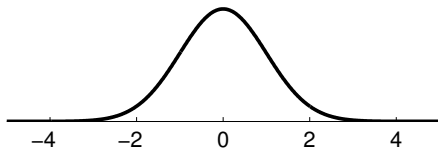


# Chapter 3

- 3.1 Marginalization
- 3.2 Normal distribution with a noninformative prior (important)
- 3.3 Normal distribution with a conjugate prior (important)
- 3.4 Multinomial model (can be skipped)
- 3.5 Multivariate normal with known variance (useful for chapter 4)
- 3.6 Multivariate normal with unknown variance (glance through)
- 3.7 Bioassay example (very important, related to one of the exercises)
- 3.8 Summary (summary)

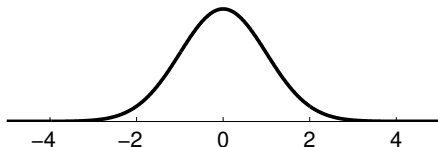
## Normal / Gaussian

$$p(y|\mu) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y - \mu)^2\right)$$



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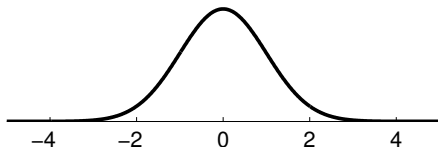
$$p(y|\mu) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y - \mu)^2\right)$$



- Normal: some prominent statistician started to use this term systematically in the end of 19th century (but it's a bit of exaggeration)

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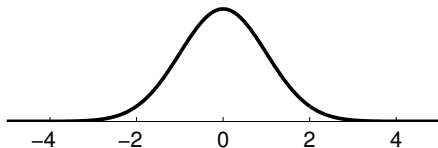
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- Gaussian: favored by Germans (but it was studied earlier by De Moivre and Laplace, and Gauss avoided using it)
- Shorthand notation
  - $y \sim N(\mu, \sigma^2)$  with variance  $\sigma^2$   
(useful in derivations)
  - $y \sim \text{normal}(\mu, \sigma)$  with deviation  $\sigma$   
(useful for interpreting prior and posterior scales, used in Stan)

# Normal / Gaussian

- Normal linear regression  
connection to least squares regression via  $(y - \mu)^2$

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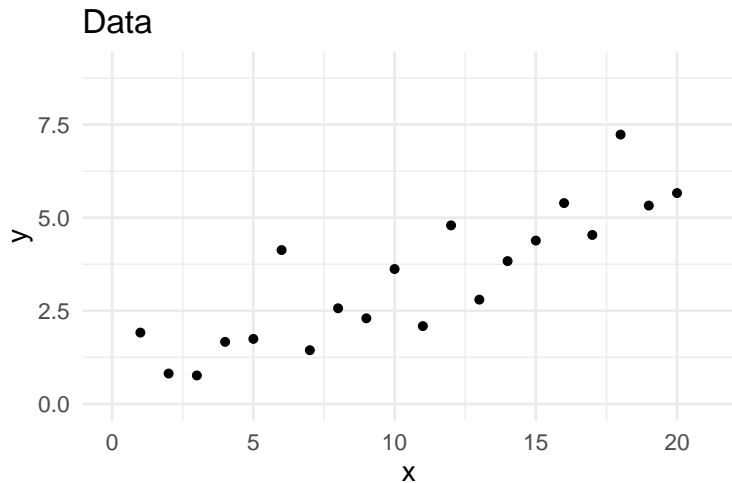
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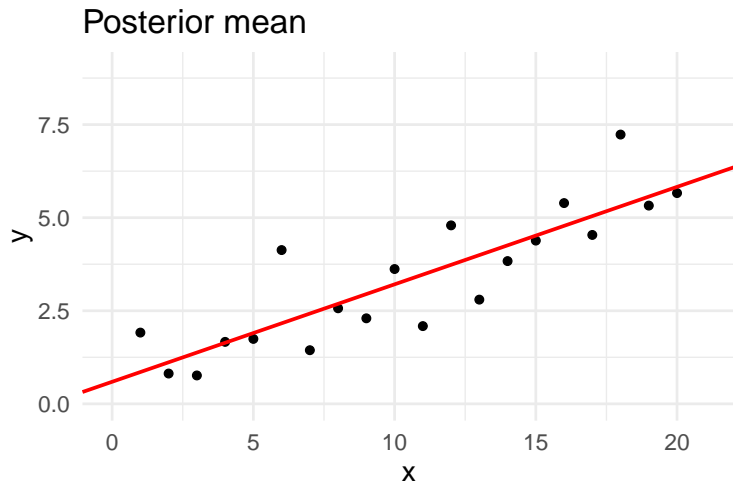
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- Gaussian processes are in practice multivariate normals
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- Posterior distribution approximation with Laplace, variational inference, expectation propagation

# Example of uncertainty in modeling

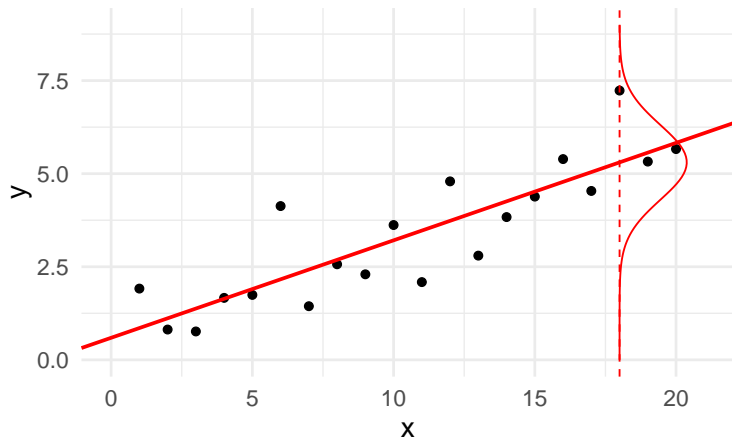


# Example of uncertainty in modeling

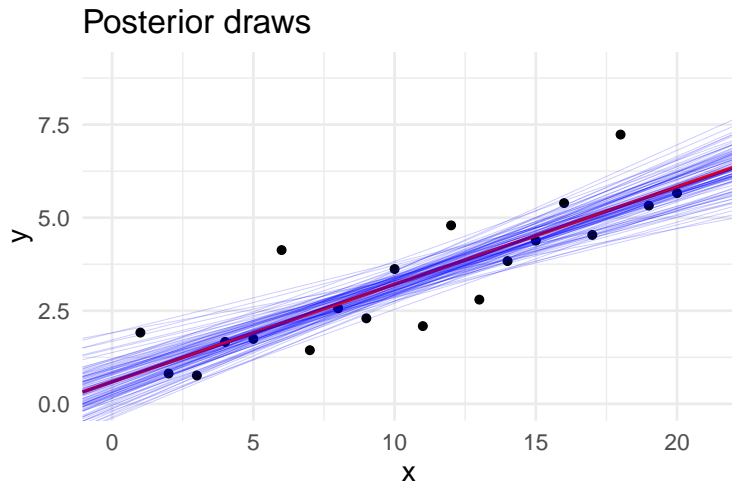


# Example of uncertainty in modeling

Predictive distribution given posterior mean



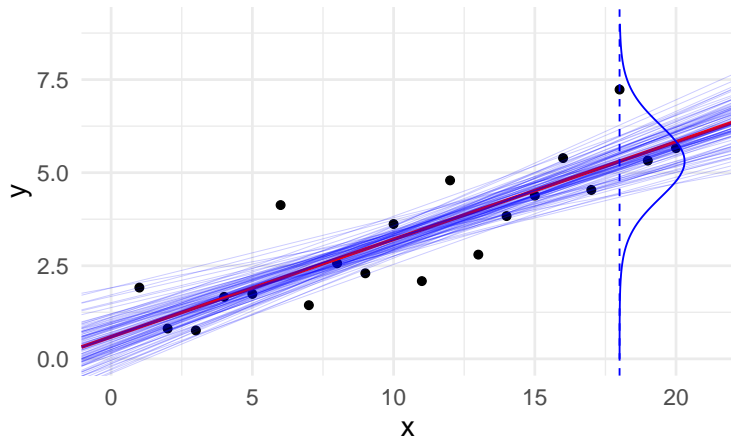
# Example of uncertainty in modeling





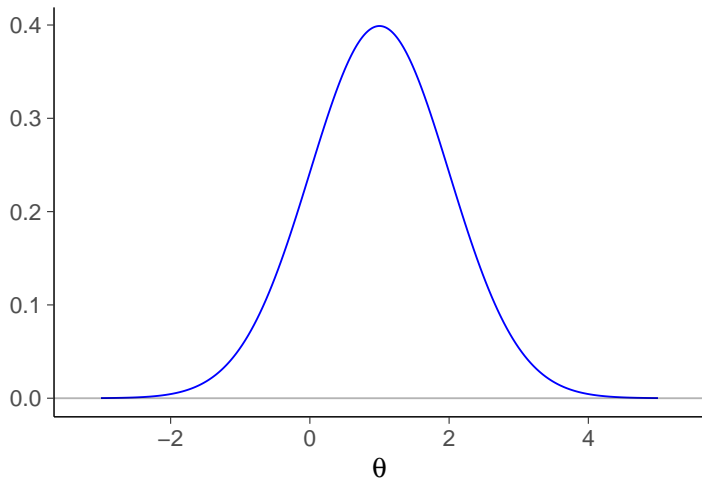
# Example of uncertainty in modeling

Posterior draws and predictive distribution



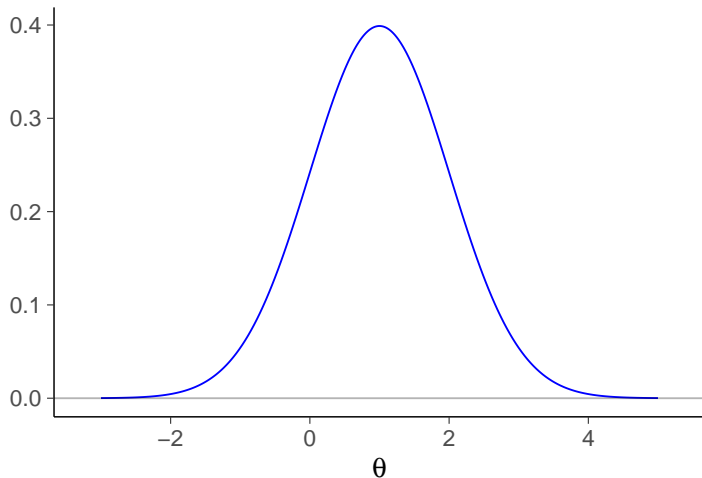
# Monte Carlo and posterior draws

$$\text{Density } p(\theta|\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(\theta - \mu)^2\right)$$



# Monte Carlo and posterior draws

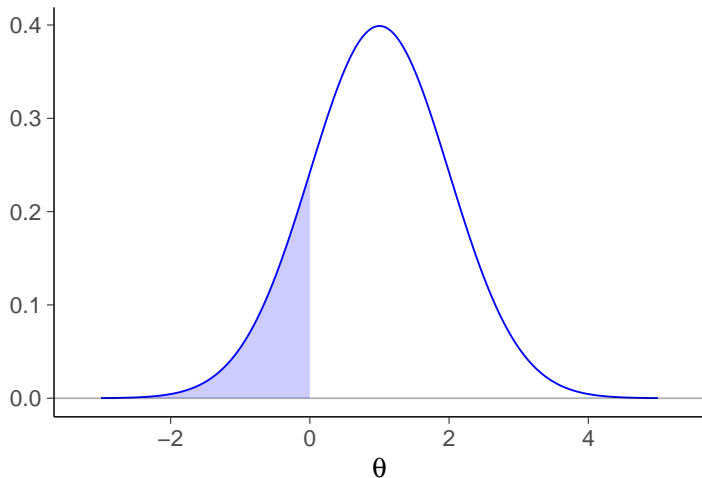
$$\text{Density } p(\theta|\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(\theta - \mu)^2\right)$$



$$E(\theta) = \int \theta p(\theta|\mu, \sigma) d\theta = \mu$$

# Monte Carlo and posterior draws

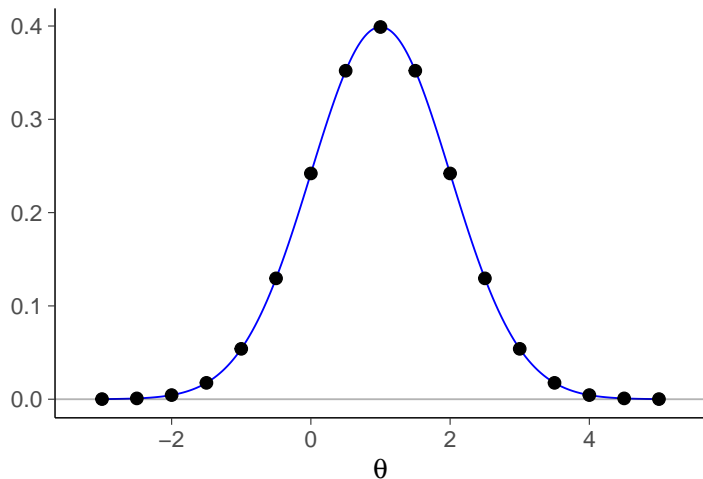
$$\text{Density } p(\theta|\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(\theta - \mu)^2\right)$$



$$p(\theta \leq 0) = \int_{-\infty}^0 p(\theta|\mu, \sigma) d\theta, \text{ many numerical approximations}$$

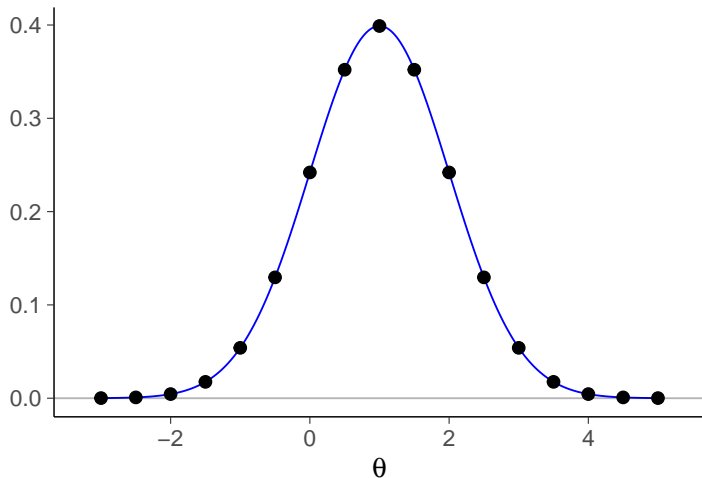
# Monte Carlo and posterior draws

In practice evaluate in finite number of locations



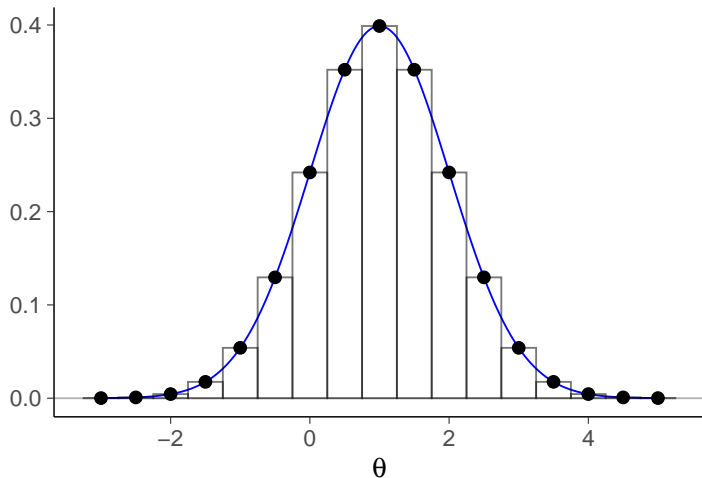
# Monte Carlo and posterior draws

Here evaluated in grid with bin width 0.5



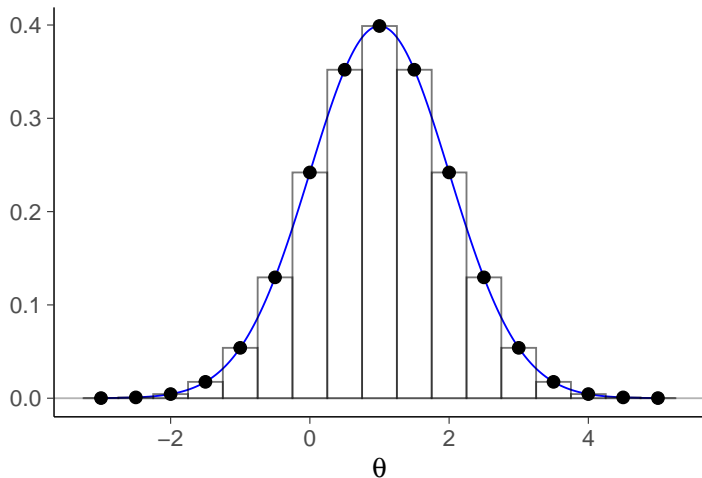
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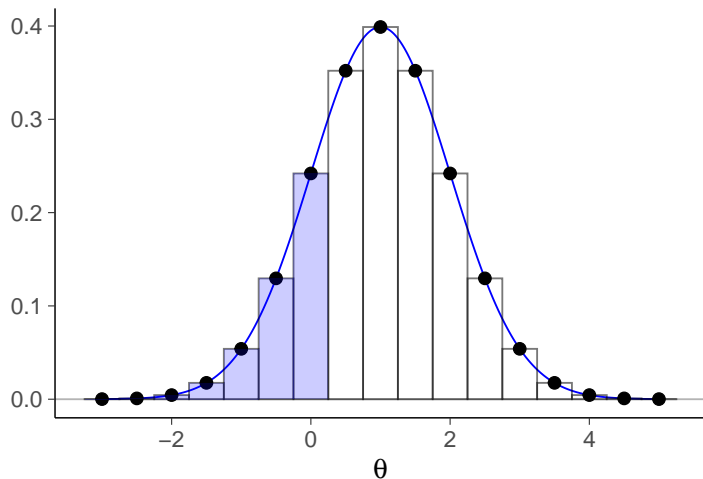


$$E(\theta) = \int \theta p(\theta) d\theta \approx \sum_s^S \theta^{(s)} w_s \approx 1, \text{ where } w_s = 0.5 p(\theta)$$



# Monte Carlo and posterior draws

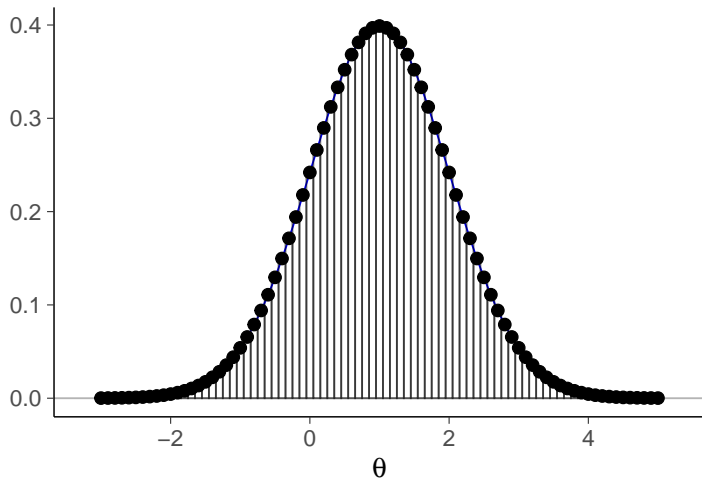
Here evaluated in grid with bin width 0.5



$$p(\theta \leq 0) = \int_{-\infty}^0 p(\theta) d\theta \approx \sum_s^S \mathbb{I}(\theta^{(s)} \leq 0) w_s \approx 0.22$$

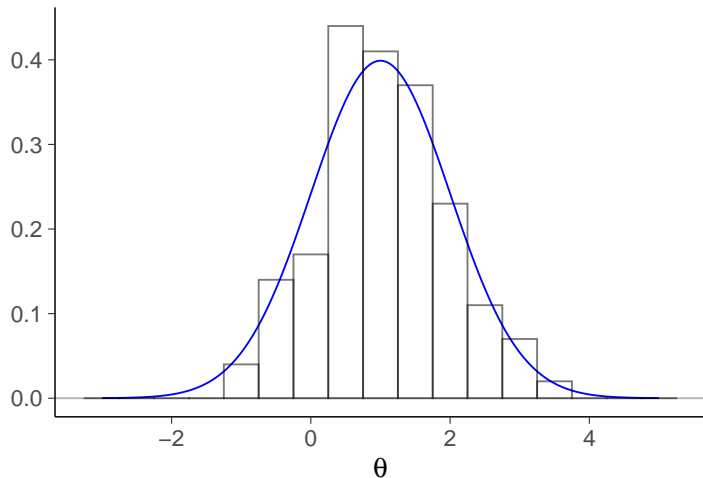
# Monte Carlo and posterior draws

Here evaluated in grid with bin width 0.1



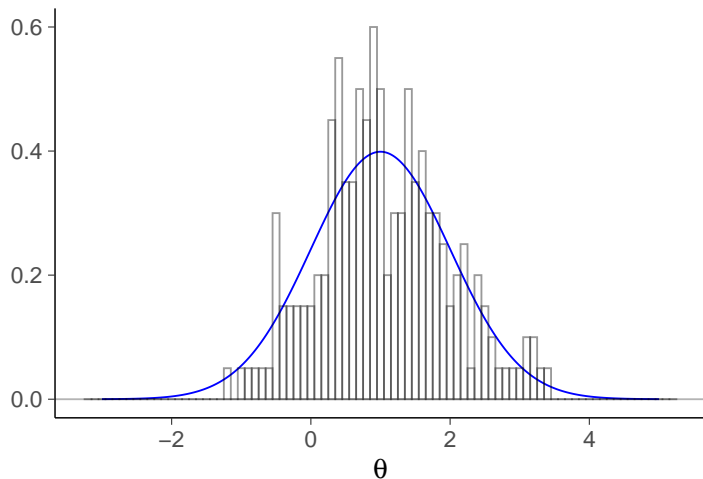
# Monte Carlo and posterior draws

Histogram of 200 random draws, bin width 0.5



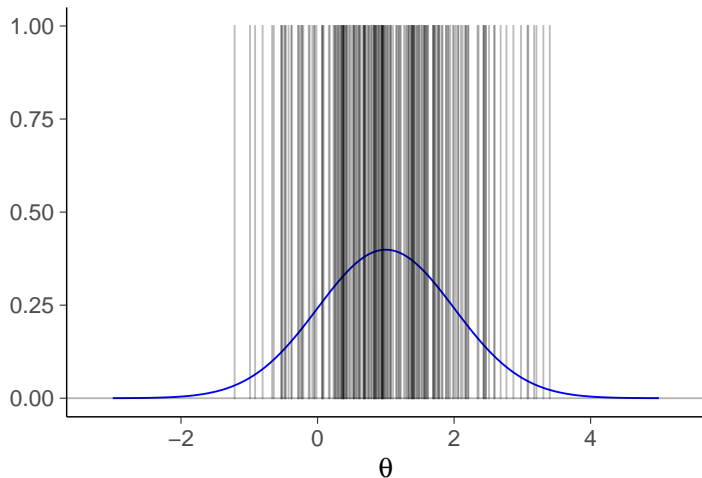
# Monte Carlo and posterior draws

Histogram of 200 random draws, bin width 0.1



# Monte Carlo and posterior draws

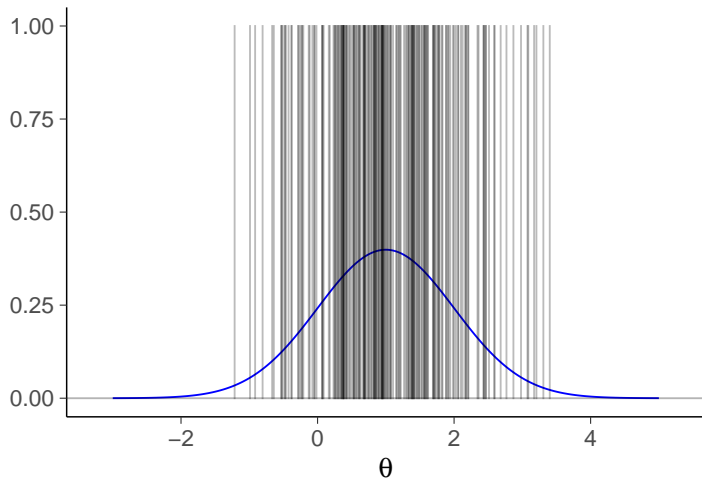
Histogram of 200 random draws, bin width 0



each bin has either 0 or 1 draw (and 0's can be ignored)

# Monte Carlo and posterior draws

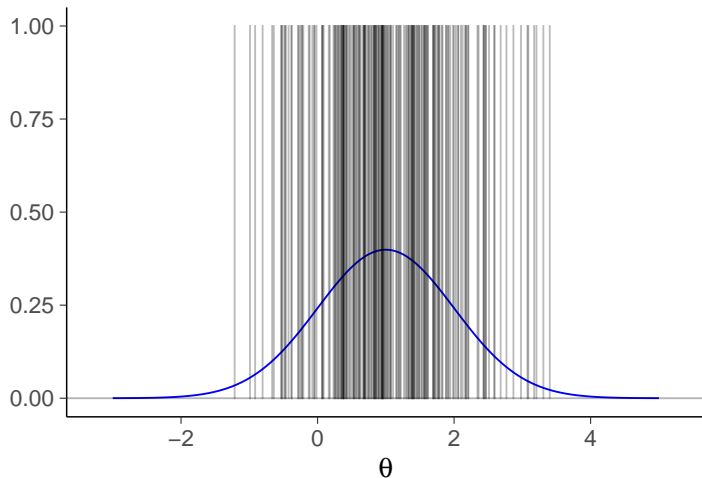
Histogram of 200 random draws, bin width 0



each bin with 1 draw has weight  $1/S$

# Monte Carlo and posterior draws

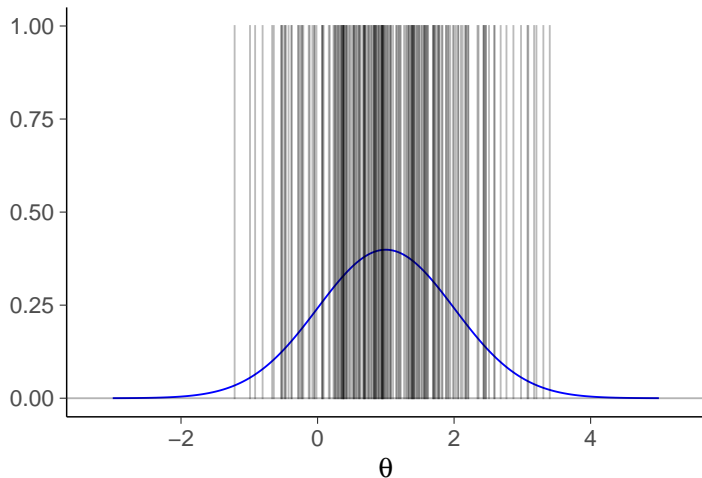
Histogram of 200 random draws, bin width 0



$$E(\theta) \approx \frac{1}{S} \sum_s^S \theta^{(s)} \approx 1$$

# Monte Carlo and posterior draws

Histogram of 200 random draws, bin width 0

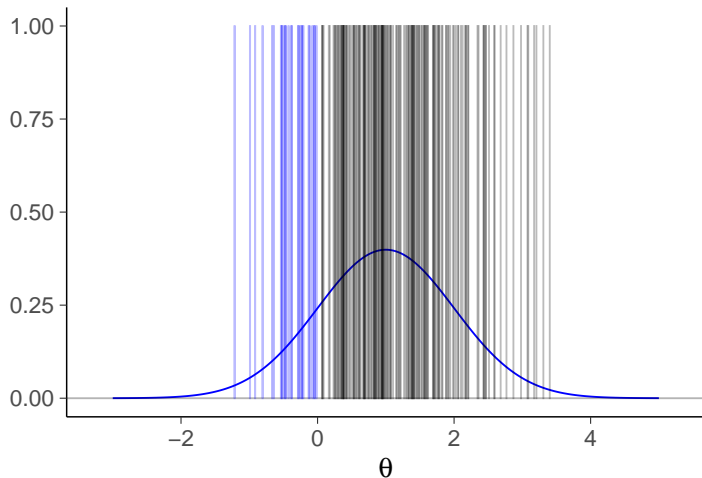


$$E(\theta) \approx \frac{1}{S} \sum_s^S \theta^{(s)} \approx 1, \text{ Monte Carlo estimate}$$



# Monte Carlo and posterior draws

Histogram of 200 random draws, bin width 0



$$p(\theta \leq 0) \approx \frac{1}{S} \sum_s^S \mathbb{I}(\theta^{(s)} \leq 0) \approx 0.14$$

# Monte Carlo and posterior draws

- $\theta^{(s)}$  draws from  $p(\theta | y)$  can be used
  - for visualization

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$$E_{p(\theta|y)}[\theta] = \int \theta p(\theta | y) \approx \frac{1}{S} \sum_{s=1}^S \theta^{(s)}$$

- easy to approximate expectations of functions

$$E_{p(\theta|y)}[g(\theta)] = \int g(\theta) p(\theta | y) \approx \frac{1}{S} \sum_{s=1}^S g(\theta^{(s)})$$

# Marginalization

- Joint distribution of parameters

$$p(\theta_1, \theta_2 \mid y) \propto p(y \mid \theta_1, \theta_2)p(\theta_1, \theta_2)$$

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- Marginalization

$$p(\theta_1 \mid y) = \int p(\theta_1, \theta_2 \mid y) d\theta_2$$

$p(\theta_1 \mid y)$  is a marginal distribution

- Monte Carlo approximation

$$p(\theta_1 \mid y) \approx \frac{1}{S} \sum_{s=1}^S p(\theta_1 \mid \theta_2^{(s)}, y),$$

where  $\theta_2^{(s)}$  are draws from  $p(\theta_2 \mid y)$

# Marginalization - predictive distribution

- Posterior predictive distribution is obtained by marginalizing out the posterior distribution

$$p(\tilde{y} | y) = \int p(\tilde{y}, \theta | y) d\theta$$

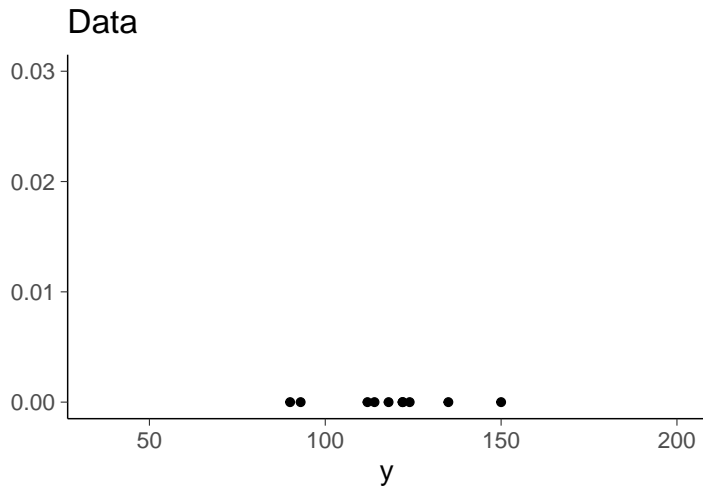


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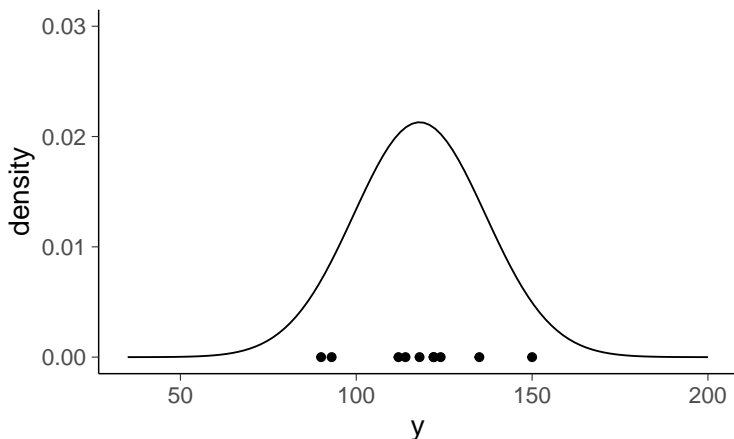
$$\begin{aligned} p(\tilde{y} | y) &= \int p(\tilde{y}, \theta | y) d\theta \\ &= \int p(\tilde{y} | \theta) p(\theta | y) d\theta \end{aligned}$$

# Normal distribution example



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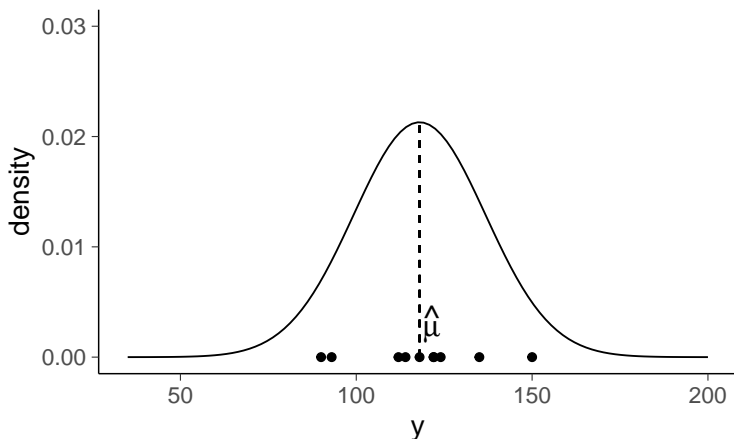
Normal fit with posterior mean



$$p(\textcolor{red}{y} \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(\textcolor{red}{y} - \mu)^2\right)$$

# Normal distribution example

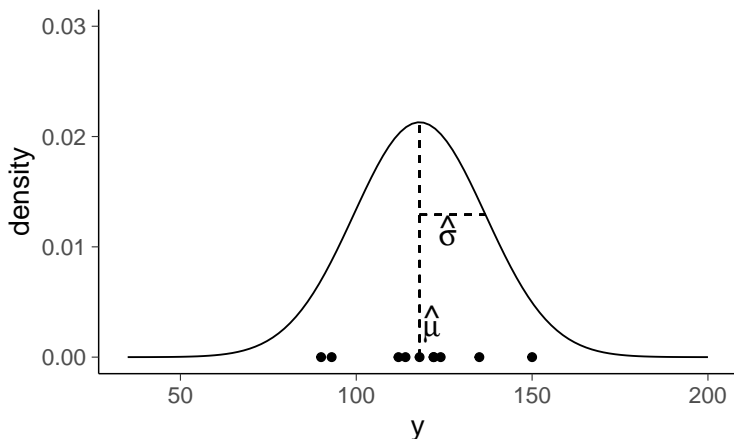
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# Normal distribution example

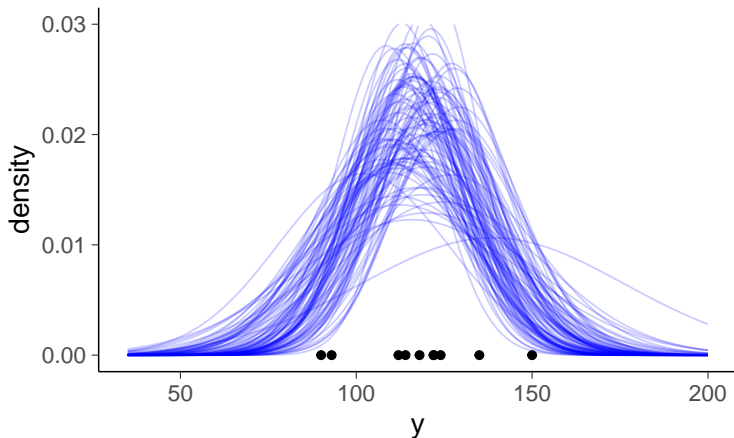
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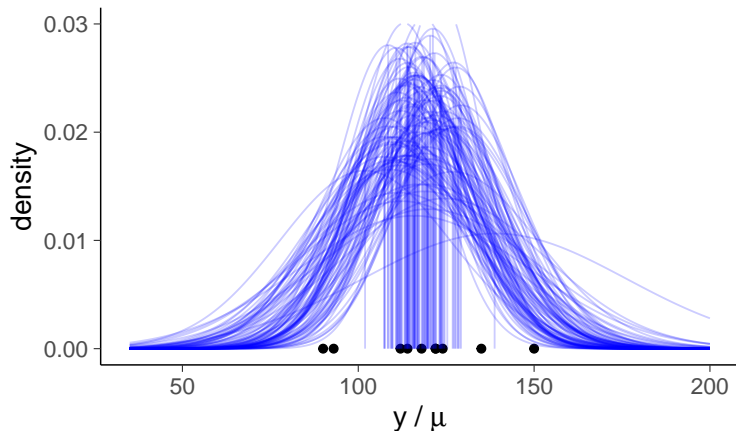
Normals with posterior draw parameters



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

# Normal distribution example

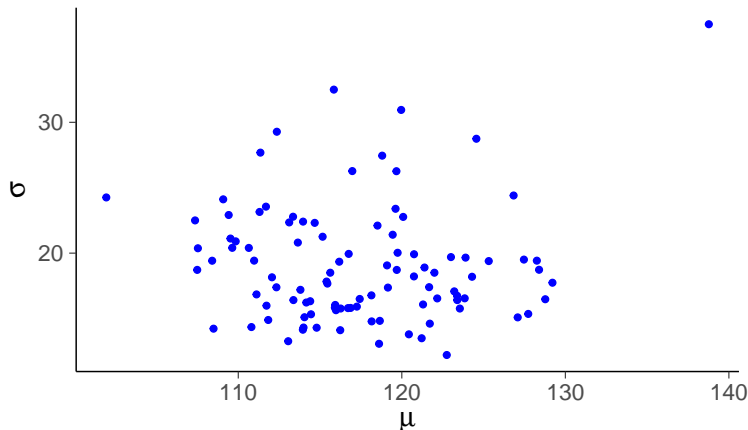
Normals with posterior draw parameters



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# Normal distribution example

Draws from the joint posterior distribution

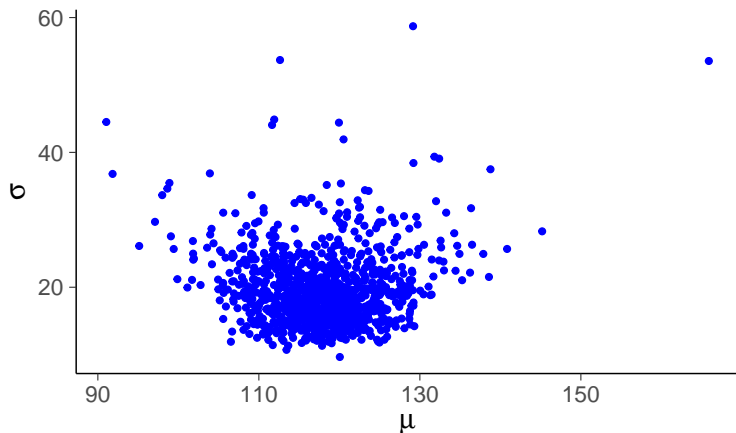


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# Normal distribution example

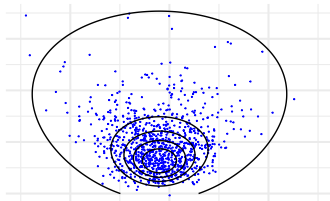
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Joint posterior

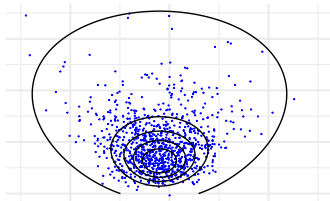
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Joint posterior

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with  $p(\mu, \sigma^2) \propto \sigma^{-2}$

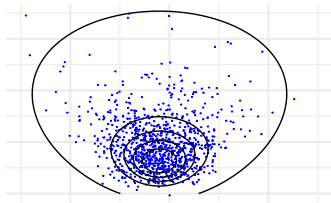


Joint posterior

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with  $p(\mu, \sigma) \propto \sigma^{-1}$  (see BDA3 p. 21 transformation of variables)

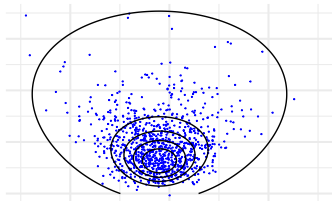


Joint posterior

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with  $p(\mu, \sigma^2) \propto \sigma^{-2}$

$$p(\mu, \sigma^2 | y) \propto \sigma^{-2} \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y_i - \mu)^2\right)$$

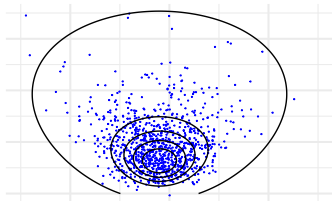


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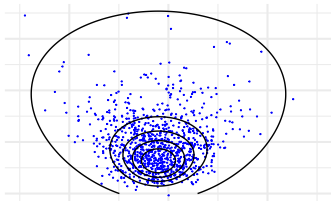
$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp \left( -\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2 \right)$$



Joint posterior

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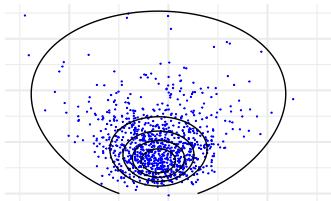
$$\begin{aligned} p(\mu, \sigma^2 \mid y) &\propto \sigma^{-n-2} \exp \left( -\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2 \right) \\ &= \sigma^{-n-2} \exp \left( -\frac{1}{2\sigma^2} \left[ \sum_{i=1}^n (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2 \right] \right) \end{aligned}$$

$$\text{where } \bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$$

## Joint posterior

$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

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$$\text{where } \bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$$

$$= \sigma^{-n-2} \exp \left( -\frac{1}{2\sigma^2} \left[ (n-1)s^2 + n(\bar{y} - \mu)^2 \right] \right)$$

$$\text{where } s^2 = \frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y})^2$$



## Normal - non-informative prior

$$\sum_{i=1}^n (y_i - \mu)^2$$

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$$\sum_{i=1}^n (y_i^2 - 2y_i\mu + \mu^2)$$

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$$\sum_{i=1}^n (y_i^2 - 2y_i\bar{y} + \bar{y}^2) + \sum_{i=1}^n (\mu^2 - 2y_i\mu - \bar{y}^2 + 2y_i\bar{y})$$

## Normal - non-informative prior

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$$\sum_{i=1}^n (y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

$$\sum_{i=1}^n (y_i^2 - 2y_i\bar{y} + \bar{y}^2) + \sum_{i=1}^n (\mu^2 - 2y_i\mu - \bar{y}^2 + 2y_i\bar{y})$$

$$\sum_{i=1}^n (y_i - \bar{y})^2 + n(\mu^2 - 2\bar{y}\mu - \bar{y}^2 + 2\bar{y}\bar{y})$$

## Normal - non-informative prior

$$\sum_{i=1}^n (y_i - \mu)^2$$

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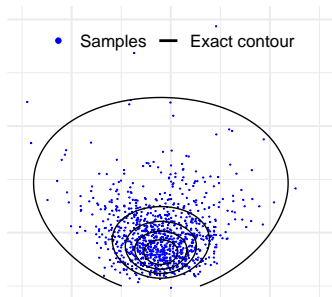
$$\sum_{i=1}^n (y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

$$\sum_{i=1}^n (y_i^2 - 2y_i\bar{y} + \bar{y}^2) + \sum_{i=1}^n (\mu^2 - 2y_i\mu - \bar{y}^2 + 2y_i\bar{y})$$

$$\sum_{i=1}^n (y_i - \bar{y})^2 + n(\mu^2 - 2\bar{y}\mu - \bar{y}^2 + 2\bar{y}\bar{y})$$

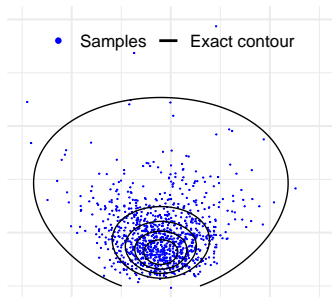
$$\sum_{i=1}^n (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2$$

## Joint posterior

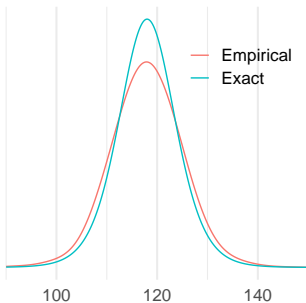


$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

## Joint posterior



## Marginal of mu



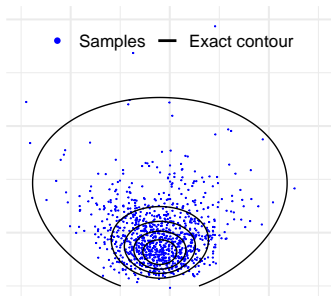
$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma | y)$$

marginals

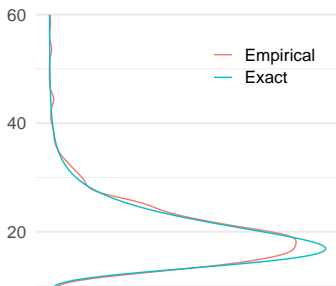
$$p(\mu | y) = \int p(\mu, \sigma | y) d\sigma$$



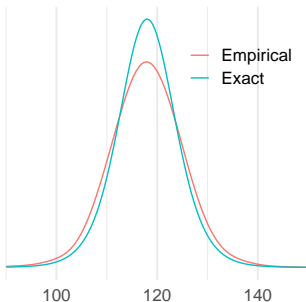
### Joint posterior



### Marginal of sigma



### Marginal of mu



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma | y)$$

marginals

$$p(\mu | y) = \int p(\mu, \sigma | y) d\sigma$$

$$p(\sigma | y) = \int p(\mu, \sigma | y) d\mu$$

Marginal posterior  $p(\sigma^2 \mid y)$  (easier for  $\sigma^2$  than  $\sigma$ )

$$p(\sigma^2 \mid y) \propto \int p(\mu, \sigma^2 \mid y) d\mu$$

Marginal posterior  $p(\sigma^2 \mid y)$  (easier for  $\sigma^2$  than  $\sigma$ )

$$\begin{aligned} p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\ &\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right) d\mu \end{aligned}$$

## Marginal posterior $p(\sigma^2 \mid y)$ (easier for $\sigma^2$ than $\sigma$ )

$$\begin{aligned} p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\ &\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right) d\mu \\ &\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} (n-1)s^2\right) \\ &\quad \int \exp\left(-\frac{n}{2\sigma^2} (\bar{y} - \mu)^2\right) d\mu \end{aligned}$$

## Marginal posterior $p(\sigma^2 | y)$ (easier for $\sigma^2$ than $\sigma$ )

$$\begin{aligned} p(\sigma^2 | y) &\propto \int p(\mu, \sigma^2 | y) d\mu \\ &\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right) d\mu \\ &\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} (n-1)s^2\right) \\ &\quad \int \exp\left(-\frac{n}{2\sigma^2} (\bar{y} - \mu)^2\right) d\mu \\ &\quad \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2} (y - \theta)^2\right) d\theta = 1 \end{aligned}$$

## Marginal posterior $p(\sigma^2 | y)$ (easier for $\sigma^2$ than $\sigma$ )

$$\begin{aligned} p(\sigma^2 | y) &\propto \int p(\mu, \sigma^2 | y) d\mu \\ &\propto \int \sigma^{-n-2} \exp \left( -\frac{1}{2\sigma^2} \left[ (n-1)s^2 + n(\bar{y} - \mu)^2 \right] \right) d\mu \\ &\propto \sigma^{-n-2} \exp \left( -\frac{1}{2\sigma^2} (n-1)s^2 \right) \\ &\quad \int \exp \left( -\frac{n}{2\sigma^2} (\bar{y} - \mu)^2 \right) d\mu \\ &\quad \int \frac{1}{\sqrt{2\pi}\sigma} \exp \left( -\frac{1}{2\sigma^2} (y - \theta)^2 \right) d\theta = 1 \\ &\propto \sigma^{-n-2} \exp \left( -\frac{1}{2\sigma^2} (n-1)s^2 \right) \sqrt{2\pi\sigma^2/n} \end{aligned}$$

## Marginal posterior $p(\sigma^2 | y)$ (easier for $\sigma^2$ than $\sigma$ )

$$\begin{aligned} p(\sigma^2 | y) &\propto \int p(\mu, \sigma^2 | y) d\mu \\ &\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right) d\mu \\ &\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} (n-1)s^2\right) \\ &\quad \int \exp\left(-\frac{n}{2\sigma^2} (\bar{y} - \mu)^2\right) d\mu \\ &\quad \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2} (y - \theta)^2\right) d\theta = 1 \\ &\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} (n-1)s^2\right) \sqrt{2\pi\sigma^2/n} \\ &\propto (\sigma^2)^{-(n+1)/2} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \end{aligned}$$

## Marginal posterior $p(\sigma^2 | y)$ (easier for $\sigma^2$ than $\sigma$ )

$$\begin{aligned} p(\sigma^2 | y) &\propto \int p(\mu, \sigma^2 | y) d\mu \\ &\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right) d\mu \\ &\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} (n-1)s^2\right) \\ &\quad \int \exp\left(-\frac{n}{2\sigma^2} (\bar{y} - \mu)^2\right) d\mu \\ &\quad \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2} (y - \theta)^2\right) d\theta = 1 \\ &\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} (n-1)s^2\right) \sqrt{2\pi\sigma^2/n} \\ &\propto (\sigma^2)^{-(n+1)/2} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \\ p(\sigma^2 | y) &= \text{Inv-}\chi^2(\sigma^2 | n-1, s^2) \end{aligned}$$



# Normal - non-informative prior

Known mean

$$\sigma^2 \mid y \sim \text{Inv-}\chi^2(n, \nu)$$

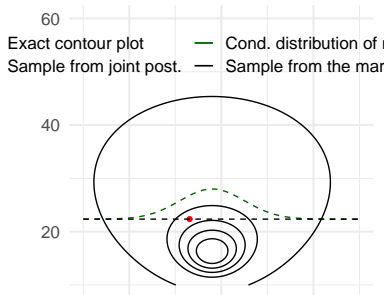
$$\text{where } \nu = \frac{1}{n} \sum_{i=1}^n (y_i - \theta)^2$$

Unknown mean

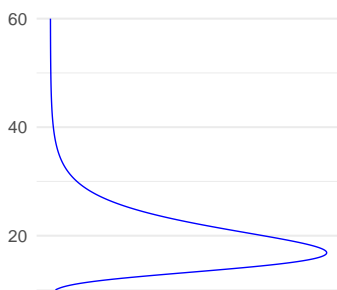
$$\sigma^2 \mid y \sim \text{Inv-}\chi^2(n-1, s^2)$$

$$\text{where } s^2 = \frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y})^2$$

## Joint posterior



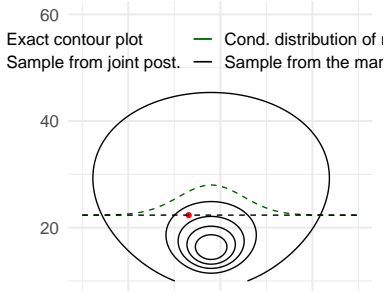
## Marginal of sigma



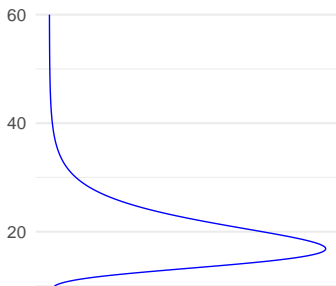
## Factorization

$$p(\mu, \sigma^2 | y) = p(\mu | \sigma^2, y) p(\sigma^2 | y)$$

## Joint posterior



## Marginal of sigma



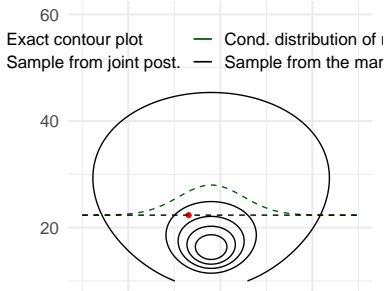
## Factorization

$$p(\mu, \sigma^2 | y) = p(\mu | \sigma^2, y) p(\sigma^2 | y)$$

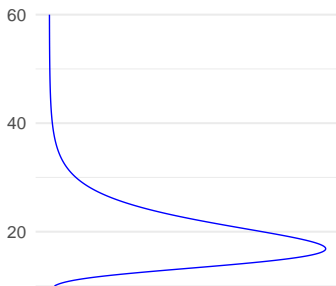
$$p(\sigma^2 | y) = \text{Inv-}\chi^2(\sigma^2 | n - 1, s^2)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 | y)$$

## Joint posterior



## Marginal of sigma



## Factorization

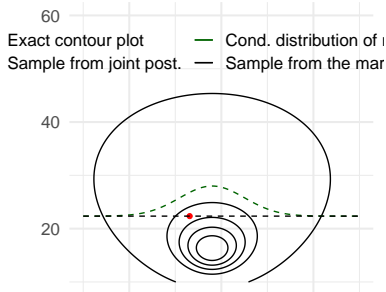
$$p(\mu, \sigma^2 | y) = p(\mu | \sigma^2, y) p(\sigma^2 | y)$$

$$p(\sigma^2 | y) = \text{Inv-}\chi^2(\sigma^2 | n - 1, s^2)$$

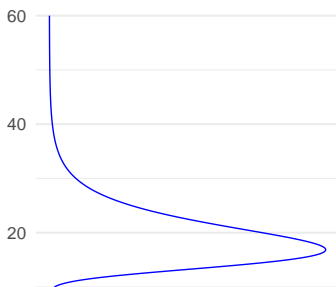
$$(\sigma^2)^{(s)} \sim p(\sigma^2 | y)$$

$$p(\mu | \sigma^2, y) = N(\mu | \bar{y}, \sigma^2/n)$$

## Joint posterior



## Marginal of sigma



## Factorization

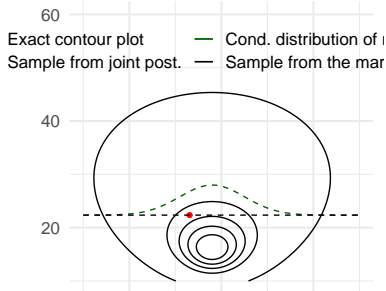
$$p(\mu, \sigma^2 | y) = p(\mu | \sigma^2, y) p(\sigma^2 | y)$$

$$p(\sigma^2 | y) = \text{Inv-}\chi^2(\sigma^2 | n - 1, s^2)$$

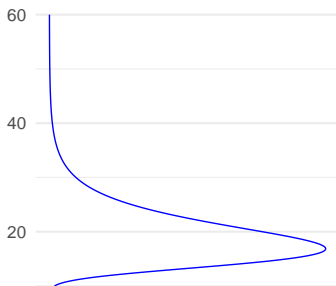
$$(\sigma^2)^{(s)} \sim p(\sigma^2 | y)$$

$$p(\mu | \sigma^2, y) = \text{N}(\mu | \bar{y}, \sigma^2/n) \propto \exp\left(-\frac{n}{2\sigma^2}(\bar{y} - \mu)^2\right)$$

## Joint posterior



## Marginal of sigma



## Factorization

$$p(\mu, \sigma^2 | y) = p(\mu | \sigma^2, y) p(\sigma^2 | y)$$

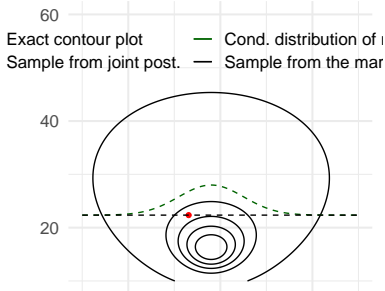
$$p(\sigma^2 | y) = \text{Inv-}\chi^2(\sigma^2 | n - 1, s^2)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 | y)$$

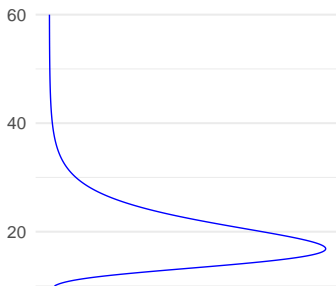
$$p(\mu | \sigma^2, y) = \text{N}(\mu | \bar{y}, \sigma^2/n)$$

$$\mu^{(s)} \sim p(\mu | \sigma^2, y)$$

## Joint posterior



## Marginal of sigma



## Factorization

$$p(\mu, \sigma^2 | y) = p(\mu | \sigma^2, y) p(\sigma^2 | y)$$

$$p(\sigma^2 | y) = \text{Inv-}\chi^2(\sigma^2 | n - 1, s^2)$$

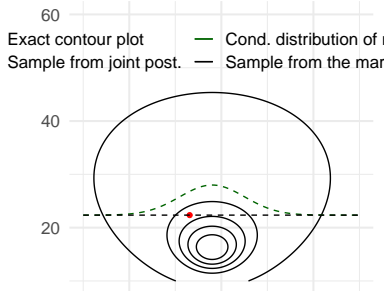
$$(\sigma^2)^{(s)} \sim p(\sigma^2 | y)$$

$$p(\mu | \sigma^2, y) = \text{N}(\mu | \bar{y}, \sigma^2/n)$$

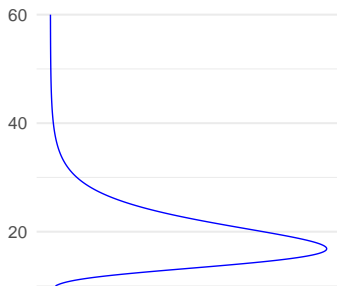
$$\mu^{(s)} \sim p(\mu | \sigma^2, y)$$

$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma | y)$$

## Joint posterior



## Marginal of sigma

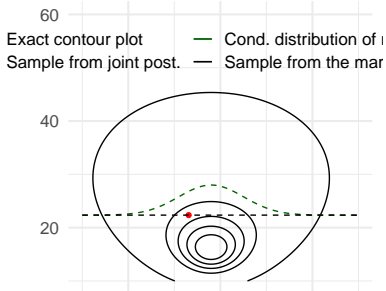


## Factorization

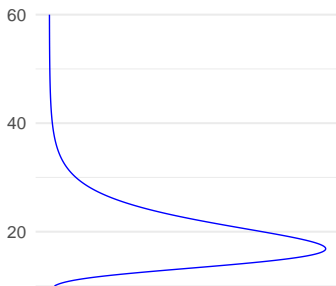
$$p(\mu, \sigma^2 | y) = p(\mu | \sigma^2, y) p(\sigma^2 | y)$$



## Joint posterior



## Marginal of sigma

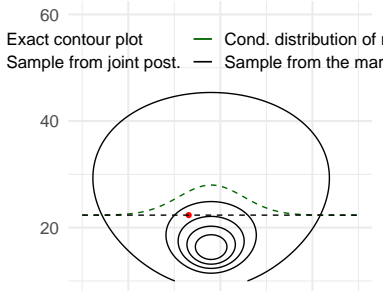


## Factorization

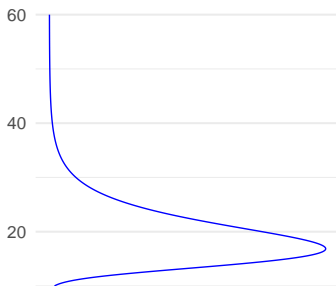
$$p(\mu, \sigma^2 | y) = p(\mu | \sigma^2, y) p(\sigma^2 | y)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 | y)$$

## Joint posterior



## Marginal of sigma



## Factorization

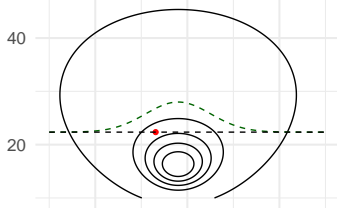
$$p(\mu, \sigma^2 | y) = p(\mu | \sigma^2, y) p(\sigma^2 | y)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 | y)$$

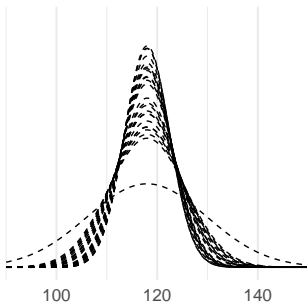
$$p(\mu | (\sigma^2)^{(s)}, y) = \text{N}(\mu | \bar{y}, (\sigma^2)^{(s)}/n)$$

## Joint posterior

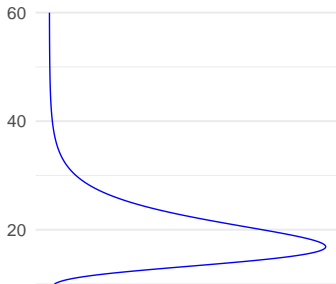
Exact contour plot      — Cond. distribution of  $\mu$   
 Sample from joint post.      — Sample from the mar



## Cond distr of mu for 25 draws



## Marginal of sigma



## Factorization

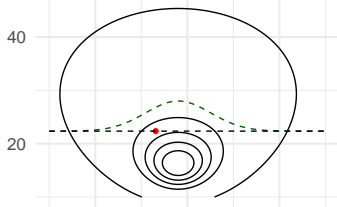
$$p(\mu, \sigma^2 | y) = p(\mu | \sigma^2, y) p(\sigma^2 | y)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 | y)$$

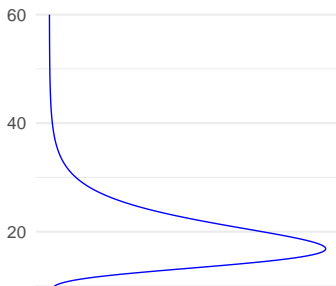
$$p(\mu | (\sigma^2)^{(s)}, y) = \text{N}(\mu | \bar{y}, (\sigma^2)^{(s)} / n)$$

## Joint posterior

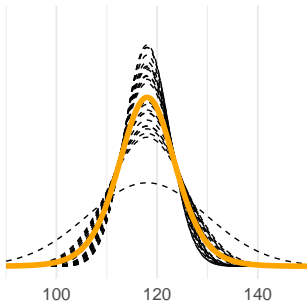
Exact contour plot      — Cond. distribution of  $\mu$   
 Sample from joint post.      — Sample from the mar



## Marginal of sigma



## Cond distr of mu for 25 draws



## Factorization

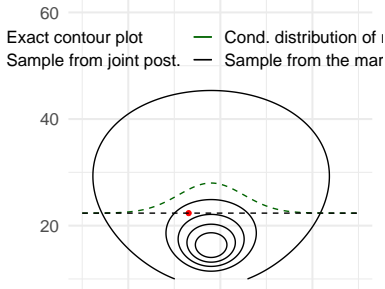
$$p(\mu, \sigma^2 | y) = p(\mu | \sigma^2, y) p(\sigma^2 | y)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 | y)$$

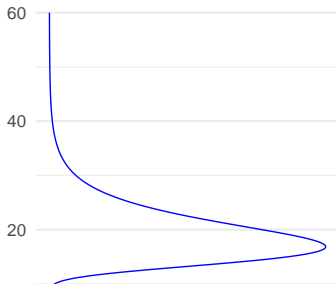
$$p(\mu | (\sigma^2)^{(s)}, y) = N(\mu | \bar{y}, (\sigma^2)^{(s)}/n)$$

$$p(\mu | y) \approx \frac{1}{S} \sum_{s=1}^S N(\mu | \bar{y}, (\sigma^2)^{(s)}/n)$$

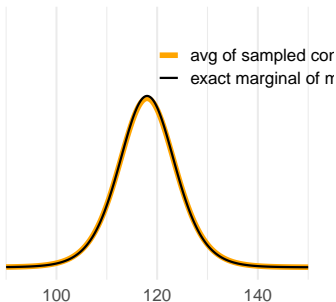
## Joint posterior



## Marginal of sigma



## Cond. distr of mu



## Factorization

$$p(\mu, \sigma^2 | y) = p(\mu | \sigma^2, y) p(\sigma^2 | y)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 | y)$$

$$p(\mu | (\sigma^2)^{(s)}, y) = N(\mu | \bar{y}, (\sigma^2)^{(s)}/n)$$

$$p(\mu | y) \approx \frac{1}{S} \sum_{s=1}^S N(\mu | \bar{y}, (\sigma^2)^{(s)}/n)$$

Marginal posterior  $p(\mu \mid y)$

$$p(\mu \mid y) = \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2$$

## Marginal posterior $p(\mu \mid y)$

$$\begin{aligned} p(\mu \mid y) &= \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2 \\ &\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right) d\sigma^2 \end{aligned}$$

## Marginal posterior $p(\mu | y)$

$$\begin{aligned} p(\mu | y) &= \int_0^\infty p(\mu, \sigma^2 | y) d\sigma^2 \\ &\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right) d\sigma^2 \end{aligned}$$

Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2$$



## Marginal posterior $p(\mu \mid y)$

$$\begin{aligned} p(\mu \mid y) &= \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2 \\ &\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right) d\sigma^2 \end{aligned}$$

Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}$$

## Marginal posterior $p(\mu | y)$

$$\begin{aligned} p(\mu | y) &= \int_0^\infty p(\mu, \sigma^2 | y) d\sigma^2 \\ &\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right) d\sigma^2 \end{aligned}$$

Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}$$

$$p(\mu | y) \propto A^{-n/2} \int_0^\infty z^{(n-2)/2} \exp(-z) dz$$

## Marginal posterior $p(\mu | y)$

$$\begin{aligned} p(\mu | y) &= \int_0^\infty p(\mu, \sigma^2 | y) d\sigma^2 \\ &\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right) d\sigma^2 \end{aligned}$$

Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}$$

$$p(\mu | y) \propto A^{-n/2} \int_0^\infty z^{(n-2)/2} \exp(-z) dz$$

Recognize gamma integral  $\Gamma(u) = \int_0^\infty x^{u-1} \exp(-x) dx$

## Marginal posterior $p(\mu | y)$

$$\begin{aligned} p(\mu | y) &= \int_0^\infty p(\mu, \sigma^2 | y) d\sigma^2 \\ &\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right) d\sigma^2 \end{aligned}$$

Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}$$

$$p(\mu | y) \propto A^{-n/2} \int_0^\infty z^{(n-2)/2} \exp(-z) dz$$

Recognize gamma integral  $\Gamma(u) = \int_0^\infty x^{u-1} \exp(-x) dx$

$$\propto [(n-1)s^2 + n(\mu - \bar{y})^2]^{-n/2}$$

## Marginal posterior $p(\mu | y)$

$$\begin{aligned} p(\mu | y) &= \int_0^\infty p(\mu, \sigma^2 | y) d\sigma^2 \\ &\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right) d\sigma^2 \end{aligned}$$

Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}$$

$$p(\mu | y) \propto A^{-n/2} \int_0^\infty z^{(n-2)/2} \exp(-z) dz$$

Recognize gamma integral  $\Gamma(u) = \int_0^\infty x^{u-1} \exp(-x) dx$

$$\begin{aligned} &\propto [(n-1)s^2 + n(\mu - \bar{y})^2]^{-n/2} \\ &\propto \left[1 + \frac{n(\mu - \bar{y})^2}{(n-1)s^2}\right]^{-n/2} \end{aligned}$$

## Marginal posterior $p(\mu | y)$

$$\begin{aligned} p(\mu | y) &= \int_0^\infty p(\mu, \sigma^2 | y) d\sigma^2 \\ &\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right) d\sigma^2 \end{aligned}$$

Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}$$

$$p(\mu | y) \propto A^{-n/2} \int_0^\infty z^{(n-2)/2} \exp(-z) dz$$

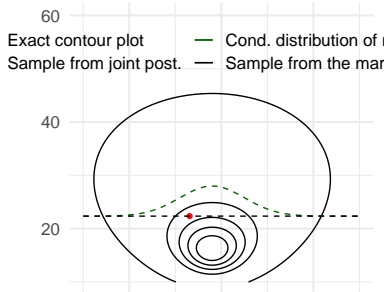
Recognize gamma integral  $\Gamma(u) = \int_0^\infty x^{u-1} \exp(-x) dx$

$$\propto [(n-1)s^2 + n(\mu - \bar{y})^2]^{-n/2}$$

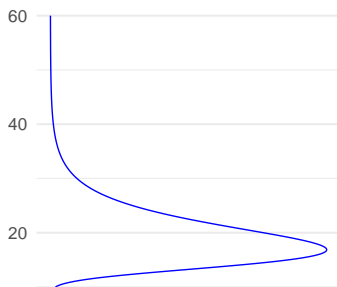
$$\propto \left[1 + \frac{n(\mu - \bar{y})^2}{(n-1)s^2}\right]^{-n/2}$$

$$p(\mu | y) = t_{n-1}(\mu | \bar{y}, s^2/n) \quad \text{Student's } t$$

# Joint posterior



# Marginal of sigma

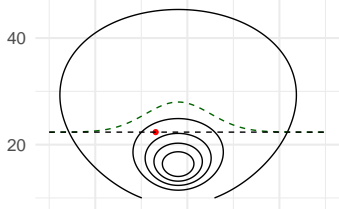


Predictive distribution for new  $\tilde{y}$

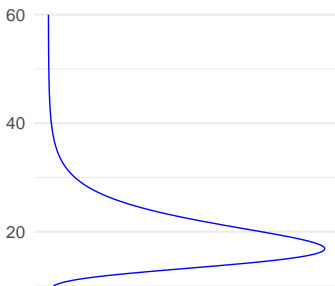
$$p(\tilde{y} | y) = \int p(\tilde{y} | \mu, \sigma) p(\mu, \sigma | y) d\mu d\sigma$$

## Joint posterior

Exact contour plot      — Cond. distribution of  $\mu$   
 Sample from joint post.      — Sample from the mar



## Marginal of sigma



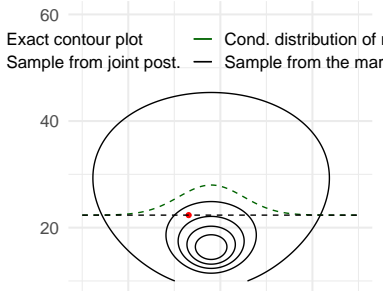
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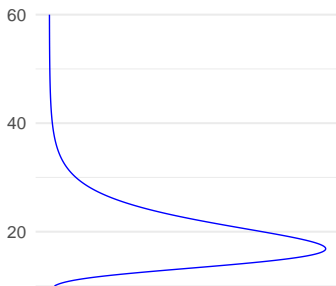
$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma | y)$$



## Joint posterior



## Marginal of sigma



## Predictive distribution for new $\tilde{y}$

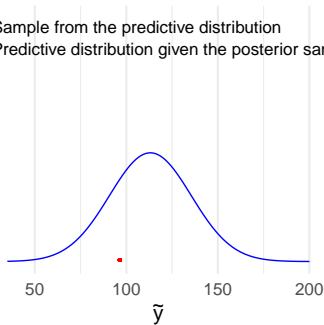
$$p(\tilde{y} | y) = \int p(\tilde{y} | \mu, \sigma) p(\mu, \sigma | y) d\mu d\sigma$$

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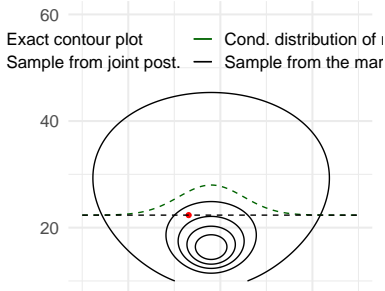
$$\tilde{y}^{(s)} \sim p(\tilde{y} | \mu^{(s)}, \sigma^{(s)})$$

## Posterior predictive distribution

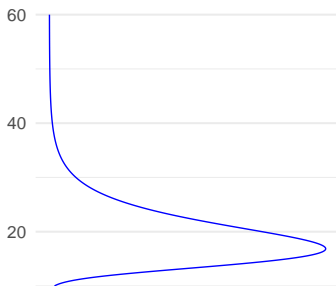
Sample from the predictive distribution  
Predictive distribution given the posterior sam



## Joint posterior



## Marginal of sigma



## Predictive distribution for new $\tilde{y}$

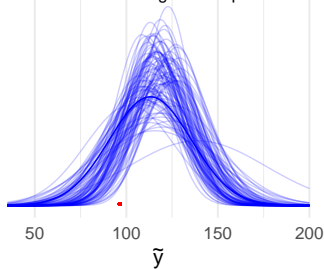
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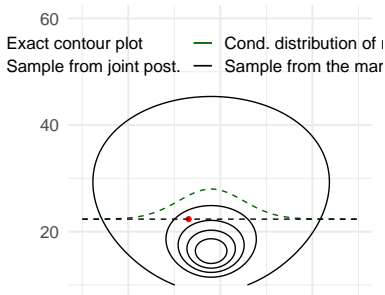
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## Posterior predictive distribution

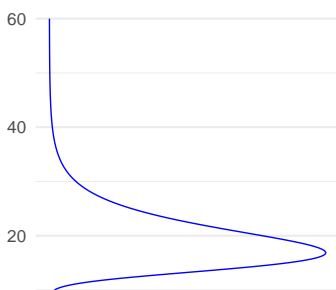
Sample from the predictive distribution  
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## Joint posterior



## Marginal of sigma



## Predictive distribution for new $\tilde{y}$

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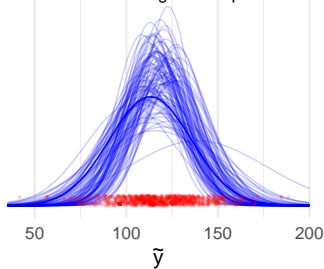
$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma | y)$$

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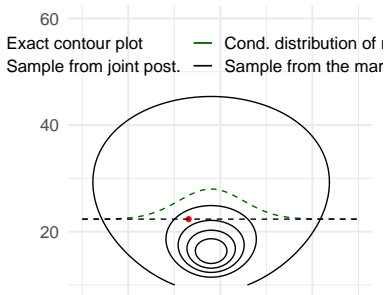
## Posterior predictive distribution

Sample from the predictive distribution

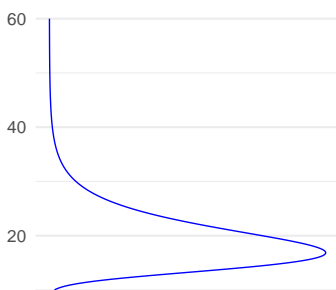
Predictive distribution given the posterior sam



## Joint posterior



## Marginal of sigma



## Predictive distribution for new $\tilde{y}$

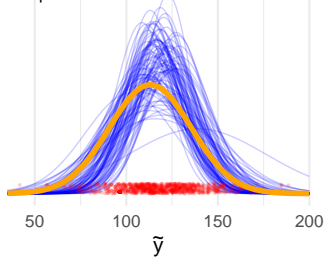
$$p(\tilde{y} | y) = \int p(\tilde{y} | \mu, \sigma) p(\mu, \sigma | y) d\mu d\sigma$$

$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma | y)$$

$$\tilde{y}^{(s)} \sim p(\tilde{y} | \mu^{(s)}, \sigma^{(s)})$$

## Posterior predictive distribution

- Sample from the predictive distribution
- Predictive distribution given the posterior sam
- Exact predictive distribution



# Normal - posterior predictive distribution

Posterior predictive distribution given known variance

$$p(\tilde{y} \mid \sigma^2, y) = \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu$$

# Normal - posterior predictive distribution

Posterior predictive distribution given known variance

$$\begin{aligned} p(\tilde{y} \mid \sigma^2, y) &= \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu \\ &= \int N(\tilde{y} \mid \mu, \sigma^2) N(\mu \mid \bar{y}, \sigma^2/n) d\mu \end{aligned}$$

# Normal - posterior predictive distribution

Posterior predictive distribution given known variance

$$\begin{aligned}p(\tilde{y} \mid \sigma^2, y) &= \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu \\&= \int N(\tilde{y} \mid \mu, \sigma^2) N(\mu \mid \bar{y}, \sigma^2/n) d\mu \\&= N(\tilde{y} \mid \bar{y}, (1 + \frac{1}{n})\sigma^2)\end{aligned}$$

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Posterior predictive distribution given known variance

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this is up to scaling factor same as  $p(\mu \mid \sigma^2, y)$



## Normal - posterior predictive distribution

Posterior predictive distribution given known variance

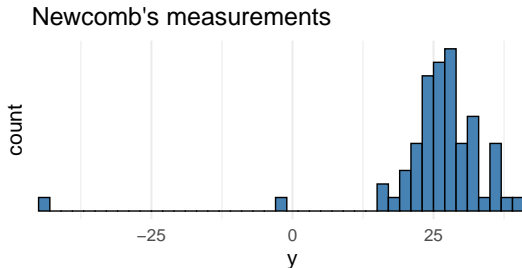
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this is up to scaling factor same as  $p(\mu \mid \sigma^2, y)$

$$p(\tilde{y} \mid y) = t_{n-1}(\tilde{y} \mid \bar{y}, (1 + \frac{1}{n})s^2)$$

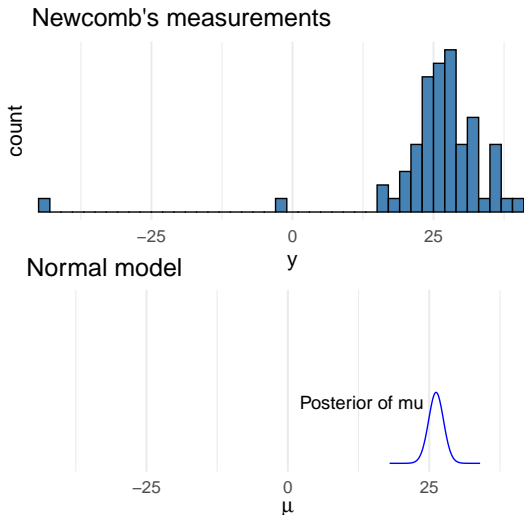
## Simon Newcomb's light of speed experiment in 1882

Newcomb measured ( $n = 66$ ) the time required for light to travel from his laboratory on the Potomac River to a mirror at the base of the Washington Monument and back, a total distance of 7422 meters.



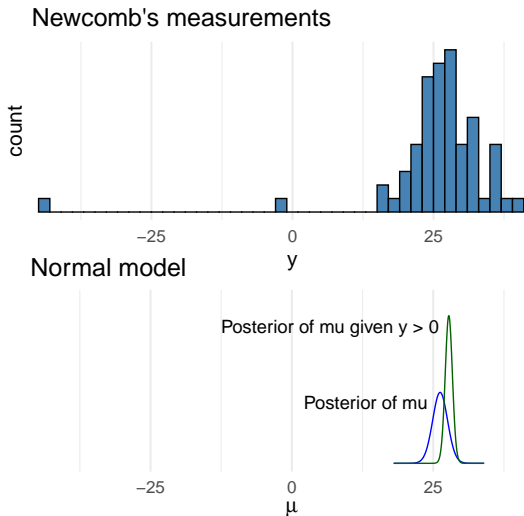
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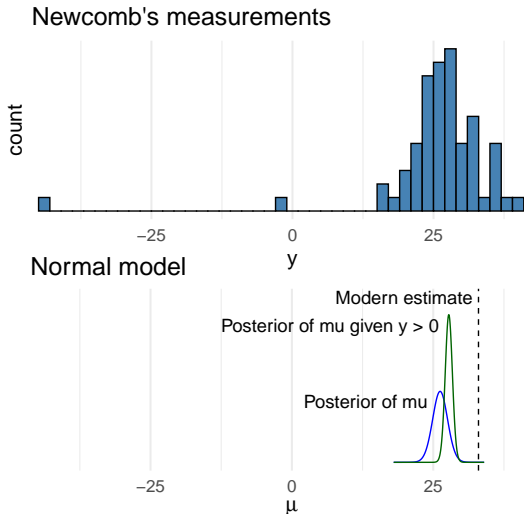
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## Normal - conjugate prior

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- Handy parameterization

$$\begin{aligned}\mu | \sigma^2 &\sim \text{N}(\mu_0, \sigma^2 / \kappa_0) \\ \sigma^2 &\sim \text{Inv-}\chi^2(\nu_0, \sigma_0^2)\end{aligned}$$

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$$p(\mu, \sigma^2) = \text{N-Inv-}\chi^2(\mu_0, \sigma_0^2 / \kappa_0; \nu_0, \sigma_0^2)$$

- $\mu$  and  $\sigma^2$  are a priori dependent
  - if  $\sigma^2$  is large, then  $\mu$  has wide prior



# Normal - conjugate prior

Joint posterior (exercise 3.9)

$$p(\mu, \sigma^2 \mid y) = \text{N-Inv-}\chi^2(\mu_n, \sigma_n^2/\kappa_n; \nu_n, \sigma_n^2)$$

where

$$\mu_n = \frac{\kappa_0}{\kappa_0 + n} \mu_0 + \frac{n}{\kappa_0 + n} \bar{y}$$

$$\kappa_n = \kappa_0 + n$$

$$\nu_n = \nu_0 + n$$

$$\nu_n \sigma_n^2 = \nu_0 \sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_0 + n} (\bar{y} - \mu_0)^2$$

# Comparison of means of two normals

- The difference of two normally distributed variables is normally distributed
- The difference of two  $t$  distributed variables with different variances and degrees of freedom doesn't have an easy form
  - easy to sample from the two distributions, and obtain samples of the differences

$$\delta^{(s)} = \mu_1^{(s)} - \mu_2^{(s)}$$

# Multivariate normal

- Observation model

$$p(y \mid \mu, \Sigma) \propto |\Sigma|^{-1/2} \exp \left( -\frac{1}{2} (y - \mu)^T \Sigma^{-1} (y - \mu) \right),$$

- BDA3 p. 72–
- New recommended LKJ-prior mentioned in Appendix A, see more in Stan manual

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- BDA3 p. 72–
- New recommended LKJ-prior mentioned in Appendix A, see more in Stan manual
- Gaussian process and Gaussian Markov random field models are in practice computed with multivariate normals
  - GPs in BDA3 Chapter 21, and a course in spring
  - GPs and GMRFs often used also as priors for latent functions and combined with non-normal observation models

# Normal linear regression

- $y_i \sim \mathcal{N}(\alpha + \beta x_i, \sigma^2), \quad i = 1, \dots, N$

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- with unknown prior scales and  $\sigma^2$ , numerical integration needed
- more in BDA3 Chapter 14 (not part of the course) and Regression and Other Stories book

# Scale mixture of normals

- Many useful distributions can be presented as scale mixture of normals, e.g.
  - Student's  $t$
  - Cauchy
  - Double exponential aka Laplace
  - Horseshoe
  - R2-D2

# Multinomial model for categorical data

- Extension of binomial
- Observation model

$$p(y \mid \theta) \propto \prod_{j=1}^k \theta_j^{y_j},$$

- BDA3 p. 69–

# Generalized linear model (GLM)

- $y_i \sim p(g^{-1}(\alpha + \beta x_i), \phi), \quad i = 1, \dots, N$ 
  - where  $p$  is non-normal (in original definition in exponential family)
  - and  $g$  is a link function

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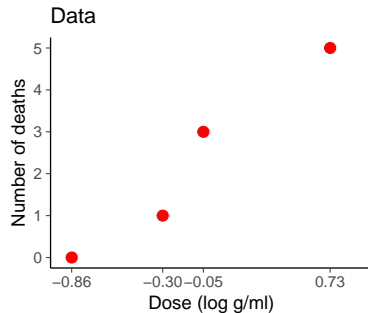
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- Bioassay analysis is used as an example

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- Bioassay analysis is used as an example
- More in BDA3 Chapter 16 and Regression and other stories book

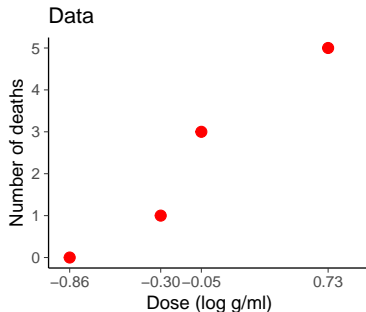
# Bioassay

Dose, $x_i$ (log g/ml)	Number of animals, $n_i$	Number of deaths, $y_i$
-0.86	5	0
-0.30	5	1
-0.05	5	3
0.73	5	5



# Bioassay

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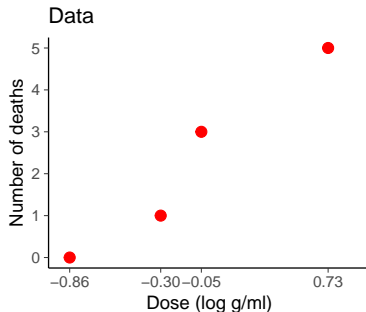
Find out lethal dose 50% (LD50)

- used to classify how hazardous chemical is
- 1984 EEC directive has 4 levels (see the chapter notes)



# Bioassay

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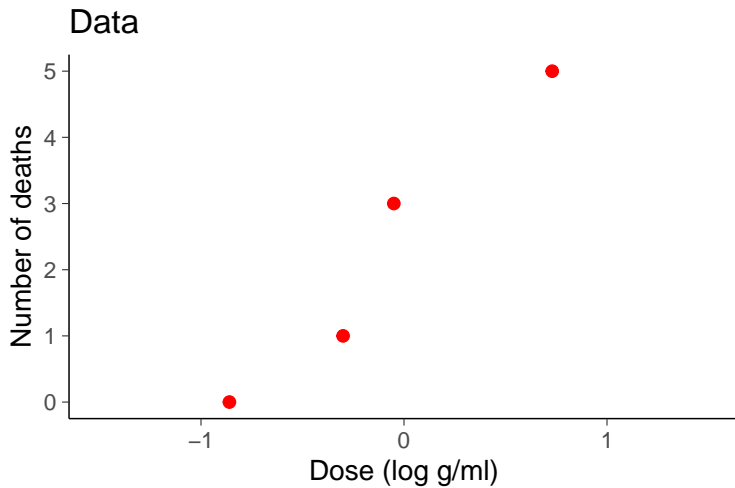
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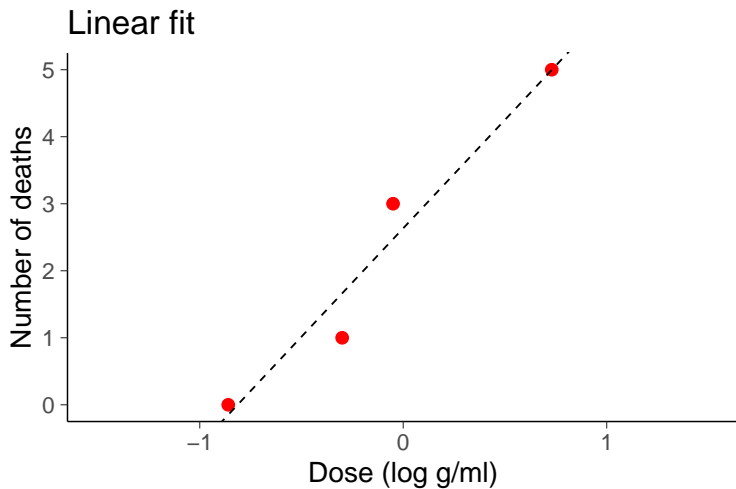
Bayesian methods help to

- reduce the number of animals needed
- easy to make sequential experiment and stop as soon as desired accuracy is obtained

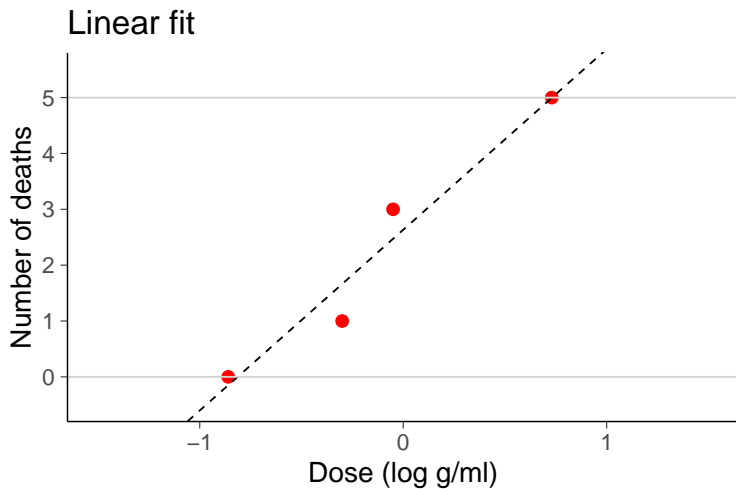
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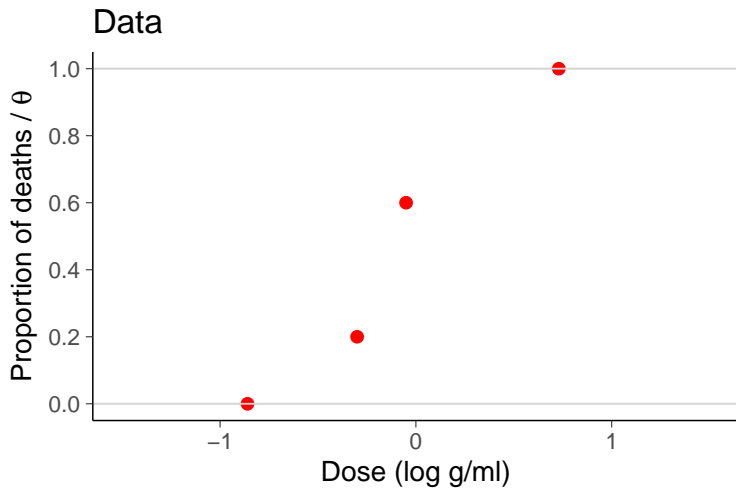
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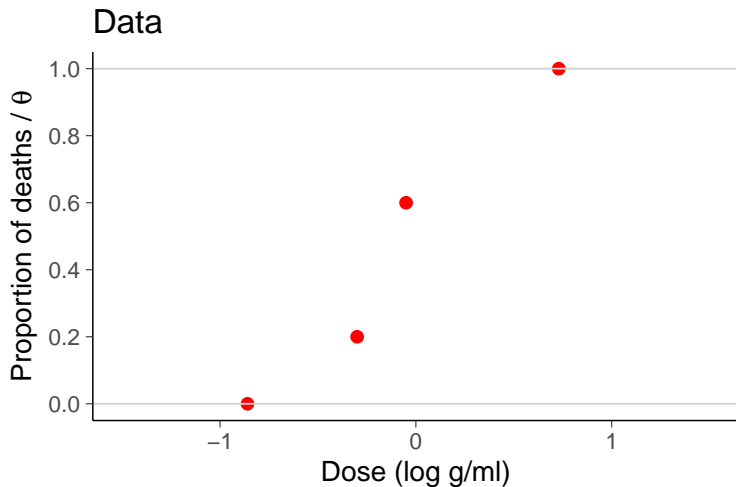
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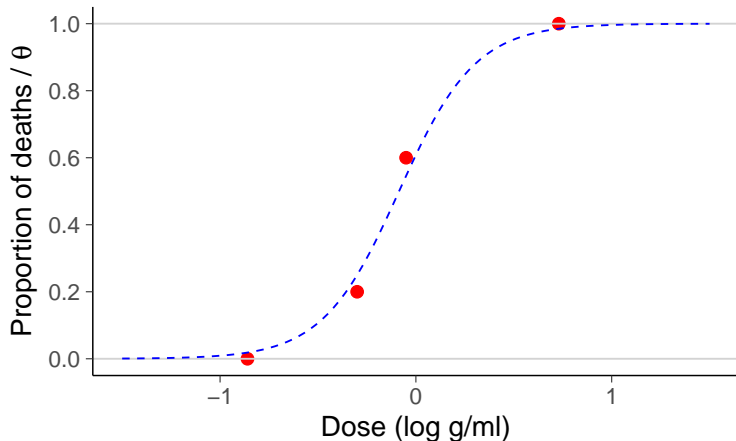


Binomial model

$$y_i \mid \theta_i \sim \text{Bin}(\theta_i, n_i)$$

# Bioassay

## Logistic regression fit



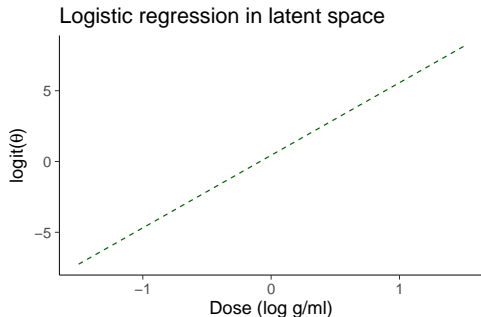
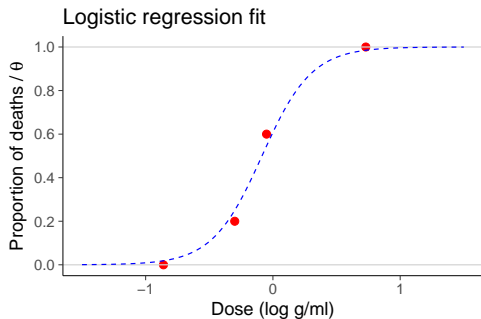
## Binomial model

$$y_i \mid \theta_i \sim \text{Bin}(\theta_i, n_i), \quad \text{logit}(\theta_i) = \log \left( \frac{\theta_i}{1 - \theta_i} \right) = \alpha + \beta x_i$$

# Bioassay

$$y_i \mid \theta_i \sim \text{Bin}(\theta_i, n_i)$$

$$\begin{aligned}\text{logit}(\theta_i) &= \log\left(\frac{\theta_i}{1 - \theta_i}\right) \\ &= \alpha + \beta x_i\end{aligned}$$



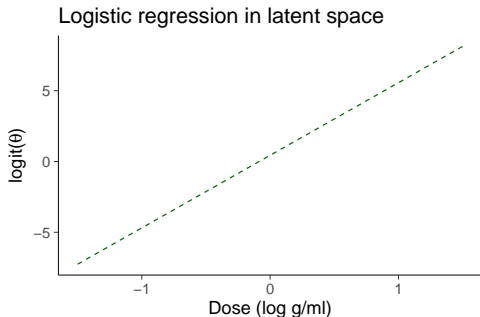
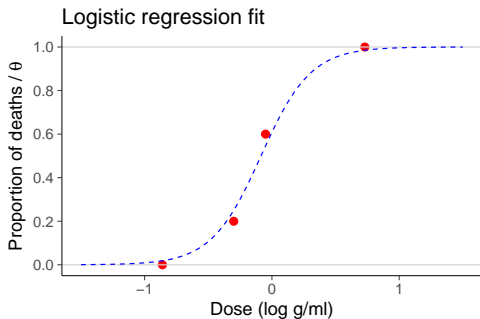


# Bioassay

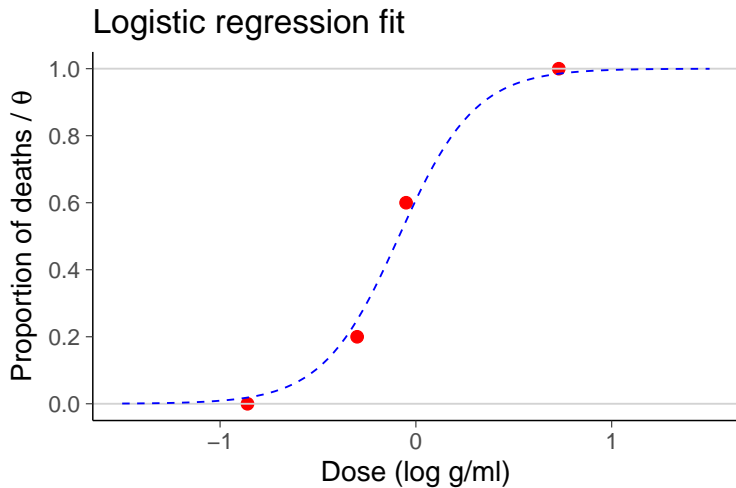
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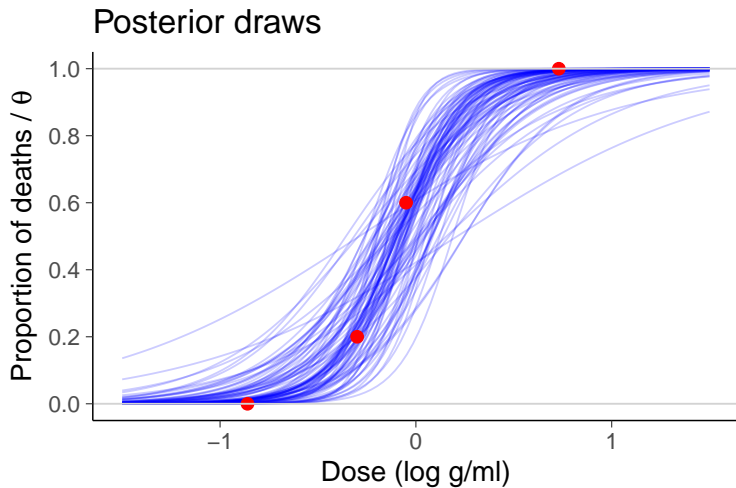
$$\theta_i = \frac{1}{1 + \exp(-(\alpha + \beta x_i))}$$



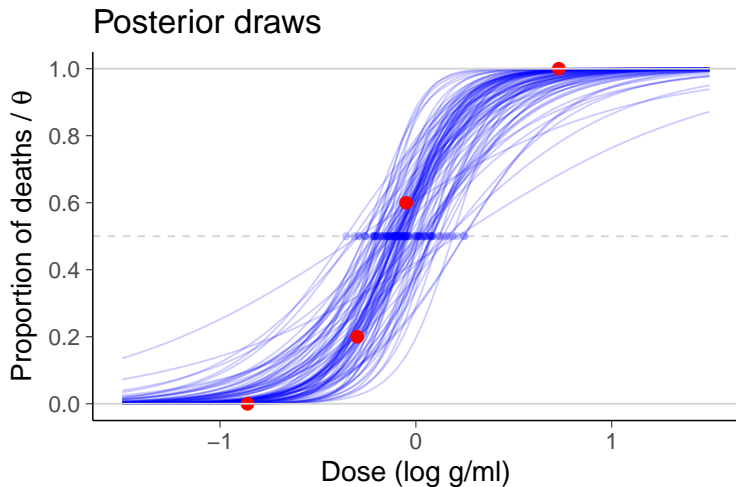
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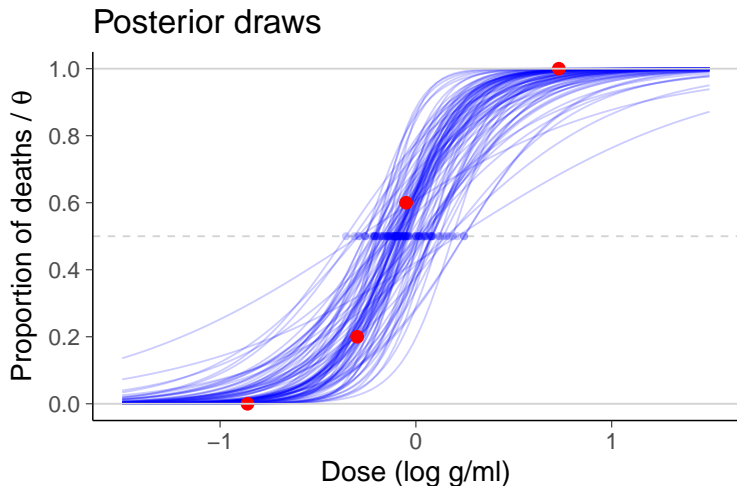


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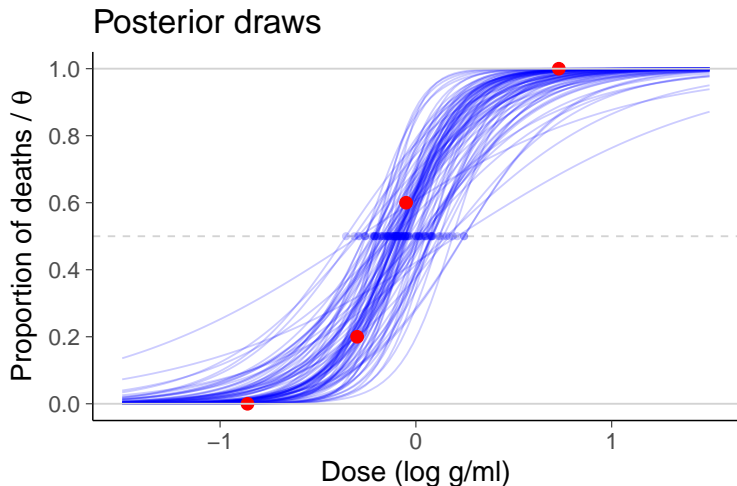
$$\text{LD50: } E\left(\frac{y}{n}\right) = \text{logit}^{-1}(\alpha + \beta x) = 0.5$$

# Bioassay



$$\text{LD50: } E\left(\frac{y}{n}\right) = \text{logit}^{-1}(\alpha + \beta x) = 0.5 \quad \Rightarrow \quad x_{\text{LD50}} = -\alpha/\beta$$

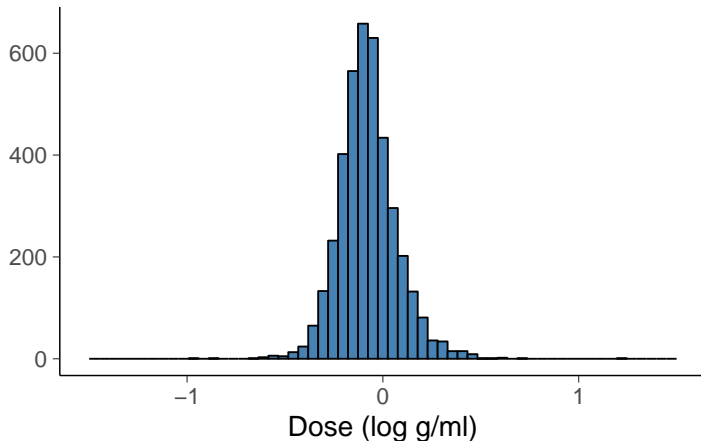
# Bioassay



$$\text{LD50: } E\left(\frac{y}{n}\right) = \text{logit}^{-1}(\alpha + \beta x) = 0.5 \quad \Rightarrow \quad x_{\text{LD50}} = -\alpha/\beta$$
$$x_{\text{LD50}}^{(s)} = -\alpha^{(s)}/\beta^{(s)}$$

# Bioassay

## Bioassay LD50



$$\text{LD50: } E\left(\frac{y}{n}\right) = \text{logit}^{-1}(\alpha + \beta x) = 0.5 \quad \Rightarrow \quad x_{\text{LD50}} = -\alpha/\beta$$
$$x_{\text{LD50}}^{(s)} = -\alpha^{(s)}/\beta^{(s)}$$

# Bioassay posterior

Binomial model

$$y_i \mid \theta_i \sim \text{Bin}(\theta_i, n_i)$$

Link function

$$\text{logit}(\theta_i) = \alpha + \beta \mathbf{x}_i$$



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## Likelihood

$$p(y_i \mid \alpha, \beta, n_i, x_i) \propto \theta_i^{y_i} [1 - \theta_i]^{n_i - y_i}$$

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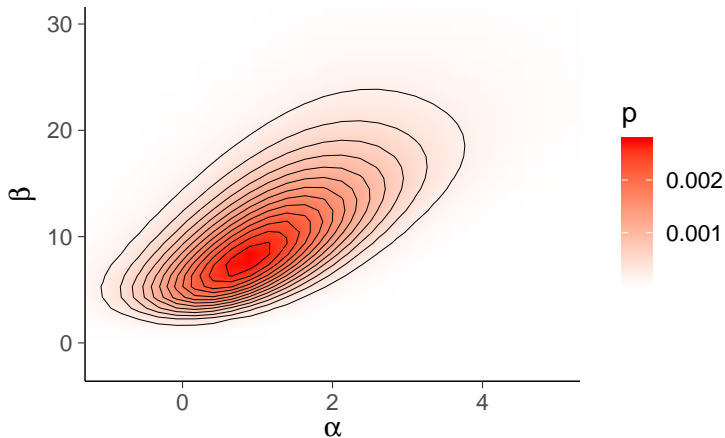
$$p(y_i \mid \alpha, \beta, n_i, x_i) \propto [\text{logit}^{-1}(\alpha + \beta x_i)]^{y_i} [1 - \text{logit}^{-1}(\alpha + \beta x_i)]^{n_i - y_i}$$

## Posterior (with uniform prior on $\alpha, \beta$ )

$$p(\alpha, \beta \mid y, n, x) \propto p(\alpha, \beta) \prod_{i=1}^n p(y_i \mid \alpha, \beta, n_i, x_i)$$

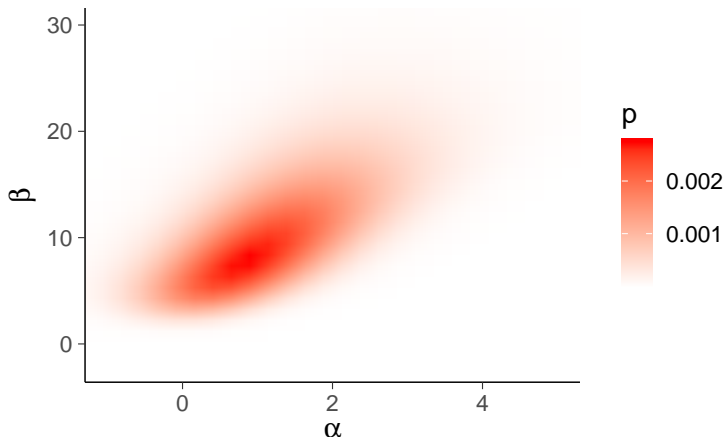
# Bioassay

Posterior density evaluated in a grid



# Bioassay

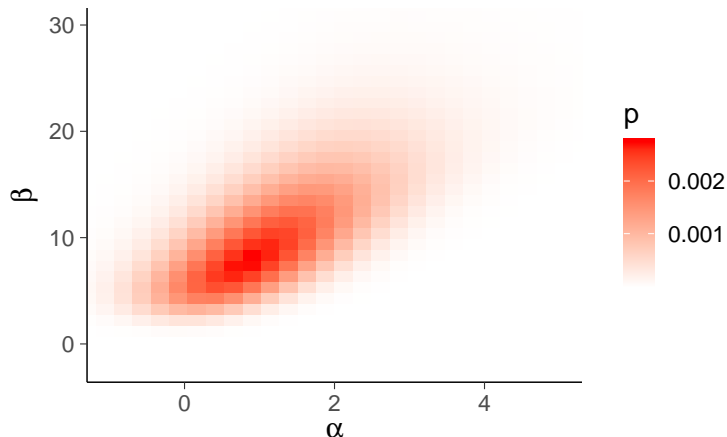
Posterior density evaluated in a grid



Density evaluated in grid, but plotted using interpolation

# Bioassay

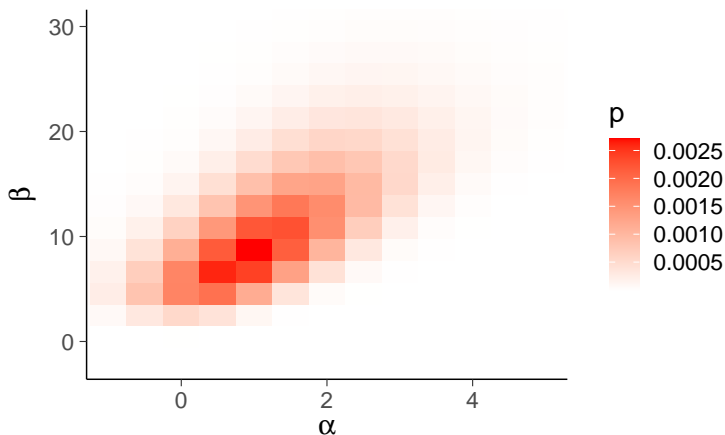
Posterior density evaluated in a grid



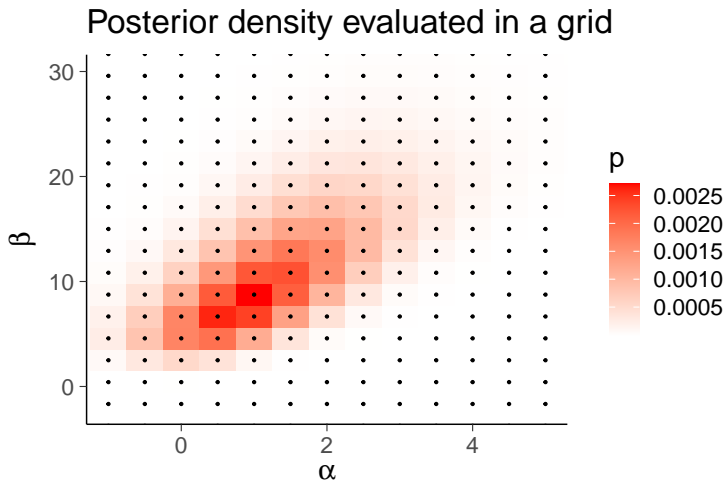
Density evaluated in grid, and plotted without interpolation

# Bioassay

Posterior density evaluated in a grid



Density evaluated in a coarser grid

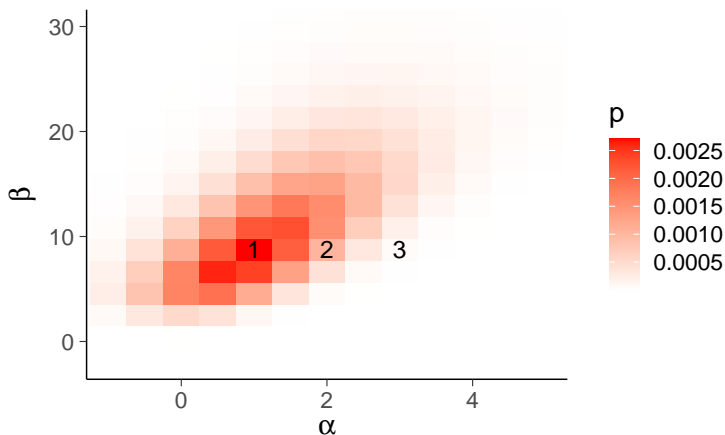


- Approximate the density as piecewise constant function
- Evaluate density in a grid over some finite region
- Density times cell area gives probability mass in each cell



# Bioassay

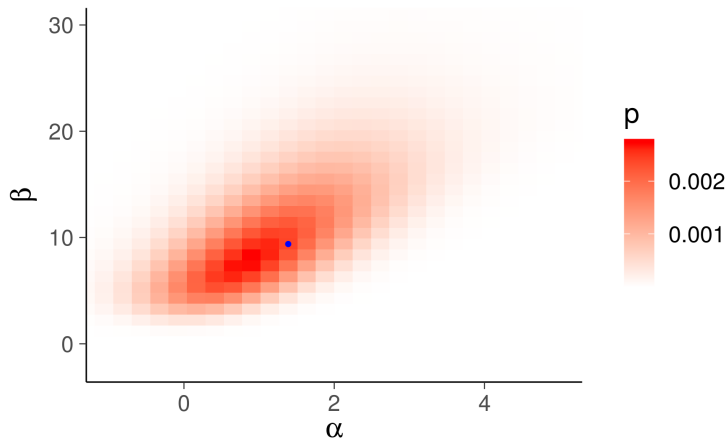
Posterior density evaluated in a grid



- Densities at 1, 2, and 3: 0.0027 0.0010 0.0001
- Probabilities of cells 1, 2, and 3: 0.0431 0.0166 0.0010
- Probabilities of cells sum to 1

# Bioassay

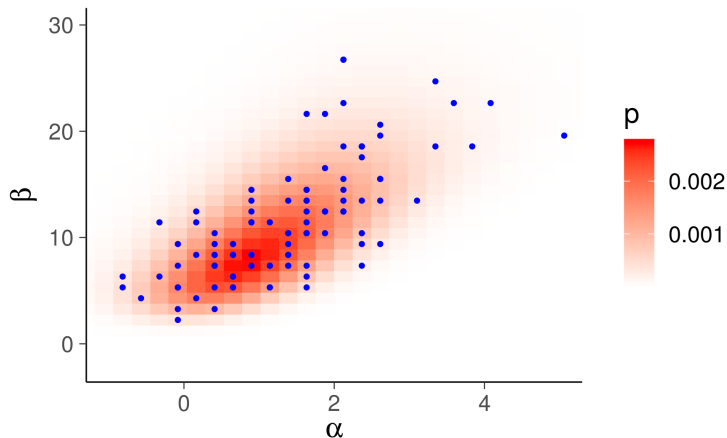
## Posterior density and draws in a grid



- Sample according to grid cell probabilities

# Bioassay

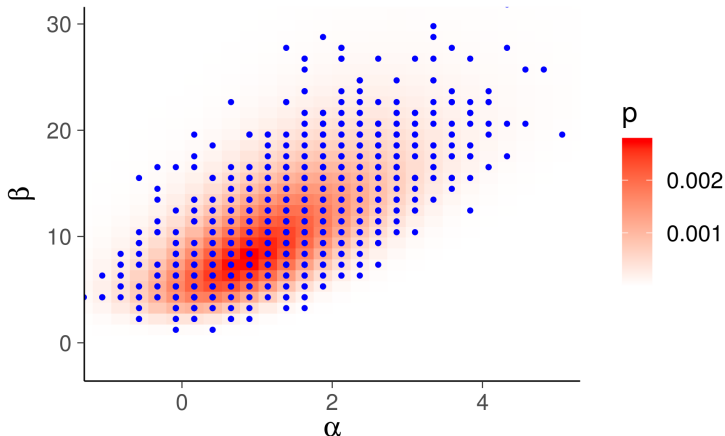
Posterior density and draws in a grid



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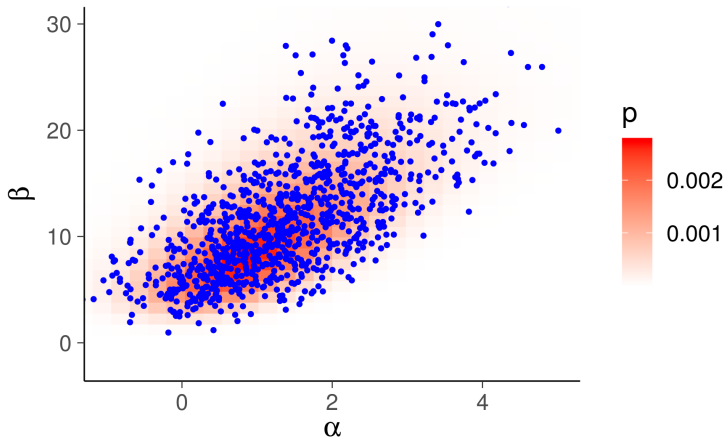
Posterior density and draws in a grid



- Sample according to grid cell probabilities
- Several draws can be from the same grid cell

# Bioassay

Posterior density in a grid and jittered draws



- Jitter can be added to improve visualization

# Grid sampling

- Draws can be used to estimate expectations, for example

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- Instead of sampling, grid could be used to evaluate functions directly, for example

$$E[-\alpha/\beta] \approx \sum_{t=1}^T -\frac{\alpha^{(t)}}{\beta^{(t)}} w_{\text{cell}}^{(t)},$$

where  $w_{\text{cell}}^{(t)}$  is the normalized probability of a grid cell  $t$ , and  $\alpha^{(t)}$  and  $\beta^{(t)}$  are center locations of grid cells

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- Grid sampling gets computationally too expensive in high dimensions