Chapter 6

- 6.1 The place of model checking in applied Bayesian statistics
- 6.2 Do the inferences from the model make sense?
- 6.3 Posterior predictive checking
- 6.4 Graphical posterior predictive checks
  - this can be skimmed, see instead the paper
    Gabry et al. (2019). *Visualization in Bayesian workflow*
- 6.5 Model checking for the educational testing example

Model checking

- demo6_1: Posterior predictive checking - light speed
- demo6_2: Posterior predictive checking - sequential dependence
- demo6_3: Posterior predictive checking - poor test statistic
- demo6_4: Posterior predictive checking - marginal predictive p-value

Model checking – overview

- Sensibility with respect to additional information not used in modeling
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  - cf. relativity theory predictions
- Internal validation
  - posterior predictive checking
  - cross-validation predictive checking

Simon Newcomb's light of speed experiment in 1882

Newcomb measured ($n = 66$) the time required for light to travel from his laboratory on the Potomac River to a mirror at the base of the Washington Monument and back, a total distance of 7422 meters.

Posterior predictive checking – example

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  - model $y \sim \mathrm{normal}(\mu, \sigma)$ with prior $(\mu, \log \sigma) \propto 1$

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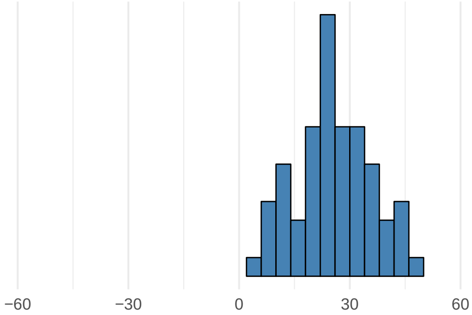

Replicates vs. future observation

- Predictive $\tilde{y}$ is the next not yet observed possible observation. y^{rep} refers to replicating the whole experiment (potentially with same values of x) and obtaining as many replicated observations as in the original data.

Posterior predictive checking – example

- Generate several replicated datasets y^{rep}

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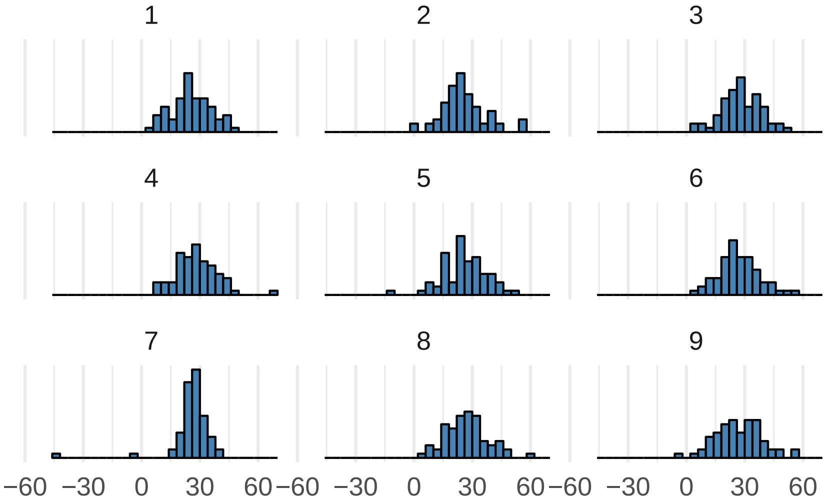
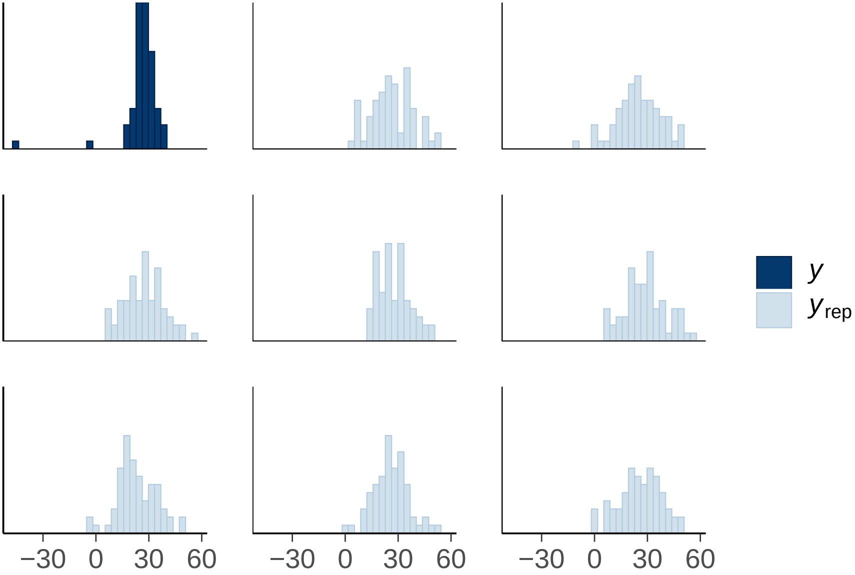


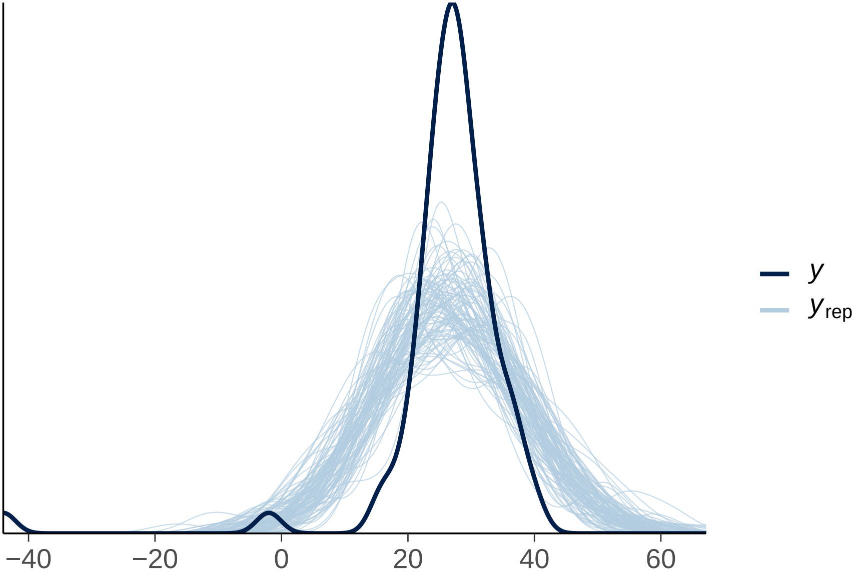
Posterior predictive checking – bayesplot

```
ppc_hist(y, yrep[1:8,])
```



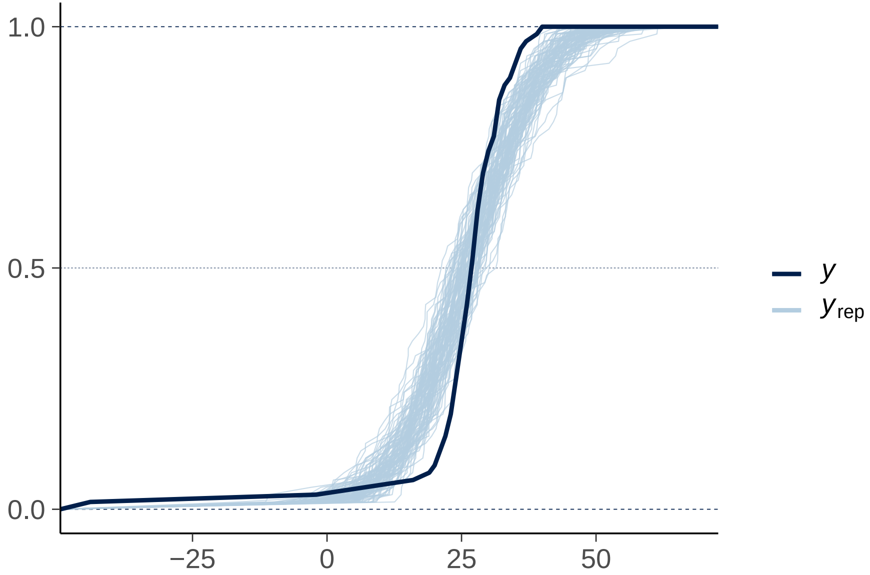
Posterior predictive checking – bayesplot

```
ppc_dens_overlay(y, yrep[1:100,])
```



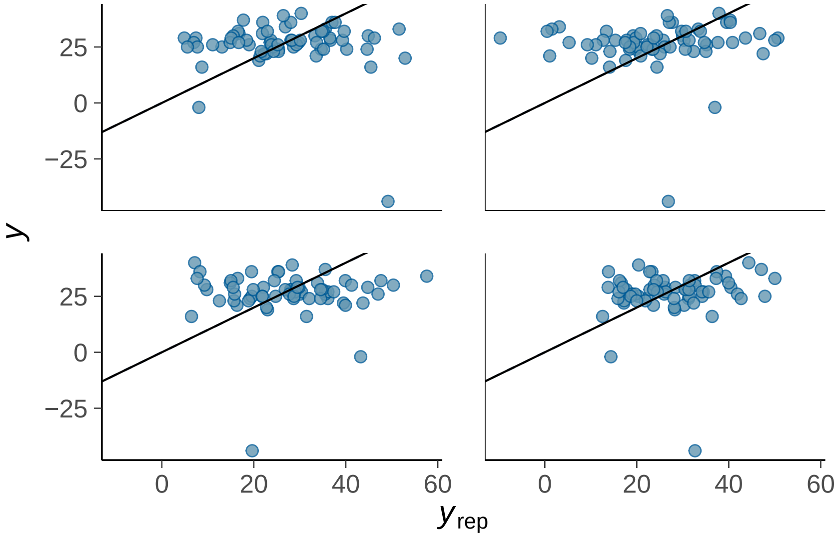
Posterior predictive checking – bayesplot

```
ppc_ecdf_overlay(y, yrep[1:100,])
```



Posterior predictive checking – bayesplot

```
ppc_scatter(y, yrep[1:4,]) + geom_abline()
```



Posterior predictive checking with test statistic

- Replicated data sets y^{rep}
- Test quantity (or discrepancy measure) $T(y, \theta)$
  - summary quantity for the observed data $T(y, \theta)$
  - summary quantity for a replicated data $T(y^{\mathrm{rep}}, \theta)$
  - can be easier to compare summary quantities than data sets

Posterior predictive checking – example

- Compute test statistic for data $T(y, \theta) = \min(y)$

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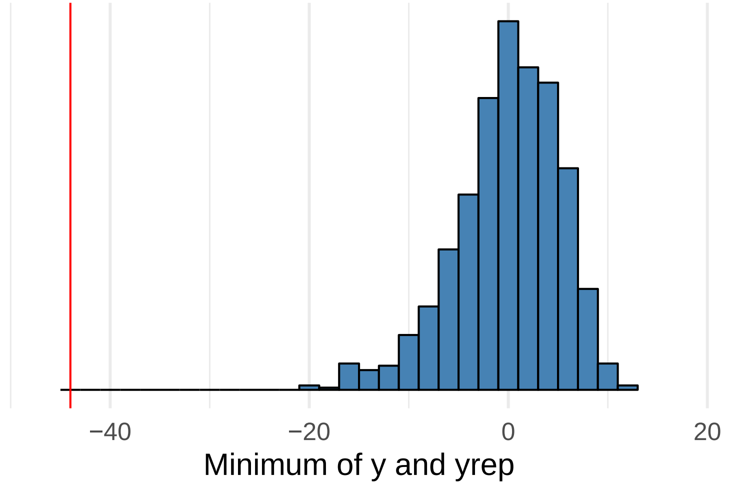

Posterior predictive checking – example

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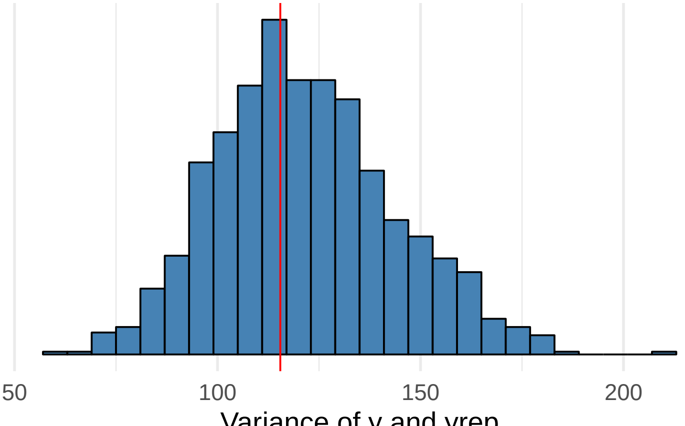


Posterior predictive checking

- *Posterior predictive p-value*

$$\begin{aligned} p &= \Pr(T(y^{\text{rep}},\theta) \geq T(y,\theta)|y) \\ &= \int\int I_{T(y^{\text{rep}},\theta)\geq T(y,\theta)} p(y^{\text{rep}}|\theta)p(\theta|y)dy^{\text{rep}}d\theta \end{aligned}$$

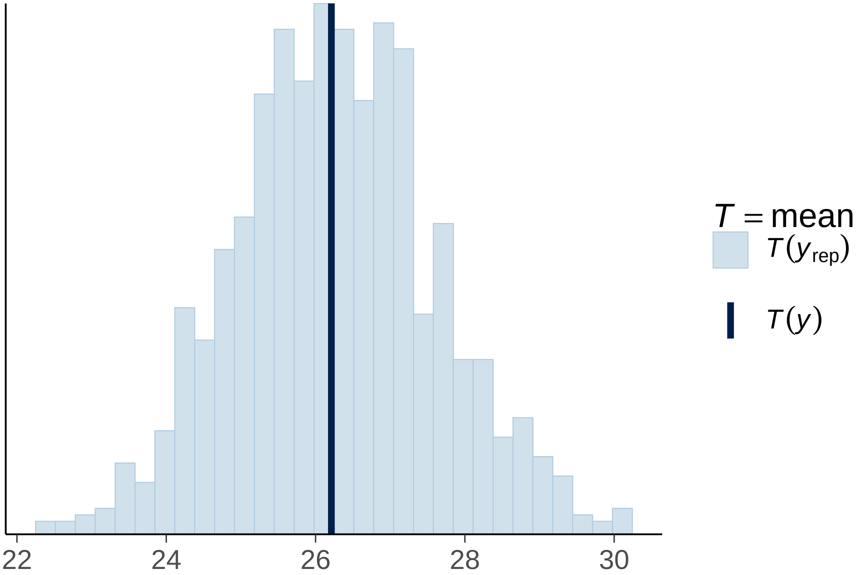
  where I is an indicator function
  - having $(y^{\text{rep}\,(s)},\theta^{(s)})$ from the posterior predictive distribution, easy to compute

$$T(y^{\text{rep}(s)},\theta^{(s)}) \geq T(y,\theta^{(s)}), \quad s = 1,\ldots,S$$

- Posterior predictive p-value (ppp-value) estimates whether difference between the model and data could arise by chance
- Not commonly used, as
  - not calibrated in case of non-ancillary statistic
  - the distribution of test statistic has more information

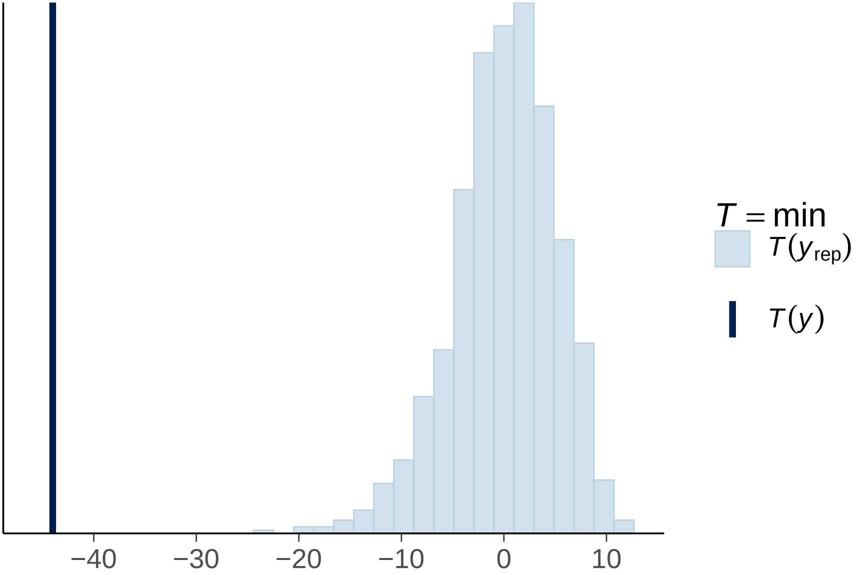
Posterior predictive checking – bayesplot

`ppc_stat(y, yrep)`, the default statistic "mean" is usually bad



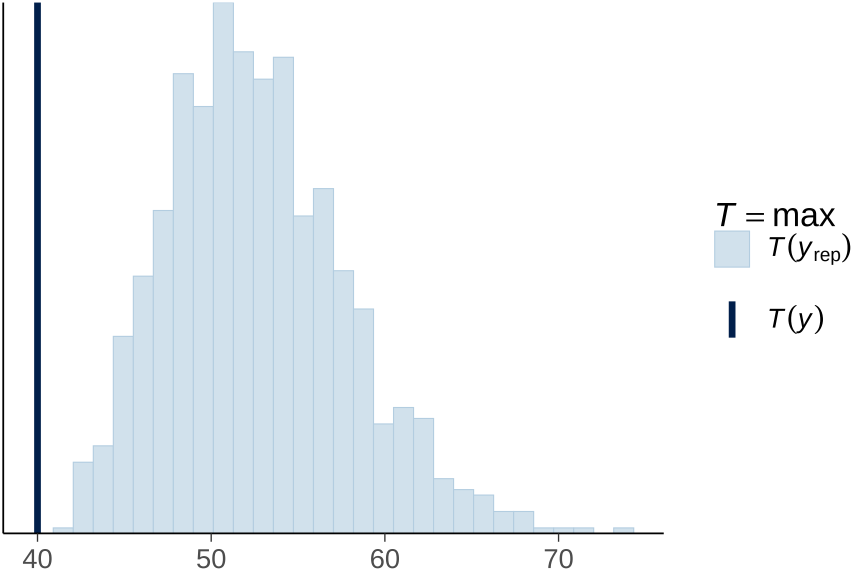
Posterior predictive checking – bayesplot

```
ppc_stat(y, yrep, stat="min")
```



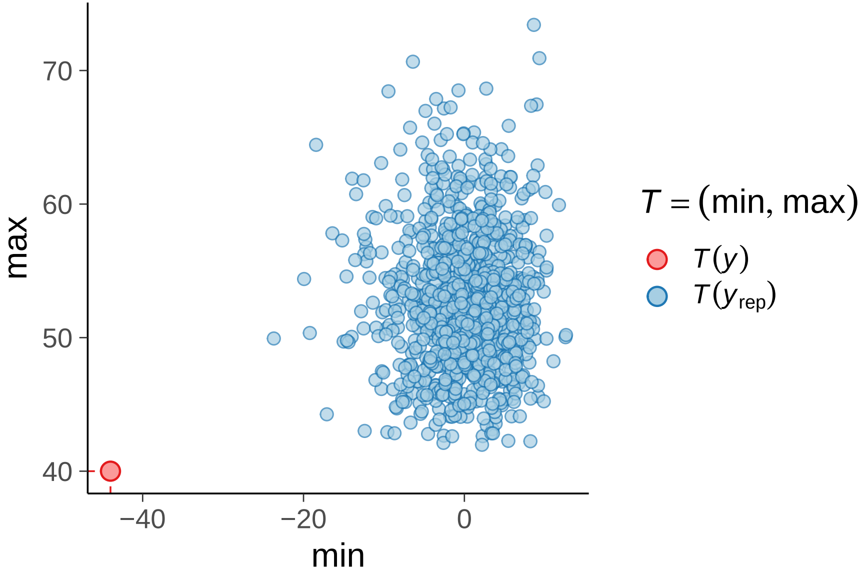
Posterior predictive checking – bayesplot

```
ppc_stat(y, yrep, stat="max")
```



Posterior predictive checking – bayesplot

```
ppc_stat2d(y, yrep, stat=c("min","max"))
```

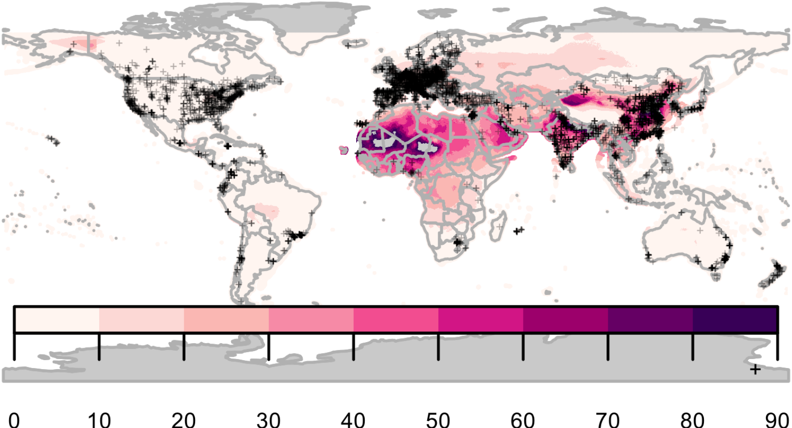


Example: Exposure to air pollution

- Example from Jonah Gabry, Daniel Simpson, Aki Vehtari, Michael Betancourt, and Andrew Gelman (2019). Visualization in Bayesian workflow. https://doi.org/10.1111/rssa.12378
- Estimation of human exposure to air pollution from particulate matter measuring less than 2.5 microns in diameter ($\mathrm{PM}_{2.5}$)
  - Exposure to $\mathrm{PM}_{2.5}$ is linked to a number of poor health outcomes and a recent report estimated that $\mathrm{PM}_{2.5}$ is responsible for three million deaths worldwide each year (Shaddick et al., 2017)
  - In order to estimate the public health effect of ambient $\mathrm{PM}_{2.5}$, we need a good estimate of the $\mathrm{PM}_{2.5}$ concentration at the same spatial resolution as our population estimates.

Example: Exposure to air pollution

- Direct measurements of PM 2.5 from ground monitors at 2980 locations
- High-resolution satellite data of aerosol optical depth



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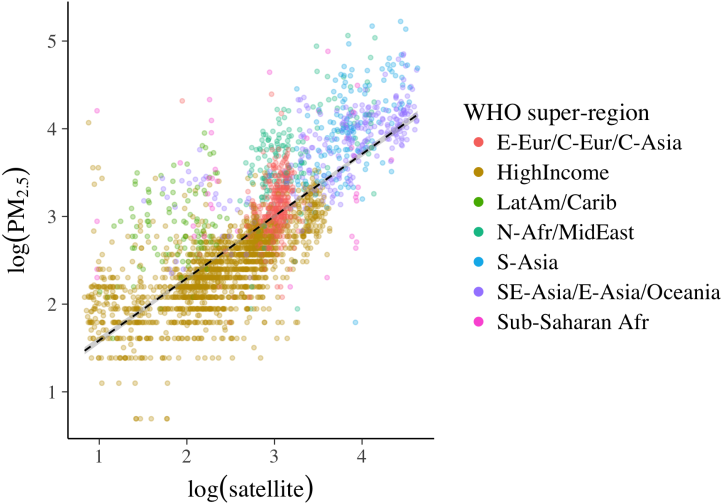

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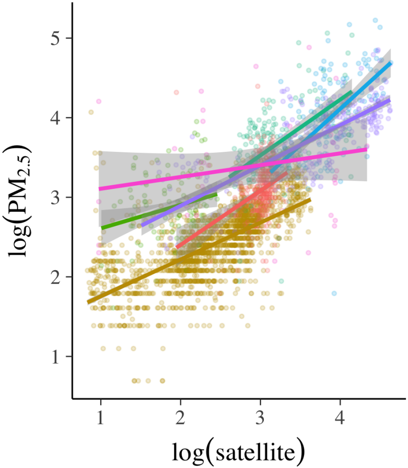
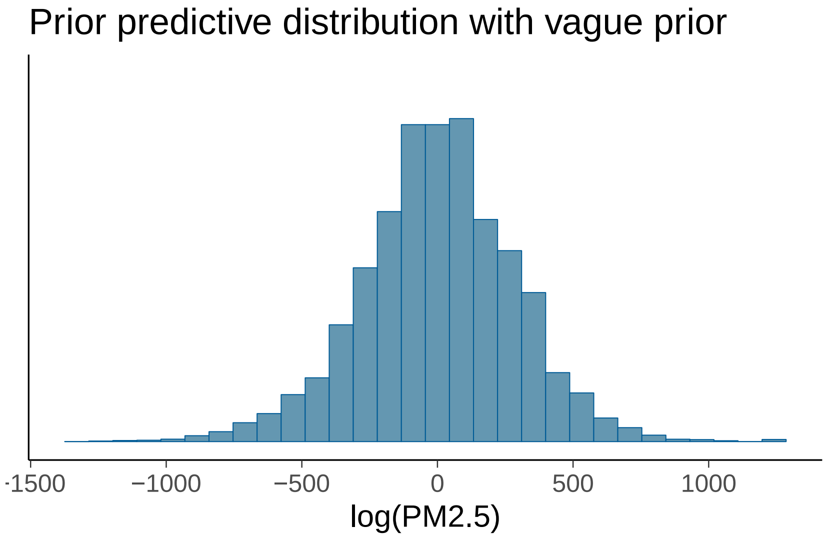


Example: Exposure to air pollution

Prior predictive checking



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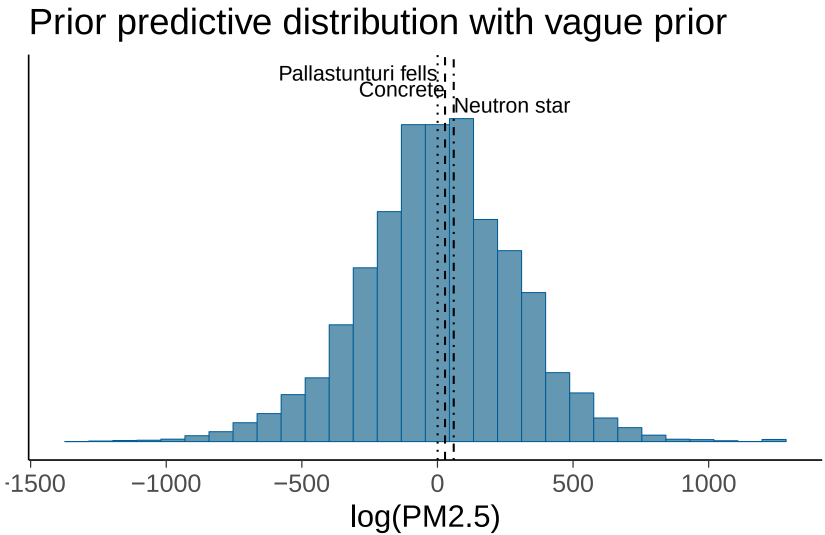
Prior predictive checking



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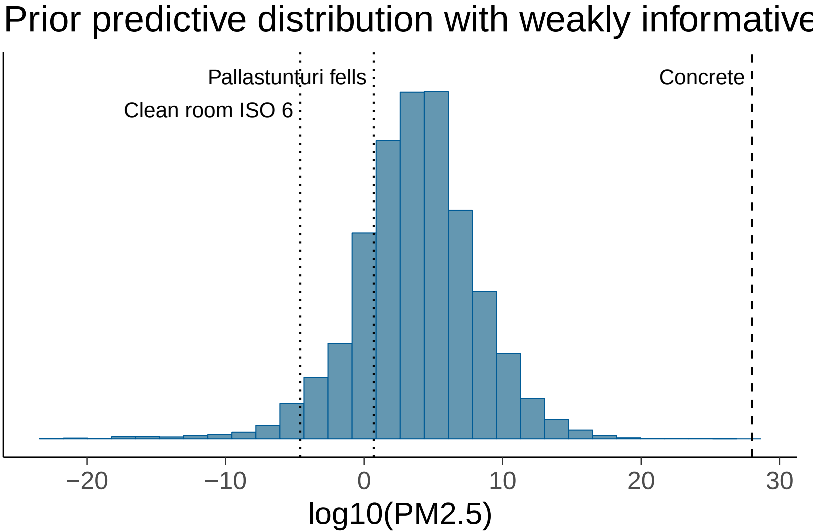
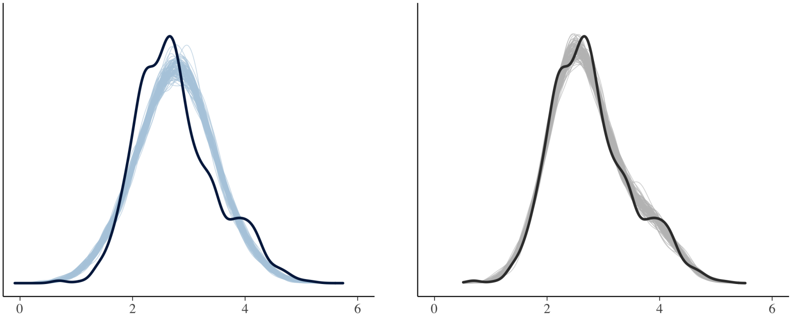
Prior predictive checking



Example: Exposure to air pollution

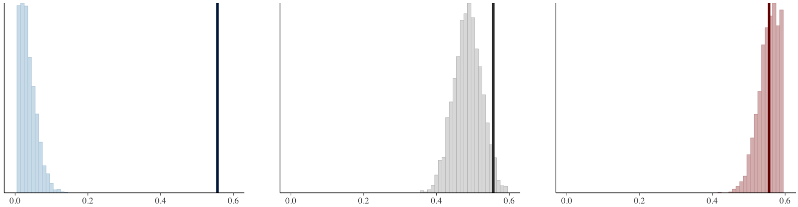
Posterior predictive checking – marginal predictive distributions



(a) Model 1 (b) Model 2

Example: Exposure to air pollution

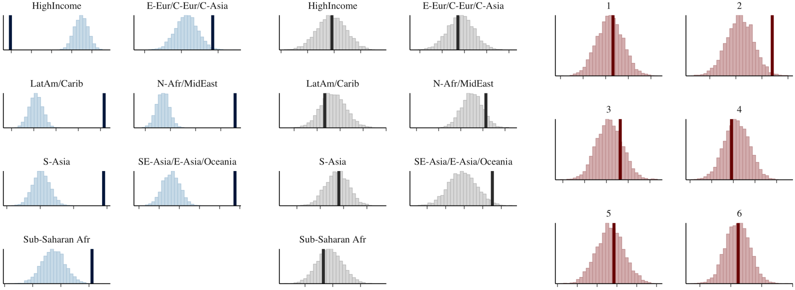
Posterior predictive checking – test statistic (skewness)



(a) Model 1 (b) Model 2 (c) Model 3

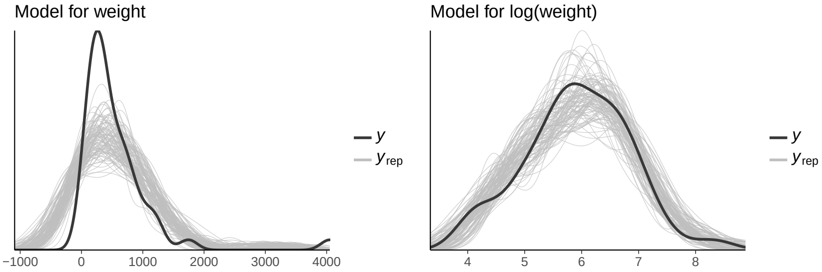
Example: Exposure to air pollution

Posterior predictive checking – test statistic (median for groups)



(a) Model 1 (b) Model 2 (c) Model 3

Positive target



Predicting the yields of mesquite bushes.

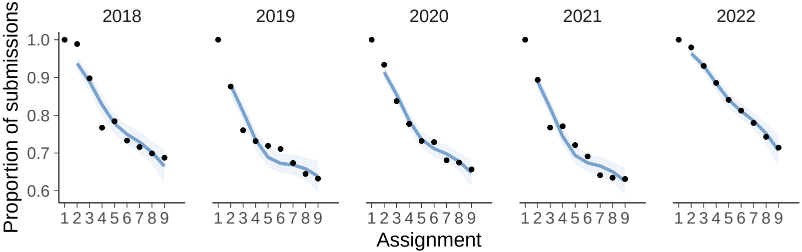
Gelman, Hill & Vehtari (2020): Regression and Other Stories, Chapter 11.

Student retention

Latent hierarchical linear + spline

```
nstudents | trials(nstudents1) ~ (assignment | year) +
  s(assignment, k=4), family=binomial()
```

Latent functions + posterior uncertainty



Student retention

1. Latent hierarchical linear model

```
nstudents | trials(nstudents1) ~ (assignment | year),
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```

2. Latent hierarchical linear model + spline

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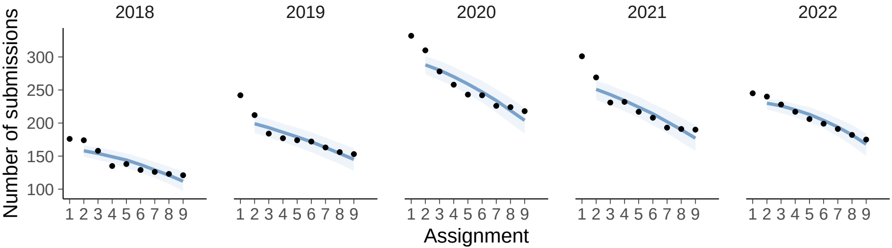
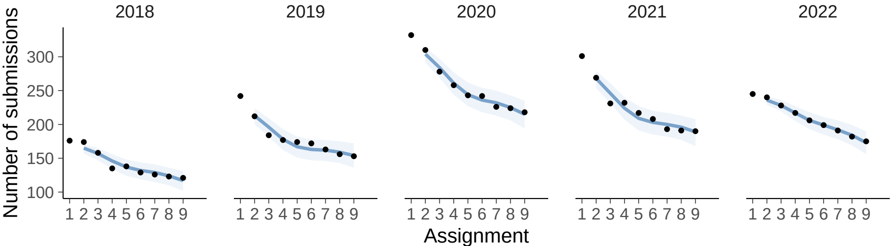
Student retention – Posterior predictive distributions

with `tidybayes`

Latent hierarchical linear model



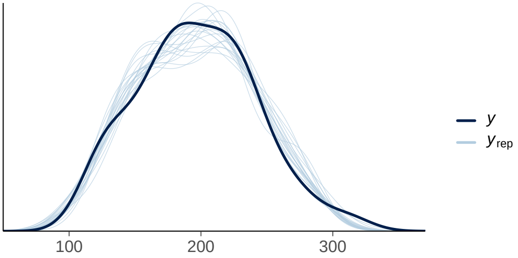
Latent hierarchical linear model + spline



Student retention – Marginal PPC

```
pp_check(fit, ndraws=100)
```

Latent hierarchical linear model



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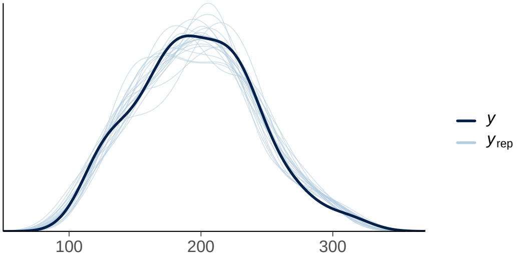
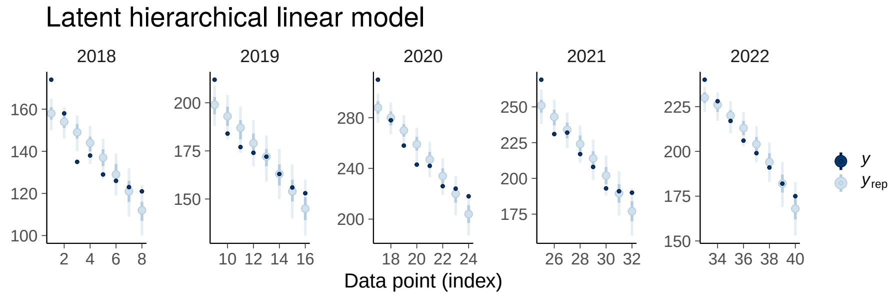
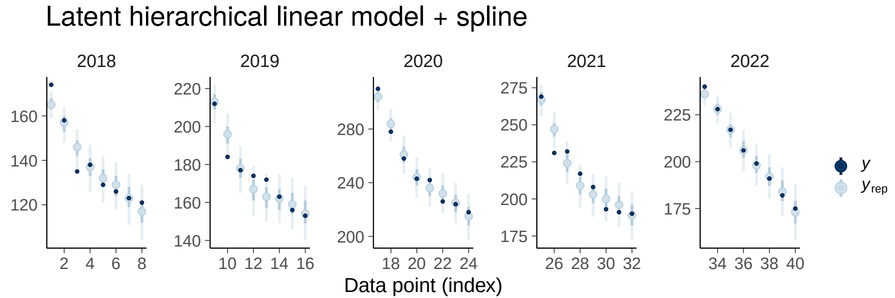


Student retention – Posterior predictive intervals

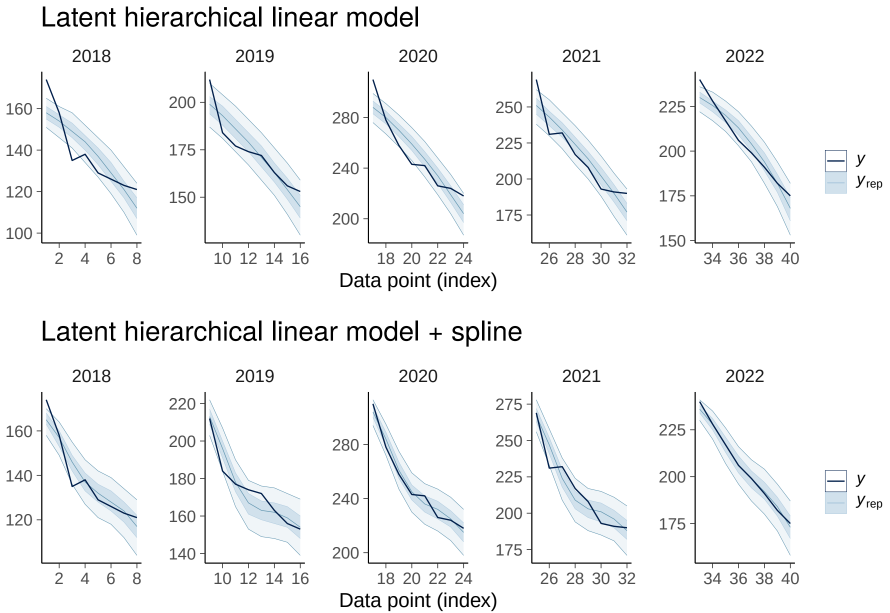
```
pp_check(fit, type = "intervals_grouped", group="year")
```





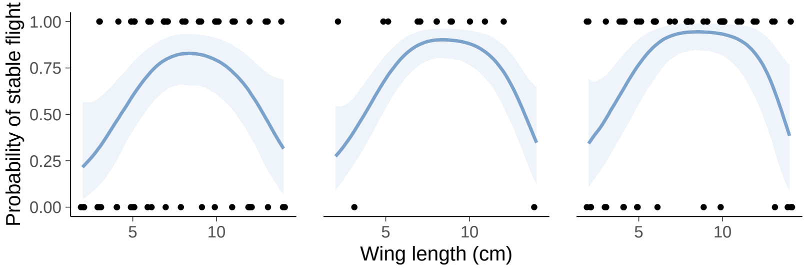
Student retention – Posterior predictive ribbon

```
pp_check(fit, type = "ribbon_grouped", group="year")
```



PPC for binary target – Helicopters

```
stable_flight ~ s(wing_length) + s(wing_length, by = nclips),
family = bernoulli()
```



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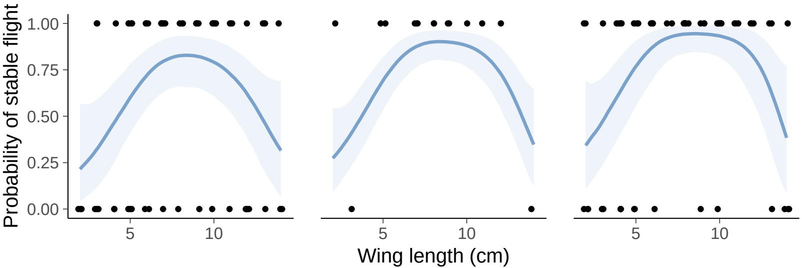
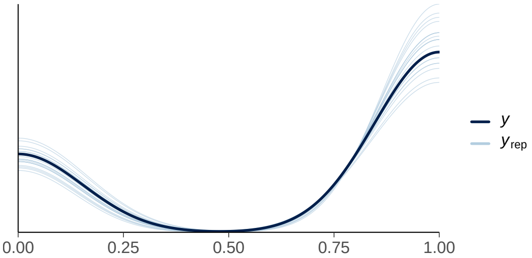



```
pp_check(fit, ndraws=20)
```



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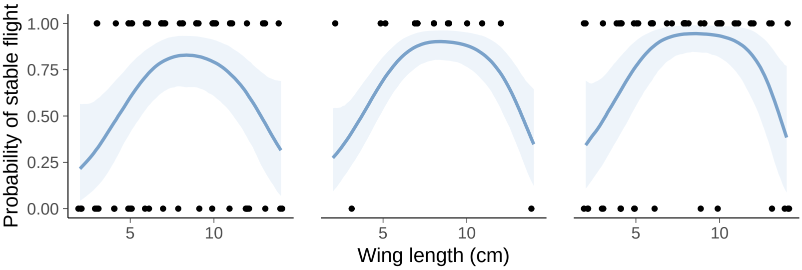
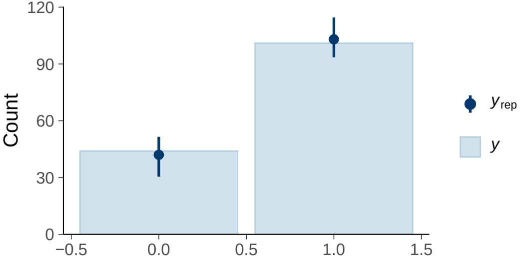



```
pp_check(fit, type="bars")
```



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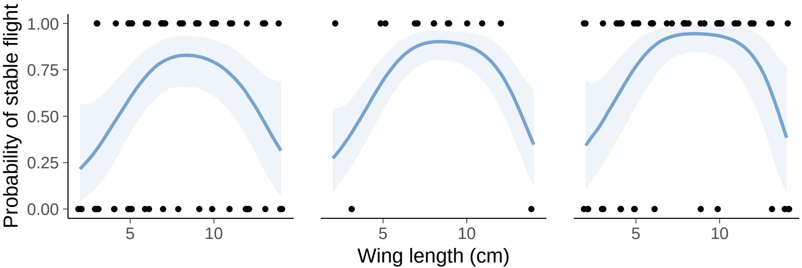
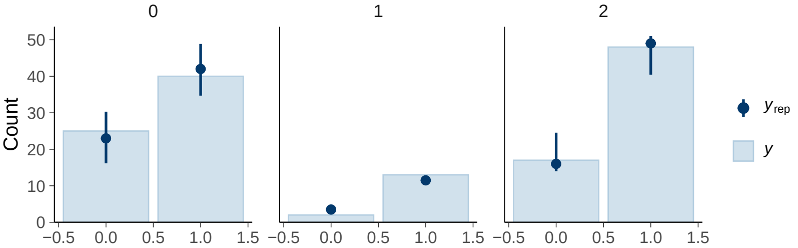



```
pp_check(fit, type="bars_grouped")
```



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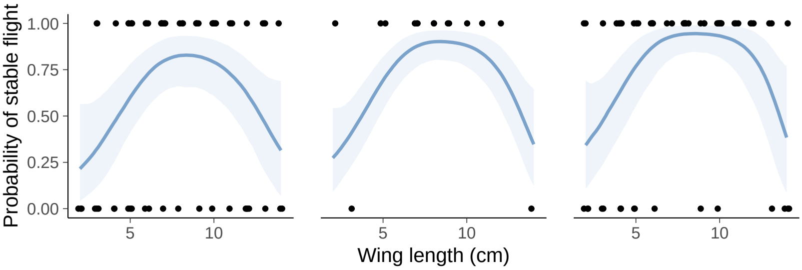
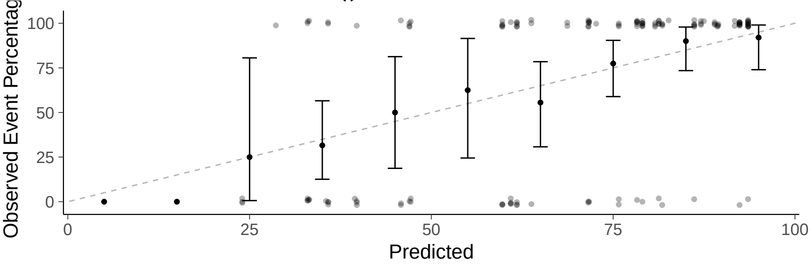


with `caret::calibration()`



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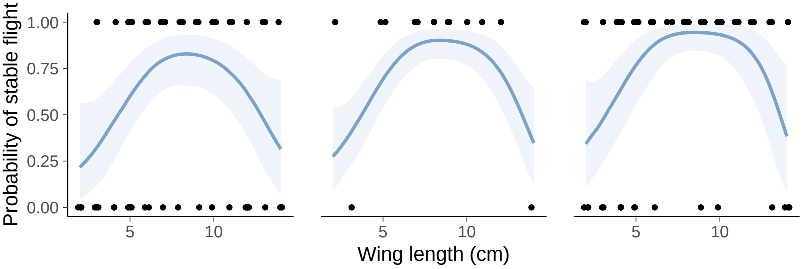
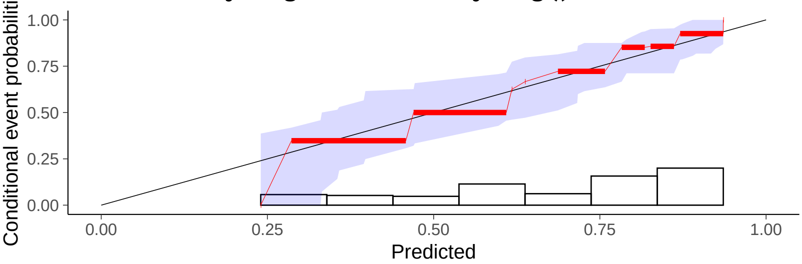


with `reliabilitydiag::reliabilitydiag()`



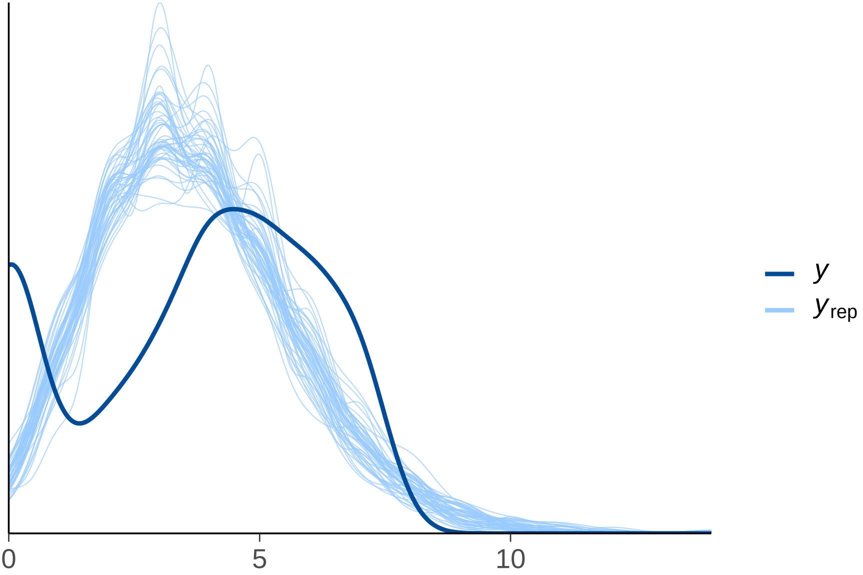
Posterior predictive checking – Stan code

- demo demos_rstan/ppc/poisson-ppc.Rmd

```
data {
  int<lower=1> N;
  int<lower=0> y[N];
}
parameters {
  real<lower=0> lambda;
}
model {
  lambda ~ exponential(0.2);
  y ~ poisson(lambda);
}
generated quantities {
  real log_lik[N];
  int y_rep[N];
  for (n in 1:N) {
    y_rep[n] = poisson_rng(lambda);
    log_lik[n] = poisson_lpmf(y[n] | lambda);
    }
}
```

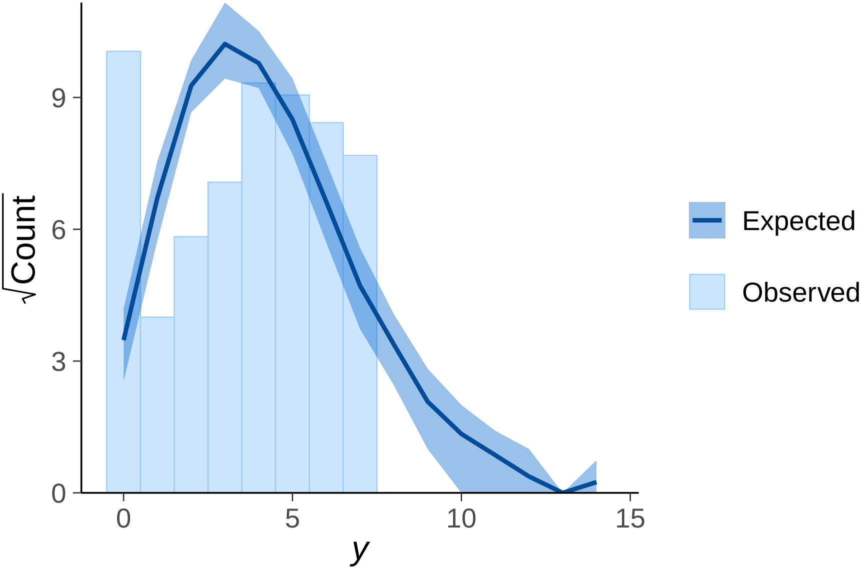
PPC for count data – Poisson model

`ppc_dens_overlay(y, yrep[1:50,])`



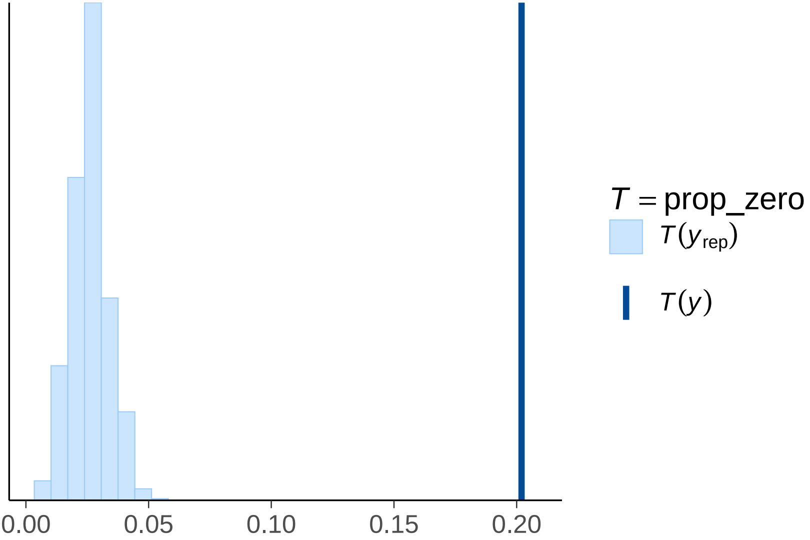
PPC for count data – Poisson model

`ppc_rootogram(y, yrep)`



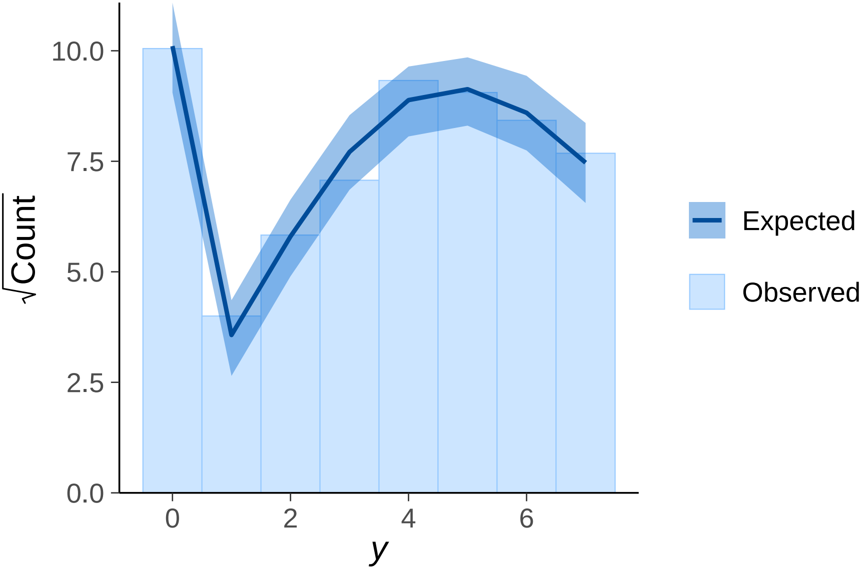
PPC for count data – Poisson model

```
prop_zero <- function(x) mean(x == 0)
ppc_stat(y, yrep, stat = "prop_zero")
```



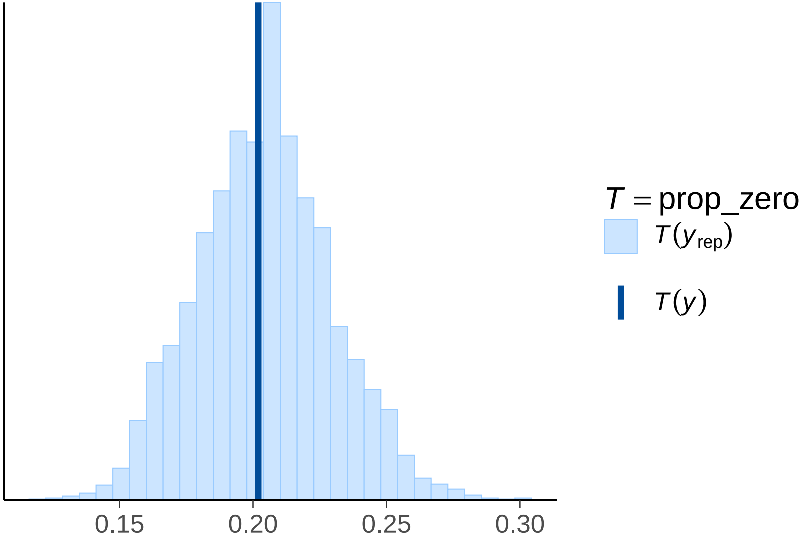
PPC for count data – hurdle truncated Poisson model

`ppc_rootogram(y, yrep2)`



PPC for count data – hurdle truncated Poisson model

```
prop_zero <- function(x) mean(x == 0)
ppc_stat(y, yrep2, stat = "prop_zero")
```



Further reading and examples

- Gabry, Simpson, Vehtari, Betancourt, and Gelman (2019). Visualization in Bayesian workflow. https://doi.org/10.1111/rssa.12378.
- Graphical posterior predictive checks using the bayesplot package http://mc-stan.org/bayesplot/articles/graphical-ppcs.html
- Another demo demos_rstan/ppc/poisson-ppc.Rmd

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