CS 437 Lecture Notes

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Fall Quarter 2025

Original lecture notes for **CS 437: Approximation Algorithms**, from Fall Quarter 2025, taught by Professor Konstantin Makarychev. This course follows ???????, ISBN ??.

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§1 September 16, 2025

I joined this class after this lecture.

§1.1 Macros

Below is an example algorithm using the macros in this repository. For simplicity, this algorithm computes the largest element of a fixed size array.

Algorithm 1.1: Algorithm to compute max(list)

```
input list
curmax ← list[0]
for n ∈ list do
    curmax ← max(n, curmax)
return curmax
```

There are also other environments, namely

Lemma 1.1 This is a lemma.

Proposition 1.2 and a proposition.

Definition 1.3 and a definition.

Example 1.4 These boxes are for examples.

Note These boxes are sparingly used, for asides.

Theorem 1.5 And finally, we've got the theorem.

As is standard, we can use the proof environment for proofs.

Proof. Trivial. □

§1.2 Set Cover

Definition 1.6 Set Cover Let V be some universe, with $|V| = n$. Let

$$S_1, \ldots, S_m \subseteq V \tag{1.1}$$

such that $\bigcup_i S_i = V$. Select the smallest $I \subseteq \{1, \ldots, m\}$ such that $\bigcup_{i \in I} S_i = V$.

Example 1.7 Let $V \equiv \{1,2,3,4,5\}$ and sets be pairs $\{i,j\}$ such that $i \neq j$. Then, an optimal solution is

$$I \equiv \{\{1,2\},\{3,4\},\{1,5\}\} \tag{1.2}$$

In this case, $\mathrm{opt}(I) = 3$

Definition 1.8 The approximation factor of an algorithm is α_n if for every I of size n, we have

$$\mathrm{alg}(I) \leq \alpha_n \cdot \mathrm{opt}(I) \tag{1.3}$$

The first theorem of this course is

Theorem 1.9 There exists a polynomial time algorithm with approximation factor $\log n$.

Algorithm 1.2: Polynomial time set cover approximation algorithm

$U_0 \leftarrow V$ `// set of not yet covered elements in` V
$t \leftarrow 0$ `// iteration counter`
for $U_t \neq \emptyset$ **do**
  Select S_i from sets that maximises $|S_i \cap U_t|$
  Include S_i in soln
  $U_t \leftarrow U_t \setminus S_i$
  $t \leftarrow t + 1$
return *soln*

1.2.1 Proof

Let $k = \mathrm{opt}$ be the number of sets in the optimal solution. Let S_{i_1} be the first selected set. Then,

$$|S_{i_1}| \geq \frac{n}{k} \tag{1.4}$$

Then it follows that

$$|U_1| = \left|\underbrace{U_0}_{V} \setminus S_{i_1}\right| = \underbrace{|U_0|}_{n} - |S_{i_1}| \tag{1.5}$$

$$\leq n - \frac{n}{k} = n\left(1 - \frac{1}{k}\right) \tag{1.6}$$

Let $S_{i_{t+1}}$ be the set chosen at iteration t.

Lemma 1.10

$$\bigcup_{i \in I^*} S_i \cap U_t = U_t \tag{1.7}$$

Proof. We can prove that LHS $\subseteq$ RHS and RHS $\subseteq$ LHS. To prove the first,

$$u \in \bigcup_{i \in I^*} S_i \cap U_t \implies u \in \text{at least one } S_i \cap U_t \implies u \in U_t \tag{1.8}$$

Thus, every element in one of the chosen sets' intersection with U_t is in U_t.

$$\begin{aligned} u \in U_t &\implies u \in \text{at least one } S_i && I^* \text{ spans universe; } S_i \text{ must exist} && (1.9)\\ &\implies u \in \text{at least one } S_i \cap U_t && && (1.10)\\ &\implies u \in \bigcup_{i \in I^*} S_i \cap U_t && && (1.11) \end{aligned}$$

And, every element in U_t is in at least one set. $\square$

It follows that, because S_i are not necessarily disjoint sets,

$$\sum_{i \in I^*} |S_i \cap U_t| \geq |U_t| \tag{1.12}$$

Thus, given there are k sets in I^*, by pigeonhole,

$$\exists i \qquad |S_i \cap U_t| \geq \frac{|U_t|}{k} \tag{1.13}$$

Then,

$$\begin{aligned} |U_{t+1}| &= \left| U_t \setminus (S_{i_{t+1}} \cap U_t) \right| && (1.14)\\ &= |U_t| - \left| S_{i_{t+1}} \cap U_t \right| && (1.15)\\ &\leq |U_t| - \frac{|U_t|}{k} = \left(1 - \frac{1}{k}\right) |U_t| && (1.16) \end{aligned}$$

Trivially,

$$|U_t| \leq \left(1 - \frac{1}{k}\right)^t \cdot n \tag{1.17}$$

Proposition 1.11 For $t = k \log n$,

$$\left(1 - \frac{1}{k}\right)^t < \frac{1}{n} \tag{1.18}$$

§2 September 18, 2025