

1 (a)

$$\begin{aligned}
\frac{\partial E(w)}{\partial w_j} &= \sum_{i=1}^N (x_j)^{(i)} y^{(i)} (1 - \sigma(w^T x^{(i)})) - (x_j)^{(i)} \sigma(w^T x^{(i)}) + (x_j)^{(i)} y^{(i)} \sigma(w^T x^{(i)}) \\
&= \sum_{i=1}^N (x_j)^{(i)} y^{(i)} - (x_j)^{(i)} \sigma(w^T x^{(i)}) \\
\frac{\partial^2 E(w)}{\partial (w_j)^2} &= - \sum_{i=1}^N (x_j)^{(i)} \sigma(w^T x^{(i)}) (1 - \sigma(w^T x^{(i)})) (x_j)^{(i)} \\
\frac{\partial^2 E(w)}{\partial w_j \partial w_k} &= - \sum_{i=1}^N (x_j)^{(i)} \sigma(w^T x^{(i)}) (1 - \sigma(w^T x^{(i)})) (x_k)^{(i)}
\end{aligned}$$

Let $X \in \mathbb{R}^{n \times m}$ be the design matrix and $w \in \mathbb{R}^m$ be the weight vector, where n is the number of observations and m is the number of features. Let \odot express the Hadamard product of its operands. We can express the second-order partial derivatives in matrix form as

$$X^T \text{diag}[-\sigma(Xw) \odot (1 - \sigma(Xw))]X$$

which gives us our Hessian matrix.

1 (b) Let $x, z \in \mathbb{R}^m$. Then

$$(x^T z)(x^T z) = \sum_{i=1}^N z_i x_i \sum_{j=1}^N x_j z_j = \sum_{i=1}^N \sum_{j=1}^N z_i x_i x_j z_j = (x^T z)^2$$

Consider

$$z^T X^T \text{diag}[\sigma(Xw) \odot (1 - \sigma(Xw))]Xz$$

Let D represent $X^T \text{diag}[\sigma(Xw)(1 - \sigma(Xw))]X$. $D \in \mathbb{R}^{m \times m}$. Let D_i be a column of D and $D^{(j)}$ be a row of D . we can now express the above as

$$\begin{aligned}
& - \sum_{i=1}^m \left[\sum_{j=1}^m z_j (D_i)^{(j)} \right] z_i \\
&= - \sum_{i=1}^m \left[\sum_{j=1}^m z_j ((D_i)^{(j)})^{\frac{1}{2}} ((D_i)^{(j)})^{\frac{1}{2}} \right] z_i = - \sum_{i=1}^m \sum_{j=1}^m z_j ((D_i)^{(j)})^{\frac{1}{2}} ((D_i)^{(j)})^{\frac{1}{2}} z_i
\end{aligned}$$

Since in general $\sum_{i=1}^N \sum_{j=1}^N z_i x_i x_j z_j = (x^T z)^2 \geq 0$,

$$\sum_{i=1}^m \sum_{j=1}^m z_j ((D_i)^{(j)})^{\frac{1}{2}} ((D_i)^{(j)})^{\frac{1}{2}} z_i \geq 0$$

therefore

$$- \sum_{i=1}^m [\sum_{j=1}^m z_j (D_i)^{(j)}] z_i \leq 0 \quad \square$$

The Hessian matrix is negative semi-definite, and our original log-likelihood function is concave.

1 (c) We can invert the sign of the log-likelihood function to get a loss function. This gives us the Hessian matrix $X^T \text{diag}[\sigma(Xw) \odot (1 - \sigma(Xw))]X$. For the gradient of the loss function with respect to w , we can take the partial derivative expression from **1 (a)**

$$\sum_{i=1}^N (x_j)^{(i)} y^{(i)} - (x_j)^{(i)} \sigma(w^T x^{(i)})$$

invert the sign, and express it in matrix form:

$$\nabla_w E = X^T \sigma(Xw) - X^T y$$

So we the update for Newton's method is

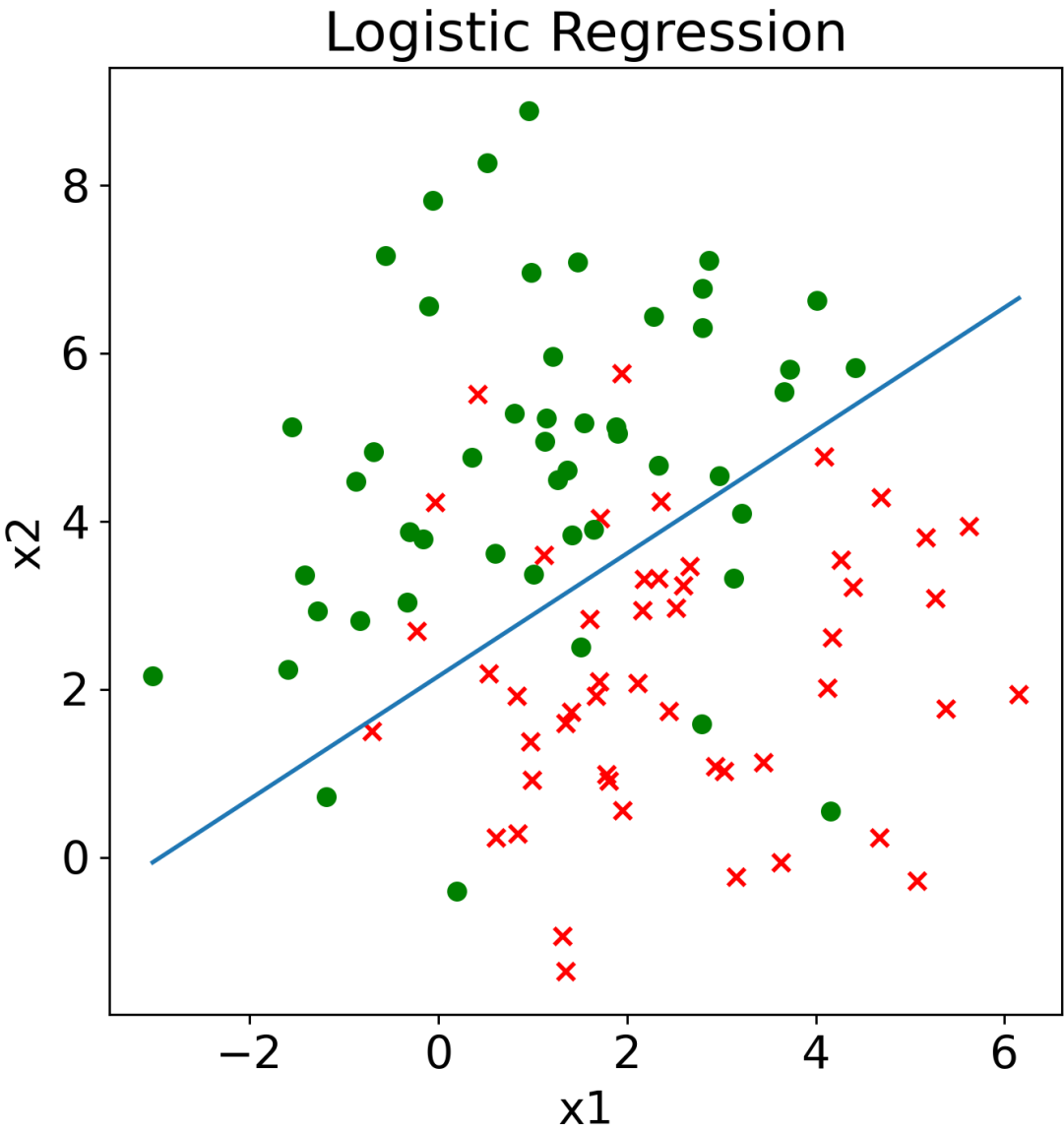
$$w_{new} = w_{old} - H^{-1} \nabla_w E$$

$$w_{new} = w_{old} - \{X^T \text{diag}[\sigma(Xw) \odot (1 - \sigma(Xw))]X\}^{-1} [X^T \sigma(Xw) - X^T y]$$

1 (e) The weights from the logistic regression are

w_0 : -1.84922892 w_1 : -0.62814188 w_2 : 0.85846843

1 (f)



2 (a) If we consider only w_m , we can express the log-likelihood as

$$\begin{aligned}
& \sum_{i=1}^N \log([p(y^{(i)} = m | x^{(i)}, w)]^{\mathbb{I}\{y^{(i)}=m\}} \\
& \sum_{i=1}^N \mathbb{I}\{y^{(i)} = m\} \log\left(\left[\frac{\exp[w_m^T \phi(x^{(i)})]}{\sum_{j=1}^{K-1} \exp[w_j^T \phi(x^{(i)})]}\right]\right) \\
\nabla w_m l(w) = & \\
& \sum_{i=1}^N \mathbb{I}\{y^{(i)} = m\} \frac{\sum_{j=1}^{K-1} \exp[w_j^T \phi(x^{(i)})]}{\exp[w_m^T \phi(x^{(i)})]} \left[\frac{\exp[w_m^T \phi(x^{(i)})] \phi(x^{(i)})}{\sum_{j=1}^{K-1} \exp[w_j^T \phi(x^{(i)})]} - \frac{(\exp[w_m^T \phi(x^{(i)})])^2 \phi(x^{(i)})}{(\sum_{j=1}^{K-1} \exp[w_j^T \phi(x^{(i)})])^2} \right] \\
& = \sum_{i=1}^N \mathbb{I}\{y^{(i)} = m\} \left[\frac{\sum_{j=1}^{K-1} \exp[w_j^T \phi(x^{(i)})] \exp[w_m^T \phi(x^{(i)})] \phi(x^{(i)})}{\exp[w_m^T \phi(x^{(i)})] \sum_{j=1}^{K-1} \exp[w_j^T \phi(x^{(i)})]} \right. \\
& \quad \left. - \frac{\sum_{j=1}^{K-1} \exp[w_j^T \phi(x^{(i)})] (\exp[w_m^T \phi(x^{(i)})])^2 \phi(x^{(i)})}{\exp[w_m^T \phi(x^{(i)})] (\sum_{j=1}^{K-1} \exp[w_j^T \phi(x^{(i)})])^2} \right] \\
& = \sum_{i=1}^N \mathbb{I}\{y^{(i)} = m\} \left[\phi(x^{(i)}) - \frac{(\exp[w_m^T \phi(x^{(i)})]) \phi(x^{(i)})}{(\sum_{j=1}^{K-1} \exp[w_j^T \phi(x^{(i)})])} \right] \\
& = \sum_{i=1}^N \mathbb{I}\{y^{(i)} = m\} \phi(x^{(i)}) \left[1 - \frac{(\exp[w_m^T \phi(x^{(i)})])}{(\sum_{j=1}^{K-1} \exp[w_j^T \phi(x^{(i)})])} \right]
\end{aligned}$$