RESULTS

4 DISCUSSION

5 CONCLUSION

References

[1] Jean-Pierre Luminet. "Image of a spherical black hole with thin accretion disk". In: *Astronomy and Astrophysics* 75 (Apr. 1979), pp. 228–235.