1 (a)

$$\frac{\partial E(w)}{\partial w_j} = \sum_{i=1}^{N}(x_j)^{(i)}y^{(i)}(1-\sigma(w^Tx^{(i)})) - (x_j)^{(i)}\sigma(w^Tx^{(i)}) + (x_j)^{(i)}y^{(i)}\sigma(w^Tx^{(i)})$$

$$= \sum_{i=1}^{N}(x_j)^{(i)}y^{(i)} - (x_j)^{(i)}\sigma(w^Tx^{(i)})$$

$$\frac{\partial^2 E(w)}{\partial (w_j)^2} = -\sum_{i=1}^{N}(x_j)^{(i)}\sigma(w^Tx^{(i)})(1-\sigma(w^Tx^{(i)}))(x_j)^{(i)}$$

$$\frac{\partial^2 E(w)}{\partial w_j w_k} = -\sum_{i=1}^{N}(x_j)^{(i)}\sigma(w^Tx^{(i)})(1-\sigma(w^Tx^{(i)}))(x_k)^{(i)}$$

Let $X \in \mathbb{R}^{n\times m}$ be the design matrix and $w \in \mathbb{R}^m$ be the weight vector, where n is the number of observations and m is the number of features. Let $\odot$ express the Hadamard product of its operands. We can express the second-order partial derivatives in matrix form as

$$X^T diag[-\sigma(Xw)\odot(1-\sigma(Xw))]X$$

which gives us our Hessian matrix.

1 (b) Let $x, z \in \mathbb{R}^m$. Then

$$(x^Tz)(x^Tz) = \sum_{i=1}^{N} z_i x_i \sum_{j=1}^{N} x_j z_j = \sum_{i=1}^{N}\sum_{j=1}^{N} z_i x_i x_j z_j = (x^Tz)^2$$

Consider

$$z^TX^T diag[\sigma(Xw)\odot(1-\sigma(Xw))]Xz$$

Let D represent $X^T diag[\sigma(Xw)(1-\sigma(Xw))]X$. $D \in \mathbb{R}^{m\times m}$. Let D_i be a column of D and $D^{(j)}$ be a row of D. we can now express the above as

$$-\sum_{i=1}^{m}[\sum_{j=1}^{m} z_j (D_i)^{(j)}]z_i$$

$$= -\sum_{i=1}^{m}[\sum_{j=1}^{m} z_j((D_i)^{(j)})^{\frac{1}{2}}((D_i)^{(j)})^{\frac{1}{2}}]z_i = -\sum_{i=1}^{m}\sum_{j=1}^{m} z_j((D_i)^{(j)})^{\frac{1}{2}}((D_i)^{(j)})^{\frac{1}{2}}z_i$$

Since in general $\sum_{i=1}^{N}\sum_{j=1}^{N} z_i x_i x_j z_j = (x^T z)^2 \geq 0$,

$$\sum_{i=1}^{m}\sum_{j=1}^{m} z_j((D_i)^{(j)})^{\frac{1}{2}}((D_i)^{(j)})^{\frac{1}{2}} z_i \geq 0$$

therefore

$$-\sum_{i=1}^{m}[\sum_{j=1}^{m} z_j(D_i)^{(j)}]z_i \leq 0 \square$$

The Hessian matrix is negative semi-definite, and our original log-likelihood function is concave.

1 (c) We can invert the sign of the log-lihelihood function to get a loss function. This gives us the Hessian matrix $X^T diag[\sigma(Xw) \odot (1 - \sigma(Xw))]X$. For the gradient of the loss function with respect to w, we can take the partial derivative expression from **1 (a)**

$$\sum_{i=1}^{N}(x_j)^{(i)}y^{(i)} - (x_j)^{(i)}\sigma(w^T x^{(i)})$$

invert the sign, and express it in matrix form:

$$\nabla_w E = X^T\sigma(Xw) - X^T y$$

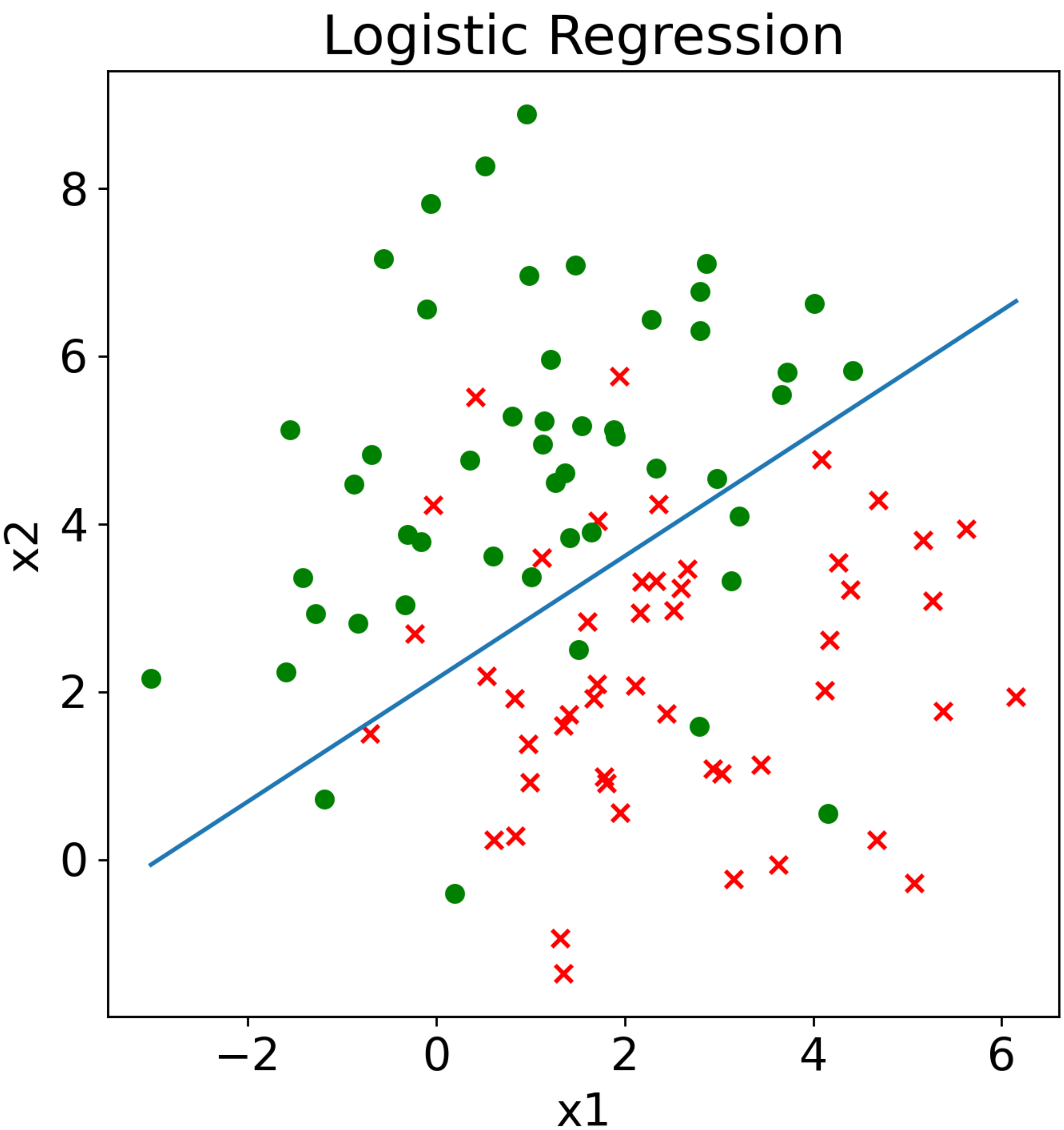
So the update for Newton's method is

$$w_{new} = w_{old} - H^{-1}\nabla_w E$$

$$w_{new} = w_{old} - \{X^T diag[\sigma(Xw) \odot (1 - \sigma(Xw))]X\}^{-1}[X^T\sigma(Xw) - X^T y]$$

1 (e) The weights from the logistic regression are
w_0: -1.84922892 w_1: -0.62814188 w_2: 0.85846843

1 (f)



2 (a) If we consider only the case where $k = w_m$, we can express the log-likelihood as

$$\sum_{i=1}^{N} log([p(y^{(i)} = m|x^{(i)}, w)]^{\mathbb{I}\{y^{(i)}=m\}})$$

$$\sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = m\} log([\frac{exp[w_m^T \phi(x^{(i)})]}{\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})]}])$$

If we look only at this case, then the corresponding part of $\nabla w_m l(w)$ is

$$\sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = m\} \frac{\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})]}{exp[w_m^T \phi(x^{(i)})]} [\frac{exp[w_m^T \phi(x^{(i)})]\phi(x^{(i)})}{\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})]} - \frac{(exp[w_m^T \phi(x^{(i)})])^2 \phi(x^{(i)})}{(\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})])^2}]$$

$$= \sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = m\} [\frac{\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})] exp[w_m^T \phi(x^{(i)})]\phi(x^{(i)})}{exp[w_m^T \phi(x^{(i)})] \sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})]}$$

$$- \frac{\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})] (exp[w_m^T \phi(x^{(i)})])^2 \phi(x^{(i)})}{exp[w_m^T \phi(x^{(i)})] (\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})])^2}]$$

$$= \sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = m\} [\phi(x^{(i)}) - \frac{(exp[w_m^T \phi(x^{(i)})])\phi(x^{(i)})}{(\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})])}]$$

$$= \sum_{i=1}^{N} \phi(x^{(i)}) \mathbb{I}\{y^{(i)} = m\} [1 - \frac{(exp[w_m^T \phi(x^{(i)})])}{(\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})])}]$$

Now, let us look at the case where $k \neq m$. Here, the corresponding part of $\nabla w_m l(w)$ is

$$\sum^{k \neq m} \mathbb{I}\{y^{(i)} = k\} \frac{\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})]}{exp[w_k^T \phi(x^{(i)})]} [- \frac{exp[w_k^T \phi(x^{(i)})] exp[w_m^T \phi(x^{(i)})]\phi(x^{(i)})}{[\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})]]^2}]$$

$$= - \sum^{k \neq m} \mathbb{I}\{y^{(i)} = k\} [\frac{exp[w_m^T \phi(x^{(i)})]\phi(x^{(i)})}{[\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})]]}]$$

If we incorporate both cases, we get

$$\sum_{i=1}^{N} \phi(x^{(i)}) \mathbb{I}\{y^{(i)} = m\} [1 - \frac{(exp[w_m^T \phi(x^{(i)})])}{(\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})])}] - \sum^{k \neq m} \mathbb{I}\{y^{(i)} = k\} [\frac{exp[w_m^T \phi(x^{(i)})]\phi(x^{(i)})}{[\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})]]}]$$

$$= \sum_{i=1}^{N} \phi(x^{(i)}) \mathbb{I}\{y^{(i)} = m\}[1 - \frac{(exp[w_m^T \phi(x^{(i)})])}{(\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})])}] - [1 - \mathbb{I}\{y^{(i)} = m\}][\frac{exp[w_m^T \phi(x^{(i)})] \phi(x^{(i)})}{[\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})]]}]$$

$$= \sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = m\} \phi(x^{(i)}) - \frac{\mathbb{I}\{y^{(i)} = m\} exp[w_m^T \phi(x^{(i)})])(\phi(x^{(i)})}{(\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})])}]$$

$$- [\frac{exp[w_m^T \phi(x^{(i)})] \phi(x^{(i)})}{[\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})]]}] + \frac{\mathbb{I}\{y^{(i)} = m\}] exp[w_m^T \phi(x^{(i)})] \phi(x^{(i)})}{[\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})]]}$$

$$= \sum_{i=1}^{N} \phi(x^{(i)}) [\mathbb{I}\{y^{(i)} = m\} - [\frac{exp[w_m^T \phi(x^{(i)})]}{[\sum_{j=1}^{K} exp[w_j^T \phi(x^{(i)})]]}]]$$

Because w_K is fixed at 0, we can write this as

$$\sum_{i=1}^{N} \phi(x^{(i)}) [\mathbb{I}\{y^{(i)} = m\} - [\frac{exp[w_m^T \phi(x^{(i)})]}{[1 + \sum_{j=1}^{K-1} exp[w_j^T \phi(x^{(i)})]]}]] \square$$

So the update rule is

$$w_m* = w_m + \alpha \sum_{i=1}^{N} \phi(x^{(i)}) [\mathbb{I}\{y^{(i)} = m\} - \frac{(exp[w_m^T \phi(x^{(i)})])}{[1 + \sum_{j=1}^{K-1} exp[w_j^T \phi(x^{(i)})]]}]$$

2 (c) Accuracy on the test set is 94%.

3 (a) With Bayes' theorem, we can express the posterior probability of the Gaussian discriminant analysis for $y = 1$ as

$$p(y=1|x;\phi,\Sigma,m_0,m_1) = \frac{p(x|y=1)p(y=1)}{p(x|y=1)p(y=1)+p(x|y=0)p(y=0)}$$

$$= \frac{\phi\frac{1}{2\pi^{\frac{M}{2}}|\Sigma|^{\frac{1}{2}}}exp[-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)]}{\phi\frac{1}{2\pi^{\frac{M}{2}}|\Sigma|^{\frac{1}{2}}}exp[-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)]+(1-\phi)\frac{1}{2\pi^{\frac{M}{2}}|\Sigma|^{\frac{1}{2}}}exp[-\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0)]}$$

$$= \frac{\phi\ exp[-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)]}{\phi\ exp[-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)]+(1-\phi)\ exp[-\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0)]}$$

$$= \frac{1}{1+\frac{(1-\phi)\ exp[-\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0)]}{\phi\ exp[-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)]}}$$

Consider

$$\frac{(1-\phi)\ exp[-\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0)]}{\phi\ exp[-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)]}$$

$$= \frac{1-\phi}{\phi}exp[-\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0)+\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)]$$

$$= \frac{1-\phi}{\phi}exp[-\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0)+\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)]$$

$$= \frac{1-\phi}{\phi}exp[x^T\Sigma^{-1}\mu_0-\frac{1}{2}\mu_0\Sigma^{-1}\mu_0-x^T\Sigma^{-1}\mu_1+\frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1]$$

$$= exp[log(\frac{1-\phi}{\phi})]exp[x^T\Sigma^{-1}\mu_0-\frac{1}{2}\mu_0\Sigma^{-1}\mu_0-x^T\Sigma^{-1}\mu_1+\frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1]$$

$$= exp[log(\frac{1-\phi}{\phi})+x^T\Sigma^{-1}\mu_0-\frac{1}{2}\mu_0\Sigma^{-1}\mu_0-x^T\Sigma^{-1}\mu_1+\frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1]$$

$$= exp[-(x^T(\Sigma^{-1}\mu_1-\Sigma^{-1}\mu_0)+\frac{1}{2}(\mu_0\Sigma^{-1}\mu_0-\mu_1^T\Sigma^{-1}\mu_1)-log(\frac{1-\phi}{\phi}))]$$

By incorporating this into our expression for $p(y=1|x;\phi,\Sigma,m_0,m_1)$ above, we get

$$\frac{1}{1+exp[-(x^T(\Sigma^{-1}\mu_1-\Sigma^{-1}\mu_0)+\frac{1}{2}(\mu_0\Sigma^{-1}\mu_0-\mu_1^T\Sigma^{-1}\mu_1)-log(\frac{1-\phi}{\phi}))]}$$

If we let w represent a function of ϕ, Σ, μ_0, and μ_1, then

$$p(y=1|x;\phi,\Sigma,m_0,m_1) = \frac{1}{1+exp[-w^T x]} \square$$

3 (b) $\ell(\phi, m_0, m_1, \Sigma) =$

$$\sum_{i=1}^{N} log[\frac{1}{2\pi^{\frac{M}{2}}|\Sigma|^{\frac{1}{2}}} exp[-\frac{1}{2}(x^{(i)} - \mu_{y_{(i)}})^T \Sigma^{-1} (x^{(i)} - \mu_{y_{(i)}})] \phi^{y^{(i)}} (1-\phi)^{1-y^{(i)}})]$$

$$\frac{\partial}{\partial \phi} \ell(\phi, \mu_0, \mu_1, \Sigma) =$$

$$\sum_{i=1}^{N} \frac{1}{\frac{1}{2\pi^{\frac{M}{2}}|\Sigma|^{\frac{1}{2}}} exp[-\frac{1}{2}(x^{(i)} - \mu_{y_{(i)}})^T \Sigma^{-1} (x^{(i)} - \mu_{y_{(i)}})] \phi^{y^{(i)}} (1-\phi)^{1-y^{(i)}}}$$

$$\frac{1}{2\pi^{\frac{M}{2}}|\Sigma|^{\frac{1}{2}}} exp[-\frac{1}{2}(x^{(i)} - \mu_{y_{(i)}})^T \Sigma^{-1} (x^{(i)} - \mu_{y_{(i)}})]$$

$$[(1-\phi)^{1-y^{(i)}} y^{(i)} \phi^{y^{(i)}-1} - \phi^{y^{(i)}} (1-y^{(i)})(1-\phi)^{-y^{(i)}}]$$

$$= \sum_{i=1}^{N} y^{(i)} \phi^{-1} - \frac{(1-y^{(i)})(1-\phi)^{-y^{(i)}}}{(1-\phi)^{1-y^{(i)}}}$$

$$= \sum_{i=1}^{N} \frac{y^{(i)}}{\phi} - \frac{1-y^{(i)}}{1-\phi}$$

$$= \sum_{i=1}^{N} \frac{y^{(i)}(1-\phi)}{\phi(1-\phi)} - \frac{\phi - y^{(i)}\phi}{\phi(1-\phi)}$$

$$0 = \sum_{i=1}^{N} \frac{y^{(i)}}{\phi(1-\phi)} - \frac{\phi}{\phi(1-\phi)}$$

$$N\phi = \sum_{i=1}^{N} y^{(i)}$$

$$\phi = \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = 1\} \square$$

$$\frac{\partial}{\partial \mu_0}\ell(\phi, \mu_0, \mu_1, \Sigma) =$$

$$\sum_{i=1}^{N} \frac{1}{\frac{1}{2\pi^{\frac{M}{2}}|\Sigma|^{\frac{1}{2}}} exp[-\frac{1}{2}(x^{(i)} - \mu_{y_{(i)}})^T \Sigma^{-1}(x^{(i)} - \mu_{y_{(i)}})]\phi^{y^{(i)}}(1-\phi)^{1-y^{(i)}}}$$

$$\frac{1}{2\pi^{\frac{M}{2}}|\Sigma|^{\frac{1}{2}}} exp[-\frac{1}{2}(x^{(i)} - \mu_{y_{(i)}})^T \Sigma^{-1}(x^{(i)} - \mu_{y_{(i)}})]\phi^{y^{(i)}}(1-\phi)^{1-y^{(i)}}$$

$$\frac{\partial}{\partial \mu_{y_{(i)}}} - \frac{1}{2}(x^{(i)} - \mu_{y_{(i)}})^T \Sigma^{-1}(x^{(i)} - \mu_{y_{(i)}})$$

$$= \frac{\partial}{\partial \mu_{y_{(i)}}} \sum_{i=1}^{N} -\frac{1}{2}(x^{(i)} - \mu_{y_{(i)}})^T \Sigma^{-1}(x^{(i)} - \mu_{y_{(i)}})$$

$$= \frac{\partial}{\partial \mu_{y_{(i)}}} \sum_{i=1}^{N} -\frac{1}{2}[x^{(i)^T} \Sigma^{-1} x^{(i)} - x^{(i)^T} \Sigma^{-1} \mu_{y_{(i)}} - \mu_{y_{(i)}}^T \Sigma^{-1} x^{(i)} + \mu_{y_{(i)}}^T \Sigma^{-1} \mu_{y_{(i)}}]$$

$$\frac{\partial}{\partial \mu_0} = 0 = \sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = 0\} x^{(i)^T} \Sigma^{-1} - \mu_{y_{(i)}}^T \Sigma^{-1}$$

$$\sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = 0\} \mu_0^T = \sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = 0\} x^{(i)^T}$$

$$\mu_0 = \frac{\sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = 0\} x^{(i)}}{\sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = 0\}} \square$$

Similarly,

$$\mu_1 = \frac{\sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = 1\} x^{(i)}}{\sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = 1\}} \square$$