Project 3: Simulation of the Solar system

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In this project, the numerical integration methods utilized are the forward Euler and velocity Verlet algorithms for solving differential equations governing the motions of celestial bodies in our Solar system. The results obtained with the velocity Verlet algorithm had a significantly higher level of precision than the forward Euler method. As the Verlet algorithm uses 9 FLOPs per interation, whereas the Euler algorithm uses 4, its CPU-time is higher, although the difference in run-time proves to be less than expected. When modeling the Solar system for n = 1,000,000 and $dt = 0.001$ the CPU-time was found to be 94.3 seconds when using the forward Euler method, while 108.1 seconds for the velocity Verlet method. Due to the improvement of accuracy, we have found the additional computational time worthwhile. Further, we found that in the calculations of celestial movements on closed orbits, both the total mechanical energy and total angular momentum of the system were conserved, thus the results obtained are in agreement with Kepler's second law. For the Earth-Sun system we found that a numerical initial velocity of 8.805 AU allowed the Earth to escape from its orbit, while the analytical limit was shown to be $\sim$ 8.886 AU. Additionally, the force of gravity was tested for different values of exponentials β for $1/r$, and as anticipated the gravitational pull became weaker the larger the β, thus resulting in a lower limit for the escape velocity. Furthermore, the effects of the mass of Jupiter on the Earth's orbit was investigated, and it was discovered that the bigger the mass, the more elliptical the orbit became. In addition, it was observed that the orbit of the Earth obtained a distinct perihelion precession when the mass of Jupiter was $M_\odot/10$. Finally, calculation of the perihelion precession of Mercury proved disappointing, as the computed relativistic perihelion angle after 100 years was found to be 384.908" while the expected value is 43". This is presumably caused by lack of precision, and we suppose that a smaller value of dt would increase the accuracy. Unfortunately, it proved too time-consuming to run this calculation for smaller values than $dt = 0.000001$.

I. INTRODUCTION

Understanding the Solar system was a prodigious part of ancient philosophy and the birth of physics. For most of history, humanity did not comprehend the concept of the Solar system. The public understanding was that the Earth was centre of the universe, where the Sun and the other planets orbited the Earth. Though ancient philosophers and scientists questioned the religious interpretation, the real scientific breakthrough of the modern understanding of gravity was introduced by the astronomers Galileo Galilei and Johannes Kepler in the 17th century. Their theories set the stage for Newton's theory of gravity published a century later.

Newton's groundbreaking theory formulated gravity with one simple equation, which describes falling objects on Earth along with the movements of massive objects in the universe. Newton's law of gravitation follows the inverse-square law, which specified means that the inverse square of the distance from a source is proportional to a specific physical entity. In addition to Newton's gravitational law, the forces between two charged particles follow this proportionality. In fact, much of Nature can physically be described by some version of the inverse-square law, such phenomena ranging from electromagnetic radiation to the intensity of sound in a gas.

Unfortunately, Newton theory of gravitation was found to have flaws. In the 19th century observational astronomers found that Mercury's trajectory could not uniquely be described by Newton's law of universal gravitation, due to its slight perturbations. The issue was resolved by Einstein's general relativity published in the early 1900s. The Einstein's theory is so far the best interpretation we have of gravity.

Our solar system consist of eight planets, Mercury, Venus, the Earth, Mars, Jupiter, Saturn, Uranus and Neptune, orbiting the sun. In addition there are so-called dwarf planets, including Pluto, also in orbit. In this report we will use data from NASAs JPL HORIZONS [1] and laws of Newtonian physics to model their orbits in three dimensions. Since Newton's understanding of gravity is not sufficient to describe Mercury's orbit, we will use the general relativistic correction in order to obtain more accurate results. Interestingly, we can compare the two theories' when studying Mercury's trajectory and study the impact of Einstein's correction.

The calculations of the orbits of the planets and Sun do not have analytical solutions when adding the gravitational interactions between the planets. The analytical solutions are limited to the two-body problem. We will not only include the forces acting on the planets from the Sun, we will further add the gravitational attraction between the planets as well as the planets' impact on the Sun's position. The solutions of the motions of the Sun and its orbiting planets do not have an analytical solution. Thus we need to use numerical tools in order to calculate the positions and velocities of the objects.

We wish to develop a code simulating the Solar system using two popular algorithms for solving coupled ordinary differential equations. The first algorithm is the forward Euler method. It is deduced by a first order Taylor expansion, and uses few FLOPs in order to solve the coupled ODE. However, the results after the integration will be of lower accuracy than the other algorithm, the velocity Verlet algorithm. Though being more precise, it pays the price of a higher number of FLOPs. In this report we will investigate the differences in the two algorithms' results and run-times.

II. THEORY

Newton's law of universal gravitation

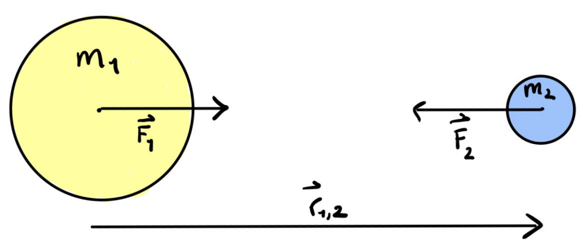


Figure 1. A sketch of the gravitational force from object 1 to object 2 and vice versa. Newton's third law states that $\vec{F_1} = -\vec{F_2}$.

The classical mechanics, involving Newton's law of gravitation, states that the gravitational force from one object with mass m_1 to another with mass m_2 is given by,

$$\vec{F}_2 = G\frac{m_1 m_2}{\mid \vec{r}_{2,1} \mid^3}\vec{r}_{2,1}. \tag{1}$$

Where G is the gravitational constant and $\vec{r}_{2,1}$ is the distance vector from object 2 to object 1 in three dimensions. The equation can be seen as an approximation of Einstein's general relativity.

According to Newton's third law, the force from m_2 to m_1 is $\vec{F_1} = -\vec{F_2}$. Thus the gravitational forces are forces of attraction. This is presented in figure 1.

If there are more than two objects interacting. The force on one of the N objects can be written as,

$$\Sigma\vec{F}_i = \Sigma_{i\neq j}^{j=\mathrm{N}} G\frac{m_i m_j \vec{r}_{i,j}}{\mid \vec{r}_{i,j} \mid^3}. \tag{2}$$

Where i is the object's index and $j \in \{0, N\}$ is the indices of the total number of objects.

Newton's second law states,

$$\Sigma\vec{F} = m\vec{a}. \tag{3}$$

Where ΣF is the sum of forces acting on an object, m the mass of the objects and $\vec{a}$ its acceleration. This is for object i obtained by 2 if all other forces than the gravitational force is negligible.

Furthermore, if we know the acceleration, we can obtain the velocities and positions of a moving object from the differential equations,

$$\frac{d}{dt}\vec{v}(t) = \vec{a}(t), \tag{4}$$

$$\frac{d}{dt}\vec{r}(t) = \vec{v}(t) \tag{5}$$

Conservation of angular momentum

The external torque of an object affected by gravitational forces from other objects is zero. The torque is obtained by the cross product between the position vector and the gravitational force vector. The two vectors are parallel, such that the torque is zero.

$$\tau_{\mathrm{ext}} = \mid \vec{r} \times \vec{F} \mid = G\frac{mM}{\mid \vec{r} \mid^3} \mid \vec{r} \times \vec{r} \mid = 0. \tag{6}$$

Due to the angular momentum's relation to the external torque, it can be shown that it is conserved,

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\mathrm{ext}} = 0. \tag{7}$$

Kepler's second law

Kepler's second law stats that the distance vector between the planets and the Sun in the Solar system sweeps out equal amount of area over an equal amount of time. This is a result of conservation of the total angular momentum of the Solar system. For a planet in a circular orbit, this is trivial, as the distance vector will be orthogonal to the velocity vector at all times. Thus, the gravitational acceleration will not affect the velocity's magnitude, only its direction, and since the magnitude of the distance and the velocity are both preserved, along with the orthogonality, the angular momentum,

$$\vec{L} = m\vec{r} \times \vec{v} \tag{8}$$

is preserved.

Conservation of energy

The non-relativistic kinetic energy K of an object with velocity $\vec{v}_1$ and mass m_1 is

$$K = \frac{1}{2}m_1 v_1^2, \tag{9}$$

where v_1 is the length of the velocity of our said object with respect to another object.

Furthermore the potential energy U of our object can be deduced by the mechanical entity $F = -\Delta U$, since the gravitational force F is conservative,

$$U = \int_0^\infty \vec{F}_{1,2} \cdot d\vec{r}_{1,2}, \tag{10}$$
$$= -G\frac{m_1 m_2}{r_{1,2}}. \tag{11}$$

Here, $r_{1,2}$ is the absolute value of the distance between the two bodies.

For circular orbits it is apparent that the potential energy is constant, due to the distance from the centre of mass $r_{1,2}$ being constant. Furthermore, the kinetic energy is also constant, as the magnitude of the velocity isn't affected by the gravitational pull. Thus, the total mechanical energy is naturally preserved.

With that being said, even for non-circular orbits, the total mechanical energy of the centre of mass is conserved, which is due to the gravitational force being conservative. As long as the only apparent forces acting on the objects in the Solar system are the N gravitational forces for the interacting objects, the total mechanical energy of the centre of mass ofSolar system will therefore remain unchanged.

Escape velocity

The total mechanical energy of an object is the sum of its potential and kinetic energy. For an object with mass m_1 and speed v_1 in the two-body problem it reads,

$$E_{\text{tot}} = K + U = \frac{1}{2}m_1 v_1^2 - G\frac{m_1 m_2}{r_{1,2}}. \tag{12}$$

The escape velocity $v_{1,\text{ esc}}$ of the object is obtained when the kinetic energy is larger than the potential energy, such that $E_{\text{tot}} > 0$. Thus, the escape velocity must be

$$v_{1,\text{ esc}} > \sqrt{\frac{2Gm_2}{r_{1,2}}}. \tag{13}$$

Specifying the problem to the two-body system containing the Earth and the Sun, it is useful to introduce alternative units. The length 1AU is defined as the distance between the Earth and the Sun. By measuring time in units of years and making the simplification that the orbit of the Earth around the Sun is circular, we further have that the orbit speed of the Earth must be

$$v_{\text{orbit}} = \frac{2\pi r_{\text{S, E}}}{T} = 2\pi\left[\frac{\text{AU}}{\text{yr}}\right], \tag{14}$$

where T is the time of one full orbit and $r_{\text{S, E}}$ is the distance between the Sun and the Earth.

Neglecting other forces acting on the Earth, using Newton's second law we have

$$m_{\text{E}} a = G\frac{m_{\text{E}} m_{\text{S}}}{r_{\text{S, E}}^2},$$

where the mass of the Earth and Sun is denoted m_{E} and m_{S} respectively, while $a = v_{\text{orbit}}^2 / r_{\text{S, E}}$ is the centripetal acceleration of the Earth. With mass measured in units of solar masses $M_\odot$, distance measured in AU and time measured in years we thus find

$$\frac{v_{\text{orbit}}^2}{r_{\text{S, E}}} = G\frac{m_{\text{S}}}{r_{\text{S, E}}^2}, \tag{15}$$
$$4\pi^2\left[\frac{\text{AU}}{\text{yr}^2}\right] = G\left[\frac{M_\odot}{\text{AU}^2}\right], \tag{16}$$

which in turn leads to

$$G = 4\pi^2\left[\frac{\text{AU}^3}{M_\odot \text{yr}^2}\right].$$

With this inserted in equation 10, we find the Earth's escape velocity to be

$$v_{\text{Earth,esc}} > 2\sqrt{2}\pi\left[\frac{\text{AU}}{\text{yr}}\right]. \tag{17}$$

Generalization of Newton's law of gravitation

We will now look at the case where the gravitational force is from a body j to another body i is,

$$F_{ij} = G\frac{m_i m_j}{r_{i,j}^\beta}\vec{e}_{i,j}. \tag{18}$$

Where $\vec{e}_{i,j}$ is the distance unit vector. By integrating over the position, as in equation 11, we obtain the potential energy,

$$U = -\frac{1}{\beta - 1}G\frac{m_j m_i}{r^{\beta-1}} = -\frac{1}{\alpha}G\frac{m_j m_i}{r^\alpha}, \tag{19}$$

where $\alpha = \beta - 1$. When $\beta > 2$, $\alpha > 1$, thus the expression for the potential energy U is smaller in magnitude (and thereby less negative) than in the strictly Newtonian case for the same value of r. This results in a lower escape velocity. It can also be shown that orbits will not be stable for $\beta \geq 3$ by using principles of the Lagrangian.

General relativistic correction

Newton's law of universal gravitation is not sufficient to describe planets and stars movement under a strong gravitational field. From [2] we have that the correction leads to a second term to Newton's theory of gravitational forces. The correction yields that the gravitational force is,

$$F_{\mathrm{GR}} = F_{\mathrm{Newton}}\Big[1 + \frac{3l^2}{r^2c^2}\Big]. \tag{20}$$

Where F_{Newton} is Newton's gravitational force, c speed of light in vacuum equal to $3.00 \cdot 10^8$m/s, r distance between the two objects and finally l is the angular momentum of the object equal to $l = \mid \vec{r} \times \vec{v} \mid$.

III. ALGORITHMS

In equation 5 the coupled first order general differential equations are presented, where the acceleration a is obtained by equation 2. In order to solve for the positions and velocities, we will utilize two different numerical integration algorithms, the forward Euler and the velocity Verlet algorithm. The two methods have a different number of floating point operations (FLOPs), which produces different run-times (see source [3]). The numerical errors, obtained by $O(h^n)$, are also different in the algorithms.

In order to numerically solve the differential equations, we need to discretize the time t and also the vectors $\vec{a}(t), \vec{v}(t)$ and $\vec{r}(t)$. This yields $t_i = t_0 + ih$, where t_0 is the initial time (often set to zero), i is the index leaping over $0, 1, 2, ..., N-1$ where N is the total number of integration points obtained by $\frac{t_{\max} - t_0}{h}$. And finally h is the step size, which is set to be a floating number $<< 1$.

Forward Euler integration

Euler's forward method is a first-order numerical algorithm which solves ordinary differential equations, or so-called ODEs, by using the two first terms of the Taylor expansion. It reads,

$$\vec{r}(t+h) = \vec{r}(t) + h\frac{d}{dt}\vec{r}(t) + O(h^2). \tag{21}$$

Where $O(h^2)$ is the deviation which is proportional to h^2.

For the coupled ODE we are studying the algorithm reads,

$$\vec{r}_{i+1} = \vec{r}_i + h\vec{v}_i \tag{22}$$
$$\vec{v}_{i+1} = \vec{v}_i + h\vec{a}_i \tag{23}$$

Thus, to achieve the positions and velocities of the planets in the Solar System we need initial values of position, velocity and acceleration. The algorithm reads,

Make a function of $\vec{a}(r_i) = \vec{a}_i$
defining $t_{\max}$, t_0, y_0, h
$N = (t_{\max} - t_0)/h$

for $i \in \{0, N-1\}$ **do**
  Compute $\vec{a}_i$
  $\vec{r}_{i+1} = \vec{r}_i + h\vec{v}_i$
  $\vec{v}_{i+1} = \vec{v}_i + h\vec{a}_i$
end for

Each iteration has 4 FLOPs, such that the total number of FLOPs is $4N$. Since the method is deduced by a first order Taylor expansion, the numerical error is proportional to h^2 the error in the Euler algorithm is also proportional to h^2. From U. Fjordholms paper *Numerical methods for ordinary differential equations*[4], the method is not energy conserving. We will therefore expect the energy of the system not to be conserved when utilizing the algorithm. However, for a sufficiently small value of h will most likely reduce the error of energies.

Velocity Verlet integration

We will also make use of the velocity Verlet integration. For an object with an unknown position $\vec{r}(t)$ as a function of time t, we can discretize this position by h and make a second order Taylor expansion which reads,

$$\vec{r}(t+h) = \vec{r}(t) + \frac{d}{dt}\vec{r}(t)h + \frac{1}{2}\frac{d^2}{dt^2}\vec{r}(t)\Delta t^2 + O(h^3), \tag{24}$$

to recover $\vec{r}(t+h)$ which is the position of the object at the next time-step. $O(h^3)$ now is the error, which is proportional to h^3.

We discretize $r(t+h)$ similar to the Forward Euler method and leap over $i \in \{0, N-1\}$. The algorithm with the gravitational forces will be,

Setting initial position and velocity, $\vec{v}_0$ and $\vec{r}_0$
Make a function of $\vec{a}_i$
Setting $t_{\max}$, t_0
$N = (t_{\max} - t_0)/h$

for $i \in \{0, N-1\}$ **do**
  Compute $\vec{a}_i$
  $\vec{r}_{i+1} = \vec{r}_i + h\vec{v}_i + \frac{h^2}{2}\vec{a}_i$
  Compute $\vec{a}_{i+1}$
  $\vec{v}_{i+1} = \vec{v}_i + \frac{h}{2}(\vec{a}_{i+1} + \vec{a}_i)$

end for

The algorithm uses 9 FLOPs per iteration such that the total number of FLOPs is $9N$. We will therefore expect the run-time to increase sufficiently from the Euler algorithm.

IV. METHOD

We will now go on to calculate the mechanical energies, angular momentum, as well as movement of the planets in the Solar System with the forward Euler and the velocity Verlet algorithm. First, we will write a code which numerically solves the differential equations of motion for the Earth-Sun system, with data of the positions and velocities from NASA (source [1]). Secondly, we will test the two algorithms' reliability and run-time by comparing with analytical results. Then we will go on and test different forms of the gravitational force with varying values of β from equation 18. Finally, we will use Einstein's general relativity in order to correctly Mercury's trajectory.

The Earth-Sun system

Testing the forward Euler and velocity Verlet integration

We start by modelling the movement of the Sun and Earth, ignoring the other planetary bodies' effect. We set the centre of mass of the two body system in the origin. Since $M_{\odot} >> M_{\mathrm{earth}}$, the position of the sun at all times will be $r_{\odot} \approx (0, 0, 0)$.

Firstly, we will test if the orbit of the Earth is circular when its tangential initial velocity is equal to $2\pi\frac{\mathrm{AU}}{\mathrm{yr}}$ and the radial velocity component is zero. Further, we will run the two different integration algorithms introduced earlier for the initial values of position and velocity that analytically should produce a circular orbit of the Earth. Since forward Euler integration is not energy conserving and has a higher degree of error than velocity Verlet integration, we expect the error to be higher than for velocity Verlet integration.

We will also check if the kinetic and potential energy is conserved. For circular orbits the kinetic and potential energy should both be constant. Thus we need to see if the initial potential and kinetic energy is preserved after a number of time steps. Again, since the Euler method is not energy conserving we expect the deviation from the initial mechanical energy to be higher using this method than for the velocity Verlet integration algorithm.

In addition we will compare the two different algorithms run-times for values of N. We have earlier obtained the number of FLOPs for the two algorithms. Since the number of FLOPs for the velocity Verlet algorithm is more than double of the number of FLOPs for the Euler algorithm, we would expect the run-time to be over two times the run-time when using the Euler algorithm.

Conservation of angular momentum

We will further use strictly the velocity Verlet algorithm as this is the energy conserving integration method, and we will investigate the conservation of the angular momentum both for circular and elliptical orbits of the Earth.

Testing forms of the gravitational force

Furthermore, we wish to observe the orbit of the Earth as a function of a varying β, where β is the exponential of the distance r from equation 18. We will simulate the movement of the Earth with the same initial position and velocity for each β, and what we would expect is that as β becomes larger, the weaker the gravitational hold from the Sun on the Earth will be. Thus we expect that at some point, for a large enough value of β, the gravitational force from the Sun is sufficiently small such that the Earth is not longer in a closed orbit around the Sun.

Escape velocity

In the theory section we found an analytical value of the escape velocity of the Earth, when modeling the Earth-Sun system. We will investigate how the analytical value, obtained in equation 17 of the escape velocity of the Earth matches the numerical results which we will find by trial and error.

Modelling the Solar system

The three-body problem

Further, we will study the three-body problem, now introducing Jupiter to the previous Earth-Sun system. By varying the mass of Jupiter, simulating the Earth's movement for the mass of Jupiter being $M_{\odot}/1000$, $M_{\odot}/100$ and $M_{\odot}/10$, we wish to investigate how the mass of Jupiter alters the Earth's orbit around the Sun. Furthermore, we will see if the mechanical energy of the system is impacted in any way, and in light of this discuss the stability of the Verlet method.

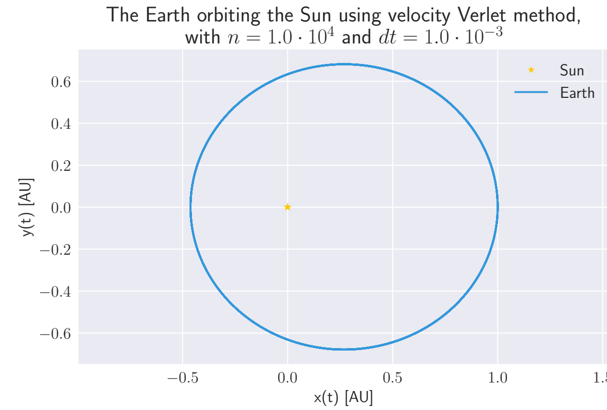
Final model of all planets

Subsequently, we will simulate the movement of all nine planets, counting Pluto, as well as the Sun. To do this we have gathered information such as initial velocities and positions of all the celestial bodies from NASAs JPL HORIZONS [1], and wish to compare the following resulting motion of the bodies with the previously produced results.

As for the other parts of our project, we will fix the center of mass such that it remains in (0, 0, 0) in our coordinate system. This is done by firstly calculating the initial position and velocity for the center of mass with the initial conditions extracted for NASA, and then subtracting it from all other positions and velocities. Thus all positions and velocities will be computed with respect to the center of mass.

The perihelion precession of Mercury

Newton's law of gravitation does not hold for Mercury's orbit. Because the distance between the Sun and Mercury is small, the gravitational pull from the Sun is substantial, compared to the other planets. It would therefore be of interest to add a relativistic correction term to our gravitational force. With this, we can see how the orbit of Mercury calculated with the pure Newtonian gravitational force differs from the more accurate orbit with the relativistic effects taken into account.



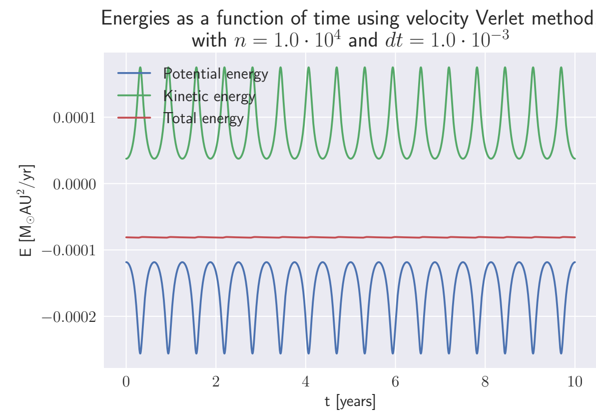


Figure 2. The figure shows mechanical energies and position for the Earth with initial distance from the Sun equal to 1 AU and initial velocity 5 AU/yr, for $n = 10000$, $dt = 0.001$.

V. RESULTS

The Earth-Sun system

Testing the forward Euler and velocity Verlet integration

Figure 3 shows plots of the Earth orbiting the Sun, calculated with forward Euler and velocity Verlet, respectively. The initial value for the velocity has been set to 2π AU/yr in order for the Earth to have a circular orbit, while $dt = 0.001$ and $n = 1000$. We observe how the position of the Earth varies for different time steps dt, and we note that the Verlet method provides perfect circular orbits, while the Euler method does not.

In figure 4 we see that the potential and kinetic energy is constant for the Verlet method, while for forward Euler it changes slightly.
The CPU-time for $n = 1000$ and $dt = 0.001$ is 0.0270 seconds when using the forward Euler method and 0.0271 seconds for the velocity Verlet method. When performing a timing of the two algorithms for $n = 10000$ we measure 0.2529 seconds when using forward Euler and 0.2506 seconds with velocity Verlet.

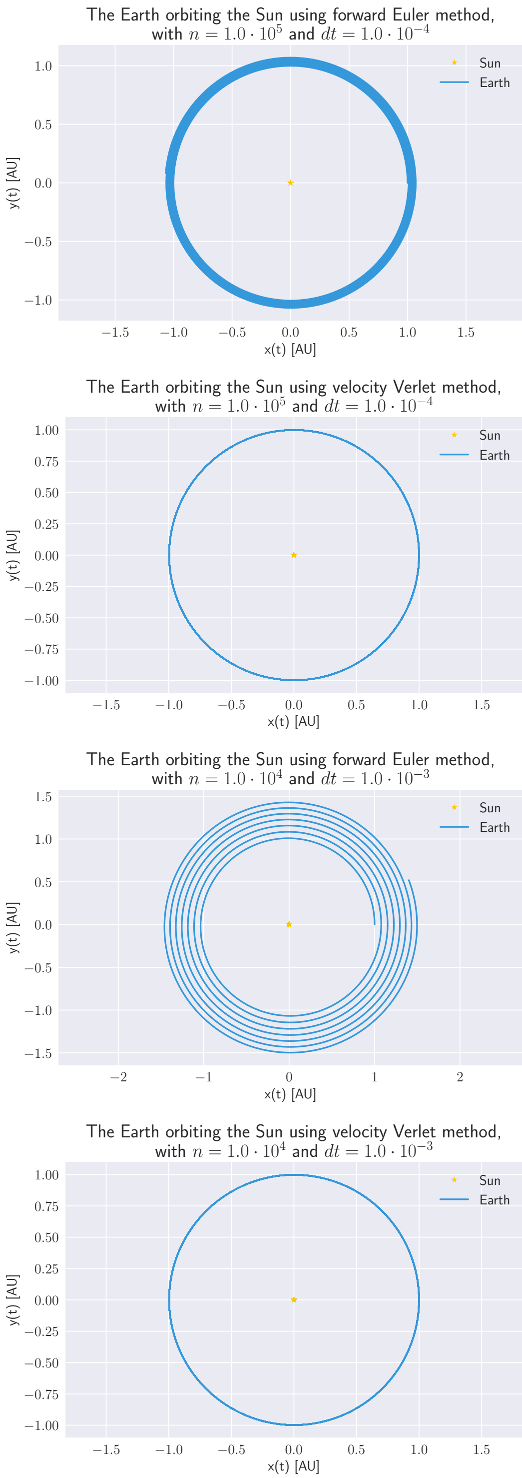


Figure 3. The figure displays the Earth orbiting the Sun for what equals 10 years on Earth, with different time steps dt, computed with the numerical integration methods forward Euler and velocity Verlet.

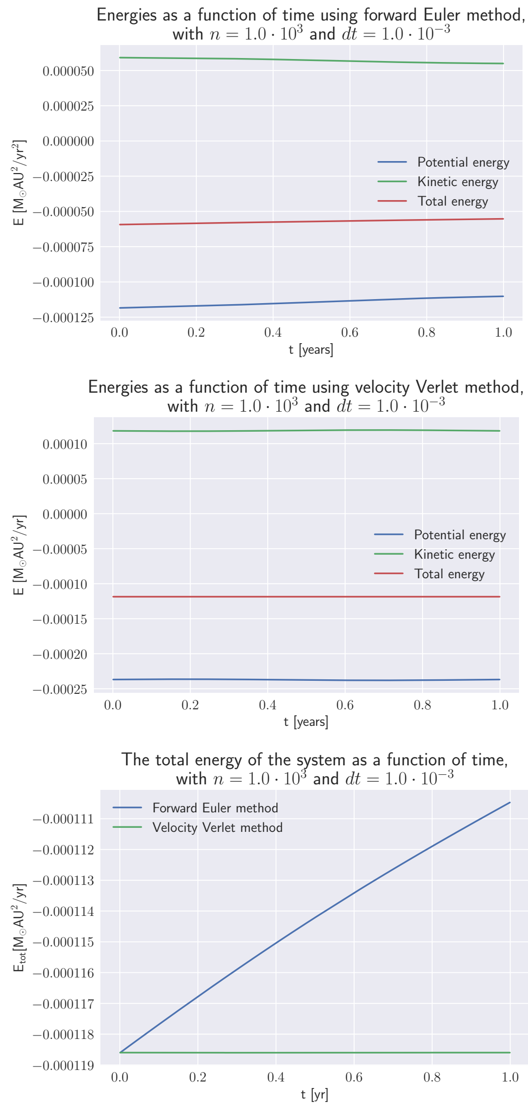


Figure 4. The figure displays mechanical energies for the two-body system with circular orbits for the Earth as a function of time, for n = 1000, dt = 0.001. In addition, it shows the total energies calculated by the Euler method and Verlet method as a function of time presented in the same plot.

Conservation of angular momentum

Figure 2 shows the movement of the Earth with initial position 1 AU from the Sun and initial velocity of 5 AU/yr. It is clear that these initial conditions makes the orbit of the Earth elliptical. In addition, the figure displays the total mechanical energy of the Earth-Sun-system. We see that the total energy is conserved, however both the kinetic and potential energy individually varies.

In figure 5 we see how the angular momentum is conserved for both elliptical and circular orbits for $n = 1000$, $dt = 0.001$.

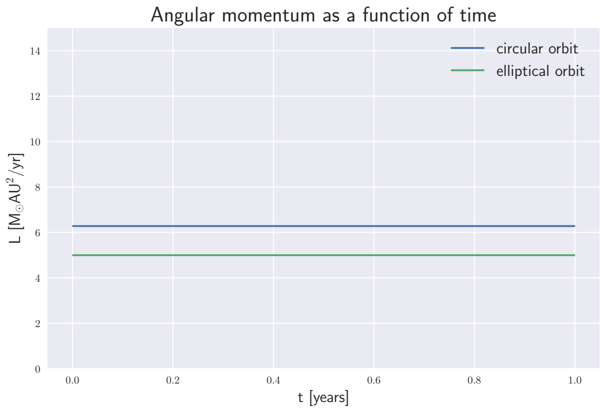


Figure 5. Graphical presentation of the total angular momentum for the Earth-Sun system, for both circular and elliptical orbits, for $n = 1000$, $dt = 0.001$.

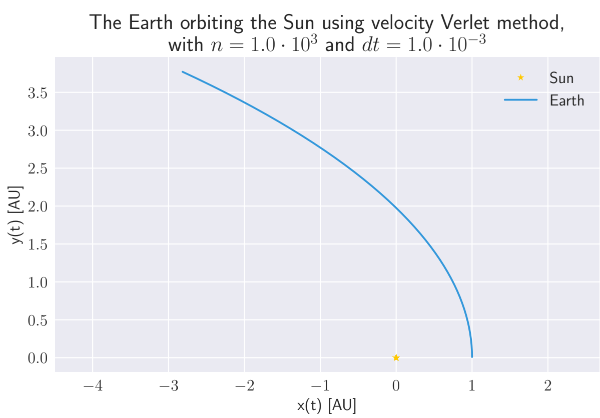


Figure 6. Motion of the Earth with initial velocity equal to 8.805 AU/yr.

Testing forms of the gravitational force

Figure 7 displays the bodies positions in the Earth-Sun system for $\beta = 2.2, 2.6$ and 3.0, with initial position of the Earth equal to 1 AU away from the Sun and initial velocity equal to 5 AU/yr, while the Sun is initially placed in the origin with zero initial velocity. We observe that for higher values of β, the gravitational force is weaker. For $\beta = 3.0$ the gravitational field of the Sun is unable to hold the Earth in a stable orbit, and the Earth escapes.

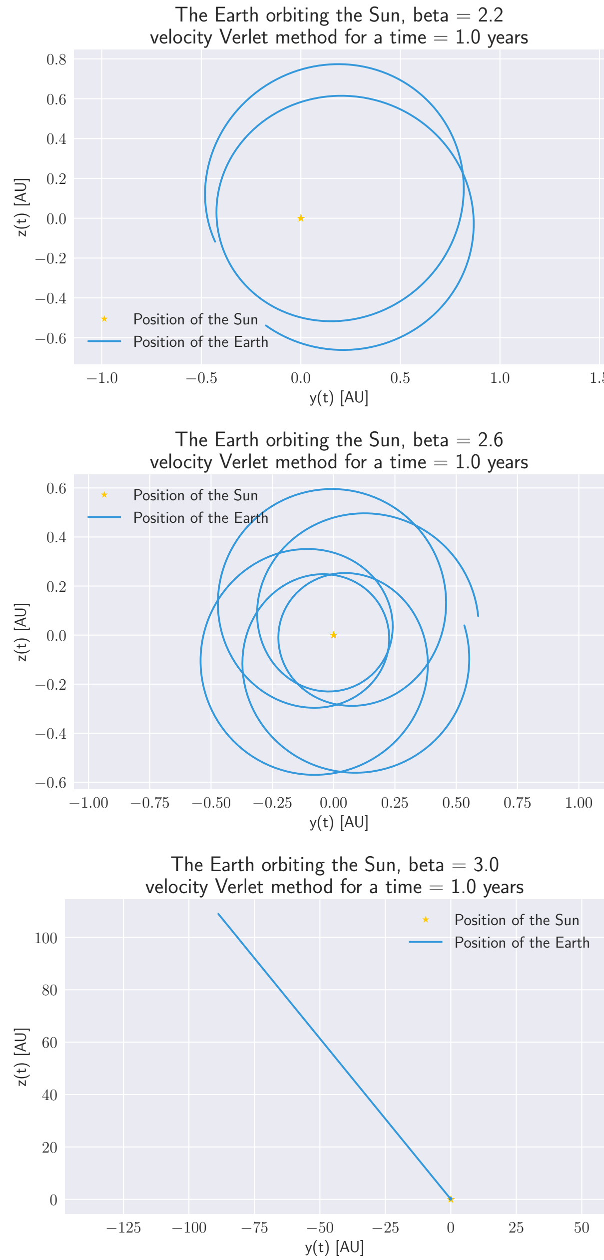


Figure 7. The Earth orbiting the Sun for n = 10000 time steps with size of time step $\Delta t = 0.001$, computed with the velocity Verlet method, displayed for $\beta = 2.2, 2.6$ and 3.0. Initial position of the Earth is equal to 1 AU away from the Sun and the initial velocity equal to 5 AU/yr.

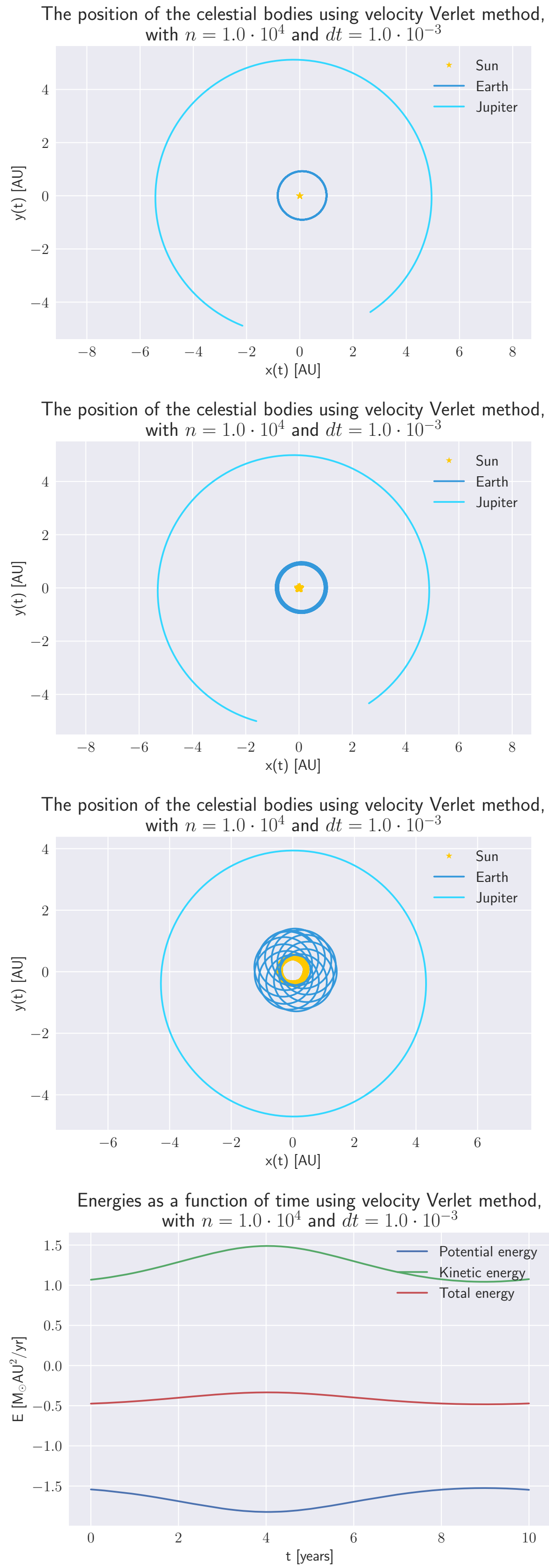


Figure 8. Graphical presentation of the xy-positions and the energies as function of time for the three-body problem, with the Sun, Earth and Jupiter. From the plots of the positions, the mass of Jupiter is 1000, 100 and 10 times smaller than the Sun respectively. In the bottom figure the mechanical energies of the system is presented as a function of time, with the mass of Jupiter 10 times smaller than the Sun.

Escape velocity

In figure 6 we observe the motion of the Earth for initial position equal to 1 AU from the Sun, with initial velocity 8.805 AU/yr. We observe that the Earth escapes from its orbit, when having a initial velocity close to the theoretical escape velocity, $v_{\mathrm{esc}} > 2\sqrt{2}\pi\mathrm{AU/yr} \approx 8.886\mathrm{AU/yr}$.

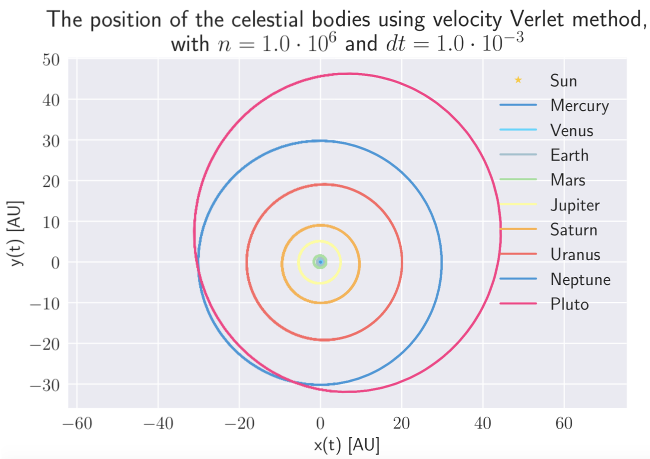
Modelling the Solar system

Three body problem

From figure 8 we observe how Jupiter alters the Earth's and the Sun's motion. From the presentation we see that the Earth stays in orbit around the Sun, although the orbit is affected greatly when the mass of Jupiter is increased to 1/10 of the solar mass. In the bottom of the figure, the mechanical energies of the system is presented, for a time interval equal to 10 years on Earth. We see that the total mechanical energy of the system is nearly conserved, although it deviates slightly from a perfectly constant line.

Final model of all planets

In figure 9 we see a graphical presentation of the movement for all the planets in the solar system, calculated for a time corresponding to a 1000 years on Earth. Further, we see that the total mechanical energy of the solar system is essentially conserved, taking into account that the amplitude of the oscillations observed for the energies is considerably small.
When tested for both integration methods, the CPU-time was found to be 94.3 seconds for $n = 1,000,000$ and $dt = 0.001$ when using the forward Euler method and 108.1 seconds for the velocity Verlet method. In other words, the CPU-time, when using velocity Verlet, is slightly higher, than for forward Euler.



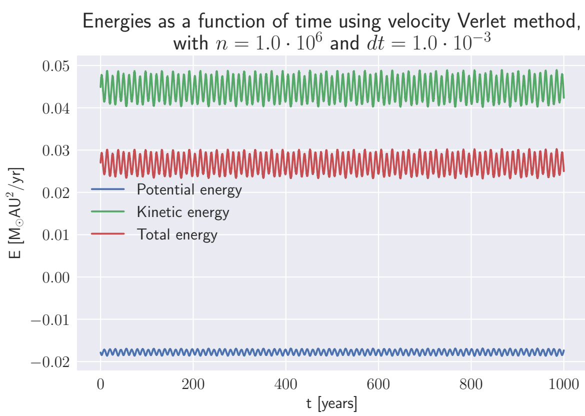


Figure 9. Final model for all planets of the solar system, produced with velocity Verlet's method. In the below presentation we see the total mechanical, kinetic and potential energy for the system, calculated with $n = 1,000,000$ and $dt = 0.001$, in which equals a 1000 years on Earth.

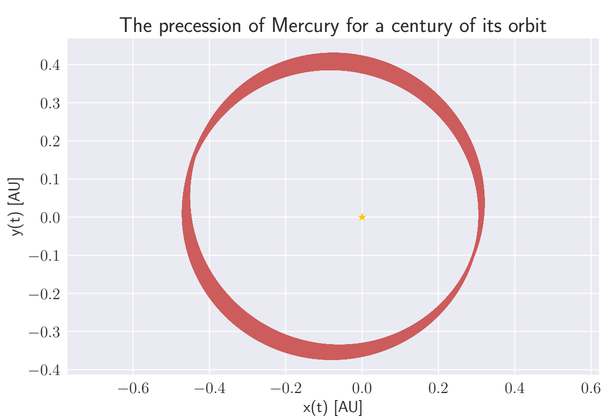


Figure 10. The figure displays the perihelion precession of Mercury for 100 years on Earth, with $n = 100,000$ and $dt = 0.001$

The perihelion precession of Mercury

Figure 10 shows the perihelion precession of Mercury for 100 years on Earth, with $n = 100,000$ and $dt = 0.001$. For a larger amount of time steps, the amount of time needed to produce the graphical presentation became too large. Still, we ran the simulation for $n = 100,000,000$ with $dt = 0.000001$, where the CPU-time ended up being 2536.92 seconds. With these values, the perihelion angles $\theta_{p,\text{Newtonian}} = -383.882$", while $\theta_{p,\text{Relativistic}} = 384.908$"" were found.

VI. DISCUSSION

From figure 3 we see that the value of the time step, dt, largely affects the motion of the Earth, when using the forward Euler method. However, with increasing dt, the level of precision seems to increase, and with a time step equal to 0.0001, the forward Euler method produces somewhat stable orbits. On the contrary the time step does not seem to affect the precision of the motion to the same degree when using the velocity Verlet method. For both $dt = 0.0001$ and $dt = 0.001$ the Verlet method produces orbits of a high level of accuracy. Due to this, in the remaining parts of the project we mostly utilize this method of integration. Moreover, since the time step $dt = 0.001$ proved sufficient in this case, most of the other parts of the project is also calculated with this time step.

When using forward Eulers method of integration with dt larger than 0.0001, we observe the orbit of the Earth to become unstable. This is coinciding with our expectations, due to the fact that this method is not symplectic. The velocity Verlet method, however, seems to conserve the energy of the system, since the method produces closed orbits. This is being confirmed when looking at the energy plots of the system, see figure 4. Moreover, the figure also agrees with the fact that the Euler integration method is not symplectic, and in the bottom plot this difference between the methods becomes even more apparent. We further see that the mechanical energy of the system in fact increases with time when using the forward Euler method.

As the number of FLOPs for the velocity Verlet algorithm is more than double the number of FLOPs for the forward Euler algorithm, we expected the CPU-time to be higher for this method than for the Euler method. Yet, we found that the run-time was only slightly higher for the Verlet algorithm, when simulating the two-body problem. The reason for this might be the fact that our algorithm also computes the energies as well as writes the computed values to files, while performing the timing. Moreover, there are many other factors one should take into account when analyzing the CPU-time, such as running other programs while measuring might disturb the results. Anyhow, this result tells us that the two algorithms uses approximately the same time when solving the differential equations 5, when studying the Earth-Sun system.

As expected, we see from figure 2 that the total mechanical energy is also approximately constant for elliptical orbits for the two-body problem. Unlike for the circular motion, we observe the kinetic and potential energies of the two-body system to oscillate individually, while the sum of the two remains almost unchanged in time. Because the radial distance between the Earth and the Sun varies for an elliptical orbit, the force of gravity will cause a retardation of the Earth's motion as the Earth moves further away from the Sun, as well as an acceleration when it moves nearer. This causes the oscillation of the kinetic energy, as it is dependent on the magnitude of the Earth's velocity. In addition, the cause for the oscillation of the potential energy is the fact that it is inversely proportional to the radial distance.

From figure 5 we see that the angular momentum is preserved, for both circular and elliptical orbits for the Earth-Sun system. When the Earth has a circular orbit, it remains at a constant distance from the Sun at all times. From equation 8, the quantity must be a constant for circular motion, as the position vector is always perpendicular to the momentum and neither the length of $\vec{r}$, nor the length of $\vec{p}$ changes as the Earth orbits the Sun. Moving on to the elliptical orbits, the angular momentum is still conserved as shown in equation 7, which is consistent with Kepler's second law. Hence, the results align with our expectations.

In figure 7 we have presented the movement of the Earth orbiting the Sun, for different values of the parameter β. We see that as β creeps towards 3, the Earth seems to fall out of its closed orbit. Setting β to 3, we see that the Earth loses its orbit completely, and escapes from the Sun. While increasing β, the gravitational force becomes weaker, and it is therefore not surprising that the Sun gets problem holding onto the Earth. This results in a lower value of the escape velocity for the Earth, as mentioned in the subsection **Generalization of Newton's law of gravitation** in the theory section. From the observation of the planetary motion it is hard to tell to which extent Nature deviates from a perfect inverse- square law. In our algorithm, only $\beta = 2.2, 2.4, 2.6, 2.8 and 3.0$ is implemented, however, from the figure 7 it is safe to say that the true value of β must lie somewhere between 2.0 and 2.2.

In figure 6 we have considered the Earth with an initial distance of 1 AU from the Sun. By trial and error we found that the numerical initial velocity in order for the Earth to escape from its orbit was equal to 8.805 AU/yr. From the theory section, we have that the analytical value of escape velocity of the Earth must be higher than approximately 8.886 AU/yr. We note that the numerical result is somewhat lower than the analytical, which implies that the numerical gravitational force is of a slightly smaller magnitude than the analytical one.

When considering the three-body problem, computing the motions of the Earth, Jupiter and the Sun using the Verlet algorithm, we find that both the total energy and angular momentum for the system is conserved, as all of the three celestial bodies moves on closed orbits. Even for the Jupiter mass equal to one tenth the mass of the Sun, the orbit of both the Earth and the Sun itself stays closed. When studying the energies of the system over a period equal to 10 years on Earth, see figure 8, we see that the variation in the total mechanical energy of the system is negligibly small. It is therefore safe to conclude that the stability of the solutions obtained from the Verlet solver is extremely high, as long as dt is sufficiently small. Moreover, from figure 8 we see that as the mass of Jupiter increases, the Earth's orbit becomes more and more elliptical, and its perihelion precess over time. This happens because the greater the mass of Jupiter, the greater its gravitational pull on the Earth becomes, which in turn affects the orbit of the Earth. We also see that as the mass of Jupiter increases, its gravitational pull becomes big enough to also significantly change the motion of the Sun, which makes sense for an object of one tenth the mass of the Sun.

In figure 9 we see a graphical presentation of the movement for all the planets in the solar system, for $n = 1,000,000$ and $dt = 0.001$. The positions of the planets have been calculated using the velocity Verlet method, and it is safe to say that the algorithm produces solutions with a great level of precision. Looking at the energy presentation of the system, we observe that the total mechanical energy of the solar system varies on a negligible small interval. When taking into account that the energies are calculated for a time interval equal to a 1000 years on Earth, our interpretation is that the total energy of the system to a large degree of precision is preserved. In alignment with our expectations, the velocity Verlet algorithm produces precise results.

When calculating the motion of the planets in the final model of the solar system, we found that the CPU-time with $n = 1,000,000$ and $dt = 0.001$ was slightly higher for the velocity Verlet algorithm than for forward Euler, which presumably is due to the difference in number of FLOPs. However, we know that the velocity Verlet method produces significantly more precise results, preserving the total energy of the system. In other words the additional computational cost is well worthwhile.

Finally, running the simulation of the perihelion precession of Mercury consumed a great amount of time, and although the resulting angles presented are not as we would expect, we presume they would have been more accurate if calculated for a smaller dt. We would then anticipate that the two angles would differ, as the angle computed for the case with the pure Newtonian force of gravity should be insignificantly small, while the angle accurately computed with the relativistic effects accounted for should be close to 43".

VII. CONCLUSION

We have utilized the numerical integration methods forward Euler and velocity Verlet for solving differential equations governing the motions of celestial bodies in our Solar system. As expected, we found that the results obtained with the velocity Verlet algorithm had a significantly higher level of precision than the forward Euler method. This is as discussed due to the symplectic nature of the Verlet algorithm, as opposed to the Euler algorithm which is not energy conserving. Meanwhile, as the Verlet algorithm has over double the amount of FLOPs compared to the Euler algorithm, its CPU-time is higher, and the greater the number of iterations, the more apparent the difference in CPU-time becomes. Still, we find the additional computational cost worthwhile due to the improvement of accuracy.

Furthermore, in the calculations of celestial movements on closed orbits, we have found the total mechanical energy to be conserved, although the kinetic and potential energy individually may oscillate for other than perfectly circular orbits. This is because of the conservative gravitational force, when all other eventual forces are set aside. The total angular momentum of the solar system is also found to be conserved, which is in line with Kepler's second law.

Testing for different values of exponentials β for $1/r$ in the gravitational force acting on the Earth, we observed as expected that the gravitational pull became weaker the larger the β. A higher value of β thus resulted in a lower limit for the escape velocity, such that the Earth showed to be able to escape its orbit with a velocity that for a lower β would keep it in a closed orbit. Moreover, the numerically tested value of the escape velocity for the Newtonian force of gravity turned out to be slightly lower than the analytical value, which indicates that the numerical force must be marginally weaker than in the analytical case.

The calculation of the perihelion precession of Mercury turned out to be somewhat unsuccessful for the tried value of timestep, as we did not produce the anticipated values of the perihelion angle. We expect the accuracy of the perihelion angle for Mercury's orbit computed with the pure Newtonian force of gravity, as well as with the gravitational force with relativistic correction would be higher if calculated for even smaller values of the time step. Unfortunately, it proved too time-consuming to run this calculation for smaller values.

Additionally, we investigated how the mass of Jupiter affected the orbit of the Earth around the Sun, and discovered that the bigger the mass, the more elliptical the orbit became. Further, the value of the mass also impacted the orbit of the Earth in the sense that the orbit obtained a distinct perihelion precession when the mass of Jupiter was one tenth of the solar mass. Finally, when modeling the entire Solar system, we noted that the velocity Verlet algorithm yet again provided results of high accuracy.

GITHUB

All data, code and plots are available at our GitHub page: `https://github.com/hedvigborgen/fys3150`. Follow the README.md file in order to run the codes.

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