Outline of the chapter 2

- 2.1 Binomial model (repeated experiment with binary outcome)
- 2.2 Posterior as compromise between data and prior information
- 2.3 Posterior summaries
- 2.4 Informative prior distributions (skip exponential families and sufficient statistics)
- 2.5 Gaussian model with known variance
- 2.6 Other single parameter models
  - the normal distribution with known mean but unknown variance is the most important
  - glance through Poisson and exponential
- 2.7 glance through this example, which illustrates benefits of prior information, no need to read all the details (it's quite long example)
- 2.8–2.9 Noninformative and weakly informative priors

Outline of the lecture 2

- Binomial model is the simplest model
  - useful to discuss likelihood, posterior, prior, integration, posterior summaries
  - very commonly used as a building block
  - examples:
    - coin tossing
    - chips from bag
    - covid tests and vaccines
    - classification / logistic regression

Binomial: known θ

- Probability of event 1 in trial is θ

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- Probability of several events in independent trials is e.g. $\theta\theta(1-\theta)\theta(1-\theta)(1-\theta)\ldots$

Binomial: known θ

- Probability of event 1 in trial is θ
- Probability of event 2 in trial is $1 - \theta$
- Probability of several events in independent trials is e.g. $\theta\theta(1-\theta)\theta(1-\theta)(1-\theta)\ldots$
- If there are n trials and we don't care about the order of the events, then the probability that event 1 happens y times is

$$p(y|\theta, n) = \binom{n}{y}\theta^{y}(1-\theta)^{n-y}$$

Binomial: known θ

- Observation model (function of y, discrete)

$$p(y|\theta, n) = \binom{n}{y} \theta^{y} (1-\theta)^{n-y}$$

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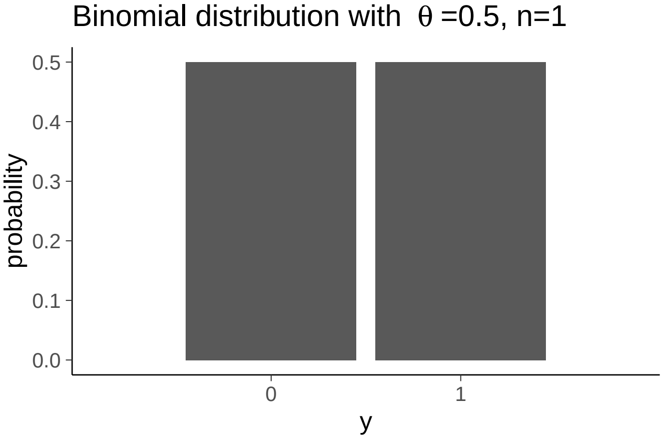

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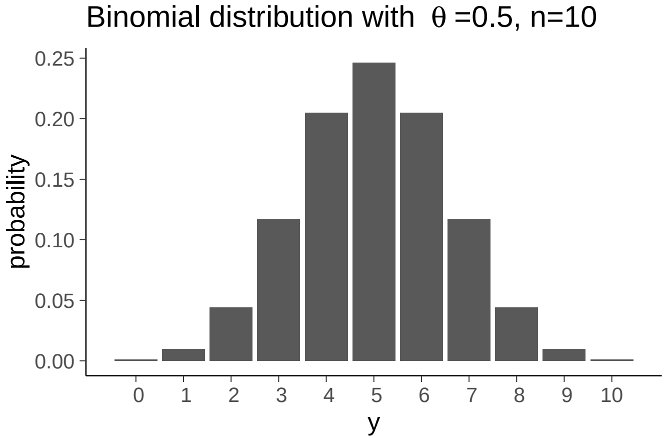

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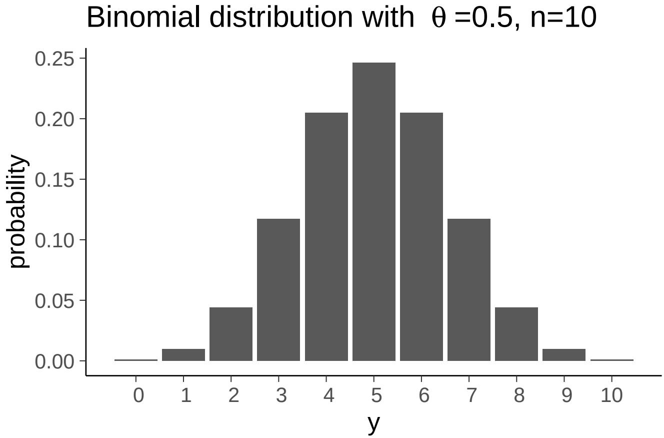

$p(y|n = 10, \theta = 0.5)$: 0.00 0.01 0.04 0.12 0.21 0.25 0.21 0.12 0.04 0.01 0.00

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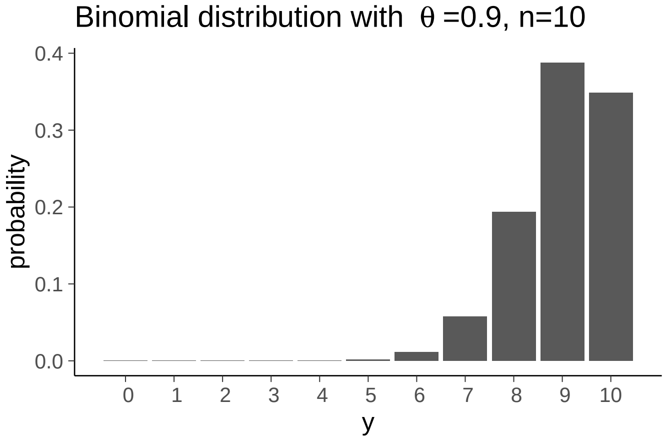

$p(y|n = 10, \theta = 0.9)$: 0.00 0.00 0.00 0.00 0.00 0.00 0.01 0.06 0.19 0.39 0.35

Binomial: known θ

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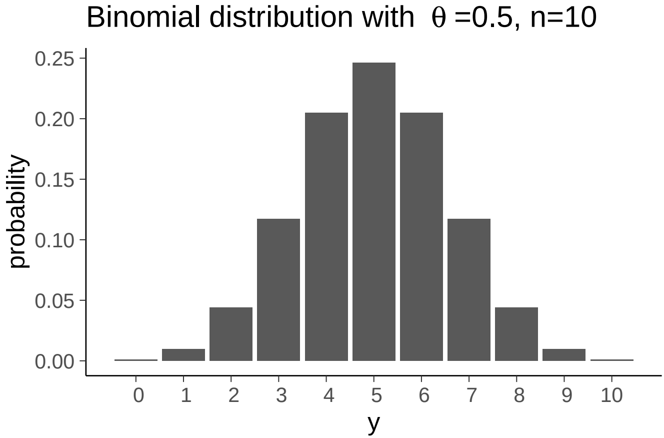

$p(y = 6|n = 10, \theta = 0.5)$: 0.00 0.01 0.04 0.12 0.21 0.25 **0.21** 0.12 0.04 0.01 0.00

Binomial: known θ

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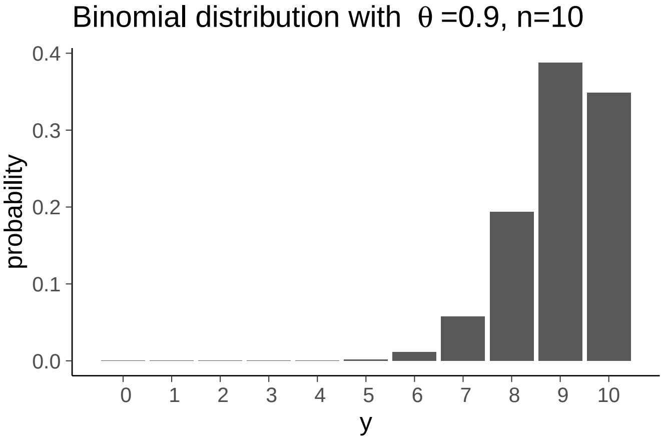
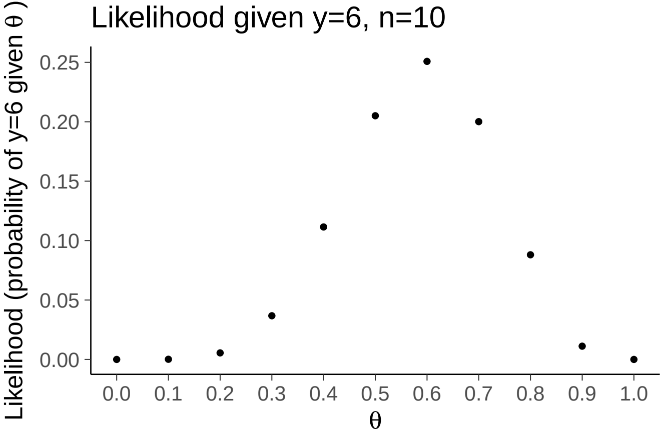


$p(y = 6|n = 10, \theta = 0.9)$: 0.00 0.00 0.00 0.00 0.00 0.00 **0.01** 0.06 0.19 0.39 0.35

Binomial: unknown θ

- Likelihood (function of θ, continuous)

$$p(y|\theta, n) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

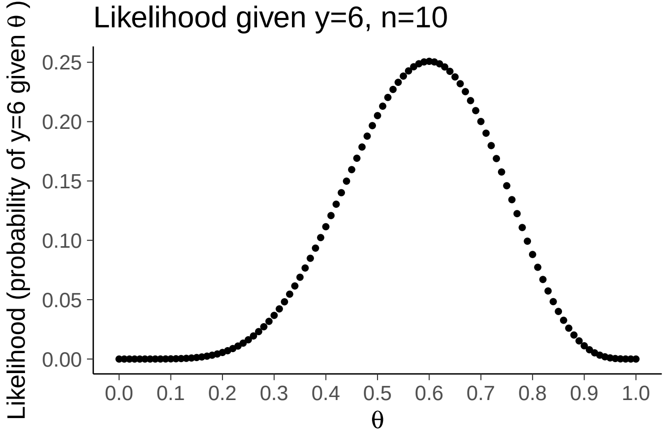


$p(y=6|n=10,\theta)$: 0.00 0.00 0.01 0.04 0.11 **0.21** 0.25 0.20 0.09 **0.01** 0.00

Binomial: unknown θ

- Likelihood (function of θ, continuous)

$$p(y|\theta, n) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$



we can compute the value for any θ, but in practice can evaluate only finite times

Binomial: unknown θ

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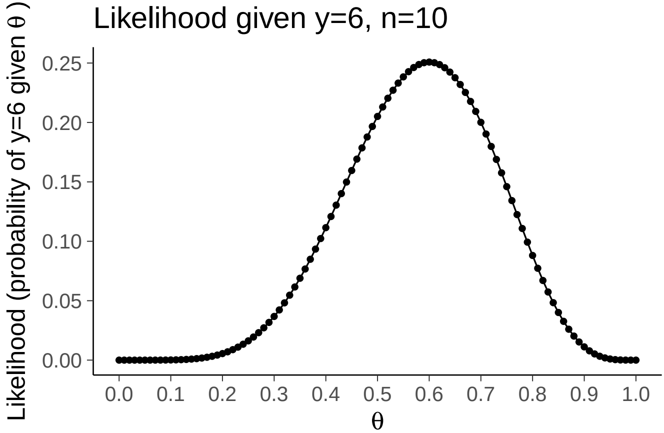
$$p(y|\theta, n) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$



with sufficient many evaluations, linearly interpolated plot looks smooth

Binomial: unknown θ

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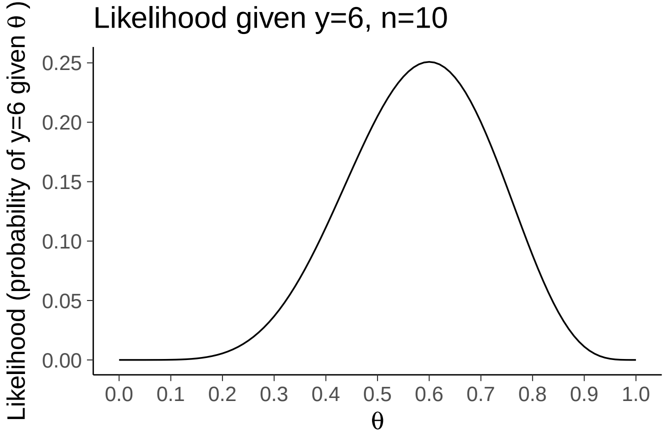

looks smooth, and we'll get back to later to computational cost issues

Binomial: unknown θ

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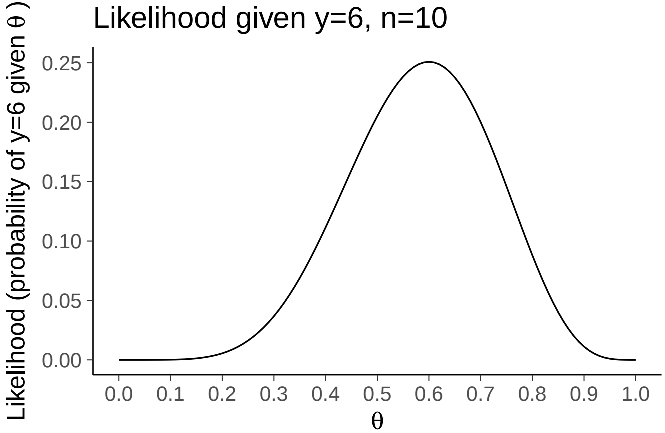

likelihood function describes uncertainty, but is not normalized distribution

Binomial: unknown θ

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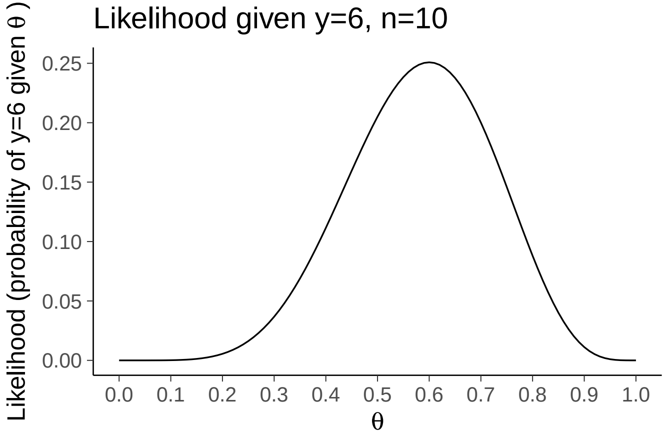

`integrate(function(theta) dbinom(6, 10, theta), 0, 1)` $\approx 0.09 \neq 1$

Binomial: unknown θ

- Posterior with Bayes rule (function of θ, continuous)

$$p(\theta|y,n) = \frac{p(y|\theta,n)p(\theta|n)}{p(y|n)}$$

Binomial: unknown θ

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$$p(\theta|y,n) = \frac{p(y|\theta,n)p(\theta|n)}{p(y|n)}$$

where $p(y|n) = \int p(y|\theta,n)p(\theta|n)d\theta$

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- Start with uniform prior

$$p(\theta|n) = p(\theta|M) = 1, \text{ when } 0 \leq \theta \leq 1$$

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- Start with uniform prior

$$p(\theta|n) = p(\theta|M) = 1, \text{ when } 0 \leq \theta \leq 1$$

- Then

$$p(\theta|y,n) = \frac{p(y|\theta,n)}{p(y|n)} = \frac{\binom{n}{y}\theta^y(1-\theta)^{n-y}}{\int_0^1 \binom{n}{y}\theta^y(1-\theta)^{n-y}d\theta}$$

$$= \frac{1}{Z}\theta^y(1-\theta)^{n-y}$$

Binomial: unknown θ

- Normalization term Z (constant given y)

$$Z = p(y|n) = \int_0^1 \theta^y (1-\theta)^{n-y} d\theta = \frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)}$$

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- Evaluate with $y = 6, n = 10$

`y<-6;n<-10;`

`integrate(function(theta) theta^y*(1-theta)^(n-y), 0, 1)` $\approx$ 0.0004329

`gamma(6+1)*gamma(10-6+1)/gamma(10+2)` $\approx$ 0.0004329

Binomial: unknown θ

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usually computed via $\log \Gamma(\cdot)$ due to the limitations of floating point presentation

Binomial: unknown θ

- Posterior is
$$p(\theta|y,n) = \frac{\Gamma(n+2)}{\Gamma(y+1)\Gamma(n-y+1)}\theta^{y}(1-\theta)^{n-y},$$

Binomial: unknown θ

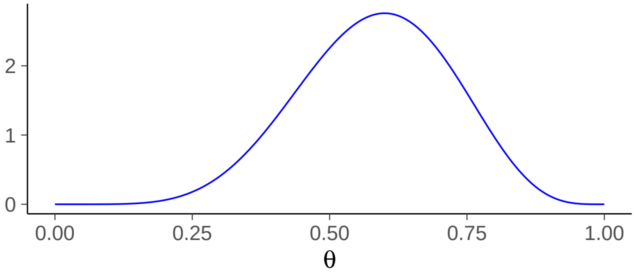
- Posterior is

$$p(\theta|y,n) = \frac{\Gamma(n+2)}{\Gamma(y+1)\Gamma(n-y+1)}\theta^y(1-\theta)^{n-y},$$

which is called Beta distribution

$$\theta|y,n \sim \text{Beta}(y+1, n-y+1)$$

p(θ | y=6, n=10, M=binom) + unif. prior)

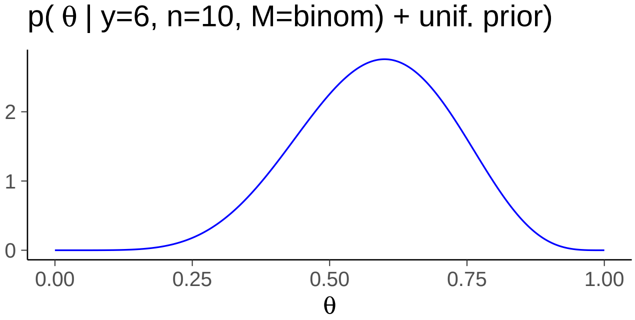


Binomial: computation

- R
  - density `dbeta`
  - CDF `pbeta`
  - quantile `qbeta`
  - random number `rbeta`
- Python
  - `from scipy.stats import beta`
  - density `beta.pdf`
  - CDF `beta.cdf`
  - prctile `beta.ppf`
  - random number `beta.rvs`

Binomial: computation

- Beta CDF not trivial to compute
- For example, `pbeta` in R uses a continued fraction with weighting factors and asymptotic expansion
- Laplace developed normal approximation (Laplace approximation), because he didn't know how to compute Beta CDF



Placenta previa

- Probability of a girl birth given placenta previa (BDA3 p. 37)
  - 437 girls and 543 boys have been observed
  - is the ratio 0.445 different from the population average 0.485?

Placenta previa

- Probability of a girl birth given placenta previa (BDA3 p. 37)
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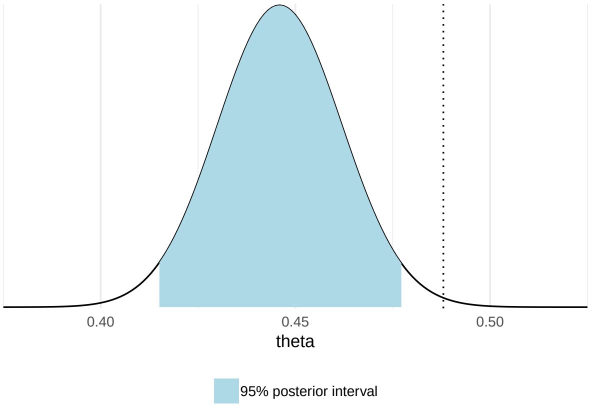
Uniform prior −> Posterior is Beta(438,544)



Binomial: unknown θ

Sometimes conditioning on the model M is explicitly shown

- Posterior with Bayes rule (function of θ, continuous)

$$p(\theta|y, n, M) = \frac{p(y|\theta, n, M)p(\theta|n, M)}{p(y|n, M)}$$

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  - makes it more clear that likelihood and prior are both part of the model
  - makes it more clear that there is no absolute probability for $p(y|n)$, but it depends on the model M
  - in case of two models, we can evaluate marginal likelihoods $p(y|n, M_1)$ and $p(y|n, M_2)$ (more in Ch 7)

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  - usually dropped to make the notation more concise

Predictive distribution – Effect of integration

- Predictive distribution for new $\tilde{y}$ (discrete)

$$p(\tilde{y} = 1 | \theta, y, n, M)$$

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$$p(\tilde{y}=1|y,n,M)=\int_0^1 p(\tilde{y}=1|\theta,y,n,M)p(\theta|y,n,M)d\theta$$

Predictive distribution – Effect of integration

- Predictive distribution for new $\tilde{y}$ (discrete)

$$
\begin{aligned}
p(\tilde{y} = 1 | y, n, M) &= \int_0^1 p(\tilde{y} = 1 | \theta, y, n, M) p(\theta | y, n, M) d\theta \\
&= \int_0^1 \theta p(\theta | y, n, M) d\theta
\end{aligned}
$$

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&= \mathrm{E}[\theta|y]
\end{aligned}
$$

Predictive distribution – Effect of integration

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\end{aligned}
$$

- With uniform prior

$$
\mathrm{E}[\theta | y] = \frac{y+1}{n+2}
$$

Predictive distribution – Effect of integration

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$$= \mathrm{E}[\theta|y]$$

- With uniform prior

$$\mathrm{E}[\theta|y] = \frac{y+1}{n+2}$$

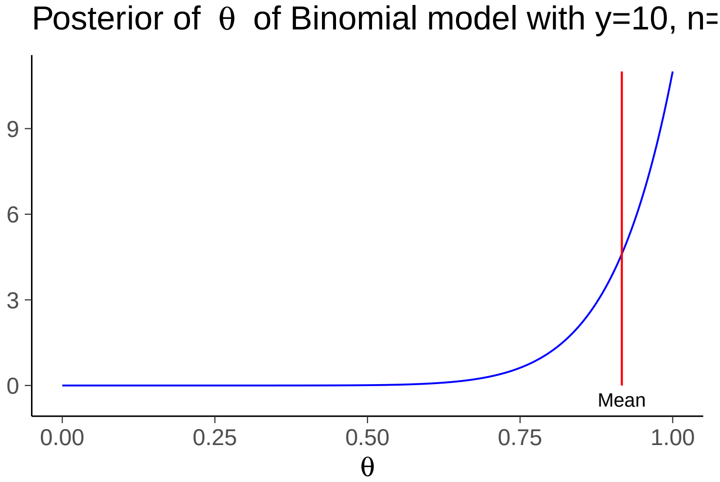
- Extreme cases

$$p(\tilde{y} = 1|y = 0, n, M) = \frac{1}{n+2}$$
$$p(\tilde{y} = 1|y = n, n, M) = \frac{n+1}{n+2}$$

- cf. maximum likelihood

Benefits of integration

Example: $n = 10, y = 10$



Predictive distribution

- Prior predictive distribution for new $\tilde{y}$ (discrete)

$$p(\tilde{y} = 1|M) = \int_0^1 p(\tilde{y} = 1|\theta, M)p(\theta|M)d\theta$$

- Posterior predictive distribution for new $\tilde{y}$ (discrete)

$$p(\tilde{y} = 1|y, n, M) = \int_0^1 p(\tilde{y} = 1|\theta, y, n, M)p(\theta|y, n, M)d\theta$$

Justification for uniform prior

- $p(\theta|M) = 1$ if
  1) we want the prior predictive distribution to be uniform

$$p(y|n, M) = \frac{1}{n+1}, \quad y = 0, \ldots, n$$

  - nice justification as it is based on observables y and n

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    - nice justification as it is based on observables y and n

  2) we think all values of θ are equally likely

Priors

- Conjugate prior (BDA3 p. 35)
- Noninformative prior (BDA3 p. 51)
- Proper and improper prior (BDA3 p. 52)
- Weakly informative prior (BDA3 p. 55)
- Informative prior (BDA3 p. 55)
- Prior sensitivity (BDA3 p. 38)

Conjugate prior

- Prior and posterior have the same form
  - only for exponential family distributions (plus for some irregular cases)
- Used to be important for computational reasons, and still sometimes used for special models to allow partial analytic marginalization (Ch 3)
  - with dynamic Hamiltonian Monte Carlo used e.g. in Stan no computational benefit

Beta prior for Binomial model

- Prior

$$\mathrm{Beta}(\theta|\alpha,\beta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

- Posterior

$$p(\theta|y,n,M) \propto \theta^{y}(1-\theta)^{n-y}\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

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after normalization

$$p(\theta|y,n,M) = \text{Beta}(\theta|\alpha+y,\beta+n-y)$$

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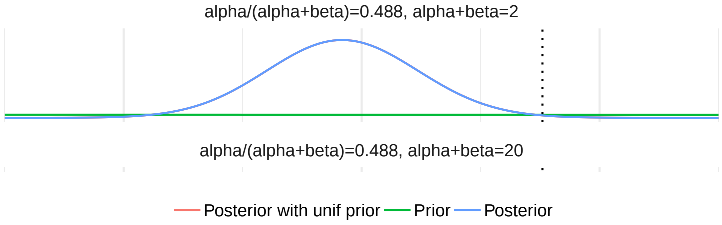
  after normalization

$$p(\theta|y, n, M) = \text{Beta}(\theta|\alpha + y, \beta + n - y)$$

- $(\alpha - 1)$ and $(\beta - 1)$ can considered to be number of prior observations
- Uniform prior when $\alpha = 1$ and $\beta = 1$

Placenta previa

- Beta prior centered on population average 0.485



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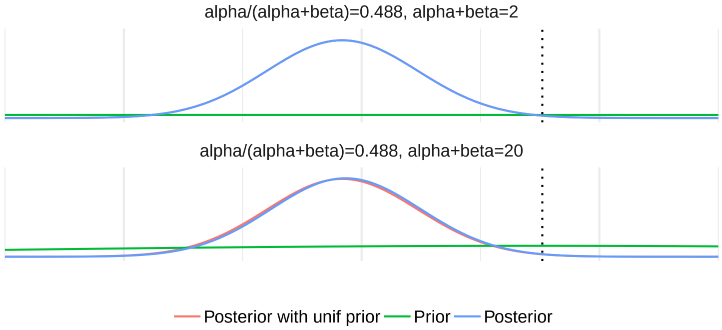

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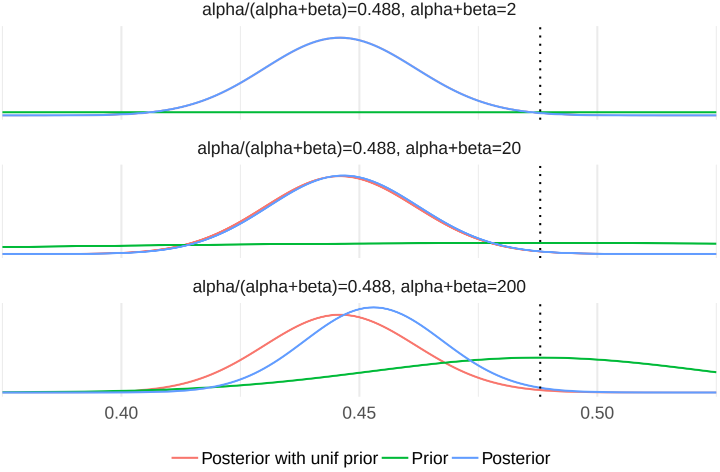
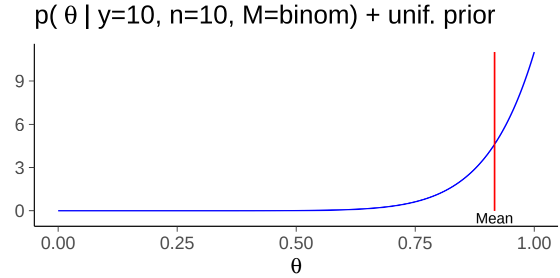


Benefits of integration and prior

Example: $n = 10, y = 10$ - uniform vs Beta(2,2) prior



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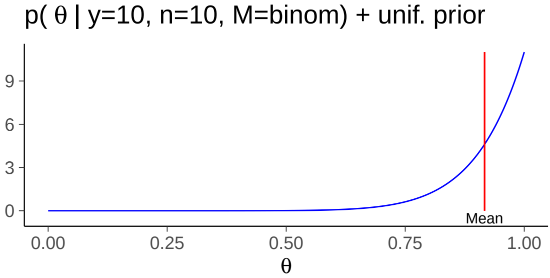

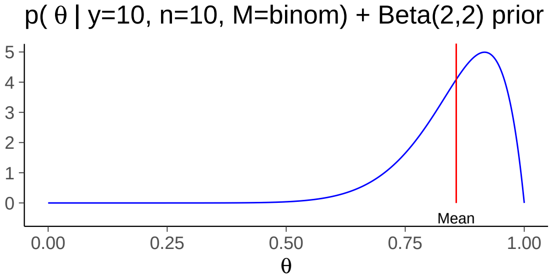

Beta prior for Binomial model

- Posterior

$$p(\theta|y, n, M) = \mathrm{Beta}(\theta|\alpha + y, \beta + n - y)$$

- Posterior mean

$$\mathrm{E}[\theta|y] = \frac{\alpha + y}{\alpha + \beta + n}$$

  - combination prior and likelihood information
  - when $n \to \infty$, $\mathrm{E}[\theta|y] \to y/n$

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- Posterior variance

$$\mathrm{Var}[\theta|y] = \frac{\mathrm{E}[\theta|y](1 - \mathrm{E}[\theta|y])}{\alpha + \beta + n + 1}$$

  - decreases when n increases
  - when $n \to \infty$, $\mathrm{Var}[\theta|y] \to 0$

Noninformative prior, proper and improper prior

- Vague, flat, diffuse, or noninformative
  - try to “to let the data speak for themselves”
  - flat is not non-informative
  - flat can be stupid
  - making prior flat somewhere can make it non-flat somewhere else
- Proper prior has $\int p(\theta) = 1$
- Improper prior density doesn’t have a finite integral
  - the posterior can still sometimes be proper

Weakly informative priors

- Weakly informative priors produce computationally better behaving posteriors
  - quite often there's at least some knowledge about the scale
  - useful also if there's more information from previous observations, but not certain how well that information is applicable in a new case uncertainty

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- Construction
  - Start with some version of a noninformative prior distribution and then add enough information so that inferences are constrained to be reasonable.
  - Start with a strong, highly informative prior and broaden it to account for uncertainty in one's prior beliefs and in the applicability of any historically based prior distribution to new data.
- Stan team prior choice recommendations https://github.com/stan-dev/stan/wiki/Prior-Choice-Recommendations

Example of informative prior

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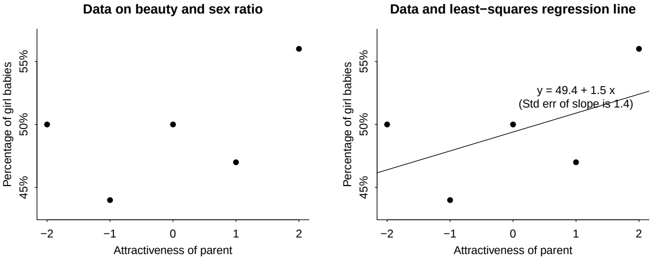

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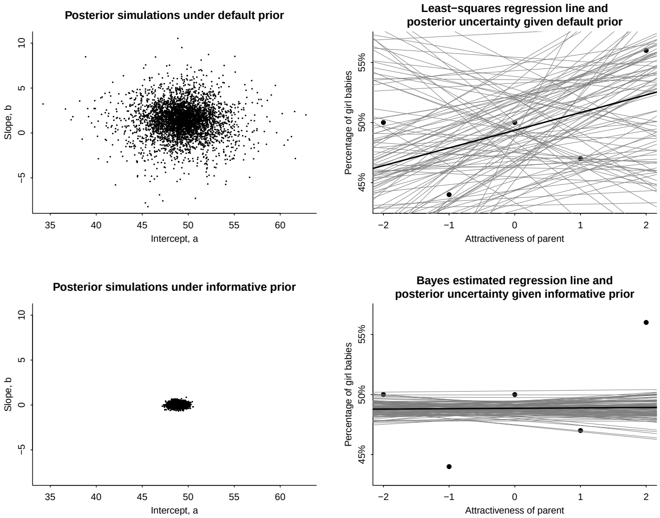

Example of weakly informative prior

- Gabry et al (2019). Visualization in Bayesian workflow.
  - Estimation of human exposure to air pollution from particulate matter measuring less than 2.5 microns in diameter ($PM_{2.5}$)
  - A recent report estimated that $PM_{2.5}$ is responsible for three million deaths worldwide each year (Shaddick et al, 2017)

Example of weakly informative prior

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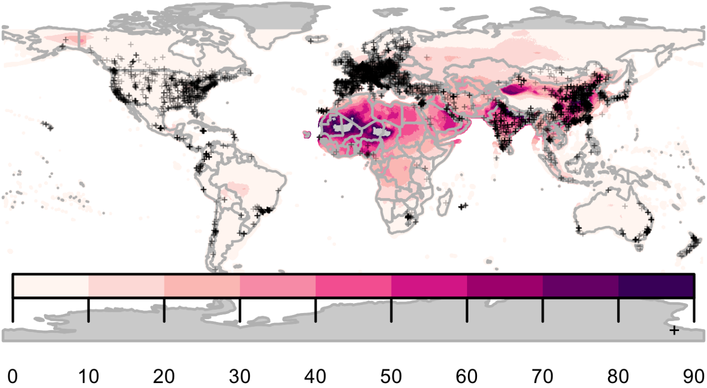

Satellite estimates and ground monitor locations

Example of weakly informative prior

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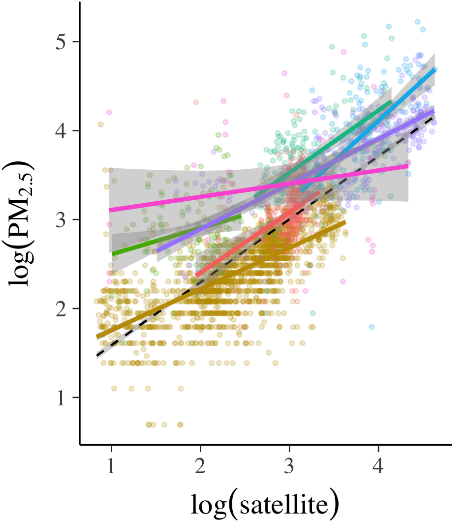

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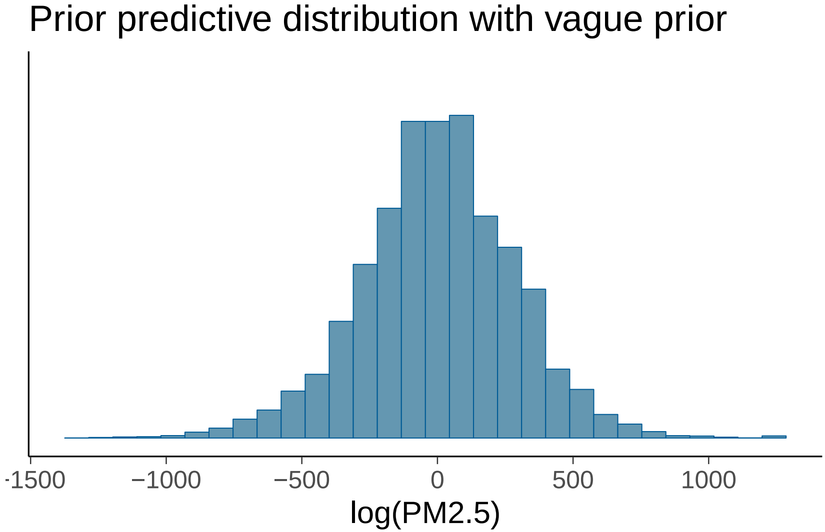

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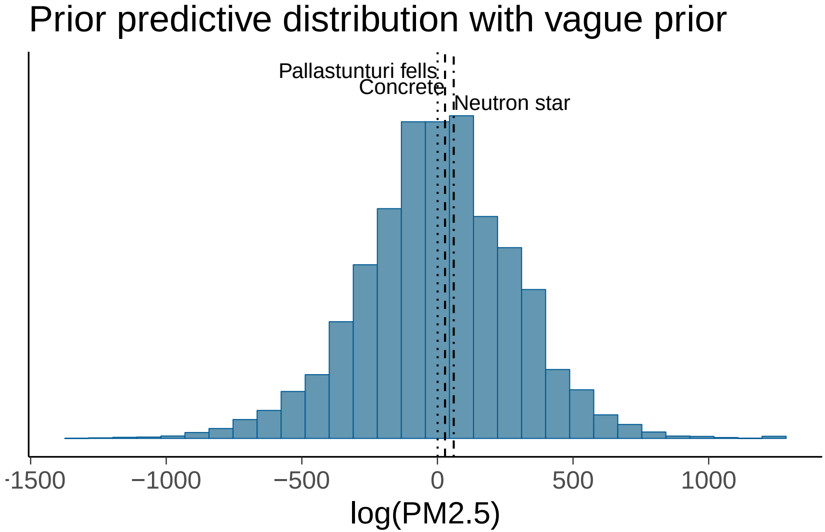

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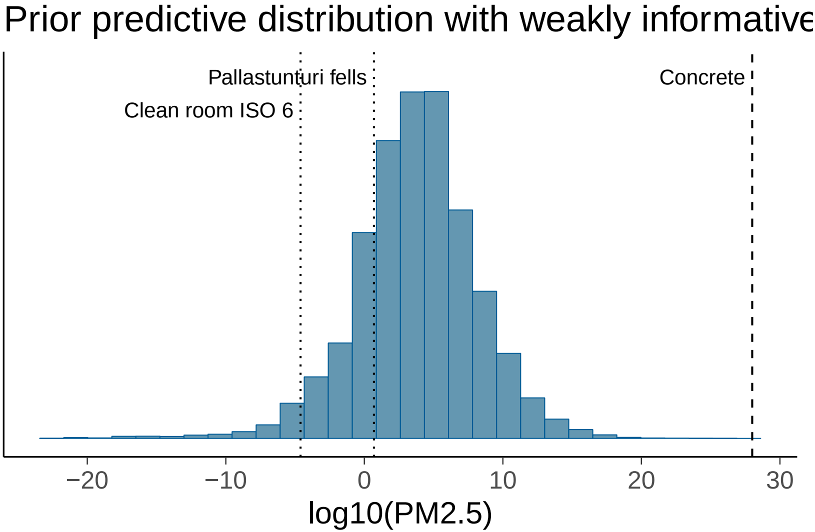
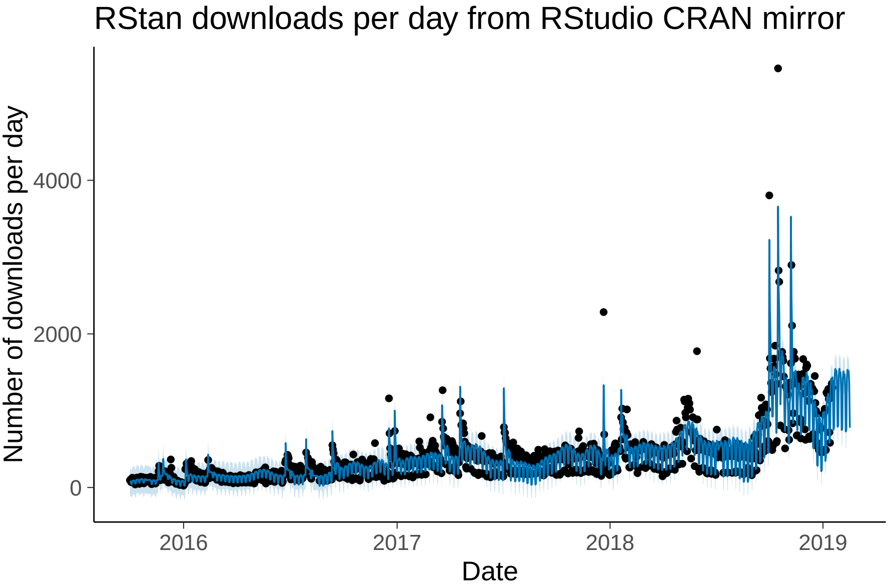


Effect of incorrect priors?

- Introduce bias, but often still produce smaller estimation error because the variance is reduced
  - bias-variance tradeoff

Structural information in predicting future



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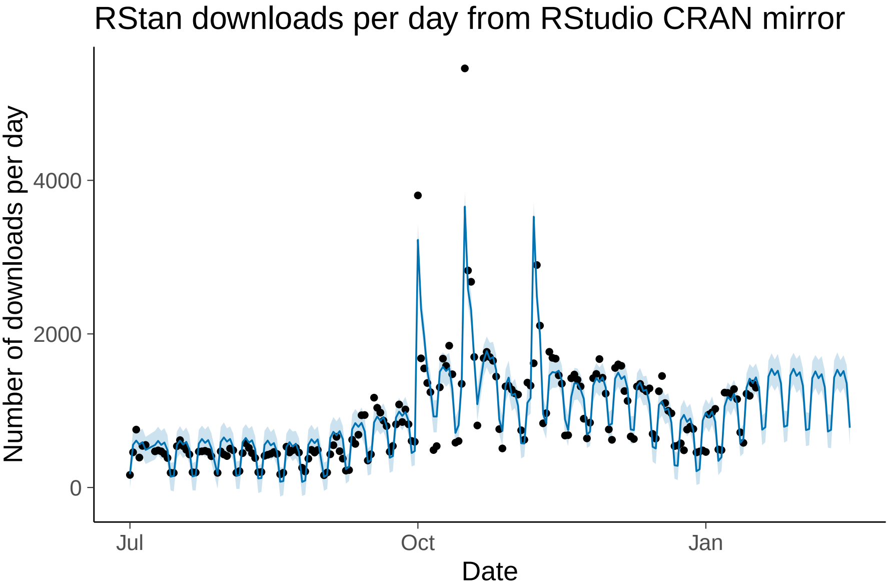
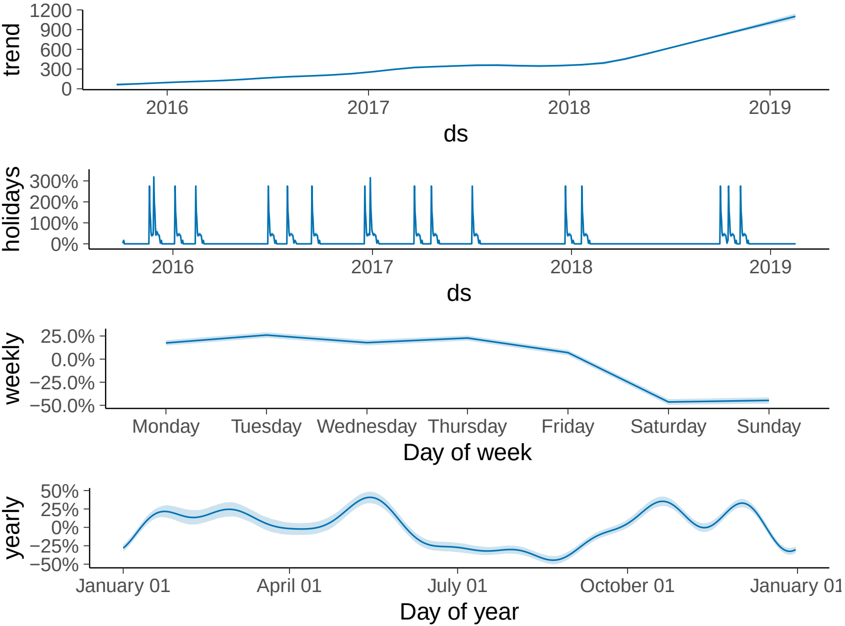


Structural information – Prophet by Facebook

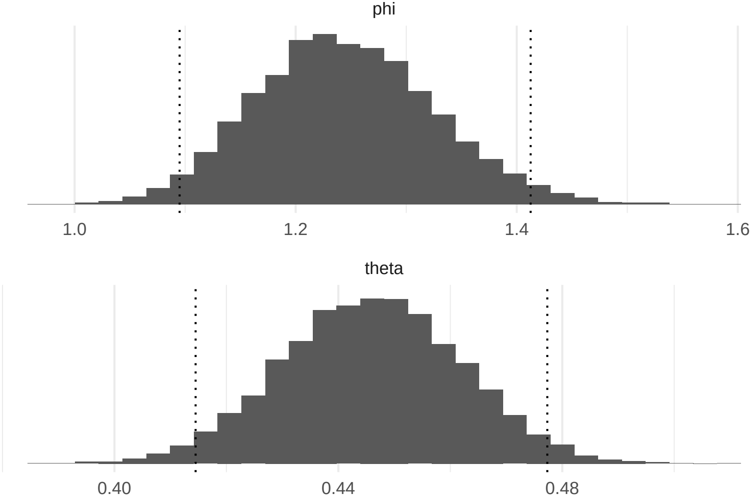


Sufficient statistics*

- The quantity $t(y)$ is said to be a *sufficient statistic* for θ, because the likelihood for θ depends on the data y only through the value of $t(y)$.

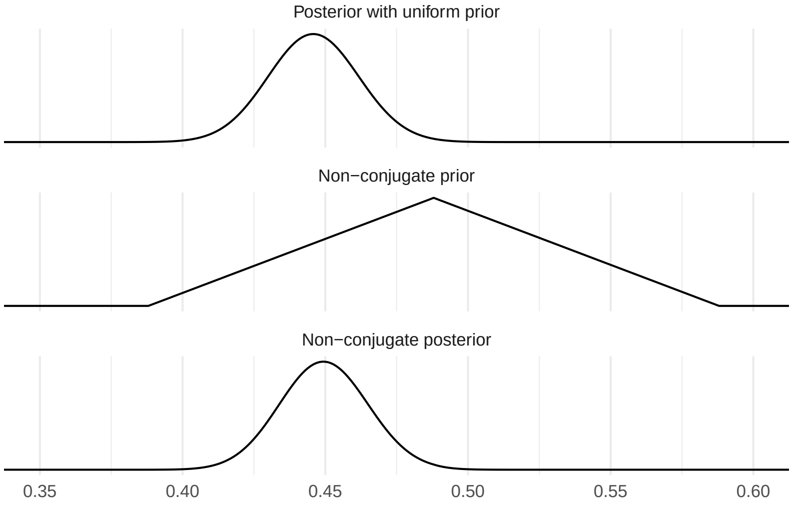
Posterior visualization and inference demos

- demo2_3: Simulate samples from Beta(438,544), and draw a histogram of θ with quantiles.



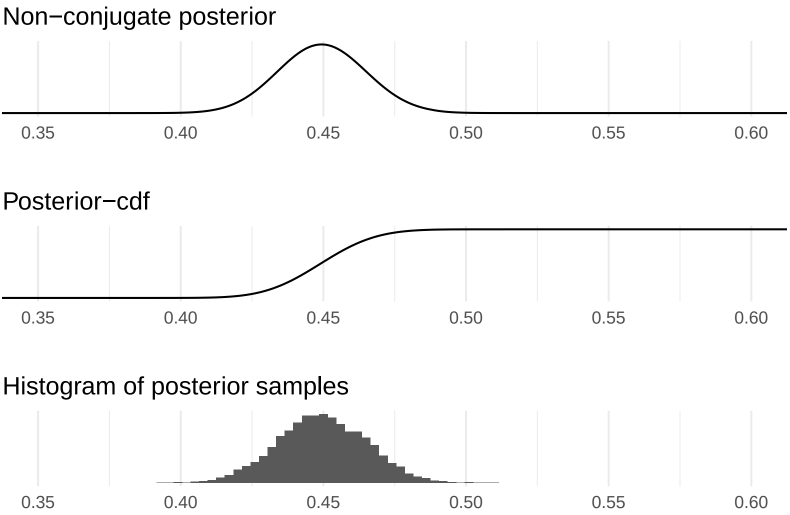
Posterior visualization and inference demos

- demo2_4: Compute posterior distribution in a grid.



Posterior visualization and inference demos

- demo2_4: Sample using the inverse-cdf method.



Algae

Assignment

Algae status is monitored in 274 sites at Finnish lakes and rivers. The observations for the 2008 algae status at each site are presented in file algae.mat ('0': no algae, '1': algae present). Let π be the probability of a monitoring site having detectable blue-green algae levels.

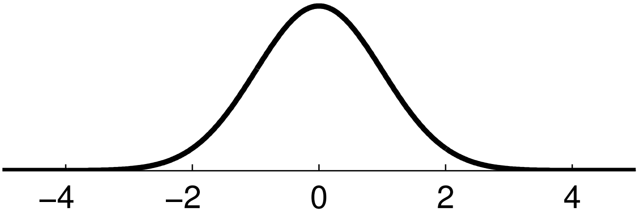
- Use a binomial model for observations and a beta(2,10) prior.
- What can you say about the value of the unknown π?
- Experiment how the result changes if you change the prior.

Normal / Gaussian

- Observations y real valued
- Mean θ and variance σ^2 (or deviation σ)
  This week assume σ^2 known (preparing for the next week)

$$p(y|\theta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right)$$

$$y \sim \mathrm{N}(\theta, \sigma^2)$$



Reasons to use Normal distribution

- Normal distribution often justified based on central limit theorem
- More often used due to the computational convenience or tradition

Central limit theorem*

- De Moivre, Laplace, Gauss, Chebysev, Liapounov, Markov, et al.
- Given certain conditions, distribution of sum (and mean) of random variables approach Gaussian distribution as $n \to \infty$
- Problems
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  - does not hold for distributions with infinite variance, e.g., Cauchy
  - may require large n,
    e.g. Binomial, when θ close to 0 or 1
  - does not hold if one the variables has much larger scale

Normal distribution - conjugate prior for θ

- Assume σ^2 known

Likelihood
$$p(y|\theta) \propto \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right)$$

Prior
$$p(\theta) \propto \exp\left(-\frac{1}{2\tau_0^2}(\theta-\mu_0)^2\right)$$

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Posterior
$$p(\theta|y) \propto \exp\left(-\frac{1}{2}\left[\frac{(y-\theta)^2}{\sigma^2} + \frac{(\theta-\mu_0)^2}{\tau_0^2}\right]\right)$$

Normal distribution - conjugate prior for θ

- Posterior (see ex 2.14a)

$$p(\theta|y) \propto \exp\left(-\frac{1}{2}\left[\frac{(y-\theta)^2}{\sigma^2}+\frac{(\theta-\mu_0)^2}{\tau_0^2}\right]\right)$$
$$\propto \exp\left(-\frac{1}{2\tau_1^2}(\theta-\mu_1)^2\right)$$

$$\theta|y \sim \mathrm{N}(\mu_1, \tau_1^2), \quad \text{where} \quad \mu_1 = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{1}{\sigma^2}y}{\frac{1}{\tau_0^2}+\frac{1}{\sigma^2}} \quad \text{and} \quad \frac{1}{\tau_1^2} = \frac{1}{\tau_0^2}+\frac{1}{\sigma^2}$$

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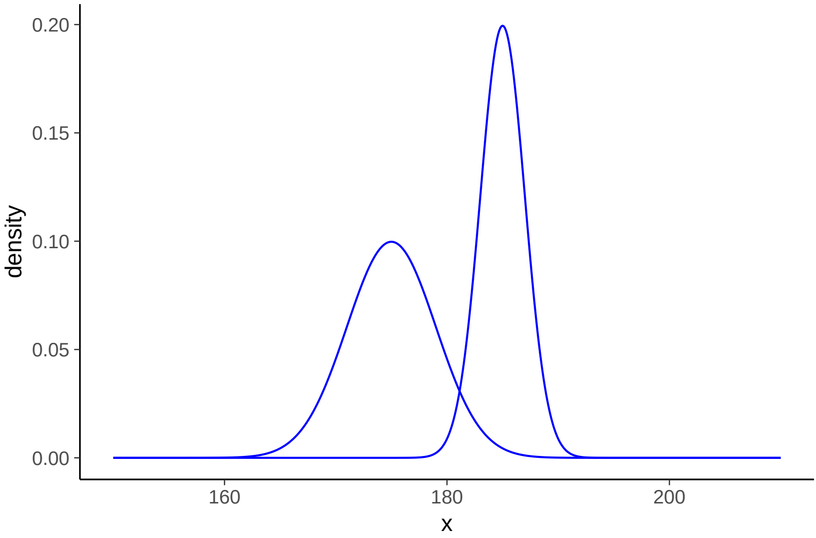
- 1/variance = precision
- Posterior precision = prior precision + data precision
- Posterior mean is precision weighted mean

Normal distribution - example



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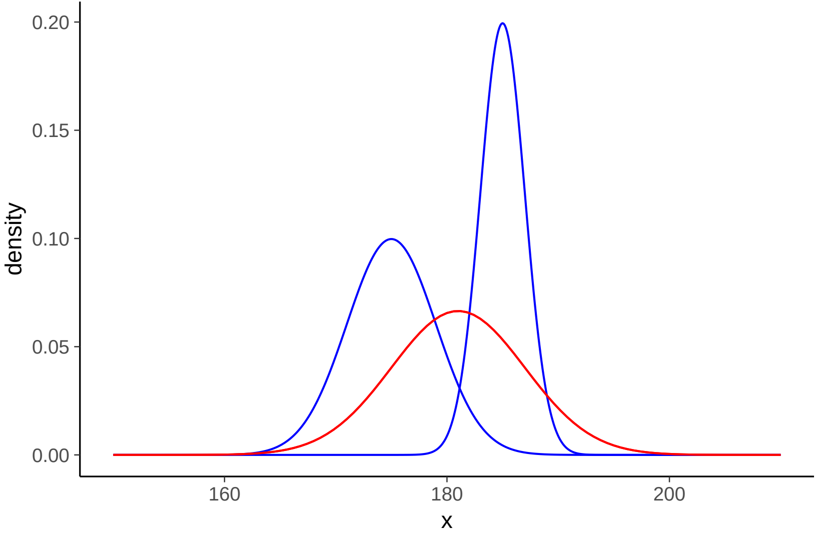

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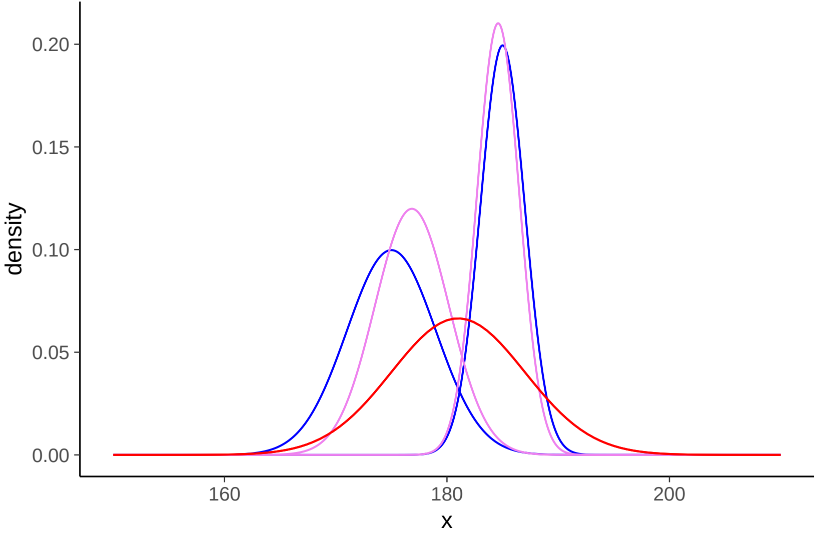

Normal distribution - conjugate prior for θ

- Several observations – use chain rule

Normal distribution - conjugate prior for θ

- Several observations $y = (y_1, \ldots, y_n)$

$$p(\theta|y) = \mathrm{N}(\theta|\mu_n, \tau_n^2)$$

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- If $\tau_0^2 = \sigma^2$, prior corresponds to one virtual observation with value μ_0
- If $\tau_0 \to \infty$ when n fixed
  or if $n \to \infty$ when τ_0 fixed

$$p(\theta|y) \approx \mathrm{N}(\theta|\bar{y}, \sigma^2/n)$$

Normal distribution - conjugate prior for θ

- Posterior predictive distribution

$$p(\tilde{y}|y) = \int p(\tilde{y}|\theta)p(\theta|y)d\theta$$

$$p(\tilde{y}|y) \propto \int \exp\left(-\frac{1}{2\sigma^2}(\tilde{y}-\theta)^2\right)\exp\left(-\frac{1}{2\tau_1^2}(\theta-\mu_1)^2\right)d\theta$$

$$\tilde{y}|y \sim \mathrm{N}(\mu_1, \sigma^2 + \tau_1^2)$$

- Predictive variance = observation model variance σ^2 + posterior variance τ_1^2

Normal model

- Gets more interesting when both mean and variance are unknown
  - next week

Normal model

- Gets more interesting when both mean and variance are unknown
  - next week
- The mean can be also a function of covariates
  - normal linear regression

Some other one parameter models

- Poisson, useful for count data (e.g. in epidemiology)
- Exponential, useful for time to an event (e.g. particle decay)