1 Richards equation (head formulation)

$$C(h)\frac{\partial h}{\partial t} = -\nabla \cdot q + Q \tag{1}$$

$$q = -K(\nabla h + \nabla z) \tag{2}$$

Insert 2 in to 1:

$$C(h)\frac{\partial h}{\partial t} = -\nabla \cdot [-K(\nabla h + \nabla z)] + Q \tag{3}$$

Initial value problem with two types of boundary conditions:

1. Neumann condition (flux form):

$$q(x,t) = q_{bnd} \qquad x \in \partial\Omega_1 \tag{4}$$

2. Robbins condition (leakage form):

$$q(x,t) = -K_{bnd}(h_o(x) - h(x)) \tag{5}$$

2 Discrete form of model (vertical dimension)

We define a finite difference grid in two dimensions (z and x). We assume that $\Delta z_i = z_{i+1} - z_i$ and $\Delta x_j = x_{j+1} - x_j$ where $z_{i+1} > z_i$ and $x_{j+1} > x_j$. Flow is negative in negative directions. We implement the model on a so-called staggerd grid where fluxes are defined on the internodes and the states are defined on the nodes. Discretisation in the vertical dimension is:

$$C_{i,j}\frac{\Delta h_{i,j}}{\Delta t} = -\frac{q_{i+\frac{1}{2},j} - q_{i-\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}} - \frac{q_{i,j+\frac{1}{2}} - q_{i,j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}} + Q_{i,j} \tag{6}$$

Expansion of this equation for the vertical gives:

$$C_{i,j}\frac{\Delta h_{i,j}}{\Delta t} = -\left[\frac{-K_{i+\frac{1}{2},j}\left(\frac{h_{i+1,j}-h_{i,j}}{\Delta z_i}+1\right) + K_{i-\frac{1}{2},j}\left(\frac{h_{i,j}-h_{i-1,j}}{\Delta z_{i-1}}+1\right)}{\Delta z_{i-\frac{1}{2}}}\right] - \left[\frac{-K_{i,j+\frac{1}{2}}\left(\frac{h_{i,j+1}-h_{i,j}}{\Delta x_j}+1\right) + K_{i,j-\frac{1}{2}}\left(\frac{h_{i,j}-h_{i,j-1}}{\Delta x_{j-1}}+1\right)}{\Delta x_{j-\frac{1}{2}}}\right] + Q_{i,j}$$

Rearranging gives:

$$\begin{aligned} C_{i,j}\frac{\Delta h_{i,j}}{\Delta t} \quad = \quad & \frac{K_{i,j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_{j-1}}h_{i,j-1} + \frac{K_{i-\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}\Delta z_{i-1}}h_{i-1,j} - \left(\frac{K_{i-\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}\Delta z_{i-1}} + \frac{K_{i+\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}\Delta z_i} + \frac{K_{i,j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_{j-1}} + \frac{K_{i,j+\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_j}\right)h_{i,j} + \\ & \frac{K_{i+\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}\Delta z_i}h_{i+1,j} + \frac{K_{i,j+\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_j}h_{i,j+1} + \frac{K_{i+\frac{1}{2},j} - K_{i-\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}} + Q_{i,j} \end{aligned}$$

Which can be rewritten as:

$$C_{i,j}\frac{\Delta h_{i,j}}{\Delta t} = a_{i,j-1}h_{i,j-1} + b_{i-1,j}h_{i-1,j} + c_{i,j}h_{i,j} + d_{i+1,j}h_{i+1,j} + e_{i,j+1}h_{i,j+1} + y_{i,j} \tag{7}$$

for all values of i and j we get a system of equations which can be written in a matrix form:
$C \cdot \Delta h = K \cdot h + y$

3 Discretisation of the boundary conditions

3.1 Neumann bottom and left condition:

$$i = 1$$

$$C_{1,j}\frac{\Delta h_{i,j}}{\Delta t} = -\frac{q_{i+\frac{1}{2},j} - q_B}{\Delta z_{i-\frac{1}{2}}} - \frac{q_{i,j+\frac{1}{2}} - q_{i,j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}} + Q_{i,j}$$

Substitution gives:

$$C_{i,j}\frac{\Delta h_{i,j}}{\Delta t} = -\left[\frac{-K_{i+\frac{1}{2},j}\left(\frac{h_{i+1,j}-h_{i,j}}{\Delta z_i}+1\right) - q_B}{\Delta z_{i-\frac{1}{2}}}\right] - \left[\frac{-K_{i,j+\frac{1}{2}}\left(\frac{h_{i,j+1}-h_{i,j}}{\Delta x_j}+1\right) + K_{i,j-\frac{1}{2}}\left(\frac{h_{i,j}-h_{i,j-1}}{\Delta x_{j-1}}+1\right)}{\Delta x_{j-\frac{1}{2}}}\right] + Q_{i,j}$$

and rearranging gives:

$$\begin{aligned} C_{i,j}\frac{\Delta h_{i,j}}{\Delta t} \quad = \quad & \frac{K_{i,j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_{j-1}}h_{i,j-1} - \left(\frac{K_{i+\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}\Delta z_i} + \frac{K_{i,j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_{j-1}} + \frac{K_{i,j+\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_j}\right)h_{i,j} + \\ & \frac{K_{i+\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}\Delta z_i}h_{i+1,j} + \frac{K_{i,j+\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_j}h_{i,j+1} + \frac{K_{i+\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}} + \frac{q_B}{\Delta z_{i-\frac{1}{2}}} + Q_{i,j} \end{aligned}$$

So in this case: $b = 0$ and c and y need to be modified. Similar approach for the left boundary condition. Beware at the corner node, both a and b drop out...

3.2 Neumann top and the right condition

$$C_{n,j}\frac{\Delta h_{n,j}}{\Delta t} = -\frac{q_T - q_{n-\frac{1}{2},j}}{\Delta z_{n-\frac{1}{2}}} - \frac{q_{n,j+\frac{1}{2}} - q_{n,j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}} + Q_{n,j}$$

Substitution gives:

$$C_{i,j}\frac{\Delta h_{1,j}}{\Delta t} = -\left[\frac{q_T + K_{\frac{1}{2},j}\left(\frac{h_{1,j}-h_{i-1,j}}{\Delta z_{i-1}}+1\right)}{\Delta z_{i-\frac{1}{2}}}\right] - \left[\frac{-K_{i,j+\frac{1}{2}}\left(\frac{h_{i,j+1}-h_{i,j}}{\Delta x_j}+1\right) + K_{i,j-\frac{1}{2}}\left(\frac{h_{i,j}-h_{i,j-1}}{\Delta x_{j-1}}+1\right)}{\Delta x_{j-\frac{1}{2}}}\right] + Q_{i,j}$$

and rearranging gives:

$$\begin{aligned} C_{i,j}\frac{\Delta h_{i,j}}{\Delta t} \quad = \quad & \frac{K_{i,j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_{j-1}}h_{i,j-1} + \frac{K_{i-\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}\Delta z_{i-1}}h_{i-1,j} - \left(\frac{K_{i-\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}\Delta z_{i-1}} + \frac{K_{i,j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_{j-1}} + \frac{K_{i,j+\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_j}\right)h_{i,j} + \\ & \frac{K_{i+\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}\Delta z_i}h_{i+1,j} + \frac{K_{i,j+\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_j}h_{i,j+1} - \frac{K_{i-\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}} - \frac{q_T}{\Delta z_{i-\frac{1}{2}}} + Q_{i,j} \end{aligned}$$

3.3 Robbins bottom and left

$$i = 1$$

$$C_{i,j}\frac{\Delta h_{i,j}}{\Delta t} = -\frac{q_{i+\frac{1}{2},j} - q_{i-\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}} - \frac{q_{i,j+\frac{1}{2}} - q_{i,j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}} + Q_{i,j} \tag{8}$$

$$q_{i-\frac{1}{2}} = -K_{S_{i,j}}(h_{i,j} - h_{Amb})$$

Expansion of this equation for the vertical gives:

$$C_{i,j}\frac{\Delta h_{i,j}}{\Delta t} = -\left[\frac{-K_{i+\frac{1}{2},j}\left(\frac{h_{i+1,j}-h_{i,j}}{\Delta z_i}+1\right)+K_{S_{i,j}}\left(h_{i,j}-h_{Amb}\right)}{\Delta z_{i-\frac{1}{2}}}\right] - \left[\frac{-K_{i,j+\frac{1}{2}}\left(\frac{h_{i,j+1}-h_{i,j}}{\Delta x_j}+1\right)+K_{i,j-\frac{1}{2}}\left(\frac{h_{i,j}-h_{i,j-1}}{\Delta x_{j-1}}+1\right)}{\Delta x_{j-\frac{1}{2}}}\right]+Q_{i,j}$$

Rearranging gives:

$$\begin{aligned} C_{i,j}\frac{\Delta h_{i,j}}{\Delta t} &= \frac{K_{i,j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_{j-1}}h_{i,j-1} - \left(\frac{K_{S_{i,j}}}{\Delta z_{i-\frac{1}{2}}} + \frac{K_{i+\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}\Delta z_i} + \frac{K_{i,j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_{j-1}} + \frac{K_{i,j+\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_j}\right)h_{i,j} + \\ & \frac{K_{i+\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}\Delta z_i}h_{i+1,j} + \frac{K_{i,j+\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_j}h_{i,j+1} + \frac{K_{i+\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}} + \frac{K_{S_{i,j}}}{\Delta z_{i-\frac{1}{2}}}h_{Amb} + Q_{i,j} \end{aligned}$$

3.4 Robbins top and right

$$i = n$$

$$C_{i,j}\frac{\Delta h_{i,j}}{\Delta t} = -\frac{q_{i+\frac{1}{2},j}-q_{i-\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}} - \frac{q_{i,j+\frac{1}{2}}-q_{i,j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}} + Q_{i,j} \tag{9}$$

$$q_{i+\frac{1}{2},j} = -K_{S_{i,j}}(h_{Amb}-h_{i,j})$$

Expansion of this equation for the vertical gives:

$$C_{i,j}\frac{\Delta h_{i,j}}{\Delta t} = -\left[\frac{-K_{S_{i,j}}(h_{Amb}-h_{i,j})+K_{i-\frac{1}{2},j}\left(\frac{h_{i,j}-h_{i-1,j}}{\Delta z_{i-1}}+1\right)}{\Delta z_{i-\frac{1}{2}}}\right] - \left[\frac{-K_{i,j+\frac{1}{2}}\left(\frac{h_{i,j+1}-h_{i,j}}{\Delta x_j}+1\right)+K_{i,j-\frac{1}{2}}\left(\frac{h_{i,j}-h_{i,j-1}}{\Delta x_{j-1}}+1\right)}{\Delta x_{j-\frac{1}{2}}}\right]+Q_{i,j}$$

Rearranging gives:

$$\begin{aligned} C_{i,j}\frac{\Delta h_{i,j}}{\Delta t} &= \frac{K_{i,j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_{j-1}}h_{i,j-1} + \frac{K_{i-\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}\Delta z_{i-1}}h_{i-1,j} - \left(\frac{K_{i-\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}\Delta z_{i-1}} + \frac{K_{S_{i,j}}}{\Delta z_{i-\frac{1}{2}}} + \frac{K_{i,j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_{j-1}} + \frac{K_{i,j+\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_j}\right)h_{i,j} + \\ & \frac{K_{i,j+\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}\Delta x_j}h_{i,j+1} - \frac{K_{i-\frac{1}{2},j}}{\Delta z_{i-\frac{1}{2}}} + \frac{K_{S_{i,j}}}{\Delta z_{i-\frac{1}{2}}}h_{Amb} + Q_{i,j} \end{aligned}$$